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PROP Profile of Poisson Geometry

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Abstract: It is shown that some classical local geometries are of infinity origin, i.e. their smooth formal germs are (homotopy) representations of cofibrant (di) operads in spaces concentrated in degree zero. In particular, they admit natural infinity generalizations when one considers homotopy representations of the (di) operads in generic differential graded spaces. Poisson geometry provides us with a simplest manifestation of this phenomenon.

0. Introduction

The first instances of algebraic and topological strongly homotopy, or infinity, structures have been discovered by Stasheff [St] long ago. Since that time infinities have acquired a prominent role in algebraic topology and homological algebra. We argue in this paper that some classical local geometries are of infinity origin, i.e. their smooth formal germs are (homotopy) representations of cofibrant PROPs $\mathcal{P}_\infty$ in spaces concentrated in degree zero; in particular, they admit natural infinity generalizations when one considers homotopy representations of $\mathcal{P}_\infty$ in generic differential graded (dg) spaces. The simplest manifestation of this phenomenon is provided by the Poisson geometry (or even by smooth germs of tensor fields!) and is the main theme of the present paper. Another example is discussed in [Mer2]. The PROPs $\mathcal{P}_\infty$ are minimal resolutions of PROPs $\mathcal{P}$ which are graph spaces built from very few basic elements, *genes*, subject to simple engineering rules. Thus to a local geometric structure one can associate a kind of a code, *genome*, which specifies it uniquely and opens a new window of opportunities of attacking differential geometric problems with the powerful tools of homological algebra.

Formal germs of geometric structures discussed in this paper are *pointed* in the sense that they vanish at the distinguished point. This is the usual price one pays for working with (di)operads without "zero terms" (as is often done in the literature). As structural equations behind the particular geometries we study in this paper are homogeneous, this restriction poses no problem: say, a generic non-pointed Poisson structure, ν, in $\mathbb{R}^n$ can be identified with the pointed one, $\hbar\nu$, in $\mathbb{R}^{n+1}$, $\hbar$ being the extra coordinate.

We introduce in this paper a dg free dioperad whose generic representations in a graded vector space V can be identified with pointed solutions of the Maurer-Cartan equations in the Lie algebra of polyvector fields on the formal manifold associated with V. The cohomology of this dioperad can not be computed directly. Instead one has to rely on some fine mathematics such as Koszulness [GiKa, G] and distributive laws [Mar1, G]. One of the main results of this paper is a proof of Theorem 3.2 which identifies the cohomology of that dg free dioperad with a surprisingly small dioperad, $\mathsf{Lie}^1\mathsf{Bi}$, of Lie 1-bialgebras, which are almost identical to the dioperad, $\mathsf{L}ieBi$, of usual Lie bialgebras except that the degree of generating Lie and coLie operations differ by 1 (compare with Gerstenhaber versus Poisson algebras). The dioperad $\mathsf{Lie}^1\mathsf{Bi}$ is proven to be Koszul. We use the resulting geometric interpretation of $\mathsf{Lie}^1\mathsf{Bi}_\infty$ algebras to give their homotopy classification (see Theorem 3.4.5) which is an extension of Kontsevich's homotopy classification [Ko1] of L_∞ algebras.

As a side remark we also discuss graph and geometric interpretations of strongly homotopy Lie bialgebras using Koszulness of the latter which was established in [G].

1. Geometry ⇒ PROP profile ⇒ Geometry $_\infty$

Let $\mathcal{P}$ be an operad, or a dioperad, or even a PROP admitting a minimal dg resolution. Let $\mathcal{P}\mathsf{A}lg$ be the category of finite dimensional dg $\mathcal{P}$-algebras, and $\mathbf{D}(\mathcal{P}\mathsf{A}lg)$ the associated derived category (which we understand here as the homotopy category of $\mathcal{P}_\infty$-algebras, $\mathcal{P}_\infty$ being the minimal resolution of $\mathcal{P}$).

For any locally defined geometric structure Geom (say, Poisson, Riemann, Kähler, etc.) it makes sense talking about the category of formal Geom-manifolds. Its objects are formal pointed manifolds (non-canonically isomorphic to $(\mathbb{R}^n, 0)$ for some n) together with a germ of formal Geom-structure at the distinguished point.

Definition 1.1. *The operad/dioperad/*PROP $\mathcal{P}$ *is called a* **PROP-profile**, *or* **genome**, *of a geometric structure* Geom *if*

- *the category of formal* Geom*-manifolds is equivalent to a full subcategory of the derived category* $\mathbf{D}(\mathcal{P}\mathsf{A}lg)$ *, and*
- *there is no sub-(di)operad of* $\mathcal{P}$ *having the above property.*

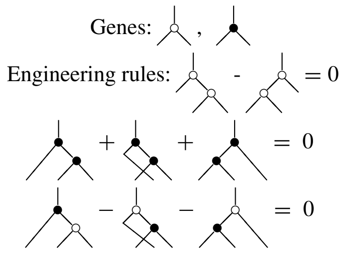
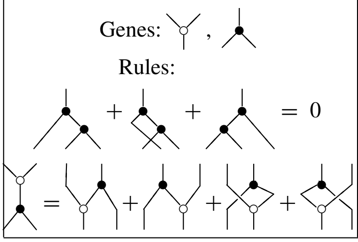
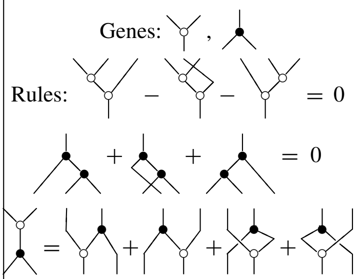
Definition 1.2. *If* $\mathcal{P}$ *is a* PROP*-profile of a geometric structure* Geom*, then a generic object of* $\mathbf{D}(\mathcal{P}\mathsf{A}lg)$ *is called a formal* Geom_∞*-manifold.*

Presumably, Geom_∞-structure is what one gets from Geom by means of the extended deformation theory.

Local geometric structures are often non-trivial and complicated creatures — the general solution of the associated defining system of nonlinear differential equations is not available; it is often a very hard job just to show existence of non-trivial solutions. Nevertheless, if such a structure Geom admits a PROP-profile, $\mathcal{P} = Free(\mathcal{E})/Ideal$[1], then Geom can be non-ambiguously characterized by its "genetic code": *genes* are, by definition, the generators of $\mathcal{E}$, and the *engineering rules* are, by definition, the generators of $Ideal$. And that code can be surprisingly simple, as Examples 1.3–1.5 and illustrate.

[1] Any operad/dioperad/etc. can be represented as a quotient of the free operad/dioperad/etc., $Free(\mathcal{E})$ generated by a collection of Σ_m-left/Σ_n-right modules $\mathcal{E} = \{\mathcal{E}(m,n)\}_{m,n\geq 1}$, by an $Ideal$. Often there exists a canonical, "common factors canceled out", representation like this.

Table 1.

Genome $\mathcal{P}$	generic representation of $\mathcal{P}_\infty$ in $\mathbb{R}^n$	generic representation of $\mathcal{P}_\infty$ in a graded vector space V
$\mathcal{P}$ is the G-operad 	smooth formal Hertling-Manin structure in $\mathbb{R}^n$ [HeMa]	smooth formal Hertling-Manin$_\infty$ structure in $\hat{V}$ [Mer1]
$\mathcal{P}$ is the dioperad TF 	smooth formal section of $\otimes^2 T_{\mathbb{R}^n}$ (variants: of $\wedge^2 T_{\mathbb{R}^n}$ or of $\odot^2 T_{\mathbb{R}^n}$) vanishing at 0	structure, $(\hat{V}, \eth \in T_{\hat{V}})$, of a smooth dg manifold together with a smooth section ϕ of $\otimes^2 T_{\hat{V}}$ (variants: of $\wedge^2 T_{\hat{V}}$ or of $\odot^2 T_{\hat{V}}$) vanishing at 0 and satisfying $Lie_\eth \phi = 0$.
$\mathcal{P}$ is the dioperad $\mathsf{Lie}^1\mathsf{Bi}$ 	smooth formal Poisson structure in $\mathbb{R}^n$ vanishing at 0	structure, $(\widehat{V \oplus V^*[1]}, \eth)$, of a smooth dg manifold together with an odd symplectic form ω_{odd} on $\widehat{V \oplus V^*[1]}$ such that the homological vector field $\eth$ is hamiltonian and vanishes on $\widehat{0 \oplus V^*[1]}$
NOTATIONS: For a graded vector space V, $\hat{V}$ stands for the formal graded manifold (non-canonically) isomorphic to the formal neighbourhood of 0 in V, and $T_{\hat{V}}$ stands for the tangent bundle on $\hat{V}$.		

