Space-Time Localization of a Class of Geometric Criteria for Preventing Blow-up in the 3D NSE

Zoran Grujić[1], **Qi S. Zhang**[2]

[1] Department of Mathematics, University of Virginia, Charlottesville, VA 22903, USA.
E-mail: zg7c@unix.mail.virginia.edu
[2] Department of Mathematics, University of California, Riverside, CA 92521, USA.
E-mail: qizhang@math.ucr.edu

Abstract: A class of local (in the space-time) conditions on the vorticity directions implying local regularity of weak solutions to the 3D Navier-Stokes equations is established. In all the preceding results, the relevant geometric conditions, although being local in nature, have been assumed uniformly throughout the spatial regions of high vorticity magnitude, and uniformly in time. In addition, similar results are obtained assuming a less restrictive integral condition on the vorticity directions.

1. Introduction

We aim to study local regularity of solutions to the 3D Navier-Stokes equations

$$u_t - \Delta u(x, t) + u \cdot \nabla u(x, t) + \nabla p = 0,$$
$$\nabla \cdot u = 0,$$
$$u(x, 0) = u_0(x) \tag{1.1}$$

for $(x, t) \in \mathbb{R}^3 \times (0, \infty)$, where Δ is the standard Laplacian, a vector field u represents the velocity of the fluid, and a scalar field p the pressure. (The viscosity is normalized, $\nu = 1$.)

Since the seminal work of Leray [L], it is known that (1.1) has a global weak solution whenever $u_0 \in L^2(\mathbb{R}^3)$ is divergence-free. However, the questions of regularity (smoothness) and uniqueness of weak solutions are still open.

In the years to follow, an array of conditions implying the regularity has been discovered. Perhaps the most classical ones are Foias-Prodi-Serrin conditions [P, S] on the space-time integrability of u. Some critical cases, as well as the analogous results in the weak Lebesgue spaces, have been treated in, e.g., [ESS, CP, K, KK]. There are also similar results expressed in terms of the vorticity $\omega = \nabla \times u$, and in particular Beale-Kato-Majda condition, $\|\omega(\cdot)\|_{L^\infty(\mathbb{R}^3)} \in L^1(0, T)$, originally derived for the 3D Euler equations. (Actually, the time-integrability of the BMO-norm suffices [KoT].)

A more geometric way of approaching the problem was brought to light by Constantin in [Co]. The vorticity formulation of the 3D NSE reads

$$\omega_t - \triangle\omega + u \cdot \nabla\omega = \omega \cdot \nabla u.$$

Notice that in the 2D-case, the right-hand side term $\omega \cdot \nabla u$–the vortex stretching term–is identically zero. The other part of the nonlinearity is the advective part, i.e., a transport of the vorticity by the velocity. For a vector v, denote by $\widetilde{v}$ the unit vector in the direction of v. The symmetric part of the velocity gradient is denoted by S, $S = \frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$, and a key quantity α is defined by $\alpha = S\widetilde{\omega} \cdot \widetilde{\omega}$. A direct calculation shows that the vortex stretching term equals to $S\omega$, and that $S\omega \cdot \omega = \alpha|\omega|^2$. Hence, multiplying the vorticity equations by ω in $L^2(\mathbb{R}^3)$ yields

$$\frac{1}{2}\frac{d}{dt}\int |\omega|^2\, dx + \int |\nabla\omega|^2\, dx = \int S\omega \cdot \omega\, dx = \int \alpha|\omega|^2\, dx.$$

Since the boundedness of the enstrophy $\mathbb{E}$, $\mathbb{E} = \int |\omega|^2\, dx$, is enough to ensure regularity, it is plain that α indeed holds a key to understanding the nonlinear effects in the 3D NSE. Constantin [Co] discovered an integral representation of α which revealed a geometric depletion of the nonlinearity,

$$\alpha(x) = \frac{3}{4\pi} P.V. \int D\big(\widetilde{y}, \widetilde{\omega}(x + y), \widetilde{\omega}(x)\big)|\omega(x + y)|\frac{dy}{|y|^3},$$

where the geometric factor D is proportional to the volume spanned by the unit vectors $\widetilde{y}, \widetilde{\omega}(x + y)$ and $\widetilde{\omega}(x)$. More precisely, $D(e_1, e_2, e_3) = (e_1 \cdot e_3)\,\mathrm{Det}(e_1, e_2, e_3)$ for any triplet of unit vectors $\{e_1, e_2, e_3\}$. A simple geometric observation shows

