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Vortex Equations in Abelian Gauged σ-Models

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Abstract: We consider nonlinear gauged σ-models with Kähler domain and target. For a special choice of potential these models admit Bogomolny (or self-duality) equations — the so-called vortex equations. Here we describe the space of solutions and energy spectrum of the vortex equations when the gauge group is a torus T^n, the domain is compact, and the target is $\mathbb{C}^n$ or $\mathbb{CP}^n$. We also obtain a large family of solutions when the target is a compact Kähler toric manifold.

Contents

1. Introduction

The most general bosonic theories without gravity are the so-called nonlinear gauged σ-models, also known as general Yang-Mills theories with matter. These theories have been studied in the theoretical physics literature for a long time now, and have recently entered the mathematics literature as well. To define them we need roughly the following data: two Riemannian manifolds M and F, a fibre bundle E over the base M with typical fibre F, and a group G acting on F by isometries. The fields of the theory are then a section $\phi : M \to E$ of the bundle and a G-connection A. The energy functional is defined as

$$\mathcal{E}(A, \phi) \;=\; \int_M \frac{1}{2} \|F_A\|^2 + \|\mathrm{d}^A \phi\|^2 + V(\phi) \;, \tag{1}$$

where F_A is the curvature of A, $\mathrm{d}^A \phi$ is a covariant derivative and $V(\phi)$ is a potential term. Notice that when the bundle E is trivial the section ϕ can be regarded as a map $\phi : M \to F$, and so for $A = 0$ this energy reduces to the usual one for (non-gauged) σ-models.

In this paper we will be concerned with the case where M and F are complex Kähler manifolds and the action of G on F is holomorphic and hamiltonian. In this case there is a very special choice of potential V, namely

$$V(\phi) \;=\; 2\|\mu \circ \phi\|^2 \;, \tag{2}$$

where μ is a moment map for the G-action on F. This potential is special for two reasons, and it is a remarkable (though not uncommon) fact that they occur simultaneously. One reason is that with this choice the theory admits a supersymmetric extension, at least when M is an appropriate euclidean space. This is an important fact well known in the physics literature (see for example [13]), but we will not make any use of it here. The other reason is that with the choice (2) the energy functional admits Bogomolny equations, or in other words has a self-duality property. This fact appears to be less well known in the physics literature, at least when the σ-model is nonlinear, and apparently was first found in [23, 11]. When M is a Riemann surface these Bogomolny equations are

$$\begin{aligned} \bar{\partial}^A \phi \;&=\; 0, \\ *F_A + \mu \circ \phi \;&=\; 0 \;, \end{aligned}$$

and can be generalized to any Kähler M. In this context these equations are usually called vortex equations, because when $F = \mathbb{C}$ and $G = U(1)$ they reduce to the usual vortex equations of the abelian Higgs model.

The solutions of the vortex equations are exactly the global minima (within each topological sector) of the energy functional $\mathcal{E}$. In fact they are also BPS states of the supersymmetric theory, although we will not justify this here. Hence it is usually interesting to know how many solutions these equations admit up to gauge transformation, i.e. to describe the space of gauge equivalence classes of vortex solutions. For instance in the abelian Higgs model ($F = \mathbb{C}$ and $G = U(1)$) this was originally done by Taubes for $M = \mathbb{C}$ [26] and by Bradlow for M compact Kähler [7]. In the more difficult non-abelian case considerable progress has been made (e.g. [8, 3, 23]), especially in the case where F is a vector space, G a unitary group and M a Riemann surface.

As a recent mathematical application, the vortex equations have been used to define the so-called Hamiltonian Gromov-Witten invariants [11, 12]. This is described from a topological field theory point of view in [4].

In this paper we will study the space of solutions of the vortex equations for M any compact Kähler manifold and G an abelian torus. In some cases we are able to completely describe this space, namely when $G = T^n$ and $F = \mathbb{C}^n$ or $F = \mathbb{CP}^n$. In some other cases a (big) family of non-trivial solutions is found, namely when F is a compact Kähler toric manifold. The results obtained show an interesting interplay between the space of vortex solutions and the geometry of the moment polytope $\mu(F)$ obtained from the T^n-action on F. We will now give a brief description of the content and results of each section.

Section 2 is just a review of the model, where we try to carefully describe all the notions involved in the definition of the energy functional and of the equations. Since this nonlinear version of gauge theory on fibre bundles with arbitrary fibres (as opposed to vector space fibres) is not the most standard, we felt that this may be useful. At the end of the section we also recall some standard facts about complex gauge transformations and torus principal bundles, which will be necessary further ahead.

In the short Sect. 3 we give the space of solutions and energy spectrum of the vortex equations in the case $G = T^n$ and $F = \mathbb{C}^n$. When $n = 1$ these are the classical vortex equations, defined on line bundles over Kähler manifolds, and the solutions were described by Bradlow in [7] and by Garcí a-Prada in [15]. When $n > 1$, following work of Schroers [24], Yang has computed the space of solutions in the case where the base is the complex plane or a compact Riemann surface [27, p. 121]. The results contained in this section are for $n \geq 1$ and any compact Kähler base. Their derivation follows quite straightforwardly from work in [3]. In the rest of the paper we will concentrate on the more delicate case where F is a compact manifold.

In Sect. 4 we study the relation between the spaces of solutions up to real gauge transformations and up to complex gauge transformations. In fact, since the target F is Kähler, the usual G-gauge transformations can be extended to $G_{\mathbb{C}}$-gauge transformations. Then the first vortex equation is invariant under the $G_{\mathbb{C}}$-transformations whereas the second equation is invariant only under the G-transformations. Thus it makes sense to ask if, given a solution of the first equation, there exists a $G_{\mathbb{C}}$-transformation that takes it to a solution of the full equations. This question was addressed by Mundet i Riera in [23], and a general "stability" criterion was found. This criterion, however, is generally not easy to evaluate in practice. In Sect. 4 we find that for $G = T^n$ and for suitable conditions on F this criterion is hugely simplified, and a direct evaluation becomes possible. In particular, whenever F is Kähler toric the answer to the question is essentially yes. The precise results are stated in Sect. 4.1.

In Sect. 5 we determine the space of solutions and energy spectrum of the vortex equations for $G = T^n$, $F = \mathbb{CP}^n$ and any compact Kähler M. The results obtained generalize the ones in [23 and 25], where the authors determine the same quantities in the case where M is a Riemann surface and $n = 1$. The calculations in this section require the results of Sect. 4. The main results are stated in 5.1 and the proofs are contained in 5.2 and 5.3.

Section 6 is mainly preparatory. We study some general properties of the vortex equations under quotients of the target manifold F. Although we deal with a general group G, the results will be mainly applied to $G = T^n$.

In Sect. 7 we use the results of Sect. 4 and 6 to find non-trivial solutions of the vortex equations for $G = T^n$ and F a compact Kähler toric manifold. This family is big enough so that when $F = \mathbb{CP}^n$ it coincides with the full space of solutions calculated in Sect. 5. It is therefore natural to ask if, for the other compact toric F, the solutions exhibited in this section also exhaust the set of vortex solutions.

Finally in Sect. 8 we make a few informal comments about the results obtained. It may be helpful for the interested reader to have a look at those before delving into the technicalities of the theorems.

2. Review of the Model

2.1. The energy functional. The data we need to define the σ-model are the following:

- Two Kähler manifolds M and F, with respective Kähler forms ω_M and ω_F.
- A connected compact Lie group G, with Lie algebra $\mathfrak{g}$, and an Ad-invariant positive-definite inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$.
- An effective, hamiltonian, left action ρ of G on F such that, for every $g \in G$, the transformations $\rho_g : F \to F$ are holomorphic, and a moment map for this action $\mu : F \to \mathfrak{g}^*$.
- A principal G-bundle $\pi_P : P \to M$.

We remark that, in the fullest generality, the complex structure on F need not be assumed integrable, but we will assume that here. Using the elements above one can define the associated bundle $E = P \times_\rho F$, which is a bundle over M with typical fibre F. It is defined as the quotient of $P \times F$ by the equivalence relation $(p, q) \sim (p \cdot g, g^{-1} \cdot q)$, for all $g \in G$. The bundle projection $\pi_E : E \to M$ is determined by $\pi_E \circ \chi(p, q) = \pi_P(p)$, where $\chi : P \times F \to E$ is the quotient map. As a matter of notation, we will sometimes denote the equivalence class $\chi(p, q)$ simply by $[p, q]$.

Definition. *The convention used here is that a moment map for the action ρ of G on (F, ω_F) is a map $\mu : F \to \mathfrak{g}^*$ such that*

(i) $\mathrm{d}\,(\mu, \xi) = \iota_{\xi^\flat}\, \omega_F$ *in $\Omega^1(F)$ for all $\xi \in \mathfrak{g}$, where $\xi^\flat$ is the vector field on F defined by the flow $t \mapsto \rho_{\exp(t\xi)}$.*
(ii) $\rho_g^* \mu = \mathrm{Ad}_g^* \circ \mu$ *for all $g \in G$, where Ad_g^* is the coadjoint representation of G on $\mathfrak{g}^*$.*

If a moment map μ exists, it is not in general unique, but all the other moment maps are of the form $\mu + a$, where $a \in [\mathfrak{g}, \mathfrak{g}]^0 \subset \mathfrak{g}^*$ is a constant in the annihilator of $[\mathfrak{g}, \mathfrak{g}]$. Recall also that under the identification $\mathfrak{g}^* \simeq \mathfrak{g}$ provided by an Ad-invariant inner product on $\mathfrak{g}$, the annihilator $[\mathfrak{g}, \mathfrak{g}]^0$ is taken to the centre of $\mathfrak{g}$.

The fields of the theory are a connection A on the principal bundle P and a smooth section ϕ of E. Calling $\mathcal{A}$ the space of such connections and $\Gamma(E)$ the space of such sections, we define the energy functional $\mathcal{E} : \mathcal{A} \times \Gamma(E) \to \mathbb{R}_{\geq 0}$ of the σ-model by

$$\mathcal{E}(A, \phi) = \int_M \left\{ \frac{1}{a^2} \| F_A \|^2 + \| \mathrm{d}^A \phi \|^2 + a^2 \| \mu \circ \phi \|^2 \right\} \omega_M^{[m]} , \qquad a \in \mathbb{R}_{>0}. \quad (3)$$

In this formula, as throughout the paper, m is the complex dimension of M, and we use the notation $\omega_M^{[k]} := \omega_M^k / k!$ for any $k \in \mathbb{N}$. In particular $\omega_M^{[m]}$ is the metric volume form on M.

The various terms under the integral sign have the following meaning. F_A is the curvature of the connection A. The norm $\| F_A \|^2$ is then the natural one, induced simultaneously by the Kähler metric on M and by the inner product $\langle, \rangle$ on $\mathfrak{g}$. In the third term of (3), the norm $\|\cdot\|$ on $\mathfrak{g}^*$ comes from the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. This term is well defined because of the G-equivariance of the moment map and the Ad_G-invariance of $\langle\ ,\ \rangle$.

As for the second term, its description is a little longer, since one should first explain the meaning of the covariant derivative $\mathrm{d}^A\phi$. This is an extension of the usual notion of covariant derivatives on vector bundles. We start by considering the differential of the quotient map, $\mathrm{d}\chi : TP \times TF \to TE$. A connection A on P induces a horizontal distribution H_A on P. Defining $\mathcal{H}_A = \mathrm{d}\chi(H_A)$, it is not difficult to show that the restrictions

$$\mathrm{d}\pi_E : \mathcal{H}_A \longrightarrow TM \qquad \text{and} \qquad \mathrm{d}\chi_{(p,q)} : T_qF \longrightarrow \ker(\mathrm{d}\pi_E)_{\chi(p,q)} \tag{4}$$

are isomorphisms, and in particular we get the splitting

$$TE = \mathcal{H}_A \oplus \ker \mathrm{d}\pi_E \ . \tag{5}$$

The covariant derivative of a section $\phi : M \to E$ is then defined as the composition

$$\mathrm{d}^A\phi : \stackrel{\mathrm{d}\phi}{\to} TE = \mathcal{H}_A \oplus \ker \mathrm{d}\pi_E \stackrel{\mathrm{proj}_2}{\to} \ker \mathrm{d}\pi_E\ ,$$

where proj_2 is just the projection. Notice that the image of $\mathrm{d}^A\phi$ is in the tangent space to the fibres of E, which are isomorphic to F. Thus when F is a vector space, the canonical isomorphism $T_vF \simeq F$ allows us to regard $\mathrm{d}^A\phi$ as a map of vector bundles $TM \to E$, that is a section of $T^*M \otimes E$, which is the usual notion of covariant derivative on a vector bundle. The norm $\|\mathrm{d}^A\phi\|^2$ is defined in the usual way, using the metric g_M on M and the metric g_F – transported by the second isomorphism of (4) – on $\ker \mathrm{d}\pi_E$.

Finally notice that the constant a^2 can be absorbed by rescaling the inner product on $\mathfrak{g}$.

2.2. The vortex equations. Having explained the meaning of the energy functional (3), we will now see how to manipulate it in order to get Bogomolny equations. First of all, using the isomorphisms (4) and the splitting (5), one can transport the complex structures J_M and J_F of M and F, respectively, as well as the Kähler metrics g_M and g_F, to the tangent bundle TE, thus defining a complex structure and a metric on TE by

$$J(A) = J_M \oplus J_F \qquad \text{and} \qquad g(A) = g_M \oplus g_F . \tag{6}$$

These depend on the connection A. Because the metrics g_M and g_F are Kähler, $J(A)$ is always compatible with $g(A)$, and so $(E, J(A), g(A))$ is an almost-Hermitian manifold. Using this complex structure on E and the one on M, one obtains a splitting $\mathrm{d}^A\phi = \partial^A\phi + \bar\partial^A\phi$ by the usual formulae

$$\bar\partial^A\phi = \frac{1}{2}(\mathrm{d}^A\phi + J_F \circ \mathrm{d}^A\phi \circ J_M) = \frac{1}{2}\,\mathrm{proj}_2 \circ (\mathrm{d}\phi + J(A) \circ \mathrm{d}\phi \circ J_M), \tag{7}$$

$$\partial^A\phi = \frac{1}{2}(\mathrm{d}^A\phi - J_F \circ \mathrm{d}^A\,\phi \circ J_M). \tag{8}$$

For later convenience we also record here the local (i.e. trivialization-dependent) formulae for $\mathrm{d}^A\phi$ and $\bar\partial^A\phi$. Let $s : \mathcal{U} \to P$ be a local section of P over a domain $\mathcal{U}$ in M. Since $E = P \times_\rho F$ is an associated bundle, this determines a trivialization of $E|_{\mathcal{U}}$ by

$$\mathcal{U} \times F \to E|_{\mathcal{U}}\ , \qquad (x, q) \to [s(x), q]\ . \tag{9}$$

With respect to these trivializations a section ϕ of E can be locally identified with a map $\hat{\phi} : \mathcal{U} \to F$, and a connection A on P can be identified with the connection form $s^* A = \alpha \in \Omega^1(\mathcal{U}; \mathfrak{g})$. Then the covariant derivatives $\mathrm{d}^A\phi$ and $\bar{\partial}^A\phi$ in $\Gamma(T^*M \otimes \phi^* \ker \mathrm{d}\pi_E)$ are locally given by

$$\begin{aligned} (\mathrm{d}^A\phi)_q &= (\mathrm{d}\,\hat{\phi})_q + (\alpha^l)_q\ \xi_l^\flat|_{\hat{\phi}(q)}, \\ (\bar{\partial}^A\phi)_q &= (\bar{\partial}\,\hat{\phi})_q + (\alpha^l)_q^{0,1}\ \xi_l^\flat|_{\hat{\phi}(q)} \qquad \forall\, q \in \mathcal{U}, \end{aligned} \tag{10}$$

which are 1-forms on T_qM with values in $T_{\hat{\phi}(q)}F$. In these formulae $\{\xi_l\}$ is any basis for $\mathfrak{g}$, the $\xi_l^\flat$ are the vector fields on F described in the definition of moment map (Sect. 2.1), and we have decomposed $\alpha = \alpha^l\, \xi_l$.

