Concentrations of Solutions to Time-Discretizied Compressible Navier-Stokes Equations

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Abstract: The compactness properties of solutions to time-discretization of compressible Navier-Stokes equations are investigated in three dimensions. The existence of generalized solutions is established.

1. Introduction

Problem formulation. A flow of compressible Newtonian fluid in bounded domain $\Omega \subset \mathbb{R}^3$ is governed by the Navier-Stokes equations

$$\frac{\partial(\varrho\mathbf{u})}{\partial t} + \mathrm{div}\,(\varrho\mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \varrho\mathbf{F} + \mathrm{div}\,\Sigma(\mathbf{u}), \tag{1.1a}$$

$$\frac{\partial\varrho}{\partial t} + \mathrm{div}\,(\varrho\mathbf{u}) = 0, \tag{1.1b}$$

where $p(\varrho) = \varrho^\gamma$, ϱ, and $\mathbf{u}$ are pressure, density, and velocity of fluid, $\mathbf{F} \in C(\Omega)^3$ is a given vector field, Σ is a viscous stress tensor defined by

$$\Sigma(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^\top) + \nu\,\mathrm{div}\,\mathbf{u}\,\mathbf{I}, \quad \mathrm{div}\,\Sigma(\mathbf{u}) = \Delta\mathbf{u} + (1+\nu)\nabla\mathrm{div}\,\mathbf{u}, \tag{1.2}$$

where $\nu > -2$ is the viscosity ratio. The existence of generalized solutions to initial boundary-value problems for Eqs. (1.1) was proved by Lions in [19] and by Feireisl, Matušů-Nečasová, Petzeltová, Straškraba in [13] for all $\gamma > 9/5$. These results were essentially improved by Feireisl, Novotný, and Petzeltová in paper [14], where the existence of solutions was established for all $\gamma > 3/2$. For the range $\gamma > 3/2$, the mathematical theory of compressible Navier-Stokes equations is covered in the book by E. Feireisl [15]. Recall that $\gamma = 5/3$ for monoatomic gases, $\gamma = 7/5$ for air, and $\gamma = 1$ in the isothermal case. The problem of the existence of solutions to (1.1) for $\gamma < 3/2$ was listed among other unsolved problems of fluid mechanics in [20]. In the paper we

consider a boundary-value problem for the stationary equations, obtained by the time discretization of Eqs. (1.1),

$$\alpha \varrho \mathbf{u} + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \varrho \mathbf{F} + \operatorname{div} \Sigma(\mathbf{u}) \text{ in } \mathcal{D}'(\Omega), \tag{1.3a}$$

$$\alpha \varrho + \operatorname{div}(\varrho \mathbf{u}) = f \text{ in } \mathcal{D}'(\Omega), \quad u = 0 \text{ on } \partial\Omega. \tag{1.3b}$$

Here $\alpha > 0$ is a parameter of the time discretization, $f \in C(\Omega)$ is a given non-negative function.

Let $\mathbb{K}$ denote the cone of non-negative functions $\varrho \in L^1(\Omega)$. A pair $(\varrho, \mathbf{u}) \in \mathbb{K} \times H_0^{1,2}(\Omega)$ is said to be a generalized solution to problem (1.3) if it satisfies the moment and the mass balance equations, which are understood in the sense of distributions.

Notation. In the paper the standard notation is used for the function spaces. The space $H^{1,p}(\Omega)$ is the Sobolev space of functions integrable along with the first order generalized derivatives in $L^p(\Omega)$ equipped with its natural norm. For $p = 2$ we use the notation $H^{1,2}(\Omega)$ rather than $H^1(\Omega)$, and for real $m > 0$ we denote the Sobolev space of order m by $H^{m,2}(\Omega)$. $H_0^{1,2}(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ in the norm of the Sobolev space $H^{1,2}(\Omega)$. We use also the summation convention over repeated indices $i, j = 1, 2, 3$, e.g., $\varphi_{i,j}\xi_i\xi_j = \sum_{i,j=1}^{3} \varphi_{i,j}\xi_i\xi_j$. The support of a function φ is denoted by $\operatorname{spt}\varphi$. For a vector function $\varphi \in L^2(\Omega)^3$ its norm is denoted by $\|\varphi\|_{L^2(\Omega)}$, the same notation is used for the tensor functions. We denote by $B(x, R) \subset \mathbb{R}^3$ the ball of the radius R with the centre at x, and by $\mathbb{S}^2 = \partial B(0, 1) = \{x \in \mathbb{R}^3 : |x| = 1\}$ the unit sphere in $\mathbb{R}^3$. A function defined as the mapping $G : \Omega \times \mathbb{R}_\lambda \mapsto \mathbb{R}$ means $G(x, \lambda) \in \mathbb{R}$ for $x \in \Omega$ and $\lambda \in \mathbb{R}$. For a positive parameter $\varepsilon > 0$ we denote e.g., by $\varphi_\varepsilon \in B$ the *sequence* as $\varepsilon \searrow 0$, and will speak about e.g., the strong (norm) convergence, possibly for a subsequence, i.e., $\lim_{\varepsilon \searrow 0} \|\varphi_\varepsilon - \varphi\|_B = 0$. We can consider as well the weak convergence or weak star convergence for the sequence φ_ε in the function space B.

In addition, by c is denoted a generic constant in all estimates given in the paper.

Generalized solutions. The local existence and uniqueness theorems for viscous steady compressible flows were proved in the pioneering paper by M. Padula [23]. P.L. Lions, cf. [19], established the existence result for problem (1.3) for all $\gamma \geq 5/3$ in the case $\alpha > 0$ and $\gamma > 5/3$ in the case $\alpha = 0$. Note that these results can be extended to the range $\gamma > 3/2$ by a method developed by E. Feireisl et al in [13, 14]. The solvability of problem (1.3) in the two-dimensional case was proved in [19] for all $(\gamma, \alpha) \in [1, \infty) \times (0, \infty) \cup (1, \infty) \times \{0\}$. The limiting case $\gamma = 1, \alpha = 0$ was considered by the authors in [25]. Recall that the standard energy estimate gives the following bounds for quantities involved in the equations:

$$\|\varrho^\gamma\|_{L^1(\Omega)} + \|\varrho|\mathbf{u}|^2\|_{L^1(\Omega)} + \|\mathbf{u}\|_{H^{1,2}(\Omega)} \leq C(\alpha, \Omega, \|\mathbf{F}\|_{C(\Omega)}, \|f\|_{C(\Omega)}) \text{ for } \gamma \geq 1, \tag{1.4a}$$

$$\|\varrho \ln(1 + \varrho)\|_{L^1(\Omega)} \leq C(\alpha, \Omega, \|\mathbf{F}\|_{C(\Omega)}, \|f\|_{C(\Omega)}) \text{ for } \gamma = 1. \tag{1.4b}$$

It is easy to see that in the three-dimensional case the energy estimates and embedding theorems guarantee the inclusion $\varrho|\mathbf{u}|^2 \in L^s(\Omega)$ with $s > 1$ if and only if $\gamma > 3/2$. Hence, for $\gamma \leq 3/2$ we have only an L^1 estimate for the density of the kinetic energy. The question is: under what conditions will a weak limit of approximate solutions to Eqs. (1.3) be a solution? If a sequence of approximate solutions $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$, $\varepsilon > 0$, to

problem (1.3) satisfies the inequalities (1.4) with a constant C, independent of ε, we can assume, after passing to a subsequence, that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } H_0^{1,2}(\Omega), \quad \varrho_\varepsilon \to \varrho \text{ weakly in } L^\gamma(\Omega), \tag{1.5}$$

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \mathcal{M} \text{ star weakly in the space of Radon measures as } \varepsilon \to 0, \tag{1.6}$$

where $\mathcal{M} = (\mathcal{M}_{i,j})_{3\times 3}$ denotes a 3×3 matrix-valued Radon measure in Ω. In the general case the weak star defect measure $\mathcal{S} = \mathcal{M} - \varrho \mathbf{u} \otimes \mathbf{u} \neq 0$. This leads to the so-called concentration problem, which was widely discussed in the mathematical literature in connection with vortex sheet dynamics, cf. [9, 10, 27]. According to DiPerna and Majda, we say Ω_{sing} is a concentration set if $\mathcal{S}(\Omega \setminus V) = 0$ for every open $V \supset \Omega_{sing}$; the cancellation concentration phenomenon is the case when div $\mathcal{S} = 0$. Hence the question is to describe the structure of the defect measure and to find conditions under which it is equal to 0.

Results. We intend to propose a new approach to this problem which is based on the following result on the compactness properties of solutions to the generalized momentum equation. Suppose that the tensor fields $\Pi_\varepsilon = \Pi_\varepsilon^{(1)} + \Pi_\varepsilon^{(2)}$ and the vector-valued functions g_ε are defined on a bounded domain $\Omega \subset \mathbb{R}^3$ and satisfy the conditions

$$\|\Pi_\varepsilon^{(1)}\|_{L^q(\Omega)} + \|g_\varepsilon\|_{L^1(\Omega)} \le c, \tag{1.7}$$

$$\Pi_\varepsilon^{(1)} \to \Pi \text{ weakly in } L^q(\Omega)^9, \quad g_\varepsilon \to g \text{ weakly in } L^1(\Omega)^3, \tag{1.8}$$

$$\|\Pi_\varepsilon^{(2)}\|_{L^{3/2}(\Omega)} + \sup_{B(x,R) \subset \Omega} \sup_{0 < r \le R} |\mathfrak{S}\Pi_\varepsilon^{(2)}(x,r,R)| = \xi_\varepsilon(R) \to 0 \text{ as } \varepsilon \to 0. \tag{1.9}$$

Here the constants $q > 3/2$ and c is independent of ε, the integral operator $\mathfrak{S}$ is defined by

$$\mathfrak{S}\Pi(x,r,R) = \int\limits_{r<|y|<R} \frac{1}{|x-y|}(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) : \Pi(y)dy, \tag{1.10}$$

where $\mathbf{n} = |x-y|^{-1}(y-x)$. Suppose also that a sequence $(\varrho_\varepsilon, \mathbf{u}_\varepsilon) \in \mathbb{K} \times H_0^{1,2}(\Omega)$ satisfies the energy inequality (1.4a), relations (1.5), and the equations

$$\text{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla p(\varrho_\varepsilon) = \text{div} \, \Pi_\varepsilon + g_\varepsilon \text{ in } \mathcal{D}'(\Omega). \tag{1.11}$$

We assume that the total energy density satisfies the weak regularity condition

$$\lim_{r \searrow 0} r^{-1} \int\limits_{B(x,r) \cap \Omega} (\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + p(\varrho_\varepsilon))dx = 0 \text{ for all } x \in \Omega. \tag{1.12}$$

The first main result of this work is the following theorem.

Theorem 1. *(i) If $\gamma > 1$ and $\alpha > 0$, then $\mathcal{M} = \varrho \mathbf{u} \otimes \mathbf{u}$. Moreover, there exists $\kappa > 0$ such that for all $\Omega' \Subset \Omega$, the sequences $\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2$ and ϱ_ε are bounded in $L^{1+\kappa}(\Omega')$ and $L^{\gamma(1+\kappa)}(\Omega')$, respectively.*

(ii) If $\gamma = 1$ and $\alpha > 0$, then $\mathcal{M} = \varrho\mathbf{u} \otimes \mathbf{u} + \mathcal{S}$, where the matrix-valued defect measure $\mathcal{S}$ has the representation

$$\int_{\Omega} \varphi(x) : d\mathcal{S}(x) = \int_{\Omega_{sing}} \mathbf{s}(x) \otimes \mathbf{s}(x) : \varphi(x)\sigma(x)d\mathcal{H}^1 \ \textit{ for all } \ \varphi \in C_0(\Omega)^9. \ (1.13)$$

Here Ω_{sing} is a Borel set, in which every compact subset is countable $(\mathcal{H}^1, 1)$ rectifiable, $\mathbf{s}(x)$ is the unit tangent vector to Ω_{sing} at a point x, $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, and σ is a non-negative, locally bounded function.

Recall that Ω_{sing} is countable $(\mathcal{H}^1, 1)$ rectifiable if there is a family of C^1 one-dimensional manifolds Γ_k such that $\mathcal{H}^1(\Omega_{sing} \setminus \cup_k \Gamma_k) = 0$.

Theorem 1 leads to new results on a solvability of problem (1.3). Suppose that Ω is a bounded domain with $\partial\Omega \in C^{1+\beta}$, $\beta \in (0, 1)$, and consider a family of boundary-value problems, depending on a small positive parameter ε,

$$(\alpha\varrho + \sqrt{\varepsilon}\varrho^5)\mathbf{u} + \text{div}\left(\mathbf{u} \otimes (\varrho\mathbf{u} - \varepsilon\nabla\varrho)\right) + \nabla p = \varrho\mathbf{F} + \text{div}\,\Sigma(\mathbf{u}) \ \text{ in } \ \Omega, \quad (1.14a)$$

$$\alpha\varrho + \sqrt{\varepsilon}\varrho^5 + \text{div}\,(\varrho\mathbf{u}) - \varepsilon\Delta\varrho = f \ \text{in } \Omega, \quad u = 0, \ \partial_n\varrho = 0 \ \text{on } \partial\Omega. \quad (1.14b)$$

Theorem 2. *For any $\gamma \geq 1$, $\alpha > 0$, $\mathbf{F} \in C^{\beta}(\Omega)^3$, and non-negative $f \in C^{\beta}(\Omega)$, problem (1.14) has a solution $\mathbf{u}_\varepsilon \in C^{2+\beta}(\Omega)^3$, $\varrho_\varepsilon \in C^{2+\beta}(\Omega)$, $\varrho_\varepsilon > 0$. There is a constant c independent of ε such that*

$$\|\varrho_\varepsilon\|_{L^\gamma(\Omega)} + \|\mathbf{u}_\varepsilon\|_{H^{1,2}(\Omega)} + \||\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1(\Omega)} \leq c, \quad (1.15a)$$

$$\varepsilon^{1/2}\|\varrho_\varepsilon^{4+\gamma}\|_{L^1(\Omega)} + \varepsilon^{1/2}\|\varrho_\varepsilon^5|\mathbf{u}_\varepsilon|^2\|_{L^1(\Omega)} \leq c, \quad (1.15b)$$

$$\varepsilon^{1/8}\|\varrho_\varepsilon\nabla\mathbf{u}_\varepsilon\|_{L^{8/5}(\Omega)} + \varepsilon^{1/8}\|\varrho_\varepsilon\mathbf{u}_\varepsilon\|_{L^{24/7}(\Omega)} \leq c, \quad (1.15c)$$

$$\varepsilon^{3/4}\|\nabla\varrho_\varepsilon\|_{L^2(\Omega)} + \varepsilon\|(1 + \varrho_\varepsilon)^{\gamma-2}|\nabla\varrho_\varepsilon|^2\|_{L^1(\Omega)} \leq c. \quad (1.15d)$$

In the case $\gamma = 1$ any solution to problem (1.14) also satisfies the inequalities

$$\|\varrho_\varepsilon \ln(1 + \varrho_\varepsilon)\|_{L^1(\Omega)} + \varepsilon^{1/2}\|\varrho_\varepsilon^5 \ln(1 + \varrho_\varepsilon)\|_{L^1(\Omega)} \leq c. \quad (1.15e)$$

This result along with Theorem 1 implies.

Theorem 3. *(i) If $\gamma > 1$ and $\alpha > 0$, then problem (1.3) admits at least one weak solution $\varrho \in L^\gamma(\Omega)$, $\mathbf{u} \in H_0^{1,2}(\Omega)$, satisfying (1.4a).*

(ii) If $\gamma = 1$ and $\alpha > 0$, then there are $\varrho \in L^1(\Omega)$ and $\mathbf{u} \in H_0^{1,2}(\Omega)$ which satisfy (1.3b) and the modified momentum balance equation

$$\alpha\varrho\mathbf{u} + \text{div}\,(\varrho\mathbf{u} \otimes \mathbf{u}) + \nabla\varrho + \text{div}\,\mathcal{S} = \varrho\mathbf{F} + \text{div}\,\Sigma \ \text{ in } \ \mathcal{D}'(\Omega),$$

where the measure-valued tensor $\mathcal{S}$ meets all requirements of Theorem 1.

Theorem 3 yields the alternative: Either the concentration set is empty or its Hausdorff dimension is equal to one. Whether concentrations are cancelled or a non-trivial singular set really exists is a question which we cannot decide with certainty. Note only that if approximate solutions and a flow region are also invariant under the action of some group $x \to x'$, then a concentration set and a measure density θ also are invariant under

the action of this group. The precise definition of the concentration set and the measure density are given below. In particular, the velocity field and the pressure are invariant with respect to the shift $x_3 \to x_3 +$ const, in the case of a two-dimensional flow in the plane (x_1, x_2). Therefore, in this case Ω_{sing} is the union of a countable set of straight lines and θ is a constant along each of those. From this we conclude that div $\mathcal{S} = 0$ and concentrations are cancelled in agreement with results of [19] and [25]. The same results hold true for axially symmetric flows. On the other hand, the simple examples show that singularities definitely exist for solutions of the pressureless Navier-Stokes equations, which are used in astrophysics. Finally, let us point out that the results of the present paper can be used, in the same way as in [25], to establish the existence of solutions for the associated shape optimization problems.

Mathematical background. There are three aspects of our method which deserve brief description.

1. The first concerns the estimates for the trace of non-negative, measure-valued tensor fields $\mathcal{E}$ with div $\mathcal{E} \in H_0^{1,q}(\Omega)^*$, which play the key role in the proof of Theorem 1.
2. The second is the observation that the boundedness of the potential $|x|^{-1} * \mu$ of the measure μ in Ω implies the boundedness of the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega, d\mu)$.
3. The third is the use of the kinetic formulation of the mass balance equation (1.3b) to obtain the convergence $p(\varrho_\varepsilon) \to p(\varrho)$.

Now, we can explain the organization of the paper. Sections 2-5 are devoted to the proof of Theorem 1. In Sect. 2 we derive the estimates for a non-negative, matrix-valued Radon measure $\mathcal{E}$ satisfying the equation div $(\mathcal{E} - \Pi) = g$, in which a tensor field Π and a vector-valued function g satisfy the assumptions of Theorem 1. We show that the measure density

$$\theta(x) = \lim_{r \to 0} \frac{1}{r} \int_{B(x,r)} d\mu \tag{1.16}$$

of the Radon measure $\mu = \text{Tr } \mathcal{E}$ and the potential

$$x \mapsto \int_\Omega |x - y|^{-1}(d\mu(y) - \mathbf{n} \otimes \mathbf{n} : d\mathcal{E}(y))$$

are well defined and bounded on every compact subset of Ω. Applying these results to the measure $d\mathcal{E}_\varepsilon = (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + p(\varrho_\varepsilon)\mathbf{I})dx$ we conclude that the Newtonian potentials $|x|^{-1} * p(\varrho_\varepsilon)$ are uniformly bounded on every compact subset of Ω. In the next section we consider the embedding of the Sobolev space into the space of functions, which are square integrable with respect to some measure μ. The main result is that the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^2(K, d\mu)$, $K \Subset \Omega$, is continuous, if

$$\omega(t) = \sup_{x \in K} \int_{B(x,t) \cap K} |x - y|^{-1} d\mu(y) < \infty.$$

Moreover, the embedding is compact provided that $\omega(t) \to 0$ as $t \to 0$. These results yield the statement of Theorem 1 for $\gamma > 1$. The proof of Theorem 1 in the case $\gamma = 1$ is based on the connection between mathematical theory of compressible fluids and mathematical problems arising in geometric measure theory. The key observation is

that the tensor fields $\varrho_\varepsilon(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + \mathbf{I})$ converge weakly to a matrix-valued measure $\mathcal{E}$ satisfying the equation div $(\mathcal{E} - \Pi) = g$ with $\Pi \in L^2(\Omega)^9$ and $g \in L^1(\Omega)^3$. The results of Sect. 2 imply the existence of the measure density $\theta(x) = \lim_{r \to 0} r^{-1} \int_{B(x,r)} d\mathrm{Tr}\,\mathcal{E}$ at each point $x \in \Omega$. In Sect. 5 we introduce the weak star defect measure $\mathcal{S} = \mathcal{E}\lfloor_{\Omega_{sing}}$, where $\Omega_{sing} = \{x : \theta(x) > 0\}$. Since the linear form $\varphi \mapsto \int_\Omega \nabla \varphi : d\mathcal{E}$ can be regarded as the first variation of a one-dimensional varifold, the rectifiability of Ω_{sing} and representation (1.6) follow from the classical results of the theory of varifolds, see [4]. The remaining part of the paper is devoted to the existence theory for boundary-value problem (1.3). In Sect. 6 we show that the statement of Theorem 3 holds true under the assumption that, for $\gamma > 1$, the sequence of solutions ϱ_ε to problem (1.14) converges almost everywhere in Ω. To obtain this result we apply the technique developed by Lions and Feireisel et al (cf. [19, 12, 14]). In Sect. 7 we recall the famous results of P.L. Lions and E. Feireisel et al on compactness properties of the *viscous flux* and the basic facts from the theory of the *oscillations defect measure* proposed by E. Feireisel in [12].