1.3. Hertling-Manin's geometry and the G-operad. A *Gerstenhaber* algebra is, by definition, a graded vector space V together with two linear maps,

$$\circ: \begin{array}{ccc} \odot^2 V & \longrightarrow & V \\ a \otimes b & \longrightarrow & a \circ b \end{array}, \qquad [\bullet]: \begin{array}{ccc} \odot^2 V & \longrightarrow & V[1] \\ a \otimes b & \longrightarrow & (-1)^{|a|}[a \bullet b] \end{array}$$

satisfying the identities,

(i) $a \circ (b \circ c) - (a \circ b) \circ c = 0$ (associativity);
(ii) $[[a \bullet b] \bullet c] = [a \bullet [b \bullet c]] + (-1)^{|b||a|+|b|+|a|}[b \bullet [a \bullet c]]$ (Jacobi identity);
(iii) $[(a \circ b) \bullet c] = a \circ [b \bullet c] + (-1)^{|b|(|c|+1)}[a \bullet c] \circ b$ (Leibniz type identity).

The operad whose algebras are Gerstenhaber algebras is often called the G-operad. It has a relatively simple structure, $Free(E)/Ideal$, with E spanned by two corollas,

$E = span\Big\{\circ =$  $, [\bullet] =$  $\Big\}$

and with engineering rules (i)–(iii). The minimal resolution of the G-operad has been constructed in [GetJo] and is often called a G_∞-operad. The derived category of Gerstenhaber algebras is equivalent to the category whose objects are isomorphism classes of minimal G_∞-structures on graded vector spaces V. Let $(M, *)$ be the formal pointed graded manifold whose tangent space at the distinguished point is isomorphic to a vector space V, and let us choose an arbitrary torsion-free affine connection ∇ on M. With this choice a structure of G_∞ algebra on a graded vector space V can be suitable described as

- a degree 1 smooth vector field $\eth$ on M satisfying the integrability condition $[\eth, \eth] = 0$ and vanishing at the distinguished point $*$; (if $\eth$ has zero at $*$ of second order, then the G_∞-structure is called minimal);
- a collection of homogeneous *tensors*,

$$\left\{\mu_{n_1,\ldots,n_k} : T_M^{\otimes n_1} \otimes T_M^{\otimes n_2} \otimes \cdots \otimes T_M^{\otimes n_k} \to T_M[k+1-n_1-\cdots-n_k]\right\}_{n_i,k\geq 1, n_i+k\geq 2}$$

satisfying an infinite tower of quadratic algebraic and differential equations. The first two floors of this tower read as follows: the data $\{\mu_n\}_{n\geq 1}$ (with $\mu_1 := Lie_\eth$) makes the tangent sheaf T_M into a sheaf of C_∞ algebras[2] satisfying an "integrability" condition,

$$[\mu_\bullet, \mu_\bullet]_{G_\infty} = Lie_\eth \mu_{\bullet,\bullet}$$

for a certain bi-differential operator $[\ ,\ ]_{G_\infty}$ whose leading term is just the usual vector field bracket of values of $\mu_\bullet$. It is also required that each tensor $\mu_{\bullet,\ldots,\bullet}$: $T_M^{\otimes\bullet} \otimes \cdots \otimes T_M^{\otimes\bullet} \to T_M$ vanishes if the input contains at least one pure shuffle product,

$$(v_1 \otimes \cdots \otimes v_k)\bigstar(v_{k+1} \otimes \cdots \otimes v_n) := \sum_{\substack{\text{Shuffles } \sigma \\ \text{of type (k,n)}}} (-1)^{Koszul(\sigma)} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)},$$
$$v_i \in T_M.$$

A change of the connection ∇ alters the tensors $\mu_{\bullet_1,\ldots,\bullet_k}$, $k \geq 2$, but leaves the homotopy class of the G_∞-structure on V invariant.

If the vector space V is concentrated in degree 0, i.e. $V \simeq \mathbb{R}^n$, then a G_∞-structure on V reduces just to a single tensor field $\mu_2 : T_M^{\otimes 2} \to T_M$ which makes the tangent sheaf into a sheaf of commutative associative algebras, and satisfies the differential equations,

$$[\mu_2, \mu_2]_{G_\infty} = 0.$$

[2] C_∞ stands for the minimal resolution of the operad of commutative associative algebras.

The explicit form for the bracket $[\ ,\ ]_{G_\infty}$ can be read off from the G_∞ operad structural equations rather straightforwardly (see [Mer1] for details),

$$\begin{aligned}[\mu_2, \mu_2]_{G_\infty}(X, Y, Z, W) = {} & [\mu_2(X, Y), \mu_2(Z, W)] \\ & -\mu_2([\mu_2(X, Y), Z], W) - \mu_2(Z, [\mu_2(X, Y), W]) \\ & -\mu_2(X, [Y, \mu_2(Z, W)]) - \mu_2[X, \mu_2(Z, W)], Y) \\ & +\mu_2(X, \mu_2(Z, [Y, W])) + \mu_2(X, \mu_2([Y, Z], W)) \\ & +\mu_2([X, Z], \mu_2(Y, W)) + \mu_2([X, W], \mu_2(Y, Z)).\end{aligned}$$

The resulting geometric structure is precisely the one discovered earlier by Hertling and Manin [HeMa] in their quest for a weaker notion of Frobenius manifold; they call it an *F-manifold* structure on V.

Hertling-Manin's geometric structures arise naturally in the theory of singularities [He] and the deformation theory [Mer1].

1.4. Germs of tensor fields. A TF *bialgebra* is, by definition, a graded vector space V together with two linear maps,

$\delta \equiv$  $: V \longrightarrow \otimes^2 V$, $\quad a \longrightarrow \sum a_1 \otimes a_2$, $\qquad [\bullet] \equiv$  $: \odot^2 V \longrightarrow V[1]$, $\quad a \otimes b \longrightarrow (-1)^{|a|}[a \bullet b]$

satisfying the identities,

(i) $[[a \bullet b] \bullet c] = [a \bullet [b \bullet c]] + (-1)^{|b||a|+|b|+|a|}[b \bullet [a \bullet c]]$ (Jacobi identity);
(ii) $\delta[a \bullet b] = \sum a_1 \otimes [a_2 \bullet b] + [a \bullet b_1] \otimes b_2 + (-1)^{|a||b|+|a|+|b|}([b \bullet a_1] \otimes a_2 + b_1 \otimes [b_2 \bullet a])$ (Leibniz type identity).

There are obvious versions of the above notion with δ taking values in $\wedge^2 V$ and $\odot^2 V$, i.e. with the gene  realizing either the trivial or sign representations of Σ_2.

The dioperad whose algebras are TF bialgebras is denoted by TF. This quadratic dioperad is Koszul so that one can construct its minimal resolution using the results of [G, GiKa, Mar1]. It turns out that the structure of TF_∞-algebra on a graded vector space V is the same as a pair of collections of linear maps,

$$\left\{\mu_n : \odot^n V \to V[1]\right\}_{n \geq 1},$$

and

$$\left\{\phi_n : \odot^n V \to V \otimes V\right\}_{n \geq 1},$$

satisfying a system of quadratic equations which are best described using a geometric language. Let M be the formal graded manifold associated to V. If $\{e_\alpha, \alpha = 1, 2, \ldots\}$ is a homogeneous basis of V, then the associated dual basis t^α, $|t^\alpha| = -|e_\alpha|$, defines a coordinate system on M. The collection of tensors $\{\mu_n\}_{n\geq 1}$ can be assembled into a germ, $\eth \in T_M$, of a degree 1 smooth vector field,

$$\eth := \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^\epsilon t^{\alpha_1} \cdots t^{\alpha_n} \mu_{\alpha_1, \ldots, \alpha_n}^{\ \ \beta} \frac{\partial}{\partial t^\beta},$$

where

$$\epsilon = \sum_{k=1}^{n} |e_{\alpha_k}|(1 + \sum_{i=1}^{k} |e_{\alpha_i}|),$$

the numbers $\mu_{\alpha_1 \dots \alpha_n}^{\beta}$ are defined by

$$\mu_n(e_{\alpha_1}, \dots, e_{\alpha_n}) = \sum \mu_{\alpha_1 \dots \alpha_n}^{\beta} e_\beta,$$

and we assume here and throughout the paper summation over repeated small Greek indices.

