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Orbital Stability of Double Solitons for the Benjamin-Ono Equation

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Abstract: In this paper we prove the orbital stability of double solitons for the Benjamin-Ono equation. In the case of the KdV equation, this stability has been proved in [17]. Parts of the proof given there rely on the fact that the Euler-Lagrange equations for the conserved quantities of the KdV equation are ordinary differential equations. Since this is not the case for the Benjamin-Ono equation, new methods are required. Our approach consists in using a new invariant for multi-solitons, and certain new identities motivated by the Sylvester Law of Inertia.

1. Introduction

A common feature of integrable evolution equations describing nonlinear wave motion is that they have an infinite sequence of conservation quantities (first integrals) $V_1(u)$, $V_2(u)$, $V_3(u)$, ... In concrete examples, the solutions of the equation $V_2'(u) + cV_1'(u) = 0$ are one-solitons (traveling waves, standing waves). These solutions evolve without changing their shape and their stability has been proved in [2, 4 and 10] for the KdV equation and in [1] for the Benjamin-Ono (BO) equation. On the other hand, the higher order Euler-Lagrange equation

$$V_3'(u) + \alpha V_2'(u) + \beta V_1'(u) = 0 \tag{1.1}$$

gives rise to more complicated solutions called double solitons (or two-solitons). In some sense, these solutions represent the superposition of two one-solitons and their speeds are related to the multipliers α and β through an algebraic equation. As in the case of the one-soliton, to prove that the double soliton is stable we have to show that they locally minimize V_3 subject to given values of V_1 and V_2. This proof involves the spectral analysis of the one-parameter family of self-adjoint operators $L(t)$, which are the linearization of (1.1) at the double soliton.

An alternative way to construct double solitons is to use the self-adjoint operator M that appears in the Lax pair associated to the integrable equation (see [7] for the construction of the Lax pair for the Benjamin-Ono equation). The double solitons are the potentials for which the self-adjoint operator M has two eigenvalues.

Although the discreteness of the spectrum of M combined with Inverse Scattering Theory might suggest some stability property of the double solitons, the first rigorous proof of the Liapunov stability of double solitons for the KdV equation has been given in [17] using the variational approach introduced above. In this case, the linearized operator $L(t)$ is a fourth order self-adjoint linear ordinary differential operator. In [17] the authors first give spectral conditions for critical points of abstract constrained variational problems to be local minimizers. As a consequence of this abstract result, they show that the stability of the double soliton follows from the following spectral conditions:

C.1 for any t, $L(t)$ has exactly one negative eigenvalue;
C.2 for any t, zero is a double eigenvalue of $L(t)$.

Their method to show that these two conditions are indeed satisfied relies very heavily on the ODE structure of $L(t)$. More specifically, they use the concept of stable and unstable subspaces for linear equations to show that the multiplicity of zero as an eigenvalue of $L(t)$ is exactly two for any t. They also use some results from the Calculus of Variations that are related to the concept of conjugate points (for unbounded intervals) to count the number of negative eigenvalues of $L(t)$.

In this paper we prove the stability of double solitons for the BO equation. As in the case of the KdV equation, to prove this stability we have to show that the family of self-adjoint operators $L(t)$ satisfies Conditions **C.1** and **C.2** above. However, in the case of BO equation, $L(t)$ is not an ordinary differential operator anymore because the Hilbert transform appears in it. This makes the spectral analysis more complicated and then a new approach is required.

The first step of our method (which is presented in Sects. 2 and 3) consists in making a simplification in the spectral problem to reduce the spectral analysis of the one-parameter family $L(t)$ to the analysis of the spectra of two stationary operators L_1 and L_2. We now describe this simplification in more detail.

We begin by describing how the operators L_1 and L_2 are constructed. The double soliton $u(t)$ appears in the coefficient of the self-adjoint operator $L(t)$. For instance, in the case of the KdV equation, $L(t)$ is given by

$$L(t)h = h^{(4)} + 10uh_{xx} + 10u_{xx}h + 10u_xh_x + 30u^2h + \alpha(-h_{xx} - 6uh) + \beta h,$$

where $u = u(t)$ is the double soliton. Let u_1 and u_2 be the one-solitons associated to that double soliton $u(t)$ and let L_1 and L_2 be the stationary "limit operators" that we get when we replace $u(t)$ by u_1 and u_2 in $L(t)$, respectively. The notation is such that u_1 is the soliton with lower speed.

Suppose we can show the following properties of the spectra of L_1 and L_2:

C.3 L_1 has one negative eigenvalue and L_2 has no negative eigenvalue;
C.4 zero is a simple eigenvalue of L_1 and of L_2.

Our reduction procedure consists in proving that conditions **C.3** − **C.4** imply **C.1** − **C.2**. In other words, what we do is to reduce the proof of the stability of the double soliton to the study of the stationary "limit" uncoupled operators L_1 and L_2, whose coefficients depend only on the one-solitons u_1 and u_2, respectively. This fact is implicit in [17], but that is so because they use techniques which are very specific to variational problems whose Euler-Lagrange equations are ODE's.

To carry out this simplification we use a new invariant for multi-solitons (part two of Theorem 3) which has been introduced by one of the authors in [16]. We also use a theorem about the asymptotic behavior of the spectrum of a sequence of self-adjoint operators (Theorem 4). This simplification is done in an abstract framework. Hence, in principle, it works for any integrable equation, independently of the structure of the Euler-Lagrange equations for the conserved quantities. However, the procedure to verify that the spectral conditions **C.3** − **C.4** are satisfied is specific to the integrable equation under study.

In the case of BO equation our method to verify those conditions is based on certain new identities (and some variants of them) of the type

$$MQM^t = NQ_0N^t. \tag{1.2}$$

In (1.2) Q is a certain self-adjoint operator, M and N are auxiliary operators and Q_0 is another self-adjoint operator which is "simpler" than Q. Identities of type

$$MQ = Q_0M \tag{1.3}$$

are well-known and they are related to the Darboux factorization ([5] and [6]). If the spectrum of Q_0 is known, the kernel of M has dimension one and (1.3) holds, then the spectrum of Q can be calculated. We will come back to this point later in this paper. However, for some linear operators that we encounter in this paper, identities of type (1.3) do not seem to exist, while identities of type (1.2) can still be found. Identities of type (1.2) are motivated by the Sylvester Law of Inertia (Theorems 1 and 2). As we will see, if (1.2) holds and we have some information about the spectrum of Q_0 and about the kernel and image of M and of N, then certain qualitative properties of the spectrum of Q can be obtained. In other words, identities of type (1.3) are stronger in the sense that they allow us to calculate spectra while identities of type (1.2) give qualitative information only. Fortunately, this qualitative information is enough for our needs.

To illustrate how our method works, we first use it to provide an alternative proof of the stability of double solitons for the KdV equation. Then, we extend the method to the new case of the BO equation.

As in [17], our approach is purely variational. In particular, Inverse Scattering Theory is not used here. Inverse scattering has been used in [21] to prove stability results for the KdV equation but, in that case, the distances between the initial conditions and the corresponding solutions at later times are measured by different norms. In [17] and in the present paper, the phase space where stability is proved is the largest Sobolev space where the first integrals V_1, V_2 and V_3 make sense ($H^2(I\!R)$ for the KdV equation and $H^1(I\!R)$ for the BO equation.)

It is likely that our method can be extended to multi-solitons of the BO equation and of its hierarchy but the algebra may become prohibitive. The hierarchy for the BO equation has been constructed in [8]. The possibility of extending our method to show stability of double solitons (and one-solitons also) for other integrable equations depends only on finding identities of type (1.2) for these equations.

Perhaps we should expect that, for general integrable equations, the stability of multi-solitons should be a consequence of stability of one-solitons. Unfortunately, such a result has not been proved yet. Using the method we develop in this paper it can conceivably be shown that if the one-solitons are stable for all equations in its hierarchy, then multi-solitons are also stable.

In [18] some results of stability and asymptotic stability are proved for the generalized KdV equations for subcritical powers of the nonlinearity. To be more specific, it is shown in [18] that if the profile of an initial condition for the generalized KdV equation is close to the profile of N traveling waves which are far apart, then this profile is preserved for *positive* time. This type of stability (which holds also for non-integrable nonlinearities) does not include the results of [17] (neither does ours.) The reason is that when we minimize conserved quantities, we get the stability of the whole orbit of the double soliton (and not just of its tail) for both positive and negative times.

This paper is organized as follows: in Sect. 2 we recall a new invariant for multisolitons which has been introduced in [16]. In Sect. 3 we prove a result about the asymptotic behavior of the spectrum of a certain sequence of self-adjoint operators. With these two abstract results we will be able to make the simplification of the spectral problem. In Sect. 4 we recall a result that gives the spectral condition for critical points of constrained variational problems to be local minimizers. In Sect. 5 we introduce identities of type (1.3) for the KdV equation and we prove the stability of double solitons for that equation. Finally, in Sect. 6 we extend the method to the BO equation. We also include an appendix where we recall some properties of the Hilbert transform; the proofs of some lemmas which depend on long calculations involving Hilbert transform are also left to the appendix.

2. A New Invariant for Multi-Solitons

We begin by recalling the definition of inertia of a real symmetric matrix.

Definition 1. *If A is a real $N \times N$ symmetric matrix then the inertia $in(A)$ is a triplet (n, z, p) of nonnegative integers where n, z and p are the number of negative, zero and positive elements (counted according to their multiplicity) of the spectrum of A .*

The next result is known as the Sylvester Law of Inertia (see [9]).

Theorem 1. *If A is a real symmetric $N \times N$ matrix and M is a real nonsingular (not necessarily orthogonal) $N \times N$ matrix, then $in(MAM^*) = in(A)$.*

The unbounded self-adjoint operators L we will be dealing with in this paper satisfy the following property: there is a $\delta > 0$ such that the spectrum of L to the left of δ consists of a finite number of eigenvalues and the corresponding spectral projections have finite dimensional range. For that class of self-adjoint operators we give the following:

Definition 2. *The inertia $in(L)$ of a self-adjoint operator as above is the pair (n, z) of nonnegative integers where, as in the case of matrices, n is the dimension of the negative subspace of L and z is the dimension of the null space of L.*

Now we can state the Generalized Sylvester Law of Inertia:

Theorem 2. *If L with domain $D(L)$ is a self-adjoint operator as above and M is an invertible bounded operator, then $in\ (MLM^*) = in\ (L)$, where MLM^* is the self-adjoint operator with domain $(M^*)^{-1}(D(L))$.*

The proof of Theorem 2 follows exactly as in the matrix case [9] because the only thing it only uses is the variational characterization of the eigenvalues of self-adjoint operators.

We now introduce certain one-parameter families of self-adjoint operators $L(t)$ that are isoinertial, that is, have a constant inertia. The families of self-adjoint operators that arise in the stability theory of multi-solitons fit in that class. Let us consider an abstract evolution equation

$$\dot{u} = f(u) \tag{2.1}$$

in a Hilbert space X. We suppose that it has a first integral $V(u)$. In order to be precise, we consider three Hilbert spaces X, X_1 and X_2 and we make the following hypotheses:

H.1 $X_2 \subset X_1 \subset X$ with continuous embedding; the embedding from X_2 into X_1 will denoted by i;
H.2 $V : X_1 \to R$ is a C^3 functional;
H.3 $f : X_2 \to X_1$ is a C^2 function;
H.4 for any $u \in X_2$ we have

$$V'(i(u))f(u) = 0. \tag{2.2}$$

If $u(t)$ is a strong solution of (2.1) and $\langle , \rangle$ is the scalar product of X, we suppose that there is a self-adjoint operator $L(t) : D(L) \subset X \to X$ with constant domain $D(L)$ such that $\langle L(t)h, k\rangle = V''(u(t))(h, k)$ for h and k in a subspace $Z \subset D(L) \cap X_2$ which is dense in X. We consider also another operator $B(t) : D(B) \subset X \to X$ such that $B(t)h = -f'(u(t))h$ for $h \in Z$ and we make the final assumption

H.5 The closed operators $B(t)$ and $B^*(t)$ have a common domain independent of t, and the Cauchy problems

$$\dot{u} = B(t)u \qquad \dot{u} = B^*(t)u$$

are well posed for both positive and negatives times in the space X.

Theorem 3. *Let $u(t)$ be a strong solution of (2.1) and suppose* $\mathbf{H_1}$ *to* $\mathbf{H_5}$ *are satisfied.*

- *(P.Lax [15]) If for some t_0 we have $V'(u(t_0)) = 0$ then $V'(u(t)) = 0$ for any t; in other words, the set of the critical points of the first integral $V(u)$ is invariant under (2.1).*
- *If $u(t)$ is as in the first part (that is, $u(t)$ is a solution of (2.1) satisfying $V'(u(t)) = 0$ for all $t \in R$), then the inertia $in(L(t))$ of the self-adjoint operator $L(t)$ that represents $V''(u(t))$ as above is independent of t.*

The proof of Theorem 3 is given in [16] but we recall what is the main idea for the second part. A very famous device for finding isospectral families of self-adjoint operators (families with constant spectrum) is the Lax pair ([14]). The motivation is the following: let $L(t)$ be a family of self-adjoint operators and let us impose that

$$L(t) = M(t)L(0)M^*(t) \tag{2.3}$$

for any t, where $M(t)$ is orthogonal and satisfies

$$\dot{M}(t) = B(t)M(t) \quad M(0) = I \tag{2.4}$$

with $B(t)$ skew-adjoint. Differentiating (2.3) with respect to t and using (2.4) we get

$$\dot{L}(t) = B(t)L(t) - L(t)B(t). \tag{2.5}$$

Conversely, if (2.5) holds with $B(t)$ skew-adjoint then $L(t) = M(t)L(0)M^*(t)$, where $M(t)$ is orthogonal and this implies that the spectrum of $L(t)$ is constant.

