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Wavelet Analysis of Fractal Boundaries. Part 1: Local Exponents

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Abstract: Let Ω be a domain of $\mathbb{R}^d$. In Part 1 of this paper, we introduce new tools in order to analyse the *local* behavior of the boundary of Ω. Classifications based on geometric accessibility conditions are introduced and compared; they are related to analytic criteria based either on local L^p regularity of the characteristic function 1_Ω, or on its wavelet coefficients. Part 2 deals with the *global* analysis of the boundary of Ω. We develop methods for determining the dimensions of the sets where the local behaviors previously introduced occur. These methods are based on analogies with the thermodynamic formalism in statistical physics and lead to new classification tools for fractal domains.

1. Introduction

1.1. Raleigh-Taylor instability and multifractal analysis . The initial motivation of this paper was to understand the paradoxical results of numerical experiments performed on Raleigh-Taylor instability. Let us first recall the context of these experiments. Raleigh-Taylor instability occurs when two fluids which are not miscible are initially placed on top of each other with the heaviest one at the top; if the viscosities are small enough, then the two fluids get mixed without interpenetrating each other and develop mushroom-type structures that degenerate into extremely thin and twisted filaments, see [26]. Raleigh-Taylor instability happens in many physical situations and an important issue is to understand the optical properties of this mixture and to relate them to some geometric (perhaps fractal) properties of the interface. In order to obtain additional information concerning these properties, several authors proposed to determine what the *multifractal formalism*

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yields when applied to the characteristic function of one of these two media, see [27, 28]. Before recalling the numerical results of this investigation, let us recall the purpose of multifractal analysis. It was initially introduced in order to analyse functions whose pointwise regularity can change abruptly. The following definition collects the different notions attached to multifractal analysis.

Definition 1. *Let $x_0 \in \mathbb{R}^d$ and let $\alpha \geq 0$. A locally bounded function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $C^\alpha(x_0)$ if there exists a constant $C > 0$ and a polynomial P satisfying $deg(P) < \alpha$ and such that in a neighbourhood of x_0,*

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha. \tag{1}$$

The **Hölder exponent** *of f at x_0 is*

$$h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}.$$

We denote by E_H the set of points where the Hölder exponent takes the value H (note that H can take the value $+\infty$). The **spectrum of singularities** *of f (denoted by $d_f(H)$) is the Hausdorff dimension of E_H.*

The support of the spectrum of singularities is the set of finite values of H that are Hölder exponents of f.

A function f is called *multifractal* if its spectrum of singularities is supported, at least, by an interval of non-empty interior. The spectrum of singularities yields local information on the behavior of f. One can also consider the global information supplied by the *scaling function of* f, $\eta_f(p)$, which can be derived from the wavelet coefficients of f. In order to define the scaling function, we need to recall some definitions and results concerning wavelet expansions.

One can construct a function ϕ and wavelets $\psi^{(i)}$, $i = 1, \cdots, 2^d - 1$, all in the Schwartz class, and such that the functions

$$\begin{cases} \phi(x-k), \quad k \in \mathbb{Z}, \\ 2^{\frac{dj}{2}} \psi^{(i)}(2^j x - k),\ j \geq 0,\ k \in \mathbb{Z}^d,\ i = 1...2^d - 1 \end{cases}$$

form an orthonormal basis of $L^2(\mathbb{R}^d)$, see [8, 18, 23]. Therefore, any L^2 function f can be written

$$f(x) = \sum_{k \in \mathbb{Z}^d} C_k \phi(x-k) + \sum_{i, j \geq 0, k \in \mathbb{Z}^d} C_{j,k}^{(i)}\ \psi^{(i)}(2^j x - k), \tag{2}$$

where $C_k = \int_{\mathbb{R}^d} f(x)\ \phi(x-k)\, dx$ and $C_{j,k}^{(i)} = 2^{dj} \int_{\mathbb{R}^d} f(x)\ \psi^{(i)}(2^j x - k)\, dx$. (Note that we choose an L^∞ normalisation for the wavelets.)

We will need the following wavelet characterization of the homogeneous Hölder spaces, see [23]:

Proposition 1. *Let $s > 0$. A function f belongs to the homogeneous Hölder space $C^s(\mathbb{R}^d)$ if and only if there exists $C > 0$ such that $\forall i, j, k$, $|C_{j,k}^{(i)}| \leq C 2^{-sj}$.*

Let us now define the scaling function of a distribution f.

Definition 2. *If f is a compactly supported L^1 function and if $p \neq 0$, then the* **scaling function** *of f is defined by*

$$\eta_f(p) = \liminf_{j \to +\infty} \left(\frac{\log \left(2^{-dj} \sum_{i,k}^{\star} \left| C_{j,k}^{(i)} \right|^p \right)}{\log(2^{-j})} \right), \tag{3}$$

where $\sum_{i,k}^{\star}$ means that the sum is taken over the coefficients $C_{j,k}^{(i)}$ that do not vanish.

When a signal, or an image, is stored by its wavelet coefficients, the scaling function is easily computable numerically. The multifractal formalism asserts that the spectrum of singularities of f can be recovered from its scaling function by a Legendre transform

$$d_f(H) = \inf_p (d - \eta_f(p) + Hp). \tag{4}$$

This formula was obtained using heuristic arguments that do not constitute a mathematical proof, see [3, 30]; and indeed, many examples for which (4) is wrong are known, see [15] for instance.

Let us now come back to Raleigh-Taylor instability. We denote by Ω one of the two domains obtained when the mixing has sufficiently been developed, and we consider $f = 1_\Omega$. The purpose of the numerical computations performed by S. Mimouni in [26] was to determine the scaling function $\eta_f(p)$. Clearly, a characteristic function is not multifractal; the support of its spectrum of singularities is restricted to one point: $H = 0$. Thus, if Δ denotes the dimension of the interface, it follows that

$$\begin{aligned} d(H) &= \Delta \quad \text{ if } H = 0 \\ &= -\infty \text{ otherwise.} \end{aligned}$$

Therefore, if the multifractal formalism holds, according to (4), we expect that $\forall p$, $\eta_f(p) = d - \Delta$. Surprisingly, the numerical results obtained show that the scaling function is far from being constant: It is a strictly concave increasing function, see [26–28], which one is tempted to interpret as the signature of a multifractal behavior.