Another collection of linear maps, $\{\phi_n\}$, can be assembled into a smooth germ, $\phi \in \otimes^2 T_M$, of a degree zero contravariant tensor field on M,

$$\phi := \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^\epsilon t^{\alpha_1} \cdots t^{\alpha_n} \phi_{\alpha_1 \dots \alpha_n}^{\beta_1 \beta_2} \frac{\partial}{\partial t^\beta} \otimes \frac{\partial}{\partial t^\beta},$$

where

$$\epsilon = |e_{\beta_2}|(|e_{\beta_1}| + 1) + \sum_{k=1}^{n} \sum_{i=1}^{k} |e_{\alpha_k}||e_{\alpha_i}|$$

and the numbers $\mu_{\alpha_1 \dots \alpha_n}^{\beta_1 \beta_2}$ are defined by

$$\mu_n(e_{\alpha_1}, \dots, e_{\alpha_n}) = \sum \mu_{\alpha_1, \dots, \alpha_n}^{\beta_1 \beta_2} e_{\beta_1} \otimes e_{\beta_2}.$$

Proposition 1.4.1. *The collections of tensors,*

$$\left\{\mu_n : \odot^n V \to V[1]\right\}_{n \geq 1} \quad and \quad \left\{\phi_n : \odot^n V \to V \otimes V\right\}_{n \geq 1},$$

define a structure of TF_∞*-algebra on* V *if and only if the associated smooth vector field* $\eth$ *and the contravariant tensor field* ϕ *satisfy the equations,*

$$[\eth, \eth] = 0$$

and

$$Lie_\eth \phi = 0,$$

where $[\ ,\ ]$ *stands for the usual bracket of vector fields and* $Lie_\eth$ *for the Lie derivative along* $\eth$.

If V is finite dimensional and concentrated in degree zero, then a TF_∞-structure in V is just a germ of a smooth rank 2 contravariant tensor on V vanishing at 0.

1.5. Poisson geometry and the dioperad of Lie 1-bialgebras. A *Lie 1-bialgebra* is, by definition, a graded vector space V together with two linear maps,

$$\begin{array}{rcl} \delta : V & \longrightarrow & \wedge^2 V \\ a & \longrightarrow & \sum a_1 \wedge a_2 \end{array} \quad , \quad \begin{array}{rcl} [\bullet] : \odot^2 V & \longrightarrow & V[1] \\ a \otimes b & \longrightarrow & (-1)^{|a|}[a \bullet b] \end{array}$$

satisfying the identities,

(i) $(\delta \otimes \mathrm{Id})\delta a + \tau(\delta \otimes \mathrm{Id})\delta a + \tau^2(\delta \otimes \mathrm{Id})\delta a = 0$, where τ is the cyclic permutation (123) represented naturally on $V \otimes V \otimes V$ (co-Jacobi identity);
(ii) $[[a \bullet b] \bullet c] = [a \bullet [b \bullet c]] + (-1)^{|b||a|+|b|+|a|}[b \bullet [a \bullet c]]$ (Jacobi identity);
(iii) $\delta[a \bullet b] = \sum a_1 \wedge [a_2 \bullet b] - (-1)^{|a_1||a_2|} a_2 \wedge [a_1 \bullet b] + [a \bullet b_1] \wedge b_2 - (-1)^{|b_1||b_2|}[a \bullet b_2] \wedge b_1$ (Leibniz type identity).

The dioperad whose algebras are Lie 1-bialgebras is denoted by $\mathsf{Lie}^1\mathsf{Bi}$. The superscript 1 in the notation is used to emphasize that the two basic operations

$\delta =$  , $[\bullet] =$ 

have homogeneities which differ by 1.

Similarly one can introduce the notion of *Lie n-bialgebras*: coLie algebra structure on V plus Lie algebra structure on $V[-n]$ plus an obvious Leibniz type identity. Homotopy theory of Lie n-bialgebras splits into two stories, one for n even, and one for n odd. The even case (more precisely, the case $n = 0$) has been studied by Gan [G]. In this paper we study the odd case, more precisely, the case $n = 1$.

The dioperad $\mathsf{Lie}^1\mathsf{Bi}$ is Koszul. Hence one can use the machinery of [G, GiKa, Mar1] to construct its minimal resolution, the dioperad $\mathsf{Lie}^1\mathsf{Bi}_\infty$. The structure of a $\mathsf{Lie}^1\mathsf{Bi}_\infty$ algebra on a graded vector space V is a collection of linear maps,

$$\left\{\mu_{m,n} : \odot^n V \to \wedge^m V[2-m]\right\}_{m \geq 1, n \geq 1},$$

satisfying a system of quadratic equations which can be described as follows. Let M be the formal graded manifold associated to V. If $\{e_\alpha, \alpha = 1, 2, \ldots\}$ is a homogeneous basis of V, then the associated dual basis t^α, $|t^\alpha| = -|e_\alpha|$, defines a coordinate system on M. For a fixed m the collection of tensors $\{\mu_{m,n}\}_{n \geq 1}$ can be assembled into a germ, $\Gamma_m \in \wedge^m T_M$, of a smooth polyvector field (vanishing at $0 \in M$),

$$\Gamma_m := \sum_{n=1}^{\infty} \frac{1}{m!n!} (-1)^{\epsilon} t^{\alpha_1} \cdots t^{\alpha_n} \mu_{\alpha_1 \ldots \alpha_n}{}^{\beta_1 \ldots \beta_m} \frac{\partial}{\partial t^{\beta_1}} \wedge \cdots \wedge \frac{\partial}{\partial t^{\beta_m}},$$

where

$$\epsilon = \sum_{k=1}^{n} |e_{\alpha_k}|(2 - m + \sum_{i=1}^{k} |e_{\alpha_i}|) + \sum_{k=1}^{n} (|e_{\beta_k}| + 1) \sum_{i=k+1}^{n} |e_{\beta_i}|$$

and the numbers $\mu_{\alpha_1, \ldots, \alpha_n}{}^{\beta_1, \ldots, \beta_m}$ are defined by

$$\mu_{m,n}(e_{\alpha_1}, \ldots, e_{\alpha_n}) = \sum \mu_{\alpha_1, \ldots, \alpha_n}{}^{\beta_1, \ldots, \beta_m} e_{\beta_1} \wedge \cdots \wedge e_{\beta_m}.$$

Proposition 1.5.1. *A collection of tensors,* $\{\mu_{m,n} : \odot^n V \to \wedge^m V[2-m]\}_{m\geq 1, n\geq 1}$, *defines a structure of* $\mathsf{Lie}^1\mathsf{Bi}_\infty$*-algebra on* V *if and only if the associated smooth polyvector field,*

$$\Gamma := \sum_{m\geq 1} \Gamma_m \in \wedge^\bullet T_M,$$

satisfies the equation

$$[\Gamma, \Gamma] = 0,$$

where [,] *stands for the Schouten bracket of polyvector fields.*

In particular, if V is concentrated in degree zero, then the only non-zero summand in Γ is $\Gamma_2 \in \wedge^2 T_M$. Hence a $\mathsf{Lie}^1\mathsf{Bi}_\infty$-algebra structure on $\mathbb{R}^n$ is nothing but a germ of a smooth Poisson structure on $\mathbb{R}^n$ vanishing at 0.

1.6. On the content of the rest. Section 2 is a reminder about PROPs, dioperads and Koszulness [G, GiKa, Mar1]. In Sects. 3 and 4 we prove Koszulness of the dioperads $\mathsf{Lie}^1\mathsf{Bi}$ and TF, apply the machinery reviewed in Sect. 2 to give explicit graph descriptions of their minimal resolutions, $\mathsf{Lie}^1\mathsf{Bi}_\infty$ and TF_∞, prove Propositions 1.4.1 and 1.5.1 and introduce and study the notion of $\mathsf{Lie}^1\mathsf{Bi}_\infty$ morphisms. Section 5 is a comment on a geometric description of algebras over the dioperad of strongly homotopy Lie bialgebras, and their strongly homotopy maps.

2. PROPs and Dioperads [G]

Let S_f be the groupoid of finite sets. It is equivalent to the category whose objects are natural numbers, $\{m\}_{m\geq 1}$, and morphisms are the permutation groups $\{\Sigma_m\}_{m\geq 1}$.

