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Dispersive Estimates for Schrödinger Equations with Threshold Resonance and Eigenvalue

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Abstract: Let $H = -\Delta + V(x)$ be a three dimensional Schrödinger operator. We study the time decay in L^p spaces of scattering solutions $e^{-itH}P_c u$, where P_c is the orthogonal projection onto the continuous spectral subspace of $L^2(\mathbf{R}^3)$ for H. Under suitable decay assumptions on $V(x)$ it is shown that they satisfy the so-called L^p-L^q estimates $\|e^{-itH}P_c u\|_p \le (4\pi|t|)^{-3(1/2-1/p)}\|u\|_q$ for all $1 \le q \le 2 \le p \le \infty$ with $1/p + 1/q = 1$ if H has no threshold resonance and eigenvalue; and for all $3/2 < q \le 2 \le p < 3$ if otherwise.

1. Introduction

The present paper is concerned with the time decay in L^p spaces of solutions of three dimensional Schrödinger equations,

$$i\partial_t u = (-\Delta + V(x))u, \quad x \in \mathbf{R}^3. \tag{1.1}$$

Throughout the paper we assume that potentials $V(x)$ are real valued and decay at infinity at least as rapidly as

$$|V(x)| \le C\langle x\rangle^{-\beta}, \quad \text{for some } \beta > 5/2, \tag{1.2}$$

where $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$.

Under this condition, the operator $H = -\Delta + V$ is selfadjoint in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^3)$ with domain $D(H) = H^2(\mathbf{R}^3)$, the Sobolev space of order 2, and the solution in $\mathcal{H}$ of (1.1) which satisfies the initial condition $u(0) = \varphi \in \mathcal{H}$ is uniquely given by $u(t) = e^{-itH}\varphi$ in terms of the unitary operator e^{-itH} defined by the functional calculus. The spectrum of H consists of a finite number of non-positive eigenvalues of finite multiplicities and the absolutely continuous part $[0, \infty)$. If φ is an eigenfunction of H, $u(t) = e^{-itH}\varphi$ is a stationary solution and never decays in time in any sense; however, if $\varphi \in L^2_c(H)$, the continuous spectral subspace for H, it is a scattering solution in the sense that for a unique $\varphi_\pm \in \mathcal{H}$,

$$\|u(t) - e^{-itH_0}\varphi_\pm\|_2 \to 0 \text{ as } t \to \pm\infty \tag{1.3}$$

(cf. [13, 23, 24]), where $H_0 = -\Delta$ is the free Schrödinger operator.

For the free Schrödinger equation it has long been known (see e.g. [15]) that, although e^{-itH_0} is unitary in L^2, the solution $e^{-itH_0}u$ decays as $t \to \pm\infty$ in L^p if $p > 2$ and it satisfies

$$\|e^{-itH_0}u\|_p \le (4\pi|t|)^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|u\|_q, \quad u \in L^2 \cap L^q(\mathbf{R}^3), \tag{1.4}$$

where $1 \le q < 2$ is the dual exponent of p: $1/p + 1/q = 1$ and L^p is the Lebesgue L^p space with the norm $\|u\|_p$. This decay estimate is known as an L^p-L^q estimate and it has been a very useful and important tool for studying linear and nonlinear Schrödinger equations (see e.g. [16]). In view of the relation (1.3), it is natural to expect that scattering solutions of (1.1) also decay in L^p if $p > 2$. Indeed, under the condition that V satisfies (1.2) with $\beta > 3$ and that H is of generic type, viz. H satisfies a spectral condition at the threshold 0 (see Definition 1.1 below), estimate (1.4) with $e^{-itH}P_c$ in place of e^{-itH_0}, P_c being the orthogonal projection onto $L^2_c(H)$,

$$\|e^{-itH}P_c u\|_p \le C_p t^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|u\|_q, \quad u \in L^2 \cap L^q, \tag{1.5}$$

has recently been proved by Goldberg-Schlag ([8], see [12, 2, 30, 30, 31, 28, 25, 27] for earlier and related works). It is also known that (1.5) cannot hold for all $2 \le p \le \infty$ if H is of exceptional type as it would contradict the local decay estimate of Jensen-Kato[10] or Murata[19].

In this paper, we show, when H is of exceptional type, how (1.5) is violated and propose a new estimate which replaces (1.5); when H is of generic type, we prove that (1.5) is satisfied under the assumption (1.2), relaxing the decay condition of Goldberg and Schlag [8] (see, however, the note at the end of the introduction).

To state the main results of the paper we introduce some notation and recall some known facts (see also the beginning of Sects. 3 and 4). For $1 \le p, q \le \infty$, $L^{p,q}$ is the Lorentz space with the norm $\|u\|_{p,q}$ ([3, 21]). For $\gamma \in \mathbf{R}$, $\mathcal{H}_\gamma = L^2(\mathbf{R}^3, \langle x\rangle^{2\gamma}dx)$ is the weighted L^2 space. The spaces $\mathcal{H}_{-\gamma}$ and $\mathcal{H}_\gamma$ are duals of each other with respect to the coupling

$$\langle u, v\rangle = \int_{\mathbf{R}^3} u(x)\overline{v(x)}dx.$$

We write $R_0(z) = (H_0 - z)^{-1}$ and $R(z) = (H - z)^{-1}$ for the resolvents of H_0 and H respectively. We define for $\lambda \in \mathbf{C}$,

$$G_0(\lambda)u(x) = \frac{1}{4\pi}\int \frac{e^{i\lambda|x-y|}}{|x-y|}u(y)dy. \tag{1.6}$$

We have $R_0(\lambda^2) = G_0(\lambda)$ for $\Im\lambda > 0$. The integral kernel of $G_0(\lambda)$ is an entire function of $\lambda \in \mathbf{C}$ and, using its derivatives at $\lambda = 0$, we define

$$D_j u(x) = \frac{1}{4\pi j!}\int |x-y|^{j-1}u(y)dy, \quad j = 0, 1, \dots, \tag{1.7}$$

so that $G_0(\lambda) = D_0 + i\lambda D_1 + (i\lambda)^2 D_2 + \cdots$ at least formally.

For any $1/2 < \gamma < \beta - 1/2$, the operator $D_0 V$ is of Hilbert-Schmidt type in $\mathcal{H}_{-\gamma}$ and we denote the null space of $1 + D_0 V$ by $\mathcal{M}$:

$$\mathcal{M} = \left\{ \phi \in \mathcal{H}_{-\gamma} : \phi(x) + \frac{1}{4\pi} \int \frac{V(y)\phi(y)}{|x-y|} dy = 0 \right\}. \tag{1.8}$$

The space $\mathcal{M}$ is finite dimensional and is independent of $1/2 < \gamma < \beta - 1/2$. All $\phi \in \mathcal{M}$ satisfy the stationary Schrödinger equation

$$-\Delta\phi(x) + V(x)\phi(x) = 0 \tag{1.9}$$

and, conversely, any function $\phi \in \mathcal{H}_{-\frac{3}{2}}$ which satisfies (1.9) belongs to $\mathcal{M}$. The eigenspace $\mathcal{E}$ of H with eigenvalue 0 is therefore a subspace of $\mathcal{M}$. The function $\phi \in \mathcal{M}$ is in $\mathcal{E}$ if and only if $\langle V, \phi \rangle = 0$ and $\mathrm{codim}_{\mathcal{M}}\mathcal{E} \le 1$. The sesquilinear form $-(u, Vv)$ is an inner product in $\mathcal{M}$.

Definition 1.1. *We say H or V is of generic type if $\mathcal{M} = \{0\}$ and is of exceptional type otherwise. H is of exceptional type of the first kind if $\mathcal{M} \neq \{0\}$ and $\mathcal{E} = 0$; of the second kind if $\mathcal{E} = \mathcal{M} \neq \{0\}$; and of the third kind if $\{0\} \subset \mathcal{E} \subset \mathcal{M}$ with strict inclusions. A function $\phi \in \mathcal{M} \setminus \mathcal{E}$ is called a resonance of H.*

Note that most V are of generic type: If V is of exceptional type, then λV is of generic type for all $\lambda \neq 1$ near $\lambda = 1$ because $D_0 V$ is compact. It is easy to see from (1.8) that the resonance $\phi(x)$ satisfies

$$\phi(x) - C|x|^{-1} \in \mathcal{H} \text{ for some constant } C \neq 0$$

and that the eigenfunctions $\phi \in \mathcal{E}$ may decay as $|x| \to \infty$ as slowly as $C\langle x \rangle^{-2}$ in contrast to the ones with negative eigenvalues, which generally decay exponentially. We write P_0 for the orthogonal projection in $\mathcal{H}$ onto $\mathcal{E}$. As $\phi \in \mathcal{E}$ satisfy $|\phi(x)| \le C\langle x \rangle^{-2}$, P_0 defined on $L^2 \cap L^q$ can be extended to a bounded operator from L^q to L^p for all $1 \le q < 3$ and $3/2 < p \le \infty$. We abuse notation and denote such extensions also by P_0.

When H is of exceptional type of the third kind, we let $\phi_1 \in \mathcal{M}$ be a (uniquely determined) resonance such that $\langle V, \phi_1 \rangle > 0$, $-\langle \phi_1, V\phi_1 \rangle = 1$ and $-\langle \phi_1, V\phi_j \rangle = 0$ for all $\phi_j \in \mathcal{E}$ and define the *canonical resonance* ([10]) by

$$\varphi(x) = \phi_1(x) + P_0 V D_2 V \phi_1(x). \tag{1.10}$$

Using $\varphi(x)$, we define a constant a and a function $\zeta(t, x)$ by

$$a = 4\pi i |\langle V, \varphi \rangle|^{-2}, \quad \zeta(t, x) = e^{i\frac{x^2}{4t}} \varphi(x). \tag{1.11}$$

We define a function $\mu(t, x)$, which plays a special role in what follows, by

$$\mu(t, x) = \frac{i}{|x|} \int_0^1 (e^{\frac{i|x|^2}{4t}} - e^{\frac{i\theta^2|x|^2}{4t}}) d\theta; \tag{1.12}$$

$\mu(t)$ is multiplication with $\mu(t, x)$. We use the notation $|f\rangle\langle g|$ interchangeably with $f \otimes g$ to denote the rank one operator defined by the integral kernel $f(x)g(y)$ (not $f(x)\overline{g(y)}$).

Definition 1.2. *We define the operators $R(t)$ and $S(t)$ respectively by*

$$R(t) = \frac{ae^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}} \zeta(t,\cdot) \otimes \zeta(t,\cdot), \tag{1.13}$$

$$S(t) = \frac{e^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}} \left(-iP_0 V D_3 V P_0 + \mu(t) D_2 V P_0 + P_0 V D_2 \mu(t)\right). \tag{1.14}$$

When H is of exceptional type of the first or the second kind, we use the same notation, setting, of course, $S(t) = 0$ or $R(t) = 0$ respectively.

We remark that for a constant $C > 0$,

$$|\zeta(t,x) - \varphi(x)| + |\mu(t,x)| \le C \min\left(\frac{1}{\sqrt{t}}, \frac{1}{|x|}, \frac{|x|}{|t|}\right). \tag{1.15}$$

As remarked above, eigenfunctions $\phi \in \mathcal{E}$ satisfy $\int V(x)\phi(x)dx = 0$. It follows that $(D_2 V\phi)(x)$ are bounded and, if $\{\phi_2, \dots, \phi_d\}$ is an orthonormal basis of $\mathcal{E}$ and $w_j(t,x) = \mu(t,x)(D_2 V \phi_j)(x)$, $j = 2, \dots, d$, then $w_j(t,x)$ are bounded by (1.15) and $S(t)$ may be written in the form

$$\frac{e^{\frac{i\pi}{4}}}{\sqrt{\pi t}} \left(\sum_{j,k=2}^{d} a_{jk} \phi_j \otimes \phi_k + \sum_{j=2}^{d} (w_j(t) \otimes \phi_j + \phi_j \otimes w_j(t)) \right).$$

Theorem 1.3. (1) *Let V satisfy $|V(x)| \le C\langle x\rangle^{-\beta}$ for some $\beta > 5/2$. Suppose that H is of generic type. Then, for any $1 \le q \le 2 \le p \le \infty$ such that $1/p + 1/q = 1$,*

$$\|e^{-itH} P_c u\|_p \le C_p t^{-3\left(\frac{1}{2}-\frac{1}{p}\right)} \|u\|_q, \quad u \in L^2 \cap L^q. \tag{1.16}$$

(2) *Let V satisfy $|V(x)| \le C\langle x\rangle^{-\beta}$ for some $\beta > 11/2$. Suppose that H is of exceptional type. Then the following statements are satisfied:*

(i) *Estimate* (1.16) *holds when p and q are restricted to $3/2 < q \le 2 \le p < 3$ and $1/p + 1/q = 1$.*

(ii) *Estimate* (1.16) *holds when $p = 3$ and $q = 3/2$ provided that L^3 and $L^{\frac{3}{2}}$ are respectively replaced by Lorentz spaces $L^{3,\infty}$ and $L^{\frac{3}{2},1}$.*

(iii) *When $3 < p \le \infty$ and $1 \le q < 3/2$ are such that $1/p + 1/q = 1$, there exists a constant C_{pq} such that for any $u \in L^2 \cap L^q$,*

$$\left\| \left(e^{-itH} P_c - R(t) - S(t)\right) u \right\|_p \le C_{pq} t^{-3\left(\frac{1}{2}-\frac{1}{p}\right)} \|u\|_q. \tag{1.17}$$

If H is of exceptional type of the first kind, statement (2) *holds under a weaker decay condition $|V(x)| \le C\langle x\rangle^{-\beta}$ with $\beta > 9/2$.*

We remark that $\|(R(t) + S(t))u\|_p \le C|t|^{-\frac{1}{2}} \|u\|_q$ for p, q such that $3 < p \le \infty$ and $1 \le q < 3/2$ and that $\|(R(t) + S(t))u\|_{3,\infty} \le C|t|^{-\frac{1}{2}} \|u\|_{3/2,1}$; however, $R(t)$ is not bounded from L^q to L^p for any other pairs and that P_c is, although an orthogonal projection in $\mathcal{H}$, bounded in L^p only for $3/2 < p < 3$ in general. Combining Theorem 1.3 and the estimate (1.15), we immediately obtain the following theorem.

Theorem 1.4. *Let V satisfy $|V(x)| \le C\langle x\rangle^{-\beta}$ for some $\beta > 11/2$. Suppose that H is of exceptional type. Then, for $3 < p \le \infty$ and $1 \le q < 3/2$ such that $1/p + 1/q = 1$, there exists a constant C such that*

$$\|e^{-itH}P_c u\|_p \le Ct^{-3(\frac{1}{2}-\frac{1}{p})}(\|u\|_q + \|\langle x\rangle^{\frac{6}{q}-5}u\|_1) \tag{1.18}$$

for any $u \in L^2 \cap L^q$ which satisfies $\langle \phi, u\rangle = 0$ for all $\phi \in \mathcal{M}$ and $\langle x\rangle^{\frac{6}{q}-5}u \in L^1$. If H is of exceptional type of the first kind, the same statement holds under the weaker decay condition $|V(x)| \le C\langle x\rangle^{-\beta}$ with $\beta > 9/2$.

We display here the plan of the paper, explaining the idea of the proof of Theorem 1.3 using a slightly sloppy argument. We refer the readers to the text for a more rigorous treatment. We say that a family of operator $\{T(t) : t \in \mathbf{R}\}$ is *regularly dispersive* if it is a strongly continuous family of bounded operators in $\mathcal{H}$ and, in addition, it satisfies the estimate (1.16) for all $1 \le q \le 2 \le p \le \infty$ such that $1/p + 1/q = 1$.

In Sect. 2, we collect some results, well known as the limiting absorption principle (LAP for short), on the behavior of resolvents $R_0(z)$ and $R(z)$ near the reals. We state them for $G_0(\lambda)$ and $G(\lambda)$ which is defined by $G(\lambda) = R(\lambda^2)$ on the upper half plane $\Im\lambda > 0$. We also record some results on certain integrals. Lemma 2.4 and Lemma 2.7 are the main tools and are frequently used in the paper. We prove the first statement of Theorem 1.3 for the generic case in Sect. 3, following basically the argument of [25] and [8] but more concisely. We use the well known representation formula of the propagator:

$$e^{-itH}P_c = \lim_{\delta\downarrow 0}\frac{1}{i\pi}\int_{|\lambda|>\delta} e^{-it\lambda^2}G(\lambda)\lambda d\lambda. \tag{1.19}$$

Here the principle value is taken to remove the contribution from P_0. We write as $G(\lambda) = (1 + G_0(\lambda)V)^{-1}G_0(\lambda)$ and expand $(1 + G_0(\lambda)V)^{-1}$:

$$G(\lambda) = \sum_{n=0}^{2}(-1)^n G_0(\lambda)(VG_0(\lambda))^n - G_0(\lambda)VG(\lambda)VG_0(\lambda)VG_0(\lambda).$$

Then $e^{-itH}P_c = \Omega_0(t) - \Omega_1(t) + \Omega_2(t) + W_3(t)$. An explicit computation using Lemma 2.4 shows that the integral kernel of $\Omega_n(t)$ is given by

$$\Omega_n(t,x,y) = \frac{\sqrt{\pi}}{2\sqrt{i}t^{\frac{3}{2}}}\int_{\mathbf{R}^{3j}} e^{i\frac{A_j^2}{4t}}\frac{A_j\prod_{j=1}^{n}V(x_j)}{\prod_{j=1}^{n+1}|x_j - x_{j-1}|}dx_1,\ldots,dx_n$$

with $x_0 = x$ and $x_{n+1} = y$ and $A_j = \sum_{j=1}^{n+1}|x_j - x_{j-1}|$. As is shown by [25],

$$|\Omega_n(t,x,y)| \le C|t|^{-\frac{3}{2}}$$

and $\Omega_n(t)$ is regularly dispersive. We write $N(\lambda) = G(\lambda)VG_0(\lambda)$ and apply integration by parts with respect to λ, which gives

$$W_3(t) = \frac{1}{2\pi t}\int_{\mathbf{R}} e^{-it\lambda^2}(G_0(\lambda)VN(\lambda)VG_0(\lambda))'d\lambda.$$

Out of three integrals produced after differentiation, we explain here how to treat the one with $G_0(\lambda)VN'(\lambda)VG_0(\lambda)$ as a prototype, which we denote by $W_{31}(t)$. It is important to notice that, if we denote the integral kernel of $L(\lambda) = \langle x\rangle^{\sigma}VN'(\lambda)V\langle x\rangle^{\sigma}$ by

$L(\lambda, z_2, z_1)$, then that of $W_{31}(t)$ may be given by using the solution of the one-dimensional free Schrödinger equation by

$$W_{31}(t,x,y) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbf{R}} \frac{1}{16\pi^2} \frac{(e^{it\Delta}\check{L}(\,\cdot\,, z_2, z_1))(A)}{\langle z_2\rangle^\sigma |x-z_2|\langle z_1\rangle^\sigma |z_1-y|} dz_1 dz_2. \tag{1.20}$$

Here $A = |x-z_2|+|z_1-y|$, Δ is the one dimensional Laplacian acting on the variable denoted by $\cdot$ and $\check{L}$ is the inverse Fourier transform of L with respect to the variable λ. We have

$$|(e^{it\Delta}\check{L}(\,\cdot\,, z_2, z_1))(A)| \le Ct^{-\frac{1}{2}} \|L(\cdot, z_2, z_1)\|_{H^s}, \tag{1.21}$$

provided $s > 1/2$. The LAP stated in Sect. 2 implies that $L(\lambda, z_2, z_1)$ is indeed an $L^2(\mathbf{R}^6_{z_2,z_1})$-valued $H^s(\mathbf{R}_\lambda)$ function of λ for some $\sigma > 1/2$ and $s > 1/2$. Applying the Schwarz inequality to (1.20) and using (1.21), we then obtain $|W_{31}(t,x,y)| \le C|t|^{-\frac{3}{2}}$. Other integrals may be estimated similarly and we obtain $|W_3(t,x,y)| \le C|t|^{-\frac{3}{2}}$. This proves statement (1) of Theorem 1.3 by the help of interpolation theory.