Suppose that instead of constructing isospectral families, we want to construct isoinertial families. Then in view of Theorem 2 we impose

$$L(t) = M(t)L(0)M^*(t) \tag{2.6}$$

and we assume that $M(t)$ evolves in time satisfying a linear equation

$$\dot{M}(t) = B(t)M(t) \quad M(0) = I. \tag{2.7}$$

Differentiating (2.6) with respect to t and using (2.7) we get

$$\dot{L}(t) = B(t)L(t) + L(t)B^*(t). \tag{2.8}$$

Conversely, if (2.8) is satisfied then (2.6) holds and, if $M(t)$ is invertible (which is guaranteed by assumption **H.5**), then according to Theorem 2 the inertia $in(L(t))$ of $L(t)$ is constant. Therefore, without imposing that $B(t)$ is skew-adjoint, Eq. (2.8) governs isoinertial families of operators.

The proof of part two of Theorem 3 consists in showing that the self-adjoint operator $L(t)$ satisfies Eq. (2.8) and this follows easily differentiating (2.2) one and two times with respect to u (details are given in [16].)

As we have pointed out in the introduction, to prove the stability of double solitons we have to verify the spectral conditions **C.1** and **C.2** for a certain one-parameter family of self-adjoint operators $L(t)$ whose coefficients depend on the double soliton $u(t)$. These two conditions mean precisely that the inertia of $L(t)$ is the pair (1,2). Since multi-solitons fit in the framework of Theorem 3, we conclude that the inertia $in(L(t))$ of $L(t)$ is independent of t. Therefore, we can choose a convenient t to calculate the inertia and the best thing we can do is to calculate the inertia $in(L(t))$ as t goes to ∞. In that case the double soliton splits into two one-solitons u_1 and u_2 far apart. If L_1 and L_2 are the operators that we get when we replace $u(t)$ by u_1 and u_2 in $L(t)$, respectively, then in the next section we show that, as t goes to ∞, the spectrum $\sigma(L(t))$ of $L(t)$ converges to the union of the spectra $\sigma(L_1)$ and $\sigma(L_2)$ of L_1 and of L_2.

3. Asymptotic Behavior of the Spectrum of Certain Sequences of Self-Adjoint Operators

As a model for the main result of this section we take $X = L^2(R)$ and we denote by τ_n the family of isometries $(\tau_n h)(x) = h(x-n)$. We consider also the following symmetric operators:

$$(Ah)(x) = h^{(4)}(x) + \alpha h''(x) + \beta h(x),$$

where α and β are such that A is invertible;

$$\begin{aligned}
(Bh)(x) &= b(x)h''(x) + b'(x)h'(x) + b_0(x)h(x);\\
(Ch)(x) &= c(x)h''(x) + c'(x)h'(x) + c_0(x)h(x);\\
(C_n h)(x) &= (\tau_n^{-1} C \tau_n h)(x) = c(x+n)h''(x) + c'(x+n)h'(x) + c_0(x+n)h(x);\\
(D_n h)(x) &= d_n(x)h''(x) + d_n'(x)h'(x) + d_{0,n}(x)h(x).
\end{aligned}$$

The functions $b(x)$, $b_0(x)$, $c(x)$ and $c_0(x)$ are smooth and, together with some derivatives, tend to zero at infinite; $d_n(x)$ and $d_{0,n}(x)$ are sequences of smooth functions whose L_∞ norm and of some of their derivatives tend to zero as n tends to ∞.

In the model above, the operator $L_n = A + B + C_n + D_n$ is the sum of an operator A which is translation invariant, an operator D_n whose coefficients tend to zero as n goes to ∞ and two other operators B and C_n whose coefficients have "support" far apart when n tends to infinity. We will show that as n gets large, the spectrum of L_n tends to the union of the spectra of $A + B$ and of $A + C$.

The abstract framework is the following: let X be a real Hilbert space and τ_n be a sequence of isometries of X. Suppose also that A, B, C are linear operators, C_n, D_n are two sequences of linear operators with $C_n = \tau_n^{-1} C \tau_n$ and that the following assumptions are satisfied:

A.1 A, $A+B$, $A+C$, $A+C_n$, $A+B+C_n+D_n$ are self-adjoint with the same domain $D(A)$;
A.2 A is invertible and A commutes with τ_n, that is, $\tau_n^{-1} A \tau_n = A$;
A.3 there is a number $\delta > 0$ such that the spectra of all self-adjoint operators A, $A + B$, $A + C$, $A + B + C_n + D_n$ to the left of δ consists of a finite number of eigenvalues and the spectral projection corresponding to any such eigenvalue has finite dimensional range;
A.4 for any $\lambda \in \rho(A + B) \cap \rho(A + C)$, the operators $A(A + C - \lambda I)^{-1}$ and $A(A + B - \lambda I)^{-1}$ are bounded ;
A.5 $|D_n A^{-1}|$ tends to zero as n tends to ∞;
A.6 for any element $u \in D(A)$, $|C_n u| \to 0$ as n tends to ∞;
A.7 for any element $u \in X$, $\tau_n u$ tends to zero weakly in X as n tends to infinity;
A.8 for any λ in the resolvent set $\rho(A + C)$ of $A + C$, the operator $B(\lambda I - A - C)^{-1}$ is compact.

Theorem 4. *Under assumptions* **A.1** *to* **A.8***, if* $\lambda < \delta$ *and* $\lambda \in \rho(A+B) \cap \rho(A+C)$*, then there is a number* N_0 *such that for* $n \geq N_0$*,* λ *belongs to the resolvent set* $\rho(L_n)$ *of* $L_n = A+B+C_n+D_n$*. Moreover, if* $\lambda_0 \in \sigma(A+B) \cup \sigma(A+C)$ *and* $\epsilon > 0$ *is given, then there is an integer* N_1 *such that for* $n \geq N_1$*, the dimension of the range of the spectral projection of* L_n *corresponding to the circle centered at* λ_0 *and radius* ϵ *is equal to the sum of the dimensions of the ranges of the spectral projections of* $A + B$ *and* $A + C$ *associated to* λ_0*.*
In particular, if the dimension of the null space of L_n *is constant and* ≥ 2 *and the sum of the dimensions of the null spaces of* $A+B$ *and* $A+C$ *is equal to 2, then the dimension of the null space of* L_n *is equal to 2; moreover, as n tends to infinity, a nonzero eigenvalue of* L_n *cannot accumulate at zero.*

Proof. First of all we notice that $\lambda \in \rho(A + C)$ if and only if $\lambda \in \rho(A + C_n)$ and

$$(A + C_n - \lambda I)^{-1} = \tau_n^{-1}(A + C - \lambda I)^{-1}\tau_n. \tag{3.1}$$

If $\lambda \in \rho(A + B) \cap \rho(A + C)$, $\lambda < \delta$ and n is large, we have to show that $\lambda \in \rho(L_n)$. From Assumption A.3, all we have to do is to show $u = 0$ is the unique solution of

$$Au + Bu + C_n u + D_n u - \lambda u = 0. \tag{3.2}$$

Therefore, assuming that u solves (3.2) we get

$$u = (A + B - \lambda I)^{-1}(-C_n u - D_n u) \tag{3.3}$$

and

$$u = (A + C_n - \lambda I)^{-1}(-Bu - D_n u). \tag{3.4}$$

If we replace the u following B in (3.4) by the value of u given by (3.3) we get:

$$u = W_n u \widehat{=} \ (A + C_n - \lambda I)^{-1} B(A + B - \lambda I)^{-1}(C_n u + D_n u) \qquad (3.5)$$
$$-(A + C_n - \lambda I)^{-1} D_n u.$$

Now we claim that

$$|(A + C - \lambda I)(A + C_n - \lambda I)^{-1}| = (A + C - \lambda I)\tau_n^{-1}(A + C - \lambda I)^{-1}\tau_n|$$

is uniformly bounded. In fact, we have $A\tau_n^{-1}(A + C - \lambda I)^{-1} = \tau_n^{-1} A(A + C - \lambda I)^{-1}$ and $C\tau_n^{-1}(A + C - \lambda I)^{-1} = CA^{-1}A\tau_n^{-1}(A + C - \lambda I)^{-1}$ and then the claim follows from Assumption A.4.

Using the fact that the norm of a bounded operator is equal to the norm of its adjoint we can write the following:

$$\begin{aligned}|(A + C_n - \lambda I)^{-1}B(A + B - \lambda I)^{-1}C_n| &= |C_n(A + B - \lambda I)^{-1}B(A + C_n - \lambda I)^{-1}| \\ &= |C_n(A + B - \lambda I)^{-1}B(A + C - \lambda I)^{-1} \\ &\quad (A + C - \lambda I)(A + C_n - \lambda I)^{-1}|.\end{aligned}$$

Since $C_n(A + B - \lambda I)^{-1} = C_n A^{-1}A(A + B - \lambda I)^{-1}$ converges to zero in the strong operator topology in view of Assumptions A.4 and A.6, and $B(A+C-\lambda I)^{-1}$ is compact in view of Assumption A.8, we see that

$$|(A + C_n - \lambda I)^{-1}B(A + B - \lambda I)^{-1}C_n|$$

tends to zero.

Putting this information together with Assumption A.5, we conclude that the norm $|W_n|$ of the operator W_n given by (3.5) tends to zero as n gets large. Then $u = 0$ for n large and this proves the first part of the theorem.

To prove the second part, suppose that $\lambda_0 < \delta$, $\lambda_0 \in \sigma(A + B) \cup \sigma(A + C)$ and $\epsilon > 0$ is given. From the first part we know that there is N_0 such that if $n \geq N_0$ and $\lambda \in \mathbb{C}$ is such that $|\lambda - \lambda_0| = \epsilon$ then $\lambda \in \rho(L_n)$. If we set $(\lambda I - L_n)^{-1} f = -u$ then u is the unique solution of

$$Au + Bu + C_n u + D_n u - \lambda u = f. \qquad (3.6)$$

To find a more convenient form for the resolvent operator $(\lambda I - L_n)^{-1} f$, we argue as in the first part in the following way: if u is the unique solution of (3.6) then u also solves

$$u = (A + B - \lambda I)^{-1}(-C_n u - D_n u + f) \qquad (3.7)$$

and

$$u = (A + C_n - \lambda I)^{-1}(-Bu - D_n u + f). \qquad (3.8)$$

If we replace the u following B in (3.8) by the value of u given by (3.7) we get:

$$u = W_n u - (A + C_n - \lambda I)^{-1}B(A + B - \lambda I)^{-1} f + (A + C_n - \lambda I)^{-1} f, \quad (3.9)$$

where $W_n u$ is given by (3.5). As we have proved in the first part, for $|\lambda - \lambda_0| = \epsilon$ the norm $|W_n|$ of W_n goes to zero as n goes to infinity and this implies that the unique solution of (3.9) is $u = Y_n(\lambda) f$ with

$$Y_n(\lambda) f = (I - W_n)^{-1}[-(A + C_n - \lambda I)^{-1}B(A + B - \lambda I)^{-1} f$$
$$+(A + C_n - \lambda I)^{-1} f] \qquad (3.10)$$

and then $(\lambda I - L_n)^{-1} f = -Y_n(\lambda) f$.

We denote by Γ the circle centered at λ_0 with radius ϵ and oriented counterclockwise. To calculate the spectral projection

$$T_n \hat{=} \frac{1}{2\pi i} \int_\Gamma (\lambda I - L_n)^{-1} \, d\lambda = -\frac{1}{2\pi i} \int_\Gamma Y_n(\lambda) \, d\lambda$$

of L_n corresponding to the disk $|\lambda - \lambda_0| = \epsilon$, we start by calculating the operator

$$S_n = \frac{1}{2\pi i} \int_\Gamma [(A + C_n - \lambda I)^{-1} - (A + C_n - \lambda I)^{-1} B (A + B - \lambda I)^{-1}] \, d\lambda. \quad (3.11)$$

The first term of S_n is simply $-Q_n$, Q_n being the spectral projection of $A + C_n$ associated to the eigenvalue λ_0. If we denote by Q the spectral projection of $A + C$ corresponding to the eigenvalue λ_0, from (3.1) we have $Q_n = \tau_n^{-1} Q \tau_n$; in particular, the ranges of Q_n and of Q have the same dimension.

To calculate the second term in (3.11) we use the following result (see [12]): if U is a self-adjoint operator and λ_0 is an isolated point of the spectrum $\sigma(U)$, then the function $(\lambda I - U)^{-1}$ has a simple pole at λ_0 and

$$(\lambda I - U)^{-1} = \frac{P}{(\lambda - \lambda_0)} + P^\perp (\lambda_0 I - U)^{-1} P^\perp + V(\lambda), \quad (3.12)$$

where P is the orthogonal spectral projection corresponding to λ_0, $P^\perp = I - P$ and $V(\lambda)$ is analytic at $\lambda = \lambda_0$ with $V(\lambda_0) = 0$.