One might wave these paradoxical results away as nonsignificant: They are just a few additional counterexamples to the validity of the multifractal formalism. Our purpose in these two joint papers will precisely be the opposite: We will try to understand why the multifractal formalism fails in this case, and how the information supplied by the scaling function can be interpreted in terms of new geometric information on Ω. This study will have two main consequences:

- These counterexamples to the multifractal formalism will allow to understand better the conditions for its validity.
- Once the information contained by the scaling function is well understood, it can be used pertinently as a classification tool for fractal domains.

Of course, this second motivation goes well beyond Raleigh-Taylor instability. Let us review briefly other fields of applications where fractal interfaces occur, and where such classification tools could be applied.

1.2. Fractal interfaces. In physics, mechanics or chemistry, many phenomena involve fractal interfaces. It is the case for fractal growth mechanisms, chemical deposition [32], fractured bodies (metals, rocks, bones,...) [12], rugosity [9], turbulent mixtures [6, 21, 33, 34], to mention just a few. Note also that natural images often contain such features as edges of mountains, edges of trees, coastlines,..., which are typical examples of fractal boundaries. Fractal curves have also been the object of several studies in mathematics; it is for instance the case for level sets of statistical processes (in particular Brownian motion or fractional Brownian motion, see [7, 37] and references therein). The study of fractal level-sets has implications in many physics and computer science problems (see [34] and references therein, and [19] in the context of turbulence).

A better understanding of these fractal structures requires the introduction and study of new mathematical tools fitted to describe and classify their geometry. Up to now the only notion used in practice was the box dimension of the interface; it was not used only as a classification parameter: In [31] the box dimension of a turbulent interface is shown to have a relevant physical interpretation since it is related to the stratification. In [33] the box dimension of an oil-water turbulent interface was determined numerically. The rugosity of rough surfaces has been studied using fractal models for the surface: It is shown in [9] that rugosity can be related with the fractal dimension of the surface. Multifractal-type arguments have also been used to derive heuristically the box-dimension of the interface in the case of intermittent turbulence, by C. Meneveau and K. Sreenivasan in [21]. However, the box dimension yields only one parameter; therefore it is a poor classification tool. Furthermore, its precise numerical estimation can be either impossible or imprecise in some practical situations; for instance, in the case of turbulent jets, see [6] where it is shown that its estimation is strongly oscillating through the scales.

1.3. Organization of the two papers. In Part 2, we will perform a close analysis of the heuristic arguments that are given as a justification of the multifractal formalism. We will establish that, in spite of the numerous computations that are currently performed under this assumption, the Legendre transform (4) cannot be expected to yield the Hausdorff dimension of the sets E_H: It rather yields the dimension of the sets of points x_0 with a given *weak-scaling exponent*, see Definition 8. This remark yields a first clue to the resolution of the paradox raised in Sect. 1.1: Of course, the Hölder exponent of a characteristic function 1_Ω can only take the value 0 at a point of the boundary of Ω; but its weak-scaling exponent can take any nonnegative value. For instance, consider in $\mathbb{R}^2$ the α-cusp ($\alpha \geq 0$),

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 \quad \text{such that} \quad x \geq 0 \quad \text{and} \quad |y| \leq |x|^{\alpha+1} \right\} \tag{5}$$

near the origin. One can show (see [20]) that the weak-scaling exponent of Ω takes the value α at the origin. Therefore, one may expect that the multifractal formalism yields the dimension of the subsets of the boundary where such a behavior occurs. In order to make this statement more precise, we have to determine which kind of geometric behaviors of the boundary near a point x_0 induces a given weak-scaling exponent. This is the initial motivation of the first of these two papers. We will examine possible pointwise behaviors that will be defined in three ways:

- geometrically,
- by a condition bearing directly on 1_Ω,
- by a condition bearing on the wavelet coefficients of 1_Ω,

and we will compare them. The two first ways will be considered in Sect. 2.1. We will introduce conditions of the third type in Sects. 2.2 to 2.4, where we will also prove the first results that allow to compare these notions. The main results of this type (Theorems 1 and 2) will be proved in Sect. 3.

Part 2 will bridge the gap between *local* and *global* analysis and determine how the local conditions introduced in the first paper can be used as the building blocks of new multifractal formalisms that are expected to yield the dimensions of the sets where such behaviors take place. These new multifractal spectra, associated with each kind of local exponent, will yield many possibilities of classification of fractal interfaces and they give very rich information on the geometry of these interfaces. Furthermore, we will prove that, in several cases, these multifractal formalisms either yield the exact dimensions required or, at least, upper bounds for these dimensions.

Let us finally mention that, though our main concern in these two papers is the investigation of fractal boundaries, we will prove several results that apply in much wider settings: For instance, Sect. 2.3 does not only deal with characteristic functions of domains, but gives the wavelet characterization of the T^p_u regularity of arbitrary functions. Similarly, in Sect. 2 of Part 2, we construct general multifractal formalisms which apply to the weak scaling exponent of tempered distributions and to the T^p_u exponent of L^p functions; in Sect. 4 of Part 2, we will show that these multifractal formalisms yield upper bounds for the corresponding spectrums.

2. Pointwise Exponents of Fractal Boundaries

Let us consider a few typical examples of local behaviors of boundaries. In the case of (5) we want to recover the exponent α which characterizes the 'thinness' of the cusp at the origin. We will also be interested in *spirals*, such as the domain between the two curves of equation (in polar coordinates)

$$r = \theta^{-\gamma} \quad \text{and} \quad r = (\theta + \pi)^{-\gamma}. \tag{6}$$

Another example which bears similarities with spirals is the one-dimensional set

$$S_\gamma = \bigcup_{n \in \mathbb{N}} \left[\frac{1}{(2n+1)^\gamma}, \frac{1}{(2n)^\gamma} \right], \tag{7}$$

called an *isolated accumulating singularity at 0* in [35, 36] (actually, the trace of a spiral on a line which passes through its center yields such a set). In these cases, we want to recover the exponent γ which characterizes the degree of 'mixing' between Ω and its complement Ω^c. We will introduce pointwise exponents that will precisely play this role and yield at the origin the exponent α in the first case, and the exponent γ in the second case. There are two ways to introduce such exponents.