$$|D\big(\widetilde{y}, \widetilde{\omega}(x + y), \widetilde{\omega}(x)\big)| \leq |\sin\varphi\big(\widetilde{\omega}(x + y), \widetilde{\omega}(x)\big)| \leq |\widetilde{\omega}(x + y) - \widetilde{\omega}(x)|;$$

hence, *local alignment* of the vorticity directions will soften up the singularity. This is very interesting since both numerical simulations and laboratory experiments reveal that the regions of high vorticity magnitude exhibit quasi low-dimensional geometry, e.g., vortex sheets and vortex tubes, in which the local alignment of the vorticity directions is evident.

The aforementioned ruminations were subsequently exploited by Constantin and C. Fefferman in [CF] where it was shown that Lipschitz regularity of $\sin\varphi$ in the regions of high vorticity magnitude, uniformly in time, suffices to control the evolution of the enstrophy. Following their approach, Beirao da Veiga and Berselli [BdVB] scaled the Lipschitz condition down to a $\frac{1}{2}$-Hölder condition. A more general class of conditions has been obtained in [GR]–essentially, an interpolation between a purely geometric $\frac{1}{2}$-Hölder condition and a purely analytical Beale-Kato-Majda condition. It states it is enough to assume that for some $q \geq 2$,

$$\|\omega(\cdot)\|_{L^q(\mathbb{R}^3)}^{\frac{q}{q-1}} \in L^1(0, T) \text{ and } |\sin\varphi\big(\widetilde{\omega}(x + y), \widetilde{\omega}(x)\big)| \leq |y|^{\frac{1}{q}}. \qquad (1.2)$$

(As in the previous works, the second condition is assumed throughout the regions of high vorticity magnitude, uniformly in time.)

In a recent work [RG], some new cancellation properties in the vortex stretching term based on the integral representation of α have been detected. This led to new geometric criteria for preventing finite time blow-up. In particular, a certain *isotropy* condition on

the velocity field u has been shown to effectively control the growth of the vorticity magnitude.

All of the aforementioned results have in common that a relevant geometric condition preventing finite time blow-up, although being essentially local in nature, is assumed uniformly in the regions of intense vorticity, and uniformly in time. Hence, the natural question arising is whether these results can be localized, both in the space and in the time variable. The localization is expected to hold from the point of view of fluid response–on the other hand, the velocity field is recovered from the vorticity field by a non-local formula and that springs a number of technical difficulties. Another, perhaps more exciting, motivation for the space-time localization is that the existing geometric scenarios for avoiding singularity formation are somewhat complementary–in particular, some are anisotropic and some are isotropic. Thus, the localization is the necessary zero step in an attempt to cover the space-time with different favorable geometric scenarios.

In this paper, we show that a full space-time localization of a class of conditions (1.2) preventing the loss of regularity is indeed possible. A limit case $q = 2$ corresponds to the localization of the purely geometric $\frac{1}{2}$-Hölder condition. Moreover, we formulate a local *integral* condition on the vorticity directions guaranteeing local regularity –this condition turns out to be less restrictive than the pointwise $\frac{1}{2}$-Hölder one.

In order to bridge some technical difficulties arising in the localization procedure, a technique developed in a recent work on the parabolic equations with divergence-free singular coefficients [Z] will be utilized.

Let us introduce another piece of notation. For a space-time point (x_0, t_0), and $r_0 > 0$, denote by $Q_{r_0}(x_0, t_0)$ (or Q_{r_0}) an open parabolic cylinder $B(x_0, r_0) \times (t_0 - r_0^2, t_0)$. For clarity of the exposition, we assume that solutions are smooth in an open parabolic cylinder, and prove that the localized enstrophy remains bounded on the closed parabolic cylinder of half the size. An alternative way would be to prove a uniform bound on a sequence of approximate solutions. This approach was used in [CF]. In this case the assumptions need to be imposed on each approximate solution.