A PROP $\mathcal{P}$ in the category, dgVec, of differential graded (shortly, dg) vector spaces is a functor $\mathcal{P} : \mathsf{S}_f \times \mathsf{S}_f{}^{op} \to \mathsf{dgVec}$ together with natural transformations,

$$\circ_{A,B,C} : \mathcal{P}(A,B) \otimes \mathcal{P}(B,C) \longrightarrow \mathcal{P}(A,C),$$
$$\otimes_{A,B,C,D} : \mathcal{P}(A,B) \otimes \mathcal{P}(C,D) \longrightarrow \mathcal{P}(A\otimes B, C\otimes D)$$

and the distinguished elements $\mathrm{Id}_A \in \mathcal{P}(A,A)$ and $s_{A,B} \in \mathcal{P}(A\otimes B, B\otimes A)$ satisfying a system of axioms [A] which just mimic the obvious properties of the following natural transformation,

$$\mathcal{E}_V : (m,n) \longrightarrow Hom(V^{\otimes n}, V^{\otimes m}),$$

canonically associated with an arbitrary dg space V. The latter fundamental example is called the *endomorphism* PROP of V.

Given a collection of dg (Σ_m, Σ_n)-bimodules, $E = \{E(m,n)\}_{m,n\geq 1}$, one can construct the associated free PROP, $Free(E)$, by decorating vertices of all possible directed graphs with a flow by the elements of E and then taking the colimit over the graph automorphism group. The composition operation $\circ$ corresponds then to gluing output legs of one graph to the input legs of another graph, and the tensor product $\otimes$ to the disjoint union of graphs. Even for a small finite dimensional collection E the resulting free PROP can be a monstrous infinite dimensional object. The notion of dioperad was introduced by Gan [G] as a way to avoid that free PROP "divergence". In the above setup, a free dioperad on E is built on graphs of genus zero, i.e. on trees.

More precisely, a *dioperad* $\mathcal{P}$ consists of data:

(i) a collection of dg (Σ_m, Σ_n) bimodules, $\{\mathcal{P}(m,n)\}_{m\geq 1, n\geq 1}$;
(ii) for each $m_1, n_1, m_2, n_2 \geq 1$, $i \in \{1, 2, \ldots, n_1\}$ and $j \in \{1, \ldots, n_1\}$ a linear map

$$ {}_i\circ_j : \mathcal{P}(m_1, n_1) \otimes \mathcal{P}(m_2, n_2) \longrightarrow \mathcal{P}(m_1 + m_2 - 1, n_1 + n_2 - 1), $$

(iii) a morphism $e : k \to \mathcal{P}(1,1)$ such that the compositions

$$ k \otimes \mathcal{P}(m,n) \xrightarrow{e\otimes Id} \mathcal{P}(1,1) \otimes \mathcal{P}(m,n) \xrightarrow{{}_1\circ_i} \mathcal{P}(m,n) $$

and

$$ \mathcal{P}(m,n) \otimes k \xrightarrow{Id\otimes e} \mathcal{P}(m,n) \otimes \mathcal{P}(1,1) \xrightarrow{{}_j\circ_1} \mathcal{P}(m,n) $$

are the canonical isomorphisms for all $m, n \geq 1$, $1 \leq i \leq m$ and $1 \leq j \leq n$.

These data satisfy associativity and equivariance conditions [G] which can be read off from the example of the *endomorphism dioperad* $\mathcal{E}nd_V$ with $\mathcal{E}nd_V(m,n) = Hom(V^{\otimes n}, V^{\otimes m})$, $e : 1 \to \mathrm{Id} \in Hom(V, V)$, and the compositions given by

$$ \begin{array}{rcl} {}_i\circ_j : \mathcal{P}(m_1, n_1) \otimes \mathcal{P}(m_2, n_2) & \longrightarrow & \mathcal{P}(m_1 + m_2 - 1, n_1 + n_2 - 1) \\ f \otimes g & \longrightarrow & (\mathrm{Id} \otimes \cdots \otimes f \otimes \cdots \otimes \mathrm{Id})\sigma(\mathrm{Id} \otimes \cdots \otimes g \otimes \cdots \otimes \mathrm{Id}), \end{array} $$

where f (resp. g) is at the j^{th} (resp. i^{th}) place, and σ is the permutation of the set $I = (1, 2, \ldots, n_1 + m_2 - 1)$ swapping the subintervals, $I_1 \leftrightarrow I_2$ and $I_4 \leftrightarrow I_5$, of the unique order preserving decomposition, $I = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4 \sqcup I_5$, of I into the disjoint union of five intervals of lengths $|I_1| = i - 1$, $|I_2| = j - 1$, $|I_3| = 1$, $|I_4| = m_2 - j$ and $|I_5| = n_1 - i$.

If $\mathcal{P}$ is a dioperad, then the collection of (Σ_m, Σ_n) bimodules,

$$ \mathcal{P}^{op}(m,n) := (\mathcal{P}(n,m), \text{ transposed actions of } \Sigma_m \text{ and } \Sigma_n), $$

is naturally a dioperad as well.

If $\mathcal{P}$ is a dioperad with $\mathcal{P}(m,n)$ vanishing for all m, n except for $(m = 1, n \geq 1)$, then $\mathcal{P}$ is called an *operad.*

A morphism of dioperads, $F : \mathcal{P} \to \mathcal{Q}$, is a collection of equivariant linear maps, $F(m,n) : \mathcal{P}(m,n) \to \mathcal{Q}(m,n)$, preserving all the structures. If $\mathcal{P}$ is a dioperad, then a *$\mathcal{P}$-algebra* is a dg vector space V together with a morphism, $F : \mathcal{P} \to \mathcal{E}nd_V$, of dioperads.

We shall consider below only dioperads $\mathcal{P}$ with $\mathcal{P}(m,n)$ being finite dimensional vector spaces (over a field k of characteristic zero) for all m, n.

The endomorphism dioperad of the vector space $k[-p]$, $p \in \mathbb{Z}$, is denoted by $\langle p \rangle$. Thus $\langle p \rangle(m,n)$ is $sgn_n^{\otimes p} \otimes sgn_m^{\otimes p}[p(n-m)]$, where sgn_m stands for the one dimensional sign representation of Σ_m. Representations of the dioperad $\mathcal{P}\langle p \rangle := \mathcal{P} \otimes \langle p \rangle$ in a vector space V are the same as representations of the dioperad $\mathcal{P}$ in $V[p]$.

If $\mathcal{P}$ is a dioperad, then $\Lambda\mathcal{P} := \{sgn_m \otimes \mathcal{P}(m,n)[2-m-n] \otimes sgn_n\}$ and $\Lambda^{-1}\mathcal{P} :== \{sgn_m \otimes \mathcal{P}(m,n)[m+n-2] \otimes sgn_n\}$ are also dioperads.

2.1. Cobar dual. If T is a directed (i.e. provided with a flow which we always assume in our pictures to go from the bottom to the top) tree, we denote by

- $Vert(T)$ the set of all vertices,
- $edge(T)$ the set of internal edges; $\det(T) := \wedge^{|edge(T)|}\mathrm{span}_k(edge(T))$;
- $Edge(T)$ is the set of all edges, i.e.

$$Edge(T) := edge(T) \sqcup \{\text{input legs (leaves)}\} \sqcup \{\text{output legs (roots)}\};$$

  $\mathrm{D}et(T) := \wedge^{|Edge(T)|}\mathrm{span}_k(Edge(T))$;
- $Out(v)$ (resp. $In(v)$) the set of outgoing (resp. incoming) edges at a vertex $v \in Vert(V)$.

An (m, n)-tree is a tree T with n input legs labeled by the set $[n] = \{1, \ldots, n\}$ and m output legs labeled by the set $[m] = \{1, \ldots, m\}$. A tree T is called *trivalent* if $|Out(v) \sqcup In(v)| = 3$ for all $v \in Vert(T)$.

Let $E = \{E(m, n)\}_{m,n\geq 1}$ be a collection of finite dimensional (Σ_m, Σ_n) bimodules with $E_{1,1} = 0$. For a pair of finite sets, $I, J \in Objects(\mathsf{S}_f)$, with $|I| = m$ and $|J| = n$, one defines

$$E(I, J) := Hom_{\mathsf{S}_f}([m], I) \times_{\Sigma_m} E(m, n)) \times_{\Sigma_n} Hom_{\mathsf{S}_\mathrm{f}}(J, [n]).$$

The *free dioperad*, $Free(E)$, generated by E is defined by

$$Free(E)(m, n) := \bigoplus_{(m,n)-\text{trees } T} E(T),$$

where

$$E(T) := \bigotimes_{v\in Vert(T)} E(Out(v), In(v)),$$

and the compositions ${}_i\circ_j$ are given by grafting the j^{th} root of one tree into the i^{th} leaf of another tree, and then taking the "unordered" tensor product [MSS] over the set of vertices of the resulting tree.