We now come to the basic fact of the theory. This was first obtained in [23] and, for M a Riemann surface, in [11].

Theorem 2.1 ([23, 11]). *For any connection $A \in \mathcal{A}$ and any section $\phi \in \Gamma(E)$,*

$$\begin{aligned} \mathcal{E}(A,\phi) = T_{[\phi]} + \int_M \Big\{ &\|\frac{1}{a}\Lambda F_A + a\,\mu\circ\phi\|^2 + 2\|\bar{\partial}^A\phi\|^2 \\ &+ \frac{4}{a^2}\|F_A^{0,2}\|^2 \Big\}\ \omega_M^{[m]}\,, \end{aligned} \tag{11}$$

where the term

$$T_{[\phi]} = \int_M \phi^*[\eta_E] \wedge \omega_M^{[m-1]} - \frac{1}{a^2} B_2(F_A, F_A) \wedge \omega_M^{[m-2]} \tag{12}$$

does not depend on A, and only on the homotopy class of ϕ.

Remark. As is usually the case with these Bogomolny-type manipulations, there is an alternative formula for $\mathcal{E}(A,\phi)$ which gives rise to the anti-Bogomolny equations. This formula can be obtained from the one above by changing the sign of the first term of $T_{[\phi]}$, substituting $\bar{\partial}^A\phi$ for $\partial^A\phi$, and changing the plus to a minus sign inside the first squared norm. The proof of [23] is still applicable, with minimal changes.

Corollary 2.2 ([23, 11]). *Within each homotopy class of the sections ϕ we have that $\mathcal{E}(A,\phi) \geq T_{[\phi]}$, and there is an equality if and only if the pair (A,ϕ) in $\mathcal{A} \times \Gamma(E)$ satisfies the equations*

$$\bar{\partial}^A\phi = 0, \tag{13a}$$

$$\Lambda F_A + a^2\,\mu\circ\phi = 0, \tag{13b}$$

$$F_A^{0,2} = 0\ . \tag{13c}$$

These first order equations are usually called vortex equations.

Besides $\bar{\partial}^A\phi$, several new terms appear in (11) when compared with (3); their meaning is the following. The operator $\Lambda : \Omega^*(M) \to \Omega^{*-2}(M)$ is the adjoint, with respect to the metric g_M, of the operator $\eta \mapsto \omega_M \wedge \eta$ on $\Omega^*(M)$. By well known formulae,

$$\Lambda F_A = *(\omega_M \wedge *F_A) = g_M(F_A, \omega_M)\,, \tag{14}$$

and so ΛF_A can be seen as a locally defined function on M with values in $\mathfrak{g}$, just as $\mu \circ \phi$. (More properly, they should be both regarded as global sections of $P \times_{\mathrm{Ad}_G} \mathfrak{g}$.) Next, $F_A^{0,2}$ is just the (0, 2)-component of F_A under the usual decomposition $\Omega^2(M) = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$. The form $B_2(F_A, F_A)$ can be explicitly written as

$$B_2(F_A, F_A) \; = \; F_A^j \wedge F_A^k \; \langle \xi_j, \xi_k \rangle \; , \tag{15}$$

where $\{\xi_j\}$ is a basis of $\mathfrak{g}$ and we have decomposed $F_A = F_A^j \; \xi_j$; it represents the characteristic class of P associated with the Ad-invariant polynomial $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$.

Finally $[\eta_E]$ is a cohomology class in $H^2(E)$, and is defined as follows. Consider the 2-form on $P \times F$

$$\eta(A) \; := \; \omega_F - \mathrm{d}(\mu, A) \; , \tag{16}$$

where we regard the connection A as a form in $\Omega^1(P, \mathfrak{g})$, in the usual sense, and $(\cdot, \cdot) : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the natural pairing. The quotient map $\chi : P \times F \to E$ is in a natural way a G-principal bundle, and it is not difficult to check that the form $\eta(A)$ is invariant under the associated G-action $(p, q) \mapsto (p \cdot g, g^{-1} \cdot q)$ on $P \times F$. Furthermore $\eta(A)$ is also a horizontal form, in the sense that it annihilates vectors in $\ker \mathrm{d}\chi$, and so $\eta(A)$ descends to E, that is $\eta(A) = \chi^* \eta_E(A)$ for some $\eta_E(A)$ in $\Omega^2(E)$. The form $\eta_E(A)$ on E is sometimes called the minimal coupling form. Now, since $\eta(A)$ is closed, $\eta_E(A)$ is also closed, and it is not difficult to show that its cohomology class in $H^2(E)$ does not depend on A. We can therefore define $[\eta_E]$ to be the cohomology class of the forms $\eta_E(A)$.

Remark. There is another way to look at the class $[\eta_E]$ on $H^2(E)$, using the Cartan complex for the G-equivariant cohomology of F. In this context, $[\eta_E]$ is just the image by the Chern-Weil homomorphism of the cohomology class in $H^2_G(F)$ determined by the equivariantly closed form $\omega_F - X^b \mu_b \; \in \; \Omega^2_G(F)$ (see for example [5, ch. VII]) .

To end this subsection we state two results that, to some extent, clarify the meaning of the first and the third vortex equations. The first proposition is well known [23]. As for the second proposition, we omit its proof, since it is a bit long and, moreover, is just a mild extension of well known calculations [20, p. 9].

Proposition 2.3. *Let $A \in \mathcal{A}$ be any connection and let ϕ be a section of E. Then $\bar{\partial}^A \phi = 0$ if and only if ϕ is holomorphic as a map $(M, J_M) \to (E, J(A))$.*

Proposition 2.4. *The condition $F_A^{0,2} = 0$ implies that the almost-complex structure $J(A)$ on E is integrable. The converse is also true if at least one point in F has a discrete isotropy group (contained in G).*

2.3. Complex gauge transformations. Here we recall the notions of complexified Lie group, complexified action, and complex gauge transformation. To any compact Lie group G one can associate a complex analytic Lie group $G_{\mathbb{C}}$, called the complexification of G. The Lie algebra of $G_{\mathbb{C}}$ can be identified with the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ of the Lie algebra of G. Both G and $\mathfrak{g}$ can be naturally embedded into $G_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$, respectively, as fixed points of natural involutions – called conjugations – in these spaces [9]. Furthermore, when the group G acts holomorphically on a compact Kähler manifold F, this action can be canonically extended to a holomorphic action of $G_{\mathbb{C}}$ on F [17].

This extension of the action on F allows us to define complex gauge transformations on the bundle $E = P \times_G F$, which extend to $G_{\mathbb{C}}$ the original G-gauge transformations. A complex gauge transformation g is a section of the bundle $P \times_{\mathrm{Ad}_G} G_{\mathbb{C}}$ over M. The set of these sections forms a group, denoted by $\mathcal{G}_{\mathbb{C}}$. Each $g \in \mathcal{G}_{\mathbb{C}}$ determines an automorphism of E by the formula

$$[p, q] \ \mapsto \ [p, \rho_{g_p}(q)] \, , \tag{17}$$

where ρ is the extended $G_{\mathbb{C}}$-action and g_p is the only element of $G_{\mathbb{C}}$ such that $g \circ \pi_P(p) = [p, g_p]$. If we compose a section $\phi \in \Gamma(E)$ with this automorphism of E we get another section, which we denote by $g(\phi)$. Complex gauge transformations can also be made to act on the space $\mathcal{A}$ of connections on P, in such a way as to extend the action of the original G-gauge transformations. This extension is defined by the formula

$$g(A) \ = \ \mathrm{Ad}_g \circ A \ - \ \pi_P^* (g^{-1} \bar{\partial} g + \bar{g}^{-1} \partial \bar{g}) \, . \tag{18}$$

An important fact about these complex gauge transformations is that both the first and the third vortex equations are invariant by them, whereas the second equation is invariant by real gauge transformations only.

2.4. Torus principal bundles. In this last subsection we will introduce some notation and recall some standard results about T^n-principal bundles used in the rest of the paper.

Let $P \to M$ be any principal T^n-bundle, let $\hat{\rho}$ be the natural action of T^n on $\mathbb{C}^n$, denote by $\hat{\rho}_j$ the restriction of $\hat{\rho}$ to the j^{th} factor $\mathbb{C}$ in $\mathbb{C}^n$, and let $\hat{L}_j = P \times_{\hat{\rho}_j} \mathbb{C}$ be the associated line bundle. We begin by stating a standard result, whose proof we omit.

Proposition 2.5. *Given any n classes $\alpha_j \in H^2(M; \mathbb{Z})$ there is exactly one principal T^n-bundle $P \to M$, up to isomorphism, such that α_j coincides with the first Chern class $c_1(\hat{L}_j)$.*

This proposition shows that the correspondence $P \mapsto \alpha(P)$, with $\alpha_j(P) = c_1(\hat{L}_j)$, defines a bijection between the set of principal T^n-bundles over M (up to isomorphism), and the n-fold cartesian product of $H^2(M; \mathbb{Z})$. Now identify the Lie algebra $\mathfrak{t}^n$ with $\mathbb{R}^n$ in such a way that the exponential map $\mathfrak{t}^n \to \mathbb{R}^n$ is

$$\exp(w_1, \ldots, w_n) \ = \ (e^{2\pi i w_1}, \ldots, e^{2\pi i w_n}) \, , \qquad w_k \in \mathbb{R}. \tag{19}$$

With this identification, for any principal T^n-bundle $P \to M$ we define

$$\deg P \ = \ - \int_M \Lambda F_A \ \omega_M^{[m]} \, , \tag{20}$$

where A is any connection on P. This constant does not depend on A. In fact, having in mind the above identification of $\mathfrak{t}^n$ with $\mathbb{R}^n$, it is clear that $2\pi i (F_A)_j$ coincides with the curvature on the base of the connection on $\hat{L}_j$ induced by A. In particular $c_1(\hat{L}_j) = -[(F_A)_j]$, and so it follows from (14) and Proposition 2.5 that

$$\deg P \ = \ \int_M \alpha(P) \wedge \omega_M^{[m-1]} \qquad \in \ \mathbb{R}^n. \tag{21}$$

We also define the constant

$$c(a, P, M) \ := \ (a^2 \operatorname{Vol} M)^{-1} \deg P \ , \tag{22}$$

which will appear often in the subsequent sections.

Finally, to end this subsection, we will state a lemma necessary for Sect. 6. Let $\beta : T^d \to T^n$ be any homomorphism of tori. This has the general form

$$\beta(g_1, \ldots, g_d) \ = \ (\ldots, \Pi_{1 \le l \le d} \ (g_l)^{\beta_{al}}, \ldots)_{1 \le a \le n} \ , \qquad \text{with } \ \beta_{al} \in \mathbb{Z} \ .$$

Given a principal T^d-bundle $P \to M$, the associated bundle $P' = P \times_\beta T^n$ is in a natural way a principal T^n-bundle over M. Then the following naturality property is easy to check.

Lemma 2.6. *The classes in $H^2(M; \mathbb{Z})$ associated with P' are $\alpha_a(P') = \sum_{l=1}^d \beta_{al} \ \alpha_l(P)$ for all $a = 1, \ldots, n$.*

3. A Simpler Case: $\mathbb{C}^n$ with T^n-Action

In this section we give the space of solutions and energy spectrum of the vortex equations in the case $F = \mathbb{C}^n$ and $G = T^n$. The results are contained in Theorems 3.1 and 3.2. Since the calculations necessary to prove these theorems are either special or simpler cases of the calculations in Sects. 4 and 5, for brevity's sake we will not give the details.

One starts with the action of T^n on $\mathbb{C}^n$ given by

$$\rho_{(g_1, \ldots, g_n)} \ (z_1, \ldots, z_n) \ = \ (\cdots, z_k \ \Pi_j \ (g_j)^{C_{kj}}, \cdots)_{1 \le k \le n} \ , \tag{23}$$

where the matrix C belongs to $SL(n; \mathbb{Z})$. It is an effective hamiltonian action. Identifying $\mathfrak{t}^n \simeq \mathbb{R}^n$ in the usual way (19), the general form of a moment map $\mu : \mathbb{C}^n \to \mathbb{R}^n$ for this action is

$$\mu(z_1, \ldots, z_n) \ = \ -\pi \left(\cdots, \ \sum_j C_{jk} \, |z_j|^2, \ \cdots \right)_{1 \le k \le n} + \ t \ , \tag{24}$$

where t is any constant in $\mathbb{R}^n$.

Now consider the associated vector bundle $E = P \times_\rho \mathbb{C}^n$. Denoting by ρ_j the restriction of the action ρ to the j^{th} component $\mathbb{C}$ of $\mathbb{C}^n$, we have that

$$E \ = \ L_1 \oplus \cdots \oplus L_n,$$

where $L_j = P \times_{\rho_j} \mathbb{C}$ is the associated line bundle. Notice that the natural hermitian products on $\mathbb{C}^n$ and $\mathbb{C}$ induce hermitian metrics on the bundles E and L_j, because the actions ρ and ρ_j are unitary. We denote by h and h_j these hermitian metrics.

Finally, an integrable connection $A \in \mathcal{A}^{1,1}(P)$ induces a metric-compatible integrable connection ∇ (resp. ∇_j) on the vector bundle E (resp. L_j). In turn, this integrable connection defines a unique holomorphic structure on E (resp. L_j) such that ∇ (resp. ∇_j) is the hermitian connection of this bundle [20]. The bundles E and L_j equipped with these holomorphic structures will be denoted by E^A and L_j^A. Notice that, also as holomorphic hermitian bundles,

$$E^A \ = \ L_1^A \oplus \cdots \oplus L_n^A . \tag{25}$$

Recalling the constants $c(P, M, a) \in \mathbb{R}^n$ and $\alpha(P) \in H^2(M; \mathbb{Z})^n$ defined in Sect. 2.4, we have the following results.

Theorem 3.1. *In the setting described above, the vortex Eq.* (13) *have solutions only if the constant* $c(P, M, a)$ *is in* $\mu(\mathbb{C}^n)$. *When this constant lies in the interior of* $\mu(\mathbb{C}^n)$, *the set of solutions can be described as follows. For each* $j = 1, \dots, n$ *pick an effective divisor* $D_j = \sum_i a^i_j \cdot Z_i$ *on* M *representing the homology class Poincaré dual to* $\sum_j C_{kj}\, \alpha_j(P)$. *Then there is a solution* (A, ϕ) *of (13) such that* D_j *is the divisor of the zero set of* ϕ_j *(the* j^{th} *component of* ϕ *under the decomposition (25)) regarded as a holomorphic section of* L^A_j. *This solution is unique up to gauge transformations, and all solutions of (13) are obtained in this way.*

Theorem 3.2. *The topological energy (12) of any solution of the vortex equations is*

$$T = \int_M \sum_k t_k\, \alpha_k(P) \wedge \omega_M^{[m-1]} - \frac{1}{a^2}\, \alpha_k(P) \wedge \alpha_k(P) \wedge \omega_M^{[m-2]} , \tag{26}$$

where $t \in \mathbb{R}^n$ *is the arbitrary constant in the moment map (24).*

In Theorem 3.1 it is of course implicit that, if it is impossible to find a suitable set of divisors D_j, then the set of vortex solutions is empty. Notice as well that the statement of these results is especially simple when M is a Riemann surface, due to the isomorphism $H^2(M; \mathbb{Z}) \simeq \mathbb{Z}$. For example the topological energy then reduces to $T = t \cdot \deg P$. Another interesting fact regarding the topological energy is that, unlike the $\mathbb{CP}^n$ case of Theorem 5.1, here the energy is completely determined by the bundle P; it does not depend on the particular solution chosen. This difference between the $\mathbb{C}^n$ and the $\mathbb{CP}^n$ cases is analogous to the fact that the degree of a line bundle completely determines the number of zeros of a holomorphic section, but not of a meromorphic section.