We study exceptional cases in Sect. 4. When H is of exceptional type, we break up (1.19) into two parts, $e^{-itH}P_c = W_h(t) + W_l(t)$, the high and the low energy parts, by inserting a partition of unity $\chi_l(\lambda) + \chi_h(\lambda) = 1$ into the integrand, where $\chi_l \in C_0^\infty(\mathbf{R})$ is even and $\chi_l(\lambda) = 1$ for $|\lambda| < \lambda_0/2$ and $\chi_l(\lambda) = 0$ for $|\lambda| > \lambda_0$ for a small positive constant λ_0. The argument of Sect. 3 for the generic case shows that the high energy part $W_h(t)$ which contains χ_h is regularly dispersive. For the low energy part $W_l(t)$ we write $G(\lambda) = G_0(\lambda) - G_0(\lambda)V(1+G_0(\lambda)V)^{-1}G_0(\lambda)$ in the integrand. The integral which contains $\chi_l(\lambda)G_0(\lambda)$ may be treated as in the generic case and it is regularly dispersive. We are left with

$$W_{l0}(t) = \lim_{\delta\downarrow 0} \frac{-1}{i\pi} \int_{|\lambda|>\delta} \chi_l(\lambda)e^{-it\lambda^2} G_0(\lambda)V(1+G_0(\lambda)V)^{-1}G_0(\lambda)\lambda d\lambda. \tag{1.22}$$

We study $W_{l0}(t)$ by examining the behavior of $(1+G_0(\lambda)V)^{-1}$ as $\lambda \to 0$. After some preparation, we study it when H is of exceptional type of the first kind in Subsect. 4.3, the second kind in Subsect. 4.4 and, synthesizing the results of previous two subsections, the third kind in Subsect. 4.5. If H is of exceptional type of the first kind we have (see Theorem 4.8)

$$(1+G_0(\lambda)V)^{-1} = I + K(\lambda) - a\lambda^{-1}|\phi\rangle\langle V\phi|,$$

where $VK(\lambda)$ satisfies the property similar to that of $L(\lambda)$ in (1.20) and a is the constant defined in (1.11). Integral (1.22) with $I + K(\lambda)$ in place of $(1+G_0(\lambda)V)^{-1}$ can then be studied by the method of Sect. 3 for $W_{31}(t)$ and it produces a regularly dispersive family of operators. On the other hand $-a\lambda^{-1}|\phi\rangle\langle V\phi|$ produces

$$W_l(t) = \frac{a}{\pi i} \int_{\mathbf{R}} \chi_l(\lambda)e^{-it\lambda^2} G_0(\lambda)|V\phi\rangle\langle V\phi|G_0(\lambda)d\lambda, \tag{1.23}$$

and its integral kernel may be computed explicitly:

$$W_l(t,x,y) = a \int_{\mathbf{R}^6} \frac{c(t,A)V(z_1)V(z_2)\phi(z_1)\phi(z_2)}{16\pi^2|x-z_2||z_1-y|} dz_1 dz_2, \tag{1.24}$$

where $A = |x - z_2| + |z_1 - y|$ and $c(t, A)$ is given by

$$c(t, A) = \frac{1}{\pi i}\int_{\mathbf{R}} \chi_l(\lambda)e^{-it\lambda^2+i\lambda A}d\lambda = \frac{e^{-\frac{i3\pi}{4}}e^{\frac{iA^2}{4t}}}{\sqrt{\pi t}}\mathcal{F}\left(e^{\frac{is^2}{4t}}\check{\chi}_l\right)\left(\frac{A}{2t}\right). \quad (1.25)$$

Here $\mathcal{F}$ is the Fourier transform. This is except for a normalization constant the well known formula for solutions of the one dimensional free Schrödinger equation. Since $|c(t, A)| \le Ct^{-\frac{1}{2}}$, we have $|W_l(t, x, y)| \le Ct^{-\frac{1}{2}}\langle x\rangle^{-1}\langle y\rangle^{-1}$. Since $\langle x\rangle^{-1} \in L^{3,\infty}$, Hölder's inequality in Lorentz spaces implies

$$\|W_l(t)u\|_{3,\infty} \le Ct^{-\frac{1}{2}}\|u\|_{3/2,1}. \quad (1.26)$$

We have shown above that $e^{-itH}P_c - W_l(t)$ is regularly dispersive and it also satisfies (1.26). Hence

$$\|e^{-itH}P_c u\|_{3,\infty} \le Ct^{-\frac{1}{2}}\|u\|_{3/2,1}, \quad (1.27)$$

and statement (2)(ii) of Theorem 1.3 follows for this case. By virtue of the interpolation theorem for Lorentz spaces, (1.27) and the obvious L^2 bound $\|e^{-itH}P_c u\|_2 \le C\|u\|_2$ imply statement (2) (i).

To prove statement (2)(iii), we first note that (1.27) and the bound $|\varphi(x)| \le C\langle x\rangle^{-1}$ imply

$$\|(e^{-itH}P_c - R(t))u\|_{3,\infty} \le Ct^{-\frac{1}{2}}\|u\|_{\frac{3}{2},1}. \quad (1.28)$$

If we replace in the right of (1.25) first $e^{\frac{is^2}{4t}}$ by 1, then $\chi_l(A/2t)$ by 1 and finally $e^{\frac{iA^2}{4t}}$ by $e^{i\frac{|x|^2+|y|^2}{4t}}$, we obtain

$$\left|c(t, A) - (\pi t)^{-\frac{1}{2}}e^{-i\frac{3\pi}{4}}e^{i\frac{|x|^2}{4t}}e^{i\frac{|y|^2}{4t}}\right| \le Ct^{-\frac{3}{2}}\langle x\rangle\langle y\rangle\langle z_1\rangle^2\langle z_2\rangle^2. \quad (1.29)$$

We insert (1.29) into (1.24) and recall that $\phi(x) = -D_0V\phi$. This produces

$$\left|W_l(t, x, y) - \frac{ae^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}}\zeta(t, x)\zeta(t, y)\right| \le Ct^{-\frac{3}{2}}.$$

Since $e^{-itH}P_c - W_l(t)$ is regularly dispersive, it then follows that

$$\|(e^{-itH}P_c - R(t))u\|_{\infty} \le Ct^{-\frac{3}{2}}\|u\|_1. \quad (1.30)$$

Interpolating (1.28) and (1.30), we obtain statement (2)(iii) of the theorem.

If H is of exceptional type of the second or the third kind, which will be discussed in Subsects. 4.2. and 4.3 respectively, $(1 + G_0(\lambda)V)^{-1}$ contains singularities also of order λ^{-2} and the argument becomes a bit more involved. However, basically the same idea works. We refer to the text for the details.

We use the following notation and conventions. For $s, \sigma \in \mathbf{R}$, $H^s(\mathbf{R}^d)$ is the Sobolev space of order s on $\mathbf{R}^d$ and $H^s_\sigma(\mathbf{R}^3) = \{u : \langle x\rangle^\sigma u \in H^s(\mathbf{R}^3)\}$ is the weighted Sobolev space. For Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, $\mathbf{B}(\mathcal{X}, \mathcal{Y})$ is the Banach space of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$, $\mathbf{B}(\mathcal{X}) = \mathbf{B}(\mathcal{X}, \mathcal{X})$ and $\mathbf{B}_2(\mathcal{X})$ is the Hilbert space of Hilbert-Schmidt operators in $\mathcal{X}$. We denote by $\mathbf{C}$ the complex plane, $\mathbf{C}^+ = \{z \in \mathbf{C} : \Im z > 0\}$ is the upper half plane and $\overline{\mathbf{C}}^+$ is the closed upper half plane: $\overline{\mathbf{C}}^+ = \{z : \Im z \geq 0\}$. For $a \in \mathbf{R}$, $a-$ (resp. $a+$) denotes any number smaller (resp. larger) than a. In what follows we always assume that V at least satisfies (1.2) although some statements hold under less stringent conditions, and after Sect. 3 we shall assume much stronger decay conditions. We occasionally use the physics notation $|v\rangle$ and $\langle u|v\rangle$ to denote vectors and the inner product.

After submission of this paper we were informed that Theorem 1.3 (1) for the generic case has recently been proved by Goldberg [7] for more general potentials $V \in L^r(\mathbf{R}^3) \cap L^s(\mathbf{R}^3)$, $r < 3/2 < s$, and that a result similar to statement (2) of Theorem 1.3 has been obtained by Erdoğan and Schlag [6] under a slightly stronger decay condition on the potentials. We thank Professor Piero D'Ancona and the anonymous referee for bringing this to our attention.

2. Preliminaries

In this section we collect some results on the resolvents, $G_0(\lambda)$ and $G(\lambda)$, and estimates on the integrals which will often appear in the sequel.

2.1. Resolvents. We recall that for $\lambda \in \mathbf{C}$,

$$G_0(\lambda)u(x) = \frac{1}{4\pi}\int \frac{e^{i\lambda|x-y|}}{|x-y|}u(y)dy. \tag{2.1}$$

For $\Im\lambda > 0$, $G_0(\lambda)$ is a $\mathbf{B}(\mathcal{H})$-valued analytic function and $R_0(\lambda^2) = G_0(\lambda)$.

Lemma 2.1. *(1) Let $\sigma, \tau > 1/2$ and $\sigma+\tau > 2$. Then, $\langle x\rangle^{-\sigma}G_0(\lambda)\langle x\rangle^{-\tau}$ is a $\mathbf{B}_2(\mathcal{H})$-valued C^ρ function of $\lambda \in \overline{\mathbf{C}}^+$ for any ρ such that $\rho < \min(\tau+\sigma-2, \tau-1/2, \sigma-1/2)$. If $\rho = j+\kappa$, $j = 0, 1, \ldots,$ and $0 \leq \kappa < 1$, we have*

$$\sup_{\lambda\in\overline{\mathbf{C}}^+}\|\langle x\rangle^{-\sigma}G_0^{(j)}(\lambda)\langle x\rangle^{-\tau}\|_{\mathbf{B}_2} + \sup_{\lambda\neq\mu}\frac{\|\langle x\rangle^{-\sigma}(G_0^{(j)}(\lambda) - G_0^{(j)}(\mu))\langle x\rangle^{-\tau}\|_{\mathbf{B}_2}}{|\lambda-\mu|^\kappa} \leq C.$$

We have $G_0(\lambda)^* = G_0(-\lambda)$ when $\lambda \in \mathbf{R}$.

(2) Let $\sigma > 1/2$. Then, $\langle x\rangle^{-\sigma}G_0(\lambda)\langle x\rangle^{-\sigma}$ is a $\mathbf{B}(\mathcal{H})$-valued $C^{\sigma-\frac{1}{2}}$ function of $\lambda \in \overline{\mathbf{C}}^+ \setminus \{0\}$. For $j = 0, 1, \ldots,$ we have

$$\|\langle x\rangle^{-\sigma-j}\partial_\lambda^j G_0(\lambda)\langle x\rangle^{-\sigma-j}\|_{\mathbf{B}(\mathcal{H})} \leq C_j|\lambda|^{-1}, \quad |\lambda| \geq 1. \tag{2.2}$$

Proof. (1) Write $m = \min(\tau+\sigma-2, \tau-1/2, \sigma-1/2)$. We may assume $\tau \leq \sigma$. Suppose first that $0 < m \leq 1$. Then, $\tau \leq 3/2$ and without losing generality we may assume $\tau < 3/2$. We then have, with $\hat{x} = x/|x|$,

$$\begin{aligned}\|\langle x\rangle^{-\sigma}G_0(\lambda)\langle x\rangle^{-\tau}\|_{\mathbf{B}_2}^2 &= \frac{1}{16\pi^2}\int_{\mathbf{R}^6}\frac{dxdy}{\langle x\rangle^{2\sigma}|x-y|^2\langle y\rangle^{2\tau}} \\ &\leq \int_{\mathbf{R}^3}\frac{dx}{\langle x\rangle^{2\sigma}}\left\{\int_{\mathbf{R}^3}\frac{dy}{|x|^{2\tau-1}|\hat{x}-y|^2|y|^{2\tau}}\right\} \\ &\leq \int_{\mathbf{R}^3}\frac{Cdx}{\langle x\rangle^{2\sigma}|x|^{2\tau-1}} < \infty\end{aligned}$$

and $\|\langle x\rangle^{-\sigma}G_0(\lambda)\langle x\rangle^{-\tau}\|_{\mathbf{B}_2}$ is uniformly bounded. Here we changed variables y to $|x|y$ and used $\langle |x|y\rangle \geq |x||y|$ in the second step, $2+2\tau > 3$ in the third and $2\sigma+2t-1>3$ in the last.

Since $0<\rho<1$, we have $|e^{ia}-e^{ib}|\leq 2^\rho|a-b|^\rho$ and we may likewise estimate as follows:

$$\|\langle x\rangle^{-\sigma}(G_0(\lambda)-G_0(\mu))\langle x\rangle^{-\tau}\|^2_{\mathbf{B}_2(\mathcal{H})} = \int_{\mathbf{R}^6}\frac{|e^{i\lambda|x-y|}-e^{i\mu|x-y|}|^2}{16\pi^2\langle x\rangle^{2\sigma}|x-y|^2\langle y\rangle^{2\tau}}dxdy$$
$$\leq \int_{\mathbf{R}^6}\frac{C|\lambda-\mu|^{2\rho}dxdy}{\langle x\rangle^{2\sigma}|x-y|^{2-2\rho}\langle y\rangle^{2\tau}} \leq \int_{\mathbf{R}^3}\frac{C_1|\lambda-\mu|^{2\rho}dx}{\langle x\rangle^{2\sigma}\langle x\rangle^{2\tau-2\rho-1}} \leq C_2|\lambda-\mu|^{2\rho}.$$

Here we used $2\tau-2\rho+2>3$ in the second step and $2\tau+2\sigma-2\rho-1>3$ in the last step. This proves (1) when $0<m\leq 1$. If $j<m\leq j+1$, $j=1,2,\dots$, we have $m=\tau-1/2$. Write $\rho=j+\kappa, 0\leq\kappa<1$. The j^{th} derivative $G_0^{(j)}(\lambda)$ has integral kernel $(4\pi)^{-1}i^je^{i\lambda|x-y|}|x-y|^{j-1}$ and $\|\langle x\rangle^{\sigma}G_0^{(j)}(\lambda)\langle x\rangle^{\tau}\|_{\mathbf{B}_2}\leq C$ follows entirely similarly as above. As $1<2(\tau-\rho)<3$ and $\sigma\geq\tau>3/2$, we have

$$\|\langle x\rangle^{-\sigma}(G_0^{(j)}(\lambda)-G_0^{(j)}(\mu))\langle x\rangle^{-\tau}\|^2_{\mathbf{B}_2(\mathcal{H})}\leq C\int_{\mathbf{R}^6}\frac{|\lambda-\mu|^{2\kappa}|x-y|^{2(\rho-1)}dxdy}{\langle x\rangle^{2\sigma}\langle y\rangle^{2\tau}}$$
$$\leq C_1\int_{\mathbf{R}^3}\frac{|\lambda-\mu|^{2\kappa}(|x|^{2\rho}+|y|^{2\rho})dxdy}{\langle x\rangle^{2\sigma}|x-y|^2\langle y\rangle^{2\tau}}\leq C_2|\lambda-\mu|^{2\kappa}.$$

Statement (1) follows. Statement (2) is well known (see [1] and [11]). □

Recall that we are assuming (1.2). The following is an obvious consequence of Lemma 2.1.

Corollary 2.2. *Let* $1/2<\gamma<\beta-1/2$. *Then,* $\langle x\rangle^{-\gamma}G_0(\lambda)V\langle x\rangle^{+\gamma}$ *is a* $\mathbf{B}_2(\mathcal{H})$*-valued* C^ρ *function of* $\lambda\in\overline{\mathbf{C}}^+$ *for any* $\rho<min(\beta-2,\gamma-\frac{1}{2},\beta-\gamma-\frac{1}{2})$. *The operator valued function* $\langle x\rangle^{+\gamma}VG_0(\lambda)\langle x\rangle^{-\gamma}$ *satisfies the same property.*

Under condition (1.2), it is well known (see [13]) that $H=-\Delta+V$ has no positive eigenvalues and the point spectral subspace $\mathcal{H}_p(H)$ for H is finite dimensional. Thus $R(\lambda^2)=(H-\lambda^2)^{-1}$ is a $\mathbf{B}(\mathcal{H})$-valued meromorphic function of $\lambda\in\mathbf{C}^+$ with possible poles $i\kappa_1,\dots,i\kappa_n$ on the imaginary axis such that $-\kappa_1^2,\dots,-\kappa_n^2$ are eigenvalues of H. The resolvent equation implies that outside those poles in the upper half plane

$$R(\lambda^2)=G_0(\lambda)(1+VG_0(\lambda))^{-1}=(1+G_0(\lambda)V)^{-1}G_0(\lambda).$$

Here $VG_0(\lambda)$ (resp.$G_0(\lambda)V$) is a $\mathbf{B}_2(\mathcal{H}_\gamma)$-valued (resp. $\mathbf{B}_2(\mathcal{H}_{-\gamma})$-valued) continuous function of $\lambda\in\overline{\mathbf{C}}^+$ if $1/2<\gamma<\beta-1/2$ by virtue of Corollary 2.2 and $-1\in\sigma(VG_0(\lambda))$ (resp. $-1\in\sigma(G_0(\lambda)V)$) if and only if λ^2 is an eigenvalue of H (see [1]). Since positive eigenvalues are absent from H as mentioned above, $R(\lambda^2)$ considered as a $\mathbf{B}(\mathcal{H}_\gamma,\mathcal{H}_{-\gamma})$ valued function is continuous up to the boundary $\mathbf{R}$ of $\mathbf{C}^+$ except possibly at $\lambda=0$. We set for $\lambda\in\mathbf{R}\setminus\{0\}$,

$$G(\lambda)=G_0(\lambda)(1+VG_0(\lambda))^{-1}=(1+G_0(\lambda)V)^{-1}G_0(\lambda). \tag{2.3}$$

Lemma 2.3. *For* $1/2 < \sigma, \tau < \beta - 1/2$ *such that* $\sigma + \tau > 2$, $\langle x\rangle^{-\sigma} G(\lambda)\langle x\rangle^{-\tau}$, *as a* $\mathbf{B}_2(\mathcal{H})$*-valued or* $\mathbf{B}(\mathcal{H})$*-valued function of* $\lambda \in \{\lambda \in \mathbf{R} : |\lambda| > \varepsilon\}$, $\varepsilon > 0$, *satisfies the same smoothness and decay properties as* $\langle x\rangle^{-\sigma} G_0(\lambda)\langle x\rangle^{-\tau}$ *as stated in* Lemma 2.1. *This is true on the whole line* $\lambda \in \mathbf{R}$, *if* $1 + VG_0(0)$ *or* $1 + G_0(0)V$ *is invertible respectively in* $\mathcal{H}_\gamma$ *or* $\mathcal{H}_{-\gamma}$ *for some, and therefore for all,* $1/2 < \gamma < \beta - 1/2$.