The peculiarity of our approach is the systematic use of representations of weak limits $\overline{\varphi} = w - \lim_{\varepsilon \to 0} \varphi(\varrho_\varepsilon)$, in the form of the Stieltjes integrals $\int_{\mathbb{R}} \varphi(\lambda) d_\lambda \Gamma(x, \lambda)$, in which a monotone distribution function $\Gamma(x, \cdot)$ does not depend on φ and has the limits $\lim_{\lambda \to \pm\infty} \Gamma(x, \lambda) = 0, 1$. In this setting the strong convergence of ϱ_ε is equivalent to the equality $\Gamma(1 - \Gamma) = 0$. In Sect. 8, Lemma 16, we show that the distribution function satisfies in the strip $\Omega \times \mathbb{R}$ the kinetic equation

$$\frac{\partial}{\partial \lambda}\left[(\lambda\alpha + \lambda\mathrm{div}\,\mathbf{u}(x) - f(x))\Gamma\right] - \mathrm{div}\,(\mathbf{u}(x)\Gamma) = \alpha\Gamma + \frac{\partial}{\partial \lambda}[\lambda M(\Gamma) + m], \quad (1.17)$$

in which m is some non-negative Borel measure and M is non-linear integro-differential operator defined by (8.2). The preference of the method of such a kinetic equation is that Eq. (1.17) allows us to tackle problems, in which the strong convergence does not take place; for instance, the problems with fast oscillating data. Note that, in contrast to the theory of conservation laws (cf. [6]), the kinetic formulation of problem (1.3) involves the non-linear term. Nevertheless, we show that Eq. (1.17) can be renormalized, which leads to the identity $\Gamma(1 - \Gamma) = 0$ and, thus, to the strong convergence of solutions to problem (1.14).

2. Tensor Fields with Integrable Divergence

In this section we obtain the basic estimates for solutions of Eqs. (1.11). The most significant of them are the bounds for the Newtonian potential $|x|^{-1} * p(\varrho_\varepsilon)$ of the pressure which are given by Proposition 2. With further applications in mind we deduce these estimates from the general statement on non-negative tensor fields with divergence from $(H_0^{1,3/2}(\Omega))^*$. Suppose that the 3×3 matrix-valued finite Radon measure $\mathcal{E} = (\mathcal{E}_{i,j})$ is symmetric and non-negative :

$$\mathcal{E}_{i,j} = \mathcal{E}_{j,i}\,, \quad \langle \mathcal{E}_{i,j}\varphi_i, \varphi_j \rangle \geq 0 \quad \text{for all} \quad \varphi_i, \varphi_j \in C_0(\Omega), \ 1 \leq i, j \leq 3.$$

In particular, $\mu = \mathrm{Tr}\,\mathcal{E}$ is a non-negative Radon measure in Ω. Suppose also that there exist a tensor field $\Pi \in L^{3/2}(\Omega)^9$ and vector-valued function $g \in L^1(\Omega)^3$ such that for every vector-function $\varphi \in C_0^\infty(\Omega)^3$,

$$\int_\Omega \partial_j\varphi_i(y)d\mathcal{E}_{i,j} = \int_\Omega \partial_j\varphi_i(y)\Pi_{i,j}(y)dy - \int_\Omega \varphi_i(y)g_i(y)dy\,. \qquad (2.1)$$

It is easy to see that the integral identities (2.1) can be equivalently assembled as the differential equation div $(\mathcal{E} - \Pi) = g$. The main result of this section is the following proposition.

Proposition 1. *Suppose that a tensor field Π admits the decomposition $\Pi = \Pi^{(1)} + \Pi^{(2)}$, in which $\Pi^{(1)} \in L^q(\Omega)^9$, $q > 3/2$, and*

$$\|\Pi^{(2)}\|_{L^{3/2}(\Omega)} + \sup_{0 < r \leq R} \mathfrak{S}\Pi^{(2)}(x, r, R) \leq C_K \tag{2.2}$$

for all x in a compact $K \Subset \Omega$ and with $R \leq \mathrm{dist}(K, \partial\Omega)$. Then there exists a constant c_K depending only on R, C_K, $\mu(\Omega)$, $\|\Pi^{(1)}\|_{L^q(\Omega)}$, and $\|g\|_{L^1(\Omega)}$ such that

$$\frac{1}{r} \int_{B(x,r)} d\mu(y) + \lim_{s \to 0} \int_{s \leq |x-y| \leq r} \frac{1}{|x - y|} (d\mu(y) - \mathbf{n} \otimes \mathbf{n} : d\mathcal{E}(y)) \leq c_K, \tag{2.3}$$

$$\frac{1}{r} \int_{B(x,r)} d\mu(y) = \zeta(x, r) + \psi(x, r) \ \text{ for } x \in K \ \text{ and } \ r \in (0, R]. \tag{2.4}$$

Here $\zeta(x, r)$ is a function which increases in r and the remainder is bounded $|\psi(x, r)| \leq \Psi(r) + C_K$, with

$$\Psi(r) = c\|\Pi^{(1)}\|_{L^q(\Omega)} r^{2-3/q} + \sup_{x \in \Omega} \int_{B(x,r) \cap \Omega} |g| dx \to 0 \quad \text{as } r \to 0.$$

Note that $\mu - \mathbf{n} \otimes \mathbf{n} : \mathcal{E} \geq 0$ in $B(x, R) \setminus \{0\}$.

The proof of Proposition 1 is based on the following lemma.

Lemma 1. *Under the assumptions of Proposition 1 for each $x \in K$,*

$$\frac{M_1(x, r)}{r} + \int_{r \leq |x-y| \leq R} \frac{1}{|x - y|} (d\mu(y) - \mathbf{n} \otimes \mathbf{n} : d\mathcal{E}(y)) + \zeta_0(x, r)$$

$$= \frac{M_1(x, R)}{R} - \frac{1}{R} \int_{B(x,R)} Tr \, \Pi dy$$

$$+ \frac{1}{r} \int_{B(x,r)} Tr \, \Pi dy + \mathfrak{S}\Pi(x, r, R) - \int_{B(x,R)} k_{r,R}(y - x) dg(y), \tag{2.5}$$

where $k_{r,R}(x) = \min\{r^{-1}, |x|^{-1}\} - R^{-1}$ and non-negative, non-increasing in r functions ζ_0, M_i, are defined by

$$\zeta_0(x, r) = \int_{(r,R]} s^{-1} d_s M_0(x, s) - \int_{(r,R]} s^{-1} M'_{0,s}(x, s) ds,$$

$$M_0(x, s) = \int_{B(x,s)} \mathbf{n} \otimes \mathbf{n} : d\mathcal{E}(y), \quad M_1(x, s) = \int_{B(x,s)} d\mu(y).$$

Proof. By abuse of notation we write simply $\zeta_0(r)$, $M_i(r)$ for $\zeta_0(x, r)$ and $M_i(x, r)$, respectively. Select positive $s < r$ and $\delta < R - s$ such that $\mu(\partial B(x, s + \delta)) = 0$ and $M'_j(s) < \infty$, $j = 0, 1$. Choose also a sequence of smooth functions h_k satisfying the conditions

$$h_k(t) = 1 \text{ for } t \leq s, \quad h_k(t) = 0 \text{ for } t \geq s + \delta = 0, \quad -2/\delta \leq h'_k \leq 0,$$
$$h'_k(t) \to -1/\delta \text{ as } k \to \infty \text{ for each } t \in (s, s + \delta).$$

Substituting the vector-valued function $\varphi(y) = h_k(|y - x|)(y - x)$ into (2.1) we derive the identity

$$\int_{B(x,s+\delta)} h_k(|y - x|)(d\mu(y) - \text{Tr } \Pi dy) + \int_{B(x,s)} h_k(|y - x|)(y - x) \cdot g(y)dy$$

$$+ \int_{s<|y-x|<s+\delta} h'_k(|y - x|)|y - x|\mathbf{n} \otimes \mathbf{n} : (d\mathcal{E}(y) - \Pi(y)dy) = 0 .$$

Passing to the limit as $k \to \infty$ and noting that

$$\lim_{k\to\infty} h'_k(|y - x|)\mathbf{n} \otimes \mathbf{n} = -\delta^{-1}\mathbf{n} \otimes \mathbf{n}$$

everywhere in the annulus $s < |y - x| < s + \delta$ we arrive at the equality

$$M_1(s) - \frac{s}{\delta}\Big(M_0(s + \delta) - M_0(s)\Big) = \int_{B(x,s)} (\text{Tr } \Pi - (y - x)g(y))dy$$

$$- \frac{s}{\delta} \int_{B(x,s+\delta)\backslash B(x,s)} \mathbf{n} \otimes \mathbf{n} : \Pi dy + \Delta ,$$

where

$$|\Delta| \leq \frac{1}{\delta} \int_{s<|y-x|<s+\delta} (|y - x| - s)\mathbf{n} \otimes \mathbf{n} : d\mathcal{E}(y)$$

$$+ \int_{s<|y-x|<s+\delta} (d\mu(y) + |\text{Tr } \Pi|dy + |y - x||g(y)|dy) \to 0 \quad \text{as } \delta \to 0.$$

Letting $\delta \to 0$ we obtain that for almost every $s \in (0, R]$,

$$M_1(s) - sM'_0(s) = \int_{B(x,s)} \text{Tr } \Pi dy - s \int_{\partial B(x,s)} \mathbf{n} \otimes \mathbf{n} : \Pi dS - \int_{B(x,s)} (y - x) \cdot g(y)dy.$$

Multiplying both sides by s^{-2} and integrating the result over the interval $(r, R]$ we arrive at

$$\int_{(r,R]} \left(\frac{M_1(s)}{s^2} - \frac{M'_0(s)}{s}\right) ds = \frac{1}{r} \int_{B(x,r)} \text{Tr } \Pi dy$$

$$- \frac{1}{R} \int_{B(x,R)} \text{Tr } \Pi dy + \mathfrak{S}\Pi(x, r, R) - \int_{B(x,R)} k_{r,R}(y - x)dg(y). \quad (2.6)$$

It is easy to see that

$$\int\limits_{(r,R]} \left(s^{-2} M_1(s) - s^{-1} M_0'(s) \right) ds = \int\limits_{(r,R]} s^{-2} M_1(s) ds - \int\limits_{(r,R]} s^{-1} dM_1(s)$$

$$+ \int\limits_{(r,R]} s^{-1} d(M_1(s) - M_0(s)) + \zeta_0(x, r), \quad (2.7)$$

where dM_i are the Stieltjes measures generated by the monotone functions $M_i, i = 0, 1$. Integrating by parts the Stieltjes integral in the right-hand side of (2.7) gives

$$\int\limits_{(r,R]} s^{-2} M_1(s) ds - \int\limits_{(r,R]} s^{-1} dM_1(s) = \frac{1}{r} \int\limits_{B(x,r)} d\mu(y) - \frac{1}{R} \int\limits_{B(x,R)} d\mu(y). \quad (2.8)$$

On the other hand, Fubini's theorem yields the identity

$$\int\limits_{(r,R]} s^{-1} d(M_1(s) - M_0(s)) = \int\limits_{r \le |x-y| \le R} \frac{1}{|x-y|} (d\mu(y) - \mathbf{n} \otimes \mathbf{n} : d\mathcal{E}(y)). \quad (2.9)$$

Combining (2.6)–(2.9) we obtain (2.5) and the lemma follows.   □

We are now in a position to complete the proof of Proposition 1. Since $|\mathfrak{S}\Pi(x, r, R)| \le \int_{B(x,r)} |x - y|^{-1} |\Pi(y)| dy$, the Hölder inequality implies

$$\frac{1}{s} \int\limits_{B(x,s)} |\text{Tr } \Pi^{(1)}| dy \le cs^{2-3/q} \|\Pi^{(1)}\|_{L^q(\Omega)},$$

$$\frac{1}{s} \int\limits_{B(x,s)} |\text{Tr } \Pi^{(2)}| dy \le c \|\Pi^{(2)}\|_{L^{3/2}(\Omega)},$$

$$|\mathfrak{S}\Pi^{(1)}(x, r, R)| \le cR^{2-3/q} \|\Pi^{(1)}\|_{L^q(\Omega)},$$

which along with (2.5) yields (2.3). Proceeding towards the representation (2.4) we set

$$\zeta(x, r) = - \int\limits_{r \le |x-y| \le R} \frac{1}{|x-y|} (d\mu(y) - \mathbf{n} \otimes \mathbf{n} : d\mathcal{E}(y)) - \zeta_0(x, r)$$

$$- \frac{M_1(R)}{R} + \int\limits_{B(x,R)} K(y - x) : \Pi dy - \frac{1}{R} \int\limits_{B(x,R)} (y - x) \cdot g(y) dy,$$

$$\psi(x, r) = \frac{1}{r} \int\limits_{B(x,r)} \text{Tr } \Pi dy - \int\limits_{B(x,r)} K(x - y) : \Pi dy$$

$$+ \int\limits_{B(x,r)} \left(\frac{1}{r} - \frac{1}{|x-y|} \right) (y - x) \cdot g(y) dy,$$

where $K(y - x) = |x - y|^{-1}(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$. It remains to note that since $|K(x)| \leq c|x|^{-1}$,

$$
|\psi(x, r)| \leq c \int_{B(x,r)} \frac{|\Pi^{(1)}|dy}{|y - x|} + \left| \int_{B(x,r)} K(x - y)\Pi^{(2)}dy \right| + \frac{1}{r} \int_{B(x,r)} |\operatorname{Tr} \Pi^{(2)}|dy
$$

$$
+ \int_{B(x,r)} |g|(y)dy \leq c\|\Pi^{(1)}\|_{L^q(\Omega)} r^{2-3/q}
$$

$$
+ \sup_{x \in K} \int_{B(x,r)} |g|(y)dy + C_K = \Psi(r) + C_K,
$$

and the proposition follows.

We apply Proposition 1 to obtain estimates for the solutions to Eqs. (1.11). It is easy to see that, under the assumptions of Theorem 1, the measures $d\mathcal{E}_\varepsilon = (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + p(\varrho_\varepsilon)\mathbf{I})dy$ and $d\mu_\varepsilon = (\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + 3p(\varrho_\varepsilon))dy$ meet all requirements of Proposition 1 which implies the following result.

Proposition 2. *Under the assumptions of Theorem 1 for any $R > 0$ there exists a constant c, which does not depend on ε, such that*

$$
\frac{1}{r} \int_{B(x,r)} d\mu_\varepsilon + \lim_{s \to 0} \int_{s \leq |x-y| \leq R} \frac{\varrho_\varepsilon|\mathbf{u}_\varepsilon^t|^2 + 2p(\varrho_\varepsilon)}{|x - y|}dy \leq c, \tag{2.10}
$$

$$
\frac{1}{r} \int_{B(x,r)} d\mu_\varepsilon = \int_{B(x,r)} \frac{\varrho_\varepsilon|\mathbf{u}_\varepsilon^t|^2 + 2p(\varrho_\varepsilon)}{|y - x|}dy + \psi_\varepsilon(x, r), \tag{2.11}
$$

for all $r \in (0, R]$, where $\mathbf{u}^t = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}$. The remainder is bounded $|\psi_\varepsilon| \leq \Psi(r) + c\xi_\varepsilon(R)$, where $\xi_\varepsilon(R)$ defined by (1.9) and

$$
\Psi(r) \leq cr^{2-3/q} + \sup_{x \in \Omega} \sup_{\varepsilon > 0} \int_{B(x,r) \cap \Omega} |g_\varepsilon|dy \to 0 \quad as \ r \to 0. \tag{2.12}
$$

Proof. Inequality (2.10) obviously follows from the assumptions of Theorem 1 and Proposition 1. Fix an arbitrary $s < r < R$, $R = \operatorname{dist}(x, \partial\Omega)$. Substituting $\mathcal{E}_\varepsilon$, μ_ε, Π_ε, and g_ε into the identity (2.5), noting that $\zeta_0 = 0$ for $\varepsilon > 0$, replacing r by s and R by r in the resulting equation we obtain

$$
\frac{1}{r} \int_{B(x,r)} d\mu_\varepsilon = \frac{1}{s} \int_{B(x,s)} d\mu_\varepsilon + \int_{s \leq |x-y| \leq r} \frac{1}{|x - y|}\left(\varrho_\varepsilon|\mathbf{u}_\varepsilon^t|^2 + 2p(\varrho_\varepsilon)\right)dy
$$

$$
+ A^1(x, s, r) + A^2(x, s, r) + a(x, s, r), \tag{2.13}
$$

where

$$
A^i(x, s, r) = \frac{1}{r} \int_{B(x,r)} \operatorname{Tr} \Pi^{(i)}dy - \frac{1}{s} \int_{B(x,s)} \operatorname{Tr} \Pi_\varepsilon^{(i)}dy - \mathfrak{S}\Pi_\varepsilon^{(i)}(x, s, r),
$$

$$
a(x, s, r) = \int_{B(x,r)} k_{s,r} g_\varepsilon dy.
$$

It follows from (1.7), (1.9), and (1.10) that

$$|A^1(x, s, r) + a(x, s, r)| \le \Psi(r), \quad |A^2(x, s, r)| \le \xi_\varepsilon(R).$$

Letting $s \to 0$ in (2.13) and noting that, by hypothesis of Theorem 1,

$$\lim_{s \to 0} s^{-1} \int_{B(x,s)} d\mu_\varepsilon = 0 \text{ for } x \in \Omega$$

we obtain identity (2.11) and the proposition follows.   □

3. Embedding Theorem

The objective of this section is to investigate the inequalities that allow the weighted $L^2(\Omega, \varrho dx)$ norm of a function to be estimated by the L^2 norm of its partial derivatives.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and μ be a finite non-negative Radon measure in Ω. Without loss of generality we can assume that the measure μ is extended by 0 outside of Ω. Denote by μ_K a compactly supported, finite Radon measure in $\mathbb{R}^3$ such that $\mu_K(E) = \mu(E \cap K)$ for every Borel set E.

V. Maz'ja established, cf. [21, 22], that the embedding $H^{1,2}(\Omega) \hookrightarrow L^2(\Omega, d\mu)$ is continuous if and only if $\mu(E) \le c$ cap (E) for all Borel sets $E \subset \Omega$. On the other hand, D. Adams, cf. [1, 2], proved that the embedding is continuous if $\|(-\Delta)^{1/2}\mu_K\|^2_{L^2(\mathbb{R}^3)} \le c\mu(K)$ for each compact $K \subset \mathbb{R}^3$. Note that formally we have

$$\|(-\Delta)^{1/2}\mu_K\|^2_{L^2(\mathbb{R}^3)} = -\int_{\mathbb{R}^3} (\Delta^{-1}\mu_K) d\mu_K \le \sup_{\mathbb{R}^3} |\Delta^{-1}\mu_K| \mu(K).$$

Hence one can expect that the boundedness of the potential $-\Delta^{-1}\mu$ provides the continuity of the embedding. In this section we develop these arguments and present a simple condition, sufficient to guarantee that the embedding $H^{1,2}(\Omega) \hookrightarrow L^2(\Omega, d\mu)$ is continuous and compact.

Theorem 4. *Suppose that for some compact $K \Subset \Omega$ and $t \in (0, dist(K, \partial\Omega))$,*

$$\omega(t) = \sup_{x \in K} \int_{B(x,t) \cap K} |x - y|^{-1} d\mu(y) < \infty.$$

Then there exists a constant c_0 depending only on Ω such that for all $v \in H_0^{1,2}(\Omega)$,

$$\int_K |v(y)|^2 d\mu(y) \le c_0 \big(\omega(t)\|v\|^2_{H^{1,2}(\Omega)} + \|v\|^2_{L^1(\Omega)}(1 + t^{-3})^2\big). \tag{3.1}$$

The proof relies on the following lemmas. Introduce the quadratic form $\mathcal{L} : H_0^{1,2} \mapsto \mathbb{R}$ defined by $\mathcal{L}(v) = \omega(t)\|\nabla v\|^2_{L^2(\Omega)} + (1 + t^{-3})^2\|v\|^2_{L^1(\Omega)}$ and set

$$\text{Cap}_{\mathcal{L}}(E) := \inf\{\mathcal{L}(v) : v \in C_0^\infty(\Omega) \text{ and } v(x) \ge 1 \text{ for all } x \in E\}.$$

Lemma 2. *Under the assumptions of Theorem 4 there is a constant c_1 depending only on Ω such that $\mu_K(E) \le c_1 Cap_{\mathcal{L}}(E)$ for all Borel sets $E \subset \Omega$.*

Proof. Recall that for any $v \in H_0^{1,2}(\Omega)$,

$$v(x) = \frac{1}{4\pi} \int_\Omega \frac{x-y}{|x-y|^3} \cdot \nabla v(y) dy \quad \text{in } \Omega \quad \text{and} \quad v(x) = 0 \quad \text{outside of } \Omega.$$

Choose a decreasing function $\eta \in C^\infty(\mathbb{R})$ such that $\eta(s) = 0$ for $s < t/8$, $\eta(s) = 1$ for $s > t/4$, and $\eta'(s) < 16/t$. Now rewrite the last identity in the form

$$v(x) = \frac{1}{4\pi} \int_\Omega \left(1 - \eta(|x-y|)\right) \frac{x-y}{|x-y|^3} \cdot \nabla v(y) dy$$

$$+ \frac{1}{4\pi} \int_\Omega \eta(|x-y|) \frac{x-y}{|x-y|^3} \cdot \nabla v(y) dy$$

$$\equiv I_1(x) + I_2(x). \tag{3.2}$$

It is easy to see that

$$\int_E |I_1(x)| d\mu_K(x) \leq c \int_{E \cap K} g_1 * |\nabla v|(x) d\mu(x)$$

$$= c \int_\Omega |\nabla v|(x) \left\{ \int_{K \cap E} g_1(y-x) d\mu(y) \right\} dx$$

$$\leq c \|\nabla v\|_{L^2(\Omega)} \left\{ \int_\Omega \left\{ \int_{K \cap E} g_1(y-x) d\mu(y) \right\}^2 dx \right\}^{1/2},$$

where $g_1(x) = |x|^{-2}$ for $|x| \leq t/4$ and $g_1(x) = 0$ otherwise. Since the function $g_1 * g_1(x) \leq c|x|^{-1}$ vanishes for $|x| \geq t$, we have

$$\left\{ \int_{K \cap E} g_1(y-x) d\mu(y) \right\}^2 = \int_{K \cap E} \left\{ \int_{K \cap E} g_1 * g_1(y-x) d\mu(y) \right\} d\mu(x)$$

$$\leq c \int_{K \cap E} \left\{ \int_{\{|x-y| \leq t\} \cap K} |x-y|^{-1} d\mu(y) \right\} d\mu(x) \leq c\omega(t) \mu(E \cap K),$$

which yields the inequality

$$\int_E |I_1(x)| d\mu_K(x) \leq c \|\nabla v\|_{L^2(\Omega)} \sqrt{\omega(t)} \sqrt{\mu_K(E)}. \tag{3.3}$$

On the other hand, since the function $\eta(|y|)|y|^{-3} y$ is smooth and the norm of its gradient is bounded by $c(1 + t^{-3})$, we have $|I_2| \leq c(1 + t^{-3}) \|v\|_{L^1(\Omega)}$. Thus, we get

$$\int_E |I_2(x)| d\mu_K(x) \leq c(1 + t^{-3}) \|v\|_{L^1(\Omega)} \mu_K(E) \leq c(1 + t^{-3}) \|v\|_{L^1(\Omega)} \sqrt{\mu_K(E)}. \tag{3.4}$$

Combining (3.3) and (3.4) we conclude that for every Borel set $E \subset \Omega$ and for all functions $v \in H_0^{1,2}(\Omega)$,

$$\int_E |v| d\mu_K \leq c\sqrt{\mathcal{L}(v)\mu_K(E)}. \tag{3.5}$$

If $v \geq 1$ on a Borel set E, then we have

$$\mu_K(E) \leq \int_E |v| d\mu_K(x) \leq c\sqrt{\mathcal{L}(v)}\sqrt{\mu_K(E)},$$

and the lemma follows. $\quad\square$

The next lemma is a version of a well-known Maz'ja result, see [22].