Therefore, if we denote by P the spectral projection associated to the operator $A + B$ at λ_0 and Q_n is as before we can write:

$$\begin{aligned} R_n(\lambda) &\hat{=} (\lambda I - A - C_n)^{-1} B (\lambda I - A - B)^{-1} \\ &= [\frac{Q_n}{\lambda - \lambda_0} \\ &\quad + Q_n^\perp (\lambda_0 I - A - C_n)^{-1} Q_n^\perp \\ &\quad + V_n(\lambda)] B \left[\frac{P}{\lambda - \lambda_0} + P^\perp (\lambda_0 I - A - B)^{-1} P^\perp + V(\lambda) \right], \end{aligned}$$

where $V_n(\lambda)$ and $V(\lambda)$ are analytic at $\lambda = \lambda_0$ and vanish there. Then the residue of $R_n(\lambda)$ at λ_0 is equal to

$$Q_n B P^\perp (\lambda_0 I - A - B)^{-1} P^\perp + Q_n^\perp (\lambda_0 I - A - C_n)^{-1} Q_n^\perp B P. \quad (3.13)$$

Since P projects on the null space of $A + B - \lambda_0 I$, we have $BP = (A - \lambda_0 I) P$ and then the second term of (3.13) can be written as

$$\begin{aligned} Q_n^\perp (\lambda_0 I - A - C_n)^{-1} Q_n^\perp B P &= Q_n^\perp (\lambda_0 I - A - C_n)^{-1} Q_n^\perp (A - \lambda_0 I) P \\ &= Q_n^\perp (\lambda_0 I - A - C_n)^{-1} Q_n^\perp [(A + C_n - \lambda_0 I) - C_n] P \\ &= Q_n^\perp P - Q_n^\perp (\lambda_0 I - A - C_n)^{-1} Q_n C_n P \\ &= P - Q_n P - Q_n^\perp (\lambda_0 I - A - C_n)^{-1} Q_n C_n P. \end{aligned}$$

Moreover, if P and Q are expressed as

$$Pu = \sum_{i=1}^{N} \langle u, \phi_i \rangle \phi_i, \quad Qu = \sum_{j=1}^{M} \langle u, \psi_j \rangle \psi_j,$$

where ϕ_i and ψ_j are unity vectors, from Assumption A.7 we see that the norm of the operator

$$Q\tau_n Pu = \sum_{i,j} \langle u, \phi_i \rangle \langle \tau_n \phi_i, \psi_j \rangle \psi_j$$

goes to zero as n goes to ∞ and then the norm of $Q_n P = \tau_n^{-1} Q\tau_n P$ also goes to zero.

According to Assumption A.6 the norm of $C_n P$ goes to zero. The norm of W_n, as we have seen in the proof of the first part, also goes to zero. Therefore, we see that the norm $|T_n - (-S_n)|$ goes to zero and putting all these things together we conclude that the norm of $T_n - Q_n - P$ tends to zero and this proves the second part of the theorem. The final assertions follow immediately from the first and second parts and the theorem is proved.

4. Local Minimizers for Constrained Variational Problems

In this section we recall a result proved in [17] and [10] which gives the spectral condition for a critical point of a constrained variational problem to be a local minimizer. For simplicity we consider the case of two constraints only. Therefore, we consider three smooth functionals $V_1(u)$, $V_2(u)$ and $V_3(u)$ in a Hilbert space X and we assume that for real numbers α and β in a certain range, $u = u(\alpha, \beta)$ is a smooth family of solution of the Euler-Lagrange equation

$$V_3'(u) + \alpha V_2'(u) + \beta V_1'(u) = 0. \tag{4.1}$$

We consider also the second derivative $V_3''(u) + \alpha V_2''(u) + \beta V_1''(u)$ of the augmented lagrangian $V_3(u) + \alpha V_2(u) + \beta V_1(u)$ and we calculate it at $u = u(\alpha, \beta)$. That second derivative is a continuous bilinear form in the Hilbert space X that can be represented by an (in general) unbounded self-adjoint operator $L = L(\alpha, \beta)$ with a certain domain. We assume that there is a $\delta > 0$ such that the essential spectrum of L is contained in $[\delta, +\infty)$ and that the family $u(\alpha, \beta)$ is nondegenerate in the sense that zero is not an eigenvalue of L. Considering also the real valued function $V(\alpha, \beta) = V_3(u(\alpha, \beta)) + \alpha V_2(u(\alpha, \beta)) + \beta V_1(u(\alpha, \beta))$ we can state the following result:

Theorem 5. *The family of critical points $u(\alpha, \beta)$ is a local minimizer for V_3 subject to $V_2(u) = k_2$ and $V_1(u) = k_1$ iff the number of negative eigenvalues of $L = L(\alpha, \beta)$ is equal to the number of positive eigenvalues of the hessian matrix $V''(\alpha, \beta)$.*

To prove the orbital stability of double solitons, we have to show that a certain two dimensional manifold $u(t, \tau)$ is a local minimizer for a certain constrained variational problem. In the notation $u(t, \tau)$, t will be time and τ denotes translation in the space variable x. Then, instead of having a single operator L we have a family $L(t, \tau)$ of operators. As it has been remarked in [17], the manifold $u(t, \tau)$ is made of degenerate critical points because $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x}$ are eigenfunctions of L associated to the zero eigenvalue. Since the definition of orbital stability takes those degeneracies in account, the condition for the critical points $u(t, \tau)$ to be nondegenerate is that the kernel of $L(t, \tau)$ has dimension two; in other words, the kernel of $L(t, \tau)$ is spanned by the tangent vectors $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial \tau}$ to the orbit $u(t, \tau)$.

Due to the noncompactness of the orbit $u(t, \tau)$, another question we have to worry about is to verify that a nonzero eigenvalue of $L(t, \tau)$ cannot accumulate at zero as t goes to infinity. The fact that this cannot happen in our case will become clear from our arguments and it has already been considered in Theorem 4.

5. Stability of Double Solitons for the KdV Equation

The stability of double solitons of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{5.1}$$

has been proved in [17]. In this section, using the results we have proved in Sects. 2 and 3 and further arguments, we give an alternative proof for that result. In Sect. 6 we extend that method to the BO equation. The first three conserved quantities for (5.1) are

$$V_1(u) = \frac{1}{2}\int_{-\infty}^{+\infty} u^2\, dx,$$
$$V_2(u) = \frac{1}{2}\int_{-\infty}^{+\infty} (u_x^2 - 2u^3)\, dx,$$
$$V_3(u) = \frac{1}{2}\int_{-\infty}^{+\infty} (u_{xx}^2 - 10uu_x^2 + 5u^4)\, dx.$$

Defining $\eta_i = -b_i x + 4b_i^3 t$ for $i = 1, 2$ and $A = (b_2 - b_1)^2/(b_1 + b_2)^2$, the double soliton is given by

$$u(t, x) = 2\frac{d^2}{dx^2}\log(1 + e^{2\eta_1(t,x)} + e^{2\eta_2(t,x)} + Ae^{(2\eta_1(t,x)+2\eta_2(t,x))}) \tag{5.2}$$

(see [11]) and the speeds are $c_i = 4b_i^2$. The traveling wave with speed $c = 4b^2$ is

$$u_c = u(t, x) = 2b^2\mathrm{sech}^2(b(x - 4b^2t)) = \frac{1}{2}\mathrm{sech}^2(\frac{\sqrt{c}}{2}(x - ct)). \tag{5.3}$$

As t tends to $+\infty$, the double soliton splits in two separate one-solitons that are far apart in the following sense: if we define

$$w(t, x) = u(t, x) - 2b_1^2\mathrm{sech}^2 b_1(x - b_1^2 t) - 2b_2^2\mathrm{sech}^2 b_2(x - b_2^2 t + \Delta_2),$$

where $\Delta_2 = \log A$ is the phase shift, then

$$\lim_{t\to+\infty} \|w(t)\|_{k,p} = 0 \quad 1 \le p \le \infty \quad k \in N \tag{5.4}$$

and

$$\lim_{t\to+\infty} \|u_1^n(t)u_2^m(t)\|_{k,p} = 0 \quad 1 \le p \le +\infty \quad m, n \in N, m, n \ge 1, \tag{5.5}$$

where $|.|_{k,p}$ denotes the norm in the Sobolev space $W_{k,p}(R)$.

The Euler-Lagrange equation $V_3'(u) + \alpha V_2'(u) + \beta V_1'(u) = 0$ is

$$u^{(4)} + 10uu_{xx} + 5u_x^2 + 10u^3 + \alpha(-u_{xx} - 3u^2) + \beta u = 0 \tag{5.6}$$

and the soliton (5.2) solves (5.6) if

$$c_1 + c_2 = \alpha \quad \text{and} \quad c_1 c_2 = \beta; \tag{5.7}$$

(see [14]). In other words, the constants c_1 and c_2 are related to the multipliers through the quadratic equation

$$c^2 - \alpha c + \beta = 0. \tag{5.8}$$

The linearized operator of (5.6) at the double soliton $u(t, \cdot)$ is

$$L(t)h = h^{(4)} + 10uh_{xx} + 10u_{xx}h + 10u_x h_x + 30u^2 h + \alpha(-h_{xx} - 6uh) + \beta h. \tag{5.9}$$

Definition 3. *We say that the double soliton (5.2) is orbitally stable if defining the two dimensional set $S = \{u(t, \cdot + \tau), t, \tau \in R\}$, where $u(t, \cdot)$ is the double soliton, then for any $\epsilon > 0$, there is $\delta > 0$ such that if $u_0 \in H^2(I\!R)$ and $d(u_0, S) < \delta$, then $d(u(t, u_0), S) < \epsilon$, where d is the distance taken in the $H^2(I\!R)$ norm and $u(t, u_0)$ is the solution of (5.1) such that $u(0, u_0) = u_0$.*

This section is devoted to prove the following:

Theorem 6. *The double soliton (5.2) is orbitally stable.*

The proof of Theorem 6 will be done as follows: we write a chain of statements S.i, $i = 0, \dots, 4$ where S.0 is Theorem 6 and S.4 is Theorem 7 below. Using the theory that we have developed in the previous sections together with simple arguments, we show that each statement implies the previous one and at the end we prove Theorem 7.

Since the Cauchy problem for (5.1) is well posed in the space $H^2(I\!R)$ and V_1, V_2 and V_3 are conserved quantities (see [13]), then, as in [17], Theorem 6 follows from:

Statement S.1: *The set S is made of local minimizers of V_3 for given values of V_2 and of V_1.*

Defining $V(\alpha, \beta) = V_3(u(t)) + \alpha V_2(u(t)) + \beta V_1(u(t))$, where $u(t)$ is the double soliton, it has been proved in [17] that $det(V''(\alpha, \beta)) < 0$. Therefore the hessian matrix $V''(\alpha, \beta)$ has one positive eigenvalue and one negative one. In view of Theorem 5 and the comments following it, we see that Statement $S.1$ is a consequence of

Statement S.2: *For every $t \in I\!R$, the operator $L(t)$ defined by (5.9) has exactly one negative eigenvalue, it has zero as a double eigenvalue and there is no accumulation of the spectrum of $L(t)$ at zero as t tends to infinity.*

As we have pointed out in the introduction, Statement 2 has been proved in [17] using ODE methods. We now present a different proof that can be extended to BO equation.

The stationary limit operators obtained by replacing in (5.9) the double soliton $u(t, x)$ by the one-solitons u_i are:

$$\begin{aligned} L_i h = h^{(4)} &+ 10u_i h_{xx} + 10u_{i,xx}h \\ &+ 10u_{i,x}h_x + 30u_i^2 h + \alpha(-h_{xx} - 6u_i h) + \beta h. \end{aligned} \tag{5.10}$$

The notation is such that $c_1 < c_2$. According to Theorem 3, part ii, the inertia of $L(t)$ is constant; moreover, from (5.4) and (5.5) we see that the assumptions of Theorem 4 are satisfied for any sequence t_n that goes to infinity; therefore Statement S.2 follows from the following:

Statement S.3:

- *L_1 has zero as a simple eigenvalue and no negative eigenvalues;*
- *L_2 has zero as a simple eigenvalue and exactly one negative eigenvalue.*

In order to simplify the calculations, we normalize one speed taking $c = 4$; in this case, from (5.8) we get $\alpha = 4 + \frac{\beta}{4}$. The profile of the traveling wave with speed $c = 4$ is $2\text{sech}^2 x$; from now on, this profile will be denoted by $u = u(x)$ and we use capitals to denote the multiplication operator by the corresponding function. The operators L_1 and L_2 given by (5.10) calculated at this particular wave is

$$L = D^4 + 10UD^2 + 10U_{xx} + 10U_x D + 30U^2 + \alpha(-D^2 - 6U) + \beta, \quad (5.11)$$

where $D = \dfrac{d}{dx}$. If in this last equation we replace α by $4 + \dfrac{\beta}{4}$ as above, we get

$$L = L_\beta = Q + \frac{\beta}{4}K, \quad (5.12)$$

where

$$Qh = D^4 + (10U - 4)D^2 + 10U_{xx} + 10U_x D + (30U^2 - 24U) \quad (5.13)$$

and

$$K = -D^2 - 6U + 4I. \quad (5.14)$$

Notice that $c = 4$ is the lower speed if $\beta < 16$ and $c = 4$ is the higher speed if $\beta > 16$. Finally, using a simple scaling argument it is easy to show that Statement S.3 follows from

Theorem 7.