Proof. We use the same notation as in the proof of Lemma 2.1. Let $0 < m \le 1$ first. By virtue of (2.3), Lemma 2.1 and Corollary 2.2 we have

$$\|\langle x\rangle^{-\sigma} G(\lambda)\langle x\rangle^{-\tau}\|_{\mathbf{B}_2} \le C.$$

By telescoping the difference, we may estimate as follows:

$$\begin{aligned}
&\|\langle x\rangle^{-\sigma}(G(\lambda) - G(\mu))\langle x\rangle^{-\tau}\|_{\mathbf{B}_2}\\
&\quad \le \|\langle x\rangle^{-\sigma}(1 + G_0(\lambda)V)^{-1}\langle x\rangle^{\sigma}\|_{\mathbf{B}}\|\langle x\rangle^{-\sigma}(G_0(\lambda) - G_0(\mu))\langle x\rangle^{-\tau}\|_{\mathbf{B}_2}\\
&\qquad + \|\langle x\rangle^{-\sigma}(1 + G_0(\lambda)V)^{-1}\langle x\rangle^{\sigma}\|_{\mathbf{B}}\|\langle x\rangle^{-\sigma}(G_0(\lambda) - G_0(\mu))\langle x\rangle^{-\tau}\|_{\mathbf{B}_2}\\
&\qquad \times \|\langle x\rangle^{\tau} V(1 + G_0(\mu)V)^{-1}\langle x\rangle^{\beta-\tau}\|_{\mathbf{B}}\|\langle x\rangle^{\tau-\beta} G_0(\mu)\langle x\rangle^{-\tau}\|_{\mathbf{B}} \le C|\lambda - \mu|^{\rho},
\end{aligned}$$

and the lemma follows for this case. When $1 < m \le 2$, we differentiate (2.3) and use the resolvent equation. We obtain

$$G'(\lambda) = (1 - G(\lambda)V)G_0'(\lambda)(1 - VG(\lambda)).$$

We then repeat the argument above using the previous result for $0 < m \le 1$. We omit repetitious details also for general m. □

By the functional calculus for selfadjoint operators, the propagator e^{-itH} may be expressed in terms of $G(\lambda)$ in the following form:

$$e^{-itH}P_c = \lim_{\delta\downarrow 0} \frac{1}{i\pi}\int_{|\lambda|>\delta} e^{-it\lambda^2} G(\lambda)\lambda d\lambda. \tag{2.4}$$

Equation (2.4) is the starting point for the proof of the main theorem.

2.2. Integrals. We collect here some formulae and estimates on integrals which will be of frequent use in what follows. We begin with the following lemma on the Gauss integral:

Lemma 2.4. *Let* $s > 1/2$. *Then, there exists a constant* C_s *depending only on* s *such that for any* $\chi \in H^s(\mathbf{R})$, $A \in \mathbf{R}$, $t > 0$ *and* $L > 0$,

$$\left|\int_{\mathbf{R}} e^{-it\lambda^2 + i\lambda A}\chi(\lambda/L)d\lambda\right| \le C_s\|\chi\|_{H^s} t^{-\frac{1}{2}}. \tag{2.5}$$

As $L \to \infty$ *we have*

$$\int_{\mathbf{R}} e^{-it\lambda^2 + i\lambda A}\chi(\lambda/L)d\lambda \to e^{-i\frac{\pi}{4}} e^{\frac{iA^2}{4t}}\sqrt{\frac{\pi}{t}}\chi(0). \tag{2.6}$$

Suppose in addition that χ *is even and* $\lambda\chi'(\lambda) \in H^s(\mathbf{R})$ *then*

$$\left|\int_{\mathbf{R}} e^{-it\lambda^2 + i\lambda A}\chi(\lambda/L)\lambda d\lambda\right| \le C_s(\|\chi\|_{H^s} + \|\lambda\chi'\|_{H^s})|A| t^{-\frac{3}{2}}. \tag{2.7}$$

As $L \to \infty$ we have

$$\int_{\mathbf{R}} e^{-it\lambda^2 + i\lambda A} \chi(\lambda/L)\lambda d\lambda \to \frac{Ae^{-i\frac{\pi}{4}} e^{\frac{iA^2}{4t}}}{2} \sqrt{\frac{\pi}{t^3}} \chi(0). \tag{2.8}$$

Proof. We first prove estimate (2.5). If we write $\check{\chi}(\kappa) = \hat{\chi}(-\kappa)$ for the conjugate Fourier transform of χ, the integral on the left in (2.5) is equal to $\sqrt{2\pi}(e^{it\Delta}\check{\chi}_L)(A)$, where Δ is the one dimensional Laplacian, and by virtue of the well known formula for the kernel of the propagator $e^{it\Delta}$,

$$\sqrt{2\pi}(e^{it\Delta}\check{\chi}_L)(A) = \frac{e^{-i\frac{\pi}{4}} e^{\frac{iA^2}{4t}}}{\sqrt{2t}} \int e^{-i\frac{Ar}{2tL} + i\frac{r^2}{4tL^2}} \check{\chi}(r)dr. \tag{2.9}$$

If $s > 1/2$, this is bounded in modulus by

$$(2t)^{-\frac{1}{2}} \|\check{\chi}\|_1 \le C_s (2t)^{-\frac{1}{2}} \|\langle r\rangle^s \check{\chi}\|_2 = C_s (2t)^{-\frac{1}{2}} \|\chi\|_{H^s},$$

and (2.5) follows. Taking the limit $L \to \infty$ in (2.9) we obtain (2.6).

Since $\lambda e^{-it\lambda^2} = \frac{i}{2t}(d/d\lambda)e^{-it\lambda^2}$, integration by parts shows that the integral in the left of (2.7) is equal to

$$\frac{A}{2t}\int_{\mathbf{R}} e^{-it\lambda^2+i\lambda A}\chi(\lambda/L)d\lambda + \frac{1}{2it}\int_{\mathbf{R}} e^{-it\lambda^2+i\lambda A}\chi'(\lambda/L)L^{-1}d\lambda.$$

The argument of the first part shows that the first summand satisfies (2.7) and it converges to the right-hand side of (2.8) as $L \to \infty$. Since $\chi'(\lambda/L)$ is odd, the second summand may be written in the form

$$\frac{1}{4it}\int_{\mathbf{R}} e^{-it\lambda^2}(e^{i\lambda A} - e^{-i\lambda A})\chi'(\lambda/L)\frac{d\lambda}{L} = \frac{A}{4t}\int_{-1}^{1}\left(\int_{\mathbf{R}} e^{-it\lambda^2+i\lambda\theta A}\zeta(\lambda/L)d\lambda\right)d\theta,$$

where $\zeta(\lambda) = \chi'(\lambda)\lambda$. Applying again (2.5) and (2.9) to the λ-integral, we see that the second summand is bounded in modulus by $C_\sigma |A|(2t)^{-\frac{3}{2}}\|\zeta\|_{H^s}$ and converges to zero as $L \to \infty$. This completes the proof. □

We recall the Kato norm:

$$\|V\|_{\mathcal{K}} = \sup_{a\in\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|V(z)|dz}{|z-a|}.$$

Lemma 2.5. *Let $x_{n+1} = x$ and $x_0 = y$. Then, for $n = 1, 2, \ldots,$*

$$\int_{\mathbf{R}^3}\frac{\prod_{j=1}^{n}|V(x_j)|}{\prod_{j=1}^{n+1}|x_j - x_{j-1}|}dx_1,\ldots,dx_n \le \frac{(4\|V\|_{\mathcal{K}})^n}{|x-y|}. \tag{2.10}$$

Proof. By induction, it suffices to show the case $n = 1$:

$$\int_{\mathbf{R}^3}\frac{|V(z)|dz}{|x-z||z-y|} \le \frac{4\|V\|_{\mathcal{K}}}{|x-y|}.$$

Change variables z to $z+y$ and write $w=x-y$. We have

$$\int_{|z|\ge|w|/2}\frac{|V(z+y)|dz}{|w-z||z|}\le\frac{2}{|w|}\int\frac{|V(z+y)|dz}{|w-z|}\le\frac{2}{|w|}\|V\|_{\mathcal{K}}.$$

If $|z|<|w|/2$, then $|w-z|\ge|w|/2$ and

$$\int_{|z|<|w|/2}\frac{|V(z+y)|dz}{|w-z||z|}\le\frac{2}{|w|}\int\frac{|V(z+y)|dz}{|z|}\le\frac{2}{|w|}\|V\|_{\mathcal{K}}.$$

The lemma follows. □

Following is a result of the celebrated Kato smoothness theorem ([15]):

Lemma 2.6. *Let $T(\lambda)$, $\lambda\in\mathbf{R}$, be a weakly measurable family of bounded operators in $\mathcal{H}$ such that $\|\langle x\rangle^{\sigma}T(\lambda)\langle x\rangle^{\sigma}\|_{\mathbf{B}(\mathcal{H})}\le C$ for some $\sigma>1$. Then, for $t\in\mathbf{R}$, the weak integral*

$$U(t)=\int_{-\infty}^{\infty}e^{-it\lambda^2}G_0(\lambda)T(\lambda)G_0(\lambda)\lambda d\lambda$$

converges in $\mathcal{H}$ and defines a bounded operator in $\mathcal{H}$. The family $\{U(t):t\in\mathbf{R}\}$ is strongly continuous and uniformly bounded in $\mathbf{B}(\mathcal{H})$.

Proof. When $\sigma>1$, the multiplication operator by $\langle x\rangle^{-\sigma}$ is H_0-smooth ([15]): $\int_{-\infty}^{\infty}\|\langle x\rangle^{-\sigma}G_0(\lambda)u\|_2^2|\lambda|d\lambda\le C\|u\|_2^2$. It follows by the Schwarz inequality that $U(t)$ is uniformly bounded in $\mathcal{H}$. It also follows by the Schwarz inequality that

$$\|(U(t)-U(s))u\|^2\le C\int_{-\infty}^{\infty}|e^{-it\lambda^2}-e^{-is\lambda^2}|^2\|\langle x\rangle^{-\sigma}G_0(\lambda)u\|^2|\lambda|d\lambda$$

and Lebesgue's dominated convergence theorem implies the lemma. □

Lemma 2.7. *Let $s,\sigma>1/2$ and let $\mathbf{R}\ni\lambda\to\mathcal{G}_\sigma(\lambda)\equiv\langle x\rangle^{\sigma}N(\lambda)\langle x\rangle^{\sigma}$ be a $\mathbf{B}_2(\mathcal{H})$-valued $H^s(\mathbf{R})$ function of λ. Define*

$$\mathcal{N}(t)=\int_{\mathbf{R}}e^{-it\lambda^2}G_0(\lambda)N(\lambda)G_0(\lambda)d\lambda,\quad t\ne0.$$

Then $\mathcal{N}(t)$ has a bounded continuous integral kernel $\mathcal{N}(t,x,y)$ and it satisfies

$$|\mathcal{N}(t,x,y)|\le C_s|t|^{-\frac12}\|\mathcal{G}_\sigma\|_{H^s(\mathbf{R},\mathbf{B}_2(\mathcal{H}))}. \tag{2.11}$$

If $\sigma>3/2$, then $\mathcal{N}(t,x,y)$ satisfies the stronger estimate,

$$|\mathcal{N}(t,x,y)|\le C_s|t|^{-\frac12}\langle x\rangle^{-1}\langle y\rangle^{-1}\|\mathcal{G}_\sigma\|_{H^s(\mathbf{R},\mathbf{B}_2(\mathcal{H}))}. \tag{2.12}$$

If $\mathcal{G}_{\sigma1}(\lambda)=\langle x\rangle^{\sigma+1}N(\lambda)\langle x\rangle^{\sigma}$ (resp. $\mathcal{G}_{\sigma2}(\lambda)=\langle x\rangle^{\sigma}N(\lambda)\langle x\rangle^{\sigma+1}$) *is $\mathbf{B}_2(\mathcal{H})$-valued $H^s(\mathbf{R})$, then for any $t\ne0$,*

$$\mathcal{N}_1(t)=\int_{\mathbf{R}}e^{-it\lambda^2}G_0'(\lambda)N(\lambda)G_0(\lambda)d\lambda$$

$$(\text{resp. }\mathcal{N}_2(t)=\int_{\mathbf{R}}e^{-it\lambda^2}G_0(\lambda)N(\lambda)G_0'(\lambda)d\lambda)$$

has a continuous integral kernel $\mathcal{N}_1(t,x,y)$ (resp. $\mathcal{N}_2(t,x,y)$) *and it satisfies* (2.11) *with obvious modifications.*

Proof. We take $\chi \in C_0^\infty(\mathbf{R})$ such that $\chi(\lambda) = 1$ for $|\lambda| \le 1$ and define

$$\mathcal{N}_L(t) = \int_{\mathbf{R}} e^{-it\lambda^2} \chi(\lambda/L) G_0(\lambda) N(\lambda) G_0(\lambda) d\lambda.$$

If $\gamma > \frac{3}{2}$, $\|G_0(\lambda)N(\lambda)G_0(\lambda)\|_{\mathbf{B}(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})} \le C\langle\lambda\rangle^{-2}\|\mathcal{G}_\sigma(\lambda)\|_{\mathbf{B}_2}$ by virtue of Lemma 2.1(3) and $\|\mathcal{N}_L(t) - \mathcal{N}(t)\|_{\mathbf{B}(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})} \to 0$ as $L \to \infty$. Denote the integral kernel of $\mathcal{G}_\sigma(\lambda)$ by $\mathcal{G}_\sigma(\lambda, x, y)$ and $A = |x - z_2| + |z_1 - y|$. Then,

$$\mathcal{N}_L(t, x, y) = \int_{\mathbf{R}} \int_{\mathbf{R}^6} e^{it\lambda^2 + i\lambda A} \chi(\lambda/L) \frac{\langle z_2\rangle^{-\sigma}}{|x - z_2|} \mathcal{G}_\sigma(\lambda, z_2, z_1) \frac{\langle z_1\rangle^{-\sigma}}{|z_1 - y|} dz_1 dz_2 d\lambda.$$

For almost all (z_1, z_2), $\|(\chi(\lambda/L) - 1)\mathcal{G}_\sigma(\lambda, z_2, z_1)\|_{H^s(\mathbf{R}_\lambda)} \to 0$ as $L \to \infty$ and (2.5) implies that

$$\int_{\mathbf{R}} e^{it\lambda^2 + i\lambda A} \chi(\lambda/L) \mathcal{G}_\sigma(\lambda, z_2, z_1) d\lambda \to \int_{\mathbf{R}} e^{it\lambda^2 + i\lambda A} \mathcal{G}_\sigma(\lambda, z_2, z_1) d\lambda$$

and that the left side is bounded by $C|t|^{-\frac{1}{2}}\|\mathcal{G}_\sigma(\cdot, z_2, z_1)\|_{H^s}$ uniformly with respect to $L \ge 1$. By the Schwarz inequality,

$$\begin{aligned} &\int \frac{\langle z_2\rangle^{-\sigma}}{|x - z_2|} \|\mathcal{G}_\sigma(\cdot, z_2, z_1)\|_{H^s} \frac{\langle z_1\rangle^{-\sigma}}{|z_1 - y|} dz_1 dz_2 \\ &\quad \le \left\| \frac{\langle z_2\rangle^{-\sigma}}{|z_2 - x|} \right\|_{L^2_{z_2}} \left\| \frac{\langle z_1\rangle^{-\sigma}}{|z_1 - y|} \right\|_{L^2_{z_1}} \|\mathcal{G}_\sigma\|_{H^s(\mathbf{R}, \mathbf{B}_2(\mathcal{H}))}. \end{aligned} \tag{2.13}$$

It follows that $|\mathcal{N}_L(t, x, y)| \le C/\sqrt{t}$ for all $x, y \in \mathbf{R}^3$ and, by Lebesgue's dominated convergence theorem, that $\mathcal{N}_L(t, x, y)$ converges to the integral kernel $\mathcal{N}(t, x, y)$ of $\mathcal{N}(t)$ as $L \to \infty$:

$$\mathcal{N}(t, x, y) = \int_{\mathbf{R}^6} \left(\int e^{it\lambda^2 + i\lambda A} \mathcal{G}_\sigma(\lambda, z_2, z_1) d\lambda \right) \frac{\langle z_2\rangle^{-\sigma}}{|x - z_2|} \frac{\langle z_1\rangle^{-\sigma}}{|z_1 - y|} dz_1 dz_2.$$

Here, $\int e^{it\lambda^2 + i\lambda A} \mathcal{G}_\sigma(\lambda, z_2, z_1) d\lambda$ is an $L^2(\mathbf{R}^6_{z_1, z_2})$-valued continuous function of (t, x, y), $t \ne 0$, since it is bounded in modulus by $C|t|^{-\frac{1}{2}}\|\mathcal{G}_\sigma(\cdot, z_2, z_1)\|_{H^s}$ and, for almost all (z_1, z_2), it is continuous with respect to (t, x, y), $t \ne 0$, as can be seen from (2.9). Then, since $\langle z_1\rangle^{-\sigma}\langle z_2\rangle^{-\sigma}/|x - z_2||y - z_1|$ is also a continuous function of (x, y) with values in $L^2(\mathbf{R}^6_{z_1, z_2})$, $\mathcal{N}(t, x, y)$ is continuous with respect to (t, x, y) if $t \ne 0$. By virtue of (2.13), $\mathcal{N}(t, x, y)$ satisfies the estimate (2.11). If $\sigma > 3/2$, the right side of (2.13) is bounded by $C\langle x\rangle^{-1}\langle y\rangle^{-1}\|\mathcal{G}_\sigma\|_{H^s(\mathbf{R}, \mathbf{B}_2(\mathcal{H}))}$ and (2.12) is satisfied. This proves the lemma for $\mathcal{N}(t)$. Modifications necessary for the proof for $\mathcal{N}_1(t)$ and $\mathcal{N}_2(t)$ are obvious and we omit the details. □

3. The Case of Generic Type

In this section, we prove statement (1) of Theorem 1.3. Thus we assume that $|V(x)| \le C\langle x\rangle^{-\beta}$ with $\beta > 5/2$ and that H is of generic type. We recall that $D_0, D_1, \ldots,$ are selfadjoint integral operators defined by

$$D_j u(x) = \frac{1}{4\pi j!} \int |x-y|^{j-1} u(y) dy. \tag{3.1}$$

If j is odd, D_j is of finite rank. We have a formal expansion

$$G_0(\lambda) = D_0 + (i\lambda)D_1 + (i\lambda)^2 D_2 + \cdots . \tag{3.2}$$

We denote the null spaces of $1 + VD_0$ and $1 + D_0 V$ considered respectively as operators in $\mathcal{H}_{-\gamma}$ or in $\mathcal{H}_\gamma$ by

$$\mathcal{M} = N(1 + D_0 V), \quad \mathcal{N} = N(1 + VD_0).$$

Since $D_0 V$ and VD_0 are compact and $D_0 V = (VD_0)^*$, $\dim \mathcal{M} = \dim \mathcal{N} < \infty$. Moreover, $\mathcal{M}$ and $\mathcal{N}$ are independent of $1/2 < \gamma < \beta - 1/2$ because $\mathcal{M}$ (resp. $\mathcal{N}$) decreases (resp. increases) with γ (see [10]).