Lemma 3. *There exists a positive constant c, which depends only on Ω and does not depend on t, such that the inequality*

$$\int_0^\infty \mathrm{Cap}_{\mathcal{L}}(N_t)dt^2 \leq c\omega(t)\|\nabla v\|_{L^2(\Omega)}^2 dx + c(1+t^{-3})^2(1 + \|v\|_{L^1(\Omega)}\|v\|_{L^2(\Omega)}^2) \tag{3.6}$$

holds for every function $v \in C_0^\infty(\Omega)$. Here $N_t = \{x \in \Omega : |v(x)| \geq t\}$.

Proof. The proof imitates the Maz'ja proof. We start with the observation that since $\mathrm{Cap}_{\mathcal{L}}(N_t)$ decreases in t,

$$\int_0^\infty \mathrm{Cap}_{\mathcal{L}}(N_t)dt^2 \leq 3 \sum_{j=-\infty}^{j=\infty} 2^{2j}\mathrm{Cap}_{\mathcal{L}}(N_{2^j}). \tag{3.7}$$

Now choose a non-decreasing function $\eta \in C^\infty(\mathbb{R})$ such that $\eta(s) = 0$ for $s \leq 0$, $\eta(s) = 1$ for $s \geq 1$ and $\eta'(s) \leq 2$ for all s. Set $v_j = \eta(2^{1-j}|v(x)| - 1)$. Note that $v_j = 0$ when $|v(x)| < 2^{j-1}$ and $v_j(x) = 1$ when $|v(x)| \geq 2^j$. Using v_j as a test function we get the inequality

$$\mathrm{Cap}_{\mathcal{L}}(N_{2^j}) \leq \omega(t) \int_{N_{2^{j-1}} \backslash N_{2^j}} |\nabla v_j|^2 dx + c(1+t^{-3})^2\left(\int_{N_{2^{j-1}}} |v_j|dx \right)^2. \tag{3.8}$$

Obviously $|\nabla v_j| \leq 2^{2-j}|\nabla v|$. Moreover, if $j \geq 0$ and $x \in N_{2^{j-1}}$, then we have $|v_j(x)| \leq 2^{1-j}|v(x)|$. It follows from this and (3.8) that

$$\mathrm{Cap}_{\mathcal{L}}(N_{2^j}) \leq 2^{4-2j}\omega(t) \int_{N_{2^{j-1}} \backslash N_{2^j}} |\nabla v|^2 dx + c(1+t^{-3})^2 \quad \text{for } j \leq 0,$$

$$\mathrm{Cap}_{\mathcal{L}}(N_{2^j}) \leq 2^{4-2j}c\omega(t) \int_{N_{2^{j-1}} \backslash N_{2^j}} |\nabla v|^2 + 2^{2-2j}c(1+t^{-3})^2\left(\int_{N_{2^{j-1}}} |v|dx \right)^2$$

$$\text{for } j \geq 0.$$

Substituting these inequalities into (3.7) we obtain

$$\int\limits_0^\infty \mathrm{Cap}_{\mathcal{L}}(N_t)dt^2 \le c\omega(t)\int\limits_\Omega |\nabla v|^2 dx + c(1+t^{-3})^2 \sum_{j\ge 0}\Big(\int\limits_{N_{2j-1}} |v|dx\Big)^2. \quad (3.9)$$

Obviously we have

$$\Big(\int\limits_{N_{2j-1}} |v|dx\Big)^2 \le \mathrm{meas}\,(N_{2j-1})\int\limits_\Omega |v|^2 dx, \quad \text{and}\ \ \mathrm{meas}\,(N_{2j-1}) \le 2^{1-j}\int\limits_\Omega |v|dx,$$

which implies

$$\Big(\int\limits_{N_{2j-1}} |v|dx\Big)^2 \le 2^{1-j}\|v\|^2_{L^2(\Omega)}\|v\|_{L^1(\Omega)}.$$

Combining this inequality with (3.9) gives (3.6) and the lemma follows. □

We are now in a position to complete the proof of Theorem 4. It suffices to prove inequality (3.1) for functions $v \in C_0^\infty(\Omega)$ normalized by the condition $\|v\|_{L^2(\Omega)} = 1$. Since by Lemma 2, $\mu_K(E) \le c\mathrm{Cap}_{\mathcal{L}}(E)$ for every compact $K \Subset \Omega$, we have

$$\int\limits_K |v|^2 d\mu(x) \equiv \int\limits_0^\infty \mu_K(N_t)dt^2 \le c\int\limits_0^\infty \mathrm{Cap}_{\mathcal{L}}(N_t)dt^2.$$

Estimating the right side of this inequality, by an application of (3.6) from Lemma 3 and noting that $\|v\|_{L^2(\Omega)} = 1$, $\|v\|_{L^1(\Omega)} \le \sqrt{\mathrm{meas}(\Omega)} \le c$, we obtain (3.1), which completes the proof.

4. Proof of Theorem 1 for $\gamma > 1$

We start with the observation that, by Proposition 2, for any compact $\Omega' \Subset \Omega$,

$$\sup_{x\in\Omega'}\int\limits_\Omega |x-y|^{-1}p_\varepsilon(y)dy \le C(\Omega').$$

From this, in view of the identity $p_\varepsilon = \varrho_\varepsilon^\gamma$, and by an application of Theorem 4 we obtain the estimate

$$\|\varrho_\varepsilon^\gamma|\mathbf{u}_\varepsilon|^2\|_{L^1(\Omega')} \le c(\Omega'),$$

which along with the energy estimate (1.4a) and the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ gives

$$\int\limits_{\Omega'} (\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2)^{1+\kappa}dx \le \Big(\int\limits_{\Omega'} \varrho_\varepsilon^\gamma|\mathbf{u}_\varepsilon|^2 dx\Big)^{(1+\kappa)/\gamma}\Big(\int\limits_{\Omega'} |\mathbf{u}_\varepsilon|^6 dx\Big)^{(\gamma-\kappa-1)/\gamma} \le C(\Omega').$$

Here $\kappa = 2(\gamma - 1)/(2+\gamma)$. On the other hand, Eq. (1.11) obviously yields, for $r < 3/2$, the estimate

$$\|\varrho_\varepsilon^\gamma\|_{L^r(\Omega')} \leq c(\Omega')\|\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^r(\Omega')} + \|\Pi\|_{L^r(\Omega')} + \|g\|_{L^1(\Omega')} + \|\varrho_\varepsilon^\gamma\|_{L^1(\Omega')}.$$

From this, (1.4a), and the assumptions of the theorem we conclude that $\|\varrho_\varepsilon^\gamma\|_{L^{1+\kappa}(\Omega')} \leq c$ with c independent of ε for each $\kappa < \min\{3/2, 2(\gamma-1)/(2+\gamma)\}$. It remains to note that since $\varrho_\varepsilon \to \varrho$ weakly in $L^\gamma(\Omega)$, $\mathbf{u}_\varepsilon \to \mathbf{u}$ a.e. in Ω, and the sequence $\varrho_\varepsilon|\mathbf{u}_n|^2$ is bounded in $L^{1+\kappa}(\Omega')$, the sequence $\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ converges to $\varrho\mathbf{u} \otimes \mathbf{u}$ weakly in $L^{1+\kappa}(\Omega')^9$.

5. Proof of Theorem 1 for $\gamma = 1$

The proof of Theorem 1 for $\gamma = 1$ is more delicate. We split the proof into two steps. First, we define the concentration set and show that either it is empty or its Hausdorff dimension is equal to 1.

Concentration set. Let us consider the sequences $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ and $\Pi_\varepsilon, g_\varepsilon$ satisfying the hypothesis of Theorem 1. Recall that $\varrho_\varepsilon \to \varrho$ weakly in $L^1(\Omega)$, and $\varrho_\varepsilon\mathbf{u}_{\varepsilon,i}\mathbf{u}_{\varepsilon,j}$ are uniformly bounded in $L^1(\Omega)$. After passing to a subsequence we can assume that there exists a 3×3 matrix-valued Radon measure $\mathcal{E}$ such that $\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + \varrho_\varepsilon\mathbf{I} \to \mathcal{E}$ star weakly in the space of Radon measures on Ω. Obviously the measure $\mathcal{E}$ and its trace μ meet all requirements of Proposition 1 with $\Pi^{(1)} = \Pi \in L^q(\Omega)$, $\Pi^{(2)} = 0$, $g = \varrho\mathbf{F}$, and $C_K = 0$. It follows from Proposition 1 that $r^{-1}\mu(B(x,r)) = \zeta(x,r) + \psi(x,r)$, where $\zeta(x,r)$ increases in r and $|\psi(x,r)| \leq \Psi(r) \to 0$ as $r \to 0$. Hence the Borel function $\theta(x) := \lim_{r \to 0} r^{-1}\mu(B(x,r))$ is well-defined and bounded on every compact subset of Ω. Split Ω into two disjoint parts Ω_{reg} and Ω_{sing} given by

$$\Omega_{reg} = \{x : x \in \Omega, \; \theta(x) = 0\}, \quad \Omega_{sing} = \{x : x \in \Omega, \; \theta(x) > 0\}. \quad (5.1)$$

The following theorem shows that Ω_{sing} is a concentration set for the sequence $\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$.

Theorem 5. *Under the assumptions of Theorem 1, for every matrix-valued function $\varphi \in C_0(\Omega)^9$ we have*

$$\int_\Omega \varphi : d\mathcal{E} := \lim_{\varepsilon \to 0} \int_\Omega \varrho_\varepsilon(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + \mathbf{I}) : \varphi \, dy$$

$$= \int_{\Omega_{reg}} \varrho(\mathbf{u} \otimes \mathbf{u} + \mathbf{I}) : \varphi \, dy + \int_{\Omega_{sing}} \varphi : d\mathcal{E}. \quad (5.2)$$

We divide the proof into a sequence of lemmas. In the sequel the notation $K_\delta = \{x| \; \text{dist}(x, K) \leq \delta\}$ stands for the compact δ-vicinity of a compact $K \Subset \Omega$.

Lemma 4. *If a compact $K \subset \Omega_{reg}$, then for any $\eta > 0$ there exist positive $\delta(\eta)$ and $c(\eta)$ such that the inequality*

$$\int_{K_\delta} \varrho_\varepsilon|v|^2 \leq \eta\|v\|_{H^{1,2}(\Omega)}^2 + c(\eta)\|v\|_{L^2(\Omega)}^2 \quad (5.3)$$

holds for all $v \in H^{1,2}(\Omega)$ and for any $\delta \leq \delta(\eta)$. Here the constants $\delta(\eta)$, $c(\eta)$ depend on K, η and do not depend on ε and v.

Proof. First we show that there exist positive $\delta(\eta)$ and $t(\eta)$ such that

$$\sup_{x \in K_{\delta(\eta)}} \sup_{\varepsilon} \int_{|y-x|<t(\eta)} |y-x|^{-1} \varrho_\varepsilon(y)dy \leq \eta. \tag{5.4}$$

If the assertion is false, then there are $k > 0$ and sequences $x_\varepsilon \in \Omega$, $t_\varepsilon > 0$, such that

$$\lim_{\varepsilon \to 0} \text{dist } (x_\varepsilon, K) = \lim_{\varepsilon \to 0} t_\varepsilon = 0 \text{ and } \int_{|y-x|<t_\varepsilon} |x-y|^{-1} \varrho_\varepsilon(y)dy \geq k.$$

Set $d\mu_\varepsilon(x) = \varrho_\varepsilon(|\mathbf{u}_\varepsilon|^2 + 3)dx$, $R = 2^{-1}\text{dist } (K, \partial\Omega)$, and recall that, by Proposition 2,

$$\int_{B(x,t_\varepsilon)} |y-x|^{-1} \varrho_\varepsilon(y)dy \leq \frac{1}{t_\varepsilon} \int_{B(x_\varepsilon,t_\varepsilon)} d\mu_\varepsilon(y)$$

$$-\psi_\varepsilon(x_\varepsilon, t_\varepsilon), \text{ where } |\psi_\varepsilon(x,t)| \leq \Psi(t) + \xi_\varepsilon(R).$$

Since, by (1.9) and (2.12), $\Psi(t_\varepsilon) + \xi_\varepsilon(R) \to 0$ for $\varepsilon \searrow 0$, the inequality $\mu_\varepsilon(B(x_\varepsilon, t_\varepsilon) > t_\varepsilon k/2$ holds for all sufficiently small ε. After passing to a subsequence we can assume that x_ε converge to some $x \in K$ as $\varepsilon \to 0$, and hence

$$\frac{1}{2t_\varepsilon} \int_{B(x,2t_\varepsilon)} d\mu_\varepsilon(y) > k/4 > 0 \text{ for all sufficiently small } \varepsilon. \tag{5.5}$$

On the other hand, Proposition 2 implies the representation $\mu_\varepsilon(B(x,t)) = t\zeta_\varepsilon(x,t) + t\psi_\varepsilon(x,t)$, in which the function ζ_ε is non-decreasing in t. From this and (5.5) we conclude that for all $t > 2t_\varepsilon$,

$$\frac{k}{4} < \frac{1}{2t_\varepsilon} \int_{B(x,2t_\varepsilon)} d\mu_\varepsilon(y) \leq \frac{1}{t} \int_{B(x,t)} d\mu_\varepsilon(y) + \Psi(t) + \Psi(2t) + 2\xi_\varepsilon(R).$$

Note that $\mu_\varepsilon(B(x,t)) \to \mu(B(x,t))$ for almost every positive t. Letting $\varepsilon \to 0$ in the last inequality and noting that $\xi_\varepsilon \to 0$ by Proposition 2, we conclude that $k \leq 4t^{-1}\mu(B(x,t)) + 4\Psi(t) + 4\Psi(2t)$ for almost every positive t. Therefore, $\theta(x) \geq k/4$ which contradicts to the inclusion $x \in \Omega_{reg}$, and the assertion follows. Now inequality (5.4) implies that the measures $d\mu_\varepsilon = \varrho_\varepsilon dx$ satisfy the hypothesis of Theorem 4 with $K = K_{\delta(\eta)}$, $t = t(\eta)$, and $\omega(t) = \eta$. Applying this theorem we obtain (5.3) which completes the proof. $\square$

Lemma 5. *For any compact $K \Subset \Omega$, $\varphi \in C(\Omega)$, and $f, g \in H_0^{1,2}(\Omega)$,*

$$\lim_{\varepsilon \to 0} \int_K \varphi(\varrho_\varepsilon - \varrho)fgdx = 0.$$

Proof. Fix $\delta > 0$ and choose $\bar{f}, \bar{g} \in C_0^\infty(\Omega)$ such that

$$\|f - \bar{f}\|_{H^{1,2}(\Omega)} + \|g - \bar{g}\|_{H^{1,2}(\Omega)} < \delta.$$

Since the Newtonian potentials of ϱ and ϱ_ε are uniformly bounded on K, Theorem 4 yields the inequality

$$\int_K (\varrho_\varepsilon + \varrho)|u||v|dx \le c\|v\|_{H^{1,2}(\Omega)}\|u\|_{H^{1,2}(\Omega)} \quad \text{for all } u, v \in H_0^{1,2}(\Omega).$$

From this we conclude that

$$\limsup_{\varepsilon \to 0}\left|\int_K \varphi(\varrho_\varepsilon - \varrho)fgdx\right| \le \limsup_{\varepsilon \to 0}\left|\int_K \varphi(\varrho_\varepsilon - \varrho)\bar{f}\bar{g}dx\right|$$

$$+c(\varphi)\limsup_{\varepsilon \to 0}\int_K (\varrho_\varepsilon + \varrho)(|g||f - \bar{f}| + |\bar{f}||g - \bar{g}|)dx$$

$$\le c\delta(\|f\|_{H^{1,2}(\Omega)} + \|g\|_{H^{1,2}(\Omega)}) \to 0 \text{ as } \delta \to 0,$$

which proves the lemma. □

Lemma 6. *Under the assumptions of Theorem 5 for any $\eta > 0$, a compact $K \Subset \Omega$, and $\varphi \in C(\Omega)$, there exists $\delta(\eta) > 0$ such that*

$$\limsup_{\varepsilon \to 0}\left|\int_{K_\delta} \varphi\varrho_\varepsilon(|\mathbf{u}_\varepsilon|^2 - |\mathbf{u}|^2)dx\right| \le \eta \quad \text{for all } \delta \le \delta(\eta). \tag{5.6}$$

Proof. Note that for every $\Delta \ge 0$,

$$\left|\int_{K_\Delta} \varphi\varrho_\varepsilon(|\mathbf{u}_\varepsilon|^2 - |\mathbf{u}|^2)dx\right| \le c(\varphi)\left(\frac{1}{\sqrt{\eta}}\int_{K_\Delta} \varrho_\varepsilon|\mathbf{u}_\varepsilon - \mathbf{u}|^2dx + \sqrt{\eta}\int_{K_\Delta} \varrho_\varepsilon|\mathbf{u}_\varepsilon + \mathbf{u}|^2dx\right)$$

$$\le c\sqrt{\eta} + c\frac{1}{\sqrt{\eta}}\int_{K_\Delta} \varrho_\varepsilon|\mathbf{u}_\varepsilon - \mathbf{u}|^2dx, \tag{5.7}$$

where the constant c depends on φ and K. Set $\Delta = \delta(\eta)$, where $\delta(\eta)$ is defined in Lemma 4. Inequality (5.3) now implies

$$\int_{K_{\delta(\eta)}} \varrho_\varepsilon|\mathbf{u}_\varepsilon - \mathbf{u}|^2 \le \eta\|\mathbf{u}_\varepsilon - \mathbf{u}\|_{H^{1,2}(\Omega)}^2 + c(\eta)\|\mathbf{u}_\varepsilon - \mathbf{u}\|_{L^2(\Omega)}^2.$$

Substituting this estimate into (5.7) we finally obtain

$$\limsup_{\varepsilon \to 0}\left|\int_{K_{\delta(\eta)}} \varphi\varrho_\varepsilon(|\mathbf{u}_\varepsilon|^2 - |\mathbf{u}|^2)dx\right| \le c\sqrt{\eta} + \limsup_{\varepsilon \to 0}\frac{c(\eta)}{\sqrt{\eta}}\|\mathbf{u}_\varepsilon - \mathbf{u}\|_{L^2(\Omega)}^2 = c\sqrt{\eta},$$

which completes the proof. □

Lemma 7. *The equality*

$$\int_K \varphi d\mu = \int_K \varphi(|\mathbf{u}|^2 + 3)\varrho dy \tag{5.8}$$

holds for all compacts $K \subset \Omega_{reg}$ and all functions $\varphi \in C_0(\Omega)$.