- *Geometric* properties, based on accessibility conditions, can be used: The exponent α in (5) can be recovered by estimating the measure of the set $B(0, r) \cap \Omega$ when $r \to 0$. The exponent γ in (6) can be obtained by estimating the largest possible size of balls included in $B(0, r) \cap \Omega$ when $r \to 0$. The corresponding geometric definitions will be introduced in Sect. 2.1.
- An *analytic* approach can be based on functional properties of the characteristic function 1_Ω of the domain Ω near x_0. We will investigate analytic classifications based on decay estimates of the wavelet transform of 1_Ω near the point x_0 in Sects. 2.2 and 2.3.

In the case of *isolated singularities*, both approaches have been related previously: H. K. Moffat showed in [29] that the Fourier transform of the characteristic function of the one-dimensional spiral (7) decays as $|k|^{-2+\frac{1}{1+\gamma}}$, and J. C. Vassilicos and J. C. R. Hunt remarked that the exponent in this power-law is directly related to the box dimension of the spiral since this box dimension is precisely $\frac{1}{1+\gamma}$, see [36]. However, the drawback of properties based on Fourier analysis is that they give clear information only for one isolated singularity. Indeed, since Fourier analysis is non-local, the information concerning different local behaviors at different locations is completely mixed-up. This is an additional reason for rather using wavelet analysis when dealing in applications with experimental data, where many such behaviors are expected to occur.

There exist some intrinsic limitations on any analysis of the geometry of a domain Ω based on local regularity conditions of 1_Ω. Indeed, a regularity condition satisfied by a function f means that f locally belongs to some function spaces; let Ω^c denote the complement of Ω. Since $1_{\Omega^c} + 1_\Omega = 1$, which is a smooth function, it follows that 1_{Ω^c} and 1_Ω belong locally to the same function spaces. Thus the knowledge of the function spaces to which 1_Ω locally belongs can only give information that cannot draw a distinction between Ω and its complement. For instance, cusps that point inside or outside Ω cannot be separated, and it is the same for accessibility conditions from the inside and from the outside of Ω. The second restriction is that, for the same reason, we should use a notion of boundary invariant by subtracting or adding to Ω a set of measure 0. Thus the relevant notion here is the *essential boundary* defined as follows.

Definition 3. *The* **essential boundary** *of Ω is the set of points that remain in the boundary if we subtract or add to Ω any set of measure 0.*

If Ω has measure 0, its essential boundary is empty, which fits the fact that, in this case, $1_\Omega = 0$ a.e. and thus belongs to all function spaces.

The following lemma shows that, after modifying Ω by a set of measure 0, we can always make the assumption that the boundary of Ω coincides with its essential boundary (and therefore we make this assumption from now on).

Lemma 1. *Let Ω be a bounded subset of $\mathbb{R}^d$. There exists Ω' which differs from Ω by a set of measure 0, and such that the boundary of Ω' is its essential boundary.*

Proof of Lemma 1. We denote by $meas(A)$ the Lebesgue measure of the set A. Consider the countable collection of open balls whose centers have rational coordinates and which have rational radii, and let us order them in some way. For each such ball B_i, if $meas(B_i \cap \Omega) = 0$, we remove from Ω the points inside B_i, and if $meas(B_i \cap \Omega^c) = 0$, we remove from Ω^c the points inside B_i. When this operation has been performed for all balls B_i, clearly, each point has been moved at most in one way, and these moves affect only a set of measure 0. Each point x of the boundary of the set Ω' thus obtained satisfies

$$\forall r 0, \quad meas(\Omega \cap B(x,r)) > 0 \quad \text{and} \quad meas(\Omega^c \cap B(x,r)) > 0 \tag{8}$$

hence belongs to the essential boundary.

2.1. Geometric approach: Accessibility conditions. If the boundary of Ω is smooth at x, a result much more precise than (8) holds:

$$meas(\Omega \cap B(x,r)) \sim r^d \quad \text{and} \quad meas(\Omega^c \cap B(x,r)) \sim r^d; \tag{9}$$

on the other hand, the cusp (5) satisfies $meas(\Omega \cap B(x,r)) \sim r^{\alpha+d}$. These scalings suggest to consider the following geometric parameter which is fitted to describe points of the boundary where a weak form of accessibility condition holds.

Definition 4. *A point x of the boundary of Ω is weak α-accessible if there exist $C > 0$ and $r_0 > 0$ such that $\forall r \leq r_0$,*

$$\min\left[meas\left(\Omega \cap B(x,r)\right)\,,\, meas(\Omega^c \cap B(x,r))\right] \leq C r^{\alpha+d}. \tag{10}$$

The supremum of all values of α such that (10) holds is called the **weak accessibility exponent at** x. *We denote it by $E^w_\Omega(x)$.*

The weak accessibility exponent is a nonnegative number, and it can be infinite. It can be equivalently defined with the help of local L^p conditions introduced by Calderón and Zygmund in [5] as substitutes of the pointwise Hölder exponents.