As H is of generic type, $\lambda \to G(\lambda) \in \mathbf{B}(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})$ is continuous on $\mathbf{R}$ for $\gamma > 1$ by Lemma 2.3 and, by the spectral theorem,

$$e^{-itH} P_c = \frac{1}{i\pi} \int_{\mathbf{R}} e^{-it\lambda^2} G(\lambda) \lambda d\lambda = \lim_{L\to\infty} \frac{1}{i\pi} \int_{\mathbf{R}} e^{-it\lambda^2} \chi_L(\lambda) G(\lambda) \lambda d\lambda \tag{3.3}$$

as strong convergence in $\mathcal{H}$, where $\chi_L(\lambda) = \chi(\lambda/L)$ and $\chi \in C_0^\infty(\mathbf{R})$ is even, $\chi(\lambda) = 1$ for $|\lambda| \le 1$ and $\chi(\lambda) = 0$ for $|\lambda| \ge 2$. Iterating the resolvent equation $G(\lambda) = G_0(\lambda) - G_0(\lambda) V G(\lambda)$, we insert in the right of (3.3),

$$G(\lambda) = \sum_{n=0}^{2} (-1)^n G_0(\lambda)(VG_0(\lambda))^n - G_0(\lambda)VG_0(\lambda)VG(\lambda)VG_0(\lambda).$$

The result is $e^{-itH} P_c = \Omega_0(t) - \Omega_1(t) + \Omega_2(t) + W_3(t)$, where for $n = 0, 1, 2$,

$$\Omega_n(t) = \lim_{L\to\infty} \frac{1}{i\pi} \int_R e^{-it\lambda^2} \chi_L(\lambda) G_0(\lambda)(VG_0(\lambda))^n \lambda d\lambda. \tag{3.4}$$

We have $\Omega_0(t) = e^{-itH_0}$. Lemma 2.1 and Lemma 2.6 imply

$$\sup_{t\in\mathbf{R}} \|\Omega_n(t)\|_{\mathbf{B}(\mathcal{H})} \le C, \quad n = 0, 1, 2. \tag{3.5}$$

Lemma 3.1. *There exists a constant $C > 0$ such that*

$$\|\Omega_n(t)u\|_\infty \le C(n+1)|t|^{-\frac{3}{2}} \left(\frac{\|V\|_{\mathcal{K}}}{4\pi} \right)^n \|u_0\|_1, \quad n = 0, 1, 2, \ldots . \tag{3.6}$$

Proof. We follow the argument due to Rodnianski-Schlag [25]. The integral kernel of the operator defined by the integral on the right side of (3.4) is given with $C_1 = 1/4\pi$, $A = \sum_{j=1}^{n+1} |x_{j-1} - x_j|$ and $dx_1, \ldots, dx_n = dX$ by

$$C_1^n \int dX \left(\frac{1}{i\pi} \int_{\mathbf{R}} e^{-it\lambda^2 + i\lambda A} \lambda \chi_L(\lambda) d\lambda \right) \frac{\prod_{j=1}^n V(x_j)}{\prod_{j=1}^{n+1} |x_{j-1} - x_j|}. \tag{3.7}$$

Note that the integrand is absolutely convergent by virtue of (2.10):

$$\int_{\mathbf{R}^{3n}} \int_{\mathbf{R}} |(\text{integrand of (3.7)})| dX d\lambda \le \frac{2^n \|V\|_{\mathcal{K}}^n}{|x-y|} \|\lambda \chi_L\|_1$$

and by the help of the Fubini theorem the computation (3.7) is legitimate. Moreover, with $x = x_{n+1}$ and $y = x_0$,

$$\int \frac{A\prod_{j=1}^{n}|V(x_j)|dX}{\prod_j |x_{j-1}-x_j|} = \sum_{k=1}^{n+1}\int \frac{\prod_{j=1}^{n}|V(x_j)|dX}{\prod_{j\neq k}|x_{j-1}-x_j|} \le (n+1)\|V\|_{\mathcal{K}}^n.$$

Hence, Lemma 2.4 implies that (3.7) converges as $L \to \infty$ to

$$\frac{C_1^{n-1}}{(4i\pi t)^{\frac{3}{2}}}\int e^{i\frac{A^2}{4t}}\frac{A\prod_{j=1}^{n}V(x_j)}{\prod_{j=1}^{n+1}|x_{j-1}-x_j|}dX,$$

which is bounded by $C(n+1)|t|^{-\frac{3}{2}}\left(\frac{\|V\|_{\mathcal{K}}}{4\pi}\right)^n$. This implies the lemma. □

Define $N(\lambda) = VG_0(\lambda)VG(\lambda)V$. If $0 < \varepsilon < 1/2$, by virtue of Lemma 2.1, we have $\|\langle x\rangle^{1+\varepsilon}N(\lambda)\langle x\rangle^{1+\varepsilon}\|_{\mathbf{B}(\mathcal{H})} \le C\langle\lambda\rangle^{-2}$. It follows by virtue of Lemma 2.6 that

$$W_3(t) = -\frac{1}{i\pi}\int_{\mathbf{R}} e^{-it\lambda^2}G_0(\lambda)N(\lambda)G_0(\lambda)\lambda d\lambda \tag{3.8}$$

is a strongly continuous family of uniformly bounded operators in $\mathcal{H}$. By integration by parts, we may write

$$W_3(t) = \frac{1}{2t\pi}\int_{\mathbf{R}} e^{-it\lambda^2}\{G_0(\lambda)N(\lambda)G_0(\lambda)\}'d\lambda. \tag{3.9}$$

Differentiation in the right side produces three integrals which respectively contain

$$G_0'(\lambda)N(\lambda)G_0(\lambda),\ G_0(\lambda)N'(\lambda)G_0(\lambda),\ \text{and } G_0(\lambda)N(\lambda)G_0'(\lambda).$$

Thus, in view of Lemma 2.7, Theorem 1.3(1) is a consequence of the following lemma and the interpolation theorem for L^p spaces.

Lemma 3.2. *Let $|V(x)| \le C\langle x\rangle^{-\beta}$, $\beta > 5/2$. Then, for some $\sigma, s > 1/2$,*

$$\langle x\rangle^{1+\sigma}N(\lambda)\langle x\rangle^{\sigma},\quad \langle x\rangle^{\sigma}N(\lambda)\langle x\rangle^{1+\sigma},\quad \langle x\rangle^{\sigma}N'(\lambda)\langle x\rangle^{\sigma}$$

are $\mathbf{B}_2(\mathcal{H})$-valued H^s functions of $\lambda \in \mathbf{R}$.

Proof. We estimate the operators by using Lemma 2.1 and Lemma 2.3. We first deal with $\langle x\rangle^{1+\sigma}N(\lambda)\langle x\rangle^{\sigma}$. If $\sigma > 1/2$ is sufficiently close to $1/2$,

$$\begin{aligned}&\|\langle x\rangle^{1+\sigma}N(\lambda)\langle x\rangle^{\sigma}\|_{\mathbf{B}_2}\\ &\quad\le \|\langle x\rangle^{1+\sigma}VG_0(\lambda)\langle x\rangle^{-\sigma-1}\|_{\mathbf{B}_2}\|\langle x\rangle^{1+\sigma}VG(\lambda)V\langle x\rangle^{\sigma}\|_{\mathbf{B}} \le C\langle\lambda\rangle^{-1}.\end{aligned} \tag{3.10}$$

We show for some $s > 1/2$ that for λ and $\mu \in \mathbf{R}$ such that $|\lambda-\mu| \le 1$,

$$\|\langle x\rangle^{1+\sigma}(N(\lambda)-N(\mu))\langle x\rangle^{\sigma}\|_{\mathbf{B}_2} \le C\langle\lambda\rangle^{-1}|\lambda-\mu|^s. \tag{3.11}$$

By reducing s by an arbitrarily small amount, two estimates (3.10) and (3.11) will imply that $\langle x\rangle^{1+\sigma}N(\lambda)\langle x\rangle^{\sigma} \in H^s(\mathbf{R}, \mathbf{B}_2(\mathcal{H}))$ for some $s > 1/2$ (cf. [18], Theorem 10.2).

In what follows in the proof we choose and fix parameters σ, τ and the exponent s in such a way that

$$\frac{3}{2} < \tau < \sigma + 1 < 2, \quad \tau + \sigma < \beta - \frac{1}{2}, \quad \frac{1}{2} < s < \min\{\beta - \sigma - \frac{3}{2}, \tau - 1\}$$

hence, $\beta - \sigma > 2$ and $\beta - \tau > 1$. We write $N(\lambda) - N(\mu)$ in the form

$$V(G_0(\lambda) - G_0(\mu))VG(\lambda)V + VG_0(\mu)V(G(\lambda) - G(\mu))V.$$

Since $\beta - \sigma > 2$ and $\beta - \tau > 1$, Lemma 2.3 implies

$$\|\langle x\rangle^{\tau} VG(\lambda)V\langle x\rangle^{\sigma}\|_{\mathbf{B}} \le C\langle\lambda\rangle^{-1}, \quad \|\langle x\rangle^{1+\sigma} VG_0(\mu)\langle x\rangle^{-1-\sigma}\|_{\mathbf{B}} \le C\langle\lambda\rangle^{-1}.$$

It follows by the choice of s that

$$\begin{aligned} &\|\langle x\rangle^{1+\sigma} V(G_0(\lambda) - G_0(\mu))VG(\lambda)V\langle x\rangle^{\sigma}\|_{\mathbf{B}_2(\mathcal{H})} \\ &\quad \le \|\langle x\rangle^{1+\sigma} V(G_0(\lambda) - G_0(\mu))\langle x\rangle^{-\tau}\|_{\mathbf{B}_2}\|\langle x\rangle^{\tau} VG(\lambda)V\langle x\rangle^{\sigma}\|_{\mathbf{B}} \\ &\quad \le C|\lambda - \mu|^{s}\langle\lambda\rangle^{-1}. \end{aligned} \tag{3.12}$$

As $\tau < \beta - \sigma < \beta - 1/2$ and $G(\lambda)$ and $G_0(\lambda)$ satisfy similar regularity and decay properties,

$$\begin{aligned} &\|\langle x\rangle^{1+\sigma} VG_0(\mu)V(G(\lambda) - G(\mu))V\langle x\rangle^{\sigma}\|_{\mathbf{B}_2} \\ &\quad \le \|\langle x\rangle^{1+\sigma} VG_0(\mu)\langle x\rangle^{-1-\sigma}\|_{\mathbf{B}} \cdot \|\langle x\rangle^{1+\sigma} V(G(\lambda) - G(\mu))V\langle x\rangle^{\sigma}\|_{\mathbf{B}_2} \\ &\quad \le C|\lambda - \mu|^{s}\langle\lambda\rangle^{-1}. \end{aligned} \tag{3.13}$$

The two estimates (3.12) and (3.13) imply (3.11).

The operator $\langle x\rangle^{\sigma} N(\lambda)\langle x\rangle^{1+\sigma}$ satisfies estimates corresponding to (3.10) and (3.11) because it is obtained from $\langle x\rangle^{\sigma+1} N(\lambda)\langle x\rangle^{\sigma}$ by taking the adjoint after replacing $G_0(\lambda)$ and $G(\lambda)$ respectively by $G(-\lambda)$ and $G_0(-\lambda)$.

Finally we deal with $\langle x\rangle^{\sigma} N'(\lambda)\langle x\rangle^{\sigma}$ which may be written as

$$\langle x\rangle^{\sigma} VG_0'(\lambda)VG(\lambda)V\langle x\rangle^{\sigma} + \langle x\rangle^{\sigma} VG_0(\lambda)VG'(\lambda)V\langle x\rangle^{\sigma}.$$

Since $\beta - \sigma > 2$ and $\beta - \tau > 1$, we have

$$\begin{aligned} &\|\langle x\rangle^{\sigma} VG_0'(\lambda)VG(\lambda)V\langle x\rangle^{\sigma}\|_{\mathbf{B}_2} \\ &\quad \le \|\langle x\rangle^{\sigma} VG_0'(\lambda)\langle x\rangle^{-\tau}\|_{\mathbf{B}_2}\|\langle x\rangle^{\tau} VG(\lambda)V\langle x\rangle^{\sigma}\|_{\mathbf{B}} \le C\langle\lambda\rangle^{-1}. \end{aligned}$$

Replacing $G_0(\lambda)$ and $G(\lambda)$ and taking the adjoint in the estimate above yield

$$\|\langle x\rangle^{\sigma} VG_0(\lambda)VG'(\lambda)V\langle x\rangle^{\sigma}\|_{\mathbf{B}_2} \le C\langle\lambda\rangle^{-1}.$$

It follows that

$$\|\langle x\rangle^{\sigma} N'(\lambda)\langle x\rangle^{\sigma}\| \le C. \tag{3.14}$$

Since $s < \beta - \sigma - 3/2 < \beta - \tau - 1/2$ and $\min(\tau, \beta - \sigma) > 3/2$, we have

$$\begin{aligned} &\|\langle x\rangle^{\sigma} V(G_0(\lambda) - G_0(\mu))VG'(\lambda)V\langle x\rangle^{\sigma}\|_{\mathbf{B}_2} \\ &\quad \le \|\langle x\rangle^{\sigma} V(G_0(\lambda) - G_0(\mu))\langle x\rangle^{\tau} V\|_{\mathbf{B}_2}\|\langle x\rangle^{-\tau} G'(\lambda)V\langle x\rangle^{\sigma}\|_{\mathbf{B}} \\ &\quad \le C|\lambda - \mu|^{s}\langle\lambda\rangle^{-1}. \end{aligned} \tag{3.15}$$

Likewise, since $s < \beta - \sigma - 3/2$ and $|\lambda - \mu| < 1$,

$$\begin{aligned}
&\|\langle x\rangle^\sigma V G_0(\mu) V (G'(\lambda) - G'(\mu)) V \langle x\rangle^\sigma\|_{\mathbf{B}_2} \\
&\quad \le \|\langle x\rangle^\sigma V G_0(\mu)\langle x\rangle^{-\sigma}\|_{\mathbf{B}} \|\langle x\rangle^\sigma V (G'(\lambda) - G'(\mu)) V \langle x\rangle^\sigma\|_{\mathbf{B}_2} \\
&\quad \le C\langle\lambda\rangle^{-1}|\lambda - \mu|^s. \qquad (3.16)
\end{aligned}$$

Symmetrically we have

$$\begin{aligned}
&\|\langle x\rangle^\sigma V G_0'(\lambda) V (G(\lambda) - G(\mu)) V \langle x\rangle^\sigma\|_{\mathbf{B}_2} \le C\langle\lambda\rangle^{-1}|\lambda - \mu|^s, \\
&\|\langle x\rangle^\sigma V (G_0'(\lambda) - G_0'(\mu)) V G(\mu) V \langle x\rangle^\sigma\|_{\mathbf{B}_2} \le C\langle\lambda\rangle^{-1}|\lambda - \mu|^s. \qquad (3.17)
\end{aligned}$$

The combination of (3.15), (3.16) and (3.17) yields

$$\|\langle x\rangle^\sigma (N'(\lambda) - N'(\mu))\langle x\rangle^\sigma\|_{\mathbf{B}_2} \le C\langle\lambda\rangle^{-1}|\lambda - \mu|^s. \qquad (3.18)$$

The estimates (3.14) and (3.18) imply that $\langle x\rangle^\sigma V N'(\lambda) V \langle x\rangle^\sigma \in H^s(\mathbf{R}, \mathbf{B}_2)$ for some $s > 1/2$. This completes the proof of the lemma. □

4. The Cases of Exceptional Type

In this section we prove statement (2) of Theorem 1.3 for the case that H is of exceptional type. We first reduce the proof to the analysis of a simpler operator $W_{0l}(t)$ to be defined by (4.1) below. Then, because of the reasons stated in the introduction, we study it according to the type of exceptionality of H separately in Subsects. 4.3, 4.4 and 4.5.

4.1. Reduction to low energy analysis. For an even function $\chi_l \in C_0^\infty(\mathbf{R})$ such that $\chi_l(\lambda) = 1$ near $\lambda = 0$ we define

$$W_{0l}(t) = \lim_{\delta\downarrow 0} \frac{-1}{i\pi} \int_{|\lambda|>\delta} e^{-it\lambda^2} \chi_l(\lambda) G_0(\lambda) V G(\lambda) \lambda d\lambda. \qquad (4.1)$$

Recall that a family $\{T(t) : t \in \mathbf{R}\}$ of bounded operators in $\mathcal{H}$ is said to be *regularly dispersive* if it is strongly continuous and, in addition, it satisfies

$$\|T(t)u\|_p \le Ct^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|u\|_q, \quad u \in L^2 \cap L^q \qquad (4.2)$$

for all $1 \le q \le 2 \le p \le \infty$ such that $1/p + 1/q = 1$. In this case we shall often say simply that $T(t)$ is regularly dispersive.