Theorem 1.1. *Let u be a Leray-Hopf solution of (1.1), and suppose $\omega_0 = \mathrm{curl}\, u_0 \in L^1(\mathbb{R}^3)$. Given $(x_0, t_0) \in \mathbb{R}^3 \times (0, \infty)$, assume that there exist $r_0, d, c > 0$, and $q \geq 2$, such that*

(i) $\omega \in L^{q,\, q/(q-1)}(Q_{3r_0}(x_0, t_0))$,

(ii) $|\sin \varphi(\widetilde{\omega}(x + y, t), \widetilde{\omega}(x, t))| \leq c|y|^{1/q}$ *for* $(x, t) \in \{|\omega| \geq d\} \cap Q_{2r_0}(x_0, t_0)$
  and $|y| \leq r_0$.

Then, if u is smooth in the open cylinder $Q_{2r_0}(x_0, t_0)$, the localized enstrophy remains bounded on $\overline{Q_{r_0}(x_0, t_0)}$, i.e.,

$$\sup_{t\in(t_0-r_0^2,t_0)} \int_{B(x_0,r_0)} |\omega|^2\, dx \leq M < \infty.$$

The next theorem is a generalizaton of Theorem 1.1 in case $q = 2$ where assumption (i) is redundant (a purely geometric case).

Theorem 1.2. *Let u be a Leray-Hopf solution of (1.1), and suppose $\omega_0 = \mathrm{curl}\, u_0 \in L^1(\mathbb{R}^3)$. Given $(x_0, t_0) \in \mathbb{R}^3 \times (0, \infty)$, assume that there exist $r_0, d > 0$ such that the function*

$$\lambda(y) \equiv \sup_{(x,t)\in\{|\omega|\geq d\}\cap Q_{2r_0}(x_0,t_0)} \left|\widetilde{y} \cdot (\widetilde{\omega}(x + y, t) \times \widetilde{\omega}(x, t))\right| \frac{1}{|y|^3} \tag{1.3}$$

is in $L_w^{6/5}(B(0, 3r_0))$, *the weak $L^{6/5}$ space. Then, if u is* *smooth in the open cylinder* $Q_{2r_0}(x_0, t_0)$, *the localized enstrophy remains bounded on* $\overline{Q_{r_0}(x_0, t_0)}$. *In other words, if*

$$\sup_{t \in (t_0 - r_0^2, t_0)} \int_{B(x_0, r_0)} |\omega|^2 \, dx = \infty,$$

then $\|\lambda\|_{L_w^{6/5}(B(0, 3r_0))} = \infty$.

Remark 1. If conditions (i) and (ii) with $q = 2$ in Theorem 1.1 hold, then the function λ in (1.3) is bounded above by $\frac{c}{|y|^{5/2}}$. Hence, it is in $L_w^{6/5}(B(0, 3r_0))$ and Theorem 1.2 applies.

The rest of the paper is organized as follows. We start the proofs of Theorems 1.1 and 1.2 in Sect. 2, and finish in Sect. 3. The proof of Theorem 1.2 is almost the same as that of Theorem 1.1. We present the proof of the latter in detail, and indicate a minor change needed in the proof of Theorem 1.2 in Sect. 3.

2. Local Control of Advection

Proof of Theorem 1.1. We will work on the vorticity equations

$$\omega_t - \Delta\omega + u \cdot \nabla\omega = \omega \cdot \nabla u, \tag{2.1}$$

and show that local enstrophy remains bounded. Comparing with the previous works, where the advective term $u \cdot \nabla\omega$ is simply integrated away, we have to deal with both the advective term and the vortex stretching term $\omega \cdot \nabla u$.

Let (x_0, t_0) be a point in the space-time, and let Q_r be a parabolic cylinder $B(x_0, r) \times (t_0 - r^2, t_0)$ for some $r \le r_0$. Choose $\psi = \phi(y)\eta(s)$ to be a refined cut-off function satisfying

$$\text{supp } \phi \subset B(x_0, 2r); \quad \phi(y) = 1, \quad y \in B(x_0, r); \quad \frac{|\nabla\phi|}{\phi^\delta} \le \frac{C}{r}, \quad 0 \le \phi \le 1;$$

here, $\delta \in (0, 1)$ will be chosen later (such a function is easily constructed by a scaling argument), and

$$\text{supp } \eta \subset (t_0 - (2r)^2, t_0]; \quad \eta(s) = 1, \quad s \in [t_0 - r^2, t_0]; \quad |\eta'| \le 2/r^2; \quad 0 \le \eta \le 1.$$