Definition 5. *Let $p \geq 1$; a function $f \in L^p_{loc}(\mathbb{R}^d)$ belongs to $T^p_u(x)$ if there exist $R > 0$ and a polynomial P, such that $deg(P) < u$, satisfying*

$$\forall r \leq R \quad \left(\frac{1}{r^d} \int_{B(x,r)} |f(y) - P(y-x)|^p dy \right)^{1/p} \leq C r^u. \tag{11}$$

The p**-exponent** *of f at x is*

$$u^p_f(x) = \sup\{u : f \in T^p_u(x)\}. \tag{12}$$

Remarks.
- The polynomial P is clearly unique. It is called the (generalized) Taylor expansion of f at x.
- As a consequence of the condition $f \in L^p_{loc}$, if (11) holds for a given $R > 0$, then it holds for any $R' > 0$.
- The usual Hölder condition $C^u(x)$ corresponds to $p = \infty$, therefore the ∞-exponent is the usual Hölder exponent.
- If f belongs to $C^u(x)$, then, $\forall p \geq 1$, f belongs to $T^p_u(x)$; more generally, if $p' < p$, then $T^p_u(x) \hookrightarrow T^{p'}_u(x)$. It follows that

$$\text{if} \quad 1 \leq p \leq p' \quad \text{then} \quad u^p_f(x) \geq u^{p'}_f(x). \tag{13}$$

- Since the functions of L^p_{loc} belong to $T^p_{-\frac{d}{p}}(x)$, the p-exponent is always larger than $-d/p$.

The purpose of Theorem 2 is to characterize the p-exponent by conditions on the wavelet coefficients. For characteristic functions, the following lemma shows that the T^p_u condition coincides with the weak accessibility condition.

Lemma 2. *Let $p \geq 1$; the domain Ω is weak α-accessible at x if and only if its characteristic function belongs to $T^p_{\alpha/p}(x)$.*

Proof. Let $f = 1_\Omega$; if $P = 0$, then

$$\int_{B(x,r)} |f(y) - P(y-x)|^p dy = meas(\Omega \cap B(x,r))$$

and, if $P = 1$, then

$$\int_{B(x,r)} |f(y) - P(y-x)|^p dy = meas(\Omega^c \cap B(x,r)).$$

Suppose now that Ω is weak α-accessible for an $\alpha > 0$. First, note that the smallest of the two quantities $meas(\Omega \cap B(x,r))$ and $meas(\Omega^c \cap B(x,r))$ remains the same for r small enough (by continuity of these two functions of r). Therefore, if (10) holds for an $\alpha > 0$, it follows that 1_Ω belongs to $T^p_{\alpha/p}(x)$.

Conversely, suppose that 1_Ω belongs to $T^p_{\alpha/p}(x)$ for an $\alpha > 0$; then

$$\int_{B(x,r)} |1_\Omega(y) - P(y-x)|^p dy \leq C r^{\alpha+d}. \tag{14}$$

Let us first show that the term of order 0 of P is either 0 or 1. Indeed, if it is not the case, let us denote this term by the constant C. In a neighbourhood of x,

$$|1_\Omega(y) - P(y-x)| \geq \frac{1}{2} \inf(|C|, |1-C|),$$

so that (14) cannot hold for an $\alpha > 0$. If $\alpha \leq p$, the result is obtained; else, one proves by the same argument that the term of order 1 of P must necessarily vanish, and a straightforward recursion yields that the following terms vanish up to the order $[\alpha/p]$. It follows that Ω is weak α-accessible.

If the boundary of Ω is smooth at x, the following stronger accessibility condition holds: For any $r > 0$, we can find at distances $\sim r$ two balls of radius r, one included in Ω and one included in Ω^c. This remark leads naturally to the following notion of *strong accessibility.*

Definition 6. *A point $x \in \partial\Omega$ is strong α-accessible if there exist $C > 0$, a sequence $r_n \to 0$ and balls $B_n^1 \subset \Omega$ and $B_n^2 \subset \Omega^c$ of radii r_n such that*

$$dist(B_n^1, x) \leq C r_n^{\frac{d}{\alpha+d}} \quad and \quad dist(B_n^2, x) \leq C r_n^{\frac{d}{\alpha+d}}. \tag{15}$$

The infimum of all values of α such that (15) holds is called the **strong accessibility exponent at** *x. We denote it by $E^s_\Omega(x)$. (If there exists no such α, we take $E^s_\Omega(x) = +\infty$).*

The strong accessibility exponent is larger than the weak accessibility exponent because, if (15) holds and if $r = 3C r_n^{\frac{d}{\alpha+d}}$, then

$$\inf\big(meas(\Omega \cap B(x,r)), meas(\Omega^c \cap B(x,r))\big) \geq C r_n^d \geq C' r^{\alpha+d}.$$

Besides the example (5) of cusp singularities, an example of strong accessibility is supplied by domains above or below the graph of Hölder continuous functions: Let $f : \mathbb{R}^{d-1} \to \mathbb{R}$ be a function in C^h with $0 < h \leq 1$; each point $(x_0, f(x_0))$ clearly is a strong $d(\frac{1}{h} - 1)$-accessible point of the boundary of the domain above the graph of f.

The weak and strong exponents differ when the two domains Ω and Ω^c are very mixed together. For example, let us compare the domains defined by (5) and (6): The cusp (5) has a weak and strong accessibility exponent α at the origin. As regards the spiral (6), the weak accessibility exponent is 0, whereas the strong accessibility exponent is $2/\gamma$. Similarly, the accumulating singularity (7) has weak accessibility exponent 0, and strong accessibility exponent $2/\gamma$. Proposition 2 will draw another distinction between strong accessibility and weak accessibility.

Remark. The fact that a domain has a fractal boundary does not necessarily imply that it displays accessibility exponents larger than 0. For instance, one immediately checks that the "Van Koch snowflake" (see [10]) has a (weak or strong) accessibility exponent $\alpha = 0$ at every point of its boundary.

2.2. Analytic approach: two-microlocal analysis. In this section, we estimate the size of the wavelet coefficients of 1_Ω in the neighbourhood of points with a given (strong or weak) accessibility exponent.