Lemma 4.1. *The operator $\Omega(t) = e^{-itH}P_c - W_{0l}(t)$ is regularly dispersive.*

Proof. As in the generic case, we decompose $e^{-itH}P_c$ in the form

$$e^{-itH}P_c = \sum_{n=0}^{2} (-1)^n \Omega_n(t) + W_3(t).$$

Recall the definition (3.8) of $W_3(t)$. As was shown in (3.5) and (3.6), $\Omega_n(t)$ are regularly dispersive. We define the low and the high energy parts $W_h(t)$ and $W_l(t)$ of $W_3(t) = W_h(t) + W_l(t)$ by

$$W_{h,l}(t) = \lim_{\delta\downarrow 0} \frac{-1}{i\pi} \int_{|\lambda|>\delta} e^{-it\lambda^2} \chi_{h,l}(\lambda) G_0(\lambda) N(\lambda) G_0(\lambda) \lambda d\lambda, \qquad (4.3)$$

where $\chi_h(\lambda) = 1 - \chi_l(\lambda)$ and $N(\lambda) = VG_0(\lambda)VG(\lambda)V$. Since $G(\lambda)$ has no singularities on the support of χ_h, it follows, by virtue of Lemma 2.7 and Lemma 3.2, and in view of the argument in Sect. 3 for the generic case, that $W_h(t)$ is also regularly dispersive. Using the resolvent equation, we write

$$G_0(\lambda)N(\lambda)G_0(\lambda) = G_0(\lambda)VG(\lambda) + \sum_{j=1}^{2}(-1)^j (G_0(\lambda)V)^j G_0(\lambda)$$

in (4.3) and further decompose $W_l(t) = W_{0l}(t) - W_{1l}(t) + W_{2l}(t)$:

$$W_{nl}(t) = \frac{-1}{i\pi}\int_{\mathbf{R}} e^{-it\lambda^2}\chi_l(\lambda)(G_0(\lambda)V)^n G_0(\lambda)\lambda d\lambda, \quad 1 \le n \le 2. \tag{4.4}$$

The operator $W_{nl}(t)$ is the same as the one defined by the integral in the right of (3.4) with $-\chi_l$ replacing χ_L and the proof of Lemma 3.1 implies

$$\|W_{nl}(t)u_0\|_\infty \le C|t|^{-\frac{3}{2}}\|V\|_{\mathcal{K}}^j\|u_0\|_1, \quad 1 \le n \le 2. \tag{4.5}$$

Lemma 2.6 clearly implies that $W_{nl}(t)$ are strongly continuous families of uniformly bounded operators in $\mathcal{H}$. Thus, $W_{1l}(t)$ and W_{2l} are regularly dispersive and so is $\Omega(t) = \sum_{n=0}^{2}(-1)^n\Omega_n(t) + W_h(t) - W_{1l}(t) + W_{2l}(t)$. This proves the lemma. □

4.2. Low energy resolvent analysis. Preliminary. In the following subsections we study $W_{0l}(t)$ separately according to the kind of exceptionality. In each case, we need to investigate the behavior of $G(\lambda)$ near $\lambda = 0$. We do it mostly following Jensen-Kato [10] and we collect here some preliminary information.

The following two lemmas collect Lemmas 2.4, 2.5, 2.6, 3.1, 3.2 and 3.3 of [10]. We recall the operators D_j, $j = 0, 1, \ldots,$ are defined by (3.1) and

$$\mathcal{M} = N(1 + D_0V), \quad \mathcal{N} = N(1 + VD_0).$$

Lemma 4.2. (1) *If $v \in \mathcal{H}_\gamma$, $\frac{1}{2} < \gamma \le \frac{5}{2}$, and $\langle v, 1\rangle = 0$, $D_0v \in H^2_{\gamma-2}(\mathbf{R}^3)$.*
(2) *For $u, v \in \mathcal{H}_{\frac{5}{2}+0}$ such that $\langle u, 1\rangle = \langle v, 1\rangle = 0$, $\langle D_2u, v\rangle = -\langle D_0u, D_0v\rangle$.*

Lemma 4.3. *Let $\frac{1}{2} < \gamma < \beta - \frac{1}{2}$. Then the following statements hold:*

(1) *$\mathcal{M} \subset H^2_{-\frac{1}{2}-}(\mathbf{R}^3)$ and $(H_0 + V)\mathcal{M} = \{0\}$. If $\frac{1}{2} < \gamma < \frac{3}{2}$, $N(H_0 + V) = \mathcal{M}$ as an operator from $H^2_{-\gamma}$.*
(2) *H_0 and V are isomorphisms $\mathcal{M} \to \mathcal{N}$. D_0 is an isomorphism $\mathcal{N} \to \mathcal{M}$.*
(3) *For $u \in \mathcal{M}$, $u \in \mathcal{H}$ if and only if $\langle u, V\rangle = 0$. In this case, $u \in H^2_{\frac{1}{2}-}(\mathbf{R}^3)$.*
(4) *For $v \in \mathcal{N}$, $D_0v \in \mathcal{H}$ if and only if $\langle 1, v\rangle = 0$. In this case, $v \in \mathcal{H}_{\beta+\frac{1}{2}-}$.*

In what follows γ *is always assumed to satisfy* $1/2 < \gamma < \beta - 1/2$. Notice that (u, D_0v) is a strictly positive quadratic form on $\mathcal{H}_\gamma$ and that VD_0 is real and formally selfadjoint with respect to this form. It follows that all eigenvalues λ of VD_0 are real and the eigenspaces are semi-simple: $N(VD_0 - \lambda) = N((VD_0 - \lambda)^2)$. By the duality, the same is true for D_0V.

Lemma 4.4. *There exist operators Q and K which are bounded in $\mathcal{H}_{-\gamma}$ for any $1/2 < \gamma < \beta - 1/2$ such that $Q^2 = Q$, $QK = KQ = 0$ and*

$$(1 + D_0 V)Q = Q(1 + D_0 V) = 0, \quad (1 + D_0 V)K = K(1 + D_0 V) = 1 - Q.$$

(1) *The projector Q is of finite rank and $K - I \in \mathbf{B}_2(\mathcal{H}_{-\gamma})$.*
(2) *We have the identities $VK = K^*V$, $KD_0 = D_0K^*$.*

Proof. The first statement is a result of the separation of the spectrum theorem ([14], p. 178). By the same theorem $1 + D_0V + Q$ is invertible and

$$(I + D_0 V + Q)^{-1} - I = -(D_0 V + Q)(1 + D_0 V + Q)^{-1} \in \mathbf{B}_2(\mathcal{H}_{-\gamma}).$$

Since $K = (1 + D_0V + Q)^{-1}(1 - Q)$, $K - I \in \mathbf{B}_2(\mathcal{H}_{-\gamma})$. Statement (2) may be found in Lemma 3.5 of [10]. □

If $u = D_0\tilde{u}$ and $v = D_0\tilde{v}$, $\tilde{u}, \tilde{v} \in \mathcal{N}$, $-(Vu, v) = (D_0\tilde{u}, \tilde{v})$. It follows that $-(Vu, v)$ defines an inner product in $\mathcal{M}$ and the spectral projection

$$Q = -\frac{1}{2\pi i}\int_{|z+1|=\delta} (D_0 V - z)^{-1} dz$$

satisfies $Q^*V = VQ$. The next lemma follows. Note that D_0V and VD_0 are real operators and we may choose a real basis of $\mathcal{M}$ and $\mathcal{N}$.

Lemma 4.5. *Let $\{\phi_1, \ldots, \phi_d\}$ be an orthonormal basis of $\mathcal{M}$ with respect to the inner product $-(Vu, v)$. Define $\psi_j = -V\phi_j$. Then $\{\psi_1, \ldots, \psi_d\}$ is the dual basis of $\mathcal{N}$ with natural coupling $\langle \phi_j, \psi_k \rangle = \delta_{jk}$ and, simultaneously, is orthonomal with respect to the inner product (D_0u, v). With these bases*

$$Q = \sum_{j=1}^{d} |\phi_j\rangle\langle\psi_j|, \quad Q^* = \sum_{j=1}^{d} |\psi_j\rangle\langle\phi_j|$$

and Q^* is the spectral projection onto $\mathcal{N}$ with respect to $1 + VD_0$. We have the identity $QD_0 = D_0Q^*$.

By virtue of Lemma 4.3 (3), the 0 eigenspace $\mathcal{E}$ of $H = -\Delta + V$ is a subspace of $\mathcal{M}$ of codimension at most one.

We write $\overline{Q} = 1 - Q$. If we define closed subspaces $\mathcal{X}_{-\gamma} = \overline{Q}\mathcal{H}_{-\gamma}$ and $\mathcal{Y}_{-\gamma} = Q\mathcal{H}_{-\gamma}$, the map $\mathcal{X}_{-\gamma}\dot{+}\mathcal{Y}_{-\gamma} \ni \{u, v\} \mapsto u + v \in \mathcal{H}_{-\gamma}$ is an isomorphism between Banach spaces. In the direct sum decomposition $\mathcal{H}_{-\gamma} = \mathcal{X}_{-\gamma}\dot{+}\mathcal{Y}_{-\gamma}$,

$$M(\lambda) = 1 + G_0(\lambda)V$$

may be written in the matrix form:

$$M(\lambda) = \begin{pmatrix} \overline{Q}M(\lambda)\overline{Q} & \overline{Q}M(\lambda)Q \\ QM(\lambda)\overline{Q} & QM(\lambda)Q \end{pmatrix} \equiv \begin{pmatrix} M_{00}(\lambda) & M_{01}(\lambda) \\ M_{10}(\lambda) & M_{11}(\lambda) \end{pmatrix}. \tag{4.6}$$

We often consider operators $M_{jk}(\lambda)$ and etc. also as operators in $\mathcal{H}_{-\gamma}$ by extending them to the complementary subspaces as zero operators.

Lemma 4.6. *There exists* λ_0 *such that* $M_{00}(\lambda) : \mathcal{X}_{-\gamma} \to \mathcal{X}_{-\gamma}$ *is invertible for* $|\lambda| < \lambda_0$ *and* $M_{00}(\lambda)^{-1} - I \in \mathbf{B}_2(\mathcal{X}_{-\gamma})$*. As a* $\mathbf{B}_2(\mathcal{X}_{-\gamma})$*-valued function of* $|\lambda| < \lambda_0$*,* $M_{00}(\lambda)^{-1} - I$ *is of class* C^δ *for* $\delta < \min(\beta - \gamma - \frac{1}{2}, \gamma - \frac{1}{2}, \beta - 2)$.

Proof. By virtue of Lemma 2.1, $M_{00}(\lambda) - 1$ is a $\mathbf{B}_2(\mathcal{X}_{-\gamma})$–valued C^δ function of λ and $M_{00}(0) = \overline{Q}(1 + D_0 V)\overline{Q}$ is invertible by Lemma 4.4. The lemma follows by a Neumann series expansion. □

The following well known lemma is very useful.

Lemma 4.7. *Let* $\mathcal{X} = \mathcal{X}_0 \dot{+} \mathcal{X}_1$ *be a direct sum decomposition of a vector space* $\mathcal{X}$*. Suppose that a linear operator* L *in* $\mathcal{X}$ *is written in the form*

$$L = \begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{pmatrix}$$

in this decomposition and that L_{00}^{-1} *exists. Set*

$$C = L_{11} - L_{10} L_{00}^{-1} L_{01}.$$

Then, L^{-1} *exists if and only if* C^{-1} *exists. In this case*

$$L^{-1} = \begin{pmatrix} L_{00}^{-1} + L_{00}^{-1} L_{01} C^{-1} L_{10} L_{00}^{-1} & -L_{00}^{-1} L_{01} C^{-1} \\ -C^{-1} L_{10} L_{00}^{-1} & C^{-1} \end{pmatrix}. \tag{4.7}$$

4.3. Exceptional type of the first kind. In this subsection we prove Theorem 1.3 (2) when H is of exceptional type of the first kind. In this case $\dim \mathcal{M} = 1$ and nontrivial $\phi \in \mathcal{M}$ satisfies

$$\phi(x) - \frac{c}{|x|} \in \mathcal{H}, \quad \phi \in H^2_{-\frac{1}{2}-} \tag{4.8}$$

for a constant $c \neq 0$. We take a uniquely determined $\phi \in \mathcal{M}$ such that $-\langle \phi, V\phi \rangle = 1$ and $\langle V, \phi \rangle > 0$ so that $Q = -|\phi\rangle\langle V\phi|$.

Theorem 4.8. *Let* $|V(x)| \le C\langle x \rangle^{-\beta}$ *with* $\beta > 9/2$*. Assume* H *is of exceptional type of the first kind. Let* $\phi \in \mathcal{M}$ *be as above. Then, in a small punctured neighbourhood* $0 < |\lambda| < \lambda_0$ *of zero,* $(1 + G_0(\lambda)V)^{-1}$ *may be written in the form*

$$(1 + G_0(\lambda)V)^{-1} = I + K(\lambda) + a\lambda^{-1} Q, \quad a = \frac{4\pi i}{|\langle V, \phi \rangle|^2}, \tag{4.9}$$

where $\langle x \rangle^{1+\sigma} V K(\lambda) \langle x \rangle^{1+\sigma}$ *is a* $\mathbf{B}_2(\mathcal{H})$*–valued* $C^{1+\rho}$ *function of* $\lambda \in (-\lambda_0, \lambda_0)$ *for some* $\sigma > 1/2$ *and* $\rho > 1/2$.

Proof. We may assume $9/2 < \beta < 5$ without losing the generality. We have $\beta - 3 < (\beta - 1)/2$. We apply Lemma 4.7 to (4.6). We need to study $C(\lambda) \equiv M_{11}(\lambda) - M_{10}(\lambda) M_{00}^{-1}(\lambda) M_{01}(\lambda)$ first. Recall $D_1 = (1/4\pi)(1 \otimes 1)$. We define the operator $J(\lambda)$ by the equation

$$\lambda^2 J(\lambda) = M(\lambda) - (1 + D_0 V + i\lambda D_1 V) \quad \lambda \neq 0.$$

We have $(M(\lambda)-(1+D_0V+i\lambda D_1V))\phi \in C^{\gamma-\frac{1}{2}-}(\mathbf{R},\mathcal{H}_{-\gamma})$ for any $3/2<\gamma<\beta-\frac{1}{2}$ by virtue of (4.8) and Lemma 2.1 and it vanishes at $\lambda=0$ along with its derivative. Hence, $J(\lambda)\phi$ and $\lambda J(\lambda)\phi$ are, as $\mathcal{H}_{-\gamma}$-valued functions, respectively of class $C^{\gamma-\frac{5}{2}-}$ and $C^{\gamma-\frac{3}{2}-}$ including $\lambda=0$. It follows by choosing $\gamma<\beta-1/2$ arbitrarily close to $\beta-1/2$ that

$$M_{11}(\lambda)=\left(\frac{\lambda|\langle V\phi|1\rangle|^2}{4i\pi}-\lambda^2\langle V\phi|J(\lambda)\phi\rangle\right)Q=-\lambda c_0(\lambda)Q, \tag{4.10}$$

where $\lambda c_0(\lambda)$, $c_0(\lambda)$ and $\langle V\phi|J(\lambda)\phi\rangle$ are functions respectively of class $C^{\beta-1-}$, $C^{\beta-2-}$ and $C^{\beta-3-}$ on $\mathbf{R}$. Likewise we have

$$\begin{aligned}\tilde\psi(\lambda)&\equiv M(\lambda)\phi=(iD_1V+\lambda J(\lambda))\phi\in C^{\gamma-\frac{3}{2}-}(\mathbf{R},\mathcal{H}_{-\gamma}),\\ \tilde\psi(\lambda)^*&\equiv M(\lambda)^*V\phi=V(-iD_1V+\lambda J(-\lambda))\phi\in C^{\gamma-\frac{3}{2}-}(\mathbf{R},\mathcal{H}_{\beta-\gamma})\end{aligned} \tag{4.11}$$

for any $3/2<\gamma<\beta-1/2$. Using these functions, we may write

$$\begin{aligned}M_{01}(\lambda)&=-\lambda(\tilde\psi(\lambda)+c_0(\lambda)\phi)\otimes V\phi,\\ M_{10}(\lambda)&=-\lambda\phi\otimes(\tilde\psi^*(\lambda)+\overline{c_0(\lambda)}V\phi)\end{aligned} \tag{4.12}$$

and $-M_{10}(\lambda)M_{00}^{-1}(\lambda)M_{01}(\lambda)=\lambda^2c_1(\lambda)Q$, where

$$c_1(\lambda)=\langle\tilde\psi^*(\lambda)+\overline{c_0(\lambda)}V\phi,\,M_{00}^{-1}(\lambda)(\tilde\psi(\lambda)+c_0(\lambda)\phi)\rangle. \tag{4.13}$$

Then, (4.11) and Lemma 4.6 for $M_{00}(\lambda)^{-1}$ to (4.13) imply that $c_1(\lambda)\in C^{\beta-3-}$. Combining this with (4.10), we have

$$C(\lambda)=\left(\frac{\lambda|\langle V\phi|1\rangle|^2}{4i\pi}+\lambda^2c_2(\lambda)\right)Q \text{ with } c_2 \text{ of class } C^{\beta-3-}, \tag{4.14}$$

and $C(\lambda)^{-1}$ exists for small $0<|\lambda|<\lambda_0$. Moreover,

$$C^{-1}(\lambda)=\left(\frac{a}{\lambda}+d(\lambda)\right)Q,\quad d(\lambda)\in C^{\beta-3-},\ \ a=\frac{4\pi i}{|\langle V,\phi\rangle|^2}. \tag{4.15}$$

It follows from Lemma 4.7 that $M(\lambda)^{-1}$ may be written in the form (4.7) with obvious modifications. Using (4.12) and (4.15), we write

$$\begin{aligned}-M_{00}^{-1}M_{01}C^{-1}&=-(a+\lambda d(\lambda))|\xi_1(\lambda)\rangle\langle V\phi|,\\ -C^{-1}M_{10}M_{00}^{-1}&=-(a+\lambda d(\lambda))|\phi\rangle\langle\xi_2(\lambda)|,\\ M_{00}^{-1}M_{01}C^{-1}M_{10}M_{00}^{-1}&=-(a+\lambda d(\lambda))|\xi_1(\lambda)\rangle\langle\xi_2(\lambda)|\end{aligned} \tag{4.16}$$

with $\xi_1=M_{00}(\lambda)^{-1}(\tilde\psi(\lambda)+c_0(\lambda)\phi)$ and $\xi_2(\lambda)=M_{00}(\lambda)^{*-1}(\tilde\psi^*(\lambda)+\overline{c_0(\lambda)}V\phi)$ and, by virtue of (4.11) and Lemma 4.6, $\langle x\rangle^{1+\sigma}V(x)\xi_1(\lambda)$ and $\langle x\rangle^{1+\sigma}\xi_2(\lambda)$ are $\mathcal{H}$-valued $H^{1+\rho}$ functions of $|\lambda|<\lambda_0$ for σ and ρ such that $1+\sigma,1+\rho<\beta-3$. Thus, putting the operators in (4.16), $(d(\lambda)-1)Q$ and $M_{00}(\lambda)^{-1}-\overline{Q}$ into $K(\lambda)$, we obtain the theorem. □