Denote by Q_{2r}^t the parabolic subcylinder of Q_{2r}, $Q_{2r}^t = B(x_0, 2r) \times (t_0 - (2r)^2, t)$ for t in $(t_0 - (2r)^2, t_0)$. Multiplying (2.1) by $\omega\psi^2$ and integrating over Q_{2r}^t (u is assumed to be smooth on Q_{2r_0}), one obtains

$$\int_{Q_{2r}^t} (\Delta\omega - u \cdot \nabla\omega + \omega \cdot \nabla u - \partial_s\omega) \, \omega\psi^2 dyds = 0.$$

Integration by parts yields

$$\int_{Q_{2r}^t} \nabla(\omega\psi^2)\nabla\omega dyds = -\int_{Q_{2r}^t} u \cdot \nabla\omega(\omega\psi^2)dyds + \int_{Q_{2r}^t} \omega \cdot \nabla u \, \omega\psi^2 dyds$$

$$- \int_{Q_{2r}^t} (\partial_s\omega)\omega\psi^2 dyds. \tag{2.2}$$

By a direct calculation,

$$\int_{Q_{2r}^t} \nabla(\omega\psi^2)\nabla\omega dyds = \int_{Q_{2r}^t} \nabla[(\omega\psi)\psi]\nabla\omega dyds$$

$$= \int_{Q_{2r}^t} [\nabla(\omega\psi)(\nabla(\omega\psi) - (\nabla\psi)\omega) + \omega\psi\nabla\psi\nabla\omega]dyds$$

$$= \int_{Q_{2r}^t} [|\nabla(\omega\psi)|^2 - |\nabla\psi|^2|\omega|^2]dyds.$$

Substituting this into (2.2),

$$\int_{Q_{2r}^t} |\nabla(\omega\psi)|^2 dyds = \int_{Q_{2r}^t} u \cdot \nabla\omega(\omega\psi^2)dyds + \int_{Q_{2r}^t} \omega \cdot \nabla u \, \omega\psi^2 dyds$$

$$- \int_{Q_{2r}^t} (\partial_s\omega)\omega\psi^2 dyds + \int_{Q_{2r}} |\nabla\psi|^2|\omega|^2 dyds. \quad (2.3)$$

Next, notice that

$$\int_{Q_{2r}^t} (\partial_s\omega)\omega\psi^2 dyds = \frac{1}{2}\int_{Q_{2r}^t} (\partial_s|\omega|^2)\psi^2 dyds$$

$$= -\int_{Q_{2r}^t} |\omega|^2\phi^2\eta\partial_s\eta dyds + \frac{1}{2}\int_{B(x_0,2r)} |\omega|^2(y,t)\phi^2(y)dy.$$

Combining this with (2.3), the following bound appears:

$$\int_{Q_{2r}^t} |\nabla(\omega\psi)|^2 dyds + \frac{1}{2}\int_{B(x_0,2r)} |\omega|^2(y,t)\phi^2(y)dy$$

$$\leq \int_{Q_{2r}} (|\nabla\psi|^2 + |\eta||\partial_s\eta|) \, |\omega|^2 dyds$$

$$+ \left|\int_{Q_{2r}^t} u \cdot \nabla\omega \, (\omega\psi^2)dyds\right| + \left|\int_{Q_{2r}^t} \omega \cdot \nabla u \, (\omega\psi^2)dyds\right|$$

$$\equiv T_e + T_a + T_s.$$

Notice that

$$\int_{Q_{2r}^t} u \cdot (\nabla\omega)(\omega\psi^2)dyds$$

$$= \frac{1}{2}\int_{Q_{2r}^t} u \cdot \psi^2\nabla|\omega|^2 dyds = -\frac{1}{2}\int_{Q_{2r}^t} \text{div}(u \cdot \psi^2)|\omega|^2 dyds$$

$$= -\frac{1}{2}\int_{Q_{2r}^t} \text{div} u \, |\psi\omega|^2 dyds - \frac{1}{2}\int_{Q_{2r}^t} u \cdot \nabla(\psi^2)|\omega|^2 dyds$$

$$= -\int_{Q_{2r}^t} u \cdot (\nabla\psi)\psi |\omega|^2 dyds;$$

hence,

$$T_a \leq \int_{Q_{2r}^t} |u||\nabla\psi||\psi||\omega|^2 dyds \leq \int_{Q_{2r}} |u||\nabla\psi||\psi||\omega|^2 dyds \equiv T_a'.$$