Let us introduce simplifying notations; wavelets will be indexed by the dyadic cubes: If λ is the cube

$$\lambda = \left\{ x \in \mathbb{R}^d : \; 2^j x - k \in [0,1)^d \right\}, \tag{16}$$

then we use the notations $\psi^{(i)}_{j,k}(x) := \psi^{(i)}_\lambda(x) := \psi^{(i)}(2^j x - k)$. Thus

$$f(x) = \sum_{i,\lambda} C^{(i)}_\lambda \psi^{(i)}_\lambda(x), \tag{17}$$

where the wavelet coefficients of f are given by

$$C^{(i)}_{j,k} = C^{(i)}_\lambda = \int_{\mathbb{R}^d} 2^{dj} \psi^{(i)}_\lambda(t) f(t) dt.$$

We will forget the index (i) in the notations, and write the wavelet coefficients either $C_{j,k}$ or C_λ.

The other tool we will use is the continuous wavelet transform; in this case, we start with one compactly supported wavelet ψ, that can be arbitrarily smooth and with an arbitrary large number of vanishing moments. Let θ be a rotation in $\mathbb{R}^d$, $a \in \mathbb{R}^+$ and $b \in \mathbb{R}^d$. The continuous wavelet transform of f is

$$C(a,b,\theta) = \frac{1}{a^d} \int_{\mathbb{R}^d} f(x) \psi\left(\theta\left(\frac{x-b}{a}\right)\right) dx.$$

Note that the function f can be reconstructed from the values of $C(a,b,\theta)$, see [22].

Definition 7. *Let ψ_λ be a smooth wavelet basis. A distribution f belongs to the two-microlocal space $C^{s,s'}(x_0)$ if its wavelet coefficients satisfy*

$$|C_{j,k}| \le C 2^{-sj} (1 + |2^j x_0 - k|)^{-s'}.$$

In this definition one has to use wavelets in C^m for an m larger than $\sup(|s|, |s'|)+1$, and which have all their moments of order up to $\sup(|s|, |s'|) + 1$ vanishing. If such is the case, this definition is independent of the wavelet basis which is chosen; the characterization using the continuous wavelet transform is similar, and obtained by replacing 2^{-j} by a, $k2^{-j}$ by b, and letting the estimates be uniform in θ.

Yves Meyer showed that two-microlocal conditions yield precise information concerning the pointwise oscillatory behavior of the function near x_0, see [16, 24]. The following lemma shows a first relationship between weak accessibility and a condition on the wavelet coefficients. A stronger result will be given in Theorem 1.

Lemma 3. *If x_0 is a weak α-accessible point of $\partial\Omega$, then 1_Ω belongs to $C^{\alpha,-\alpha-d}(x_0)$.*

Proof of Lemma 3. Suppose for instance that $meas(\Omega \cap B(x_0, r)) \leq Cr^{\alpha+d}$; then, if the wavelets are compactly supported,

$$|C_\lambda| \leq C2^{dj} \int_\Omega |\psi_\lambda(x)|dx \leq 2^{dj} meas\,((C\lambda) \cap \Omega) \leq C2^{dj}(2^{-j} + |x_0 - \lambda|)^{\alpha+d}.$$

The following proposition is a first application of analytic methods in the study of geometric properties of boundaries. A second consequence of Lemma 3 will be given in Proposition 4.

Proposition 2. *For any domain Ω, the set of weak 0-accessible points is always dense in $\partial\Omega$, whereas there exist domains such that, for any $\alpha \geq 0$, $\partial\Omega$ contains no strong α-accessible point.*

Proof of Proposition 2. We start with the first assertion. We can suppose that $\partial\Omega$ is not empty. We use a basis of compactly supported wavelets. Let $x \in \partial\Omega$ and $\epsilon, \eta > 0$. Since 1_Ω does not belong to $C^\epsilon(B(x, \eta/2))$, following Proposition 1 there exists an arbitrarily large j and a k such that

$$k2^{-j} + [0, 2^{-j}]^d \subset B(x, \eta) \quad \text{and} \quad |C_{j,k}| \geq 2^{-\epsilon j}.$$

The support of the wavelet $\psi_{j,k}$ intersects $\partial\Omega$ (else the corresponding wavelet coefficient would vanish). Let x_1 belong to this intersection. We continue by induction, starting with x_1 instead of x, $\epsilon/2$ instead of ϵ and $\eta/2$ instead of η .... We thus obtain a Cauchy sequence x_n of points of $\partial\Omega$, and its limit point x' satisfies

$$|x - x'| \leq 2\eta, \exists j_n, k_n \;\text{ such that }\; |x' - k_n 2^{-j_n}| \leq C2^{-j_n} \;\text{ and }\; |C_{j_n,k_n}| \geq 2^{-\epsilon 2^{-n} j_n}.$$

Thus, by Lemma 3, the weak exponent at x' vanishes, and x' is arbitrarily close to x.

For the second part of Proposition 2, it suffices to consider the interval $[0, 1]$ from which we subtract all the subintervals $[k2^{-j} - \frac{1}{4}2^{-2j}, k2^{-j} + \frac{1}{4}2^{-2j}]$, for $j \geq 1$ and $k = 1, ..., 2^j - 1$; the result follows because the boundary of Ω is its essential boundary, and Ω contains no interval. Generalizations in several dimensions are straightforward.

Lemma 3 implies that the weak α-accessibility can be related to the *weak scaling exponent* introduced by Y. Meyer; recall that $\mathcal{S}_0(\mathbb{R}^d)$ is the set of functions in the Schwartz class whose moments of any order vanish.

Definition 8. *A tempered distribution f belongs to $\Gamma^s(x_0)$ if $\forall\psi \in \mathcal{S}_0(\mathbb{R}^d)$ $\exists C(\psi)$ such that*

$$\forall a \in (0,1), \quad \left| \int a^d f(x) \psi\left(\frac{x-b}{a}\right) dx \right| \leq C(\psi) a^s.$$

The **weak scaling exponent** *at x_0 is*

$$h_f^{ws}(x_0) = \sup\{s : f \in \Gamma^s(x_0)\}.$$

The following characterization is proved in [24] (Theorem 1.2):

$$f \in \Gamma^s(x_0) \Longleftrightarrow \exists s' \in \mathbb{R} : \ f \in C^{s,s'}(x_0). \tag{18}$$

We will show in Part 2 that the wavelet formulation of the multifractal formalism naturally leads to estimates on the Hausdorff dimension of the points where the weak scaling exponent takes a given value; this is an additional reason for considering the weak scaling exponent as a classification tool for singularities. The following result follows directly from (18) and Lemma 3.