- *For $0 < \beta \neq 16$, zero is a simple eigenvalue eigenvalue of L_β;*
- *for $0 < \beta < 16$, L_β has one negative eigenvalue and for $\beta > 16$, L_β has no negative eigenvalue.*

We use the subscript *odd* to denote space of odd functions and the subscript *ev* to denote space of even functions. Notice that K and Q map even functions in even functions and odd functions in odd functions. We now state three lemmas from which the proof of Theorem 7 follows.

Lemma 1.

1. *For $h \in H^1_{odd}(I\!R)$ we have $\langle Kh, h\rangle \geq 0$ and $\langle Kh, h\rangle = 0$ if and only if h is a multiple of $v = u'$;*
2. *in H^1_{ev} the operator K has exactly one negative eigenvalue and zero is not an eigenvalue.*

Lemma 2. *For any $h \in H^2(I\!R)$ we have $\langle Qh, h\rangle \geq 0$ and $\langle Qh, h\rangle = 0$ iff h is a multiple of $u' = v$.*

Lemma 3. *For $\beta = 16$ the function*

$$\frac{1}{2}(x\tanh x - 1)\mathrm{sech}^2 x = \left.\frac{du_c(x)}{dc}\right|_{c=4},$$

where $u_c(x)$ is the one-soliton (5.3), is an (even) eigenfunction of L_{16} associated to the zero eigenvalue.

Remark 1. The spectral conditions given by Lemma 1 are precisely the conditions that are needed to prove the orbital stability of one-solitons.

Admitting Lemmas 1, 2 and 3 we prove Theorem 7.

Proof of Theorem 7. First we analyse the operator L_β in the space $H^2_{odd}(I\!R)$. As we have pointed out, $Qu' = 0 = Ku'$. Moreover, from Lemmas 1 and 2 we see that, in the space $H^2_{odd}(I\!R)$, the operators Q and K are positive in the subspace of functions orthogonal to u'; hence, zero is an eigenvalue of L_β associated to the eigenfunction $v = u'$ and L_β is also positive in the subspace orthogonal to u' . Therefore, in $H^2_{odd}(I\!R)$, for any $\beta > 0$, L_β has zero as a simple eigenvalue with eigenfunction u' and all other eigenvalues are positive.

Now we turn to $H^2_{ev}(I\!R)$. If we divide L_β by $\beta/4$ the sign of the eigenvalues are preserved. Then we define $\gamma = 4/\beta$ and the operator

$$\tilde{L}_\gamma = \gamma Q + K \tag{5.15}$$

and we denote by $\lambda_1(\gamma)$ the lowest eigenvalue of $\tilde{L}_\gamma$ in $H^2_{ev}(I\!R)$. For $\gamma = 0$, from Lemma 1 we conclude that $\lambda_1(\gamma)$ is negative and all the other eigenvalues of $\tilde{L}_0$ are positive. Since $\langle Qh, h\rangle > 0$ for $h \in H^2_{ev}(I\!R)$, $h \neq 0$, from the variational characterization of the eigenvalues for self-adjoint operators, we conclude that all eigenvalues of $\tilde{L}_\gamma$ move strictly to the right as γ increases. Moreover, from Lemma 3 we see that $\lambda_1(1/4) = 0$; we conclude that $\lambda_1(\gamma) < 0$ for $\gamma < 1/4$ and $\lambda_1(\gamma) > 0$ for $\gamma > 1/4$ and this proves the theorem.

Then, all is left is to prove Lemmas 1, 2 and 3. The proof of Lemma 3 follows by elementary calculation and the rest of this section will be devoted to prove Lemmas 1 and 2. We start defining two auxiliary linear operators that will play a crucial role in the proof of Lemma 1 and Lemma 2 and later in our method:

$$Mh = h'(x) + 2\tanh x h(x), \qquad M^t h = -h'(x) + 2\tanh x h(x). \tag{5.16}$$

We also define

$$K_0 = -D^2 + 4I. \tag{5.17}$$

□

Lemma 4. *The following identity holds:*

$$MKM^t = M^t K_0 M. \tag{5.18}$$

The proof of (5.18) follows by expansion of both sides. Before using that identity to prove Lemma 1, we make a few comments. First let us recall that if

$$Kh = -h_{xx} + p(x)h \quad \text{and} \quad K_0 h = -h_{xx} + p_0(x)h \tag{5.19}$$

are second order ordinary differential operators and the spectrum of K_0 is known, then the spectrum of K can be calculated if an identity of type

$$MK = K_0 M \tag{5.20}$$

holds. In (5.20) M is an auxiliary operator whose kernel and range are known. This method has been used in [5] (see also [6], Sect. 2.4 for a presentation of it). Starting with an operator with constant coefficients and imposing (5.20) recursively, it can be shown that for the reflectionless potentials $p(x) = -m(m+1)\text{sech}^2 x$, where m is an integer, the spectrum of K given by (5.19) can be calculated explicitly. The auxiliary operator M in (5.20) is given by $Mh = h'(x) + n \tanh x$, where n is an integer related to m. Since in the definition (5.14) of the operator K u is given by $u = 2\text{sech}^2(x)$, we see that the potential $-6u$ in (5.14) is equal to $-12\text{sech}^2(x)$. Then it is the reflectionless potential with $m = 3$ and we conclude that the spectrum of K can be calculated explicitly. In other words, (5.20) gives information which is much better than Lemma 1. However, as we have pointed out in the introduction, in the study of the stability of double solitons, we will have to make the spectral analysis of some higher order operators for which, at least apparently, there is no identity of type (5.20) but there is an identity of type (5.18). Therefore, we give the proof of Lemma 1 using identity (5.18) to illustrate how our method works. The first thing is to find the kernel and the range of the auxiliary operators M and M^t.

Lemma 5. *If*

$$M, M^t : D(M) = D(M^t) = H^1(R) \subset L^2(R) \to L^2(R)$$

are given by (5.16) then

i) the null space of M is spanned by u and M^t is injective;
ii) M is onto and the image of M^t is the subspace orthogonal to u.

Proof. The general solution of

$$h'(x) = -2\tanh(x)h(x) \quad \text{and} \quad h'(x) = 2\tanh(x)h(x)$$

are

$$h(x) = C\text{sech}^2(x) \quad \text{and} \quad h(x) = C\cosh^2(x)$$

respectively, and this proves part one. If $h \in L^2(R)$ is given, it is easy to see that $g(x) = \text{sech}^2(x)\int_0^x \cosh^2(s)h(s)\,ds$ solves $g'(x) + 2\tanh(x)g(x) = h(x)$ and it remains to show that the operator $(Th)(x) = g(x)$ maps $L^2(R)$ boundedly into itself. Since $|\int_0^x \cosh^2(s)\,ds \le \cosh(x)\sinh(|x|)$ we see that T maps $L^\infty(R)$ boundedly into itself. Moreover, for $b > 0$ we have

$$\begin{aligned}\int_0^b \operatorname{sech}^2(x) \int_0^x \cosh^2(s)|h(s)|\,ds\,dx &= \int_0^b \left(\int_0^x \cosh^2(s)|h(s)|\,ds)(\tanh(x)\right)' dx \\ &= \tanh(b) \int_0^b \cosh^2(x)|h(x)\,dx \\ &\quad - \int_0^b \tanh(x)\cosh^2(x)|h(x)|\,dx \\ &= \int_0^b (\tanh(b) - \tanh(x))\cosh^2(x)|h(x)|\,dx \\ &\leq \int_0^b (1 - \tanh(x))\cosh^2(x)|h(x)|\,dx.\end{aligned}$$

Furthermore, from $(1 - \tanh(x))\cosh^2(x) = (1 + \exp^{-x})/2 \leq 1$ for $x \geq 0$, we get

$$\int_0^b \operatorname{sech}^2(x) \int_0^x \cosh^2(s)|h(s)|\,ds\,dx \leq \int_0^b |h(x)\,dx.$$

The case $b < 0$ is treated in a similar way. This shows that T maps $L^1(R)$ boundedly into itself and then, by interpolation, T maps $L^2(R)$ boundedly into itself.

Finally, if h belongs to the range of M^t then it has to be orthogonal to the null space of Q, that is, h has to be orthogonal to u. Conversely, suppose that $\int_{-\infty}^{+\infty} \operatorname{sech}^2(x)h(x)\,dx = 0$ and let us define

$$g(x) = -\int_{-\infty}^x \frac{\cosh^2 x}{\cosh^2 s} h(s)\,ds = \int_x^{+\infty} \frac{\cosh^2 x}{\cosh^2 s} h(s)\,ds.$$

An easy calculation shows that $g(x)$ solve the equation $-g'(x) + 2\tanh(x)g(x) = h(x)$ and then, it remains to show that the operator $S(h) = g$ maps $L^2(R)$ boundedly into itself. First we define $(S_1 h)(x) = -\int_{-\infty}^x \frac{\cosh^2 x}{\cosh^2 s} h(s)\,ds$ for $x \leq 0$. Arguing exactly as above we can show that S_1 maps $L^\infty((-\infty, 0])$ boundedly into itself and also $L^1((-\infty, 0])$ boundedly itself. For $x \geq 0$ we get the same estimates for the operator $(S_2 h)(x) = \int_x^{+\infty} \frac{\cosh^2 x}{\cosh^2 s} h(s)\,ds$ and this proves the lemma. □

Proof of Lemma 1. First we notice that M and M^t map odd functions in even functions and even functions in odd functions. According to Lemma 5, any $h \in H^2(I\!R)$ can be written as $h = \alpha u + M^t k$. We consider first the case $h \in H^2_{odd}(I\!R)$. In this case, $\alpha = 0$, k is even and using (5.18) we get:

$$\langle Kh, h\rangle = \langle KM^t k, M^t k\rangle = \langle MKM^t k, k\rangle = \langle M^t K_0 M k, k\rangle = \langle K_0 M^t k, M^t k\rangle.$$

Since $\langle K_0 s, s\rangle \geq 0$ for any $s \in H^2(I\!R)$ and $\langle K_0 s, s\rangle = 0$ iff $s = 0$, we conclude that if $h \in H^2_{odd}(I\!R)$, we have $\langle Kh, h\rangle \geq 0$ and $\langle Kh, h\rangle = 0$ iff $M^t k = 0$. Moreover, according to Lemma 5, $M^t k = 0$ implies that k has to be a multiple of u and then $h = M^t k$ is also a multiple of u'.

Suppose now that h is even; then $h = \alpha u + M^t k$, with k odd. In the hyperplane $\alpha = 0$ we have $\langle Kh, h\rangle = \langle K_0 Mk, Mk\rangle \geq 0$ and $\langle Kh, h\rangle = 0$ iff $Mk = 0$ and, according to Lemma 5, this implies $k = 0$ because k is even. Therefore $\langle Kh, h\rangle > 0$ in a hyperplane and this implies that K can have at most one nonpositive eigenvalue. Since $Ku = -3u^2$ we see that $\langle Ku, u\rangle < 0$ and this implies that K has exactly one negative eigenvalue in $H^2_{ev}(I\!R)$ and the lemma is proved.

It remains to deal with the more complicated higher order operator Q defined by (5.13). First we need a lemma whose proof follows expanding both sides of the identity. □

Lemma 6. *Defining*

$$Q_0 = D^4 - 4D^2 \tag{5.21}$$

we have

$$MQM^t = M^t Q_0 M, \tag{5.22}$$

where M and M^t are given (5.16)

Remark 2. From (5.18) and (5.22), we see that the auxiliary operator M that conjugates K with K_0 is the same that conjugates Q with Q_0.

Proof of Lemma 2. According to Lemma 5, any $h \in H^2(I\!R)$ we can be written as $h = \alpha u + M^t k$ and then

$$\begin{aligned}\langle Qh, h\rangle &= \alpha^2\langle Qu, u\rangle + 2\alpha\langle MQu, k\rangle + \langle MQM^t k, k\rangle\\ &= \alpha^2\langle Qu, u\rangle + 2\alpha\langle MQu, k\rangle + \langle Q_0 Mk, Mk\rangle.\end{aligned}$$

If we define s by $k = \alpha q + s$ and we denote by $T = M^t Q_0 M$ the right-hand side of (5.22) we get

$$\begin{aligned}\langle Qh, h\rangle = \alpha^2(&\langle Qu, u\rangle - \langle Tq, q\rangle - 2\langle MQu - Tq, q\rangle)\\ &+2\alpha\langle (MQu - Tq), s\rangle + \langle Ts, s\rangle.\end{aligned}$$

To eliminate the cross term of this quadratic form in α and s we have to choose q in such way that

$$MQu = Tq \tag{5.23}$$

and, in this case,

$$\langle Qh, h\rangle = \alpha^2(\langle Qu, u\rangle - \langle Tq, q\rangle) + \langle Ts, s\rangle.$$

We can verify that $q = \frac{1}{3}x\,\mathrm{sech}^2 x$ satisfies (5.23) and then performing some calculation we get:

$$\langle Qh, h\rangle = \frac{36608}{315}\alpha^2 + \langle Ts, s\rangle = \frac{36608}{315}\alpha^2 + \langle Q_0 Ms, Ms\rangle.$$

Since $\langle Q_0 y, y\rangle \geq 0$ and $\langle Q_0 y, y\rangle = 0$ iff $y = 0$, we conclude that $\langle Qh, h\rangle \geq 0$ and $\langle Qh, h\rangle = 0$ iff $\alpha = 0$ and $Ms = 0$. Moreover, according to Lemma 5, $Ms = 0$ implies that s has to be a multiple of u and then $k = s$ is also a multiple of u. Hence, $h = M^t s$ is a multiple of u' and this proves the lemma.