Let $\mathcal{P} = \{\mathcal{P}(m, n)\}_{m,n\geq 1}$ be a collection of graded (Σ_m, Σ_n) bimodules. We denote by $\bar{\mathcal{P}}$ the collection $\{\bar{\mathcal{P}}(m, n)\}_{m,n\geq 1}$ given by $\bar{\mathcal{P}}(m, n) := \mathcal{P}(m, n)$ for $m + n \geq 3$ and $\bar{\mathcal{P}}(1, 1) = 0$. The collection of dual vector spaces, $\bar{\mathcal{P}}^* = \{\bar{\mathcal{P}}(m, n)^*\}_{m,n\geq 1}$, is naturally a collection of (Σ_m, Σ_n)-bimodules with the transposed actions. We also set $\mathcal{P}^\vee = \{\bar{\mathcal{P}}(m, n)^\vee := sgn_m \otimes \bar{\mathcal{P}}(m, n)^* \otimes sgn_n\}$.

Let $\mathcal{P}$ be a graded dioperad with zero differential. The *cobar dual* of $\mathcal{P}$ is the dg dioperad $\mathbf{D}\mathcal{P}$ defined by

(i) as a dioperad of graded vector spaces, $\mathbf{D}\mathcal{P} = \Lambda^{-1}Free(\bar{\mathcal{P}}^*[-1]) = Free(\Lambda^{-1}\bar{\mathcal{P}}^*[-1])$;
(ii) as a complex, $\mathbf{D}\mathcal{P}$ is non-positively graded, $\mathbf{D}\mathcal{P}(m, n) = \sum_{i=0}^{m+n-3} \mathbf{D}\mathcal{P}^{-i}(m, n)$ with the differential given by dualizations of the compositions ${}_\bullet\circ_\bullet$ and edge contractions [G, GiKa],

$$\begin{array}{ccccccc}
\mathbf{D}\mathcal{P}^{3-m-n}(m,n) & \stackrel{d}{\to} & \mathbf{D}\mathcal{P}^{4-m-n}(m,n) & \stackrel{d}{\to} & \mathbf{D}\mathcal{P}^{3-m-n}(m,n) & \stackrel{d}{\to}\cdots\stackrel{d}{\to} & \mathbf{D}\mathcal{P}^{0}(m,n) \\
\| & & \| & & \| & & \| \\
\bar{\mathcal{P}}^\vee(m,n) & \stackrel{d}{\to} & \bigoplus_{|edge(T)|=1} \bar{\mathcal{P}}^*\otimes \mathrm{D}et(T) & \stackrel{d}{\to} & \bigoplus_{|edge(T)|=2} \bar{\mathcal{P}}^*\otimes \mathrm{D}et(T) & \stackrel{d}{\to}\cdots\stackrel{d}{\to} & \bigoplus_{\substack{\text{trivalent}\\ \text{trees } T}} \bar{\mathcal{P}}^*\otimes \mathrm{D}et(T)
\end{array}$$

where the sums are taken over (m, n)-trees.

Remark 2.2. The vector space $\mathbf{D}\mathcal{P}$ is bigraded: one grading comes from the grading of $\mathcal{P}$ as a vector space and another one from trees as in (ii) just above. The differential preserves the first grading and increases by 1 the second one. The $\mathbb{Z}$-grading of $\mathbf{D}\bar{\mathcal{P}}$ is always understood to be the associated total grading. In particular, $deg_{\mathbf{D}\mathcal{P}}\bar{\mathcal{P}}^\vee(m,n) = deg_{\mathsf{V}ect}(\bar{\mathcal{P}}^\vee(m,n)[m+n-3])$.

2.3. Koszul dioperads. A *quadratic dioperad* is a dioperad $\mathcal{P}$ of the form

$$\mathcal{P} = \frac{Free(E)}{Ideal < R >},$$

where $E = \{E(m,n)\}$ is a collection of finite dimensional (Σ_m, Σ_n)-bimodules with $E(m,n) = 0$ for $(m,n) \neq (1,2), (2,1)$, and the $Ideal$ in $Free(E)$ is generated by a collection, R, of three sub-bimodules $R(1,2) \subset Free(E)(1,2)$, $R(2,1) \subset Free(E)(2,1)$ and $R(2,2) \subset Free(E)(2,2)$. The *quadratic dual* dioperad, $\mathcal{P}^!$, is then defined by

$$\mathcal{P}^! = \frac{Free(E^\vee)}{Ideal < R^\perp >},$$

where $R^\perp$ is the collection of the three sub-bimodules $R^\perp(i,j) \subset Free(E^\vee)(i,j)$ which are annihilators of $R(i,j)$, $(i,j) = (1,2), (2,2), (2,1)$.

Clearly, $\mathbf{D}\mathcal{P}^0 = Free(E^\vee)$ so that there is a natural epimorphism

$$\mathbf{D}\mathcal{P}^0 \longrightarrow \mathcal{P}^!.$$

Its kernel is precisely $\mathsf{I}m\, d(\mathbf{D}\mathcal{P}^{-1})$. Hence $H^0(\mathbf{D}\mathcal{P}) = \mathcal{P}^!$. The quadratic operad $\mathcal{P}$ is called *Koszul* if the above morphism is a quasi-isomorphism, i.e. $H^i(\mathbf{D}\mathcal{P}) = 0$ for all $i < 0$. In that case the operad $\mathbf{D}\mathcal{P}^!$ provides us with a *minimal resolution* of the operad $\mathcal{P}$ and is often denoted by $\mathcal{P}_\infty$. Algebras over $\mathcal{P}_\infty$ are often called *strong homotopy* $\mathcal{P}$-algebras; their most important property is that they can be transferred via quasi-isomorphisms of complexes [Mar2].

2.4. Koszulness criterion. An (m,n)-tree T is called *reduced* if each vertex has

- either an outgoing root or at least two outgoing internal edges, and/or
- either an incoming leaf or at least two incoming internal edges.

For a collection, $E = \{E_{m,n}\}_{m,n\geq 1}$, of (Σ_m, Σ_n)-bimodules define another collection of (Σ_m, Σ_n)-bimodules as follows,

$$\underline{Free}(E)(m,n) := \bigoplus_{\substack{\text{reduced}\\ (m,n)-\text{trees}}} E(T).$$

Let $\mathcal{P}$ be a quadratic dioperad, i.e. $\mathcal{P} = Free(E)/Ideal <R>$
for some generators $E = \{E(1,2), E(2,1)\}$ and relations $R = \{R(1,3), R(2,2), R(3,1)\}$. With $\mathcal{P}$ one can canonically associate two quadratic operads, $\mathcal{P}_L$ and $\mathcal{P}_R$, such that

$$\mathcal{P}_L = \frac{Free(E(1,2))}{Ideal < R(1,3) >}, \quad \mathcal{P}_R^{op} = \frac{Free(E(2,1))}{Ideal < R(2,1) >}.$$

Let us denote by $\mathcal{P}_L \diamond \mathcal{P}_R^{op}$ the collection of (Σ_m, Σ_n)-bimodules given by

$$\mathcal{P}_L \diamond \mathcal{P}_R^{op}(m,n) := \begin{cases} \mathcal{P}_L(1,n) & if\ m=1, n\geq 1; \\ \mathcal{P}_L^{op}(m,1) & if\ n=1, m\geq 1; \\ 0 & otherwise. \end{cases}$$

Theorem 2.4.1. *[G, Mar1, MV]. A quadratic dioperad $\mathcal{P}$ is Koszul if the operads $\mathcal{P}_L$ and $\mathcal{P}_R$ are Koszul and*

$$\mathcal{P}(i,j) = \underline{Free}(\mathcal{P}_L \diamond \mathcal{P}_R^{op})(i,j)$$

for $(i,j) = (1,3), (2,2), (3,1)$. *Moreover, in this case* $\mathcal{P}(m,n) = \underline{Free}(\mathcal{P}_L \diamond \mathcal{P}_R^{op})(m,n)$ *for all* $m,n \geq 1$.