Corollary 1. *Let Ω be a domain of $\mathbb{R}^d$. Then, for any $x \in \partial\Omega$, the weak scaling exponent of 1_Ω at x is larger than the weak accessibility exponent of Ω at x.*

2.3. Wavelet characterizations of T_u^p regularity. In this section, we show how to derive T_u^p regularity from estimates on the wavelet coefficients. In particular, Theorem 2 together with Lemma 2 will show that the weak accessibility exponent at a point $x \in \partial\Omega$ can be derived from the wavelet coefficients of the characteristic function of Ω. We will use the spaces $X_{x_0}^{s,s',p,q}$, which are weighted Besov spaces; let us start by recalling the wavelet characterization of the homogeneous Besov spaces $B_p^{s,q}$, see [23]:

$$f \in B_p^{s,q} \Longleftrightarrow \sum_j \left(\sum_k |C_\lambda 2^{(s-\frac{d}{p})j}|^p \right)^{q/p} < \infty. \tag{19}$$

(It follows from Proposition 1 that $C^\gamma(\mathbb{R}^d) = B_\infty^{\gamma,\infty}$).

Definition 9. *Let s, s' be real number and let p and q be positive real numbers. A tempered distribution f belongs to $X_{x_0}^{s,s',p,q}$ if its wavelet coefficients satisfy*

$$\sum_{j\in\mathbb{Z}} 2^{(s-\frac{d}{p})qj} \left(\sum_{k\in\mathbb{Z}^d} |C_\lambda|^p (1+|k-2^j x_0|)^{s'p} \right)^{\frac{q}{p}} < +\infty. \tag{20}$$

The spaces $X_{x_0}^{s,s',p,q}$ were introduced in order to study local oscillating behaviours, see [25]. They coincide with more classical spaces in several cases:

- If $s' = 0$, $X_{x_0}^{s,0,p,q}$ is independent of x_0 and coincides with the Besov space $B_p^{s,q}$.
- If $p = q = +\infty$, $X_{x_0}^{s,s',\infty,\infty}$ coincides with the two-microlocal space $C^{s,s'}(x_0)$.

These spaces have a local version defined as follows.

Definition 10. *A tempered distribution f belongs to $\dot{X}_{x_0}^{s,s',p,q}$ if there exists $A > 0$ such that*

$$\sum_{j\geq 0} 2^{(s-\frac{d}{p})qj} \left(\sum_{|k-2^j x_0|\leq A2^j} |C_\lambda|^p (1+|k-2^j x_0|)^{s'p} \right)^{\frac{q}{p}} < +\infty.$$

The spaces $X_{x_0}^{s,s',p,q}$ and their local versions $\dot{X}_0^{s,s',p,q}$ do not depend on the choice of the wavelet basis, see [25]. The following theorem shows that $T_u^p(x_0)$ regularity is closely related with these conditions; it will be proved in Sect. 3.

Theorem 1. *Let $p \geq 1$, $s \geq 0$, $x_0 \in \mathbb{R}^d$ and $f \in L^p_{loc}$.*

1. *If f belongs to $\dot{X}_{x_0}^{s,-s,p,1}$, then f belongs to $T^p_{s-\frac{d}{p}}(x_0)$.*
2. *If $f \in T^p_{s-\frac{d}{p}}(x_0)$, then $\exists A, C > 0$ such that the wavelet coefficients of f satisfy*

$$\exists C \ \forall j \quad 2^{j(sp-d)} \sum_{|k-2^j x_0|\leq A2^j} |C_{j,k}|^p (1+|k-2^j x_0|)^{-sp} \leq Cj. \tag{21}$$

Let $p \geq 1$, and $f \in L^p_{loc}$; if A is small enough, let

$$\Sigma_j^p(s, A) = 2^{j(sp-d)} \sum_{|k-2^j x_0|\leq A2^j} |C_{j,k}|^p (1+|k-2^j x_0|)^{-sp} \tag{22}$$

and

$$i_p(x_0) = \sup \left\{ s : \liminf \frac{\log\left(\Sigma_j^p(s, A)^{1/p}\right)}{-j\log 2} \geq 0 \right\}. \tag{23}$$

The following theorem shows that the p-exponent (see Definition 5) can be derived from the wavelet coefficients; it will be proved in Section 4.

Theorem 2. *Let $p \geq 1$ and let $f \in L^p_{loc}$; then*

1. *$i_p(x_0)$ is positive, independent of the value of A, and of the wavelet basis;*
2. *the following inequality always holds*

$$u_f^p(x_0) \leq i_p(x_0) - \frac{d}{p}; \tag{24}$$

3. *if there exists $\delta > 0$ such that $f \in B_p^{\delta,p}$, then*

$$u_f^p(x_0) = i_p(x_0) - \frac{d}{p}. \tag{25}$$

Remark. The Hölder exponent can be characterized by a condition on the modulus of the wavelet coefficients if $f \in C^\epsilon(\mathbb{R}^d)$ for an $\epsilon > 0$, see [14, 16]; the global regularity assumption $f \in B_p^{\delta,p}$ in Part 3 of Theorem 2 plays a similar role; however, it *does not* imply that f has some uniform Hölder regularity, or even that f is locally bounded. Thus Theorems 1 and 2 can be applied to functions that are discontinuous, or even that are not locally bounded; this is very important for applications in several fields; for instance, the velocity of turbulent fluids is now known not to be bounded near vorticity filaments, see [1]; most natural images are discontinuous, and it is also often the case for medical images (mammography data for instance, see [1]). A new multifractal analysis has to be developed for applications in these fields; Theorem 2 shows that it can be based on the p-exponent and its wavelet characterization (an important requirement since, in practice, signals are often stored through their wavelet coefficients).