Remark 3. For the operator K, zero is an isolated eigenvalue; however, for the operator Q, zero belongs to its essential spectrum. □

6. Stability of Double Solitons for the BO Equation

In this section we extend the method we used in Sect. 5 to prove the stability of double solitons of the Benjamin-Ono equation

$$\frac{\partial u(t,x)}{\partial t} + 4u(t,x)u_x(t,x) + H(u_{xx}(t,x)) = 0, \tag{6.1}$$

where H is the Hilbert transform. In the appendix we give the definition and some properties of the Hilbert transform that will be used. We also develop a calculus involving the Hilbert transform and the function $u = \dfrac{1}{1+x^2}$. The proofs of some identities are also left to the appendix.

The first three first integrals of (6.1) are:

$$V_1(u) = \frac{1}{2}\int_{-\infty}^{+\infty} u^2\,dx,$$
$$V_2(u) = \frac{1}{2}\int_{-\infty}^{+\infty} (-uHu_x - \frac{4}{3}u^3)\,dx,$$
$$V_3(u) = \frac{1}{2}\int_{-\infty}^{+\infty} (u_x^2 + 3u^2Hu_x + 2u^4)\,dx.$$

These three first integrals will be considered in the space $H^1(R)$ and they can be found in [20] (a change of scale has to be made because the coefficients of BO equation in [20] are different from ours).

Formulae (6.2)–(6.7) can be found in [19]. The one-soliton with speed $c > 0$ for (6.1) is:

$$u(t,x) = \frac{c}{c^2(x-ct)^2+1}$$

and its profile

$$u_c(x) = \frac{c}{c^2x^2+1} \tag{6.2}$$

satisfies

$$-Hu_c' - 2u_c^2 + cu_c = 0. \tag{6.3}$$

The double soliton with speeds $c_1 > 0$ and $c_2 > 0$, $c_1 < c_2$ is

$$u(t,x) = \frac{c_2\theta_1^2 + c_1\theta_2^2 + (c_1+c_2)c_{12}}{(\theta_1\theta_2 - c_{12})^2 + (\theta_1+\theta_2)^2}, \tag{6.4}$$

where $\theta_n = c_n(x - c_nt)$, $n = 1, 2$ and $c_{12} = \left(\dfrac{c_1+c_2}{c_1-c_2}\right)^2$.

If we define

$$f = -\theta_1\theta_2 + i(\theta_1+\theta_2) + c_{12}, \tag{6.5}$$

then the double soliton (6.4) is also given by

$$u(t,x) = \frac{i}{2}\frac{\partial}{\partial x}\ln\frac{f^*(t,x)}{f(t,x)} \tag{6.6}$$

and

$$Hu(t,x) = -\frac{1}{2}\left[\frac{1}{f(t,x)}\frac{\partial f(t,x)}{\partial x} + \frac{1}{f^*(t,x)}\frac{\partial f^*(t,x)}{\partial x}\right]. \tag{6.7}$$

The double soliton is a superposition of two one-solitons in the following sense:

- if we define $w(t,x) = u(t,x) - u_1(t,x) - u_2(t,x)$, where u_1 and u_2 are the one-solitons with speeds c_1 and c_2, respectively, then

$$\lim_{t\to+\infty} \|w(t)\|_{k,p} = 0 \quad 1 \le p \le +\infty \quad k \in N, \tag{6.8}$$

where $|.|_{k,p}$ denotes the norm in the Sobolev space $W_{k,p}(R)$ (as it has been remarked in [19], unlike the KdV equation, no phase shift appears as the result of collisons between solitons of BO equation);

-

$$\lim_{t\to+\infty} \|u_1^n(t)u_2^m(t)\|_{k,p} = 0 \quad 1 \le p \le +\infty \quad m,n \in N, m,n \ge 1. \tag{6.9}$$

If α and β are constants the Euler-Lagrange equation

$$V_3'(u) + \alpha V_2'(u) + \beta V_1'(u) = 0$$

is

$$-u_{xx} + 3uHu_x + 3H(uu_x) + 4u^3 + \alpha(-Hu_x - 2u^2) + \beta u = 0. \tag{6.10}$$

As in the case of the KdV, we have to find the relationship between the speeds c_1 and c_2 and the multipliers α and β in such way that the double soliton (6.4) satisfies (6.10). Applying the operator H on both sides of Eq. (6.1) and taking in account that $H^2 = -I$ we see that

$$H(uu_x) = \frac{1}{4}\left[u_{xx}(t,x) - \frac{\partial Hu(t,x)}{\partial t}\right].$$

Therefore, all terms in (6.10) can be calculated in terms of f given by (6.5) and performing the calculation we find that (6.10) is satisfied if

$$c_1 + c_2 = \frac{4\alpha}{3} \quad \text{and} \quad c_1c_2 = \frac{4\beta}{3}. \tag{6.11}$$

In other words, for given multipliers α and β in a certain range, the speeds of the double soliton are the positive solutions of the quadratic equation

$$c^2 - \frac{4\alpha}{3}c + \frac{4\beta}{3} = 0. \tag{6.12}$$

The profiles of the corresponding one-solitons are:

$$u_1(x) = \frac{c_1}{c_1^2x^2+1} \quad u_2(x) = \frac{c_2}{c_2^2x^2+1}. \tag{6.13}$$

Definition 4. *We say that the double soliton (6.4) is orbitally stable if defining the two dimensional set* $S = \{u(t, \cdot + \tau), t, \tau \in R\}$, *where* $u(t, \cdot)$ *is the double soliton, then for any* $\epsilon > 0$ *there is* $\delta > 0$ *such that if* $u_0 \in H^1(I\!R)$ *and* $d(u_0, S) < \delta$ *then* $d(u(t, u_0), S) < \epsilon$, *where d is the distance taken in the* $H^1(I\!R)$ *norm and* $u(t, u_0)$ *is the solution of (6.1) such that* $u(0, u_0) = u_0$.

This section is devoted to prove the following:

Theorem 8. *The double soliton (6.4) is orbitally stable.*

The proof follows the same steps as in the case of the KdV equation. However, the proof of Theorem 9 below is much more diffficult than the proof of Theorem 7. The linearized operator for (6.10) at the double soliton is

$$\begin{aligned} L(t)h &= -h_{xx} + 3hHu_x(t,x) + 3u(t,x)Hh_x + 3H(u_x(t,x)h) + 3u(t,x)h_x \\ &\quad + 12u(t,x)^2h + \alpha(-Hh_x - 4u(t,x)h) + \beta h. \end{aligned} \tag{6.14}$$

We denote by L_1 and L_2 the reduced operators obtained by replacing the double soliton $u(t, .)$ in (6.14) by the one-solitons u_1 and u_2 given by (6.13). If we normalize one of the speeds to be equal to one then, in view of (6.12), we have

$$\alpha = \beta + \frac{3}{4}. \tag{6.15}$$

Therefore, the reduced operator calculated at the one-soliton $u = \dfrac{1}{1+x^2}$ with speed one is

$$L_\beta = Q + \beta K, \tag{6.16}$$

where

$$Q = -D^2 + 6U^2 + 3UHD + 3HV + 3HUD - \frac{3}{4}HD \tag{6.17}$$

and

$$K = -HD - 4U + I. \tag{6.18}$$

In (6.17) and (6.18), $v = u' = -2xu^2$, the capitals U and V denote multiplication operators and $D = \dfrac{d}{dx}$. Moreover, $c = 1$ is the lower speed if $\beta < 3/4$ and it is the higher speed if $\beta > 3/4$.

To formulate the spectral condition for stability of the double soliton, according to Sect. 4, first we have to calculate

$$\begin{aligned} V(\alpha, \beta) &= V(u(t, x, \alpha, \beta)) \\ &= V_3(u(t, x, \alpha, \beta)) + \alpha V_2(u(t, x, \alpha, \beta)) + \beta V_1(u(t, x, \alpha, \beta)), \end{aligned}$$

where $u(t, x, \alpha, \beta)$ is the double soliton. Since V_3, V_2 and V_1 are first integrals, to calculate $V(\alpha, \beta)$ we can pass to the limit as $t \to +\infty$. Using estimates (6.8) and (6.9) we see that

$$V(\alpha, \beta) = V_3(u_1) + V_3(u_2) + \alpha(V_2(u_1) + V_2(u_2)) + \beta(V_1(u_1) + V_1(u_2)),$$

where u_1 and u_2 are given by (6.13). If, as before, $u_c(x) = \dfrac{c}{c^2x^2+1}$ is the profile of a simple wave with speed c and we define $I_n = \displaystyle\int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^n}$, we have $I_1 = \pi$, $I_2 = \pi/2$, $I_3 = 3\pi/8$, $I_4 = 15\pi/48$ and

$$\int_{-\infty}^{+\infty} u_c^2(x)\,dx = cI_2, \qquad \int_{-\infty}^{+\infty} u_c^3(x)\,dx = c^2 I_3, \qquad \int_{-\infty}^{+\infty} u_c^4(x)\,dx = c^3 I_4.$$

Morever, since u_c satisfies (6.3) we have

$$\int_{-\infty}^{+\infty} u_c(x) H u_c'(x)\,dx = -2\int_{-\infty}^{+\infty} u_c^3(x)\,dx + c\int_{-\infty}^{+\infty} u_c^2(x)\,dx$$

and

$$\int_{-\infty}^{+\infty} u_c^2(x) H u_c'(x)\,dx = -2\int_{-\infty}^{+\infty} u_c^4(x)\,dx + c\int_{-\infty}^{+\infty} u_c^3(x)\,dx.$$

Furthermore

$$\begin{aligned}\int_{-\infty}^{+\infty} u_c'(x)^2(x)\,dx &= 4c^6\int_{-\infty}^{+\infty} \frac{x^2}{(c^2x^2+1)^4}\,dx = 4c^3\int_{-\infty}^{+\infty} \frac{y^2}{(y^2+1)^4}\,dy \\ &= 4c^3\int_{-\infty}^{+\infty} \frac{(y^2+1)-1}{(y^2+1)^4}\,dy = 4c^3(I_3 - I_4).\end{aligned}$$

Collecting all calculations we get

$$\begin{aligned} V_3(u_c) &= c^3(-2I_4 + 3I_3), \\ V_2(u_c) &= c^2(-2I_3 + 2I_2), \\ V_1(u_c) &= cI_2/2. \end{aligned} \tag{6.19}$$

From (6.11) we also get

$$\begin{aligned} c_1^2 + c_2^2 &= (c_1+c_2)^2 - 2c_1c_2 = \frac{16}{9}\alpha^2 - \frac{8}{3}\beta, \\ c_1^3 + c_2^3 &= (c_1+c_2)(c_1^2 - c_1c_2 + c_2^2) = \frac{64}{27}\alpha^3 - \frac{16}{3}\alpha\beta, \end{aligned} \tag{6.20}$$

and, finally, using (6.19), we find that

$$\begin{aligned} V(\alpha,\beta) &= (c_1^3+c_2^3)(-2I_4+3I_3) + \alpha(c_1^2+c_2^2)(2I_3+2I_2) + \beta(c_1+c_2)I_2/2 \\ &= \frac{\pi}{27}(116\alpha^3 - 189\alpha\beta). \end{aligned}$$

The determinant of the hessian matrix $V''(\alpha,\beta)$ is independent of α and β and it is equal to $-49\pi^2 < 0$; this implies that $V''(\alpha,\beta)$ has exactly one positive and one negative eigenvalue.

The well-posedness of the Cauchy problem for (6.1) in the space $H^1(I\!R)$ has been proved in [23].

From the comments above we see that the proof of Theorem 8 goes exactly as in the case of the KdV equation and the only thing that we have to prove is the counterpart of Theorem 7 which is the following:

Theorem 9.

1. *For* $0 < \beta \neq 3/4$ *the operator* L_β *defined by (6.16) has zero as a simple eigenvalue;*
2. *for* $0 < \beta < 3/4$, L_β *has one negative eigenvalue and for* $3/4 < \beta$ *the operator* L_β *has no negative eigenvalue.*

We use the subscript *odd* to denote space of odd functions and the subscript *ev* to denote space of even functions. Notice that K and Q map even functions in even functions and odd functions in odd functions. As in the case of the KdV equation, the proof of Theorem 9 is a consequence of the following three lemmas:

Lemma 7.

1. *For* $h \in H^1_{odd}(\mathbb{R})$ *we have* $\langle Kh, h\rangle \geq 0$ *and* $\langle Kh, h\rangle = 0$ *if and only if* h *is a multiple of* $v = u'$;
2. *in* H^1_{ev} K *has exactly one negative eigenvalue and zero is not an eigenvalue.*

Lemma 8. *For any* $h \in H^1(\mathbb{R})$ *we have* $\langle Qh, h\rangle \geq 0$ *and* $\langle Qh, h\rangle = 0$ *iff* h *is a multiple of* $u' = v$.