Proof. It suffices to prove the lemma for $\varphi \geq 0$. Choose an arbitrary non-negative $\varphi \in C_0(\Omega)$. In this case Lemma 6 implies the inequalities

$$\int\limits_{K_{\delta(\eta)}} \varphi(|\mathbf{u}|^2 + 3)\varrho dy - \eta \leq \liminf_{\varepsilon \to 0} \int\limits_{K_{\delta(\eta)}} \varphi(|\mathbf{u}_\varepsilon|^2 + 3)\varrho_\varepsilon dy$$

$$\leq \limsup_{\varepsilon \to 0} \int\limits_{K_{\delta(\eta)}} \varphi(|\mathbf{u}_\varepsilon|^2 + 3)\varrho_\varepsilon dy \leq \int\limits_{K_{\delta(\eta)}} \varphi(|\mathbf{u}|^2 + 3)\varrho dy + \eta . \qquad (5.9)$$

On the other hand, since $\varrho_\varepsilon(|\mathbf{u}_\varepsilon|^2 + 3)$ converges star weakly to μ in the space of the Radon measures on Ω and $K_{\delta(\eta)}$ is a compact, we have

$$\int\limits_{\text{int}(K_{\delta(\eta)})} \varphi d\mu \leq \liminf_{\varepsilon \to 0} \int\limits_{\text{int}(K_{\delta(\eta)})} \varphi(|\mathbf{u}_\varepsilon|^2 + 3)\varrho_\varepsilon dy$$

$$\leq \limsup_{\varepsilon \to 0} \int\limits_{K_{\delta(\eta)}} \varphi(|\mathbf{u}_\varepsilon|^2 + 3)\varrho_\varepsilon dy \leq \int\limits_{K_{\delta(\eta)}} \varphi d\mu .$$

From this and (5.9) we conclude that

$$\int\limits_{\text{int}(K_{\delta(\eta)})} \varphi d\mu \leq \int\limits_{K_{\delta(\eta)}} \varphi(|\mathbf{u}|^2 + 3)\varrho dy + \eta \ \text{ and } \ \int\limits_{K_{\delta(\eta)}} \varphi d\mu \geq \int\limits_{K_{\delta(\eta)}} \varphi(|\mathbf{u}|^2 + 3)\varrho dy - \eta.$$

$$(5.10)$$

Since $\cap_{\eta > 0}\text{int}(K_{\delta(\eta)}) = \cap_{\eta > 0}K_{\delta(\eta)} = K$, we have

$$\lim_{\eta \to 0} \int\limits_{\text{int}(K_{\delta(\eta)})} \varphi d\mu = \lim_{\eta \to 0} \int\limits_{K_{\delta(\eta)}} \varphi d\mu = \int\limits_{K} \varphi d\mu.$$

Passing to the limit in (5.10) as $\eta \to 0$ we obtain (5.8), which completes the proof.   □

Lemma 8. *The inequality*

$$\int\limits_{K} (\varrho\mathbf{u} \otimes \mathbf{u} + \varrho\mathbf{I}) : \varphi \otimes \varphi dx \leq \int\limits_{K} \varphi \otimes \varphi : d\mathcal{E} \qquad (5.11)$$

holds for every compact set $K \Subset \Omega$ and all vector-valued functions $\varphi \in C_0(\Omega)^3$.

Proof. Recall that the sequence $\mathbf{u}_\varepsilon$ converges to $\mathbf{u}$ almost everywhere in Ω. By Egoroff's theorem, there exists a sequence of measurable sets $A_k \subset K$ such that $\mathbf{u}_\varepsilon \to \mathbf{u}$ uniformly on each A_k and meas $(K \setminus A_k) \to 0$ as $k \to \infty$. Lemma 5 and weak L^1 convergence of ϱ_ε to ϱ imply the relations

$$\lim_{\varepsilon \to 0} \int\limits_{K} (\varrho_\varepsilon - \varrho)\mathbf{u} \otimes \mathbf{u} : \varphi \otimes \varphi dy = 0, \quad \lim_{\varepsilon \to 0} \int\limits_{A_k} \varrho_\varepsilon(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - \mathbf{u} \otimes \mathbf{u}) : \varphi \otimes \varphi dy = 0,$$

which along with the inequality $\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \varphi \otimes \varphi \geq 0$ yields

$$\limsup_{\varepsilon \to 0} \int_K (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + \varrho_\varepsilon \mathbf{I}) : \varphi \otimes \varphi dy - \int_K (\varrho \mathbf{u} \otimes \mathbf{u} + \varrho \mathbf{I}) : \varphi \otimes \varphi dy$$

$$= \limsup_{\varepsilon \to 0} \int_{K \backslash A_k} \varrho_\varepsilon (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - \mathbf{u} \otimes \mathbf{u}) : \varphi \otimes \varphi dy \geq - \int_{K \backslash A_k} \varrho \mathbf{u} \otimes \mathbf{u} : \varphi \otimes \varphi dy.$$

$$(5.12)$$

Since $\varrho |\mathbf{u}|^2$ belongs to $L^1(\Omega)$, the last term tends to 0 as $k \to \infty$, which implies

$$\limsup_{\varepsilon \to 0} \int_K (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + \varrho_\varepsilon \mathbf{I}) : \varphi \otimes \varphi dy - \int_K (\varrho \mathbf{u} \otimes \mathbf{u} + \varrho \mathbf{I}) : \varphi \otimes \varphi dy \geq 0. \quad (5.13)$$

Since the non-negative L^1-functions $(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + \varrho_\varepsilon \mathbf{I}) : \varphi \otimes \varphi$ converge to $\varphi \otimes \varphi : \mathcal{E}$ star weakly in the space of Radon measures on Ω, we have

$$\limsup_{\varepsilon \to 0} \int_K (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + 3\varrho_\varepsilon \mathbf{I}) : \varphi \otimes \varphi dy \leq \int_K \varphi \otimes \varphi : d\mathcal{E},$$

which along with (5.13) gives (5.11), and the lemma follows.   □

We are now in a position to complete the proof of Theorem 5. Choose an arbitrary compact $K \subset \Omega_{reg}$, a non-negative function $h \in C_0(\Omega)$, and a vector $\xi \in \mathbb{R}^3$. Set $\varphi(x) = \sqrt{h(x)}\xi$. Applying Lemma 8 we get the inequality

$$B\xi \cdot \xi \equiv \int_K \varphi \otimes \varphi : d\mathcal{E} - \int_K (\varrho \mathbf{u} \otimes \mathbf{u} + \varrho \mathbf{I}) : \varphi \otimes \varphi dx \geq 0,$$

where the symmetric matrix B is given by

$$B = \int_K h d\mathcal{E} - \int_K h(\varrho \mathbf{u} \otimes \mathbf{u} + \varrho \mathbf{I}) dx .$$

By Lemma 7, $\mathrm{Tr} B = 0$, which implies $B = 0$. Hence the equality

$$\int_K h(\varrho \mathbf{u}_i \mathbf{u}_j + \varrho \delta_{i,j}) dx = \int_K h d\mathcal{E}_{i,j} \qquad (5.14)$$

holds for all $h \in C_0(\Omega)$. Next, choose an increasing sequence $\{K_m\}_{m \geq 1}$ of compact subsets of Ω such that $\mu(\Omega_{reg} \setminus \cup K_m) = \mathrm{meas}(\Omega_{reg} \setminus \cup K_m) = 0$. From (5.14) we conclude that for every matrix-valued function $\varphi \in C_0(\Omega)^9$,

$$\int_{\Omega_{reg}} \varphi : d\mathcal{E} = \lim_{m \to \infty} \int_{K_m} \varphi : d\mathcal{E} = \lim_{m \to \infty} \int_{K_m} \varphi : (\varrho \mathbf{u} \otimes \mathbf{u} + \varrho \mathbf{I}) dx$$

$$= \int_{\Omega_{reg}} \varphi : (\varrho \mathbf{u} \otimes \mathbf{u} + \varrho \mathbf{I}) dx ,$$

and the theorem follows.

Rectifiability of concentration set. In this paragraph we show that the set Ω_{sing} is count-ably $\mathcal{H}^1$ rectifiable, and investigate in more details the structure of the measure $\mathcal{E}$. Recall some basic facts from geometric measure theory which will be used throughout this section, cf. [11, 4].

Let μ be a non-negative Radon measure in Ω and $a \in \Omega$. Then VarTan (μ, a) is a set of those Radon measures υ on $\mathbb{R}^3$ for which there is a sequence $\{\varepsilon_i\}_{i\geq 1}$ of positive real numbers such that $\lim\limits_{i\to\infty} \varepsilon_i = 0$ and

$$\int\limits_{\mathbb{R}^3} \varphi d\upsilon = \lim\limits_{i\to\infty} \varepsilon_i^{-1} \int\limits_{\Omega} \varphi\left(\varepsilon_i^{-1}(x-a)\right) d\mu. \tag{5.15}$$

If there exists $\theta(a) = \lim\limits_{r\to 0} r^{-1}\mu(B(a,r)) < \infty$, then VarTan (μ, a) is non-empty, and $\upsilon(B(0,r)) = r\upsilon(B(0,1)) = r\theta(a)$ for every member υ of VarTan (μ, a). The tangential cone to a measure μ at the point a is the set

$$\text{Tan}\,(\mu, a) = \bigcap\limits_{\mathcal{A}}\{\text{Tan}\,(\mathcal{A}, a) : \mathcal{A} \subset \Omega \text{ and } \lim\limits_{r\to 0} r^{-1}\mu((\Omega \setminus \mathcal{A}) \cap B(a,r)) = 0\}\,,$$

where for a subset $\mathcal{A} \subset \Omega$ the notation Tan $(\mathcal{A}, a)$ stands for the cone

$$\text{Tan}\,(\mathcal{A}, a) = \{v \in \mathbb{R}^3 : \forall \varepsilon > 0\,\exists x \in \mathcal{A},\ r \in (0, R)$$
$$\text{such that } |x - a| < \varepsilon \text{ and } |r(x - a) - v| < \varepsilon\}\,.$$

If $0 < \theta(a) < \infty$ and VarTan (μ, a) consists of the only element $\upsilon = 2^{-1}\theta(a)\mathcal{H}^1\lfloor_{\mathcal{T}}$ concentrated on a one-dimensional subspace $\mathcal{T} \subset \mathbb{R}^3$, then Tan $(\mu, a) = \mathcal{T}$. The following rectifiability result, cf. [4] Sect. 2.8, is a straightforward consequence of the definitions.

Proposition 3. *Suppose that μ is a non-negative Radon measure in Ω with $\theta(x) \in \mathbb{R}$ for each $x \in \Omega$, and Tan (μ, x) is a one-dimensional subspace of $\mathbb{R}^3$ for μ-almost every $x \in \Omega_{sing} = \{x : 0 < \theta(x) < \infty\}$. Then every compact subset of Ω_{sing} is $(\mathcal{H}^1, 1)$-rectifiable and*

$$\int\limits_{\Omega_{sing}} f(x)d\mu(x) = \frac{1}{2} \int\limits_{\Omega_{sing}} f(x)\theta(x)d\mathcal{H}^1(x) \quad \textit{for all } f \in C_0(\Omega).$$

The main result of this section is the following theorem.

Theorem 6. *Under the assumptions of Theorem 1,*

  (i) every compact subset of Ω_{sing} is $(\mathcal{H}^1, 1)$-rectifiable;
  (ii) for μ-almost every $x \in \Omega_{sing}$ there is $\mathbf{s}(x) \in \mathbb{S}^2$ such that Tan $(\mu, x) = \text{span}\,\{\mathbf{s}(x)\}$;
(iii) the measure $\mathcal{E}$ has the representation

$$\int\limits_{\Omega_{sing}} \varphi(x) : d\mathcal{E}(x) = \frac{1}{2} \int\limits_{\Omega_{sing}} \varphi(x) : \mathbf{s}(x) \otimes \mathbf{s}(x)\theta(x)d\mathcal{H}^1(x) \tag{5.16}$$

for all $\varphi \in C_0(\Omega)^9$.

Proof. We start with the observation that the components of measure $\mathcal{E}$ are absolutely continuous with respect to μ which implies the representation

$$\int\limits_{\Omega_{sing}} \psi(x)d\mathcal{E}(x) = \int\limits_{\Omega_{sing}} \psi(x)M(x)d\mu(x) \qquad \text{for all } \psi \in C_0(\Omega)^9. \qquad (5.17)$$

Here the matrix-valued function $M \in L^1(\Omega_{sing}, \mu)^9$ is non-negative and $\mathrm{Tr}M(x) = 1$ μ-almost everywhere in Ω_{sing}. Note also that

$$\lim_{r\to 0} \frac{1}{r} \int\limits_{B(a,r)} |M(a)-M(x)|d\mu(x) = \lim_{r\to 0} \frac{1}{\mu(B(a,r))} \int\limits_{B(a,r)} |M(a)-M(x)|d\mu(x) = 0$$

$$(5.18)$$

for μ-almost every $a \in \Omega_{sing}$. Next, represent the concentration set in the form $\Omega_{sing} = \bigcup_{k\geq 1} \Omega_k$, $\Omega_k = \{x : 1/k < \theta(x) < k\}$. The estimates from the geometric measure theory imply that $1/(2k)\mathcal{H}^1(A) \leq \mu(A) \leq k\mathcal{H}^1(A)$ for any Borel set $A \subset \Omega_k$. Hence, $\mu(A) = 0$ for every set A of zero $\mathcal{H}^1$-measure, and the measure μ is absolutely continuous with respect to the one-dimensional Hausdorff measure. Next recall, cf. [26], that for any function $f \in L^1(\Omega)$, there is a set E of zero $\mathcal{H}^1$-measure such that

$$\lim_{r\to 0} \frac{1}{r} \int\limits_{B(a,r)} |f(x)|dx = 0$$

everywhere in Ω possibly except of the set E. Since μ is absolutely continuous with respect to the Hausdorff $\mathcal{H}^1$, this relation holds true for μ-almost every $a \in \Omega$. In particular, the equalities

$$\lim_{r\to 0} \frac{1}{r} \int\limits_{B(a,r)} \varrho|\mathbf{u}|^2 dx = \lim_{r\to 0} \frac{1}{r} \int\limits_{B(a,r)} \varrho dx$$

$$= \lim_{r\to 0} \frac{1}{r} \int\limits_{B(a,r)} |\Pi|dx = \lim_{r\to 0} \frac{1}{r} \int\limits_{B(a,r)} |g|dx = 0 \qquad (5.19)$$

holds true for μ-almost all $a \in \Omega$. Passing to the limit in Eqs. (1.11) and using the equalities (5.2), (5.17) we conclude that for every $\varphi \in C_0^1(\Omega)^3$,

$$\int\limits_{\Omega} (\nabla\varphi : \varrho\mathbf{u} \otimes \mathbf{u} + \mathrm{div}\,\varphi\varrho)\,dx + \int\limits_{\Omega_{sing}} \nabla\varphi : M(x)d\mu(x) = \int\limits_{\Omega} \nabla\varphi : \Pi dx - \int\limits_{\Omega} \varphi \cdot g dx.$$

$$(5.20)$$

Now fix an arbitrary $a \in \Omega_{sing}$ satisfying (5.18), (5.19) and a vector-valued function $\varphi \in C_0^1(\mathbb{R}^3)^3$. Substituting $\varphi(\varepsilon^{-1}(x-a))$ in (5.20) we arrive at

$$\frac{1}{\varepsilon} \int\limits_{\Omega_{sing}} \nabla\varphi\left(\frac{x-a}{\varepsilon}\right) : M(a)d\mu(x)$$

$$= \frac{1}{\varepsilon} \int\limits_{\Omega_{sing}} \nabla\varphi\left(\frac{x-a}{\varepsilon}\right) : (M(a) - M(x))d\mu(x) + \frac{1}{\varepsilon} \int\limits_{\Omega} \nabla\varphi\left(\frac{x-a}{\varepsilon}\right) : (\Pi - \varrho\mathbf{u} \otimes \mathbf{u})\,dx$$

$$- \frac{1}{\varepsilon} \int\limits_{\Omega} \mathrm{div}\,\varphi\left(\frac{x-a}{\varepsilon}\right)\varrho dx - \frac{1}{\varepsilon} \int\limits_{\Omega} \varphi\left(\frac{x-a}{\varepsilon}\right) \cdot g dx . \qquad (5.21)$$

Note that, by (5.18) and (5.19), the right side of this equality tends to 0 as $\varepsilon \to 0$. Choose an element $\upsilon \in$ VarTan (μ, a) and the sequence ε_i satisfying (5.15). Substituting $\varepsilon = \varepsilon_i$ into (5.21) and passing to the limit we obtain

$$\int\limits_{\mathbb{R}^3} \nabla \varphi(x) : M(a)d\upsilon(x) = 0 \ \text{ for all } \ \varphi \in C_0^1(\mathbb{R}^3)^3. \tag{5.22}$$

It follows from this that the matrix-valued measure $M(a)\upsilon$ satisfies all assumptions of Lemma 1 with $\Pi = 0$, $g = 0$, and $\Omega = \mathbb{R}^3$. Replacing $\mathcal{E}$ by $M(a)\upsilon$ in identity (2.5) and noting that Tr $M(a) = 1$ we obtain

$$\frac{1}{r} \int\limits_{B(0,r)} d\upsilon + \int\limits_{r \le |y| \le R} \frac{1}{|y|}(1 - \mathbf{n} \otimes \mathbf{n} : M(a))d\upsilon + \zeta_0(0, r) = \frac{1}{R} \int\limits_{B(0,R)} d\upsilon, \tag{5.23}$$

where $\zeta_0 \ge 0$ and $\mathbf{n} = y/|y|$. Since $r^{-1}\upsilon(B(0, r)) = R^{-1}\upsilon(B(0, R))$, we conclude from this that

$$\int\limits_{r \le |y| \le R} \frac{1}{|y|}(1 - \mathbf{n} \otimes \mathbf{n} : M(a))d\upsilon = 0 \ \text{ for all } \ r, R.$$

Therefore, $|y| - y \otimes y : M(a) = 0$ for υ-almost every $y \in \mathbb{R}^3$. It is possible if and only if there exists $\mathbf{s}(a) \in \mathbb{S}^2$ such that $M(a) = \mathbf{s}(a) \otimes \mathbf{s}(a)$ and $\upsilon = 2^{-1}\theta(a)\mathcal{H}^1 \lfloor \text{span} \{\mathbf{s}(a)\}$. Hence Tan $(\mu, a) = \text{span} \{\mathbf{s}(a)\}$ for μ-almost every $a \in \Omega_{sing}$. From this we conclude that the measure μ satisfies the hypothesis of Proposition 3. Applying this proposition we obtain (5.16) which completes the proof of Theorem 6.   □

We return to the proof of Theorem 1. It remains to note that the statement of Theorem 1 for $\gamma = 1$ is an obvious consequence of Theorems 5 and 6.

6. Proof of Theorems 2 and 3

Proof of Theorem 2. In order to avoid repetitions we give only the proof for $\gamma > 1$. Fix an arbitrary $\varepsilon > 0$ and consider the family of boundary-value problems, depending on parameter $t \in [0, 1]$,

$$(\alpha\varrho + \sqrt{\varepsilon}\varrho^5)\mathbf{u} + \text{div} \left(\mathbf{u} \otimes (t\varrho\mathbf{u} - \varepsilon\nabla\varrho)\right) + t\nabla p(\varrho) = t\varrho\mathbf{F} + \text{div} \, \Sigma(\mathbf{u}) \ \text{ in } \ \Omega, \tag{6.1a}$$

$$\alpha\varrho + \sqrt{\varepsilon}\varrho^5 + t\text{div} \, (\varrho\mathbf{u}) - \varepsilon\Delta\varrho = tf \ \text{ in } \ \Omega, \quad u = 0, \ \partial_n\varrho = 0 \ \text{ on } \ \partial\Omega. \tag{6.1b}$$

We begin with proving a priori estimates for solution to problem (6.1). Multiplying both sides of Eqs. (6.1a) and (6.1b) by $\frac{\gamma-1}{\gamma}\mathbf{u}$ and $\varrho^{\gamma-1}$, respectively, integrating over Ω, and combining the obtained results we arrive at the identity

$$\frac{\gamma - 1}{\gamma} \int\limits_\Omega \left(|\nabla\mathbf{u}|^2 + (1 + v)|\text{div} \, \mathbf{u}|^2 + \frac{\varrho|\mathbf{u}^2|}{2}(\alpha + \sqrt{\varepsilon}\varrho^4)\right)dx$$

$$+ \int\limits_\Omega \left(\alpha\varrho^\gamma + \sqrt{\varepsilon}\varrho^{4+\gamma} + \varepsilon(\gamma - 1)\varrho^{\gamma-2}|\nabla\varrho|^2\right)dx$$

$$= \int\limits_\Omega \left(t\frac{\gamma - 1}{\gamma}\varrho\mathbf{Fu} + f(\varrho^{\gamma-1} - \frac{|\mathbf{u}|^2}{2})\right)dx \, ,$$

which holds true for every smooth solution $(\varrho, \mathbf{u})$, with a positive density ϱ, to problem (6.1). Noting that for $\nu > -2$,

$$\int_{\Omega} \left(|\nabla \mathbf{u}|^2 + (1+\nu)|\text{div } \mathbf{u}|^2 \right) dx \geq c(\nu) \int_{\Omega} |\nabla \mathbf{u}|^2 dx, \quad \text{where } c(\nu) > 0,$$

we obtain the estimates

$$\|\varrho\|_{L^{\gamma}(\Omega)} + \|\mathbf{u}\|_{H^{1,2}(\Omega)} + \|\varrho|\mathbf{u}|^2\|_{L^1(\Omega)} \leq c, \tag{6.2}$$

$$\varepsilon^{1/2}\|\varrho^{4+\gamma}\|_{L^1(\Omega)} + \varepsilon^{1/2}\|\varrho^5|\mathbf{u}|^2\|_{L^1(\Omega)} + \varepsilon\|(1+\varrho)^{2-\gamma}|\nabla\varrho|^2\|_{L^1(\Omega)} \leq c \tag{6.3}$$

with a constant c independent of t and ε. Next, multiplying both sides of (6.1b) by $\sqrt{\varepsilon}\varrho^{m-1}$, $1 \leq m \leq 4$, and integrating the result over Ω we obtain

$$\int_{\Omega} \left(\varepsilon^{3/2}(m-1)\varrho^{m-2}|\nabla\varrho|^2 + \varepsilon^{1/2}\alpha\varrho^m + \varepsilon\varrho^{4+m} \right) dx$$

$$= \sqrt{\varepsilon} \int_{\Omega} \left(\varrho^{m-1} f - t\frac{m-1}{m}\varrho^m \text{div } \mathbf{u} \right) dx .$$