The global regularity condition $f \in B_p^{\delta,p}$ for a $\delta > 0$ is necessary to obtain (25), as shown by Proposition 3 below; Corollary 2 will show that this condition is satisfied for characteristic functions of domains Ω under a very weak assumption on Ω. As regards applications in image modelling, note that the assumption that images belong to BV is often made; however this assumption is valid only for simple synthesis images, but is known to be wrong for natural images, see [11] for instance. The global regularity assumption $1_\Omega \in B_p^{\delta,p}$ for a $\delta > 0$ is, of course, much weaker.

Proposition 3. *Let $(\psi_{j,k})_{j,k}$ be an orthogonal wavelet basis on $\mathbb{R}$ such that the wavelet ψ is compactly supported. Let f be defined as follows: If $\varepsilon_j = (j(\log j)^2)^{-1/p}$ for $j > 0$, and $k_j = [2^j \varepsilon_j]$, then the wavelet coefficients of f are*

$$\left.\begin{aligned} C_{j,k} &= 2^{j/p}\varepsilon_j \quad \textit{if} \quad k = k_j, \\ &= 0 \qquad\quad \textit{otherwise} \end{aligned}\right\} . \tag{26}$$

Then $f \in L^p_{loc}$ and

$$u_f^p(x_0) = \frac{-1}{p} \quad \textit{whereas} \quad i_p(x_0) \geq 1.$$

It follows that $u_f^p(x_0) \neq i_p(x_0) - \frac{1}{p}$, so that the global regularity condition $f \in B_p^{\delta,p}$ for a $\delta > 0$ is necessary.

Proof of Proposition 3. For j large enough, the wavelets indexed by couples (j, k) such that $C_{j,k} \neq 0$ have disjoint supports so that

$$\| f \|_{L^p}^p \leq C \sum_{j \geq 0} |C_{j,k_j}|^p 2^{-j} \leq \sum_{j \geq 0} \varepsilon_j^p \leq C.$$

First, note that $\Sigma_j^p(1, A) \leq C$ so that $i_p(x_0) \geq 1$. Let us now estimate $u_f^p(x_0)$. Because of the lacunarity of the wavelet series, the quantity $\int_{B(0,r)} |f(y) - P(y-x)|^p dy$ is minimal if $P = 0$. If $r = 2\epsilon_J$, then

$$\int_{B(0,r)} |f(y)|^p dx \geq \int \sum_{j \geq J} |C_{j,k_j} \psi_{j,k_j}|^p dy \geq \sum_{j \geq J} 2^{-j} |C_{j,k_j}|^p \sim \frac{1}{\log J},$$

from which it follows that $u_f^p(x_0) \leq \frac{-1}{p}$.

Definition 11. *Let $A \subset \mathbb{R}^d$; if $\varepsilon > 0$, let $N_\epsilon(A)$ be the smallest number of sets of radius ϵ required to cover A. The* **upper box dimension** *of A is*

$$\overline{dim}_B(A) = \limsup_{\epsilon \to 0} \left(\frac{\log N_\epsilon(A)}{-\log \epsilon} \right).$$

The following corollary shows that the condition $\overline{dim}_B(\partial\Omega) < d$ implies the regularity assumption of Theorem 2 ($f \in B_p^{\delta,p}$ for a $\delta > 0$). Therefore, this geometric condition plays the role of a uniform regularity condition. When it is satisfied, the weak accessibility exponent of Ω at every point can be deduced from the wavelet coefficients of 1_Ω.

Corollary 2. *Let Ω be a bounded domain of $\mathbb{R}^d$; then*

$$\forall \delta < \frac{d - \overline{dim}_B(\partial\Omega)}{p}, \quad 1_\Omega \in B_p^{\delta,p}.$$

If $\overline{dim}_B(\partial\Omega) < d$, then $\forall x \in \partial\Omega$, $E_\Omega^w(x) = pi_p(x) - d$.

Proof. Let $\Delta = \overline{dim}_B(\partial\Omega)$; using compactly supported wavelets, $\forall \epsilon > 0$ the number of nonvanishing wavelet coefficients at each scale is bounded by $C2^{(\Delta+\epsilon)j}$; since 1_Ω is bounded, these wavelet coefficients satisfy $|C_\lambda| \le C$. The first statement follows immediately. The second statement follows from Theorem 2 and Lemma 2.

2.4. Strong accessibility and the oscillation exponent. The purpose of this section is to study how strong accessibility at a point of $\partial\Omega$ can be estimated by conditions on the wavelet coefficients of 1_Ω. The following lemma goes in the direction opposite to Lemma 3: It shows that large wavelet coefficients can be found close to strong α-accessible points. We use an orthonormal basis of compactly supported wavelets, see [8].

Lemma 4. *Suppose that $x \in \partial\Omega$ is strong α-accessible. Using the notations of Definition 6, let $j_n = -[\log_2(r_n)]$; then there exist l, which depends only on the wavelet chosen, a sequence $J_n \in [j_n d/(d+\alpha) - l, j_n]$, and K_n such that*

$$\left.\begin{array}{l} |K_n 2^{-J_n} - x| \le C2^{-j_n d/(\alpha+d)} \\ |C_{J_n,K_n}| \ge \dfrac{C'}{J_n} \end{array}\right\}. \tag{27}$$

Proof of Lemma 4. Orthonormal wavelet decompositions can be constructed through a multiresolution analysis; it means that there exists a compactly supported function φ, arbitrarily smooth and such that $\int \varphi = 1$ and, if

$$P_j(f) = \sum_{j'<j} \sum_k \langle f|\psi_{j',k}\rangle \psi_{j',k},$$

$P_j(f)$ can also be written

$$P_j(f)(x) = \sum_k \left(\int 2^{dj} f(y)\varphi(2^j y - k)dy \right) \varphi(2^j x - k). \tag{28}$$