Recall now the pointwise relation

$$\omega \cdot \nabla u \cdot \omega = \alpha |\omega|^2,$$

where

$$\alpha(x, t) = \frac{3}{4\pi} P.V. \int_{\mathbb{R}^3} D\big(\widetilde{y}, \widetilde{\omega}(x + y), \widetilde{\omega}(x)\big) |\omega(x + y, t)| \frac{dy}{|y|^3}.$$

The vortex stretching term T_s is then estimated via the splitting into the regions of low and high vorticity,

$$T_s \leq \int_{Q_{2r}^t \cap \{|\omega| < d\}} |\omega|^2 |\nabla u| \psi^2 dy ds + \int_{Q_{2r}^t \cap \{|\omega| \geq d\}} |\alpha| |\omega|^2 \psi^2 dy ds$$

$$\leq \int_{Q_{2r} \cap \{|\omega| < d\}} |\omega|^2 |\nabla u| \psi^2 dy ds + \int_{Q_{2r} \cap \{|\omega| \geq d\}} |\alpha| |\omega|^2 \psi^2 dy ds \equiv T_s^1 + T_s^2.$$

Collecting the above bounds, (2.4) yields

$$\int_{Q_{2r}} |\nabla(\omega \psi)|^2 dy ds + \sup_{t \in (t_0 - (2r)^2, t_0)} \frac{1}{2} \int_{B(x_0, 2r)} |\omega|^2(y, t) \phi^2(y) dy$$

$$\leq T_e + T_a' + (T_s^1 + T_s^2). \tag{2.4$'$}$$

Notice that the first term, T_e, is already in good shape. The vortex stretching terms T_s^1 and T_s^2 will be estimated in the next section.

The following argument is the key step in localizing advection. Notice that, for $\delta \in (0, 1)$, $a \in (0, 2)$ and $m \in (1, 2]$,

$$T_a' = \int_{Q_{2r}} |u| \, \psi^{1+\delta} |\omega|^{2-a} \frac{|\nabla \psi|}{\psi^\delta} |\omega|^a dy ds$$

$$\leq \left[\int_{Q_{2r}} |u|^m \psi^{(1+\delta)m} |\omega|^{(2-a)m} dy ds \right]^{1/m}$$

$$\times \left[\int_{Q_{2r}} \left(\frac{|\nabla \psi|}{\psi^\delta} \right)^{m/(m-1)} |\omega|^{am/(m-1)} dy ds \right]^{(m-1)/m}.$$

Choose a and δ so that

$$(2 - a)m = 2, \qquad (1 + \delta)m = 2.$$

Then,

$$am/(m - 1) = a \frac{2}{2 - a} / \left(\frac{2}{2 - a} - 1 \right) = 2, \qquad \delta = (2/m) - 1 < 1,$$

which paired with the assumptions on the cut-off function ψ yields

$$T_a' \leq \left[\int_{Q_{2r}} |u|^m |\psi \omega|^2 dy ds \right]^{1/m} \left[\int_{Q_{2r}} \frac{c}{r^{m/(m-1)}} |\omega|^2 dy ds \right]^{(m-1)/m}.$$

This implies that, for any $\epsilon > 0$,

$$T_a' \leq \epsilon^m \int_{Q_{2r}} |u|^m |\psi \omega|^2 dy ds + C \frac{\epsilon^{-m/(m-1)}}{r^{m/(m-1)}} \int_{Q_{2r}} |\omega|^2 dy ds. \tag{2.5}$$

Since u is a Leray-Hopf solution,

$$\int_{Q_{2r}} |u|^{4/3} |\psi\omega|^2 dy ds \leq k \int_{Q_{2r}} |\nabla(\psi\omega)|^2 dy ds;$$

indeed,

$$\int_{t_0-4r^2}^{t_0} \int_{B_{2r}} |u|^{4/3} h^2 dx dt$$

$$\leq \int_{t_0-4r^2}^{t_0} \left(\int_{B_{2r}} |u|^2 dx \right)^{2/3} \left(\int_{B_{2r}} h^6 dx \right)^{1/3} dt$$