The key ingredient to prove Theorem 3.1 is the following proposition, which follows quite straightforwardly from the "stability" criterion of [3].

Proposition 3.3. *Assume that* $c(P, M, a)$ *lies in the interior of* $\mu(\mathbb{C}^n)$, *and let* $(A, \phi) \in \mathcal{A}^{1,1}(P) \times \Gamma(E)$ *be any pair such that* $\bar{\partial}^A \phi = 0$ *and* ϕ_j *is not identically zero for any* j. *Then there exists a complex gauge transformation* $g : M \to (\mathbb{C}^*)^n$, *unique up to multiplication by real gauge transformations, such that the pair* $(g(A), g(\phi))$ *is a solution of the vortex equations.*

When the target of the σ-model is a compact manifold, instead of $\mathbb{C}^n$, things get rather more complicated. In the next section we will try to find results analogous to Proposition 3.3 in the compact setting. For this we will use results of [2] and a more general "stability" criterion of [23].

4. The 2^{nd} Vortex Equation as an Imaginary-Gauge Fixing Condition

4.1. Main results. As was mentioned in Sect. 2.3, an important fact about the complex gauge transformations is that both the first and the third vortex equations are invariant by them, whereas the second equation is not. Hence, given a pair (A, ϕ) that solves (13a) and (13c), it makes sense to ask whether there is a complex gauge transformation g such that $(g(A), g(\phi))$ solves (13b), and therefore all the vortex equations [23]. The ideal answer would be that such a transformation always exists and is unique up to real gauge transformations. This would mean that Eq. (13b) acts as a sort of imaginary-gauge fixing condition, and that the set of solutions of (13) up to real gauge transformations is the same as the set of solutions of (13a) and (13c) up to complex gauge transformations.

The purpose of this section is to study this problem when the gauge group is T^n and the target F is compact. The basic results obtained are expressed in Theorems 4.1 and 4.2. As a kind of corollary, we find that although the ideal answer stated above is not in general true, it comes very close to being completely true when the target manifold F is a "simple" one — for example when F is toric (see Corollary 4.3 and the following remark). This will eventually allow us to compute the moduli space of solutions when $F = \mathbb{CP}^n$ (Sect. 5), and to find non-trivial solutions when F is toric (Sect. 7).

In order to state the basic results of this section we first need to establish some notation. The complexified torus is $T^n_{\mathbb{C}} \simeq (\mathbb{C}^*)^n$, and its Lie algebra is identified with $\mathfrak{t}^n \oplus i\mathfrak{t}^n \simeq \mathbb{C}^n$ in such a way that the exponential map is

$$\exp(w_1, \ldots, w_n) \;=\; (e^{2\pi i w_1}, \ldots, e^{2\pi i w_n}), \qquad w_k \in \mathbb{C}. \tag{27}$$

The inner product on $\mathfrak{t}^n \simeq \mathbb{R}^n$ is just the euclidean one. For any point p in F we call $\mathcal{O}_p$ and $\mathcal{O}_p^{\mathbb{C}}$ its T^n- and $T^n_{\mathbb{C}}$-orbit, respectively; similarly, the real and complex isotropy groups of p are denoted by G_p and $G_p^{\mathbb{C}}$.

Also a word about polytopes. By the convexity theorem, if $\mu : F \to \mathfrak{t}^n \simeq \mathbb{R}^n$ is a moment map for a torus action on F, which is assumed compact, its image $\mu(F)$ is a convex polytope in $\mathbb{R}^n$ (see for example [2 or 22]). As a set, $\mu(F)$ is the disjoint union of its k-dimensional open faces, or k-cells, for $k = 0, \ldots, \dim \mu(F)$. Thus for example $\mu(F)$ has only one open face of maximal dimension, and the 0-dimensional open faces are the vertices of $\mu(F)$. We are now ready to state the main results of this section.

Theorem 4.1. *A necessary condition for the equation $\Lambda F_A + a^2\, \mu \circ \phi = 0$ to have a solution is that the constant $c(a, P, M)$ (cf. (22)) lies in $\mu(F)$. If this is satisfied, let σ_c be the only open face of the polytope $\mu(F)$ that contains this point, and let $\bar{\sigma}_c$ denote its closure. Then for any (A, ϕ) solution of (13b), the image $\mu \circ \phi(M)$ is contained in $\bar{\sigma}_c$ and is not contained in any of the closed faces of $\bar{\sigma}_c \setminus \sigma_c$.*

Theorem 4.2. *Let $(A, \phi) \in \mathcal{A} \times \Gamma(E)$ be a pair such that, for all x in some open dense subset of M, the conditions*

(i) $\mathcal{O}^{\mathbb{C}}_{\phi(x)} \cap \mu^{-1}(c) \neq \emptyset$;
(ii) $G_{\phi(x)}$ has dimension $n - \dim \sigma_c$;

are satisfied. Then there exists a complex gauge transformation $g : M \to T^n_{\mathbb{C}}$ that takes (A, ϕ) to a solution of (13b). This transformation is unique up to multiplication by transformations whose imaginary part is a constant in $\exp(i\sigma_c^{\perp})$.

Remark. Here we will only prove this theorem in the generic case where the constant $c(P, M, a)$ lies in the interior of $\mu(F)$, i.e. when $\dim \sigma_c = n$. This is the only case needed in the subsequent sections. The proof is based on a very general criterion of [23]. For a hint of the proof in the general case see the remark in Sect. 4.3.

Corollary 4.3. *Assume that the orbit $\mathcal{O}_p^{\mathbb{C}}$ of any point $p \in \mu^{-1}(\sigma_c)$ satisfies $\mu(\mathcal{O}_p^{\mathbb{C}}) = \sigma_c$. Then, given any pair $(A, \phi) \in \mathcal{A}^{1,1} \times \Gamma(E)$ such that $\bar{\partial}^A \phi = 0$, there exists a complex gauge transformation that takes (A, ϕ) to a solution of (13b) if and only if the image $\mu \circ \phi(M)$ is contained in $\bar{\sigma}_c$ but not in any of the closed faces of $\bar{\sigma}_c \setminus \sigma_c$. Furthermore, when it exists, this transformation is unique up to multiplication by transformations whose imaginary part is a constant in $\exp(i\sigma_c^{\perp})$.*

Remark. The condition of this corollary, namely $\mu(\mathcal{O}_p^{\mathbb{C}}) = \sigma_c$ for any $p \in \mu^{-1}(\sigma_c)$, is very restrictive. It is satisfied, however, when the action is effective and $\dim_{\mathbb{C}} F = n = \dim_{\mathbb{R}} T^n$. In this case F becomes a compact Kähler toric manifold, and it is well known that for such manifolds there is a one-to-one correspondence between open faces of $\mu(F)$ and $T^n_{\mathbb{C}}$-orbits in F, which is given by $\sigma \mapsto \mu^{-1}(\sigma)$. This is valid for all Kähler toric manifolds, not just the canonical ones described in Sect. 7.1.

4.2. Proof of Theorem 4.1 and Corollary 4.3. We begin this subsection with the proof of Theorem 4.1. After stating two auxiliary lemmas, we end it with the proof of Corollary 4.3.

Proof of Theorem 4.1. Let (A, ϕ) be a solution of (13b). Integrating this equation over M and using (20) and (22) one has that

$$\int_M (\mu \circ \phi - c)\, \omega_M^{[m]} \;=\; 0 \qquad \in \;\mathbb{R}^n. \tag{28}$$

If $c \notin \mu(F)$, from the convexity of $\mu(F)$ it is clear that for all $v \in \mu(F)$ the vectors $v - c$ will lie in the same open half-space of $\mathbb{R}^n$. In particular the same thing happens with the vectors $\mu \circ \phi(x) - c$ for all $x \in M$, and thus it is impossible for (28) to hold — a contradiction.

Now suppose that c lies in some open face σ of $\mu(F)$. If σ is n-dimensional, it is obvious that $\mu \circ \phi(M) \subseteq \bar{\sigma} = \mu(F)$. If the dimension of σ is $k < n$, let $A_1, \ldots, A_{n-k}$ be the closed $(n-1)$-dimensional faces of $\mu(F)$ whose intersection is $\bar{\sigma}$, and let n_j be an outward normal vector to A_j. Then, from the convexity of $\mu(F)$, one has that $n_j \cdot (v - c) \leq 0$ for all $v \in \mu(F)$, and the equality holds iff $v \in A_j$. But (28) implies that

$$\int_{x \in M} n_j \cdot (\mu \circ \phi(x) - c)\;\; \omega_M^{[m]} \;=\; 0\,,$$

and so we conclude that $\mu \circ \phi(x) \in A_j$ for all $x \in M$. Since this is true for all j we actually have that $\mu \circ \phi(M) \subseteq \bar{\sigma}$, as required.

On the other hand, let B be any closed face of $\bar{\sigma} \setminus \sigma$ — which is also a $(k-1)$-dimensional closed face of $\mu(F)$ — and let u be a vector normal to B, parallel to $\bar{\sigma}$, and pointing outward of $\mu(F)$. Then, because $c \in \sigma$ and $\bar{\sigma}$ is convex, one has that $u \cdot (v - c) > 0$ for all $v \in B$. In particular it is impossible that $\mu \circ \phi(M) \subseteq B$, otherwise one would have that

$$\int_{x \in M} u \cdot (\mu \circ \phi(x) - c)\;\; \omega_M^{[m]} \;>\; 0\,,$$

which contradicts (28). $\square$

Lemma 4.4. *Let σ be any open face of the polytope $\mu(F)$, and denote by $\bar{\sigma}$ its closure. Then*

(i) $\mu^{-1}(\bar{\sigma})$ is a connected complex submanifold of F;
(ii) $\mu^{-1}(\sigma)$ is invariant under the $T^n_{\mathbb{C}}$-action.

This lemma is a well known result. Statement (i) follows rather straightforwardly from Lemmas 5.53 and 5.54 of [22] and their proof; statement (ii) follows from Theorem 2 of [2].

Lemma 4.5. *Let $\phi : M \to E$ be a section of E such that $\bar{\partial}^A \phi = 0$ for some connection $A \in \mathcal{A}^{1,1}(P)$. Then for any open face σ of $\mu(F)$ the inverse image $(\mu \circ \phi)^{-1}(\bar{\sigma})$ is an analytic subvariety of M.*

Proof. To avoid any confusion, in this proof we will use different symbols for the moment map $\mu : F \to \mathbb{R}^n$ and its lift $\tilde{\mu} : E \to \mathbb{R}^n$. To start with, notice that $\bar{\sigma}$ is a disjoint union of open faces of $\mu(F)$, possibly with different dimensions, and so it follows from Lemma 4.4 that $\mu^{-1}(\bar{\sigma})$ is a complex submanifold of F which is invariant by the T^n-action. It is not difficult to check that this implies that $E' = P \times_{T^n} \mu^{-1}(\bar{\sigma})$ is a complex submanifold of $E = P \times_{T^n} F$, where the complex structure on E is $J(A)$. Furthermore, from the definition $\tilde{\mu} \circ \chi(p, q) = \mu(q)$ (see Sect. 2.1), we also have that $E' = \tilde{\mu}^{-1}(\bar{\sigma})$.

On the other hand, by Proposition 2.3, the section ϕ is a holomorphic map from M to $(E, J(A))$. This map is proper because M is compact, and since a section is always an immersion, we conclude that $\phi(M)$ is actually a complex submanifold of E, and $\phi : M \to \phi(M)$ is a biholomorphism. It is then clear that $E' \cap \phi(M)$, being an intersection of complex submanifolds, is an analytic subvariety of $\phi(M)$. Hence $\phi^{-1}(\tilde{\mu}(\bar{\sigma})) = \phi^{-1}(E' \cap \phi(M))$ is an analytic subvariety of M. □

Proof of Corollary 4.3. We denote by $(*)$ the condition "$\mu \circ \phi(M)$ is contained in $\bar{\sigma}_c$ but not in any of the closed faces of $\bar{\sigma}_c \setminus \sigma_c$". The proof of necessity is fast, due to Theorem 4.1. In fact, it follows from part (ii) of Lemma 4.4 that, if a section $\phi \in \Gamma(E)$ satisfies $(*)$, so does its entire complex gauge equivalence class. (We are using that the closed faces of $\bar{\sigma}_c$ are themselves a union of open faces of $\mu(F)$, and so their inverse image by μ is also $T^n_{\mathbb{C}}$-invariant.) The necessity of $(*)$ is then a direct consequence of Theorem 4.1.

To prove the sufficiency and uniqueness statements we will use Theorem 4.2. If $\bar{\partial}^A \phi = 0$ and $(*)$ is satisfied, by Lemma 4.5 it is true that for each closed face B of $\bar{\sigma}_c \setminus \sigma_c$, the inverse image $(\mu \circ \phi)^{-1}(B)$ is an analytic subvariety of M which is not the entire M; in particular this set has zero measure in M. Since this is true for all the faces of $\bar{\sigma}_c \setminus \sigma_c$, we conclude that $(\mu \circ \phi)^{-1}(\sigma_c)$ is open and dense in M. Now if $x \in (\mu \circ \phi)^{-1}(\sigma_c)$, that is $\phi(x) \in \mu^{-1}(\sigma_c)$, by assumption $\mu(\mathcal{O}^{\mathbb{C}}_{\phi(x)}) = \sigma_c$. Hence on the one hand, since $c(P, M, a) \in \sigma_c$, this implies that condition (i) of Theorem 4.2 is satisfied; on the other hand, using Lemma 4.6, this implies that condition (ii) of Theorem 4.2 is satisfied as well. Applying this theorem we obtain the sufficiency and uniqueness parts. □

4.3. Proof of Theorem 4.2. We first derive an auxiliary lemma and then prove Theorem 4.2 in the case where $c(P, M, a)$ lies in the interior of $\mu(F)$. At the end of the subsection we make a remark about the proof in the general case.

The lemma is the following. Given $p \in F$, let σ_p denote the only open face of $\mu(F)$ that contains the point $\mu(p)$. Then by Lemma 4.4 and Theorem 2 of [2], the image $\mu(\mathcal{O}^{\mathbb{C}}_p)$ is a convex open polytope contained in σ_p.