Noting that

$$\left| \int_{\Omega} \varrho^{m-1} f \, dx \right| + \left| \int_{\Omega} \varrho^m \text{div } \mathbf{u} \, dx \right| \leq c_1 \left(\int_{\Omega} \varrho^{2m} \right)^{1/2} \leq c_2 \left(\int_{\Omega} \varrho^{4+m} \right)^{1/2} + c_3,$$

we arrive at the inequality

$$\int_{\Omega} \left(\varepsilon^{3/2}(m-1)\varrho^{m-2}|\nabla\varrho|^2 + \varepsilon^{1/2}\alpha\varrho^m + \varepsilon\varrho^{4+m} \right) dx \leq c_4 \sqrt{\varepsilon} \left(\int_{\Omega} \varrho^{4+m} \right)^{1/2} + c_5,$$

which with $m = 2$ and $m = 4$ gives the estimates

$$\varepsilon^{3/4}\|\nabla\varrho\|_{L^2(\Omega)} + \varepsilon^{1/8}\|\varrho\|_{L^8(\Omega)} \leq c. \tag{6.4}$$

Next note that (6.2)–(6.4) imply the estimates

$$\|\varrho\nabla\mathbf{u}\|_{L^{8/5}(\Omega)} \leq \|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\varrho\|_{L^8(\Omega)} \leq c\varepsilon^{-1/8}, \tag{6.5a}$$

$$\||\mathbf{u}||\nabla\varrho|\|_{L^{3/2}(\Omega)} \leq \|\mathbf{u}\|_{L^6(\Omega)}\|\nabla\varrho\|_{L^2(\Omega)} \leq c\varepsilon^{-3/4}, \tag{6.5b}$$

$$\|\varrho\mathbf{u}\|_{L^{24/7}(\Omega)} \leq \|\mathbf{u}\|_{L^6(\Omega)}\|\varrho\|_{L^8(\Omega)} \leq c\varepsilon^{-1/8}, \tag{6.5c}$$

$$\|\varrho^5|\|_{L^{8/5}(\Omega)} \leq \|\varrho\|_{L^8(\Omega)}^5 \leq c\varepsilon^{-5/8}. \tag{6.5d}$$

It is easy to see that a solution to (6.1a) has the representation $\varrho = \varrho_1 + \varrho_2$ in which ϱ_i satisfy the equations

$$(\varepsilon\Delta - \alpha)\varrho_1 = t\text{div }(\varrho\mathbf{u}), \quad (\varepsilon\Delta - \alpha)\varrho_2 = \sqrt{\varepsilon}\varrho^5,$$

and the homogeneous Neumann boundary conditions. From this, in view of the embeddings $H^{1,24/7}(\Omega) \hookrightarrow C(\Omega)$, $H^{2,8/5}(\Omega) \hookrightarrow C(\Omega)$, and inequalities (6.5c), (6.5d), we

conclude that $\varrho \le c_\varepsilon$, where a constant c_ε does not depend on t. Since ϱ is bounded from above, Eq. (6.1a) can be rewritten in the form

$$\Delta \mathbf{u} + (1 + \nu)\nabla \mathrm{div}\, \mathbf{u} = \mathrm{div}\, G_2 + G_1 \quad \text{in}\ \Omega,$$

with the terms in the right side satisfying the inequalities $|G_i| \le c_\varepsilon(1 + |\mathbf{u}|^i)$, $i = 1, 2$. Embedding $H^{1,3}(\Omega) \hookrightarrow L^r(\Omega)$, $r > 1$, inequality (6.2) and a priori estimates for elliptic equations yield $\|\mathbf{u}\|_{L^r(\Omega)} \le c_\varepsilon(r)$. Arguing as before we conclude that $\|\mathbf{u}\|_{H^{1,r}(\Omega)} \le c_\varepsilon(r)$ for every $r > 1$ and hence $|\mathbf{u}| \le c_\varepsilon$. Therefore, ϱ and $|\mathbf{u}|$ are bounded by a constant independent of t. From this and the results from the theory of weakly nonlinear elliptic equations, see Theorem 13.1 in [3], we conclude that the inequality

$$\|(\varrho, \mathbf{u})\|_{C^{2+\beta}(\Omega)} < C^*(\varepsilon, \Omega, \|(f, \mathbf{F})\|_{C^\beta(\Omega)}, \alpha, \nu) \tag{6.6}$$

holds for every solution $(\varrho, \mathbf{u}) \in C^{1+\beta}(\Omega)$, $\varrho > 0$, to problem (6.1).

To tackle the existence question we need to reformulate problem (6.1) as a nonlinear operator equation in the form $(\varrho, \mathbf{u}) = \Phi_t(\varrho, \mathbf{u})$. Introduce the mapping $\Phi_t : (\hat{\varrho}, \hat{\mathbf{u}}) \mapsto (\varrho, \mathbf{u})$ defined as a solution to the boundary-value problem

$$\mathrm{div}\, \Sigma(\mathbf{u}) = (\alpha\hat{\varrho} + \sqrt{\varepsilon}\hat{\varrho}^5)\hat{\mathbf{u}} + \mathrm{div}\,\big((t\hat{\varrho}\hat{\mathbf{u}} - \varepsilon\nabla\hat{\varrho}) \otimes \hat{\mathbf{u}}\big) + t\nabla p(\hat{\varrho}) - t\hat{\varrho}\mathbf{F}, \tag{6.7a}$$

$$\varepsilon\Delta\varrho - \alpha\varrho = \sqrt{\varepsilon}\hat{\varrho}^4\varrho + t\,\mathrm{div}\,(\hat{\varrho}\hat{\mathbf{u}}) - f \quad \text{in}\ \Omega, \quad u = 0, \quad \partial_n\varrho = 0 \text{ on } \partial\Omega. \tag{6.7b}$$

Obviously, the mapping $(\hat{\varrho}, \hat{\mathbf{u}}, t) \mapsto (\varrho, \mathbf{u})$ denoted by $\Phi : C^{1+\beta}(\Omega)^4 \times [0, 1] \mapsto C^{2+\beta}(\Omega)^4$ is continuous. The remaining part of the proof relies on the following lemma, but first we recall an abstract result useful for our purposes.

The proof of Lemma 9 is based on the following result from the theory of positive operators, cf. [18], ch.7.3.10. Let E_1, E_2 be Banach spaces with the cones $\mathbb{K}_i \subset E_i$, $i = 1, 2$. A bounded operator $A : E_2 \mapsto E_1$ is said to be positive if $A\mathbb{K}_2 \subset \mathbb{K}_1$.

Proposition 4. *Suppose that bounded operators B_1, $B_2 : E_1 \mapsto E_2$ satisfy the following conditions:*

1. *The operator B_1 has the bounded, positive inverse.*
2. *The operator $B_1^{-1}(B_1 - B_2)$ is compact.*
3. *There is an element $\varrho_0 \in \mathrm{Int}\, \mathbb{K}_1$ and a positive constant $\varkappa$ such that $B_2\varrho_0 \ge \varkappa B_1\varrho_0$.*
4. *$(B_1 - B_2)u \in \mathbb{K}_2$ for all $u \in \mathbb{K}_1$, in other words, $B_1 \ge B_2$.*

Then the operator B_2 has a positive inverse.

The following lemma is an application of the abstract result.

Lemma 9. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with the boundary $\partial\Omega \in C^{2+\beta}$, vector field $\mathbf{u} \in C^{1+\beta}(\Omega)^3$ vanishes on $\partial\Omega$, and a function $b \in C^\beta(\Omega)$ is strictly positive in the closure $\mathrm{cl}\,\Omega$. Then for any non-negative $f \in C^\beta(\Omega)$ the problem*

$$-\varepsilon\Delta\varrho + \mathrm{div}\,(\varrho\mathbf{u}) + b\varrho = f \quad \text{in}\ \Omega, \ \partial_n\varrho = 0 \ \text{on}\ \partial\Omega, \tag{6.8}$$

has a unique strictly positive solution $\varrho \in C^{2+\beta}(\Omega)$.

Proof. Now denote by E_1 the closed subspace of $C^{2+\beta}(\Omega)$ which consists of all functions satisfying the homogeneous boundary Neumann condition on $\partial\Omega$, and set $E_2 = C^\beta(\Omega)$. Let $\mathbb{K}_i \subset E_i$, $i = 1, 2$, be the cones of non-negative functions. Let $L : E_1 \mapsto E_2$ be a linear operator defined by $L\varrho \equiv -\varepsilon\Delta\varrho + \mathrm{div}\,(\varrho\mathbf{u})$. Hence our task is to prove that there exists the bounded positive operator $(L + b)^{-1} : E_2 \mapsto E_1$. We start with proving that

the equation $L\varrho_0 = 0$ has a strictly positive solution $\varrho_0 \in E_1$. It follows from the general theory of boundary-value problems for the second order elliptic differential equations [16] that for fixed $k > \sup_{\Omega}(|\text{div}\,\mathbf{u}(x)| + b(x))$ and all $f \in E_2$, the Neumann problem

$$(L + k)\varrho = f \in E_2, \quad \varrho \in E_1 \tag{6.9}$$

has the unique non-negative solution. By the maximum principle ϱ is non-negative if f is non-negative. Hence the compact positive operator is defined by $A = (L + k)^{-1}$: $E_2 \mapsto E_2$. We can apply the maximum principle ([16], Theorem 9.6) for (6.9), to obtain that ϱ does not attain the non-positive minimum in Ω. Hence a solution to problem (6.9) is strictly positive inside Ω for all $f \in \mathbb{K}_2 \setminus \{0\}$. Moreover, the solution is strictly positive in the closure of Ω. In order to prove the last statement suppose, on the contrary to our claim, that $Af(x_0) = 0$ at some point $x_0 \in \partial\Omega$. Observe that the function $\varrho = Af \in E_1$ is positive inside Ω and satisfies the inequality $(L + k)\varrho \geq 0$ in Ω. Furthermore, ϱ takes the zero value for the minimum at x_0. By Lemma 3.4 in [16] the interior normal derivative of ϱ at point x_0 is strictly positive, which is in contradiction with the homogeneous Neumann boundary condition.

Set $\varrho_1 = A\mathbf{1}$, where $\mathbf{1}$ is the characteristic function of Ω. Since Af continuous and strictly positive, for every $f \in \mathbb{K}_2 \setminus \{0\}$ there exist positive constants α, β depending on f such that $\alpha\varrho_1 \leq Af \leq \beta\varrho_1$. Hence $A : E_2 \mapsto E_2$ is a compact ϱ_1-positive operator, [17]. Classical results from the theory of positive operators, see Theorems 2.8, 2.10 and 2.13 from [17] for example, imply that A has a positive simple eigenvalue λ_0 such that the corresponding eigenfunction is strictly positive and $\lambda_0 > |\lambda|$ for any eigenvalue $\lambda \neq \lambda_0$.

Our next task is to show that $\lambda_0 = (k)^{-1}$. We begin with observation that the operator equation $(k)^{-1}\varrho - A\varrho = \mathbf{1}$ is equivalent to the boundary-value problem $L\varrho = k, \varrho \in E_1$, which has no solutions. Therefore, by the Fredholm alternative, k^{-1} is an eigenvalue of the operator A. Let us prove that k^{-1} is the maximal eigenvalue. If this assertion is false, then there exists the positive eigenvalue $\lambda_0 > k^{-1}$. By the definition of A, the eigenfunction $\varrho_0 \in E_1$ satisfying the equation $\lambda_0\varrho_0 - A\varrho_0 = 0$ is a solution to the boundary-value problem $(L + \tau)\varrho_0 = 0, \varrho_0 \in E_1$, where $\tau = k - \lambda_0^{-1} > 0$. Let us consider the parabolic boundary-value problem

$$\frac{\partial v}{\partial t} - \varepsilon\Delta v + \text{div}\,(v\mathbf{u}) = 0 \text{ in } \Omega \times (0, \infty), \quad \nabla v \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \times (0, \infty),$$

$$v(x, 0) = v_0(x) \text{ in } \Omega.$$

For any $v_0 \in C^\beta(\Omega)$ this problem has the unique smooth solution, which is positive if v_0 is non-negative. Introduce the operator $V(t) : v_0(\cdot) \mapsto v(\cdot, t)$. Obviously $V(t)$ preserves the order and the charge, i.e., $V(t)v_0' \geq V(t)v_0$ for any $v_0' \geq v_0$ and $\int_\Omega V(t)v_0(x)dx = \int_\Omega v_0 dx$. Since for every $u, v \in C^\beta(\Omega)$ the function $\max\{u, v\} \in C^\beta(\Omega)$ we can apply the Crandall-Tartar Theorem [7] which implies that $V(t)$ is a non-expansive operator in the metric of $L^1(\Omega)$. In particular, we have $\|V(t)\varrho_0\|_{L^1(\Omega)} = \|\varrho_0\|_{L^1(\Omega)}$. On the other hand, equation $(L + \tau)\rho_0 = 0$ implies that $V(t)\varrho_0 = e^{\tau t}\varrho_0$. Hence $\tau = 0$ which gives $\lambda_0 = k^{-1}$ and the assertion follows. Recall that ϱ_0 satisfies the equation $L\varrho_0 = 0$.

Therefore, the operators $B_1 = (L + k)$, $B_2 = (L + b) : E_2 \mapsto E_1$ are continuous and the inverse $(L + k)^{-1} = A : E_1 \mapsto E_2$ is positive. Moreover, $L + k \geq L + b$ and $(L + b)\varrho_0 = b\varrho_0 \geq \varkappa k\varrho_0 = \varkappa(L + k)\varrho_0$ for some positive $\varkappa$. Obviously $\varrho_0 > 0$ is the interior point of the cones $\mathbb{K}_1$ and $\mathbb{K}_2$. From this we conclude that the operator $L + b$ has the bounded positive inverse, which completes the proof. $\square$

Let us turn back to the proof of Theorem 2. Denote by $\mathcal{J}$ a subset of $C^{1+\beta}(\Omega)^4$ defined by the inequalities $\{(\varrho, \mathbf{u}) : \varrho \geq 0, \|(\varrho, \mathbf{u})\|_{C^{1+\beta}(\Omega)} \leq C^*\}$. It follows from Lemma 9 that every fixed point $(\varrho, \mathbf{u})$ of Φ_t satisfies inequality (6.6). Moreover, $\varrho \in \mathcal{J}$ is strictly positive. Hence there are no fixed points of Φ_t at $\partial\mathcal{J}$ for all $t \in [0, 1]$. On the other hand, the mapping Φ_0 has the unique fixed point inside $\mathcal{J}$. By the Leray-Schauder fixed point Theorem, problem (6.1) has a solution $(\varrho, \mathbf{u}) \in \mathrm{int}\, \mathcal{J}$. It remains to note that estimates (1.15) follows from (6.2)–(6.5d) and the proof of Theorem 2 is completed.

Proof of Theorem 3. The proof is based on the following lemmas.

Lemma 10. *Let* $(\mathbf{u}_\varepsilon, \varrho_\varepsilon)$ *be a sequence of solutions to problem* (1.14)*. Then there is a subsequence, still denoted by* $(\mathbf{u}_\varepsilon, \varrho_\varepsilon)$*, such that*

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } H_0^{1,2}(\Omega), \varrho_\varepsilon \to \varrho \text{ weakly in } L^\gamma(\Omega), \varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho\mathbf{u} \text{ weakly in } L^1(\Omega),$$
$$(6.10)$$

$$\sqrt{\varepsilon}\|\varrho_\varepsilon^5\|_{L^1(\Omega)} + \sqrt{\varepsilon}\|\varrho_\varepsilon^5\mathbf{u}_\varepsilon\|_{L^1(\Omega)} + \varepsilon\||\mathbf{u}_\varepsilon|\nabla\varrho_\varepsilon\|_{L^{3/2}(\Omega)} \to 0 \text{ as } \varepsilon \to 0. \qquad (6.11)$$

Proof. The first two convergences in (6.10) obviously follows from Theorem 2. In order to prove the third convergence note that by Egoroff's theorem for every $\eta > 0$ there exists $E \subset \Omega$ such that $\mathbf{u}_\varepsilon \to \mathbf{u}$ uniformly on E, and $\mathrm{meas}(\Omega \setminus E) < \eta$ which yields

$$\limsup_{\varepsilon\to 0}\left|\int_\Omega h(\varrho_\varepsilon\mathbf{u}_\varepsilon - \varrho\mathbf{u})dx\right| \leq 2\|h\|_{L^\infty(\Omega)}\limsup_{\varepsilon\to 0}\int_E \varrho_\varepsilon|\mathbf{u}_\varepsilon|dx$$

$$\leq c\limsup_{\varepsilon\to 0}\left(\int_E \varrho_\varepsilon dx\right)^{1/2}\left(\int_E \varrho_\varepsilon|\mathbf{u}_\varepsilon|^2dx\right)^{1/2}$$

$$\leq c\limsup_{\varepsilon\to 0}\left(\int_E \varrho_\varepsilon dx\right)^{1/2} \to 0 \quad \text{as } \eta \to 0.$$

Since, by (1.15b) and (1.15e), the sequence $\sqrt{\varepsilon}\varrho_\varepsilon^5\ln(1 + \varrho_\varepsilon)$ is bounded in $L^1(\Omega)$, we have

$$\limsup_{\varepsilon\to 0}\sqrt{\varepsilon}\int_\Omega \varrho_\varepsilon^5 dx \leq \limsup_{\varepsilon\to 0}\sqrt{\varepsilon}\int_{\varrho_\varepsilon > N}\varrho_\varepsilon^5 dx$$

$$\leq \ln(1 + N)^{-1}\limsup_{\varepsilon\to 0}\sqrt{\varepsilon}\int_{\varrho_\varepsilon > N}\varrho_\varepsilon^5\ln(1 + \varrho_\varepsilon)dx$$

$$\leq c\ln(1 + N)^{-1} \to 0 \text{ as } N \to \infty,$$

which implies the convergence of the first term in (6.11). Noting that

$$\sqrt{\varepsilon}\|\varrho_\varepsilon^5\mathbf{u}_\varepsilon\|_{L^1(\Omega)} \leq N\sqrt{\varepsilon}\|\varrho_\varepsilon^5\|_{L^1(\Omega)} + N^{-1}\sqrt{\varepsilon}\|\varrho_\varepsilon^5|\mathbf{u}_\varepsilon|^2\|_{L^1(\Omega)}$$
$$\leq N\sqrt{\varepsilon}\|\varrho_\varepsilon^5\|_{L^1(\Omega)} + cN^{-1}$$

we obtain $\limsup\limits_{\varepsilon\to 0}\sqrt{\varepsilon}\|\varrho_\varepsilon^5\mathbf{u}_\varepsilon\|_{L^1(\Omega)} \leq cN^{-1}$, which gives the convergence of the second term in (6.11). It remains to note that, by (1.15d),

$$\varepsilon\||\mathbf{u}_\varepsilon|\nabla\varrho_\varepsilon\|_{L^{3/2}(\Omega)} \leq \varepsilon\|\nabla\varrho_\varepsilon\|_{L^2(\Omega)}\|\mathbf{u}_\varepsilon\|_{L^6(\Omega)} \leq c\varepsilon^{1/4},$$

and the lemma follows. $\square$

Lemma 11. *If* $\gamma = 1$, *then* $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \varrho\mathbf{u} \otimes \mathbf{u} + \mathcal{S}$ *in* $\mathcal{D}'(\Omega)$, *where* $\mathcal{S}$ *meets all requirements of Theorem 1. If* $\gamma > 1$ *then for any* $\Omega' \Subset \Omega$,

$$\varrho_\varepsilon \to \varrho \text{ weakly in } L^{\gamma(1+\kappa)}(\Omega'), \quad \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \varrho\mathbf{u} \otimes \mathbf{u} \text{ weakly in } L^{(1+\kappa)}(\Omega').$$

Proof. We start with the observation that Eq. (1.14a) can be written in the form (1.11) with the terms in the right side defined by

$$\Pi_\varepsilon^{(1)} = \Sigma(\mathbf{u}_\varepsilon), \quad \Pi_\varepsilon^{(2)} = -\varepsilon\nabla\varrho_\varepsilon \otimes \mathbf{u}_\varepsilon, \quad g_\varepsilon = \varrho_\varepsilon \mathbf{F} - \varrho_\varepsilon \mathbf{u}_\varepsilon - \sqrt{\varepsilon}\varrho_\varepsilon^5 \mathbf{u}_\varepsilon. \quad (6.12)$$

Hence the lemma will be proved if we show that the functions satisfy all assumptions of Theorem 1. Since the sequence $\Pi_\varepsilon^{(1)}$ is bounded in $L^2(\Omega)^9$, it follows from (6.10) and (6.11) that it suffices to show that $\Pi_\varepsilon^{(2)}$ satisfy (1.9). To this end note that formulae (1.10) and (6.12) yield the representation

$$\mathfrak{S}\Pi_\varepsilon^{(2)}(x, r, R) = \int_r^R \frac{dt}{t} \int_{|x-y|=t} \delta_i \varrho_\varepsilon \mathbf{u}_{\varepsilon,i} dS,$$

where $\delta_i = \partial_i - \mathbf{n}_i \mathbf{n}_k \partial_k$ denotes the tangential derivatives on the sphere $\{y \in \mathbb{R}^3 : |x - y| = t\}$ for a fixed x. Integrating by parts over this sphere we obtain the identity

$$\mathfrak{S}\Pi_\varepsilon^{(2)}(x, r, R) = -\int_r^R \frac{dt}{t} \int_{|x-y|=t} \varrho_\varepsilon \delta_i \mathbf{u}_{\varepsilon,i} dS - \int_r^R \frac{dt}{t^2} \int_{|x-y|=t} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{n} dS,$$

which along with (1.15c) implies the estimate

$$|\mathfrak{S}\Pi_\varepsilon^{(2)}(x, r, R)| \le \varepsilon \int_{B(x,R)} \left(\frac{|\varrho_\varepsilon \nabla\mathbf{u}_\varepsilon|}{|y|} + \frac{|\varrho_\varepsilon \mathbf{u}_\varepsilon|}{|y|^2} \right) dy$$

$$\le c\varepsilon\|\varrho_\varepsilon \nabla\mathbf{u}_\varepsilon\|_{L^{8/5}(\Omega)} + c\varepsilon\|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^{27/4}(\Omega)} \le c\varepsilon^{7/8} \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence $\Pi_\varepsilon^{(i)}$ and g_ε satisfy all conditions of Theorem 1 which completes the proof. □

It follows from Lemma 11 that for every $\Omega' \Subset \Omega$ the sequence p_ε converges weakly in $L^{1+\kappa}(\Omega')$ to some function $\overline{p} \in L^1(\Omega) \cap L_{loc}^{1+\kappa}(\Omega)$. The following theorem plays an important role in the proof of Theorem 3.