Lemma 9. *For* $\beta = 3/4$ *the function*

$$-u + 2u^2 = \left.\frac{du_c(x)}{dc}\right|_{c=1},$$

where u_c *is given by (6.2), is an (even) eigenfunction of* $L_{3/4}$ *associated to the zero eigenvalue.*

Remark 4. As in the case of the KdV equation, the spectral conditions given by Lemma 7 are precisely the conditions that are needed to prove the orbital stability of one-solitons for the BO equation.

The proof of Theorem 9 admitting Lemmas 7, 8 and 9, follows exactly as the proof of Theorem 7 for the KdV equation and we will not repeat it.

To prove Lemma 9 we simply replace $-u + 2u^2$ in $L_{3/4}$ and we make the calculation according to formulae given in the appendix. The rest of this section will be dedicated to the proofs of Lemmas 7 and 8. As we will see, the proofs of them are much more difficult than the proofs of Lemmas 1 and 2. We start analysing the operator K given by (6.18) which is precisely the linearized operator for the one-soliton. The one-soliton with speed $c = 1$ is

$$u = \frac{1}{1 + x^2} \tag{6.21}$$

and it solves

$$-HDu - 2u^2 + u = 0. \tag{6.22}$$

The linearized operator for (6.22) is the operator K given by (6.18).

Remark 5. In [3] a resolution of the identity for the operator K has been found; in particular, the spectrum of K has been calculated explicitly and Lemma 7 actually follows from that. However, as in the case of the KdV equation, we give a proof for Lemma 7 because the method will be used to prove the much more complicated Lemma 8.

To extend the method we have used for the KdV equation, we have to find identities similar to (5.18). We start defining the following functions:

$$a = a(x) = x^2 - 1; \quad b = b(x) = -2x; \quad u = u(x) = \frac{1}{1+x^2};$$
$$v = v(x) = u'(x) = -2xu^2; \quad w = w(x) = u - 2u^2;$$

and we denote by A, B, U, V and W the operators defined by multiplication by a, b, u, v and w, respectively. Notice that $au = 1 - 2u$. We also define the following bounded operators from $L^2(I\!R)$ into itself:

$$M = AUH + BU \quad N = -AUH + BU, \tag{6.23}$$

and their adjoint

$$M^t = -HAU + BU \quad N^t = HAU + BU. \tag{6.24}$$

Lemma 10. *Setting*

$$K_0 = -HD + I, \tag{6.25}$$

then the following identity holds

$$MKM^t = NK_0N^t. \tag{6.26}$$

The proof of Lemma 10 will be given in the appendix.

Remark 6. If we look at identity (5.18), instead of (6.26) it would be more natural to look for an identity of the type:

$$MKM^t = M^tK_0M \tag{6.27}$$

(that is the way we started). Formally, (6.27) holds with $M = H + \dfrac{2x}{1-x^2}$. To eliminate the singularity we multiply both sides of (6.27) on the right and on the left by $1 - x^2$. We also compose the resulting equation with U to get bounded auxiliary operators and the result is identity (6.26).

To use identity (6.26) to prove Lemma 7, we need to calculate the image and the null space of the auxiliary operators appearing there. Notice that if h is an even (odd) function, then $M(h)$, $N(h)$, $M^t(h)$, $N^t(h)$ are odd (even). Denoting by $Ker(T)$ and $Im(T)$ the null space and the image of a bounded operator T then the following is true:

Lemma 11. *If M and N are considered from $L^2(I\!R)$ into itself then*

1. $Ker(M) = [w, xw]$;
2. $Ker(N^t) = [u, xu]$;
3. $Im(M^t) = [w, xw]^\perp$;
4. $Im(N) = [u, xu]^\perp$;
5. $MN = I$;
6. $N^tM^t = I$.

In particular, M^t and N are one-to-one and M and N^t are onto.

The proof of Lemma 11 will be given in the appendix. We denote by $H^1_{ev}(I\!R)$ the set of the elements of $H^1(I\!R)$ which are even and by $H^1_{odd}(I\!R)$ the odd ones.

Proof of Lemma 7. First we notice that $\langle K_0 s, s\rangle \geq 0$ and $\langle K_0 s, s\rangle = 0$ iff $s = 0$ because $\widehat{-Hs'}(\xi) = |\xi|\hat{s}(\xi)$ (see the appendix.) If h is odd then, according to Lemma 11, Part 3, h can be decomposed as $h = \alpha x w + M^t k$, where k is even and then, using (6.26) we get

$$\begin{aligned}\langle Kh, h\rangle &= \alpha^2\langle K(xw), xw\rangle + 2\alpha\langle MK(xw), k\rangle + \langle MKM^t k, k\rangle \\ &= \frac{\pi\alpha^2}{4} + \langle NK_0N^t k, k\rangle\end{aligned}$$

because $MK(xw) = M(xw) = 0$ and $\langle K(xw), xw\rangle = \pi/4$. Therefore, $\langle Kh, h\rangle \geq 0$ and $\langle Kh, h\rangle = 0$ iff $\alpha = 0$ and $N^t k = 0$ and this implies $k = cu$ (Lemma 11, Part 2). Hence, $h = cM^t u = -4cxu^2 = 2cu'$ and this proves the first part of Lemma 7.

To prove the second, we first notice that $\langle Ku, u\rangle < 0$ because $K(u) = -2u^2$. This implies that K has at least one negative eigenvalue in the space of the even functions. Moreover, if h is even then, according to Lemma 11, Part 3, we can make the decomposition $h = \alpha w + M^t k$, k odd. Taking $\alpha = 0$ and using identity (6.26) we see that $\langle Kh, h\rangle = \langle NK_0N^t k, k\rangle \geq 0$ and $\langle Kh, h\rangle = 0$ only if $N^t k = 0$ and this implies $k = cxu$ (Lemma 11, Part 2). We conclude that $\langle Kh, h\rangle \geq 0$ for h belonging to the codimension one subspace $Im(M^t)_{ev}$ and then K cannot have two negative eigenvalues. It remains to prove that zero is not an eigenvalue of K in $H^1_{ev}(I\!R)$. By contradiction, suppose K has one negative eigenvalue with eigenfunction ϕ_1 and one zero eigenvalue with eigenfunction ϕ_2. In this case, the codimension one subspace $Im(M^t)_{ev}$ has to intercept the subspace spanned by ϕ_1 and ϕ_2; since $\langle Kh, h\rangle \leq 0$ in $S = [\phi_1, \phi_2]$ and $\langle Kh, h\rangle = 0$ for $h \in S$ only if h is a multiple of ϕ_2 we see that $Im(M^t)_{ev}$ has to intercept S in $[\phi_2]$. However, as before, $\langle Kh, h\rangle = 0$ for $h \in Im(M^t)_{ev}$ only if h is a multiple of $M^t(xu) = 4(u^2 - u)$; since the function $u^2 - u$ is not an eigenfunction of K associated to the zero eigenvalue we conclude that K cannot have a zero eigenvalue in the space of the even functions and Lemma 7 is proved.

Now we concentrate on the proof of Lemma 8 which will be broken in several lemmas. As in the case of the KdV equation, the spectrum of Q considered in both $H^1_{odd}(I\!R)$ and $H^1_{ev}(I\!R)$ accumulate at zero (from the right.) Arguing as in the case of the KdV equation, we start trying to find an identity of type (5.22) involving the operator Q under consideration and a "better" operator Q_0. In view of identities (5.18), (5.22) and (6.26), perhaps we should expect that the following identity holds

$$MQM^t = N(-D^2 - \frac{3}{4}HD)N^t,$$

because $-D^2 - \frac{3}{4}HD$ is the part of Q that has constant coefficients. Regretfully, this identity is false! The next attempt is to find coefficients k_1, k_2 and k_3 in such way that

$$Q_0 = -D^2 + k_1U + k_2U^2 + k_3UHD + k_3HV + k_3HUD - \frac{3}{4}HD$$

is "better" than Q and

$$MQM^t = NQ_0N^t. \tag{6.28}$$

We show how to accomplish that. According to formulae presented in the appendix, for any integer m the operators

$$H(u^m h) - u^m Hh \quad \text{and} \quad H(xu^m h) - xu^m Hh$$

have finite dimensional range. Therefore, modulo a finite dimensional range operator, H commutes with U^m and xU^m. With this fact in mind and expanding both sides of (6.28), we see that the infinite dimensional part of (6.28) is satisfied if $k_1 = 3$, $k_2 = -2$ and $k_3 = -1$. Using these coefficients to define the operator

$$Q_0 = -D^2 + 3U - 2U^2 - UHD - HV - HUD - \frac{3}{4}HD \tag{6.29}$$

we start with □

Lemma 12. *The following identity holds:*

$$Q_0 = (HD - U)^2 + 3U(1 - U) - \frac{3}{4}HD. \tag{6.30}$$

In particular $\langle Q_0 h, h\rangle \geq 0$ *and* $\langle Q_0 h, h\rangle = 0$ *iff* $h = 0$.

Proof. The verification of (6.30) is trivial. The final statement follows from $0 < u(x) \leq 1$ and $u(x) = 1$ only for $x = 0$ and the lemma is proved.

Since Q_0 is positive and we want to prove that Q is positive, some progress has been made. However, using Q_0 defined by (6.29) to calculate $MQM^t - NQ_0N^t$ we see that this difference is a nonzero operator with finite dimensional range. In other words, the infinite dimensional part of both sides of (6.28) are equal but the difference of them contains a residue (an operator with finite dimensional range). We state this as □

Lemma 13. *If* M, N, Q *and* Q_0 *are as above then*

$$MQM^t = NQ_0N^t + R, \tag{6.31}$$

where

$$Rk = \frac{1}{\pi}\Big[\langle 9u - 12u^2, k\rangle u + \langle -12u + 16u^2, k\rangle u^2 \tag{6.32}$$

$$+\langle xu + 8xu^2, k\rangle xu + \langle 8xu - 16xu^2, k\rangle xu^2\Big]. \tag{6.33}$$

The proof of Lemma 13 will be given in the appendix. For reasons that will be given later, identity (6.31) still is not convenient. To prove Lemma 8 we will use variants of (6.31) which are obtained perturbing M by an operator with finite dimensional range; in other words, we will use identities like

$$M_1QM_1^t = NQ_0N^t + R_1. \tag{6.34}$$

Suppose we have proved an identity of type (6.34) in such way that for any $h \in H^1(I\!R)$, there is a decomposition $h = \alpha\phi + M_1^t k$ where ϕ spans $ker(M_1)$. Then

$$\begin{aligned}\langle Qh, h\rangle &= \alpha^2\langle Q\phi, \phi\rangle + 2\alpha\langle Q\phi, M_1^t k\rangle + \langle QM_1^t k, M_1^t k\rangle \\ &= \alpha^2\langle Q\phi, \phi\rangle + 2\alpha\langle M_1Q\phi, k\rangle + \langle M_1QM_1^t k, k\rangle.\end{aligned}$$

Defining $s = k + \alpha q$, where q will be found, and denoting by T the right-hand side of (6.34) we have

$$\begin{aligned}\langle Qh, h\rangle = \alpha^2 &(\langle Q\phi, \phi\rangle - \langle Tq, q\rangle - 2\langle M_1Q\phi - Tq, q\rangle) \\ &+2\alpha\langle(M_1Q\phi - Tq), s\rangle + \langle Ts, s\rangle.\end{aligned} \tag{6.35}$$

We look at this last equality as a quadratic form in α and s and to eliminate the cross term we have to choose ϕ and q in such way that

$$M_1 Q\phi = Tq. \tag{6.36}$$

In this case,

$$\langle Qh, h\rangle = \alpha^2(\langle Q\phi, \phi\rangle - \langle Tq, q\rangle) + \langle Ts, s\rangle. \tag{6.37}$$

Since we want to show that Q is positive, the solution q of (6.36) has to be known explicitly because we need to know that the coefficient of α^2 in (6.37) is positive. Also, we have to choose certain parameters appearing in the definition of M_1 in such way that we can conclude that the operator T given by the right side of (6.34) is positive. We analyse the case of even and odd functions separately. As we will see, the case of even functions is much more difficult.