Lemma 4.6. *Given $p \in F$, the Lie algebra of the isotropy subgroup $G_p \subseteq T^n$ is the subspace of $\mathfrak{t}^n$ formed by the vectors orthogonal to $\mu(\mathcal{O}^{\mathbb{C}}_p)$. In particular* Lie G_p *contains the subspace $\sigma_p^{\perp}$.*

Proof. Theorem 2 of [2] guarantees that the restriction of μ to $\mathcal{O}^{\mathbb{C}}_p$ induces a homeomorphism $\mathcal{O}^{\mathbb{C}}_p / T^n \to \mu(\mathcal{O}^{\mathbb{C}}_p)$. Since the dimension of the isotropy subgroup $G^{\mathbb{C}}_p \subseteq T^n_{\mathbb{C}}$

is twice the dimension of G_p, we conclude that $\mathcal{O}_p^{\mathbb{C}}/T^n$, and therefore $\mu(\mathcal{O}_p^{\mathbb{C}})$, have dimension $n - \dim G_p$. On the other hand, for any $v \in \mathfrak{t}^n$, property (i) of the definition of a moment map (see Sect. 2.1) implies that

$$v \perp \operatorname{Image}(\mathrm{d}\mu)_p \quad \Longleftrightarrow \quad (v^\flat)_p = 0 \quad \Longleftrightarrow \quad v \in \operatorname{Lie} G_p \, .$$

Since

$$\operatorname{Image}(\mathrm{d}\,\mu)_p \quad \supseteq \quad (\mathrm{d}\,\mu)_p(T_p\mathcal{O}_p^{\mathbb{C}}) \;=\; T_{\mu(p)}\,\mu(\mathcal{O}_p^{\mathbb{C}})\, ,$$

after identifying $T_{\mu(p)}\mathfrak{t}^n \simeq \mathfrak{t}^n$ we obtain that $\operatorname{Lie} G_p$ is contained in the subspace of $\mathfrak{t}^n$ orthogonal to $\mu(\mathcal{O}_p^{\mathbb{C}})$. Comparing the dimensions, we conclude that $\operatorname{Lie} G_p$ is in fact equal to that subspace. □

Proposition 4.7. *Theorem 4.2 is true when* $\dim \sigma_c = n$.

Proof. To prove this proposition we will use the results of [2] and the Hitchin-Kobayashi correspondence of [23]. The latter result is hugely simplified for abelian G, which is the case that matters to us, and can be stated in the following form [23]:

Given a simple pair $(A, \phi) \in \mathcal{A}^{1,1} \times \Gamma(E)$, *there exists a complex gauge transformation that takes this pair to a solution of (13b) iff*

$$-v \cdot \deg P \;+\; a^2 \int_{x \in M} \lambda(\phi(x), v) \quad > \quad 0 \qquad for\ all\ v \in \mathbb{R}^n \, . \tag{29}$$

When it exists, this transformation is unique up to composition with real gauge transformations.

Thus to prove the lemma we only have to show that the pair (A, ϕ) of Theorem 4.2 is simple and satisfies (29). The definition of the function λ under the integral is the following. Let $\eta_t^v : F \to F$ be the gradient flow of the function $v \cdot \mu : F \to \mathbb{R}$, and write $\phi(x) = \chi(p, q)$ (see Sect. 2.1); then

$$\lambda(\phi(x), v) \;:=\; \lim_{t \to +\infty} \; v \cdot \mu(\eta_t^v(q)) \, .$$

The integral of λ over M is not in general an easy number to estimate. However, when the assumptions of Theorem 4.2 hold, this obstacle evaporates, and we will now see how. Take $x \in M$ such that (i) and (ii) hold. By (i) the constant c is in the open polytope $Q_x := \mu(\mathcal{O}_{\phi(x)}^{\mathbb{C}})$; by (ii) and Lemma 4.6, this polytope has dimension n. Therefore using Lemma 3.1 of [2], we have that

$$\lim_{t \to +\infty} \; v \cdot \mu(\eta_t^v(q)) \;=\; \sup_{p \in \mathcal{O}_{\phi(x)}^{\mathbb{C}}} \; v \cdot \mu(p) \;=\; \sup_{u \in Q_x} \; v \cdot u \;>\; v \cdot c \, ,$$

where the strict inequality follows from Q_x being open and having dimension n (in particular v cannot be orthogonal to Q_x). Since this holds for all x in an open dense subset of M, we conclude that

$$\int_{x \in M} \lambda(\phi(x), v) \quad > \quad (\operatorname{Vol} M)\; v \cdot c \qquad \text{for all } v \in \mathbb{R}^n \, ,$$

which is equivalent to (29).

To prove that the pair (A, ϕ) is simple (for the definition of simple pair see [23]), it is enough to show that any infinitesimal gauge transformation $s : M \to \operatorname{Lie} T_{\mathbb{C}}^n = \mathbb{C}^n$

that leaves (A, ϕ) fixed is necessarily zero. Let $\mathcal{A}$ be the space of connections on P, and as usual identify $T_A\mathcal{A} \simeq \Omega^1(M, \mathfrak{t}^n)$. The infinitesimal gauge transformation s produces a tangent vector in $T_A\mathcal{A}$, and by the transformation rules (18) this is given by

$$\bar{\partial}s \; + \; \partial\bar{s} \qquad \in \; \Omega^1(M, \mathfrak{t}^n) \; .$$

But if s leaves A fixed, that is $\bar{\partial}s + \partial\bar{s} = 0$, the decomposition $\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$ implies that $\bar{\partial}s = 0$, and since M is compact the function s must be constant. On the other hand for any $x \in M$ such that (ii) is satisfied we have that Lie $G_{\phi(x)} = \{0\}$, and so $s(x)$ leaves $\phi(x) \in E$ fixed iff $s(x) = 0$. By the constancy of s we finally conclude that $s = 0$, and this finishes the proof. □

Remark. The proof for the general case $\dim \sigma_c \leq n$ goes along the following lines. If $c(P, M, a)$ lies in the boundary of $\mu(F)$, by assumption (i) and Lemma 4.4 so does the image $\mu \circ \phi(M)$. Lemma 4.6 then tells us that there is a subtorus of T^n that acts trivially on ϕ. The strategy of the proof is to eliminate this subtorus by formulating the problem in terms of the quotient group and quotient principal bundle. The isotropy groups $G_{\phi(x)}$ of assumption (ii) will then have dimension zero, and we will be reduced to the case $\dim \sigma_c = n$.

5. The Vortex Solutions for Target $\mathbb{CP}^n$

5.1. The main result. We start with the natural action of T^{n+1} on $\mathbb{CP}^n$, given in homogeneous coordinates by

$$(g_0, \ldots , g_n) \cdot [z_0, \ldots , z_n] \; = \; [g_0 z_0, \ldots , g_n z_n] \; .$$

Although this is not an effective action, it induces an effective hamiltonian action of the quotient group T^{n+1}/N, where N denotes the diagonal circle inside T^{n+1}. Now, this quotient group is isomorphic to T^n but, since there is no canonical choice of isomorphism, there are several different ways of implementing the T^{n+1}/N-action as an action of T^n on $\mathbb{CP}^n$. The general formula for these T^n-actions is

$$\rho_{(g_1,\ldots,g_n)}\,([z_0, \ldots , z_n]) \; = \; [z_0\,,\; \Pi_j (g_j)^{C_{1j}} z_1\,, \ldots ,\; \Pi_j (g_j)^{C_{nj}} z_n\;]\,, \tag{30}$$

where the matrix C is in $SL(n, \mathbb{Z})$. The different choices of C correspond to the different possible isomorphisms $T^{n+1}/N \simeq T^n$. These actions are all hamiltonian and, using the identification $(\mathfrak{t}^n)^* \simeq \mathfrak{t}^n \simeq \mathbb{R}^n$ determined by (19), the general form of a moment map $\mu : \mathbb{CP}^n \to \mathbb{R}^n$ is

$$\mu([z_0, \ldots , z_n]) \; = \; \frac{-\pi}{\sum_i |z_i|^2} \left(\ldots , \sum_{j\geq 1} C_{jk} |z_j|^2, \ldots \right)_{1\leq k\leq n} \; + \; \text{const.} \tag{31}$$

We denote by Δ the image $\mu(\mathbb{CP}^n)$ in $\mathbb{R}^n$. This is clearly a convex polytope, since it is the image of the polytope

$$\{x \in \mathbb{R}^n \, : \; 0 \leq x_k \leq 1 \;\text{ and }\; \Sigma_k\, x_k \leq 1\}$$

by the linear transformation $-\pi \; C^T$, possibly composed with a translation.

The aim of this section is to prove Theorems 5.1 and 5.2, stated below. They characterize the space of solutions and energy spectrum, respectively, of the vortex equations for target $\mathbb{CP}^n$ with the T^n-action described above. Also Theorem 5.3, which appears here as an intermediate step to prove Theorem 5.1, may have some independent interest. Before stating these theorems, however, some notation must be introduced.

Let $B_0, \ldots, B_n$ be the $(n-1)$-dimensional faces of the polytope Δ. We denote by $\beta_j \in \mathbb{Z}^n$ the unique primitive normal vector to B_j that points to the exterior of Δ. For each $j = 0, \ldots, n$ define

$$F_j \;=\; \mu^{-1}(B_j) \;=\; \left\{[z_0, \ldots, z_n] \in \mathbb{CP}^n : \; z_j = 0\right\} ,$$

which is a $\mathbb{CP}^{n-1}$ inside $\mathbb{CP}^n$. Since F_j is a T^n-invariant complex submanifold of $\mathbb{CP}^n$, as in the proof of Lemma 4.5 one can define the sub-bundles $E_j = P \times_\rho F_j$ of E; these are complex submanifolds of $(E, J(A))$, where $J(A)$ is the complex structure on E induced by an integrable connection A on P. Recalling also the constants $c(P, M, a) \in \mathbb{R}^n$ and $\alpha(P) \in H^2(M; \mathbb{Z})^n$ defined in Sect. 2.4, we have the following results.

Theorem 5.1. *In the setting described above, the vortex equations (13) have solutions only if the constant $c(P, M, a)$ is in Δ. When this constant lies in the interior of Δ, the set of solutions can be described as follows. For each $j = 0, \ldots, n$ pick an effective divisor $D_j = \sum_i a^i_j \cdot Z_i$ on M such that*

(i) the intersection of hypersurfaces $\operatorname{supp} D_0 \cap \cdots \cap \operatorname{supp} D_n$ *is empty;*
(ii) the Poincaré duals (PD) of the fundamental homology cycles carried by the divisors D_j satisfy $\alpha(P) = \sum_j \beta_j\, \mathrm{PD}(D_j)$ in $H^2(M; \mathbb{Z})^n$.

Then there is a solution (A, ϕ) of (13), unique up to gauge equivalence, such that the intersection multiplicities of the complex submanifolds $\phi(M)$ and E_j satisfy

$$\mathrm{mult}_{\phi(Z_i)}(E_j, \phi(M)) \;=\; a^i_j \;.$$

Furthermore all the solutions of (13) are obtained in this way.

(Here $\operatorname{supp} D_j$ denotes the support of the divisor D_j, i.e. the subset of M formed by the union of the hypersurfaces Z_i with non-zero coefficient a^i_j.)

Theorem 5.2. *Assume that $c(P, M, a)$ lies in the interior of Δ, and let (A, ϕ) be a solution of the vortex equations characterized by divisors D_j, as in the theorem above. Then the topological energy (12) of this solution is*

$$T_{[\phi]} \;=\; e(P, M, \mu, a) \;+\; \frac{\pi}{n+1} \sum_{j=0}^{n} \int_M \mathrm{PD}(D_j) \wedge \omega_M^{[m-1]} \;,$$

where the constant e does not depend on (A, ϕ). Denoting by $b \in \mathbb{R}^n$ the barycentre of the polytope Δ, the value of this constant is

$$e \;=\; \sum_{k=1}^{n} \int_M b_k\, \alpha_k(P) \wedge \omega_M^{[m-1]} \;-\; \frac{1}{a^2}\, \alpha_k(P) \wedge \alpha_k(P) \wedge \omega_M^{[m-2]} \;.$$

Remark. The statement of these results is especially simple when M is a Riemann surface. In this case the hypersurfaces Z_i are just points in M and, under the isomorphism $H^2(M; \mathbb{Z}) \simeq \mathbb{Z}$, there is an identification $\mathrm{PD}(D_j) \simeq \sum_i a^i_j \in \mathbb{Z}$. The topological energy also reduces to

$$T_{[\phi]} \;=\; b \cdot \deg P \;+\; \frac{\pi}{n+1} \sum_{i,j} a^i_j \;.$$

Remark. When the constant $c(P, M, a)$ lies in the boundary of Δ, according to Theorem 4.1 the solutions (A, ϕ) of the vortex equations are constrained to satisfy $\phi(M) \subset E_{j_1} \cap \cdots \cap E_{j_k}$, where $n - k$ is the dimension of the open face of Δ that contains $c(P, M, a)$. Thus in some sense these are σ-models with target $\mathbb{CP}^{n-k}$ and gauge group T^n. This gauge group is too big, and has a subtorus T^k that acts trivially on the sections ϕ, so these cases are somewhat degenerate.

5.2. Proof of Theorem 5.1.

Equivalent theorem. The first statement of Theorem 5.1 follows from Theorem 4.1 and Corollary 4.3. As for the rest of Theorem 5.1, we will prove it by stating and proving the equivalent Theorem 5.3.

Let $\mathcal{S}$ be the set of solutions of the vortex equations (13), and define

$$\mathcal{B} = \left\{(A, \phi) \in \mathcal{A}^{1,1}(P) \times \Gamma(E) : \bar{\partial}^A \phi = 0 \text{ and } \phi(M) \not\subseteq E_j \text{ for all } 0 \leq j \leq n\right\} .$$

The first thing to notice is that by Theorem 4.1, Corollary 4.3 and the subsequent remark, the natural inclusion of $\mathcal{S}$ in $\mathcal{B}$ actually induces a bijection of quotient spaces

$$\mathcal{S}/(\text{real gauge transf.}) \longleftrightarrow \mathcal{B}/(\text{complex gauge transf.}) . \tag{32}$$

On the other hand the action of $T^n_{\mathbb{C}} \simeq (\mathbb{C}^*)^n$ on $\mathbb{CP}^n$ is given by (30), with $g_j \in \mathbb{C}^*$, so it is clear from the definitions of F_j and E_j that

$$\phi^{-1}(E_j) = (g \cdot \phi)^{-1}(E_j) \qquad \subset M$$

for any complex gauge transformation $g : M \to T^n_{\mathbb{C}}$. Moreover, it is a direct consequence of Propositions 5.4 and 5.5, stated below, that for any irreducible hypersurface $Z \subset M$ the intersection multiplicities satisfy

$$\text{mult}_{\phi(Z)}(E_j, \phi(M)) = \text{mult}_{(g\cdot\phi)(Z)}(E_j, (g \cdot \phi)(M)) ,$$

or in other words they are complex-gauge invariant. This fact together with the bijection (32) (which, recall, is induced by the inclusion $\mathcal{S} \hookrightarrow \mathcal{B}$) show that Theorem 5.1 is equivalent to the following result.

Theorem 5.3. *Assume that $c(P, M, a)$ lies in the interior of Δ, and for each $j = 0, \ldots, n$ pick an effective divisor $D_j = \sum_i a^i_j \cdot Z_i$ on M such that conditions (i) and (ii) of Theorem 5.1 are satisfied. Then there exists a pair $(A, \phi) \in \mathcal{B}$, unique up to complex gauge equivalence, such that*

$$\text{mult}_{\phi(Z_i)}(E_j, \phi(M)) = a^i_j . \tag{33}$$

Furthermore all pairs in $\mathcal{B}$ can be obtained in this way.