$$\leq \sup_{t \in [t_0-4r^2, t_0]} \left(\int_{B_{2r}} |u|^2(x,t) dx \right)^{2/3} \int_{t_0-4r^2}^{t_0} \left(\int_{B_{2r}} h^6 dx \right)^{1/3} dt$$

$$\leq C \sup_{t \in [t_0-4r^2, t_0]} \left(\int_{B_{2r}} |u|^2(x,t) dx \right)^{2/3} \int_{t_0-4r^2}^{t_0} \int_{B_{2r}} |\nabla h|^2 dx dt$$

for any h compactly supported in B_{2r}. (The last step is by the Sobolev imbedding.) Substituting the above into (2.5) ($m = 4/3$), we can find $k_1 < 1/2$ and $k_2 > 0$ such that

$$T_a' \leq k_1 \int_{Q_{2r}} |\nabla(\psi\omega)|^2 dy ds + k_2 \frac{1}{r^4} \int_{Q_{2r}} |\omega|^2 dy ds. \tag{2.6}$$

This establishes a local control on the advective term. In the next section, we provide a local control on the vortex stretching term.

3. Local Control of Vortex Stretching

Recall that the vortex stretching term

$$\int_{Q_{2r}} u \cdot \nabla \omega \cdot \psi^2 \omega dy ds \tag{3.1}$$

was estimated via splitting the region of integration by

$$T_s^1 + T_s^2 = \int_{Q_{2r} \cap \{|\omega| < d\}} |\omega|^2 |\nabla u| \psi^2 dy ds + \int_{Q_{2r} \cap \{|\omega| \geq d\}} |\alpha| |\omega|^2 \psi^2 dy ds, \tag{3.2}$$

where

$$\alpha(x,t) = \frac{3}{4\pi} P.V. \int_{\mathbb{R}^3} D\big(\tilde{y}, \tilde{\omega}(x+y), \tilde{\omega}(x)\big) |\omega(x+y,t)| \frac{dy}{|y|^3}. \tag{3.3}$$

Clearly, T_s^1 is bounded above since

$$T_s^1 \leq cd \int_{Q_{2r}} |\nabla u|^2 \psi^2 dx dt;$$

hence, we focus on

$$T_s^2 = \int_{Q_{2r} \cap \{|\omega(x,t)| \geq d\}} \alpha(x,t) |\omega\psi|^2(x,t) dx dt. \tag{3.4}$$

Decompose α as

$$\alpha = \alpha_1 + \alpha_2$$
$$\equiv \frac{3}{4\pi} P.V. \left(\int_{|y| \leq r} + \int_{|y| \geq r} \right) D\big(\tilde{y}, \tilde{\omega}(x+y), \tilde{\omega}(x)\big) \, |\omega(x+y,t)| \frac{dy}{|y|^3}. \quad (3.5)$$

First, observe that

$$|\alpha_2(x,t)| \leq \frac{3}{4\pi r^3} \, \|\omega(\cdot, t)\|_{L^1(\mathbb{R}^3)}, \quad (3.6)$$

and this is *a priori* bounded (cf. [Co2]).

Next, we estimate α_1. By our assumption, $\omega \in L^{q, q/(q-1)}(Q_{3r})$ and

$$\big| \sin \varphi(\tilde{\omega}(x+y,t), \tilde{\omega}(x,t)) \big| \leq c|y|^{1/q}$$

whenever $(x,t) \in Q_{2r} \cap \{|\omega| \geq d\}$ and $|y| \leq r$. Hence,

$$|\alpha_1(x,t)| \leq c \int_{|y| \leq r} \frac{|\omega(x+y,t)|}{|y|^{3-(1/q)}} dy.$$

After a change of variables, we obtain, for all $x \in \mathbb{R}^3$,

$$\chi_{B(x_0, 2r)}(x)|\alpha_1(x,t)| \leq c \int_{|x_0 - y| \leq 3r} \frac{|\omega(y,t)|}{|x-y|^{3-(1/q)}} dy$$
$$= c \int_{\mathbb{R}^3} \frac{|\omega(y,t)| \, \chi_{B(x_0, 3r)}(y)}{|x-y|^{3-(1/q)}} dy.$$