We are ready to study $W_{0l}(t)$ when H is of exceptional type of the first kind. We choose $\chi_l \in C_0^\infty(\mathbf{R})$ such that χ_l is even, $\chi_1(\lambda) = 1$ when $|\lambda| < \lambda_0/2$ and $\chi_l(\lambda) = 0$ when $|\lambda| \geq \lambda_0$. We write, using (4.9),

$$\begin{aligned} G_0(\lambda)VG(\lambda) &= G_0(\lambda)V(1+G_0(\lambda)V)^{-1}G_0(\lambda) \\ &= G_0(\lambda)VG_0(\lambda) + G_0(\lambda)VK(\lambda)G_0(\lambda) + a\lambda^{-1}G_0(\lambda)VQG_0(\lambda), \end{aligned}$$

and insert this in the right of (4.1) to obtain

$$W_{0l}(t) = W_{1l}(t) + Z_(t) + Z_2(t). \tag{4.17}$$

We know that $W_{1l}(t)$ is regularly dispersive from the proof of Lemma 4.1. Next we consider

$$Z_1(t) = \frac{-1}{i\pi}\int_{\mathbf{R}} e^{-it\lambda^2}\chi_l(\lambda)G_0(\lambda)VK(\lambda)G_0(\lambda)\lambda d\lambda. \tag{4.18}$$

Lemma 4.9. *Assume $\beta > 9/2$. Then, $Z_1(t)$ is regularly dispersive.*

Proof. Denote $K_1(\lambda) = \chi_l(\lambda)K(\lambda)$. Take $\sigma, \rho > 1/2$ as in Theorem 4.8. Then, $\| \langle x\rangle^{1+\sigma} VK_1(\lambda)\langle x\rangle^{1+\sigma}\|_{\mathbf{B}(\mathcal{H})} \leq C$ and, by virtue of Lemma 2.6, $Z_1(t)$ is strongly continuous and uniformly bounded in $\mathbf{B}(\mathcal{H})$. It is also obvious that $G_0(\lambda)VK_1(\lambda)G_0(\lambda)$ is C^1 as a $\mathbf{B}_2(\mathcal{H}_{-\sigma})$-valued function and, after integration by parts we obtain

$$Z_1(t) = \frac{1}{\pi t}\int_{\mathbf{R}} e^{-it\lambda^2}\{G_0(\lambda)VK_1(\lambda)G_0(\lambda)\}' d\lambda. \tag{4.19}$$

Lemma 2.7 then implies $\|Z_1(t)u\|_\infty \leq C|t|^{-\frac{3}{2}}\|u\|_1$ and the lemma follows by interpolation. □

Finally we study the contribution from the singular part of (4.9):

$$Z_2(t) = \frac{-a}{i\pi}\int_{\mathbf{R}} e^{-it\lambda^2}\chi_l(\lambda)G_0(\lambda)VQG_0(\lambda)d\lambda. \tag{4.20}$$

Lemma 4.10. *Let $\beta > 9/2$. Then, $Z_2(t)$ is a strongly continuous family of uniformly bounded operators in $\mathcal{H}$ and its integral kernel $Z_2(t,x,y)$ satisfies*

$$\left| Z_2(t,x,y) - \frac{ae^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}} e^{i\frac{(x^2+y^2)}{4t}}\phi(x)\phi(y)\right| \leq C\min(t^{-\frac{1}{2}}\langle x\rangle^{-1}\langle y\rangle^{-1}, t^{-\frac{3}{2}}) \tag{4.21}$$

for a constant $C > 0$. In particular, $Z_2(t)$ satisfies

$$\|Z_2(t)u\|_{3,\infty} \leq Ct^{-\frac{1}{2}}\|u\|_{\frac{3}{2},1}, \quad u \in L^2 \cap L^{\frac{3}{2},1}. \tag{4.22}$$

Proof. Since $Z_2(t) = e^{-itH}P_c - \Omega(t) - Z_1(t)$, Lemma 4.1 and Lemma 4.9 implies the first statement. The integral kernel $Z_2(t,x,y)$ is given by

$$Z_2(t,x,y) = a\int_{\mathbf{R}^6} c(t,A)\frac{V(z_2)\phi(z_2)V(z_1)\phi(z_1)}{16\pi^2|x-z_2||z_1-y|}dz_1dz_2, \tag{4.23}$$

where $A = |x - z_2| + |z_1 - y|$ and

$$c(t, A) = \frac{1}{i\pi} \int_{\mathbf{R}} e^{-it\lambda^2 + i\lambda A} \chi_l(\lambda) d\lambda = \frac{e^{-i\frac{3\pi}{4}} e^{i\frac{A^2}{4t}}}{\sqrt{\pi t}} \mathcal{F}\left(e^{\frac{is^2}{4t}} \check{\chi}_l\right)\left(\frac{A}{2t}\right). \quad (4.24)$$

We have $|c(t, A)| \leq \|\check{\chi}_l\|_1 (\pi t)^{-\frac{1}{2}}$, hence

$$|Z_2(t, x, y)| \leq \frac{C}{\sqrt{t}} \int_{\mathbf{R}^6} \frac{|V(z_2)\phi(z_2)|}{|x - z_2|} \frac{|V(z_1)\phi(z_1)|}{|z_1 - y|} dz_1 dz_2 \leq \frac{Ct^{-\frac{1}{2}}}{\langle x \rangle \langle y \rangle}. \quad (4.25)$$

Estimate (4.25) implies (4.22). We prove (4.21). Since $|e^{\frac{is^2}{4t}} - 1| \leq |s^2|/4t$ and $|\chi_l(A/t) - 1| \leq C|A/t|$, we have

$$\left|\mathcal{F}\left(e^{\frac{is^2}{4t}} \check{\chi}_l\right)\left(\frac{A}{2t}\right) - 1\right| \leq Ct^{-1}(\|s^2 \check{\chi}_l\|_{L^1} + |A|).$$

If we set $B = 2(|x - z_2||z_2| + |z_1 - y||z_1|) + |z_1|^2 + |z_2|^2$, it is easy to see that $|e^{iA^2/4t} - e^{i(x^2+y^2)/4t}| \leq B/4t$. It follows that

$$\left|c(t, A) - \frac{e^{-i\frac{3\pi}{4}} e^{i\frac{x^2}{4t}} e^{i\frac{y^2}{4t}}}{\sqrt{\pi t}}\right| \leq C(1 + A + B)t^{-\frac{3}{2}}. \quad (4.26)$$

Combine (4.23) and (4.26) and use the relation $(1 + D_0 V)\phi = 0$ and

$$\sup_{x,y} \int_{\mathbf{R}^6} \frac{(1 + A + B)|V(z_2)\phi(z_2)V(z_1)\phi(z_1)|}{|x - z_2||z_1 - y|} dz_1 dz_2 < \infty$$

which follows from $|V(x)\phi(x)| \leq C\langle x \rangle^{-\beta-1}$ with $\beta > 9/2$. We see that the left side of (4.21) is bounded by $Ct^{-\frac{3}{2}}$. Estimate (4.25) and the bound $|\phi(x)| \leq C\langle x \rangle^{-1}$ show it is also bounded by $Ct^{-\frac{1}{2}}\langle x \rangle^{-1}\langle y \rangle^{-1}$. We are done. □

Proof of Theorem 1.3 when H is exceptional type of the first kind. We recall $\Omega(t)$ of Lemma 4.1 and define $\tilde{\Omega}(t) = \Omega(t) + W_{1l}(t) + Z_1(t)$ so that $e^{-itH}P_c = \tilde{\Omega}(t) + Z_2(t)$. By virtue of Lemma 4.9, $\tilde{\Omega}(t)$ is regularly dispersive and $\|\tilde{\Omega}(t)u\|_3 \leq Ct^{-\frac{1}{2}}\|u\|_{\frac{3}{2}}$, in particular. Since $L^{\frac{3}{2},1} \subset L^{\frac{3}{2}}$ and $L^3 \subset L^{3,\infty}$, this and (4.22) imply

$$\|e^{-itH}P_c u\|_{3,\infty} \leq Ct^{-\frac{1}{2}}\|u\|_{\frac{3}{2},1}. \quad (4.27)$$

We interpolate (4.27) with the L^2-bound: $\|e^{-itH}P_c u\|_{2,2} \leq \|u\|_{2,2}$. If we set

$$\frac{1}{q} = \frac{2}{3}(1-\theta) + \frac{\theta}{2}, \quad \frac{1}{p} = \frac{1}{3}(1-\theta) + \frac{\theta}{2}, \quad 0 < \theta < 1,$$

then $2/3 < q < 2 < p < 3$ with $1/p + 1/q = 1$ and, using also $L^{p,q} \subset L^{p,p} = L^p$, we have $[L^{\frac{3}{2},1}, L^2]_{\theta,q} = L^q$, $[L^{3,\infty}, L^{2,2}]_{\theta,q} = L^{p,q} \subset L^p$ (see [3], Theorem 5.3.1) and the desired estimate for this case:

$$\|e^{-itH}P_c u\|_p \leq Ct^{-\frac{1}{2}(1-\theta)}\|u\|_q = Ct^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|u\|_q. \quad (4.28)$$

We next show the estimate corresponding to (1.17):

$$\left\| \left(e^{-itH}P_c - R(t)\right)u\right\|_p \le Ct^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|u\|_q. \tag{4.29}$$

Estimates (4.27) and $|\phi(x)| \le C\langle x\rangle^{-1}$ imply

$$\|(e^{-itH}P_c - R(t))u\|_{3,\infty} \le Ct^{-\frac{1}{2}}\|u\|_{\frac{3}{2},1}. \tag{4.30}$$

By virtue of (4.21), we have $\|(Z_2(t) - R(t))u\|_\infty \le Ct^{-\frac{3}{2}}\|u\|_1$. Combining this with the fact that $\tilde{\Omega}(t) = e^{-itH}P_c - Z_2(t)$ is regularly dispersive, we obtain

$$\|(e^{-itH}P_c - R(t))u\|_\infty \le Ct^{-\frac{3}{2}}\|u\|_1. \tag{4.31}$$

We interpolate (4.30) and (4.31). This time we set

$$\frac{1}{q} = \frac{2}{3}(1-\theta) + \frac{\theta}{1}, \quad \frac{1}{p} = \frac{1}{3}(1-\theta) + \frac{\theta}{\infty}, \quad 0 < \theta < 1,$$

so that $1 < q < 2/3$, $3 < p < \infty$ and $1/p + 1/q = 1$. Then, again using $L^{p,q} \subset L^p$, we have $[L^{\frac{3}{2},1}, L^1]_{\theta,q} = L^q$, $[L^{3,\infty}, L^\infty]_{\theta,q} = L^{p,q} \subset L^p$ and

$$\|(e^{-itH}P_c - R(t))u\|_p \le Ct^{-\frac{1}{2}(1-\theta)-\frac{3}{2}\theta}\|u\|_q = Ct^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|u\|_q, \tag{4.32}$$

which is (4.29). This completes the proof of Theorem 1.3 when H is exceptional type of the first kind.

4.4. Exceptional type of the second kind. In this subsection we prove Theorem 1.3 (2) when H is of exceptional type of the second kind. In view of Lemma 4.1, we need to study $W_{0l}(t)$ only. As previously we begin by studying the resolvent $G(\lambda)$ near $\lambda = 0$. In this case $\mathcal{M}$ coincides with the 0 eigenspace $\mathcal{E}$ of H and all $\phi \in \mathcal{E}$ satisfy

$$\langle V, \phi\rangle = 0, \quad |\phi(x)| \le C\langle x\rangle^{-2}, \text{ hence } \phi \in \mathcal{H}_{\frac{1}{2}-}. \tag{4.33}$$

Theorem 4.11. *Let $|V(x)| \le C\langle x\rangle^{-\beta}$ for some $\beta > 11/2$. Assume that H is of exceptional type of the second kind and let P_0 be the orthogonal projection in $\mathcal{H}$ onto the 0 eigenspace of $H = -\Delta + V$. Then there exists a constant $\lambda_0 > 0$ such that for $0 < |\lambda| < \lambda_0$,*

$$(1 + G_0(\lambda)V)^{-1} = I + K(\lambda) + \lambda^{-2}P_0V + i\lambda^{-1}P_0VD_3VP_0V, \tag{4.34}$$

where $\langle x\rangle^{1+\sigma}VK(\lambda)\langle x\rangle^{1+\sigma}$ is a $\mathbf{B}_2(\mathcal{H})$-valued $C^{1+\rho}$ function of $-\lambda_0 < \lambda < \lambda_0$ (including $\lambda = 0$) for some $\sigma > 1/2$ and $\rho > 1/2$.

Proof. Without losing generality we assume $11/2 < \beta < 6$, which implies $\beta - 4 < (\beta - 1)/2$. We again apply Lemma 4.7 to (4.6) and the argument is parallel to that of the proof of Theorem 4.8. We define

$$\begin{aligned} E_2(\lambda) &= (i\lambda)^{-2}(M(\lambda) - (1 + D_0 V + i\lambda D_1 V)), \\ J_4(\lambda) &= (i\lambda)^{-4}(M(\lambda) - (1 + D_0 V + \cdots + (i\lambda)^3 D_3 V)). \end{aligned} \tag{4.35}$$

It follows from (4.33) (see Lemma 2.2 (2) of [9]) that

$$\begin{aligned} &E_2(\lambda)\phi \in C^{\gamma-\frac{5}{2}-}(\mathbf{R}, \mathcal{H}_{-\gamma+1}), \quad \frac{5}{2} < \gamma < \beta + \frac{1}{2}; \\ &E_2(\lambda)^* V\phi \in C^{\gamma-\frac{5}{2}-}(\mathbf{R}, \mathcal{H}_{\beta-\gamma+1}), \quad \frac{5}{2} < \gamma < \beta + \frac{1}{2}; \\ &J_4(\lambda)\phi \in C^{\gamma-\frac{9}{2}-}(\mathbf{R}, \mathcal{H}_{-\gamma+1}), \quad \frac{9}{2} < \gamma < \beta + \frac{1}{2}. \end{aligned} \tag{4.36}$$

Since $(1 + D_0 V + i\lambda D_1 V)Q = Q(1 + D_0 V + i\lambda D_1 V) = 0$, we have

$$\begin{aligned} &M_{01}(\lambda) = (i\lambda)^2 \overline{Q} E_2(\lambda) Q, \quad M_{10}(\lambda) = (i\lambda)^2 Q E_2(\lambda) \overline{Q}, \\ &M_{11}(\lambda) = (i\lambda)^2 Q E_2(\lambda) Q, \\ &\quad E_2(\lambda) = D_2 V + i\lambda D_3 V + (i\lambda)^2 J_4(\lambda). \end{aligned} \tag{4.37}$$

Take an orthonormal basis $\{\phi_j\}$ of $\mathcal{M}$ and its dual basis $\{-V\phi_j\}$. Then, $QJ_4(\lambda)Q = \sum_{j,k} a_{jk}(\lambda)(\phi_j \otimes V\phi_k)$ with $a_{jk}(\lambda) = \langle V\phi_j, J_4(\lambda)\phi_k\rangle$ and, by choosing γ arbitrarily close to $\beta + 1/2$ in the last relation of (4.36), we see that $a_{jk}(\lambda)$ are of class $C^{\beta-4-}$. By virtue of Lemma 4.2 (2),

$$\begin{aligned} QD_2VQ &= \sum \langle V\phi_j, D_2 V\phi_k\rangle |\phi_j\rangle\langle V\phi_k| \\ &= -\sum \langle D_0 V\phi_j, D_0 V\phi_k\rangle |\phi_j\rangle\langle V\phi_k| = -\sum \langle \phi_j, \phi_k\rangle |\phi_j\rangle\langle \phi_k| V. \end{aligned}$$

The matrix $A = (\langle \phi_j, \phi_k\rangle)$ is positive definite and, if we define $B = A^{-\frac{1}{2}}$ and $\tilde{\phi}_k = \sum_j B_{jk}\phi_j$, then $\{\tilde{\phi}_1, \ldots, \tilde{\phi}_d\}$ becomes an orthonormal basis of $\mathcal{M}$ with respect to the standard L^2 inner product, and

$$(QD_2VQ)^{-1} = -\sum B_{jk}^2 |\phi_j\rangle\langle\phi_k| V = -\sum |\tilde{\phi}_j\rangle\langle\tilde{\phi}_k| V = -P_0 V. \tag{4.38}$$

Since $\langle\tilde{\phi}_j | V\phi_k\rangle = -B_{jk}$, we have $P_0 V Q = P_0 V$. It follows by a Neumann series expansion that

$$\begin{aligned} M_{11}(\lambda)^{-1} &= \lambda^{-2} P_0 V (I - i\lambda Q D_3 V P_0 V + \lambda^2 Q J_4(\lambda) P_0 V)^{-1} Q \\ &= \lambda^{-2} P_0 V + i\lambda^{-1} P_0 V D_3 V P_0 V + Q E_3(\lambda) Q. \end{aligned} \tag{4.39}$$

Here $E_3(\lambda)$ collects all remaining terms in the expansion and, as $J_4(\lambda)$ is of class $C^{\beta-4-}$, if we write as $QE_3(\lambda)Q = \sum b_{jk}(\lambda)\phi_j \otimes V\phi_k$, $b_{jk}(\lambda)$ are also of class $C^{\beta-4-}$. We have

$$M_{10}(\lambda) M_{00}^{-1}(\lambda) M_{01}(\lambda) = \lambda^4 Q E_2(\lambda) \overline{Q} M_{00}^{-1}(\lambda) \overline{Q} E_2(\lambda) Q. \tag{4.40}$$

Since $E_2(\lambda)\phi_k$ and $E_2(\lambda)^* V\phi_j$ satisfy the property (4.36) and $\overline{Q}M_{00}^{-1}(\lambda)\overline{Q}$ is a $\mathbf{B}(\mathcal{H}_{-r})$–valued C^δ function of λ for $1/2 < \delta < \min(\beta - \gamma - 1/2, \gamma - 1/2, \beta - 2)$ by virtue of Lemma 4.6, the matrix elements

$$\langle E_2(\lambda)^* V\phi_j, \overline{Q}M_{00}(\lambda)^{-1}\overline{Q}E_2(\lambda)\phi_k\rangle$$

of (4.40) with respect to these bases are of class $C^{\beta-4}$. We obtain, combining this with (4.39), that

$$M_{10}(\lambda)M_{00}^{-1}(\lambda)M_{01}(\lambda)M_{11}(\lambda)^{-1} = \lambda^2 QE_5(\lambda)Q$$

with $E_5(\lambda)$ which has $C^{\beta-4-}$ matrix elements. It follows that

$$C(\lambda) = M_{11}(\lambda) - M_{10}(\lambda)M_{00}^{-1}(\lambda)M_{01}(\lambda) = (I - \lambda^2 QE_5(\lambda)Q)M_{11}(\lambda)$$

is invertible for $\lambda \neq 0$ and

$$\begin{aligned} C^{-1}(\lambda) &= M_{11}(\lambda)^{-1}(I - \lambda^2 QE_5(\lambda)Q)^{-1} \\ &= \lambda^{-2}P_0V + i\lambda^{-1}P_0VD_3VP_0V + QE_6(\lambda)Q \end{aligned} \tag{4.41}$$

with $C^{\beta-4-}$ function $E_6(\lambda)$. From (4.36) and Lemma 4.6, it also follows that

$$\langle x\rangle^{-\beta+\frac{5}{2}-}\overline{Q}M_{00}(\lambda)^{-1}\overline{Q}E_2(\lambda)Q, \quad QE_2(\lambda)\overline{Q}M_{00}\overline{Q}\langle x\rangle^{\frac{5}{2}+} \in C^{\beta-4-}(\mathbf{R}, \mathbf{B}_2(\mathcal{H})).$$