The method that we will use to prove this theorem is not intrinsic, in the sense that it is based on the use of the usual local charts from $\mathbb{CP}^n$ to $\mathbb{C}^n$. In informal terms, we use the fact that the domains of these charts are T^n-invariant and dense in $\mathbb{CP}^n$ to transfer the problem of finding holomorphic sections of E — which has fibre $\mathbb{CP}^n$ — to the problem of finding meromorphic sections of vector bundles with fibre $\mathbb{C}^n$.

Proof of the equivalent theorem. As always, we start by introducing some notation. For each $j = 0, \ldots, n$ define the action ρ_j of T^n on $\mathbb{C}$ by restricting the action ρ of formula (30) to the j^{th} homogeneous coordinate of $\mathbb{CP}^n$. Thus for example ρ_0 is the trivial action, while for $j \neq 0$ the actions ρ_j depend on the matrix C. Define also the associated line bundles $L_j = P \times_{\rho_j} \mathbb{C}$.

Now consider the usual complex charts $\varphi_j : \mathcal{U}_j \to \mathbb{C}^n$ of $\mathbb{CP}^n$, defined by

$$\begin{aligned} \mathcal{U}_j &= \mathbb{CP}^n \setminus F_j = \left\{[z_0, \ldots, z_n] \in \mathbb{CP}^n : z_j \neq 0\right\} , \\ \varphi_j([z_0, \ldots, z_n]) &= z_j^{-1} (z_0, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) . \end{aligned} \tag{34}$$

From formula (30) and the definition of ρ_j it is clear that $\mathcal{U}_j$ is T^n-invariant and that, for any $g \in T^n$,

$$\varphi_j \circ \rho_g = \Big(\ \ldots , \ (\rho_j)_{g^{-1}} \cdot (\rho_k)_g , \ \ldots \ \Big)_{k \neq j} \circ \varphi_j .$$

Thus defining the vector bundle over M,

$$V_j = (L_j)^{-1} \otimes \big(L_0 \oplus \cdots \oplus L_{j-1} \oplus L_{j+1} \oplus \cdots \oplus L_n\big) , \tag{35}$$

and recalling that

$$E \setminus E_j = \left\{[p, x] \in P \times_\rho \mathbb{CP}^n : \ x \in \mathcal{U}_j\right\} ,$$

one has that the maps

$$\tilde{\varphi}_j : E \setminus E_j \to V_j , \qquad [p, x] \mapsto [p, \varphi_j(x)] \tag{36}$$

are well defined. These maps clearly are fibre-preserving diffeomorphisms. As a matter of notation, we will sometimes call $L_{j,k}$ the k^{th} line bundle in the direct sum decomposition (35); thus for example

$$L_{0,k} = (L_0)^{-1} \otimes L_k , \quad L_{n,k} = (L_n)^{-1} \otimes L_{k-1} \quad \text{and} \quad V_j = \oplus_{1 \leq k \leq n} L_{j,k} . \tag{37}$$

Since the actions ρ_j preserve the canonical hermitian product on $\mathbb{C}$, the line bundles $L_{j,k}$ are all equipped with a natural hermitian metric, denoted $h_{j,k}$. Another standard fact is that a connection A on P induces connections on the associated line bundles L_j and $L_{j,k}$. These connections are $h_{j,k}$-compatible. If the connection A is integrable, i.e $F_A^{0,2} = 0$, then the induced connections on the $L_{j,k}$ are integrable as well, i.e. their curvature form is in $\Omega^{1,1}(M)$.

The reason why we are interested in these integrable connections is that, according to a well known result, an integrable, metric-compatible connection ∇ on a C^∞ hermitian vector bundle $(V, h) \to M$, induces a unique holomorphic structure $\mathcal{H}$ on V such that ∇ is the hermitian connection of $(V, h, \mathcal{H})$ [20]. The bundle V equipped with this holomorphic structure will be denoted by V^∇. We will often apply this result to the line bundles $L_{j,k}$. When the integrable connection on $L_{j,k}$ comes from a connection A on P, we denote by $L_{j,k}^A$ the line bundle together with the induced holomorphic structure.

Using all these conventions we define

$$\mathcal{C} = \{(\nabla_1, \xi_1, \ldots, \nabla_n, \xi_n) : \text{ conditions (1) and (2) are satisfied}\} ,$$

where the conditions are

(1) ∇_k is an $h_{0,k}$-compatible connection on $L_{0,k}$ and ξ_k is a non-zero meromorphic section of $L_{0,k}^{\nabla_k}$;
(2) the divisors on M associated to the sections ξ_k satisfy $(\xi_1)_- = \cdots = (\xi_n)_- =: (\xi)_-$, and the intersection $\mathrm{supp}\,(\xi)_- \cap \mathrm{supp}\,(\xi_1)_+ \cap \cdots \cap \mathrm{supp}\,(\xi_n)_+$ is empty.

In the last condition we have decomposed a divisor $D = D_+ - D_-$ into its positive and negative parts. The main tools to prove Theorem 5.3 are then the following two propositions.

Proposition 5.4. *There exists a bijection $\Upsilon : \mathcal{B} \to \mathcal{C}$ determined by the following conditions:*

(i) ∇_k is the connection on $L_{0,k}$ induced by the connection A on P.
(ii) $\tilde{\varphi}_0 \circ \phi(q) = (\xi_1(q), \ldots, \xi_n(q))$ in V_0 for all $q \in M \setminus \phi^{-1}(E_0)$.

Furthermore, let (A, ϕ) and (A', ϕ') be two pairs in $\mathcal{B}$ and let $(\ldots, \nabla_k, \xi_k, \ldots)$ and $(\ldots, \nabla'_k, \xi'_k, \ldots)$ be their images by Υ. Then (A, ϕ) and (A', ϕ') are complex gauge equivalent if and only if for all k the meromorphic sections ξ_k and ξ'_k have the same associated divisor in M.

Proposition 5.5. *Given a pair (A, ϕ) in $\mathcal{B}$, let $(\ldots, \nabla_k, \xi_k, \ldots)$ be its image in $\mathcal{C}$ by the bijection Υ. Then for any irreducible analytic hypersurface $Z \subset M$ and for any $j = 0, \ldots, n$, the multiplicity of intersection of E_j and $\phi(M)$ along $\phi(Z)$ is given by*

$$\mathrm{mult}_{\phi(Z)}(E_j, \phi(M)) = \mathrm{ord}_Z(\xi_j) - \min_{0\le k\le n} \{\mathrm{ord}_Z(\xi_k)\}\,, \tag{38}$$

where we define $\mathrm{ord}_Z(\xi_0) = 0$.

Remark. Formula (38) implies that, for fixed Z, the multiplicities $\mathrm{mult}_{\phi(Z)}(E_j, \phi(M))$ are non-negative integers, with at least one of them being zero. On the other hand, given such a set of multiplicities, put

$$\mathrm{ord}_Z(\xi_j) = \mathrm{mult}_{\phi(Z)}(E_j, \phi(M)) - \mathrm{mult}_{\phi(Z)}(E_0, \phi(M)). \tag{39}$$

It is then apparent that formulae (38) and (39) define inverse maps between the set of sets of n arbitrary integers $\mathrm{ord}_Z(\xi_j)$, and the set of sets of $n+1$ non-negative, and not all positive, integers.

The proofs of these propositions, especially the first one, are rather long and uninteresting, so we will not give the details here. We are now ready to prove Theorem 5.3.

Proof of Theorem 5.3. Let $D_0, \ldots, D_n$ be divisors on M satisfying conditions (i) and (ii) of Theorem 5.1. We will first show the existence of a pair $(A, \phi) \in \mathcal{B}$ satisfying (33). By definition of the actions ρ_j, the line bundle L_0 is trivial and, for $k > 0$,

$$L_k = \bigotimes_{1\le l\le n} (\hat{L}_l)^{C_{kl}}\,,$$

where the line bundles $\hat{L}_j$ were defined in Sect. 2.4. In particular, using (37), this implies that

$$c_1(L_{0,k}) = \sum_l C_{kl}\, c_1(\hat{L}_l) = \sum_l C_{kl}\, \alpha_l(P). \tag{40}$$

Now denote by $e_1, \dots, e_n$ the standard basis of $\mathbb{R}^n$ and by e_0 the vector $-e_1 - \cdots - e_n$. It is not difficult to check directly that, for $a = 0, \dots, n$, the vector $\beta_a = C^{-1} e_a$ is a primitive vector in $\mathbb{Z}^n$ normal to one of the $(n-1)$-dimensional faces of the polytope Δ. Moreover this is an outward pointing normal vector, and so the β_a's coincide with the β_j's that appear in condition (ii) of Theorem 5.1. Hence by this condition

$$c_1(L_{0,k}) \;=\; \sum_{a,l} C_{kl}\,(C^{-1}e_a)_l\;\mathrm{PD}(D_a) \;=\; \mathrm{PD}(D_k - D_0).$$

Thus the divisor $D_k - D_0$ defines a holomorphic structure on $L_{0,k}$ together with a meromorphic section ξ_k of this line bundle (see [20 or 16]). Now denote by ∇_k the hermitian connection on the hermitian bundle $(L_{0,k},\ h_{0,k})$ equipped with that holomorphic structure. Then by construction the multiplet $(\nabla_1, \xi_1, \dots, \nabla_n, \xi_n)$ satisfies condition (1) of the definition of $\mathcal{C}$. Since $(\xi_k)_+ = D_k$ and $(\xi_k)_- = D_0$, it also satisfies condition (2), as follows from the requirement (i) of Theorem 5.1. Thus

$$(\dots, \nabla_k, \xi_k, \dots)_{1 \le k \le n} \;\in\; \mathcal{C}\ .$$

According to Propositions 5.4 and 5.5 this determines a pair $(A, \phi) \in \mathcal{B}$ such that

$$\mathrm{mult}_{\phi(Z_i)}\,(E_j, \phi(M)) \;=\; (a^i_j - a^i_0) \;-\; \min_{0\le k\le n}\{a^i_k - a^i_0\} \;=\; a^i_j \;-\; \min_{0\le k\le n}\{a^i_k\} \;=\; a^i_j\ ,$$

where in the last equality we have used again the requirement (i) on the divisors D_k. This settles the existence part of Theorem 5.3.

We will now prove the uniqueness statement. Keeping the same divisors $D_j = \sum_i a^i_j \cdot Z_i$ as above, suppose that $(A', \phi') \in \mathcal{B}$ is another pair that satisfies (33), and denote by

$$(\dots, \nabla'_k, \xi'_k, \dots) \;\in\; \mathcal{C}$$

the image of this pair by the bijection Υ. Since we are assuming that the intersection divisors of $\phi(M)$ and $\phi'(M)$ with E_j are the same, Proposition 5.5 and formula (39) in the subsequent remark imply that

$$\mathrm{ord}_{Z_i}(\xi_j) \;=\; a^i_j - a^i_0 \;=\; \mathrm{ord}_{Z_i}(\xi'_j)\ .$$

Thus the meromorphic sections ξ_j and ξ'_j have the same divisor in M, and from Proposition 5.4 we conclude that (A', ϕ') is complex gauge equivalent to (A, ϕ), as required.

To complete the proof of Theorem 5.3 we just need to justify the last assertion, i.e. that for every pair $(A, \phi) \in \mathcal{B}$ the divisors $D_j = \sum_i a^i_j \cdot Z_i$ defined by (33) satisfy conditions (i) and (ii) of Theorem 5.1. In the first place, the definition of $\mathcal{B}$ tells us that $\phi(M) \not\subseteq E_j$, so $\phi^{-1}(E_j)$ is a union of irreducible hypersurfaces of M. In particular the intersection multiplicities of (33) are finite integers, and the divisors D_j are well defined. Secondly, by the definition of D_j as the inverse image by ϕ of the intersection divisor of $\phi(M)$ and E_j, we have that $\phi(\mathrm{supp}\ D_j) \subset E_j$. This implies that (i) is satisfied, because $\cap_j E_j = \emptyset$. Finally, to recognize that the divisors D_j associated to ϕ satisfy (ii) as well, consider the sections $\xi_k = (\tilde{\varphi}_0 \circ \phi)_k$ of Proposition 5.4. From Proposition 5.5 and formula (39) it is clear that $D_k - D_0$ is just the divisor of ξ_k. But ξ_k is a meromorphic section of $L^A_{0,k}$, and so by standard results the Poincaré dual of the divisor of ξ_k is

$c_1(L_{0,k})$. Using these facts, (40), and the formula $\beta_a = C^{-1} e_a$ established earlier, we then get that

$$\sum_{0 \le a \le n} C\, \beta_a \,\mathrm{PD}(D_a) \;=\; \sum_{1 \le k \le n} e_k \,\mathrm{PD}(D_k - D_0) \;=\; \sum_{1 \le k \le n} e_k \, c_1(L_{0,k}) \;=\; C\, \alpha(P)$$

in $H^2(M;\mathbb{Z})^n$. Multiplying on the left by the matrix C^{-1} we obtain that (ii) is indeed satisfied. $\square$

5.3. Proof of Theorem 5.2. The main task is to express the cohomology class $[\eta_E] \in H^2(E;\mathbb{R})$ in terms of the Poincaré dual of $[E_j] \in H_{2(m+n)-2}(E;\mathbb{Z})$. We start by noticing that, up to exact forms, any closed 2-form on M may be written as $s_1\, \omega_M + \beta$, where $s_1 \in \mathbb{R}$, and $\beta \in \Omega^2(M)$ is such that $\beta \wedge \omega_M^{m-1} = 0$. This is a consequence of the Lefschetz decomposition on Kähler manifolds [16]. Furthermore it is apparent from expression (44) below that the class of $\eta_E(A)$, when restricted to the fibres $E_x \simeq \mathbb{CP}^n$ of E, generates the group $H^2(E_x;\mathbb{R}) \simeq \mathbb{R}$. It is then a consequence of the Leray-Hirsch theorem [6] that $\mathrm{PD}(E_j)$, and in fact any element of $H^2(E;\mathbb{R})$, is of the form

$$\mathrm{PD}(E_j) \;=\; s_0\,[\eta_E] + s_1\, \pi_E^*[\omega_M] + \pi_E^*[\beta]\,,$$

where $s_0, s_1 \in \mathbb{R}$. Hence we have that

$$s_0 \int_M \phi^*[\eta_E] \wedge \omega_M^{[m-1]} \;=\; \int_M \phi^*\mathrm{PD}(E_j) \wedge \omega_M^{[m-1]} \;-\; s_1 \int_M m\, \omega_M^{[m]}\,.$$

Now, by well known properties of the Poincaré duality, the restriction of $\mathrm{PD}(E_j)$ to $\phi(M)$ is just the Poincaré dual in $\phi(M)$ of the intersection divisor of E_j and $\phi(M)$ [1]. But because of (33) this divisor is just $\sum_i a_j^i \cdot \phi(Z_i) = \phi_*(D_j)$, and since $\phi : M \to \phi(M)$ is a biholomorphism we obtain that

$$\phi^*\,\mathrm{PD}(E_j) \;=\; \phi^*\,\mathrm{PD}(\phi_*\, D_j) \;=\; \mathrm{PD}(D_j) \qquad \text{in } \; H^2(M;\mathbb{Z})\,.$$

Thus

$$s_0 \int_M \phi^*[\eta_E] \wedge \omega_M^{[m-1]} \;=\; -s_1\, m\, (\mathrm{Vol}\, M) \;+\; \int_M \mathrm{PD}(D_j) \wedge \omega_M^{[m-1]}\,. \tag{41}$$

The task now is to compute the constants s_0 and s_1. Firstly we remark that

$$\begin{aligned} \int_{E_j} \eta_E^{[n-1]} \wedge \pi_E^*\, \omega_M^{[m]} \;&=\; \int_E \eta_E^{[n-1]} \wedge \pi_E^*\, \omega_M^{[m]} \wedge \mathrm{PD}(E_j) \\ &=\; n\, s_0 \int_E \eta_E^{[n]} \wedge \pi_E^*\, \omega_M^{[m]}\,. \end{aligned} \tag{42}$$

Also

$$\begin{aligned} \int_{E_j} \eta_E^{[n]} \wedge \pi_E^*\, \omega_M^{[m-1]} &= \int_E \eta_E^{[n]} \wedge \pi_E^*\, \omega_M^{[m-1]} \wedge \mathrm{PD}(E_j) \\ &= (n+1) s_0 \int_E \eta_E^{[n+1]} \wedge \pi_E^*\, \omega_M^{[m-1]} \;+\; m\, s_1 \int_E \eta_E^{[n]} \wedge \pi_E^*\, \omega_M^{[m]}\,. \end{aligned} \tag{43}$$

[1] I thank Dr. J.M. Woolf for explaining this to me.