Since

$$\frac{1}{q} + \frac{1}{3q/(3q-1)} = 1 + \frac{1}{3q/2},$$

a version of the weak Young's inequality implies

$$\left(\int_{B(x_0, 2r)} |\alpha_1(x,t)|^{3q/2} dx \right)^{2/3q} \leq c_0 \left\| \frac{1}{|\cdot|} \right\|_{L_w^3}^{\frac{3q-1}{q}} \left(\int_{B(x_0, 3r)} |\omega(x,t)|^q dx \right)^{1/q},$$

where c_0 is a constant independent of r, and L_w^3 is the weak L^3-norm. (This inequality can be found, e.g., on p. 107 of [LL].) Taking the $q/(q-1)$-power in the above inequality, and integrating in time,

$$\|\alpha_1\|_{L^{3q/2, q/(q-1)}(Q_{2r})} \leq c\|\omega\|_{L^{q, q/(q-1)}(Q_{3r})}. \quad (3.7)$$

Using (3.7) and Hölder's inequality, we arrive at

$$\int_{Q_{2r} \cap \{|w(x,t)| \geq d\}} |\alpha_1(x,t)| \, |\omega\psi|^2(x,t) dx dt$$
$$\leq \|\alpha_1\|_{L^{3q/2, q/(q-1)}(Q_{2r})} \, \| \, |\omega\psi|^2 \|_{L^{3q/(3q-2), \, q}(Q_{2r})}$$
$$\leq c\|\omega\|_{L^{q, q/(q-1)}(Q_{3r})} \, \||\omega\psi|^2\|_{L^{3q/(3q-2), \, q}(Q_{2r})}$$
$$= c\|\omega\|_{L^{q, q/(q-1)}(Q_{3r})} \, \||\omega\psi|\|^2_{L^{6q/(3q-2), \, 2q}(Q_{2r})}. \quad (3.8)$$

Write $a = 6q/(3q - 2)$ and $b = 2q$, and notice that

$$\frac{3}{a} + \frac{2}{b} = \frac{3q - 2}{2q} + \frac{2}{2q} = \frac{3}{2}.$$

A version of the Gagliardo-Nirenberg inequality then yields

$$\|\omega\psi\|^2_{L^{6q/(3q-2), \, 2q}(Q_{2r})} \leq C \Bigg[\int_{Q_{2r}} |\nabla(\omega\psi)|^2(x, t)dxdt$$
$$+ \sup_{t\in(t_0-(2r)^2, t_0)} \int_{B(x_0, 2r)} |\omega\psi|^2(x, t)dx \Bigg].$$

Inserting this into (3.8),

$$\int_{Q_{2r}\cap\{|\omega(x,t)|\geq d\}} |\alpha_1(x, t)| \, |\omega\psi|^2(x, t)dxdt$$
$$\leq C\|\omega\|_{L^{q,q/(q-1)}(Q_{3r})} \Bigg[\int\int_{Q_{2r}} |\nabla(\omega\psi)|^2(x, t)dxdt$$
$$+ \sup_{t\in(t_0-(2r)^2, t_0)} \int_{B(x_0, 2r)} |\omega\psi|^2(x, t)dx \Bigg]. \tag{3.9}$$

Collecting the estimates on T_s^1 and T_s^2 implies a desired bound on the vortex stretching term,

$$T_s^1 + T_s^2 \leq C_1\|\omega\|_{L^{q,q/(q-1)}(Q_{3r})}$$
$$\times \Bigg[\int_{Q_{2r}} |\nabla(\omega\psi)|^2(x, t)dxdt + \sup_{t\in(t_0-(2r)^2, t_0)} \int_{B(x_0, 2r)} |\omega\psi|^2(x, t)dx \Bigg]$$
$$+ C(r, \|\nabla u\|_{L^2(Q_{2r})}). \tag{3.10}$$

Here, C_1 is independent of r. Now, recall that $\omega \in L^{q,q/(q-1)}(Q_{3r_0})$. Therefore, there exists $r^*, 0 < r^* \leq r_0$, such that for all $r \leq r^*$, $C_1\|\omega\|_{L^{q,q/(q-1)}(Q_{3r})} < 1/2$. Substituting (3.10) and (2.6) into (2.4)', we deduce

$$\int_{Q_{2r}} |\nabla(\omega\psi)|^2 dyds + \frac{1}{2} \sup_{t\in(t_0-(2r)^2, t_0)} \int_{B(x_0, 2r)} |(\omega\psi)|^2 dyds \leq c(r) < \infty \tag{3.11}$$

for any $r \leq r^*$. This concludes the proof in the case $r^* = r_0$. If $r^* < r_0$, the conclusion is drawn by shifting the vertex of the cylinder Q_{2r}. $\square$