We start with the case of odd functions. The reason for (6.31) not to be convenient is that, apparently, Eq. (6.36) cannot be solved explicitly in q for ϕ in the Kernel of M. Our next attempt is to perturb M in order to incorporate the residue R in the perturbed operator. To be more specific we have:

Lemma 14. *Defining*

$$M_1(h) = M(h) + m_5\langle xu, h\rangle u + m_6\langle xu, h\rangle u^2 \tag{6.38}$$

for h odd and

$$M_1^t(k) = M^t(k) + m_5\langle u, k\rangle xu + m_6\langle u^2, k\rangle xu \tag{6.39}$$

for k even, with $m_5 = 12/\pi$ and $m_6 = -16/\pi$, and M and M^t given by (6.23) and (6.24), then for $k \in H^2_{ev}(I\!R)$ we have

$$M_1 Q M_1^t(k) = N Q_0 N^t(k). \tag{6.40}$$

The proof of Lemma 14 will be given in the appendix. Notice that M_1 (as M and N) maps even functions in odd functions and odd functions in even functions. To carry out the procedure above we need the following proposition whose proof is also left to the appendix:

Lemma 15. *If $\phi = -5xu^2 + 8xu^3$ then $Ker(M_1) = [\phi]$ and $Im(M_1^t) = [\phi]^\perp$, where $ker(M_1)$ is taken in the space of odd functions and $Im(M_1^t)$ is calculated for M_1^t as a map from even functions into odd functions.*

Proof of Lemma 8 for h odd. If $h \in H^1_{odd}(I\!R)$ then, according to Lemma 15, we have the decomposition $h = \alpha\phi + M_1^t k$ with k even. If we define $s = k + \alpha q$ then $\langle Qh, h\rangle$ is given by (6.35), where $T = NQ_0N^t$ (the right-hand side of (6.40)). Moreover, taking $q = -2u^2$ we have $M_1 Q\phi = Tq$ and then, according to (6.37) and performing some calculation, we get

$$\langle Qh, h\rangle = \frac{9\pi\alpha^2}{64} + \langle NQ_0N^t s, s\rangle.$$

Since Q_0 is positive definite by Lemma 12, we conclude that $\langle Qh, h\rangle \geq 0$ and $\langle Qh, h\rangle = 0$ iff $\alpha = 0$ and $N^t s = 0 = N^t k$. In this case, according to Lemma 11, Part 2, k has to be a multiple of u and this implies that h has to be a multiple of $M_1^t(u) = 2u'$ and Lemma 8 is proved for odd functions.

Now we turn to the more complicated case of even functions. We start with the following identity whose proof is left to the appendix: □

Lemma 16. *Defining*

$$M_2 h = Mh + \frac{32}{\pi}\langle u^2, h\rangle x u^2 \tag{6.41}$$

for h even and

$$M_2^t k = M^t k + \frac{32}{\pi}\langle x u^2, k\rangle u^2 \tag{6.42}$$

for k odd, where M and M^t are given by (6.23) and (6.24), then for k odd we have

$$M_2 Q M_2^t(k) = N Q_0 N^t(k) + R_2(k), \tag{6.43}$$

where

$$R_2(k) = c_{11}\langle xu, k\rangle xu + c_{12}\langle xu^2, k\rangle xu + c_{12}\langle xu, k\rangle xu^2 + c_{22}\langle xu^2, k\rangle xu^2 \tag{6.44}$$

with

$$c_{11} = \frac{1}{\pi} \qquad c_{12} = -\frac{28}{\pi} \qquad c_{22} = \frac{400}{\pi}. \tag{6.45}$$

Remark 7. If instead of (6.41) we define

$$M_2(h) = M(h) + m_1\langle u, h\rangle xu + m_2\langle u, h\rangle xu^2 + m_3\langle u^2, h\rangle xu + m_4\langle u^2, h\rangle xu^2,$$

where the parameters m_4 and m_2 are given by

$$m_4 = -4m_3 - \frac{32}{\pi} \qquad m_2 = -4m_1 + \frac{32}{\pi}$$

and m_1 and m_3 belong to the (nonempty) ellipse

$$13\pi^2 m_1^2 + 19\pi^2 m_1 m_3 + 7\pi^2 m_3^2 - 48\pi m_1 - 36\pi m_3 + 16 = 0,$$

then (6.43) holds with no residue ($R_2 = 0$). However, in that case, $Ker M_2$ has dimension two and, apparently, the equation

$$M_2 Q\phi = Tq = N Q_0 N^t q$$

cannot be explicitly solved in q for two linearly independent functions ϕ belonging to $Ker(M_2)$. Probably this is related to the fact that the infimum of $\langle Qh, h\rangle$ in $H^1_{ev}(I\!R)$ (which is zero) is not achieved (at a nonzero element). Besides the choice given by (6.41), which corresponds to $m_1 = m_2 = m_3 = 0$, $m_4 = \frac{32}{\pi}$, there may be others that may work (with different nonzero residue R_2).

Lemma 17. *If $T = N Q_0 N^t + R_2$ is the right-hand side of (6.43) and s is odd then $\langle Ts, s\rangle \geq 0$ and $\langle Ts, s\rangle = 0$ iff $s = 0$.*

Proof. We define $N^t s = p$ and then $s = \alpha x u + M^t p$ (Lemma 11, Parts 4 and 6) and

$$\langle xu, s\rangle = \langle xu, \alpha xu + M^t p\rangle = \alpha\langle xu, xu\rangle + \langle M(xu), p\rangle = \frac{\alpha\pi}{2} - \langle u, p\rangle$$

and

$$\langle xu^2, s\rangle = \langle xu^2, \alpha xu + M^t p\rangle = \alpha\langle xu^2, xu^2\rangle + \langle M(xu^2), p\rangle = \frac{\alpha\pi}{8} - \frac{1}{2}\langle u, p\rangle.$$

Therefore,

$$\langle R_2 s, s\rangle = \frac{\pi^2}{64}(16c_{11} + 8c_{12} + c_{22})\alpha^2 + \frac{\pi}{8}(-8c_{11} - 6c_{12} - c_{22})\langle u, p\rangle\alpha + \frac{1}{4}(4c_{11} + 4c_{12} + c_{22})\langle u, p\rangle^2. \quad (6.46)$$

Since $16c_{11} + 8c_{12} + c_{22} = 192/\pi$, the coefficient of α^2 in (6.46) is positive and then the minimum of (6.46) considered as a quadratic function of α is achieved at $\alpha = \frac{5}{\pi^2}\langle u, p\rangle$ and the value of this minimum is $-\frac{2}{\pi}\langle u, p\rangle^2$. Then $\langle R_2 s, s\rangle \geq -\frac{2}{\pi}\langle u, p\rangle^2$ and this implies that

$$\langle Ts, s\rangle \geq \langle Q_0 p, p\rangle - \frac{2}{\pi}\langle u, p\rangle^2 \quad (6.47)$$

and equality holds only for $\alpha = \frac{5}{\pi^2}\langle u, p\rangle$. Hence, Lemma 17 is a consequence of the following: □

Lemma 18. *Consider the operator*

$$T_0(p) = Q_0 p - \frac{2}{\pi}\langle u, p\rangle u$$

acting on even functions. Then $\langle T_0 p, p\rangle \geq 0$ *and* $\langle T_0 p, p\rangle = 0$ *if and only if* $p = 0$.

Remark 8. According to Lemma 12, the operator Q_0 is positive but it is easy to see that zero belongs to its essential spectrum. Therefore, the positivity of Q_0 is very sensitive to negative perturbation in any direction. The constant $\frac{2}{\pi}$ is optimal for Lemma 18 to be true.

Proof of Lemma 18. First let us notice that for any two functions p, z in the space $H^1(\mathbb{R})$ we have

$$\langle Q_0 z, p\rangle^2 \leq \langle Q_0 p, p\rangle\langle Q_0 z, z\rangle \quad (6.48)$$

and equality holds if and only if p and z are linearly dependent. This follows from Schwarz inequality because according to Lemma 12, the bilinear form $[p, z] = \langle Q_0 p, z\rangle$ is a scalar product.

Formally, Lemma 18 follows from (6.48) taking $z \equiv 1$ because $Q_0(1) = 2u$ and $\langle u, 1\rangle = \pi$. Since the function $z \equiv 1$ does not belong to the space $H^1(\mathbb{R})$, that procedure has to be justified. This will be done next.

Let $\phi : I\!R \to I\!R$ be an even function belonging to $H^2(I\!R)$ such that $\phi(0) = 1$ and let us define $\phi_\lambda(x) = \phi(\frac{x}{\lambda})$. Then $H\phi \in H^2(I\!R)$ and $(H\phi_\lambda)(x) = (H\phi)(x/\lambda)$; in particular $H\phi \in L_\infty$. Taking $z = \phi_\lambda$ in (6.48) we get

$$\langle Q_0\phi_\lambda, p\rangle^2 \leq \langle Q_0, p\rangle\langle Q_0\phi_\lambda, \phi_\lambda\rangle. \tag{6.49}$$

We interrupt the proof of Lemma 18 to analyse the limit of (6.49) as λ tends to infinity. □

Lemma 19. *As λ tends to infinity*

1. $\langle Q_0\phi_\lambda, p\rangle$ *tends to* $\langle 2u, p\rangle$*;*
2. $\langle Q_0\phi_\lambda, \phi_\lambda\rangle$ *tends to* $\langle 2u, 1\rangle - \frac{3}{4}\langle HD\phi, \phi\rangle = 2\pi - \frac{3}{4}\langle HD\phi, \phi\rangle$.

Proof. The first limit is easier because it is linear in ϕ_λ and so we will prove only the second. A change of variables shows that:

$$|\phi'_\lambda|_{L^2} = \lambda^{-\frac{1}{2}}|\phi|_{L^2}, \tag{6.50}$$

$$|\phi''_\lambda|_{L^2} = \lambda^{-\frac{3}{2}}|\phi|_{L^2}. \tag{6.51}$$

We expand $\langle Q_0\phi_\lambda, \phi_\lambda\rangle$ and first we collect the terms that tend to zero:

$$\begin{aligned}
|\langle\phi''_\lambda, \phi_\lambda\rangle| &= |\langle\phi'_\lambda, \phi'_\lambda\rangle|\frac{1}{\lambda}|\phi|^2_{L^2};\\
|\langle uH\phi'_\lambda, \phi_\lambda\rangle| &= |\langle\phi'_\lambda, H(u\phi_\lambda)\rangle| \leq |\phi'_\lambda|_{L^2}|\phi_\lambda|_{L^\infty}|u|_{L^2}\\
&= |\phi'_\lambda|_{L^2}|\phi|_{L^\infty}|u|_{L^2};\\
|\langle H(u\phi'_\lambda), \phi_\lambda\rangle| &= |\langle u\phi'_\lambda, H\phi_\lambda\rangle| \leq |u\phi'_\lambda|_{L^1}|H\phi_\lambda|_{L^\infty} \leq |u|_{L^2}|\phi'_\lambda|_{L^2}||H\phi_\lambda|_{L^\infty}\\
&= \lambda^{-1/2}|u|_{L^2}|\phi|_{L^2}|H\phi|_{L^\infty}.
\end{aligned}$$

The next terms:

$$\int_{-\infty}^{+\infty}(3u - 2u^2)\phi_\lambda^2\,dx \to \int_{-\infty}^{+\infty}(3u - 2u^2)\,dx \quad (Lebesgue)$$

$$\int_{-\infty}^{+\infty} H(v\phi_\lambda)\phi_\lambda\,dx = \int_{-\infty}^{+\infty} H(v\phi_\lambda)(\phi_\lambda - 1)\,dx + \int_{-\infty}^{+\infty} H(v)\phi_\lambda\,dx$$

and

$$\begin{aligned}
\left|\int_{-\infty}^{+\infty} H(v\phi_\lambda)(\phi_\lambda - 1)\,dx\right| &\leq |(v(\phi_\lambda - 1)|_{L^1}|H\phi_\lambda|_{L^\infty}\\
&= |(v(\phi_\lambda - 1)|_{L^1}|H\phi|_{L^\infty} \to 0(Lebesgue)
\end{aligned}$$

and

$$\int_{-\infty}^{+\infty} H(v)\phi_\lambda\,dx \to \int_{-\infty}^{+\infty} H(v)\,dx = \int_{-\infty}^{+\infty}(u - 2u^2)\,dx \quad (Lebesgue).$$

Furthermore,

$$\langle -H\phi'_\lambda, \phi_\lambda\rangle = \int_{-\infty}^{+\infty}\phi'_\lambda(H\phi_\lambda)\,dx = \int_{-\infty}^{+\infty}\phi'(H\phi)\,dx = \langle -H\phi', \phi\rangle\,dx$$

and the lemma is proved.

We see that the term $\langle -HD\phi_\lambda, \phi_\lambda\rangle$ is self similar with respect to dilations and then a further limit process will be required. □

End of proof of Lemma 18. According to Lemma 19, for any even function $\phi \in H^2(I\!R)$ such that $\phi(0) = 1$ and any $h \in H^1(I\!R)$ the following inequality holds:

$$4\langle u, p\rangle^2 \leq \langle Q_0 p, p\rangle(2\pi - \langle HD\phi, \phi\rangle). \tag{6.52}$$

To get rid of the term $\langle HD\phi, \phi\rangle$ we claim the following: there is a sequence of even functions $\phi_n \in H^2(I\!R)$ such that $\phi_n(0) = 1$ and $\langle HD\phi_n, \phi_n\rangle$ tends to zero. In fact, denoting by $f_n(\xi) = \hat{\phi_n}(\xi)$ the Fourier transform of ϕ and using the inverse Fourier transform we have

$$\phi_n(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi} f_n(\xi)\, d\xi$$

and then

$$\phi_n(0) = (2\pi)^{-1}\int_{-\infty}^{+\infty} f_n(\xi)\, d\xi.$$

Therefore, it is sufficient to show that there is a sequence f_n satisfying the following conditions:

1. f_n is real, even, bounded and has support in the interval $[-1, 1]$;
2. $\int_{-\infty}^{+\infty} f_n(\xi)\, d\xi = 2\pi$;
3. $\int_0^{+\infty} |\xi| f_n^2(\xi)\, d\xi$ tends to zero as n tends to infinity.