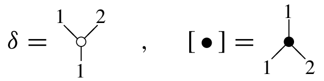
3. A Minimal Resolution of $\mathsf{Lie}^1\mathsf{Bi}$

First we present a graph description of the dioperad $\mathsf{Lie}^1\mathsf{Bi}$; it will pay off when we discuss $\mathsf{Lie}^1\mathsf{Bi}_\infty$. By definition (see Sect. 1.5), $\mathsf{Lie}^1\mathsf{Bi}$ is a quadratic dioperad,

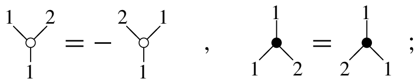
$$\mathsf{Lie}^1\mathsf{Bi} = \frac{Free(E)}{Ideal < R >},$$

where

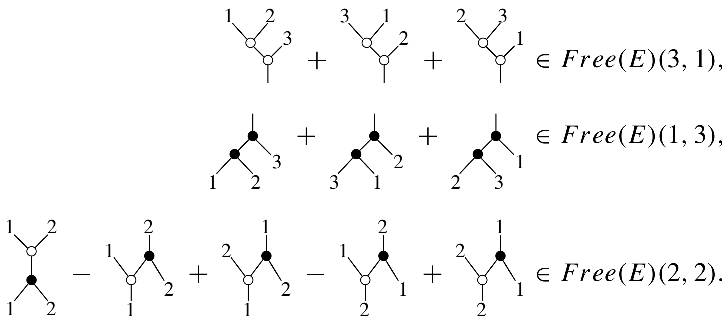
(i) $E(2,1) := sgn_2 \otimes 1_1$ and $E(1,2) := 1_1 \otimes 1_2[-1]$, where 1_n stands for the one dimensional trivial representation of Σ_n; let $\delta \in E(2,1)$ and $[\bullet] \in E(1,2)$ be basis vectors; we can represent both as directed[3] plane corollas,



with the following symmetries,



(ii) the relations R are generated by the following elements,



[3] In all our graphs the direction of edges is chosen to go from the bottom to the top.

Proposition 3.1. $\mathsf{Lie}^1\mathsf{Bi}$ *is Koszul.*

Proof. We have $\mathsf{Lie}^1\mathsf{Bi}_L = \mathsf{L}ie \otimes \{\mathsf{1}\}$ and $\mathsf{Lie}^1\mathsf{Bi}_R = \mathsf{L}ie$, where $\mathsf{L}ie$ stands for the operad of Lie algebras and

$$\{\mathsf{m}\} := \left\{\{m\}(n) := sgn_n^{\otimes m}[m(n-1)]\right\}_{n\geq 1}$$

for the endomorphism operad of $k[-m]$. As $\mathsf{L}ie$ is Koszul [GiKa], both the operads $\mathsf{Lie}^1\mathsf{Bi}_L$ and $\mathsf{Lie}^1\mathsf{Bi}_R$ are Koszul as well. Next, a straightforward analysis of all calculational schemes in $\mathsf{Lie}^1\mathsf{Bi}$ represented by directed trivalent (i, j)-trees with $i + j = 5$ shows that they generate no new relations so that

$$\mathsf{Lie}^1\mathsf{Bi}(i, j) = \underline{Free}\left(\mathsf{L}ie \otimes \{\mathsf{1}\} \diamond \mathsf{L}ie^{op}\right)(i, j)$$

for $(i, j) = (1, 3), (2, 2), (3, 1)$. Hence by Theorem 2.4.1, the dioperad $\mathsf{Lie}^1\mathsf{Bi}$ is Koszul. □

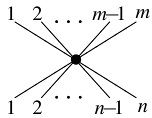
Proposition 1.5.1 is a straightforward corollary of the following

Theorem 3.2. *The minimal resolution,* $\mathsf{Lie}^1\mathsf{Bi}_\infty$*, of the dioperad* $\mathsf{Lie}^1\mathsf{Bi}$ *can be described as follows:*

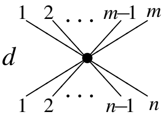
(i) As a dioperad of graded vector spaces, $\mathsf{Lie}^1\mathsf{Bi}_\infty = Free(E)$*, where the collection,* $E = \{E(m, n)\}$*, of one dimensional* (Σ_m, Σ_n)*-modules is given by*

$$E(m, n) := \begin{cases} sgn_m \otimes 1_n[m-2] & \text{if } m + n \geq 3; \\ 0 & \text{otherwise.} \end{cases}$$

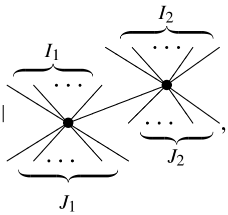
(ii) If we represent a basis element of $E(m, n)$ *by the unique (up to a sign) planar* (m, n)*-corolla,*



with skew-symmetric outgoing legs and symmetric ingoing legs, then the differential d *is given on generators by*

d 

$$= \sum_{\substack{I_1 \sqcup I_2 = (1,\dots,m) \\ J_1 \sqcup J_2 = (1,\dots,n) \\ |I_1|\geq 0, |I_2|\geq 1 \\ |J_1|\geq 1, |J_2|\geq 0}} (-1)^{\sigma(I_1 \sqcup I_2) + |I_1||I_2|}$$



,

where $\sigma(I_1 \sqcup I_2)$ *is the sign of the shuffle* $I_1 \sqcup I_2 = (1, \dots, m)$*.*

Proof. Claim (i) follows from the fact that $\mathsf{Lie}^1\mathsf{Bi}^!(m, n) = 1_m \otimes sgn_n[1 - n]$ and Remark 2.2. Claim 2 is a straightforward though tedious graph translation of the initial term,

$$(\mathsf{Lie}^1\mathsf{Bi}^!)^\vee(m, n) \xrightarrow{d} \bigoplus_{\substack{(m,n)-\text{trees } T \\ |edge(T)|=1}} (\mathsf{Lie}^1\mathsf{Bi}^!)^* \otimes \mathrm{Det}(T),$$

of Definition 2.1 of the differential d in $\mathbf{D}\mathsf{Lie}^1\mathsf{Bi}^!$. □

3.3. A geometric model for $\mathsf{Lie}^1\mathsf{Bi}_\infty$ *structures.* Let V be a finite dimensional graded vector space. Then the graded formal manifold, $\mathcal{M}$, modeled on the infinitesimal neighbourhood of 0 in the vector space $V \oplus V^*[1]$ has an odd symplectic form ω induced from the natural pairing $V \otimes V^*[1] \to k[1]$. In particular, the graded structure sheaf $\mathcal{O}_\mathcal{M}$ on $\mathcal{M}$ has a degree -1 Poisson bracket, $\{\bullet\}$, such that

$$\{f \bullet g\} = (-1)^{|f||g|+|f|+|g|}\{g \bullet f\}$$

and the Jacobi identity is satisfied. The odd symplectic manifold $(\mathcal{M}, \omega)$ has two particular Lagrangian submanifolds, $\mathcal{L} \subset \mathcal{M}$ and $\Pi\mathcal{L} \subset \mathcal{M}$ associated with, respectively, the subspaces $0 \oplus V^*[1] \subset V \oplus V^*[1]$ and $V \oplus 0 \subset V \oplus V^*[1]$.

Proposition 3.3.1. *A* $\mathsf{Lie}^1\mathsf{Bi}_\infty$ *algebra structure in a graded vector space* V *is the same as a degree two smooth function* $\Gamma \in \mathcal{O}_\mathcal{M}$ *vanishing on* $\mathcal{L} \cup \Pi\mathcal{L}$ *and satisfying the equation* $\{\Gamma \bullet \Gamma\} = 0$.

Proof. The manifold $\mathcal{M}$ is isomorphic to the total space of the shifted cotangent bundle, $T^*_M[1]$, of the manifold M of Proposition 1.5.1. Hence smooth functions on $\mathcal{M}$ are the same as smooth polyvector fields on M, and the Poisson bracket $\{\bullet\}$ on $\mathcal{M}$ is the same as the Schouten bracket on M. □

3.4. $\mathsf{Lie}^1\mathsf{Bi}_\infty$ *morphisms.* Let $(V, \{\mu_{m,n}\})$ and $(V', \{\mu'_{m,n}\})$ be two $\mathsf{Lie}^1\mathsf{Bi}_\infty$ algebras.

Definition 3.4.1. *A* $\mathsf{Lie}^1\mathsf{Bi}_\infty$ **morphism** $F : V \to V'$ *is, by definition, a symplectomorphism,* $F : (\mathcal{M}, \omega) \to (\mathcal{M}', \omega')$ *such that* $F(\mathcal{L}) \subset \mathcal{L}'$, $F(\Pi\mathcal{L}) \subset \Pi\mathcal{L}'$ *and* $F^*\Gamma' = \Gamma$.