Theorem 7. *Let* $\gamma > 1$. *Then* $\overline{p} = p(\varrho)$.

Proof. Sections 7, 8 are devoted to the proof. □

It remains to note that the statement of Theorem 3 is an obvious consequence of Lemmas 10, 11 and Theorem 7.

7. Young Measures, Viscosity Flux, and Oscillations

Young measures and distribution functions. In this paragraph we give the representations of weak limits of approximate solutions via the Young measures. Let us consider the sequence of solutions to problem (1.14). It follows from Lemma 11 that there exists a subsequence, still denoted by $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$, which enjoys the following property. Let positive γ and κ be given. For any continuous function $G : \Omega \times \mathbb{R}_\lambda \mapsto \mathbb{R}$ satisfying the conditions

$$(1 + |\lambda|)^{-\gamma(1+\kappa)} \sup_{x \in \Omega} |G(x, \lambda)| \to 0 \quad \text{as } |\lambda| \to \infty \tag{7.1}$$

and $\Omega' \Subset \Omega$, the sequence $G(x, \varrho_\varepsilon)$ converges weakly in $L^1(\Omega')$ to some function $\overline{G} \in L^1_{loc}(\Omega)$. Moreover, for any continuous bounded function $\varphi : \mathbb{R} \mapsto \mathbb{R}$, the functions $\varphi(\varrho_\varepsilon)\mathrm{div}\,\mathbf{u}_\varepsilon$ converge weakly in $L^2(\Omega)$ to some function $\overline{\varphi \mathrm{div}\,\mathbf{u}}$. The Ball's version [5] of the fundamental theorem on Young measures implies the following result.

Lemma 12. *There exists a weakly measurable family of probability measures* $\sigma_x \in C_0(\mathbb{R})^*$ *with* $\mathrm{spt}\sigma_x \subset [0, \infty)$ *such that the equality*

$$\overline{G} = \int_{[0,\infty)} G(x, \lambda) d\sigma_x(\lambda) \tag{7.2}$$

holds for any function G satisfying condition (7.1). Moreover, the function

$$x \mapsto \int_{[0,\infty)} \lambda^{\gamma(1+\kappa)} d\sigma_x(\lambda)$$

belongs to $L^1_{loc}(\Omega)$. If the function G satisfy more restrictive condition $(1 + |\lambda|)^{-\gamma}$ $\sup_{x \in \Omega} |\, G(x, \lambda)| \to 0$ for $|\lambda| \to \infty$, then $G(x, \varrho_\varepsilon) \to \overline{G}$ weakly in $L^1(\Omega)$. In particular, the functions $\overline{p} \in L^1(\Omega)$ and $\varrho \in L^\gamma(\Omega)$ are given by

$$\overline{p}(x) = \int_{[0,\infty)} \lambda^\gamma d\sigma_x(\lambda), \quad \varrho(x) = \int_{[0,\infty)} \lambda d\sigma_x(\lambda). \tag{7.3}$$

For technical reasons it is convenient to replace the measure σ_x with its distribution function $\Gamma(x, \lambda) := \sigma_x(-\infty, \lambda]$ such that the measure σ_x is the Stieltjes measure $d_\lambda \Gamma(x, \lambda)$. The distribution function is measurable with respect to $(x, \lambda) \in \Omega \times \mathbb{R}_\lambda$, monotone and continuous from the right in λ,

$$\Gamma(x, \lambda) = 0 \text{ for } \lambda < 0, \quad \Gamma(x, \lambda) \nearrow 1 \quad \text{as } \lambda \nearrow \infty. \tag{7.4}$$

Recalling that σ_x is the Stieltjes measure associated with $\Gamma(x, \cdot)$ we get the following formulae:

$$\overline{p}(x) = \gamma \int_{[0,\infty)} \lambda^{\gamma-1}(1 - \Gamma(x, \lambda))d\lambda, \quad \varrho(x) = \int_{[0,\infty)} (1 - \Gamma(x, \lambda))d\lambda. \tag{7.5}$$

Remark 1. Relations (7.5) imply that equality $\Gamma(1 - \Gamma) = 0$ a.e. in $\Omega \times \mathbb{R}_\lambda$ which yields $\overline{p} = p(\varrho)$.

In the rest of this paragraph we consider the family of Radon measures m_ε on $\Omega \times \mathbb{R}_\lambda$ defined by

$$\int_{\Omega \times \mathbb{R}_\lambda} \Phi(x, \lambda) dm_\varepsilon(x, \lambda) := \varepsilon \int_\Omega \Phi(x, \varrho_\varepsilon(x)) |\nabla \varrho_\varepsilon(x)|^2 dx \qquad (7.6)$$

for each $\Phi \in C_0(\Omega \times \mathbb{R}_\lambda)$. It follows from Theorem 2 that there exist a subsequence, still denoted by $\{\varrho_\varepsilon\}$, and a Radon measure m on $\Omega \times \mathbb{R}_\lambda$ such that

$$\int_{\Omega \times \mathbb{R}_\lambda} \Phi(x, \lambda) dm_\varepsilon(x, \lambda) \to \int_{\Omega \times \mathbb{R}_\lambda} \Phi(x, \lambda) dm(x, \lambda) \quad \text{for all } \Phi \in C_0(\Omega \times \mathbb{R}_\lambda). \quad (7.7)$$

Obviously $m(\Omega \times (-\infty, 0)) = 0$. Moreover, estimates (1.15d) from Theorem 2 yield

$$\int_{\Omega \times \mathbb{R}_\lambda} (1 + |\lambda|)^{-1} dm(x, \lambda) < \infty . \qquad (7.8)$$

The effective viscosity flux. Following [19] we introduce the quantity $V(\varrho, \mathbf{u}) = p(\varrho) - (2+\nu)\mathrm{div}\, \mathbf{u}$ called the effective viscous flux. As it was shown in [19, 12, 14] the effective viscous flux enjoys many remarkable properties. The most important is the multiplicative relation $\overline{\varphi(\varrho)V} = \overline{\varphi(\varrho)}\ \overline{V}$ which was proved in [19] for all $\gamma > 3/2$. The simple proof of this result, based on the new version of the compensated compactness principle, was given in papers [12, 14]. In our case, by Lemma 11, the critical estimate $\||\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^{(1+\kappa)}(\Omega')} \le c(\Omega')$ holds for every $\Omega' \Subset \Omega$, which leads to the following local version of the compensated compactness result from [14].

Let us consider a function $\Phi \in C(\Omega \times \mathbb{R})$ such that

$$\Phi(\cdot, \lambda) \in C_0(\Omega) \text{ for all } \lambda \in \mathbb{R}^+, \quad \Phi(\cdot, \lambda) = \Phi_\infty(\cdot) \in C_0(\Omega) \text{ for all } \lambda > N > 0. \tag{7.9}$$

Lemma 13. *Let* $(\mathbf{u}_\varepsilon, \varrho_\varepsilon)$ *be a sequence of solutions to problem* (1.14) *satisfying the hypotheses of Theorem 2. Then for* $\Phi \in C(\Omega \times \mathbb{R})$ *satisfying* (7.9),

$$\int_\Omega \overline{\Phi(\cdot, \varrho)V(\varrho, \mathbf{u})} dx = \int_\Omega \overline{\Phi}\ \overline{V} dx, \quad \text{where } \overline{V} = \overline{p} - (2+\nu)\mathrm{div}\, \mathbf{u}. \qquad (7.10)$$

Proof. We start with the observation that for every $\delta > 0$ there exists a function

$$\Phi_\delta(x, \lambda) = \sum_{k=0}^n h_k(x)\varphi_k(\lambda), \quad h_k \in C_0^\infty(\Omega), \quad \varphi_k \in C^\infty(\mathbb{R}^+), \qquad (7.11)$$

such that $\varphi_k'(\lambda) = 0$ for $\lambda > 2N$, and $|\Phi_\delta - \Phi| \le \delta$. In order to construct Φ_δ note that there are functions $\psi_k \in C^\infty(0, 2N)$, $h_k \in C_0^\infty(\Omega)$, $k = 1, .., n$ and $h_0 \in C_\Omega^\infty$ such that

$$\left| \Phi(x, \lambda) - \Phi_\infty(x) - \sum_{k=1}^n h_k(x)\psi_k(x) \right| \le \frac{\delta}{2}, \quad |\Phi_\infty(x) - h_0(x)| \le \frac{\delta}{2}$$

for all $(x, \lambda) \in \Omega \times [0, 2N]$. Set $\varphi_0 = 1$,

$$\varphi_k = \eta(\lambda)\psi_k(\lambda) \ \text{ for } \lambda \le 2N, \ \text{ and } \varphi_k = 0 \ \text{ otherwise }, 1 < k \le n, \qquad (7.12)$$

where $\eta : \mathbb{R} \mapsto [0, 1]$ is a smooth function satisfying the conditions

$$\eta(\lambda) = 1 \ \text{ for } \lambda \le N, \quad \eta(\lambda) = 0 \ \text{ for } \lambda \ge 2N.$$

It is clear that the function Φ_δ defined by (7.11), (7.12) is the desired approximation. Hence it suffices to prove the proposition for $\Phi(x, \lambda) = h(x)\varphi(\lambda)$ with $h \in C_0^\infty(\Omega)$ and $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi'(\lambda) = 0$ for large λ.

Denote by $\mathbf{1}_\Omega$ the extension operator such that $\mathbf{1}_\Omega u = u$ in Ω and $\mathbf{1}_\Omega u = 0$ outside Ω. The adjoint operator $\mathbf{1}_\Omega^*$ assigns to every function its restriction to Ω. Introduce the linear vectorial operator A and the matrix operator R with the components

$$A_i = \mathbf{1}_\Omega^* \Delta^{-1} \partial_{x_i} \mathbf{1}_\Omega, \quad R_{ij} = \mathbf{1}_\Omega^* \partial_{x_i} \Delta^{-1} \partial_{x_j} \mathbf{1}_\Omega, \quad 1 \le i, j \le 3.$$

Recall that the operators $R_{ij} : L^r(\Omega) \mapsto L^r(\Omega)$, $A_i : L^r(\Omega) \mapsto H^{1,r}(\Omega)$ are bounded for every $r > 1$. Multiplying regularized moment balance equation (1.14a) by h we arrive at

$$h(\alpha\varrho_\varepsilon + \sqrt{\varepsilon}\varrho_\varepsilon^5)\mathbf{u}_\varepsilon + \text{div}\left\{ h\mathbf{u}_\varepsilon \otimes (\varrho_\varepsilon \mathbf{u}_\varepsilon - \varepsilon\nabla\varrho_\varepsilon) - \Sigma(h\mathbf{u}_\varepsilon) + hp_\varepsilon\mathbf{I} + \mathbf{u}_\varepsilon \otimes \nabla h \right.$$

$$\left. +\nabla h \otimes \mathbf{u}_\varepsilon + \nu\nabla h \cdot \mathbf{u}_\varepsilon\mathbf{I} \right\} - \left\{ \mathbf{u}_\varepsilon \otimes (\varrho_\varepsilon \mathbf{u}_\varepsilon - \varepsilon\nabla\varrho_\varepsilon) - \Sigma(\mathbf{u}_\varepsilon) + p_\varepsilon\mathbf{I} \right\}\nabla h = h\varrho_\varepsilon\mathbf{F}.$$

Next apply to both sides of this identity the operator A to obtain

$$A \cdot \{h(\alpha\varrho_\varepsilon + \sqrt{\varepsilon}\varrho_\varepsilon^5)\mathbf{u}_\varepsilon - h\varrho_\varepsilon\mathbf{F}\} + R : \{h\mathbf{u}_\varepsilon \otimes (\varrho_\varepsilon \mathbf{u}_\varepsilon - \varepsilon\nabla\varrho_\varepsilon) - \Sigma(h\mathbf{u}_\varepsilon) + hp_\varepsilon\mathbf{I}$$

$$+2\mathbf{u}_\varepsilon \otimes \nabla h + \nu\nabla h \cdot \mathbf{u}_\varepsilon\mathbf{I}\} = A \cdot \left\{ \left(\mathbf{u}_\varepsilon \otimes (\varrho_\varepsilon \mathbf{u}_\varepsilon - \varepsilon\nabla\varrho_\varepsilon) - \Sigma(\mathbf{u}_\varepsilon) + p_\varepsilon\mathbf{I} \right)\nabla h \right\}. \tag{7.13}$$

Since h is compactly supported in Ω, we have

$$R : \{-\Sigma(h\mathbf{u}_\varepsilon) + hp_\varepsilon\mathbf{I}\} = hV(\varrho_\varepsilon, \mathbf{u}_\varepsilon) - (2 + \nu)\nabla h \cdot \mathbf{u}_\varepsilon,$$
$$R : ((\nabla h \cdot \mathbf{u}_\varepsilon)\mathbf{I}) = \nabla h \cdot \mathbf{u}_\varepsilon. \tag{7.14}$$

Multiplying both sides of (7.13) by $\varphi_\varepsilon = \varphi(\varrho_\varepsilon)$, integrating the result over Ω, and using relations (7.14) we obtain

$$\int_\Omega h\varphi_\varepsilon V(\varrho_\varepsilon, \mathbf{u}_\varepsilon)dx + \int_\Omega \varphi_\varepsilon \big(\mathcal{P}_\varepsilon + \mathcal{Q}_\varepsilon + R : (h\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)\big)dx = 0, \quad (7.15)$$

$$\mathcal{P}_\varepsilon = -2\nabla h \cdot \mathbf{u}_\varepsilon + 2R : (\mathbf{u}_\varepsilon \otimes \nabla h)$$
$$\qquad\qquad -A \cdot \big\{(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - \Sigma(\mathbf{u}_\varepsilon) + p_\varepsilon\mathbf{I})\nabla h + \varrho_\varepsilon h(\mathbf{F} - \alpha\mathbf{u}_\varepsilon)\big\},$$
$$\mathcal{Q}_\varepsilon = A \cdot \big\{h\sqrt{\varepsilon}\varrho_\varepsilon^5\mathbf{u}_\varepsilon + (\varepsilon\mathbf{u}_\varepsilon \otimes \nabla\varrho_\varepsilon)\nabla h\big\} - R : (\varepsilon h\mathbf{u}_\varepsilon \otimes \nabla\varrho_\varepsilon).$$

On the other hand, multiplying both sides of regularized mass balance equation (1.14b) by $h\varphi'(\varrho_\varepsilon)$ we get

$$h\varphi_\varepsilon'(\varrho_\varepsilon)\big(\alpha\varrho_\varepsilon + \sqrt{\varepsilon}\varrho_\varepsilon^5\big) + \text{div}\,(h\varphi_\varepsilon\mathbf{u}_\varepsilon) + h\Big(\varphi_\varepsilon'(\varrho_\varepsilon)\varrho_\varepsilon - \varphi_\varepsilon\Big)\text{div}\,\mathbf{u}_\varepsilon - \varphi_\varepsilon\nabla h \cdot \mathbf{u}_\varepsilon$$

$$= \varepsilon\Big[\Delta(h\varphi_\varepsilon) - 2\nabla h \cdot \nabla\varphi_\varepsilon - \varphi_\varepsilon\Delta h - h\varphi''(\varrho_\varepsilon)|\nabla\varrho_\varepsilon|^2\Big] + h\varphi'(\varepsilon)f. \tag{7.16}$$

Introduce the vector field

$$\mathbf{v}_\varepsilon = \varrho_\varepsilon^{(n)} \mathbf{u}_\varepsilon \text{ where } \varrho_\varepsilon^{(n)} = \min\{n, \varrho\}, n \geq 1.$$

Applying the operator $\mathbf{v}_\varepsilon \cdot A := v_{\varepsilon,i} A_i$ to both sides of (7.16), integrating the result over Ω, and using the identities

$$\int_\Omega \mathbf{v}_\varepsilon \cdot A\Delta(h\varphi_\varepsilon)dx = \int_\Omega \nabla(h\varphi_\varepsilon)R\mathbf{v}_\varepsilon dx,$$

$$\int_\Omega \mathbf{v}_\varepsilon \cdot A(h\varphi''(\varrho_\varepsilon)|\nabla\varrho_\varepsilon|^2)dx = -\int_\Omega h\varphi''(\varrho_\varepsilon)|\nabla\varrho_\varepsilon|^2 A \cdot \mathbf{v}_\varepsilon dx$$

we arrive at the equality

$$\int_\Omega \left\{\mathbf{v}_\varepsilon(\mathfrak{P}_\varepsilon + \mathfrak{Q}_\varepsilon) - h(A \cdot \mathbf{v}_\varepsilon)I_\varepsilon\right\}dx + \int_\Omega \mathbf{v}_\varepsilon R\left(h\varphi_\varepsilon\mathbf{u}_\varepsilon\right)dx = 0, \tag{7.17}$$

$$\mathfrak{P}_\varepsilon = A \cdot \left\{\alpha h\varphi'(\varrho_\varepsilon)\varrho_\varepsilon + h(\varphi'(\varrho_\varepsilon)\varrho_\varepsilon - \varphi_\varepsilon)\text{div }\mathbf{u}_\varepsilon - \varphi_\varepsilon\nabla h \cdot \mathbf{u}_\varepsilon - h\varphi'(\varrho_\varepsilon)f\right\},$$

$$\mathfrak{Q}_\varepsilon = A \cdot \{2\varepsilon\nabla h\nabla\varphi_\varepsilon + \varepsilon\varphi_\varepsilon\Delta h + h\sqrt{\varepsilon}\varrho_\varepsilon^5\} - \varepsilon R\nabla(h\varphi_\varepsilon),$$

$$I_\varepsilon = \varepsilon\varphi''(\varrho_\varepsilon)|\nabla\varrho_\varepsilon|^2.$$

Combining (7.15), (7.17) and noting that

$$\int_\Omega \left(\mathbf{v}_\varepsilon R(h\varphi_\varepsilon\mathbf{u}_\varepsilon) - \varphi_\varepsilon R : (h\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)\right)dx = \int_\Omega \left(\mathbf{v}_{\varepsilon i} R_{ij}(h\varphi_\varepsilon\mathbf{u}_{\varepsilon j}) - \varphi_\varepsilon R_{ij}(h\mathbf{v}_{\varepsilon i}\mathbf{u}_{\varepsilon j})\right)dx$$

$$+ \int_\Omega \varphi_\varepsilon R : [(\varrho_\varepsilon^{(n)} - \varrho_\varepsilon)h\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon]dx = \int_\Omega hu_{\varepsilon j}\left(\varphi_\varepsilon R_{ij}\mathbf{v}_{\varepsilon i} - \mathbf{v}_{\varepsilon i} R_{ij}\varphi_\varepsilon\right)dx$$

$$+ \int_\Omega \varphi_\varepsilon R : [(\varrho_\varepsilon^{(n)} - \varrho_\varepsilon)h\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon]dx,$$

we obtain the equality

$$\int_\Omega h\varphi_\varepsilon V(\varrho_\varepsilon, \mathbf{u}_\varepsilon)dx = \int_\Omega \left(\mathbf{v}_\varepsilon(\mathfrak{P}_\varepsilon + \mathfrak{Q}_\varepsilon) - \varphi_\varepsilon(P_\varepsilon + Q_\varepsilon) + \mathfrak{R}_\varepsilon - h(A \cdot \mathbf{v}_\varepsilon)I_\varepsilon\right)dx + \varpi_\varepsilon, \tag{7.18}$$

in which components of the vector $\mathfrak{R}_\varepsilon$ and the scalar ϖ_ε defined by

$$\mathfrak{R}_{\varepsilon,i} = hu_{\varepsilon,j}\left(\varphi_\varepsilon R_{ij}\mathbf{v}_{\varepsilon i} - \mathbf{v}_{\varepsilon i} R_{ij}\varphi_\varepsilon\right), \quad \varpi_\varepsilon = \int_\Omega \varphi_\varepsilon R : [(\varrho_\varepsilon^{(n)} - \varrho_\varepsilon)h\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon]dx. \tag{7.19}$$