The constants s_0 and s_1 are therefore determined by the value of the integrals in (42) and (43). To compute these integrals, recall from Sect. 2.2 that $[\eta_E]$ is the cohomology class of the closed 2-form $\eta_E(A)$ in E, and that

$$\chi^*\eta_E(A) \;=\; \omega_F - \mathrm{d}(\mu, A) \qquad \text{in } \; \Omega^2(P\times F)\;.$$

As in (9), a local section $s:\mathcal{U}\to\pi_P^{-1}(\mathcal{U})$ of P determines a trivialization $E|_{\mathcal{U}}\simeq\mathcal{U}\times F$, and it is not difficult to check that with respect to this trivialization we have

$$\eta_E(A)\,|_{\mathcal{U}} \;=\; \omega_F - \mathrm{d}(\mu, s^*A) \qquad \text{in } \; \Omega^2(\mathcal{U}\times F)\;. \tag{44}$$

It follows that for any $k\in\mathbb{N}$,

$$\eta_E(A)^{[k]} \;=\; \omega_F^{[k]} \;-\; \omega_F^{[k-1]}\wedge \mathrm{d}(\mu, s^*A) \;+\; \cdots \qquad \text{in } \; \Omega^{2k}(\mathcal{U}\times F)\;,$$

and integrating along the fibre [6] we get that

$$(\pi_E)_*\,(\eta_E(A)^{[k]}) \;=\; \begin{cases} \mathrm{Vol}\, F & \text{if } k=n \\ -\,F_A^l \int_F \mu_l & \text{if } k=n+1. \end{cases}$$

Using the standard properties of the homomorphism $(\pi_E)_*$ (see [6]), we therefore have that

$$\int_E \eta_E^{[n]}\wedge\pi_E^*\,\omega_M^{[m]} \;=\; \int_M (\pi_E)_*(\eta_E^{[n]})\wedge\omega_M^{[m]} \;=\; (\mathrm{Vol}\, F)\,(\mathrm{Vol}\, M)\;, \tag{45}$$

$$\begin{aligned} \int_E \eta_E^{[n+1]}\wedge\pi_E^*\omega_M^{[m-1]} \;&=\; \int_M (\pi_E)_*(\eta_E^{[n+1]})\wedge\omega_M^{[m-1]} \\ &= -\left(\int_F \mu_l\right)\,\int_M F_A^l\wedge\omega_M^{[m-1]}\;. \end{aligned} \tag{46}$$

Now consider the inclusions $i_{F_j}:F_j\hookrightarrow F$ and $i_{E_j}:E_j\hookrightarrow E$. Using the restriction $i_{F_j}^*\omega_F$ as a Kähler form on F_j, and $\mu\circ i_{F_j}$ as a moment map for the T^n-action on F_j, one can define the 2-form $\eta_{E_j}(A)$ on $E_j = P\times_{T^n} F_j$, which is the analogue of $\eta_E(A)$ on E. But

$$\chi^*\, i_{E_j}^*\;\eta_E(A) \;=\; i_{F_j}^*\;\omega_F - \mathrm{d}(\mu\circ i_{F_j},\, A) \;=\; \chi^*\;\eta_{E_j}(A)\;,$$

and so $\eta_{E_j}(A)$ is just $i_{E_j}^*\eta_E(A)$. Since $\pi_{E_j}=\pi_E\circ i_{E_j}$ as well, by analogy with (45) and (46) we have that

$$\begin{aligned} \int_{E_j} i_{E_j}^*\;(\eta_E^{[n-1]}\wedge\pi_E^*\,\omega_M^{[m]}) &= \int_{E_j}\eta_{E_j}^{[n-1]}\wedge\pi_{E_j}^*\omega_M^{[m]} \;=\; (\mathrm{Vol}\, F_j)\,(\mathrm{Vol}\, M), \\ \int_{E_j} i_{E_j}^*(\eta_E^{[n]}\wedge\pi_E^*\,\omega_M^{[m-1]}) &= \int_{E_j}\eta_{E_j}^{[n]}\wedge(\pi_{E_j})^*\omega_M^{[m-1]} \\ &= -\left(\int_{F_j}\mu_l\circ i_{F_j}\right)\int_M F_A^l\wedge\omega_M^{[m-1]}. \end{aligned}$$

From the value of these integrals it is straightforward to compute the constants s_0 and s_1; it is enough to use (42), (43) and the fact that $\mathbb{CP}^k$ with the Fubini-Study metric has volume $\pi^k/k!$. Doing this and substituting the result into (41), one obtains that

$$\int_M \phi^*[\eta_E] \wedge \omega_M^{[m-1]} = \frac{n!}{\pi^n}\left[(n+1)\int_F \mu - \pi \int_{F_j} \mu \circ i_{F_j}\right] \cdot \int_M -F_A \wedge \omega_M^{[m-1]} + \pi \int_M \mathrm{PD}(D_j) \wedge \omega_M^{[m-1]} .$$

Now on the one hand, as mentioned in Sect. 2.4, the cohomology class of $-F_A$ is just $\alpha(P)$. On the other hand, since the equality above is valid for all j, we may as well sum over j and divide by $n+1$. Doing this, applying Lemma 5.6 below and using the definitions (12) and (15), we obtain the formula of Theorem 5.2.

Lemma 5.6. *Denoting by $b \in \mathbb{R}^n$ the barycentre of the polytope Δ, one has that*

$$\frac{n!}{\pi^n}\left[(n+1)\int_F \mu - \frac{\pi}{n+1}\sum_{j=0}^{n}\int_{F_j} \mu \circ i_{F_j}\right] = \frac{1}{\mathrm{Vol}\,\Delta}\int_{v\in\Delta} v = b . \tag{47}$$

Proof. Instead of computing these integrals directly, using (31), we will evaluate them using the Duistermaat-Heckman theorem (see for instance [22, 18, 10]). Since $F = \mathbb{CP}^n$ is a toric manifold, the Duistermaat-Heckman polynomial is piecewise a constant; with our conventions it is 1 in the interior of Δ and 0 in the exterior. Therefore

$$\int_F \mu = \int_{v\in\Delta} v ,$$

where the integral on the right-hand side is taken with respect to the Lebesgue measure in $\mathbb{R}^n$. To evaluate the other integrals of Lemma 5.6, take a T^{n-1}-action on $F_j \simeq \mathbb{C}P^{n-1}$ of the same kind as (30), and let $\mu_j : F_j \to \mathbb{R}^{n-1}$ be a moment map for it. Since $\mu \circ i_{F_j}$ is T^{n-1}-invariant, it is clear that it can be written as

$$\mu \circ i_{F_j} = S \circ \mu_j + \text{const.} ,$$

where $S : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is some linear embedding. Thus using again the Duistermaat-Heckman theorem we obtain that

$$\int_{F_j} \mu \circ i_{F_j} = \int_{v\in\mu_j(F_j)} S(v) + \text{const.} = \frac{\mathrm{Vol}\,\mu_j(F_j)}{\mathrm{Vol}\,\mu(F_j)} \int_{v\in\mu(F_j)} v ,$$

where the prefactor of the last term is just the inverse of the determinant of S as a linear map from $\mathbb{R}^{n-1}$ to its image. Finally a third application of the Duistermaat-Heckman theorem shows that

$$\mathrm{Vol}\,\Delta = \mathrm{Vol}\,F = \frac{\pi^n}{n!} = \frac{\pi}{n}\,\mathrm{Vol}\,F_j = \frac{\pi}{n}\,\mathrm{Vol}\,\mu_j(F_j) .$$

Hence the left-hand side of (47) is equal to

$$\frac{n+1}{\mathrm{Vol}\,\Delta}\int_{v\in\Delta} v - \frac{n}{n+1}\sum_{j=0}^{n}\frac{1}{\mathrm{Vol}\,\mu(F_j)}\int_{v\in\mu(F_j)} v . \tag{48}$$

Now notice that the first term in the expression above is $n+1$ times the barycentre vector of Δ, while the second term is $n(n+1)^{-1}$ times the sum of the barycentre vectors of the faces $\mu(F_j)$ of Δ. But for a polytope $\Delta \subset \mathbb{R}^n$ with vertices $p_0, \dots, p_n$ the barycentre vector is

$$b \;=\; \frac{1}{\mathrm{Vol}\,\Delta} \int_{v\in\Delta} v \;=\; \frac{1}{n+1}\,(p_0 + \cdots + p_n) \qquad \in\; \mathbb{R}^n\,.$$

In particular, applying this expression to the faces $\mu(F_j)$ of Δ, we get that the last term of (48) is just

$$-\frac{n}{n+1}\sum_{j=0}^{n}\frac{1}{n}\,(p_0 + \cdots + \hat{p}_j + \cdots + p_n) \;=\; -\frac{n}{n+1}\,(p_0 + \cdots + p_n) \;=\; -n\,b\,.$$

Substituting this into (48) we get the required result. □

6. Constructing Solutions on Quotient Targets

Consider the gauged σ-model determined by the data of Sect. 2.1, and let H be a closed normal subgroup of G. In informal terms, the aim of this section is to compare the vortex equations defined for target F with G-action, and the vortex equations defined for target $F/H_{\mathbb{C}}$ with G/H-action. The main result obtained is Theorem 6.3. We point out that this operation of quotient on the target is more delicate than, for example, the product of targets. In particular, the second vortex equation does not have any natural behaviour under these quotients, and so Theorem 6.3 only concerns the set $\hat{\mathcal{S}}$ of solutions of the first and third vortex equations. Nevertheless, the results in this section are still useful, because in Sect. 4 we showed that, in the abelian case, the quotients $\hat{\mathcal{S}}/\mathcal{G}_{\mathbb{C}}$ are often very similar to the usual quotients $\mathcal{S}/\mathcal{G}$ of vortex solutions. All this will eventually be used in Sect. 7 to exhibit non-trivial solutions of the vortex equations when the target F is a Kähler toric manifold.

We now formalize the problem. Denote by $\mathfrak{h} \subset \mathfrak{g}$ the Lie algebra of H, by $G' = G/H$ the quotient group, and by $\mathfrak{g}' \simeq \mathfrak{g}/\mathfrak{h}$ the Lie algebra of G'. The invariant inner product on $\mathfrak{g}$ induces a splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ with associated projections $\pi_1 : \mathfrak{g} \to \mathfrak{h}$ and $\pi_2 : \mathfrak{g} \to \mathfrak{h}^{\perp}$; it also induces natural identifications $\mathfrak{g}^* \simeq \mathfrak{g}$, $\mathfrak{h}^* \simeq \mathfrak{h}$ and $(\mathfrak{g}')^* \simeq \mathfrak{g}' \simeq \mathfrak{h}^{\perp}$. Using these identifications it is not difficult to check that $\pi_1 \circ \mu : F \to \mathfrak{h}$ is a moment map for the action of H on F.

Suppose, moreover, that $H_{\mathbb{C}}$ acts freely on F, and that there exists an element $a \in \pi_1 \circ \mu(F)$ which is invariant by the coadjoint action of H on $\mathfrak{h}^*$. Then, by standard results [19], the quotient $F/H_{\mathbb{C}}$ is a Kähler manifold in a natural way. The Kähler structure on $F' := F/H_{\mathbb{C}}$ depends on the choice of a and can be characterized as follows. The complex structure on F' is the only one such that the projection $\pi_F : F \to F'$ is holomorphic; the symplectic form $\omega_{F'}$ on F' is determined by the condition

$$i_{Z_a}^*\,\pi_F^*\,\omega_{F'} \;=\; i_{Z_a}^*\,\omega_F\;,$$

where Z_a is the inverse image $(\pi_1 \circ \mu)^{-1}(a)$, and i_{Z_a} is the inclusion $Z_a \hookrightarrow F$. Note that it can be shown that Z_a is a H-invariant submanifold of F, and that $F' = Z_a/H$ [19].

Remark. When F is compact it is never possible to find hamiltonian H-actions such that $H_{\mathbb{C}}$ acts freely. On the other hand, denoting by μ_H the moment map of the H-action, there is a canonical choice of an Ad^*_H-invariant element $a \in \mu_H(F)$, which is $a = \int_{x\in F} \mu_H(x)$. It can then be shown that if H acts freely on $\mu_H^{-1}(a)$, then $H_{\mathbb{C}} \cdot \mu_H^{-1}(a)$ is an open subset of F where the action of $H_{\mathbb{C}}$ is free [19].

The group G' acts naturally on F' by the rule

$$\pi_G(g) \cdot \pi_F(p) \;=\; \pi_F(g \cdot p) \qquad \forall\, g \in G,\; p \in F\;, \tag{49}$$

where $\pi_G : G \to G'$ is the quotient map. It is not difficult to check that this is still a holomorphic hamiltonian action. In fact, a moment map $\mu' : F' \to \mathfrak{g}'^* \simeq \mathfrak{h}^\perp$ for this action is determined by the formula

$$i^*_{Z_a}\, \pi^*_F\, \mu' \;=\; i^*_{Z_a} (\pi_2 \circ \mu)\;. \tag{50}$$

Besides acting on F, the subgroup H also acts freely on the principal bundle P. Let P' be the quotient space P/H and let $\zeta : P \to P'$ be the quotient map. The group G' acts naturally and freely on P', and if we define the projection $\pi_{P'} : P' \to M$ by

$$\pi_{P'} \circ \zeta \;=\; \pi_P\;,$$

it is apparent that P' is the total space of a G'-bundle over M. If $A \in \Omega^1(P;\mathfrak{g})$ is a connection on P, it is clear that $(\mathrm{d}\pi_G) \circ A$ descends to a form $A' \in \Omega^1(P';\mathfrak{g}')$. This is a connection form on the bundle P' [21, p. 79]. Using that $P \times_{\mathrm{Ad}_G} \mathfrak{g}' \simeq P' \times_{\mathrm{Ad}_{G'}} \mathfrak{g}'$, the curvature form of A' is $\mathrm{d}\pi_G \circ F_A \in \Omega^2(M; P \times_{\mathrm{Ad}_G} \mathfrak{g}')$, where $F_A \in \Omega^2(M; P \times_{\mathrm{Ad}_G} \mathfrak{g})$ is the curvature form of A. Identifying $\mathfrak{g}' \simeq \mathfrak{h}^\perp$ this curvature form is just $\pi_2 \circ F_A$. In particular

$$F_A^{0,2} \;=\; 0 \quad\Rightarrow\quad F_{A'}^{0,2} \;=\; \pi_2 \circ F_A^{0,2} \;=\; 0\,. \tag{51}$$

Consider now the associated bundle $E' = P \times_{\mathrm{Ad}_G} F' = P' \times_{\mathrm{Ad}_{G'}} F'$. There is a natural bundle map $E \to E'$ determined by the formula

$$[p, q] \;\mapsto\; [p, \pi_F(q)] \qquad \forall\, p \in P,\; q \in F\;. \tag{52}$$

As always, this induces a map on the space of sections

$$\Gamma(E) \;\longrightarrow\; \Gamma(E'), \qquad \phi \mapsto \phi'\;.$$

Using the definition (7) and the holomorphy of π_F it is then not difficult to check that

$$\bar\partial^A \phi \;=\; 0 \quad\Rightarrow\quad \bar\partial^{A'} \phi' \;=\; 0\,. \tag{53}$$

Hence, in terms of the spaces of solutions

$$\mathcal{S}(P,E) = \{(A,\phi) \in \mathcal{A}(P) \times \Gamma(E) : \text{ Eqs. (13) are satisfied}\} \qquad \text{and}$$
$$\hat{\mathcal{S}}(P,E) = \{(A,\phi) \in \mathcal{A}(P) \times \Gamma(E) : \text{ Eqs. (13a) and (13c) are satisfied}\}\;,$$

we have that the correspondence $(A,\phi) \mapsto (A',\phi')$ defines a map

$$\Upsilon \;:\; \hat{\mathcal{S}}(P,E) \;\longrightarrow\; \hat{\mathcal{S}}(P',E')\;. \tag{54}$$

We will now see how Υ behaves when we quotient by the complex gauge transformations.