Let f_n be the sequence of even functions defined in the following way for $\xi \geq 0$:

$$f_n(\xi) = \frac{c_n}{\xi} \quad \text{for} \quad \frac{1}{n} \leq \xi \leq 1,$$
$$f_n(\xi) = 0 \quad \text{for} \quad \xi > 1.$$

Imposing $\int_{-\infty}^{+\infty} f_n(\xi)\, d\xi = 2\pi$ we get $c_n = \dfrac{\pi}{\log n}$ and then $\int_0^{+\infty} |\xi| f^2(\xi)\, d\xi = \dfrac{\pi^2}{4\log n}$ and this proves the claim.

Replacing ϕ by ϕ_n in (6.52) and passing to the limit in n we get $\langle T_0 p, p\rangle \geq 0$ and this proves the first part of Lemma 18.

Suppose now that for some $0 \neq p_0 \in H^1_{ev}(I\!R)$ we have $\langle T_0 p_0, p_0\rangle = 0$. In this case we must have $T_0 p_0 = Q_0 p_0 - \dfrac{2}{\pi}\langle u, p_0\rangle u = 0$ because $\langle T_0 p, p\rangle$ has a minimum at $p = p_0$. If we had $\langle u, p_0\rangle = 0$ then $Q_0 p_0 = 0$ and this would imply $p_0 = 0$ (Lemma 12). Therefore, multiplying p_0 by some constant, we may assume that $Q_0 p_0 = 2u$. Since Q_0 is injective and formally $Q_0(1) = 2u$, we try to show that $Q_0 p_0 = 2u$ implies $p_0 = 1$ a.e. In order to justify this conclusion we proceed in the following way: from $Q_0 p_0 = 2u$ we get $\langle p_0, Q_0\phi_\lambda\rangle = 2\langle u, \phi_\lambda\rangle$ and from Lemma 19 we also get $2\langle p_0, u\rangle = 2\langle u, 1\rangle$. Moreover, as λ tends to infinity

$$\langle Q_0(p_0 - \phi_\lambda), (p_0 - \phi_\lambda)\rangle = \langle Q_0 p_0, s_0\rangle - 2\langle Q_0 p_0, \phi_\lambda\rangle + \langle Q_0\phi_\lambda, \phi_\lambda\rangle$$

tends to $-\frac{3}{4}\langle HD\phi, \phi\rangle$. Furthermore, from (6.30) we conclude that for any finite real number b we have

$$\int_{-b}^{b} 3u(1-u)(p_0-\phi_\lambda)^2 \leq \langle Q_0(p_0-\phi_\lambda), (p_0-\phi_\lambda)\rangle,$$

and then, passing to the limit in λ, we get

$$\int_{-b}^{b} 3u(1-u)(p_0-1)^2 \leq -\frac{3}{4}\langle HD\phi, \phi\rangle.$$

Replacing ϕ by the sequence ϕ_n as above and passing to the limit in n, we conclude that $p_0 = 1$ a.e. and this is a contradiction because $p_0 \in H^1(I\!R)$ and Lemma 18 is proved.

The next lemma, whose proof will be given in the appendix, provides a useful decomposition of even functions. □

Lemma 20. *If* $\phi = \frac{3}{16}u - \frac{9}{8}u^2 + u^3$, *then* $Ker(M_2) = [\phi]$ *and* $Im(M_2^t) = [\phi]^\perp$, *where* $Ker(M_2)$ *is calculated on even functions and* $Im(M_2^t)$ *is calculated for* M_2^t *as a map from odd functions into even functions.*

Proof of Lemma 8 for h even. For any $h \in H^1_{ev}(I\!R)$, according to Lemma 20, we have the decomposition $h = \alpha\phi + M_2^t k$. If we set $s = k + \alpha q$, where q will be chosen, we get

$$\begin{aligned}\langle Qh, h\rangle = \alpha^2(\langle Q\phi, \phi\rangle - \langle Tq, q\rangle - 2\langle M_2Q\phi - Tq, q\rangle)\\ +2\alpha\langle(M_2Q\phi - Tq), s\rangle + \langle Ts, s\rangle,\end{aligned}$$

where T is the right-hand side of (6.43). If we take $q = -\frac{11}{64}xu + \frac{1}{4}xu^2$ we have $M_2Q\phi = Tq$ and then

$$\langle Qh, h\rangle = \alpha^2(\langle Q\phi, \phi\rangle - \langle Tq, q\rangle) + \langle Ts, s\rangle = \frac{9}{4096}\alpha^2 + \langle Ts, s\rangle,$$

and the conclusion follows from Lemma 17 and Lemma 8 is proved. □

7. Appendix

In this section we recall some properties of the Hilbert transform and we prove some lemmas involving it. Following [3], we define the Hilbert transform by:

$$(Hf)(x) = PV\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x-y)}{y}\,dy = PV\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x+y)}{y}\,dy, \tag{7.1}$$

where PV stands for principal value. In some classical books on Harmonic Analysis ([22], for instance) the definition of Hilbert transform is (7.1) with a minus sign in front of it.

Some properties of Hilbert transform.

1. $\widehat{Hf}(\xi) = i\,sign(\xi)\hat{f}(\xi)$; ([22]);
2. H is a bounded operator from $L^2(I\!R)$ into itself (follows from 1);
3. $H^2 = -I$, $H^* = -H$ and $\widehat{-HDf}(\xi) = |\xi|\hat{f}(\xi)$ (follow from 1);

4. H commutes with translation; in particular, $HD = DH$.

□

Some formulae. Let us recall the definition and some identities involving functions previously defined:

$$a = x^2 - 1; \quad b = -2x; \quad u = \frac{1}{1+x^2}; \quad au = 1 - 2u;$$
$$x^2u = 1 - 2u; \quad v = u_x = -2xu^2; \quad w = u - 2u^2 = au^2.$$

□

Theorem 10. *For a dense set of functions in $L^2(I\!R)$ the following identities hold:*

$$H(xh) = xH(h) + \frac{1}{\pi}\langle 1, h\rangle, \tag{7.2}$$
$$H(x^2h) = x^2H(h) + \frac{1}{\pi}[\langle x, h\rangle + \langle 1, h\rangle x], \tag{7.3}$$
$$H(uh) = uH(h) + \frac{1}{\pi}[-\langle xu, h\rangle u - \langle u, h\rangle xu], \tag{7.4}$$
$$H(ah) = aH(h) + \frac{1}{\pi}[\langle x, h\rangle + \langle 1, h\rangle x], \tag{7.5}$$
$$H(auh) = auH(h) + \frac{1}{\pi}[\langle 2xu, h\rangle u + \langle 2u, h\rangle xu], \tag{7.6}$$
$$H(u^2h) = u^2H(h) + \frac{1}{\pi}[-\langle xu^2, h\rangle u - \langle xu, h\rangle u^2 - \langle u^2, h\rangle xu - \langle u, h\rangle xu^2], \tag{7.7}$$
$$H(xuh) = xuH(h) + \frac{1}{\pi}[\langle u, h\rangle u - \langle xu, h\rangle xu], \tag{7.8}$$
$$H(xu^2h) = xu^2H(h) + \frac{1}{\pi}[\langle -u + u^2, h\rangle u + \langle u, h\rangle u^2 - \langle xu^2, h\rangle xu - \langle xu, h\rangle xu^2], \tag{7.9}$$
$$H(xh_x) = xH(h_x), \tag{7.10}$$
$$H(x^2h_x) = x^2H(h_x) - \frac{1}{\pi}\langle 1, h\rangle, \tag{7.11}$$
$$H(uh_x) = uH(h_x) + \frac{1}{\pi}[\langle -u + 2u^2, h\rangle u - \langle 2xu^2, h\rangle xu], \tag{7.12}$$
$$H(xuh_x) = xuH(h_x) + \frac{1}{\pi}[\langle 2xu^2, h\rangle u + \langle -u + 2u^2, h\rangle xu], \tag{7.13}$$
$$H(xh_{xx}) = xH(h_{xx}), \tag{7.14}$$
$$H(x^2h_{xx}) = x^2H(h_{xx}). \tag{7.15}$$

Proof. Equation (7.2) follows from (7.1) and (7.3) follows by iteration of (7.2). From (7.3) we get

$$H((1+x^2)h) = (1+x^2)H(h) + \frac{1}{\pi}[\langle x, h\rangle + \langle 1, h\rangle x];$$

if in this last identity we replace h by uh and divide the result by $1 + x^2$ we get (7.4). All the other formulae follow from the first three and the theorem is proved. □

The Hilbert transform of special functions.

$$H(u) = -xu \quad H(xu) = u \quad H(u^2) = -xu^2 - \frac{1}{2}xu,$$
$$H(xu^2) = u^2 - \frac{1}{2}u \quad H(u^3) = -xu^3 - \frac{1}{2}xu^2 - \frac{3}{8}xu.$$

The first formula is proved in [3] and the rest follows from formulae given in Theorem 10.

Now we prove some lemmas stated in section 6. □

Proof of Lemma 11.

Proof. Using the Hilbert transform of the special functions given above it is easy to see that

$$[w, xw] \subset Ker(M).$$

On the other hand, if $h \in Ker(M)$, we have

$$auH(h) - 2xuh = 0, \tag{7.16}$$

and then applying H to both sides of (7.16) and using the relations (7.6) and (7.8) we get

$$-auh - 2xuH(h) = \alpha u + \beta xu, \tag{7.17}$$

where α and β are constants. Since

$$det \begin{pmatrix} -2xu & au \\ -au & -2xu \end{pmatrix} = 1,$$

if we look at (7.16) and (7.17) as a system in h and $H(h)$ and solve it in h, we get $h = au(\alpha u + \beta xu) = \alpha w + \beta xw$ and this proves Part 1.

From the definitions of the operators M and N^t it follows that $M(auh) = auN^t(h)$. Therefore, Part 2 follows from Part 1.

We always have $Im(M^t) \subset Ker(M)^{\perp}$ and then we have only to prove that $[w, xw]^{\perp} = [au^2, xau^2]^{\perp} \subset Im(M^t)$. If $h \in [au^2, xau^2]^{\perp}$ then, using the Hilbert transform of the special functions u, xu and the formula (7.6) with h replaced by $H(auh)$, it is easy to see that

$$H(auH(auh)) = -a^2u^2h.$$

From the formula (7.8) with auh instead h we obtain

$$H(xau^2h) = H(xu\ auh) = xuH(auh).$$

Hence, taking $f = N^t(h) = H(auh) - 2xuh$, we have

$$\begin{aligned} M^t(f) &= -H(auH(auh) - 2xau^2h) - 2xu(H(auh) - 2xuh) \\ &= (a^2u^2 + 4x^2u^2)h = h, \end{aligned}$$

and this proves Part 3.

If $h \in [u, xu]^{\perp}$, one can verify that $N(M(h)) = h$ and this proves Part 4.

Part 5 follows from the relations (7.6) and (7.8) and Lemma 11 is proved.

The proofs of Lemmas 15 and 20 follow in a similar way. □

Proof of Lemma 10. Using that $H^2 = -I$ and $HD = DH$, we first expand both sides of (6.26):

$$\begin{aligned}
&(AUH + BU)(-HD - 4U + I)(-HAU + BU)(h)\\
&\quad = -auH(a_x uh) - auH(auh_x) + ab_x u^2 h + abu^2 h_x + 4auH(uH(auh))\\
&\qquad -4auH(bu^2 h) + a^2u^2 h + auH(buh) - ba_x u^2 h - bau^2 h_x - buH(b_x uh)\\
&\qquad -buH(buh_x) + 4bu^2 H(auh) - 4b^2u^3 h - buH(auh) + b^2u^2 h
\end{aligned} \tag{7.18}$$

and

$$\begin{aligned}
&(-AUH + BU)(-HD + I)(HAU + BU)(h)\\
&\quad = -auH(a_x uh) - auH(auh_x) - ab_x u^2 h - abu^2 h_x + a^2u^2 h - auH(buh)\\
&\qquad +ba_x u^2 h + bau^2 h_x - buH(b_x uh) - buH(buh_x) + buH(auh) + b^2u^2 h.
\end{aligned} \tag{7.19}$$

Making the difference (7.18) - (7.19) we get

$$\begin{aligned}
&2ab_x u^2 h - 2a_x bu^2 h + 4auH(uH(auh)) - 4auH(bu^2 h) + 2auH(buh)\\
&\quad +4bu^2 H(auh) - 4b^2u^3 h - 2buH(auh).
\end{aligned} \tag{7.20}$$

Using the relations (7.2)-(7.15), the expression (7.20) can be rewritten in the form

$$\begin{aligned}
&(2ab_x u^2 - 2a_x bu^2 - 4a^2u^3 - 4b^2u^3)h\\
&\quad +(-4au^3 b + 2abu^2 + 4au^3 b - 2abu^2)H(h) + R,
\end{aligned}$$

where R contains all the finite dimensional terms. If we replace the values of a, b and u in this last expression we see that the infinite dimensional part (the coefficients of h and Hh) are zero as well as the finite dimensional part R and the lemma is proved.

The proofs of the identities given by Lemmas 13, 14 and 16 involve much longer calculations but, similarly to the proof of Lemma 10, they follow by expansion of both sides and using formulae given by Theorem 10.

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