Recall that $\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\nabla h \to \varrho\mathbf{u} \otimes \mathbf{u}\nabla h$ and $p_\varepsilon\nabla h \to \overline{p}\nabla h$ weakly in $L^{1+\kappa}(\Omega)$ as $\varepsilon \to 0$. Hence,

$$P_\varepsilon \to \overline{P} \equiv -2\nabla h\mathbf{u} + 2R : (\mathbf{u} \otimes \nabla h) - A \cdot \{(\varrho\mathbf{u} \otimes \mathbf{u} - \Sigma(\mathbf{u}) + \overline{p}\mathbf{I})\nabla h$$
$$+ h\varrho(\mathbf{F} - \alpha\mathbf{u})\} \text{ in } L^r(\Omega) \text{ for some } r > 1, \tag{7.20}$$

$$\mathfrak{P}_\varepsilon \to \overline{\mathfrak{P}} \equiv A \cdot \{\alpha h\overline{\varphi\varrho} + h\overline{(\varphi'\varrho - \varphi)}\text{div }\mathbf{u} - \overline{\varphi}\nabla h \cdot \mathbf{u} - \overline{\varphi'}hf\} \text{ in } L^2(\Omega). \tag{7.21}$$

Estimates (6.11) yield

$$\mathcal{Q}_\varepsilon \to 0 \text{ in } L^1(\Omega) \text{ and } \mathfrak{Q}_\varepsilon \to 0 \text{ in } L^2(\Omega) \quad \text{as } \varepsilon \to 0. \tag{7.22}$$

Next, the functions I_ε are uniformly bounded in $L^1(\Omega)$ and converge weakly in the space of Radon measures $C_0^*(\Omega)$ to the Radon measure M_φ defined by

$$\langle M_\varphi, h \rangle = \int_{\Omega \times [0,\infty)} h(x)\varphi''(\lambda)dm(x, \lambda),$$

where the measure m is given by (7.6) . Since the sequence $\mathbf{v}_\varepsilon$ converges weakly in $L^6(\Omega)$, the continuous functions $A \cdot \mathbf{v}_\varepsilon$ converge uniformly in Ω to $A \cdot \overline{\mathbf{v}}$ which leads to

$$\lim_{\varepsilon \to 0} \int_\Omega h(A \cdot \mathbf{v}_\varepsilon)I_\varepsilon dx = \langle M_\varphi, hA \cdot \overline{\mathbf{v}} \rangle. \tag{7.23}$$

Since the sequences φ_ε and $\mathbf{v}_\varepsilon$ are bounded in $L^\infty(\Omega)$ and $L^6(\Omega)$, respectively, it follows from the compensated compactness Lemma from [14] that

$$\varphi_\varepsilon R_{ij}\mathbf{v}_{\varepsilon i} - \mathbf{v}_{\varepsilon i}R_{ij}\varphi_\varepsilon \to \overline{\varphi}R_{ij}\overline{\mathbf{v}}_i - \overline{\mathbf{v}}_i R_{ij}\overline{\varphi} \text{ weakly in } L^2(\Omega).$$

Therefore, $\mathfrak{R}_\varepsilon$ converges weakly in $L^{3/2}(\Omega)$ to $\overline{\mathfrak{R}} = \{\mathbf{u}_j(\overline{\varphi}R_{ij}\overline{\mathbf{v}}_i - \overline{\mathbf{v}}_i R_{ij}\overline{\varphi})\}$. Passing to the limit in (7.18) and using (7.20)–(7.23) we obtain

$$\int_\Omega h\overline{\varphi V}dx = \int_\Omega \left(\overline{\mathbf{v}\mathfrak{P}} - \overline{\varphi}\overline{\mathcal{P}} + \overline{\mathfrak{R}}\right)dx - \int_\Omega hA \cdot \overline{\mathbf{v}}dM_\varphi(x) + \overline{\varpi} \tag{7.24}$$

with $|\overline{\varpi}| \leq \limsup |\varpi_\varepsilon|$. On the other hand, passage to the limit in equalities (7.13) and (7.16) gives

$$h\alpha\varrho + \operatorname{div}\left\{h\varrho\mathbf{u} \otimes \mathbf{u} - \Sigma(h\mathbf{u}) + h\overline{p}I + \mathbf{u} \otimes \nabla h + \nabla h \otimes \mathbf{u} + \nabla h \cdot \mathbf{u}\right\}$$
$$- \left(\varrho\mathbf{u} \otimes \mathbf{u} - \Sigma(\mathbf{u}) + \overline{p}I\right)\nabla h = h\varrho\mathbf{F}, \tag{7.25}$$
$$\alpha h\overline{\varphi'\varrho} + \operatorname{div}(h\overline{\varphi}\mathbf{u}) + h\overline{(\varphi'\varrho - \varphi)\operatorname{div}\mathbf{u}} - \overline{\varphi}\nabla h \cdot \mathbf{u} + hM_\varphi = \overline{\varphi'}f. \tag{7.26}$$

Applying to both sides of (7.25), (7.26) the operators $\overline{\varphi}A$ and $\overline{\mathbf{v}} \cdot A$, respectively, and arguing as before we obtain

$$\int_\Omega h\overline{\varphi}\overline{V}dx = \int_\Omega \left(\overline{\mathbf{v}\mathfrak{P}} - \overline{\varphi}\overline{\mathcal{P}} + \overline{\mathfrak{R}}\right)dx + \int_\Omega hA \cdot \overline{\mathbf{v}}dM_\varphi(x) + \varpi , \tag{7.27}$$

where

$$\varpi = \int_\Omega \overline{\varphi}R : [h(\overline{\mathbf{v}} - \varrho\mathbf{u}) \otimes \mathbf{u}]dx.$$

Combining (7.24) and (7.27) we finally obtain

$$\int_\Omega h\overline{\varphi}\overline{V}dx - \int_\Omega h\overline{\varphi V}dx = \varpi - \overline{\varpi}. \tag{7.28}$$

By Lemma 11 the sequence $h\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ converges weakly in $L^{1+\kappa}(\Omega)$ to $h\varrho\mathbf{u} \otimes \mathbf{u}$; obviously $\varrho_\varepsilon^{(n)} \mathbf{u}_\varepsilon = \mathbf{v}_\varepsilon$ converges weakly in $L^6(\Omega)$ to $\overline{\mathbf{v}}$. From this we conclude that

$$|\varpi| \leq \|R : [h(\overline{\mathbf{v}} - \varrho\mathbf{u}) \otimes \mathbf{u}]\|_{L^1(\Omega)} \leq c\|h(\overline{\mathbf{v}} - \varrho\mathbf{u}) \otimes \mathbf{u}\|_{L^{1+\kappa/2}(\Omega)}$$
$$\leq c \limsup_{\varepsilon \to 0} \|h(\mathbf{v}_\varepsilon - \varrho_\varepsilon\mathbf{u}_\varepsilon) \otimes \mathbf{u}_\varepsilon\|_{L^{1+\kappa/2}(\mathrm{spt}h)}.$$

The similar arguments give the same bound for $\limsup\limits_{\varepsilon \to 0} |\varpi_\varepsilon|$. Since the sequence $\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ is bounded in $L_{loc}^{1+\kappa}(\Omega)$ and the sequence ϱ_ε is bounded in $L^1(\Omega)$, we have

$$\|h(\mathbf{v}_\varepsilon - \varrho_\varepsilon\mathbf{u}_\varepsilon) \otimes \mathbf{u}_\varepsilon\|_{L^{1+\kappa/2}(\mathrm{spt}h)} \leq c(h)\|(\varrho_\varepsilon^{(n)} - \varrho_\varepsilon)\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\|_{L^{1+\kappa/2}(\mathrm{spt}h)}$$

$$\leq c \left(\int_{\{\varrho_\varepsilon > n\} \cap \mathrm{spt}h} (\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2)^{1+\kappa/2} dx \right)^{2/(2+\kappa)}$$

$$\leq c\|\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^{1+\kappa}(\mathrm{spt}h)} \operatorname{mes} \{\varrho_\varepsilon < n\}^\iota \leq cn^{-\iota},$$

where $\iota = \kappa(1+\kappa)^{-1}(2+\kappa)^{-1} > 0$. Hence $|\varpi| + |\overline{\varpi}| \to 0$ as $n \to \infty$. It remains to note that the left side of (7.28) does not depend on n, and the lemma follows. $\square$

The oscillation defect measure. The notion of oscillation defect measure was introduced in [12] in order to justify the existence theory for isentropic flows with "small" values of the adiabatic constant γ. Following [12, 15] the oscillation defect measure associated with the sequence ϱ_ε is defined as follows:

$$\mathbf{osc}_p[\varrho_\varepsilon \to \varrho](\Omega) := \sup_{k \geq 1} \left(\limsup_{\varepsilon \to 0} \int_\Omega |T_k(\varrho_\varepsilon) - T_k(\varrho)|^p dx \right),$$

where $T_k(z) = kT(z/k)$, $T(z)$ is a smooth concave function, which is equal to z for $z \leq 1$ and is a const for $z \geq 3$. The smoothness properties of T_k are not important and we can take it in the simplest form $T_k(z) = \min\{z, k\}$. In particular, for the sequence of ϱ_ε satisfying the hypotheses of Lemma 12 we have

$$\mathbf{osc}_p[\varrho_\varepsilon \to \varrho](\Omega) = \sup_{k \geq 1} \lim_{\varepsilon \to 0} \int_\Omega |\min\{\varrho_\varepsilon, k\} - \min\{\varrho, k\}|^p dx \geq \sup_{k \geq 1} \int_\Omega |\mathcal{T}_k|^p dx,$$

where $\mathcal{T}_k = \overline{\min\{\varrho, k\}} - \min\{\varrho, k\}$. The important consequence of Lemma 13 is the following version of the result of E. Feireisl et al [12, 14] on the integrability of the oscillations defect measure. In order to formulate the result we introduce the function $\mathcal{T}_\vartheta(x)$ defined by the equality $\mathcal{T}_\vartheta(x) = \overline{\min\{\varrho, \vartheta\}}(x) - \min\{\varrho(x), \vartheta(x)\}$ for any $\vartheta \in C(\Omega)$.

Lemma 14. *Under the assumptions of Theorem 1 and Lemma 12, there is a constant c independent of ϑ such that the inequalities*

$$\|\mathcal{T}_\vartheta\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} \leq \lim_{\varepsilon \to 0} \int_\Omega |\min\{\varrho_\varepsilon(x), \vartheta(x)\} - \min\{\varrho(x), \vartheta(x)\}|^{1+\gamma} dx \leq c \quad (7.29)$$

hold for all $\vartheta \in C(\Omega)$. Recall that the limit in the right side exists by the choice of the sequence ϱ_ε.

Proof. The proof imitates the proof of Lemma 4.3 from [14]. It is easily seen that

$$\|\mathcal{T}_\vartheta\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} \leq \limsup_{\varepsilon \to 0} \int_\Omega |\min\{\varrho_\varepsilon(x), \vartheta(x)\} - \min\{\varrho(x), \vartheta(x)\}|^{1+\gamma} dx. \quad (7.30)$$

Hence, it suffices to show that the right side of this inequality has a bound independent of ϑ. Noting that

$$|\min\{s', \vartheta\} - \min\{s'', \vartheta\}|^{1+\gamma}$$
$$\leq (\min\{s', \vartheta\} - \min\{s'', \vartheta\})(s'^\gamma - s''^\gamma) \quad \text{for all } s', s'' \in \mathbb{R}^+,$$

and $\overline{\varrho^\gamma} \geq \varrho^\gamma$, $\overline{\min\{\varrho, \vartheta\}} \leq \min\{\varrho, \vartheta\}$ we get for any compactly supported, non-negative function $h \in C(\Omega)$,

$$\lim_{\varepsilon \to 0} \int_\Omega h |\min\{\varrho_\varepsilon, \vartheta\} - \min\{\varrho, \vartheta\}|^{1+\gamma} dx$$

$$\leq \lim_{\varepsilon \to 0} \int_\Omega h(\min\{\varrho_\varepsilon, \vartheta\} - \min\{\varrho, \vartheta\})(\varrho_\varepsilon^\gamma - \varrho^\gamma) dx$$

$$\leq \lim_{\varepsilon \to 0} \int_\Omega h(\min\{\varrho_\varepsilon, \vartheta\} - \min\{\varrho, \vartheta\})(\varrho_\varepsilon^\gamma - \varrho^\gamma) dx$$
$$+ \int_\Omega (\overline{\varrho^\gamma} - \varrho^\gamma)(\min\{\varrho, \vartheta\} - \overline{\min\{\varrho, \vartheta\}}) dx$$

$$= \lim_{\varepsilon \to 0} \int_\Omega h(\varrho_\varepsilon^\gamma \min\{\varrho_\varepsilon, \vartheta\} - \overline{\varrho^\gamma} \overline{\min\{\varrho, \vartheta\}}) dx$$

$$= \lim_{\varepsilon \to 0} \int_\Omega h(p_\varepsilon \min\{\varrho_\varepsilon, \vartheta\} - \overline{p} \overline{\min\{\varrho, \vartheta\}}) dx. \quad (7.31)$$

By Lemma 13 with $\Phi(\varrho, x) = h(x) \min\{\varrho, \vartheta(x)\}$, the right side of (7.31), divided by $(2 + v)$, is equal to

$$\lim_{\varepsilon \to 0} \int_\Omega h(\min\{\varrho_\varepsilon, \vartheta\} \operatorname{div} \mathbf{u}_\varepsilon - \overline{\min\{\varrho, \vartheta\} \operatorname{div} \mathbf{u}}) dx$$

$$= \lim_{\varepsilon \to 0} \int_\Omega h(\min\{\varrho_\varepsilon, \vartheta\} - \min\{\varrho, \vartheta\}) \operatorname{div} \mathbf{u}_\varepsilon dx$$

$$- \lim_{\varepsilon \to 0} \int_\Omega h(\min\{\varrho_\varepsilon, \vartheta\} - \min\{\varrho, \vartheta\}) \operatorname{div} \mathbf{u} dx$$

$$\leq \delta \limsup_{\varepsilon \to 0} \int_\Omega h |\min\{\varrho_\varepsilon, \vartheta\} - \min\{\varrho, \vartheta\}|^{1+\gamma} dx$$

$$+ \delta^{-\gamma} \limsup_{\varepsilon \to 0} \int_\Omega h(|\operatorname{div} \mathbf{u}_\varepsilon| + |\operatorname{div} \mathbf{u}|)^{(1+\gamma)/\gamma}$$

$$\leq \delta \limsup_{\varepsilon \to 0} \int_\Omega h |\min\{\varrho_\varepsilon, \vartheta\} - \min\{\varrho, \vartheta\}|^{1+\gamma} dx + c\delta^{-\gamma} \|h\|_{C(\Omega)}. \quad (7.32)$$

Combining (7.32) and (7.31), letting $h \to 1$, and choosing δ sufficiently small gives (7.29).   □

We reformulate this result in terms of the distribution function Γ. Recall that the functions $\min\{\varrho_\varepsilon, \lambda\}$ are uniformly bounded and $\min\{\varrho_\varepsilon, \lambda\}\mathrm{div}\,\mathbf{u}_\varepsilon$ converges weakly in $L^2(\Omega)$ for each non-negative λ. Introduce the functions

$$\mathcal{V}_\lambda = \overline{\left(\min\{\varrho, \lambda\}\mathrm{div}\,\mathbf{u}\right)} - \overline{\min\{\varrho, \lambda\}}\mathrm{div}\,\mathbf{u} \in L^2(\Omega), \tag{7.33}$$

$$\mathfrak{H} = \int_{[0,\infty)} \Gamma(x, s)(1 - \Gamma(x, s))ds, \quad x \in \Omega.$$

Lemma 15. *There is a constant c independent of λ such that*

$$\|\mathfrak{H}\|_{L^{1+\gamma}(\Omega)} + \sup_\lambda \|\mathcal{V}_\lambda\|_{L^1(\Omega)} \le c. \tag{7.34}$$

Proof. We begin with the observation that by Lemma 12,

$$\mathcal{T}_\vartheta(x) = \int_{[0,\infty)} \min\{\lambda, \vartheta(x)\}d_\lambda\Gamma(x, \lambda) - \min\left\{\int_{[0,\infty)} \lambda d_\lambda\Gamma(x, \lambda), \vartheta\right\} \tag{7.35}$$

for each $\vartheta \in C(\Omega)$. From this and the identity $\varrho(x) = \int_{[0,\infty)}(1 - \Gamma(x, \lambda))d\lambda$ we conclude that

$$\mathcal{T}_\vartheta(x) = \int_0^{\vartheta(x)} \Gamma(x, s)ds \text{ for } \vartheta(x) \ge \varrho(x) \text{ and } \mathcal{T}_\vartheta(x) = \int_{\vartheta(x)}^\infty (1 - \Gamma(x, s))ds \text{ otherwise.}$$
$$\tag{7.36}$$

Next choose a sequence of continuous non-negative functions $\vartheta_k \to \varrho$ as $k \to \infty$ a.e. in Ω. By Lemma 14, the functions $\mathcal{T}_{\vartheta_k}$ are uniformly bounded in $L^{1+\gamma}(\Omega)$ and converges a.e. in Ω to the function

$$\mathcal{T}_\varrho(x) = \int_0^{\varrho(x)} \Gamma(x, s)ds = \int_{\varrho(x)}^\infty (1 - \Gamma(x, s))ds,$$

which yields the inclusion $\mathcal{T}_\varrho \in L^{1+\gamma}(\Omega)$. It remains to note that estimate (7.34) for $\mathfrak{H}$ obviously follows from the inequality $\mathfrak{H} \le 2\mathcal{T}_\varrho$.

In order to estimate $\mathcal{V}_\lambda$ note that

$$\mathcal{V}_\lambda = \text{w-}\lim_{\varepsilon \to 0}\left((\min\{\varrho_\varepsilon, \lambda\} - \min\{\varrho, \lambda\})\mathrm{div}\,\mathbf{u}_\varepsilon\right) - \left(\text{w-}\lim_{\varepsilon \to 0}\min\{\varrho_\varepsilon, \lambda\} - \min\{\varrho, \lambda\}\right)\mathrm{div}\,\mathbf{u},$$

where $w - \lim$ is the weak limit in $L^1(\Omega)$. From this and the boundedness of norms $\|\mathrm{div}\,\mathbf{u}_\varepsilon\|_{L^2(\Omega)}$ we obtain

$$\|\mathcal{V}_\lambda\|_{L^1(\Omega)} \le \limsup_{\varepsilon \to 0}(\|\mathrm{div}\,\mathbf{u}_\varepsilon\|_{L^2(\Omega)}$$
$$+ \|\mathrm{div}\,\mathbf{u}\|_{L^2(\Omega)})\|\min\{\varrho_\varepsilon(x), \lambda\} - \min\{\varrho(x), \lambda\}\|_{L^2(\Omega)},$$

which along with (7.29) yields (7.34) and the lemma follows.   □

8. Proof of Theorem 7

It follows from Remark 1 that Theorem 7 will be proved if we show that $\Gamma(x, \lambda)(1 - \Gamma(x, \lambda)) = 0$ almost everywhere in $\Omega \times \mathbb{R}$. We split the proof of this equality into three steps. First, the kinetic equation for the distribution function is derived.

Kinetic formulation of the mass balance equation. In this paragraph we show that the distribution function $\Gamma(x, \lambda)$ of the Young measure, associated with a sequence of solutions to problem (1.14), satisfies some integro-differential transport equation named a kinetic equation. This result is given by the following lemma.

Lemma 16. *Suppose that all assumptions of Theorem 7 are satisfied and Γ is a distribution function of the Young measure associated with a sequence ϱ_ε of solutions to problem (1.14). Then for any $\zeta \in C_0^\infty(\Omega \times \mathbb{R})$,*

$$\int_{\Omega \times \mathbb{R}_\lambda} \zeta \{\lambda \alpha - f(x)\} d_\lambda \Gamma(x, \lambda) dx + \int_{\Omega \times \mathbb{R}_\lambda} \Gamma(x, \lambda) \nabla_{x,\lambda} \zeta \cdot \mathbf{w} d\lambda dx$$

$$+ \int_{\Omega \times \mathbb{R}_\lambda} \partial_\lambda \zeta \, dm(x, \lambda) + \int_{\Omega \times \mathbb{R}_\lambda} \partial_\lambda \zeta \lambda M(x, \lambda) dx = 0. \qquad (8.1)$$

Here m is the Radon measure on $\Omega \times \mathbb{R}_\lambda$ given by (7.7), the solenoidal vector field $\mathbf{w} = (\mathbf{u}(x), -\lambda \operatorname{div} \mathbf{u})$, and the function M is defined by the equalities

$$M = -\frac{1}{2+\nu} \int_{(-\infty, \lambda)} (s^\gamma - \overline{p}) d_s \Gamma(x, s) = \frac{1}{2+\nu} \int_{[\lambda, \infty)} (s^\gamma - \overline{p}) d_s \Gamma(x, s), \quad (8.2)$$

in which $\overline{p}(x) = \int_\mathbb{R} p(\lambda) d_\lambda \Gamma(x, \lambda)$. For almost every x the non-negative function $M(x, \cdot)$ has a bounded variation $\operatorname{Var}_\lambda M(x, \cdot)$, which belongs to $L^1(\Omega)$. Note that the integral identity (8.1) is equivalent to the kinetic equation (1.17), which is understood in the sense of distributions.