To start with, recall that the quotient map π_G can be extended to a homomorphism $(\pi_G)_{\mathbb{C}} : G_{\mathbb{C}} \to G'_{\mathbb{C}}$, and that this homomorphism induces an identification $G'_{\mathbb{C}} \simeq G_{\mathbb{C}}/H_{\mathbb{C}}$. The homomorphism $(\pi_G)_{\mathbb{C}}$ then defines a natural bundle map

$$P \times_{\mathrm{Ad}_G} G_{\mathbb{C}} \longrightarrow P \times_{\mathrm{Ad}_G} G'_{\mathbb{C}} , \qquad [p, g] \mapsto [\, p, (\pi_G)_{\mathbb{C}}(g)\,] . \tag{55}$$

As always, composition with this bundle map defines a map of sections

$$\pi_{\mathcal{G}_{\mathbb{C}}} : \mathcal{G}_{\mathbb{C}} \longrightarrow \mathcal{G}'_{\mathbb{C}} , \qquad g \mapsto g' .$$

This map clearly is a homomorphism of gauge groups. One can also check that it has the following naturality property.

Lemma 6.1. *Let (A, ϕ) be any pair in $\mathcal{A}(P) \times \Gamma(E)$ and g any gauge transformation in $\mathcal{G}_{\mathbb{C}}$. Then $g(\phi)' = g'(\phi')$ in $\Gamma(E')$ and $g(A)' = g'(A')$ in $\mathcal{A}(P')$.*

A direct consequence of this lemma is that the map Υ descends to a map of quotient spaces $\hat{\mathcal{S}}(P, E)/\mathcal{G}_{\mathbb{C}} \to \hat{\mathcal{S}}(P', E')/\mathcal{G}'_{\mathbb{C}}$. Another important property of Υ is the following.

Lemma 6.2. *Let (A_1, ϕ_1) and (A_2, ϕ_2) be two pairs in $\hat{\mathcal{S}}(P, E)$. Then $(A'_1, \phi'_1) = (A'_2, \phi'_2)$ if and only if there exists a gauge transformation $g \in \mathcal{H}_{\mathbb{C}}$ such that $\phi_2 = g(\phi_1)$ and $A_2 = g(A_1)$. When it exists, this transformation is unique.*

This is proved using the assumptions that $H_{\mathbb{C}}$ acts freely on F, the rules (17) and (18) for gauge transformations, and the fact that the (A_j, ϕ_j) satisfy the vortex equations (13a) and (13c). We omit the details. Combining the two lemmas above one directly obtains the main result of this section, which is the following.

Theorem 6.3. *The induced map $\Upsilon : \hat{\mathcal{S}}(P, E)/\mathcal{G}_{\mathbb{C}} \longrightarrow \hat{\mathcal{S}}(P', E')/\pi_{\mathcal{G}_{\mathbb{C}}}(\mathcal{G}_{\mathbb{C}})$ is injective.*

Remark. In many cases of interest the homomorphism $\pi_{\mathcal{G}_{\mathbb{C}}}$ is surjective, and so we actually get an injection $\Upsilon : \hat{\mathcal{S}}(P, E)/\mathcal{G}_{\mathbb{C}} \to \hat{\mathcal{S}}(P', E')/\mathcal{G}'_{\mathbb{C}}$. This happens, for example, when the group G can be factorized as $G = H \times W$, where W is some other subgroup of G. In this case $G_{\mathbb{C}} \simeq H_{\mathbb{C}} \times W_{\mathbb{C}}$ and $G'_{\mathbb{C}} \simeq W_{\mathbb{C}}$, and so it is clear that any section of $P \times_{\mathrm{Ad}_G} G'_{\mathbb{C}}$ can be lifted to a section of $P \times_{\mathrm{Ad}_G} G_{\mathbb{C}}$.

7. Solutions for Target a Compact Toric Manifold

7.1. The canonical Kähler toric manifolds. A compact Kähler toric manifold F is by definition a compact Kähler manifold equipped with an effective hamiltonian action of T^n — where n is the complex dimension of F — which operates by holomorphic transformations. If $\mu : F \to \mathbb{R}^n$ is a moment map for this action, it is well known that the image $\mu(F)$ is a special kind of polytope in $\mathbb{R}^n$, usually called a Delzant polytope, and that this polytope determines F up to T^n-equivariant symplectomorphisms [14].

Definition. *A Delzant polytope Δ in $\mathbb{R}^n$ is a convex polytope such that:*

- *there are n edges meeting at each vertex;*

- *the edges meeting at the vertex p are rational, in the sense that they are of the form $p + tv_i$, $t \in \mathbb{R}$, with $v_i \in \mathbb{Z}^n$;*
- *these $v_1, \dots, v_n$ can be chosen to be a basis of $\mathbb{Z}^n$.*

The symplectomorphism mentioned above, however, does not necessarily preserve the complex structure on F, and so the polytope $\mu(F)$ does not determine F as a Kähler manifold. In other words, this means that several inequivalent Kähler toric manifolds may give rise to the same image polytope $\mu(F)$. Although lacking injectivity, the correspondence between Kähler toric manifolds and Delzant polytopes is certainly surjective. This is because, given any Delzant polytope Δ in $\mathbb{R}^n$, there is a natural way to construct a Kähler toric manifold F_Δ such that $\mu(F_\Delta) = \Delta$. We will now briefly recall this construction; for more details see for example [18 or 10].

Let Δ be a Delzant polytope in $\mathbb{R}^n$ with $(n-1)$-dimensional faces, or facets, $B_1, \dots, B_d$, where $d > n$. Then one can uniquely choose vectors $u_1, \dots, u_d \in \mathbb{Z}^n$ such that u_i is a primitive, outward pointing, normal vector to B_i. The polytope Δ is then the intersection of half-spaces

$$\left\{x \in \mathbb{R}^n : \, u_i \cdot x \, \le \, \lambda_i \, , \; i = 1, \dots, d\right\} ,$$

for some $\lambda_i \in \mathbb{R}$. Denoting by $e_1, \dots, e_d$ the standard basis of $\mathbb{R}^d$, define the linear map

$$\beta : \mathbb{R}^d \longrightarrow \mathbb{R}^n \, , \qquad e_j \mapsto u_j \, ,$$

and its i-linear extension $\beta_{\mathbb{C}} : \mathbb{C}^d \to \mathbb{C}^n$. It is not difficult to show that $\beta(\mathbb{Z}^d) = \mathbb{Z}^n$, and so these maps descend to homomorphisms of tori

$$\begin{array}{ccccccc} \mathbb{R}^d & \xrightarrow{\beta} & \mathbb{R}^n & \qquad & \mathbb{C}^d & \xrightarrow{\beta_{\mathbb{C}}} & \mathbb{C}^n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T^d & \xrightarrow{\tilde\beta} & T^n & & T^d_{\mathbb{C}} & \xrightarrow{\tilde\beta_{\mathbb{C}}} & T^n_{\mathbb{C}} \, . \end{array}$$

In both these diagrams the vertical arrows represent the exponential map (27). The subspace $\mathfrak{n} = \ker \beta$ of $\mathbb{R}^d$ exponentiates to the subgroups

$$\begin{aligned} N \; &= \ker \tilde\beta \; = \; \exp(\mathfrak{n}) & &\subset T^d, \\ N_{\mathbb{C}} \; &= \ker \tilde\beta_{\mathbb{C}} \; = \; \exp(\mathfrak{n}) \times \exp(i\mathfrak{n}) & &\subset T^d_{\mathbb{C}} \, , \end{aligned}$$

and one has the short exact sequence

$$0 \longrightarrow N_{\mathbb{C}} \longrightarrow T^d_{\mathbb{C}} \longrightarrow T^n_{\mathbb{C}} \longrightarrow 0. \tag{56}$$

Now consider the natural action of T^d on the Kähler manifold $\mathbb{C}^d$ given by

$$(g_1, \dots, g_d) \cdot (z_1, \dots, z_d) \; = \; (g_1 z_1, \dots, g_d z_d) \, . \tag{57}$$

This action operates by holomorphic transformations and has moment map

$$\mu : \mathbb{C}^d \longrightarrow \mathbb{R}^d \, , \qquad (z_1, \dots, z_d) \;\mapsto\; -\pi(|z_1|^2, \dots, |z_d|^2) + (\lambda_1, \dots, \lambda_d) \, .$$

The restriction of this action to the subgroup N has moment map $\pi_1 \circ \mu : \mathbb{C}^d \to \mathfrak{n}$, where π_1 is the orthogonal projection from $\mathbb{R}^d$ to $\mathfrak{n}$. Notice also that the action (57) has a natural extension to the complexified group $T^d_{\mathbb{C}}$; this is given by the same formula, but with the g_j's belonging to $\mathbb{C}^*$. Now define the subset

$$\mathbb{C}^d_\Delta = \left\{ z \in \mathbb{C}^d : z_{j_1} = \cdots = z_{j_k} = 0 \text{ allowed only if } \bigcap_{1 \leq l \leq k} B_{j_l} \neq \emptyset \right\} . \quad (58)$$

It is shown in Appendix 1 of [18] that $\mathbb{C}^d_\Delta$ is an open dense subset of $\mathbb{C}^d$, where $N_{\mathbb{C}}$ acts freely. Furthermore the inverse image $Z = (\pi_1 \circ \mu)^{-1}(0)$ is contained in $\mathbb{C}^d_\Delta$, and in fact $\mathbb{C}^d_\Delta = N_{\mathbb{C}} \cdot Z$. Hence, by the quotient construction described in Sect. 6, the quotient manifold $F_\Delta = \mathbb{C}^d_\Delta / N_{\mathbb{C}}$ has a unique structure of Kähler manifold such that the projection $\pi_F : \mathbb{C}^d_\Delta \to F_\Delta$ is holomorphic and

$$i_Z^* \, \pi_F^* \, \omega_{F_\Delta} = i_Z^* \, \omega_{\mathbb{C}^d} .$$

Just as in Sect. 6, the quotient group T^d/N acts naturally on F_Δ by holomorphic transformations. Identifying $T^d/N \simeq T^n$ through $\tilde{\beta}$, this action has a moment map $\mu' : F_\Delta \to \mathbb{R}^n$ determined by

$$i_Z^* \, \pi_F^* \, \mu' = i_Z^* \, (\beta \circ \mu) .$$

It can be shown that $\mu'(F_\Delta) = \Delta$. The Kähler manifold F_Δ equipped with this T^n-action is the canonical Kähler toric manifold we were looking for.

Example. When Δ is the Delzant polytope

$$\Delta = \left\{ x \in \mathbb{R}^n : x_j \leq 0 \;\text{ and }\; \Sigma_j \, x_j \geq -\pi \right\} ,$$

one gets that $\mathbb{C}^d_\Delta = \mathbb{C}^{n+1} \setminus \{0\}$ and that $N \simeq T^1$ is the diagonal subgroup of $T^d = T^{n+1}$. It is then clear that $F_\Delta = \mathbb{C}^d_\Delta / N_{\mathbb{C}} = \mathbb{CP}^n$, and one can check that the induced Kähler metric on $\mathbb{CP}^n$ is the Fubini-Study one.

Besides the result $\mu'(F_\Delta) = \Delta$ described above, in the next section we will also use that for any facet B_j of Δ,

$$(\mu' \circ \pi_F)^{-1}(B_j) = \{ z \in \mathbb{C}^d_\Delta : z_j = 0 \} . \quad (59)$$

This fact also follows from the results in Appendix 1 of [18].

7.2. A family of non-trivial solutions. Let F be a compact Kähler toric manifold, let $\mu : F \to \mathbb{R}^n$ be a moment map for the associated torus action, and call Δ the image $\mu(F)$, which is a Delzant polytope in $\mathbb{R}^n$. Denote by $B_1, \ldots, B_d$ the $(n-1)$-dimensional faces of Δ, and by $\beta_j \in \mathbb{Z}^n$ the unique primitive, outward pointing, normal vector to B_j. Finally identify $\mathfrak{t}^n \simeq \mathbb{R}^n$ through (27), and take the euclidean inner product on $\mathbb{R}^n$ to identify $\mathfrak{t}^n \simeq (\mathfrak{t}^n)^*$.

Now take any principal T^n-bundle P' over the Kähler manifold M, and denote by $\alpha(P') \in H^2(M; \mathbb{Z})^n$ the vector of cohomology classes described in Sect. 2.4. Using this principal bundle one can define the associated bundle $E' = P' \times_{T^n} F$, which has base M and typical fibre F. From Lemma 4.4 we know that the subsets

$$F_j := (\mu')^{-1}(B_j)$$

are T^n-invariant complex submanifolds of F. Furthermore, as described in the proof of Lemma 4.5, the associated bundles

$$E'_j \;:=\; P' \times_{T^n} F_j$$

are complex submanifolds of $(E', J(A))$, where $J(A)$ is the complex structure on E' induced by an integrable connection A on P' (see Sect. 2.2). The aim of this section is to prove the following result.

Theorem 7.1. *In the setting described above the vortex equations (13) have solutions only if the constant $c(P', M, a)$ is in Δ. When this constant lies in the interior of Δ, a set of solutions can be described as follows. For each $j = 1, \dots, d$ pick an effective divisor $D_j = \sum_i a^i_j \cdot Z_i$ on M such that*

(i) if $\bigcap_{1\leq l\leq k} B_{j_l} = \emptyset$ for some indices $j_1, \dots, j_k$, then the intersection of hypersurfaces $\operatorname{supp} D_{j_1} \cap \cdots \cap \operatorname{supp} D_{j_k}$ is empty as well;
(ii) the Poincaré duals of the fundamental homology cycles carried by the divisors D_j satisfy $\alpha(P') = \sum_j \beta_j\, \mathrm{PD}(D_j)$ in $H^2(M; \mathbb{Z})^n$.

Then there is a solution $(A, \phi) \in \mathcal{S}(P', E')$ of the vortex equations such that the intersection multiplicities of the complex submanifolds $\phi(M)$ and E'_j satisfy

$$\mathrm{mult}_{\phi(Z_i)}(E'_j, \phi(M)) \;=\; a^i_j\ . \tag{60}$$

Different choices of divisors provide gauge inequivalent solutions.