Then, by virtue of (4.41), we see that the operators

$$\begin{aligned} &-M_{00}^{-1}(\lambda)M_{01}(\lambda)C^{-1}(\lambda), \quad -C^{-1}(\lambda)M_{01}(\lambda)M_{00}^{-1}(\lambda), \\ &M_{00}^{-1}(\lambda)M_{01}(\lambda)C^{-1}(\lambda)M_{01}(\lambda)M_{00}^{-1}(\lambda) \end{aligned} \tag{4.42}$$

are, when sandwiched by $\langle x\rangle^{-\beta+\frac{5}{2}-}$ and $\langle x\rangle^{\frac{5}{2}-}$ from the left and the right respectively, all $\mathbf{B}_2(\mathcal{H})$-valued $C^{\beta-4-}$ functions. Since $\beta > 11/2$, putting the operators in (4.42), $QE_6(\lambda)Q$ and $M_{00} - I$ into $K(\lambda)$, we obtain the theorem. □

Now we are ready to study $W_{0l}(t)$:

$$W_{0l}(t) = \lim_{\delta\downarrow 0}\frac{-1}{i\pi}\int_{|\lambda|>\delta} e^{-it\lambda^2}\chi_l(\lambda)G_0(\lambda)VG(\lambda)\lambda d\lambda \tag{4.43}$$

in the case when H is an exceptional type of the second kind. We may choose the cut off function $\chi_l(\lambda)$ such that $\chi(\lambda) = 0$ for $\lambda > \lambda_0$ and $\chi_l(\lambda) = 1$ for $|\lambda| < \lambda_0/2$ as previously. By virtue of (4.34), we have

$$\begin{aligned} G_0(\lambda)VG(\lambda) = {}& G_0(\lambda)VG_0(\lambda) + G_0(\lambda)VK(\lambda)G_0(\lambda) \\ &+\lambda^{-2}G_0(\lambda)VP_0VG_0(\lambda) + i\lambda^{-1}G_0(\lambda)VP_0VD_3VP_0VG_0(\lambda). \end{aligned} \tag{4.44}$$

The contribution of $G_0(\lambda)VG_0(\lambda)$ to $W_{0l}(t)$ is equal to $W_{1l}(t)$ and it is regularly dispersive. We denote the contribution from $G_0(\lambda)VK(\lambda)G_0(\lambda)$ by $X_1(t)$, which corresponds to $Z_1(t)$ in the first case. By virtue of Theorem 4.11 $\langle x\rangle^{\sigma+1}VK(\lambda)\langle x\rangle^{\sigma+1}$ is a $\mathbf{B}_2(\mathcal{H})$-valued $H^{\frac{3}{2}+}$ function of $|\lambda| < \lambda_0$. It follows by the argument used for studying $Z_1(t)$ of the previous subsection that $X_1(t)$ is regularly dispersive. Let $X_2(t)$ and $X_3(t)$ respectively be the contributions from the fourth and the third summands:

$$X_2(t) = \frac{-1}{\pi} \int_{\mathbf{R}} e^{-it\lambda^2} \chi_l(\lambda) G_0(\lambda) V P_0 V D_3 V P_0 V G_0(\lambda) d\lambda, \tag{4.45}$$

$$X_3(t) = \frac{-1}{i\pi} \lim_{\delta \downarrow 0} \int_{|\lambda|>\delta} e^{-it\lambda^2} \lambda^{-1} \chi_l(\lambda) G_0(\lambda) V P_0 V G_0(\lambda) d\lambda. \tag{4.46}$$

A priori we know that $X_2(t) + X_3(t)$ is a strongly continuous family of uniformly bounded operators in $\mathcal{H}$:

$$\|(X_2(t) + X_3(t))u\|_2 \le C\|u\|_2, \quad t \in \mathbf{R}, \tag{4.47}$$

as it may be written as a sum of operators which satisfy this property.

Lemma 4.12. *There exists C such that*

$$\|X_2(t)u\|_{3,\infty} \le Ct^{-\frac{1}{2}} \|u\|_{\frac{3}{2},1}, \quad u \in L^2 \cap L^{\frac{3}{2},1}, \tag{4.48}$$

$$\left\| X_2(t)u + i\frac{e^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}} P_0 V D_3 P_0 u \right\|_\infty \le Ct^{-\frac{3}{2}} \|u\|_1, \quad u \in L^2 \cap L^1. \tag{4.49}$$

Proof. We let $\{\tilde{\phi}_j\}$ be an orthonomal basis of $\mathcal{E}$ with respect to the L^2-norm. With $c_{jk} = \langle \tilde{\phi}_j, V D_3 V \tilde{\phi}_k \rangle$ we write

$$P_0 V D_3 V P_0 = \sum_{j,k=1}^{d} c_{jk} |\tilde{\phi}_j\rangle\langle\tilde{\phi}_k|, \quad c_{jk} = \langle\tilde{\phi}_j | V D_3 V | \tilde{\phi}_k\rangle.$$

We define

$$W_{jk}(t) = \frac{-1}{\pi} \int_{\mathbf{R}} e^{-it\lambda^2} \chi_l(\lambda) G_0(\lambda) V |\tilde{\phi}_j\rangle\langle\tilde{\phi}_k| V G_0(\lambda) d\lambda.$$

Notice that $W_{jk}(t)$ is exactly of the same form as $Z_2(t)$ except that a is replaced by $-i$ and the resonance ϕ by the eigenfunctions $\tilde{\phi}_j$ and $\tilde{\phi}_k$. It follows by the argument which led to (4.25) that the integral kernel $W_{jk}(t, x, y)$ of $W_{jk}(t)$ satisfies

$$\left|W_{jk}(t, x, y)\right| \le C|t|^{-\frac{1}{2}} \langle x\rangle^{-1} \langle y\rangle^{-1}, \tag{4.50}$$

which implies (4.48). It also implies

$$\left| W_{jk}(t, x, y) + i\frac{e^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}} e^{i\frac{(x^2+y^2)}{4t}} \tilde{\phi}_j(x)\tilde{\phi}_k(y) \right| \le C|t|^{-\frac{3}{2}}. \tag{4.51}$$

Here, however, as eigenfunctions decay faster than resonances and $|\tilde{\phi}_j(x)| \le C\langle x\rangle^{-2}$, we may estimate $\left|(e^{i\frac{x^2}{4t}} - 1)\tilde{\phi}_j(t, x)\right| \le Ct^{-1}$. It follows that

$$\left\| W_{jk}(t)u + i\frac{e^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}} (u, \tilde{\phi}_k)\tilde{\phi}_j \right\|_\infty \le Ct^{-\frac{3}{2}} \|u\|_1. \tag{4.52}$$

Summing up (4.52) with respect to j, k, we obtain the lemma. □

Lemma 4.13. *For $\phi \in \mathcal{E}$, a zero eigenfunction of H, define*

$$\tilde{w}(t,x) = \lim_{\delta \downarrow 0} \frac{1}{i\pi} \int_{|\lambda|>\delta} e^{-it\lambda^2} \chi_l(\lambda)(G_0(\lambda) - D_0)V\phi(x) \frac{d\lambda}{\lambda}. \tag{4.53}$$

Then $\tilde{w}(t,x)$ satisfies the following properties:

$$|\tilde{w}(t,x)| \le Ct^{-\frac{1}{2}} \langle x \rangle^{-1}, \tag{4.54}$$

$$|\tilde{w}(t,x) - e^{-i\frac{3\pi}{4}} \frac{\mu(t,x)}{\sqrt{\pi t}} (D_2 V\phi)(x)| \le Ct^{-\frac{3}{2}}, \tag{4.55}$$

where $\mu(t,x)$ is the function defined by (1.12)*:*

$$\mu(t,x) = \frac{i}{|x|} \int_0^1 (e^{\frac{i|x|^2}{4t}} - e^{\frac{i\theta^2|x|^2}{4t}}) d\theta. \tag{4.56}$$

Proof. Since ϕ satisfies $\langle 1, V\phi \rangle = 0$, we may write

$$(G_0(\lambda) - D_0)V\phi(x) = \frac{1}{4\pi} \int \left(\frac{e^{i\lambda|x-y|} - 1}{|x-y|} - \frac{e^{i\lambda|x|} - 1}{|x|} \right) V(y)\phi(y) dy. \tag{4.57}$$

We write the function inside the parenthesis under the integral sign in the form

$$\frac{i\lambda}{|x|}(|x-y| - |x|) \int_0^1 (e^{i\lambda(\theta|x-y|+(1-\theta)|x|)} - e^{i|x-y|\lambda\theta}) d\theta. \tag{4.58}$$

After rewriting $(G_0(\lambda) - D_0)V\phi(x)$ in this way, we compute the right-hand side of (4.53) by first performing the λ integral as always. If we set $A = \theta|x-y| + (1-\theta)|x|$ and $B = \theta|x-y|$, we have

$$\frac{1}{i\pi} \int_{\mathbf{R}} e^{-it\lambda^2} (e^{i\lambda A} - e^{i\lambda B}) \chi_l(\lambda) d\lambda = c(t,A) - c(t,B),$$

where $c(t,X)$ is defined by (4.24):

$$c(t,X) = \frac{e^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}} e^{\frac{iX^2}{4t}} \mathcal{F}\left(e^{\frac{is^2}{4t}} \check{\chi}_l \right) \left(\frac{X}{2t} \right),$$

and $\tilde{w}(t,x)$ may now be written in the form

$$\frac{i}{4\pi|x|} \int_0^1 \left(\int (|x-y| - |x|)(c(t,A) - c(t,B))V(y)\phi(y) dy \right) d\theta. \tag{4.59}$$

Since $|c(t,X)| \le Ct^{-\frac{1}{2}}$ and $||x-y| - |x|| \le |y|$, (4.59) clearly implies

$$|\tilde{w}(t,x)| \le C|x|^{-1} t^{-\frac{1}{2}}.$$

However, the choice of origin is arbitrary and we obtain (4.54).

Since $|A^2 - |x|^2| = \theta|(|x-y| - |x|)(\theta(|x-y| - |x|) + 2|x|)| \le 2|y|(|x| + |y|)$, the argument which leads to (4.26) implies uniformly with respect to θ,

$$\left| c(t,A) - \frac{e^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}} e^{\frac{i|x|^2}{4t}} \right| \le C(|A| + \|s^2\check{\chi}\|_1 + |y|(|x| + |y|)) t^{-\frac{3}{2}} \le C\langle x \rangle \langle y \rangle^2 t^{-\frac{3}{2}}.$$

Likewise, we have $||x-y|^2 - |x|^2| \le 2|y|(|x|+|y|)$ and

$$\left| c(t,B) - \frac{e^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}} e^{\frac{i\theta^2|x|^2}{4t}} \right| \le C\langle x\rangle\langle y\rangle^2 t^{-\frac{3}{2}}.$$

Note that $|\langle y\rangle^3 V(y)\phi(y)| \le C\langle y\rangle^{-\beta+1}$ is integrable by the assumption $\beta > 11/2$. It follows that $\tilde{w}(t,x)$ differs from

$$\frac{ie^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}|x|}\left(\int_0^1 (e^{\frac{i|x|^2}{4t}} - e^{\frac{i\theta^2|x|^2}{4t}})d\theta\right)\left(\frac{1}{4\pi}\int (|x-y|-|x|)V(y)\phi(y)dy\right)$$

by a function bounded by $Ct^{-\frac{3}{2}}$. Here the function in the second parenthesis is equal to $(D_2V\phi)(x)$ because $\langle V,\phi\rangle = 0$. We have obtained (4.55). □

Lemma 4.14. *Let $\mu(t)$ be the multiplication by $\mu(t,x)$. Then, there exists C such that*

$$\|X_3(t)u\|_{3,\infty} \le Ct^{-\frac{1}{2}}\|u\|_{3/2,1}, \tag{4.60}$$

$$\left\| X_3(t)u - \frac{e^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}}\left(\mu(t)D_2VP_0 + P_0VD_2\mu(t)\right)u \right\|_\infty \le Ct^{-\frac{3}{2}}\|u\|_1. \tag{4.61}$$

Proof. Using $D_0VP_0 = -P_0$ and $P_0VD_0 = -P_0$, which follows since the 0 eigenfunctions ϕ of H satisfy $D_0V\phi = -\phi$, we may write

$$\begin{aligned} G_0(\lambda)VP_0VG_0(\lambda) = (G_0(\lambda)-D_0)VP_0V(G_0(\lambda)-D_0)\\ -(G_0(\lambda)-D_0)VP_0 - P_0V(G_0(\lambda)-D_0) + P_0. \end{aligned}$$

This produces $X_3(t) = X_{31}(t) + X_{32}(t) + X_{33}(t)$ where

$$X_{31}(t) = \frac{i}{\pi}\lim_{\delta\downarrow 0}\int_{|\lambda|>\delta} e^{-it\lambda^2}\chi_l(\lambda)(G_0(\lambda)-D_0)VP_0V(G_0(\lambda)-D_0)\frac{d\lambda}{\lambda}, \tag{4.62}$$

$$X_{32}(t) = \frac{1}{i\pi}\lim_{\delta\downarrow 0}\int_{|\lambda|>\delta} e^{-it\lambda^2}\chi_l(\lambda)(G_0(\lambda)-D_0)VP_0\frac{d\lambda}{\lambda}, \tag{4.63}$$

$$X_{33}(t) = \frac{1}{i\pi}\lim_{\delta\downarrow 0}\int_{|\lambda|>\delta} e^{-it\lambda^2}\chi_l(\lambda)P_0V(G_0(\lambda)-D_0)\frac{d\lambda}{\lambda}. \tag{4.64}$$

Here the contribution from P_0 vanishes because $e^{-it\lambda^2}\lambda^{-1}\chi_l(\lambda)$ is an odd function of λ. We take an orthonormal basis $\{\phi_1,\ldots,\phi_d\}$ of $\mathcal{E}$ with respect to the L^2 inner product and let $\tilde{w}_j(t,x)$ be the $\tilde{w}(t,x)$ of Lemma 4.13 corresponding to ϕ_j, $j=1,\ldots,d$. Then the integral kernels of $X_{32}(t)$ and $X_{33}(t)$ are given respectively by

$$X_{32}(t,x,y) = \sum_{j=1}^d \tilde{w}_j(t,x)\phi_j(y), \quad X_{33}(t,x,y) = \sum_{j=1}^d \phi_j(x)\tilde{w}_j(t,y),$$

and, by virtue of Lemma 4.13, the lemma follows if we prove

$$\|X_{31}(t)u\|_{3,\infty} \le Ct^{-\frac{1}{2}}\|u\|_{3/2,1}, \quad \|X_{31}(t)u\|_\infty \le Ct^{-\frac{3}{2}}\|u\|_1. \tag{4.65}$$

By using (4.57) and (4.58), we write the integral kernel of $X_{31}(t)$ in the following form. We define

$$a(t, A) = \frac{i}{\pi} \int_{\mathbf{R}} e^{-it\lambda^2 + i\lambda A} \lambda \chi_l(\lambda) d\lambda$$

and use the short-hand notation

$$L(x, y) = |x - y| - |x|, \quad \psi_j(x) = -V(x)\phi_j(x), \ j = 1, \dots, d.$$

Note that $|L(x, y)| \leq |y|$. If we define $Y_{kj}(t, x, y, \theta, \theta')$ for $k = 1, \dots, 4$ and $j = 1, \dots, d$ by

$$Y_{kj} = \frac{-1}{16\pi^2 |x||y|} \int_{\mathbf{R}^6} L(x, z_2) L(y, z_1) \psi_j(z_2) \psi_j(z_1) a(t, A_k) dz_1 dz_2,$$

where the variables $A_1, \dots, A_4$ inside $a(t, A_k)$ are respectively given by

$$A_1 = \theta|x - z_2| + \theta'|y - z_1| + (1 - \theta')|y|, \quad A_2 = \theta|x - z_2| + \theta'|y - z_1|,$$
$$A_3 = \theta|x - z_2| + (1 - \theta)|x| + \theta'|y - z_1|,$$
$$A_4 = \theta|x - z_2| + (1 - \theta)|x| + \theta'|y - z_1| + (1 - \theta')|y|,$$

then, the integral kernel of $X_{31}(t)$ may be written in the form

$$X_{31}(t, x, y) = \sum_{k=1}^{4} \sum_{j=1}^{d} (-1)^k \int_0^1 \int_0^1 Y_{kj}(t, x, y, \theta, \theta') d\theta d\theta'. \tag{4.66}$$

Clearly $|A_k| \leq (\langle x \rangle + \langle z_2 \rangle + \langle z_1 \rangle + \langle y \rangle)$, $k = 1, \dots, 4$ and

$$|a(t, A)| \leq Ct^{-\frac{1}{2}}, \quad |a(t, A)| \leq Ct^{-\frac{3}{2}} |A|, \tag{4.67}$$

by virtue of (2.5) and (2.7). It follows that

$$|X_{31}(t, x, y)| \leq C \min \left(\frac{t^{-\frac{1}{2}}}{|x||y|}, \frac{t^{-\frac{3}{2}} (\langle x \rangle + \langle y \rangle)}{|x||y|} \right). \tag{4.68}$$

Here again the choice of the origin of coordinates is irrelevant for the estimate and we may replace $t^{-\frac{1}{2}}(1/|x||y|)$ by $Ct^{-\frac{1}{2}}(1/\langle x \rangle \langle y \rangle)$ and $t^{-\frac{3}{2}}(\langle x \rangle + \langle y \rangle / |x||y|)$ by $t^{-\frac{3}{2}}(\langle x \rangle + \langle y \rangle / \langle x \rangle \langle y \rangle)$ in (4.68) and (4.65) follows. This completes the proof of the lemma. □

Proof of Theorem 1.3 when H is exceptional type of the second kind. We have shown that $e^{-itH} P_c - (X_2(t) + X_3(t))$ is regular dispersive. It follows by virtue of Lemma 4.12 and Lemma 4.14,

$$\|e^{-itH} P_c u\|_{3,\infty} \leq Ct^{-\frac{1}{2}} \|u\|_{\frac{3}{2},1}, \tag{4.69}$$

$$\|(e^{-itH} P_c - R(t))u\|_{3,\infty} \leq Ct^{-\frac{1}{2}} \|u\|_{\frac{3}{2},1}, \tag{4.70}$$

$$\|(e^{-itH} P_c - R(t))u\|_{\infty} \leq Ct^{-\frac{3}{2}} \|u\|_1. \tag{4.71}$$

We interpolate (4.69) with the L^2 bound $\|e^{-itH} P_c u\|_2 \leq \|u\|_2$ and (4.70) with (4.71). The argument is virtually a repetition of the corresponding part of the previous subsection and we omit the details.