Thus a $\mathsf{Lie}^1\mathsf{Bi}_\infty$ morphism $F : V \to V'$ is a pair of collections of linear maps,

$$\left\{F_{m,n} : \odot^m V \otimes \wedge^n V^* \to V'[-n]\right\}_{m\geq 1, n\geq 0},$$
$$\left\{\bar{F}_{m,n} : \odot^m V \otimes \wedge^n V^* \to V'^*[1-n]\right\}_{n\geq 0, m\geq 1} \qquad (\star)$$

satisfying the system equations, $F^*(\omega') = \omega$ and $F^*\Gamma' = \Gamma$. In particular, the equation $F^*(\omega') = \omega$ says that the linear maps,

$$F_{1,0} : (V, \mu_{1,1}) \to (V', \mu'_{1,1}) \text{ and } \bar{F}_{0,1} : (V^*, \mu^*_{1,1}) \to (V'^*, \mu'^*_{1,1}),$$

are morphisms of complexes, while the equation $F^*(\omega') = \omega$ says that the composition,

$$F^*_{0,1} \circ \bar{F}_{1,0} : V \longrightarrow V$$

is the identity map.

Definition 3.4.2. *A* $\mathsf{Lie}^1\mathsf{Bi}_\infty$*-morphism* $F : V \to V'$ *is called a* **quasi-isomorphism** *if the morphisms of complexes*

$$F_{1,0} : (V, \mu_{1,1}) \to (V', \mu'_{1,1}) \text{ and } \bar{F}_{0,1} : (V^*, \mu^*_{1,1}) \to (V'^*, \mu'^*_{1,1})$$

induce isomorphisms in cohomology.

Remark 3.4.3. One might get an impression that the notions introduced above make sense only for *finite dimensional* $\mathsf{Lie}^1\mathsf{Bi}_\infty$ algebras. However, this is no more than an artifact of the geometric intuition we tried to rely on in our definitions. In fact, everything above (and below) make sense for infinite dimensional representations as well. For example, one can replace ($\star$) by

$$\left\{F_{m,n} : \odot^m V \to \wedge^n V \otimes V'[-n]\right\}_{m\geq 1, n\geq 0},$$
$$\left\{\bar{F}_{m,n} : \odot^m V \otimes V'[n-1] \to \wedge^n V\right\}_{n\geq 0, m\geq 1},$$

and reinterpret the equations defining the $\mathsf{Lie}^1\mathsf{Bi}_\infty$ morphism accordingly. For example, with this reinterpretation it is the morphism $F_{1,0} \circ \bar{F}_{0,1}$ which is the identity map.

3.4.4. Contractible and minimal $\mathsf{Lie}^1\mathsf{Bi}_\infty$*-structures.* Let V be a graded vector space and $(\mathcal{M} = V \oplus V^*[1], \omega)$ the associated odd symplectic manifold (as in Sect. 3.3). There is a one-to-one correspondence between differentials, $d : V \to V$, and quadratic degree 2 function, Γ_{quad}, on $(\mathcal{M}, \omega)$, vanishing on $\mathcal{L} \cup \Pi\mathcal{L}$ and satisfying $[\Gamma_{quad} \bullet \Gamma_{quad}] = 0$. If $H^*(V, d) = 0$, the associated data, $(\mathcal{M}, \omega, \Gamma_{quad})$, is called a *contractible* $\mathsf{Lie}^1\mathsf{Bi}_\infty$*-structure* on V.

A $\mathsf{Lie}^1\mathsf{Bi}_\infty$ structure, $(\mathcal{M}, \omega, \Gamma)$, on V is called *minimal* if $\Gamma = 0 \bmod I^3$, where I is the ideal of the distinguished point, $\mathcal{L} \cap \Pi\mathcal{L}$, in $\mathcal{M}$. Put another way, the formal power series Γ in some (and hence any) coordinate system on $\mathcal{M}$ begins with cubic terms at least.

Theorem 3.4.5. *(Homotopy classification of* $\mathsf{Lie}^1\mathsf{Bi}_\infty$*-structures, cf. [Ko1, Ko2]). Each* $\mathsf{Lie}^1\mathsf{Bi}_\infty$ *algebra is isomorphic to the tensor product of a contractible* $\mathsf{Lie}^1\mathsf{Bi}_\infty$ *algebra and a minimal one.*

Proof. Let $(\mathcal{M}, \omega, \Gamma)$ be the geometric equivalent of any given $\mathsf{Lie}^1\mathsf{Bi}_\infty$ algebra. To prove the statement we have to construct coordinates, $(x^a, y^a, z^\alpha, \psi_a, \phi_a, \xi_\alpha)$, on $\mathcal{M}$ such that

(i) $\omega = \underbrace{\left(\sum_a (dx^a \wedge d\psi_\alpha + dy^a \wedge d\phi_\alpha\right)}_{\omega_1} + \underbrace{\sum_\alpha dz^\alpha \wedge d\xi_\alpha}_{\omega_2},$

(ii) $\mathcal{L}$ is given by $x^a = y^a = z^\alpha = 0$ while $\Pi\mathcal{L}$ is given by $\psi_a = \phi_a = \xi_\alpha = 0$,

(iii) $\Gamma = \underbrace{\sum_a y^a \psi_a}_{\Gamma_1} + \underbrace{\Phi(z^\alpha, \xi_\alpha)}_{\Gamma_2}$ for some formal power series $\Phi(z^\alpha, \xi_\alpha)$ which begins with cubic terms at least.

For then $(\mathcal{M}, \omega, \Gamma) \simeq (\mathcal{M}_1, \omega_1, \Gamma_1) \times (\mathcal{M}_2, \omega_2, \Gamma_2)$ with the first factor being a contractible $\mathsf{Lie}^1\mathsf{Bi}_\infty$ structure while the second factor is a minimal one.

We shall establish existence of the above coordinates by induction.

As the first step of the induction procedure we choose arbitrary linear coordinates, $\{t^A\}$, $A \in \{1, \ldots, \dim V\}$, on V and the associated dual coordinates, $\{\chi_A\}$, $|\chi_A| = -|t^A| + 1$, on $V^*[1]$. The odd symplectic form is given in these coordinates as $\omega = \sum_A dt^A \wedge d\chi_A$, $\mathcal{L}$ is given by $t^A = 0$ while $\Pi\mathcal{L}$ is given by $\chi_A = 0$. Then,

$$\Gamma = \sum_{A,B} C^A_B t^A \chi_B \bmod I^3,$$

for some constants C^A_B which are nothing but the coefficients of the differential, $d : V \to V$, associated to the quadratic bit of Γ. As we work over the field of characteristic zero, we can choose a cohomological decomposition of V with respect to this differential,

$$V = H(V, d) \oplus B \oplus B[-1],$$

so that d vanishes on $H(V, d)$ and $B[-1]$ and, on the remaining summand B, it is equal to the natural isomorphism $B \to B[-1]$. Let $\{z^\alpha\}$ be some linear coordinates in $H(V, d)$, $\{y^a\}$ linear coordinates on B and $\{x^a\}$, $|x^a| = |y^a| - 1$, the associated (via the natural isomorphism) linear coordinates in $B[-1]$. Denote by $(\xi_\alpha, \psi_a, \phi_a)$ the coordinates on $V^*[1]$ dual to (z^α, x^a, y^a). In the resulting coordinate system on $\mathcal{M}$ the conditions (i)–(ii) are satisfied, while the condition (iii) is satisfied modulo I^3.