Comparing with Theorem 5.1 one recognizes that, when $F = \mathbb{CP}^n$, the set of solutions obtained in Theorem 7.1 actually coincides with the full set of solutions, up to gauge transformations. This motivates the following question.

Question. *Let F be any compact Kähler toric manifold, and suppose that the constant $c(P', M, a)$ lies in the interior of Δ. Do the solutions described in Theorem 7.1 represent the full space of vortex solutions, up to gauge equivalence?*

Proof of Theorem 7.1. We first prove the theorem in the case where F is the canonical manifold F_Δ. At the end we will deal with the case of any F such that $\mu(F) = \Delta$.

Given the divisors D_j, by Proposition 2.5 there is a principal T^d-bundle $P \to M$ such that $\mathrm{PD}(D_j) = \alpha_j(P) = c_1(L_j)$ for all $j = 1, \dots, d$. Let

$$E \;=\; P \times_{T^d} \mathbb{C}^d \;=\; L_1 \oplus \cdots \oplus L_d \tag{61}$$

be the associated bundle. Using the notation of Sect. 7.1, define also the bundle $\dot{E} = P \times_{T^d} \mathbb{C}^d_\Delta$, which is an open dense subset of E.

Now consider the spaces of solutions $\mathcal{S}(P, E)$ and $\hat{\mathcal{S}}(P, E)$ defined before (54). As in Sect. 3 (with $C = Id$), since $\mathrm{PD}(D_j) = c_1(L_j)$, there exists a pair $(A, \phi) \in \hat{\mathcal{S}}(P, E)$ such that D_j is the divisor of the zero set of ϕ_j — the j^{th} component of ϕ under the decomposition (61) — regarded as a holomorphic section of L^A_j. Notice that condition (i) on the divisors implies that if $\bigcap_{1\leq l\leq k} B_{j_l} = \emptyset$ the intersection $\bigcap_{1\leq l\leq k} \phi^{-1}_{j_l}(0)$ is empty; thus, having in mind the definition of $\mathbb{C}^d_\Delta$, we conclude that $\phi(M) \subset \dot{E}$, and the pair (A, ϕ) may be regarded as belonging to $\hat{\mathcal{S}}(P, \dot{E})$.

At this point we want to apply the results of Sect. 6 in order to obtain solutions in $\hat{\mathcal{S}}(P', E')$. Going back to this section, put $G = T^d$, $H = N$ and $F = \mathbb{C}^d_\Delta$. The homomorphism $\tilde{\beta} : T^d \to T^n$, which has kernel N, provides identifications $T^d/N \simeq T^n$ and $P/N \simeq P \times_{\tilde{\beta}} T^n$. But by Lemma 2.6 and condition (ii) we have that

$$\alpha_a(P \times_{\tilde{\beta}} T^n) \;=\; \sum_l \beta_{al}\, \alpha_l(P) \;=\; \alpha_a(P') \qquad \text{for all } a = 1, \dots, n.$$

Thus the bundles P/N and P' are isomorphic, and so the P' of Sect. 6 coincides with the P' of this section. On the other hand, since F_Δ is the Kähler quotient $\mathbb{C}^d_\Delta/N_\mathbb{C}$ with the T^n-action provided by the identification of T^d/N with T^n through $\tilde{\beta}$, the F' and E' of Sect. 6 are just the F_Δ and E' of this section. Applying the results of Sect. 6 we therefore have that the map Υ takes (A, ϕ) to a solution (A', ϕ') in $\hat{\mathcal{S}}(P', E')$. By Lemma 7.2 below, this solution satisfies the condition (ii) on the intersection multiplicities.

We now use the results of Sect. 4, namely Corollary 4.3 and the subsequent remark. These guarantee that (A', ϕ') is complex gauge equivalent to a solution $(\tilde{A}, \tilde{\phi}) \in \mathcal{S}(P', E')$ of the full vortex equations. By the proof of Lemma 7.3 below, the intersection multiplicities of $\phi'(M)$ and $\tilde{\phi}(M)$ with the submanifolds E'_j are the same, and so $(\tilde{A}, \tilde{\phi})$ satisfies condition (ii). This proves the existence part of the theorem. As for the last assertion of the theorem, it follows directly from Lemma 7.3 and the fact that the $(\tilde{A}, \tilde{\phi})$ are complex gauge equivalent to the (A', ϕ'). This finishes the proof for F_Δ.

To show that the theorem remains valid for any F with $\mu(F) = \Delta$, we first remark that such an F is equivariantly biholomorphic to F_Δ [1]. In particular, since the vortex equations (13a) and (13c) only depend on the T^n-action and complex structure on F, not on the symplectic form, the spaces $\hat{\mathcal{S}}(P', E')$ are the same in the F and F_Δ cases. This shows that the solution (A', ϕ') constructed above for the F_Δ case also provides a solution for the F case. Repeating the argument of the paragraph above we conclude that the theorem also holds for F. □

Lemma 7.2. *Let $(A, \phi) \in \hat{\mathcal{S}}(P, \dot{E})$ be the pair constructed above, and let $(A', \phi') \in \hat{\mathcal{S}}(P', E')$ be its image by the map Υ of Sect. 6. Then*

$$\mathrm{mult}_{\phi'(Z_i)}(E'_j, \phi'(M)) \;=\; a^i_j\ .$$

Proof. Denote by $\pi_{\dot{E}} : \dot{E} \to E'$ the bundle map defined in (52), and let E_j be the sub-bundle $\bigoplus_{k \neq j} L_k$ of E. It follows from (59) and the definition of E'_j that

$$\pi_{\dot{E}}^{-1}(E'_j) \;=\; E_j \cap \dot{E}\ . \tag{62}$$

Since the section ϕ' of E' is by definition $\pi_{\dot{E}} \circ \phi$, we then have that

$$\phi'^{-1}(E'_j) \;=\; \phi^{-1}(E_j \cap \dot{E}) \;=\; \phi^{-1}(E_j)\ .$$

Writing this analytic hypersurface in M as a union $\bigcup_{i \in I} Z_i$ of irreducible hypersurfaces, it is tautological that

$$E_j \cap \phi(M) \;=\; \bigcup_{i \in I} \phi(Z_i) \qquad \text{and} \qquad E'_j \cap \phi'(M) \;=\; \bigcup_{i \in I} \phi'(Z_i)\ .$$

Notice that $\phi(Z_i)$ and $\phi'(Z_i)$ are irreducible analytic hypersurfaces in $\phi(M)$ and $\phi'(M)$, respectively, for it was shown in the proof of Lemma 4.5 that ϕ and ϕ' are biholomorphisms onto their images.

Now, given any generic point $p \in Z_i$, let the submanifolds $E_j \subset E$ and $E'_j \subset E'$ be locally defined around $\phi(p)$ and $\phi'(p)$ by holomorphic functions f and f', respectively. This means that f is a locally defined holomorphic function whose germ at $\phi(p)$ is irreducible in the ring $\mathcal{O}_{\phi(p)}(E)$, and such that the zero locus of f coincides with E_j in a neighbourhood of $\phi(p)$. Similarly for f'. Then from the formulae of [16, p. 65, 130 and 395] it follows that

$$\mathrm{mult}_{\phi(Z_i)}(E_j, \phi(M)) = \mathrm{ord}_{\phi(Z_i),\,\phi(p)}\,(f|_{\phi(M)}) = \mathrm{ord}_{Z_i,\,p}(\phi^* f) \qquad \text{and} \quad (63)$$

$$\mathrm{mult}_{\phi'(Z_i)}(E'_j, \phi'(M)) = \mathrm{ord}_{\phi'(Z_i),\,\phi'(p)}\,(f'|_{\phi'(M)}) = \mathrm{ord}_{Z_i,\,p}(\phi'^* f')\ , \qquad (64)$$

where in the rightmost equalities we used that both ϕ and ϕ' are biholomorphisms onto their image. But it is shown in Lemma 7.4 below that if E'_j is locally defined by f', then E_j is locally defined by $f' \circ \pi_{\dot{E}}$. Therefore from the definition $\phi' = \pi_{\dot{E}} \circ \phi$ we obtain that

$$\mathrm{mult}_{\phi(Z_i)}(E_j, \phi(M)) = \mathrm{mult}_{\phi'(Z_i)}(E'_j, \phi'(M))\ .$$

To finish the proof, pick local holomorphic trivializations of the line bundles L^A_j with complex coordinates w_j on the fibre. These induce a holomorphic trivialization of

$$E^A = L^A_1 \oplus \cdots \oplus L^A_d$$

with complex coordinates $w_1, \ldots, w_d$ on the fibre. It is clear that the submanifold E_j is locally defined by the holomorphic function w_j, and from (63) we get that

$$\mathrm{mult}_{\phi(Z_i)}(E_j, \phi(M)) = \mathrm{ord}_{Z_i,\,p}\,(\phi_j) = a^i_j\ ,$$

where in the last equality we used that, by construction of ϕ, $D_j = \sum_i a^i_j \cdot Z_i$ is the divisor of the zero set of ϕ_j regarded as a holomorphic section of L^A_j. □

Lemma 7.3. *In the construction above, different choices of divisors D_j lead to complex-gauge inequivalent solutions $(A', \phi') \in \hat{\mathcal{S}}(P', E')$.*

Proof. Let $\{D^{(1)}_j\}$ and $\{D^{(2)}_j\}$ be two sets of divisors satisfying conditions (i) and (ii), and for $r = 1, 2$ let $(A_r, \phi_r) \in \hat{\mathcal{S}}(P_{(r)}, E_{(r)})$ and $(A'_r, \phi'_r) \in \hat{\mathcal{S}}(P', E')$ be the solutions obtained by the construction above. Suppose furthermore that there exists a complex gauge transformation $\hat{g} : M \to T^n_{\mathbb{C}}$ such that $(A'_2, \phi'_2) = \hat{g}(A'_1, \phi'_1)$.

It is shown in [18, p. 12 and 115] that there exists a subgroup H of T^d such that $T^d_{\mathbb{C}} = N_{\mathbb{C}} \times H_{\mathbb{C}}$, and so the exact sequence (56) splits. In particular there exists a gauge transformation $g : M \to T^d_{\mathbb{C}}$ such that $\hat{g} = \tilde{\beta}_{\mathbb{C}} \circ g = g'$. From Lemma 6.1 we then get that $[g(A_1, \phi_1)]' = (A'_2, \phi'_2)$. In particular, by Lemma 7.2 and its proof, $D^{(2)}_j$ is just the divisor of the zero set of $g(\phi_1)_j$ regarded as a holomorphic section of $(L_{(1)})^{g(A_1)}_j$. But it is well known that a complex gauge transformation does not change the zero set divisor of a section of a line bundle, and so D^2_j coincides with the zero set divisor of $(\phi_1)_j$ regarded as a holomorphic section of $(L_{(1)})^{A_1}_j$, which by construction of ϕ_1 is just $D^{(1)}_j$. This proves the lemma. □

Lemma 7.4. *Fix a connection $A \in \mathcal{A}^{1,1}(P)$ and take the complex structures on the bundles $E = P \times_{T^d} \mathbb{C}^d$ and $E' = P \times_{T^d} F_\Delta = P' \times_{T^d/N} F_\Delta$ to be the ones induced by A, as in Sect. 2.2. Let p be any point in $E_j \cap \dot{E}$ and suppose that, in some neighbourhood of $\pi_{\dot{E}}(p)$, the submanifold $E'_j \subset E'$ is locally defined by a holomorphic function f'. Then the submanifold $E_j \subset E$ is locally defined by the holomorphic function $f' \circ \pi_{\dot{E}}$ in some neighbourhood of p.*

This last lemma can be proved using the local form of holomorphic submersions.

8. Some Comments

In this short and last section we will just make a few informal and not completely rigorous comments about the general pattern of the vortex solutions found in Sects. 3, 5.1 and 7.2.

In all the cases the solutions are characterized by a choice of hypersurfaces in M. These cannot be arbitrary hypersurfaces, but must satisfy some topological constraints relating their Poincaré duals with the Chern numbers of the bundle P where the connection A is defined. Once an allowed choice of hypersurfaces is made, there is a unique solution of the vortex equations (up to gauge equivalence) such that the section $\phi : M \to E$ has some prescribed values along the hypersurfaces. This prescription usually means that along a given hypersurface the map ϕ, which can be locally regarded as having values on the fibre F, is forced to have values on a certain complex submanifold of F. For $F = \mathbb{C}^n$ and $F = \mathbb{CP}^n$ these complex submanifolds are just the natural $\mathbb{C}^{n-1}$'s and $\mathbb{CP}^{n-1}$'s, respectively, contained in F. When F is a compact Kähler toric manifold these submanifolds are the inverse images by the moment map $\mu : F \to \mathbb{R}^n$ of the $(n-1)$-dimensional faces of the Delzant polytope $\Delta \subset \mathbb{R}^n$ characterizing F.

The overall picture becomes clearer if one looks at simple examples, for instance $F = \mathbb{C}$ or $F = \mathbb{CP}^1$. In the former case there is only one type of hypersurface to choose in M; along these the Higgs field ϕ vanishes and they are interpreted as the locations of the usual vortices. In the latter case there are two types of hypersurface to choose in M: the ones taken by ϕ to the south-pole of $\mathbb{CP}^1$ (vortices), and the ones with image the north-pole (anti-vortices). Still in the $F = \mathbb{CP}^1$ case, Theorem 5.2 tells us that all hypersurfaces contribute equally to the total energy of a solution (A, ϕ), independently of their type.

The significance of the hypersurfaces of M that characterize the solutions of the vortex equations can be better understood by varying the real parameter a in these equations. In order to do this fix the principal bundle P where the connection A is defined, fix an allowed choice of hypersurfaces, and choose a moment map $\mu : F \to \mathbb{R}^n$ such that the origin is in the interior of the image polytope $\Delta = \mu(F)$. Then for arbitrarily large a the constant $c(a, P, M)$ of (22) is in Δ, and so solutions exist. Furthermore, by Theorem 5.2, the energy of these solutions tends to a finite constant as $a \to +\infty$. Now, if the energy is to be kept constant, it is evident from (3) that as a grows the value of $\mu \circ \phi$ should approach zero almost everywhere. On the other hand we know that along the chosen hypersurfaces $\mu \circ \phi$ has values in some face of Δ, and this is independent of a. Thus $\mu \circ \phi$ tends to zero everywhere except along the hypersurfaces.

Consider now the second vortex equation (13b). It tells us that, in the regions where $\mu \circ \phi \neq 0$ as $a \to +\infty$, the quantity ΛF_A must also become very large. Thus in some sense the curvature of A, or the magnetic field, becomes localized around the chosen hypersurfaces as $a \to +\infty$. Notice also that, becoming localized around the hypersurfaces, the curvature F_A should be related in some way to the Poincaré duals of these hypersurfaces; this is in fact what is expressed by condition (ii) of Theorems 5.1 and 7.1.

Thus as a tends to infinity the general picture is that the solutions (A, ϕ) tend to the vacuum solutions of the theory — which are characterized by P trivial, $A = 0$ and $\phi = \text{const.} \in \mu^{-1}(0)$ — except at the chosen hypersurfaces.

The opposite limit is when the parameter a tends to zero. In this case it is apparent from (3) that the energy functional tends to the pure Yang-Mills functional, and that the section ϕ does not contribute to the energy. The only relevant equations are then (13b) and (13c), which become the Hermite-Einstein equations.

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