Proof of Theorem 1.2. The proof of Theorem 1.2 is almost identical to that of Theorem 1.1. The only modification occurs after (3.6) and before (3.7). By our assumption, whenever $(x, t) \in Q_{2r} \cap \{|\omega| \geq d\}$ and $|y| \leq r$, we have, for the same (x, t) and y,

$$|\alpha_1(x, t)| \leq c \int_{|y|\leq r} K(y)|\omega(x + y, t)|dy,$$

where $K \in L_w^{6/5}$. Hence,

$$
\chi_{B(x_0,2r)}(x)|\alpha(x,t)| \leq c \int_{|x_0-y|\leq 3r} K(x-y)|\omega(y,t)|dy
$$

$$
= c \int_{\mathbb{R}^3} K(x-y)|\omega(y,t)|\chi_{B(x_0,3r)}(y)dy,
$$

and an application of the weak Young's inequality implies

$$
\left(\int_{B(x_0,2r)} |\alpha_1(x,t)|^3 dx \right)^{1/3} \leq c_0 \|K(\cdot)\|_{L_w^{6/5}}^{5/6} \left(\int_{B(x_0,3r)} |\omega(x,t)|^2 dx \right)^{1/2},
$$

where c_0 is a constant independent of r. The rest of the proof is identical to that of Theorem 1.1 past (3.7), with $q = 2$.   □

References

[BdVB]   Beirao da Veiga, H., Berselli, L.C.: On the regularizing effect of the vorticity direction in incompressible viscous flows. Differ. Integral Eqs. **15** no. 3, 345–356 (2002)
[CP]   Chen, Z-M; Price, W.G.: Blow-up rate estimates for weak solutions of the Navier-Stokes equations. R. Soc. Proc. Ser. A Math. Phys. Eng. Sci. **457**, 2625–2642 (2001)
[Co]   Constantin, P.: Geometric statistics in turbulence. SIAM Rev. **36** no. 1, 73–98 (1994)
[Co2]   Constantin, P.: Navier-Stokes equations and area of interfaces. Commun. Math. Phys. **129** no. 2, 241–266 (1990)
[CF]   Constantin, P., Fefferman, C.: Direction of vorticity and the problem of global regularity for the Navier-Stokes equations. Indiana Univ. Math. J. **42**, 775–789 (1993)
[GR]   Grujić, Z., Ruzmaikina, A.: Interpolation between algebraic and geometric conditions for smoothness of the vorticity in the 3D NSE. Indiana Univ. Math. J. **53**, 1073–1080 (2004)
[ESS]   Escauriaza, L., Seregin, G.A., Sverak, V.: $L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness. Usp. Mat. Nauk **58**, 3–44 (2003)
[K]   Kozono, H.: Removable singularities of weak solutions to the Navier-Stokes equations. Commun. PDE **23**, 949–966 (1998)
[KK]   Kim, H., Kozono, H.: Interior regularity criteria in weak spaces for the Navier-Stokes equations. Manus. Math. **115**, 85–100 (2004)
[KoT]   Kozono, H., Taniuchi, Y.: Bilinear estimates in BMO and the Navier-Stokes equations. Math Z. **235**, 173–194 (2000)
[L]   Leray, J.: Essai sur le mouvements d' un liquide visqueux emplissant l' espace. Acta Math. **63**, 193–248 (1934)
[LL]   Lieb, E.H., Loss, M.: *Analysis.* Second edition. Graduate Studies in Mathematics, **14**. Providence, RI: Amer. Math. Soc., 2001
[P]   Prodi, G.: Un teorema de unicita per le equazioni di Navier-Stokes. Annali di Mat. **48**, 173–182 (1959)
[RG]   Ruzmaikina, A., Grujić, Z.: On depletion of the vortex-stretching term in the 3D Navier-Stokes equations. Commun. Math. Phys. **247**, 601–611 (2004)
[S]   Serrin, J.: On the interior regularity of weak solutions of the Navier-Stokes equations. Arch. Rat. Mech. Anal. **9**, 187–195 (1962)
[Z]   Zhang, Q.S.: A strong regularity result for parabolic equations. Commun. Math. Phys. **244**, 245–260 (2004)