Assume by induction that we have constructed a coordinate system on $\mathcal{M}$ in which conditions (i)–(iii) are satisfied mod I^{N+1}. Then we have,

$$\Gamma = \underbrace{\sum_a y^a \psi_a}_{\Gamma_1} + \underbrace{\Phi^{\leq N}(z^\alpha, \xi_\alpha)}_{\text{polynomials of degrees from 3 to N}} + \underbrace{\Gamma^{N+1}(x, y, z, \psi, \phi, \chi)}_{\text{polynomial of degree } N+1} \mod I^{N+2}.$$

The equation $[\Gamma \bullet \Gamma] = 0 \mod I^{N+2}$ implies,

$$\delta\Gamma^{N+1} = 0,$$

where δ is the following differential on $\mathcal{O}_\mathcal{M}$,

$$\begin{aligned} \delta : \mathcal{O}_\mathcal{M} &\longrightarrow \mathcal{O}_\mathcal{M} \\ f &\longrightarrow [\textstyle\sum_a y^a \psi_a \bullet f]. \end{aligned}$$

Let $B \in \mathcal{O}_\mathcal{M}$ be an arbitrary polynomial of degree $N + 1$ and with $|B| = 1$. It gives rise to a symplectomorphism, $F : \mathcal{M} \to \mathcal{M}$, given as $\exp v_B$ with the vector field v_B defined by

$$\begin{aligned} v_B : \mathcal{O}_\mathcal{M} &\longrightarrow \mathcal{O}_\mathcal{M} \\ f &\longrightarrow [B \bullet f]. \end{aligned}$$

One has,

$$F^*\Gamma = \sum_a y^a \psi_a + \Phi^{\leq N}(z^\alpha, \xi_\alpha) + \Gamma^{N+1} + \delta B \mod I^{N+2}.$$

Thus $\Gamma^{N+1}(x, y, z, \psi, \phi, \chi)$ is a δ-cycle defined up to a δ-coboundary. As cohomology of δ in $\mathcal{O}_\mathcal{M}$ is equal to $k[[z^\alpha, \xi_\alpha]]$, one can always find Γ^{N+1} such that it is a function of $\{z^\alpha, \xi_\alpha\}$ only. □

Corollary 3.4.6. *If $F : V \to V'$ is a Lie_1Bi_∞ quasi-isomorphism, then there exists a Lie_1Bi_∞ quasi-isomorphism $G : V' \to V$ such that on the cohomology level $[F_{1,0}] = [\bar{G}_{0,1}]^*$ and $[G_{1,0}] = [\bar{F}_{0,1}]^*$.*

Proof. is exactly the same as the proof of an analogous statement for L_∞ algebras in [Ko1]. □

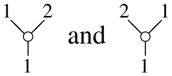
4. Minimal Resolution of the Operad TF

By definition (see Sect. 1.4), TF is a quadratic dioperad

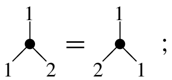
$$\mathsf{TF} = \frac{Free(E)}{Ideal < R >},$$

where

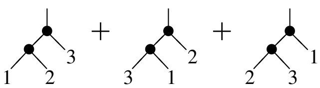
(i) $E(2, 1) := k[\Sigma_2] \otimes 1_1$ and $E(1, 2) := 1_1 \otimes 1_2[-1]$; we represent two basis vectors of $k[\Sigma_2] \otimes 1_1$ by planar corollas

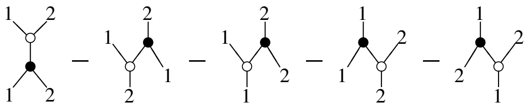


and a basis vector of $E(1, 2)$ by the symmetric corolla,



(ii) the relations R are generated by the following elements,


$\in Free(E)(1, 3)$,


$\in Free(E)(2, 2)$.

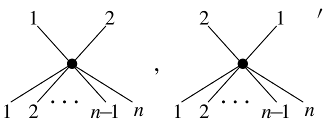
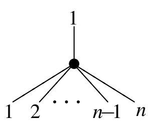
Proposition 1.4.1 follows immediately from the following

Proposition 4.1. *The minimal resolution,* TF_∞*, of the dioperad* TF *can be described as follows:*

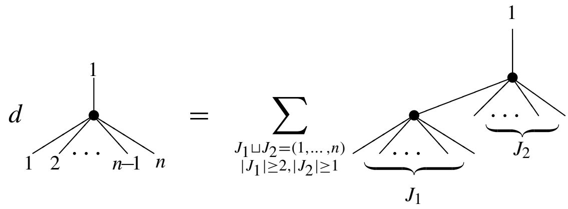
(i) As a dioperad of graded vector spaces, $\mathsf{TF}_\infty = Free(E)$*, where the collection,* $E = \{E(m, n)\}$*, of* (Σ_m, Σ_n)*-modules is given by*

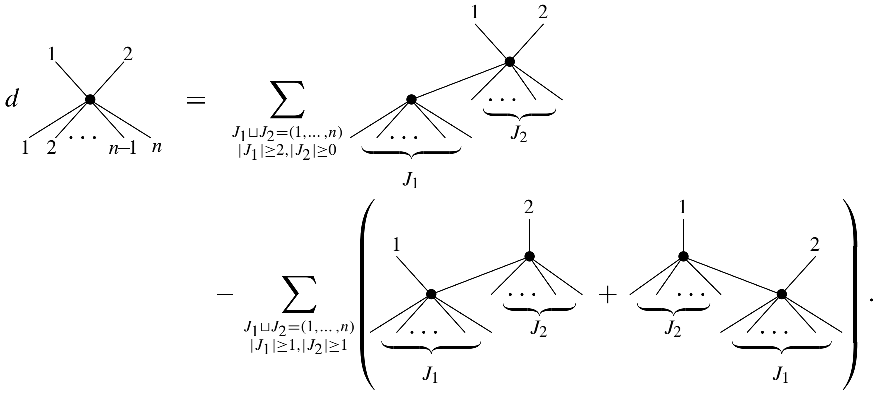
$$E(m, n) := \begin{cases} k[\Sigma_2] \otimes 1_n & if\ m = 2, n \geq 2; \\ 1_n[-1] & if\ m = 1, n \geq 2; \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If we represent two basis elements of $E(2, n)$ *by planar* $(2, n)$*-corollas, and the basis element of* $E(1, n)$ *by planar* $(1, n)$ *corolla,*


and, respectively,


with symmetric ingoing legs, then the differential d is given on generators by,





Proof. Using criterion 2.4 it is easy to see that the dioperad TF is Koszul. Then the cobar dual $\mathbf{D}\mathsf{TF}^!$ provides the required minimal resolution. The rest is a straightforward calculation. □

5. A Comment on $\mathsf{Lie}^1\mathsf{Bi}_\infty$ Algebras

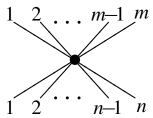
It was shown in [G] that the dioperad, LieBi_∞, of (usual) Lie bialgebras is Koszul so that its minimal resolution, LieBi_∞, can be constructed using the techniques reviewed in Sect. 2. Here we present its explicit graph description; in fact, we prefer to show $\mathsf{LieBi}_\infty\langle 1\rangle$.

Proposition 5.1. *The dioperad* $\mathsf{LieBi}_\infty\langle 1\rangle$ *can be described as follows.*

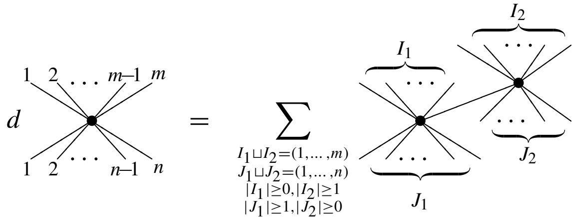
(i) As a dioperad of graded vector spaces, $\mathsf{LieBi}_\infty\langle 1\rangle = Free(E)$*, where the collection,* $E = \{E(m,n)\}$*, of one dimensional* (Σ_m, Σ_n)*-modules is given by*

$$E(m,n) := \begin{cases} 1_m \otimes 1_n[2m-3] & \text{if } m+n \geq 3; \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If we represent a basis element of $E(m,n)$ *by the unique space* (m,n)*-corolla,*



then the differential d is given on generators by,



Let V be a graded vector space, and let $\mathcal{M}$ be the graded formal manifold isomorphic to the neighbourhood of zero in $V[1] \oplus V^*[1]$. The manifold $\mathcal{M}$ has a natural even symplectic structure ω induced from the pairing $V[1] \otimes V^*[1] \to k[2]$; it also has two particular Lagrangian submanifolds, $\mathcal{L}'$ and $\mathcal{L}''$, modeled on the subspaces $0 \oplus V^*[1] \subset V[1] \oplus V^*[1]$ and, respectively, $V \oplus 0 \subset V[1] \oplus V^*[1]$. The symplectic form induces degree -2 Poisson bracket, $\{\ ,\ \}$, on the structure sheaf, $\mathcal{O}_{\mathcal{M}}$, of smooth functions on $\mathcal{M}$.

The following result has been independently obtained by Lyubashenko [Lyu].

Corollary 5.2. *A* LieBi_∞ *algebra structure in a graded vector space V is the same as a degree 3 smooth function $f \in \mathcal{O}_{\mathcal{M}}$ vanishing on $\mathcal{L}' \cup \mathcal{L}''$ and satisfying the equation* $\{f, f\} = 0$.

$\mathsf{L}ieBi_\infty$ morphisms and quasi-isomorphisms are defined exactly as in 3.4.1 and 3.4.2; then an obvious analogue of Theorem 3.4.5 holds true. We omit the details.

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