4.5. Exceptional case of the third kind. We finally consider the case when H is of exceptional type of the third kind. As usual we begin by studying $M(\lambda)^{-1} = (1+G_0(\lambda)V)^{-1}$ near $\lambda = 0$. We take the orthonormal (with respect to the inner product $-(Vu, v)$) basis $\{\phi_1, \dots, \phi_d\}$ of $\mathcal{M}$ of Lemma 4.5 in such a way that $\{\phi_2, \dots, \phi_d\}$ is a basis of $P_0\mathcal{H}$ and such that $\langle \phi_1, V\rangle > 0$. The last condition determines ϕ_1 uniquely. Define the orthogonal projections π_1 onto $\{\phi_1\}$ and π_2 onto $P_0\mathcal{H}$ with respect to this inner product, viz. $\pi_1 = -|\phi_1\rangle\langle V\phi_1|$ and $\pi_2 = -\sum_{j=2}^{d} |\phi_j\rangle\langle V\phi_j|$, and

$$Q_0 = \overline{Q} = 1 - Q, \quad Q_1 = Q\pi_1 Q, \quad Q_2 = Q\pi_2 Q.$$

We have $Q = Q_1 + Q_2$. As previously we write $\psi_j = -V\phi_j : j = 1, \dots, d$. $\{\psi_j\}$ is the basis of $\mathcal{N} = \mathcal{M}^*$ which is dual to $\{\phi_j\}$.

Lemma 4.15. *As identities in $\mathcal{H}_{-\gamma}$, we have the following:*

$$Q_j Q_k = \delta_{jk} \ \ (j, k = 0, 1, 2) \ \text{and} \ \ Q_0 + Q_1 + Q_2 = I, \tag{4.72}$$
$$(1 + D_0 V)Q_1 = (1 + D_0 V)Q_2 = 0, \tag{4.73}$$
$$Q_2 D_1 V Q_0 = 0, \quad Q_2 D_1 V Q_1 = 0, \quad Q_2 D_1 V Q_2 = 0, \tag{4.74}$$
$$Q_0 D_1 V Q_2 = 0, \quad Q_1 D_1 V Q_2 = 0. \tag{4.75}$$

Proof. Equations (4.72) and (4.73) are obvious. Since $D_1 = (1/4\pi)|1\rangle\langle 1|$, (4.74) and (4.75) follow from $Q_2|1\rangle = 0$ and $\langle 1|VQ_2 = 0$. □

We first study $[QM(\lambda)Q]^{-1}$ by using Lemma 4.7. We write $QM(\lambda)Q$ in matrix form with respect to the decomposition $\mathcal{M} = Q_1\mathcal{M} \dotplus Q_2\mathcal{M}$:

$$QM(\lambda)Q = \begin{pmatrix} Q_1 M(\lambda) Q_1 & Q_1 M(\lambda) Q_2 \\ Q_2 M(\lambda) Q_1 & Q_2 M(\lambda) Q_2 \end{pmatrix} \equiv \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix}. \tag{4.76}$$

In what follows we assume $11/2 < \beta < 6$ so that $\beta - 4 < \frac{1}{2}(\beta - 1)$ and *irrespectively denote by $E(\lambda)$ various finite dimensional operator valued functions of λ which are of class $C^{\beta-4-}$ in a neighborhood of $\lambda = 0$.*

The function $\langle V\phi_1|G_0(\lambda)|V\phi_1\rangle$ is of class $C^{\beta-1-}$ because $V\phi_1 \in \mathcal{H}_{\beta-\frac{1}{2}-}$. Since ϕ_1 satisfies $(1 + D_0 V)\phi_1 = 0$ and $\langle V, \phi_1\rangle \neq 0$, it follows as in the case of the first type that with $c_1 \in C^{\beta-3-}$,

$$M_{11}(\lambda) = c(\lambda)Q_1 \text{ with } c(\lambda) = (4\pi i)^{-1}\lambda|\langle V, \phi_1\rangle|^2 + \lambda^2 c_1(\lambda).$$

Hence $M_{11}(\lambda)$ is invertible for $0 < |\lambda| < \lambda_0$ for sufficiently small $\lambda_0 > 0$ and, with $a = 4\pi i|\langle V, \phi_1\rangle|^{-2}$ as previously,

$$M_{11}^{-1}(\lambda) = (\lambda^{-1}a + d(\lambda))Q_1, \quad d \in C^{\beta-3-}. \tag{4.77}$$

Likewise $M_{12}(\lambda)$ and $M_{21}(\lambda)$ are of $C^{\beta-1-}$ and, as $Q_2 D_1 V = D_1 V Q_2 = 0$,

$$M_{12}(\lambda) = -\lambda^2 Q_1(D_2 V + \lambda E(\lambda))Q_2,$$
$$M_{21}(\lambda) = -\lambda^2 Q_2(D_2 V + \lambda E(\lambda))Q_1,$$
$$M_{21}(\lambda)M_{11}^{-1}(\lambda)M_{12}(\lambda) = \lambda^3 Q_2(aD_2 V Q_1 D_2 V + \lambda E(\lambda))Q_2. \tag{4.78}$$

Since $V\phi_j(x) \in \mathcal{H}_{\beta+\frac{1}{2}-}$ for $2 \le j \le d$, $M_{22}(\lambda)$ is of class $C^{\beta-}$ and

$$M_{22}(\lambda) = -\lambda^2 Q_2(D_2 V + i\lambda D_3 V - \lambda^2 E(\lambda))Q_2. \tag{4.79}$$

Notice that $M_{22}(\lambda)$ is what corresponds to $M_{11}(\lambda)$ of the previous Subsect. 4.4. Hence (4.38) and (4.39) imply, with P_0 being the orthogonal projection in $\mathcal{H}$ onto $\mathcal{E}$ that $(Q_2D_2VQ_2)^{-1} = -P_0V$, $P_0VQ_2 = P_0V$ and that

$$M_{22}(\lambda)^{-1} = \lambda^{-2}P_0V + i\lambda^{-1}P_0VD_3VP_0V + P_0VE(\lambda)Q_2. \tag{4.80}$$

It follows by a Neumann series expansion that

$$\begin{aligned} C_{22}(\lambda) &= M_{22}(\lambda) - M_{21}(\lambda)M_{11}^{-1}(\lambda)M_{12}(\lambda) \\ &= M_{22}(\lambda)(1 - M_{22}(\lambda)^{-1}M_{21}(\lambda)M_{11}^{-1}(\lambda)M_{12}(\lambda)) \end{aligned}$$

is invertible and

$$\begin{aligned} C_{22}^{-1}(\lambda) &= \lambda^{-2}P_0V + i\lambda^{-1}P_0VD_3VP_0V \\ &\quad + a\lambda^{-1}P_0VD_2VQ_1D_2VP_0V + P_0VE(\lambda)P_0V. \end{aligned} \tag{4.81}$$

If we set $\tilde{\phi}_1 = P_0VD_2V\phi_1 \in P_0\mathcal{H}$, then $P_0VD_2VQ_1D_2VP_0V = -|\tilde{\phi}_1\rangle\langle\tilde{\phi}_1|V$ and the right side of (4.81) may be written in the form

$$\lambda^{-2}P_0V + i\lambda^{-1}P_0VD_3VP_0V - \lambda^{-1}a|\tilde{\phi}_1\rangle\langle\tilde{\phi}_1|V + P_0VE(\lambda)P_0V. \tag{4.82}$$

Using (4.77), (4.78), (4.81) and the definition of $\tilde{\phi}_1$, we may write

$$\begin{aligned} -M_{11}^{-1}(\lambda)M_{12}(\lambda)C_{22}^{-1}(\lambda) &= -a\lambda^{-1}|\phi_1\rangle\langle\tilde{\phi}_1|V + E(\lambda), \\ -C_{22}^{-1}(\lambda)M_{21}(\lambda)M_{11}^{-1}(\lambda) &= -a\lambda^{-1}|\tilde{\phi}_1\rangle\langle\phi_1|V + E(\lambda), \\ M_{11}^{-1}(\lambda)M_{12}(\lambda)C_{22}^{-1}(\lambda)M_{21}(\lambda)M_{11}^{-1}(\lambda) &= E(\lambda). \end{aligned} \tag{4.83}$$

Combining (4.77), (4.82) and (4.83) by means of Lemma 4.7, we see that $(QM(\lambda)Q)^{-1}$ is in matrix form given modulo an $E(\lambda)$ by

$$\begin{pmatrix} -a\lambda^{-1}|\phi_1\rangle\langle V\phi_1| & -a\lambda^{-1}|\phi_1\rangle\langle V\tilde{\phi}_1| \\ -a\lambda^{-1}|\tilde{\phi}_1\rangle\langle V\phi_1| & \lambda^{-2}P_0V + i\lambda^{-1}P_0VD_3VP_0V - \lambda^{-1}a|\tilde{\phi}_1\rangle\langle V\tilde{\phi}_1| \end{pmatrix} \tag{4.84}$$

and, therefore, if we define the canonical resonance $\varphi = \phi_1 - \tilde{\phi}_1$ as in (1.10), φ still satisfies $\varphi \in \mathcal{M}$ and $\langle\varphi, V\rangle = 1$, and we obtain

$$(QM(\lambda)Q)^{-1} = \frac{P_0V}{\lambda^2} + \frac{iP_0VD_3VP_0V}{\lambda} - \frac{a}{\lambda}|\varphi\rangle\langle\varphi|V + E(\lambda). \tag{4.85}$$

For studying $M(\lambda)^{-1}$ we repeat a similar argument. We write $M(\lambda)$ in the matrix form with respect to the decomposition $\mathcal{H}_{-\gamma} = \overline{Q}\mathcal{H}_{-\gamma} \dot{+} \mathcal{M}$:

$$M(\lambda) = \begin{pmatrix} \overline{Q}M(\lambda)\overline{Q} & \overline{Q}M(\lambda)Q \\ QM(\lambda)\overline{Q} & QM(\lambda)Q \end{pmatrix} \equiv \begin{pmatrix} L_{00}(\lambda) & L_{01}(\lambda) \\ L_{10}(\lambda) & L_{11}(\lambda) \end{pmatrix},$$

where the right-hand side is the definition. By virtue of Lemma 4.6, for any $1/2 < \gamma < \beta - 1/2$, $A(\lambda) \equiv L_{00}(\lambda)^{-1}$ exists in $\overline{Q}\mathcal{H}_{-\gamma}$ and of class C^δ for any $\delta < \min(\beta - \gamma - 1/2, \gamma - 1/2, \beta - 2)$ and $A(\lambda) - \overline{Q}$ is of Hilbert-Schmidt class. By virtue of (4.73), (4.74) and (4.75), with respect to the decomposition $Q = Q_1 + Q_2$, $L_{10}(\lambda)L_{00}^{-1}(\lambda)L_{01}(\lambda) = QM(\lambda)A(\lambda)M(\lambda)Q$ may be written as

$$\begin{pmatrix} Q_1 M(\lambda)A(\lambda)M(\lambda)Q_1 & Q_1 M(\lambda)A(\lambda)M(\lambda)Q_2 \\ Q_1 M(\lambda)A(\lambda)M(\lambda)Q_1 & Q_2 M(\lambda)A(\lambda)M(\lambda)Q_2 \end{pmatrix} = \begin{pmatrix} \lambda^2 E_{11}(\lambda) & \lambda^3 E_{12}(\lambda) \\ \lambda^3 E_{21}(\lambda) & \lambda^4 E_{22}(\lambda) \end{pmatrix},$$

where E_{ij} are of class $C^{\beta-4-}$. Since $L_{11}^{-1}(\lambda) = (QM(\lambda)Q)^{-1}$ is of the form

$$L_{11}^{-1}(\lambda) = \begin{pmatrix} \lambda^{-1}E(\lambda) & \lambda^{-1}E(\lambda) \\ \lambda^{-1}E(\lambda) & \lambda^{-2}E(\lambda) \end{pmatrix} \tag{4.86}$$

by virtue of (4.84), in the decomposition in $\mathcal{M} = Q_1\mathcal{M} \dot{+} Q_2\mathcal{M}$,

$$N(\lambda) \equiv L_{11}^{-1}(\lambda)L_{10}(\lambda)L_{00}^{-1}(\lambda)L_{01}(\lambda) = \begin{pmatrix} \lambda E(\lambda) & \lambda^2 E(\lambda) \\ \lambda E(\lambda) & \lambda^2 E(\lambda) \end{pmatrix}.$$

It follows that $C(\lambda) = L_{11}(\lambda) - L_{10}(\lambda)L_{00}^{-1}(\lambda)L_{01}(\lambda) = L_{11}(\lambda)(1 - N(\lambda))$ is invertible for $0 < |\lambda| < \lambda_0$,

$$C^{-1}(\lambda) = L_{11}^{-1}(\lambda) + (1 - N(\lambda))^{-1}N(\lambda)L_{11}^{-1}(\lambda) \tag{4.87}$$

and $(1 - N(\lambda))^{-1}N(\lambda)L_{11}^{-1}(\lambda)$ is of the form

$$\begin{pmatrix} \lambda E(\lambda) & \lambda^2 E(\lambda) \\ \lambda E(\lambda) & \lambda^2 E(\lambda) \end{pmatrix} \begin{pmatrix} \lambda^{-1}E(\lambda) & \lambda^{-1}E(\lambda) \\ \lambda^{-1}E(\lambda) & \lambda^{-2}E(\lambda) \end{pmatrix} = E(\lambda). \tag{4.88}$$

We have $L_{01}(\lambda) = \overline{Q}M(\lambda)Q = \lambda\overline{Q}F_1(\lambda)Q_1 + \lambda^2\overline{Q}F_2(\lambda)Q_2$ with

$$F_1(\lambda) = \lambda^{-1}G_0(\lambda)VQ_1, \quad F_2(\lambda) = \lambda^{-2}G_0(\lambda)VQ_2(\lambda).$$

Here $\lambda^{-1}G_0(\lambda)V\phi_1$ is an $\mathcal{H}_{-\gamma}$-valued $C^{\gamma-3/2-}$ function of λ for any $3/2 < \gamma < \beta - 1/2$ and, as in (4.36), $\lambda^{-2}G_0(\lambda)V\phi_j$, $2 \le j \le d$, are $\mathcal{H}_{-\gamma+1}$–valued $C^{\gamma-5/2-}$ functions for any $5/2 < \gamma < \beta + 1/2$. It follows by applying Lemma 4.6 for $L_{00}(\lambda)$ respectively with $\gamma = \beta - 2 - \varepsilon$ and with $\gamma = \beta - 1 - \varepsilon$ with $0 < \varepsilon < \frac{1}{2}$ that $A(\lambda)\overline{Q}F_1(\lambda)\phi_1$ and $A(\lambda)\overline{Q}F_1(\lambda)\phi_j$, $2 \le j \le d$ are $B(\mathcal{M}, \mathcal{H}_{-\beta+2+\varepsilon})$-valued $C^{\beta-4}$ functions of λ (recall that $A(\lambda) = L_{00}^{-1}(\lambda)$). Combining this with (4.86), (4.87) and (4.88), we conclude that

$$A(\lambda)L_{01}(\lambda)C^{-1}(\lambda) = \left(\lambda A(\lambda)\overline{Q}F_1(\lambda)Q_1 \;\; \lambda^2 A(\lambda)\overline{Q}F_2(\lambda)Q_2\right) \begin{pmatrix} \lambda^{-1}E(\lambda) & \lambda^{-1}E(\lambda) \\ \lambda^{-1}E(\lambda) & \lambda^{-2}E(\lambda) \end{pmatrix}$$

is a $\mathbf{B}(\mathcal{M}, \mathcal{H}_{-\beta+2+\varepsilon})$–valued $C^{\beta-4-}$ function of λ near $\lambda = 0$. By an argument dual to the previous one, we see that

$$C^{-1}(\lambda)L_{10}(\lambda)L_{00}^{-1}(\lambda) \text{ is also of class } C^{\beta-4-}$$

as a $\mathbf{B}(\mathcal{H}_{-2-\varepsilon}, \mathcal{M})$–valued function of λ near the origin. Summarizing the results by using Lemma 4.7, we have shown the following theorem:

Theorem 4.16. *Suppose V satisfies $|V(x)| \le C\langle x\rangle^{-\beta}$ with $\frac{11}{2} < \beta$ and H is of exceptional type of the third kind. Let φ be the canonical resonance and $a = 4\pi i|\langle V, \varphi\rangle|^{-2}$. Then,*

$$(I + G_0(\lambda)V)^{-1} - I = \frac{P_0 V}{\lambda^2} + \frac{i P_0 V D_3 V P_0 V}{\lambda} - \frac{a}{\lambda}|\varphi\rangle\langle\varphi|V + K(\lambda), \quad (4.89)$$

where $K(\lambda)$ is such that $\langle x\rangle^{1+\sigma} V K(\lambda)\langle x\rangle^{1+\sigma}$ is a $\mathbf{B}_2(\mathcal{H})$–valued C^{1+s} function of λ in a neighbourhood of λ for some $\sigma, s > 1/2$.

Once Theorem 4.16 is obtained, the proof of Theorem 1.3 for the case H is an exceptional type of the third kind completed by combining the arguments in the preceding two subsections. We may safely omit the repetitious proof.

4.6. Dispersive estimates. Finally we prove Theorem 1.4. We may assume H is an exceptional type of third kind. We have $|\zeta(t,x) - \varphi(x)| + |\mu(t,x)| \le C \min\left(\frac{|x|}{t}, \frac{1}{|x|}\right)$. Hence,

$$|\zeta(t,x) - \varphi(x)| + |\mu(t,x)| \le C t^{2-\frac{3}{q}} |x|^{\frac{6}{q}-5}, \quad 1 \le q \le 3/2.$$

Thus, if $\langle u, \phi\rangle = 0$ for all $\phi \in \mathcal{M}$, then, for any $p > 3$,

$$\|R(t)u\|_p \le \frac{|a|}{\sqrt{\pi t}} \|\varphi\|_p |\langle\zeta(t) - \varphi, u\rangle| \le C t^{-3\left(\frac{1}{q}-\frac{1}{2}\right)} \||x|^{\frac{6}{q}-5} u\|_1. \quad (4.90)$$

For $\phi \in \mathcal{E}$, we have $|D_2 V\phi(x)| \le C$. It follows, since $\langle\phi, V D_2 \mu(t)u\rangle = \langle D_2 V\phi, \mu(t)u\rangle$, that

$$|\langle\phi, V D_2 \mu(t)u\rangle| \le C\|\mu(t)u\|_1 \le C t^{2-\frac{3}{q}} \||x|^{\frac{6}{q}-5} u\|_1, \quad \phi \in \mathcal{E}.$$

Since $\phi \in \mathcal{E}$ belong to L^p for $p > 3$, we also have

$$\|S(t)u\|_p \le C t^{-3\left(\frac{1}{q}-\frac{1}{2}\right)} \||x|^{\frac{6}{q}-5} u\|_1. \quad (4.91)$$

We choose $p > 3$ as the dual exponent of $1 \le q < 3/2$ and combine (4.90) and (4.91) with (1.17). We obtain (1.18). This completes the proof.

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