Proof. Choose an arbitrary $\varphi \in C^\infty(\mathbb{R})$ vanishing near $+\infty$, and note that for an arbitrary $h \in C_0^\infty(\Omega)$,

$$\int_\Omega \left\{ h\varphi_\varepsilon'(\varrho_\varepsilon)(\alpha \varrho_\varepsilon + \sqrt{\varepsilon} \varrho_\varepsilon^5 - f) - \varphi_\varepsilon \nabla h \cdot \mathbf{u}_\varepsilon + h(\varphi_\varepsilon'(\varrho_\varepsilon)\varrho_\varepsilon - \varphi_\varepsilon) \operatorname{div} \mathbf{u}_\varepsilon \right\} dx$$

$$+ \int_\Omega \varepsilon \left\{ -\Delta h\varphi_\varepsilon + h\varphi''(\varrho_\varepsilon)|\nabla \varrho_\varepsilon|^2 \right\} dx = 0.$$

Letting $\varepsilon \to 0$ we obtain

$$\int_\Omega \left(\alpha \overline{\varphi' \varrho} h - \overline{\varphi} \nabla h \cdot \mathbf{u} - h f \overline{\varphi'} \right) dx$$

$$+ \int_\Omega h\overline{(\varphi'(\varrho)\varrho - \varphi)} \operatorname{div} \mathbf{u} \, dx + \int_{\Omega \times \mathbb{R}_\lambda} h(x) \varphi''(\lambda) dm(x, \lambda) = 0. \qquad (8.3)$$

Lemma 13 gives the expression for the second integral at the right side

$$\int_{\Omega} h\overline{(\varphi'(\varrho)\varrho - \varphi)\mathrm{div}\,\mathbf{u}}dx$$

$$= \int_{\Omega} h\Big[\overline{(\varphi'\varrho - \varphi)\mathrm{div}\,\mathbf{u}} + \frac{1}{2+\nu}\big(\overline{(\varphi'\varrho - \varphi)p} - \overline{(\varphi'\varrho - \varphi)}\overline{p}\big)\Big]dx \ . \quad (8.4)$$

Next choose an arbitrary smooth function $\eta(\lambda)$ vanishing near $+\infty$ and set $\varphi(\lambda) = \int_{\lambda}^{\infty} \eta(s)ds$. Lemma 12 yields the representations

$$\overline{\varphi'\varrho}(x) = -\int_{[0,\infty)} \lambda\eta(\lambda)d_{\lambda}\Gamma(x,\lambda), \quad \overline{\varphi'}(x) = -\int_{[0,\infty)} \eta(\lambda)d_{\lambda}\Gamma(x,\lambda),$$

$$\overline{\varphi}(x) = \int_{[0,\infty)} \Big\{\int_{\lambda}^{\infty} \eta(s)ds\Big\} d_{\lambda}\Gamma(x,\lambda) = \int_{[0,\infty)} \eta(\lambda)\Gamma(x,\lambda)d\lambda,$$

which are substituted into (8.3) and (8.4) and lead to the equality

$$\int_{\Omega\times\mathbb{R}_{\lambda}} h\eta\{f - \lambda\alpha - \lambda\mathrm{div}\,\mathbf{u}\}d_{\lambda}\Gamma dx - \int_{\Omega\times\mathbb{R}_{\lambda}} \eta\mathrm{div}\,(h\mathbf{u})\Gamma dxd\lambda$$

$$- \int_{\Omega\times\mathbb{R}_{\lambda}} h\eta' dm(x,t) + \frac{1}{2+\nu}\int_{\Omega} h\Big(\overline{(\varphi'\varrho - \varphi)p} - \overline{(\varphi'\varrho - \varphi)}\overline{p}\Big)dx. \quad (8.5)$$

Recall that Γ and m vanish for $\lambda < 0$, the function $G(\lambda, x) = (\varphi'(\lambda)\lambda - \varphi(\lambda))h(x)$ satisfies all conditions of Lemma 12, and h is compactly supported in Ω. It follows from this and Lemma 12 that

$$\int_{\Omega} h\Big(\overline{(\varphi'\varrho - \varphi)p} - \overline{(\varphi'\varrho - \varphi)}\overline{p}\Big)dx = -\int_{\Omega\times\mathbb{R}_{\lambda}} h\eta\lambda(\lambda^{\gamma} - \overline{p})d_{\lambda}\Gamma(x,\lambda)dx$$

$$- \int_{\Omega\times\mathbb{R}_{\lambda}} h\Big\{\int_{\lambda}^{\infty} \eta(s)ds\Big\}(\lambda^{\gamma} - \overline{p})d_{\lambda}\Gamma(x,\lambda)dx = (2+\nu)\int_{\Omega\times\mathbb{R}_{\lambda}} h\eta d_{\lambda}\big(\lambda M(x,\lambda)\big)dx.$$

$$(8.6)$$

Since $d_{\lambda}\Gamma(x, \cdot)$ is a probability measure and $p(\lambda) = \lambda^{\gamma}$, the function M is non-negative and $\mathrm{Var}_{\lambda}M \le 2\overline{p} \in L^{1}(\Omega)$. Substituting (8.6) into (8.5) and noting that the linear hull of the set of functions in the form $h\eta$ is dense in $C_{0}^{\infty}(\Omega \times \mathbb{R})$, we obtain (8.1) which completes the proof. □

Renormalization of the kinetic equation. Renormalization of kinetic equation (1.17) is a core of our method. Recall that the notion of a renormalized solution, introduced in the pioneering paper [8], plays an important role in the theory of compressible Navier-Stokes equations developed by P.L. Lions and E. Feireisl et al. Moreover, the kinetic equation itself is a result of the renormalization procedure. Formally we can renormalize Eq. (1.17) multiplying both its sides by a function $\Psi'(\Gamma)$, which leads to the transport

equation for a function $\Psi(\Gamma)$. The justification of this construction is a delicate question because we do not know whether the function $\Psi'(\Gamma)$ is integrable with respect to the measure m or not. Instead of this property we show that $\Psi'(\Gamma)$ satisfies the simple and effective integral inequality. The result is given by the following lemma. Choose an arbitrary $t > 1$ and function $\eta \in C^\infty(\mathbb{R})$ satisfying the conditions

$$\eta(\lambda) = 1 \ \text{for} \ \lambda \le t - 1, \quad \eta(\lambda) = 0 \ \text{for} \ \lambda \ge t + 1, \quad \eta'(\lambda) \le 0,$$

and fix a function $\Psi \in C^2(\mathbb{R})$ such that

$$\Psi''(\lambda) \le 0, \quad \Psi(\lambda) \ge 0 \ \text{for} \ \lambda \in [0, 1], \quad \Psi(0) = 0 .$$

Lemma 17. *Under the assumptions of Lemma 16 for all* $t > 2 + 2\|f\|_{L^\infty(\Omega)}/\alpha$,

$$\int_{\Omega \times [-\infty, t-1)} \Psi(\Gamma) dx d\lambda \le -(t+1) \int_{\Omega \times \mathbb{R}_\lambda} \eta' \Big\{ \Psi(\Gamma)(|div \, \mathbf{u}| + 2\alpha) + \sigma M \Big\} dx d\lambda$$

$$-\sigma(t+1) \int_{\Omega \times \mathbb{R}_\lambda} \eta' \lambda^{-1} dm(x, \lambda), \quad \sigma = \max_{[0,1]} |\Psi'|. \ (8.7)$$

Proof. Let θ^3 and θ^1 be regularising kernels in $\mathbb{R}^3$ and $\mathbb{R}^1$, respectively i.e.

$$\theta^j \in \mathcal{D}_+(\mathbb{R}^j), \quad \int_{\mathbb{R}^j} \theta^j(x) dx = 1, \quad \text{spt} \, \theta^j \Subset \{|x| \le 1\}.$$

The corresponding mollifiers are defined by the equalities

$$u_{k,\infty}(x, \lambda) = k^3 \int_\Omega \theta^3(k(x - y)) u(y, \lambda) dy,$$

$$u_{\infty,k}(x, \lambda) = k \int_\mathbb{R} \theta^1(k(\lambda - \mu)) u(x, \mu) d\mu,$$

for each $u \in L^1_{loc}(\Omega \times \mathbb{R})$ and dist $(x, \partial\Omega) > k^{-1}$. We will simply write $u_{k,n}$ instead of $(u_{k,\infty})_{\infty,n}$. Substituting the test function

$$\zeta(x, \lambda) = k^3 n \vartheta^3(k(x_0 - x)) \vartheta^1(n(\lambda_0 - \lambda)), \quad \text{dist} \, (x_0, \partial\Omega) > k^{-1}, \quad \lambda_0 \in \mathbb{R},$$

into (8.1) we arrive at the equality

$$\partial_\lambda \Big\{ [(\lambda(\alpha + div \, \mathbf{u}) - f] \Gamma_{k,n} \Big\} - \alpha \Gamma_{k,n} - div \, (\Gamma_{k,n} \mathbf{u}) - \partial_\lambda (m + \lambda M)_{k,n}$$
$$= (\mathfrak{r}_1)_{k,\infty} + \mathfrak{r}_2 + \partial_\lambda \mathfrak{r}_3 \equiv \mathfrak{r}, \quad (8.8)$$

which holds true in any domain $\Omega' \times \mathbb{R}_\lambda$ with dist $(\Omega', \partial\Omega) < k^{-1}$. Here the remainders $\mathfrak{r}_j$ are given by

$$\mathfrak{r}_1 = -(\alpha + div \, \mathbf{u}) \partial_\lambda [(\lambda\Gamma)_{\infty,n} - \lambda\Gamma_{\infty,n}], \quad \mathfrak{r}_2 = div \, [(\Gamma_{\infty,n}\mathbf{u})_{k,\infty} - \Gamma_{k,n}\mathbf{u}],$$
$$\mathfrak{r}_3 = \lambda\Gamma_{k,n} div \, \mathbf{u} - \lambda(\Gamma_{\infty,n} div \, \mathbf{u})_{k,\infty} + (\Gamma_{\infty,n} f)_{k,\infty} - \Gamma_{k,n} f .$$

Recall that $\Gamma_{k,n}$ and $(m + \lambda M)_{k,n}$ are smooth functions in $\Omega' \times \mathbb{R}$. Multiplying both sides of (8.8) by $\Psi'(\Gamma_{k,n})$ and noting that $\partial_\lambda \Psi'(\Gamma_{k,n}) \leq 0$, $M, m \geq 0$ we obtain

$$(\lambda \alpha - f)\partial_\lambda \Psi(\Gamma_{k,n}) - \text{div}_{x,\lambda}(\Psi(\Gamma_{k,n})\mathbf{w}) - \partial_\lambda \Big(\Psi'(\Gamma_{k,n})(m + \lambda M)_{k,n} \Big) \geq \Psi'(\Gamma_{k,n})\mathfrak{r} \,. \tag{8.9}$$

Choose an arbitrary non-negative function $h \in C_0^\infty(\Omega)$ with $\text{spt} h \Subset \Omega'$. Multiplying both sides of (8.9) by $h(x)\eta(\lambda)$ and integrating the result over $\Omega' \times \mathbb{R}_\lambda$ we arrive at the inequality

$$\int\limits_{\Omega' \times \mathbb{R}_\lambda} \Psi(\Gamma_{k,n})\eta\Big[\alpha h - \nabla h \cdot \mathbf{u}\Big]dxd\lambda + \int\limits_{\Omega' \times \mathbb{R}_\lambda} h\eta'\Psi(\Gamma_{k,n})(\lambda \text{div}\,\mathbf{u} + \lambda \alpha - f)dxd\lambda$$

$$- \int\limits_{\Omega' \times \mathbb{R}_\lambda} h\eta'\Psi'(\Gamma_{n,k})(m + \lambda M)_{k,n}dxd\lambda \leq - \int\limits_{\Omega' \times \mathbb{R}_\lambda} h\eta\Psi'(\Gamma_{k,n})\mathfrak{r}dxd\lambda \,.$$

Since $\eta' \leq 0$ and $m + \lambda M \geq 0$, we have

$$\int\limits_{\Omega' \times \mathbb{R}_\lambda} h\eta'\Psi'(\Gamma_{k,n})(m + \lambda M)_{k,n}dxd\lambda \leq \min_{[0,1]} \Psi' \int\limits_{\Omega' \times \mathbb{R}_\lambda} h\eta'(m + \lambda M)_{k,n}dxd\lambda \,,$$

which yields

$$\int\limits_{\Omega' \times \mathbb{R}_\lambda} \Psi(\Gamma_{k,n})\eta\Big[\alpha h - \nabla h \cdot \mathbf{u}\Big]dxd\lambda + \int\limits_{\Omega' \times \mathbb{R}_\lambda} h\eta'\Psi(\Gamma_{k,n})(\lambda \text{div}\,\mathbf{u} + \lambda \alpha - f)dxd\lambda$$

$$- \min_{[0,1]} \Psi' \int\limits_{\Omega' \times \mathbb{R}_\lambda} h\eta'(m + \lambda M)_{k,n}dxd\lambda \leq - \int\limits_{\Omega' \times \mathbb{R}_\lambda} h\eta\Psi'(\Gamma_{k,n})\mathfrak{r}dxd\lambda \,. \tag{8.10}$$

Now our task is to show that the right side tends to 0 as $k, n \to \infty$. Fix an arbitrary $n > 0$. Since the mapping $(x, \lambda) \to \mathbf{u}(x)$, which does not depend on λ, belongs to $L^\infty(\mathbb{R}_\lambda, H^1(\Omega))$, it follows from Lemma 2.3 in [8] that $\mathfrak{r}_2 \to 0$ in $L^1_{loc}(\Omega \times \mathbb{R})$ when $k \to \infty$. Obviously, the sequence $\mathfrak{r}_3 \to 0$ converges in $L^2_{loc}(\Omega \times \mathbb{R})$ when $k \to \infty$. It follows from this that for each fixed n,

$$\lim_{k \to \infty} \int\limits_{\Omega' \times \mathbb{R}_\lambda} h\eta\Psi'(\Gamma_{k,n})\mathfrak{r}dxd\lambda = \int\limits_{\Omega' \times \mathbb{R}_\lambda} h\Psi'(\Gamma_{\infty,n})\eta\mathfrak{r}_1 dxd\lambda \,.$$

It is easy to see that

$$\partial_\lambda[(\lambda\Gamma)_{\infty,n} - \lambda\Gamma_{\infty,n}](x, \lambda) = k \int_\mathbb{R} \theta(k(\lambda - \mu))\Gamma(x, \mu)d\mu, \quad \theta(s) = s\theta^{1'}(s) + \theta^1(s).$$

Since θ is compactly supported and $\int_\mathbb{R} \theta(s)ds = 0$, the sequence $\partial_\lambda[(\lambda\Gamma)_{\infty,n} - \lambda\Gamma_{\infty,n}]$ (x, λ) converges to 0 in $L^r_{loc}(\Omega \times \mathbb{R})$ for each $r < \infty$. Hence $\mathfrak{r}_1 \to 0$ in $L^r_{loc}(\Omega \times \mathbb{R})$, which implies

$$\lim_{n\to\infty}\lim_{k\to\infty}\int_{\Omega'\times\mathbb{R}_\lambda} h\eta\Psi'(\Gamma_{k,n})\mathfrak{r}dxd\lambda = 0.$$

Letting $k \to \infty$, $n \to \infty$ in (8.10) we conclude that the inequality

$$\int_{\Omega\times[-\infty,t-1)} h\Psi(\Gamma)dxd\lambda - \int_{\Omega\times\mathbb{R}_\lambda} \eta\Psi(\Gamma)\nabla h \cdot \mathbf{u}dxd\lambda$$

$$\leq -(t+1)\int_{\Omega\times\mathbb{R}_\lambda} h\eta'\Big(\Psi(\Gamma)(|\text{div }\mathbf{u}| + 2\alpha) + M\Big)dxd\lambda$$

$$-(t+1)\int_{\Omega\times\mathbb{R}_\lambda} h\eta'\lambda^{-1}dm(x,\lambda) \tag{8.11}$$

holds for all $t > 2\alpha^{-1}\|f\|_{L^\infty(\Omega)} + 2$ and all non-negative $h \in C_0^\infty(\Omega)$. Now choose a sequence of functions $h_j \in C_0^\infty(\Omega)$ such that $h_j \to 1$ as $j \to \infty$ in Ω and $|\nabla h_j(x)| \leq j\text{dist}(x,\partial\Omega)$. Substituting $h = h_j$ into (8.1), letting $j \to \infty$, and noting that $\|\nabla h_j \cdot \mathbf{u}\|_{L^1(\Omega)} \to 0$ we finally obtain (8.7) which completes the proof. □

The proof of equality $\Gamma(1-\Gamma) = 0$. The last part of the proof is based on the following lemma

Lemma 18. *Under the above assumptions,*

$$\int_{\mathbb{R}_\lambda} \eta'(\lambda)M(x,\lambda)d\lambda = -\int_{[0,\infty)} \eta''(\lambda)\mathcal{V}_\lambda(x)d\lambda, \tag{8.12}$$

where $\mathcal{V}_\lambda$ *is defined by* (7.33).

Proof. It is easy to see that

$$-(2+\nu)\int_{\mathbb{R}_\lambda} \eta'(\lambda)M(x,\lambda)d\lambda = \int_{[0,\infty)} \left(\int_{[\lambda,\infty)} \eta''(s)ds\right)\left(\int_{[\lambda,\infty)} (t^\gamma - \overline{p})d_t\Gamma(x,t)\right)d\lambda$$

$$= \int_{[0,\infty)} \eta''(s)\left(\int_{[0,s)} d\lambda \int_{[\lambda,\infty)} (t^\gamma - \overline{p})d_t\Gamma(x,t)\right)ds$$

$$= \int_{[0,\infty)} \eta''(s)\left(\int_{[0,\infty)} \min\{t,s\}(t^\gamma - \overline{p})d_t\Gamma(x,t)\right)ds$$

$$= \int_{[0,\infty)} \eta''(s)(\overline{\min\{\varrho,s\}p} - \overline{\min\{\varrho,s\}\overline{p}})ds.$$

On the other hand, Lemma 13 yields $\overline{\min\{\varrho,\lambda\}p} - \overline{\min\{\varrho,\lambda\}}\overline{p} = (2+\nu)\mathcal{V}_\lambda(x)$, and the lemma follows. □

Take Ψ in the simplest form $\Psi(\Gamma) = \Gamma(1-\Gamma)$ with $\sigma = 1$. Since $\eta'(\lambda)$ vanishes for $\lambda < 1$, we have

$$- \int\limits_{\Omega \times \mathbb{R}_\lambda} \eta' \Psi(\Gamma)(|\mathrm{div}\,\mathbf{u}| + 2\alpha) dx d\lambda$$

$$= \int\limits_{[1,\infty)} \eta''(\lambda) d\lambda \int\limits_{\Omega \times [1,\lambda)} \Psi(\Gamma(x,s))(|\mathrm{div}\,\mathbf{u}(x)| + 2\alpha) dx ds,$$

$$- \int\limits_{\Omega \times \mathbb{R}_\lambda} \eta' \lambda^{-1} dm(x,\lambda) = \int\limits_{[1,\infty)} \eta''(\lambda) d\lambda \int\limits_{\Omega \times [1,\lambda)} s^{-1} dm(x,s).$$

Substituting these identities along with (8.12) into (8.7) implies the inequality

$$\int\limits_{\Omega \times [-\infty, t-1)} \Psi(\Gamma) dx d\lambda \le (1+t) \int\limits_{[1,\infty)} \eta''(\lambda) \wp(\lambda) d\lambda, \qquad (8.13)$$

in which the function $\wp : [1, \infty) \mapsto \mathbb{R}$ is given by

$$\wp(\lambda) = \int\limits_{\Omega \times [1,\lambda)} \Psi(\Gamma(x,s))(|\mathrm{div}\,\mathbf{u}(x)| + 2\alpha) dx ds + \int\limits_{\Omega} \mathcal{V}_\lambda(x) dx + \int\limits_{\Omega \times [1,\lambda)} s^{-1} dm(x,s).$$

It follows from Lemma 15 that for $\lambda > 1$,

$$|\wp(\lambda)| \le c(1 + \|\mathbf{u}\|_{H^{1,2}(\Omega)}) \|\mathfrak{H}\|_{L^2(\Omega)} + \|\mathcal{V}_\lambda\|_{L^1(\Omega)} + c \int\limits_{\Omega \times [1,\infty)} (1+\lambda)^{-1} dm(x,\lambda) \le c,$$

where a constant c does not depend on λ. Next set $\eta(\lambda) = \int\limits_{\lambda}^{\infty} \omega(s - t) ds$, where ω is a smooth, even, non-negative function supported on the interval $(-1, 1)$ and such that $\int_{\mathbb{R}} \omega(s) ds = 1$. Then inequality (8.13) can be rewritten in the form

$$\int\limits_{\Omega \times [-\infty, t-1)} \Psi(\Gamma) dx d\lambda \le (1+t) \frac{d}{dt}(\omega * \wp)(t). \qquad (8.14)$$

Since the smooth function $(\omega * \wp)(t)$ is uniformly bounded on the interval $[1, \infty)$, then there is a sequence $t_k \to \infty$ such that $\lim\limits_{k \to \infty} (t_k + 1) \frac{d}{dt}(\omega * \wp)(t_k) \le 0$. Substituting $t = t_k$ into (8.14) and letting $k \to \infty$ we conclude that $\Psi(\Gamma) = 0$ a.e. in $\Omega \times \mathbb{R}_\lambda$, which completes the proof of Theorem 7.

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Communicated by P. Constantin