If x is strong α-accessible, then there exists $l \in \mathbb{N}$, which depends only on the size of the support of φ, such that $P_{j_n+l}(x_n) = 1$ and $P_{j_n+l}(y_n) = 0$ (where $x_n \in B_n^1$ and $y_n \in B_n^2$, where B_n^1 and B_n^2 are as in Definition 6). Let $J \leq j_n$; applying the mean value theorem to (28) and using that 1_Ω is bounded, we obtain

$$|P_J(x_n) - P_J(y_n)| \leq C2^J |x_n - y_n|.$$

Therefore, there exists $l' \in \mathbb{N}$, which depends only on φ such that, if $J = dj_n/(d+\alpha) - l'$, then $|P_J(x_n) - P_J(y_n)| \leq 1/2$. It follows that either $|P_{j_n+l}(x_n) - P_J(x_n)| \geq 1/4$ or $|P_{j_n+l}(y_n) - P_J(y_n)| \geq 1/4$. But

$$P_{j_n+l}(z) - P_J(z) = \sum_J^{j_n+l} \sum_k C_{j,k} \psi_{j,k}(z)$$

and for each j the sum over k has at most C terms. Thus one of the wavelet coefficients $C_{j,k}$ is larger than $C'/(j_n + l - J)$. Furthermore the support of the corresponding wavelet $\psi_{j,k}$ contains either x_n or y_n. Hence Lemma 4 holds. The result can easily be extended to the case where the wavelets have only fast decay.

Lemma 4 implies that the strong accessibility exponent can be related to the *oscillation exponent*, a notion introduced in [2] and that was motivated by the following situation often met in signal analysis: Characterizing the pointwise behavior of a function with the sole Hölder exponent yields restrictive information since it does not describe the more or less oscillatory behavior of the function near the point x_0. This oscillatory behavior can be modelled as follows. Let $t > 0$ and let $h^t_f(x_0)$ denote the Hölder exponent of the fractional primitive of order t at x_0 of a function $f \in L^\infty_{loc}$; more precisely let ϕ be a C^∞ compactly supported function satisfying $\phi(x_0) = 1$, and let $(Id - \Delta)^{-t/2}$ be the convolution operator which amounts to multiply the Fourier transform of the function with $(1 + |\xi|^2)^{-t/2}$; we denote by $h^t_f(x_0)$ the Hölder exponent at x_0 of the function $f_t = (Id - \Delta)^{-t/2}(\phi f)$. The following definition was introduced in [2] (see also [16, 17, 24] where alternative definitions are discussed).

Definition 12. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded function. If $h_f(x_0) \neq +\infty$, then the* **oscillation exponent** *of F at x_0 is defined by*

$$\beta_f(x_0) = \left(\frac{\partial}{\partial t} h^t_f(x_0)\right)_{t=0} - 1 \tag{29}$$

(where the derivative at $t = 0$ should be understood as a right-derivative).

Note that the mapping $t \longrightarrow h^t_f(x_0)$ is a concave increasing function, see [2], so that the derivative in (29) always exists (but may be infinite).

Corollary 3. *Let Ω be a domain of $\mathbb{R}^d$. If $x \in \partial\Omega$, let $\beta_\Omega(x)$ denote the oscillation exponent of 1_Ω at x. Then,*

$$\forall x \in \partial\Omega, \quad \beta_\Omega(x) \leq \frac{E^s_\Omega(x)}{d}.$$

Proposition 4 will yield a natural geometric condition under which this upper bound becomes an equality.

Proof. If $f \in L^\infty_{loc}$, then, if $t > 0$, the Hölder exponent of f_t at x is

$$\liminf_{j \to +\infty} \inf_k \left(\frac{\log(2^{-tj}|C_{j,k}|)}{\log(2^{-j} + |x - \lambda|)} \right),$$

see [2, 4]. If 1_Ω is strong α-accessible at x, choosing the particular sequence of wavelet coefficients given by (27), we obtain that, if $t > 0$, then $h^t_{1_\Omega}(x) \le t\,(1 + \alpha/d)$, so that $\beta_{1_\Omega}(x) \le \alpha/d$.

Definition 13. *The* **two-microlocal domain** *of f at x_0, denoted by $E(f(x_0))$, is the set of indices (s, s') such that f belongs to $C^{s,s'}(x_0)$. The boundary of $E(f(x_0))$ can be parametrized by a decreasing concave function $s = A_{x_0}(s')$, called the* **two-microlocal frontier**.

The fact that $A_{x_0}(s')$ is concave is proved in [13, 24]; its knowledge gives precise information about the pointwise behavior of the function; in particular, the Hölder exponent of fractional primitives of a function in L^∞_{loc} can be derived from $A_{x_0}(s')$, see [2]. The following proposition gives the precise two-microlocal behavior at the points similar to the cusp-singularities (5), i.e. at the points where the weak and strong accessibility exponents coincide.

Proposition 4. *Let $x_0 \in \partial\Omega$ be a point where $E^w_\Omega(x_0) = E^s_\Omega(x_0)$; then*

$$\forall s' \in (-E^w_\Omega(x_0) - d, 0] \quad A_{x_0}(s') = \frac{-d}{E^w_\Omega(x_0) + d} s',$$

and x_0 is an oscillating singularity, with an oscillation exponent $\beta_\Omega(x_0) = E^w_\Omega(x_0)/d$.

Proof. Since $x_0 \in \partial\Omega$, then $(0, 0) \in E(f(x_0))$; let $\alpha = E^w_\Omega(x_0) = E^s_\Omega(x_0)$; then Lemma 3 implies that $\forall \alpha' < \alpha$, $(\alpha', -\alpha' - d) \in E(f(x_0))$, and Lemma 4 implies that $(s, s') \notin E(f(x_0))$ if $s' > -(1 + \frac{d}{\alpha})s$. Since $E(f(x_0))$ is convex, the first statement follows.

It is shown in [2] that, if $f \in L^\infty_{loc}$,

$$1 + \beta(x_0) = \frac{1}{1 + (A_{x_0})'_g(-h(x_0))}, \tag{30}$$

where $(A_{x_0})'_g(t)$ denotes the left derivative of A at t (and here $h(x_0) = 0$). The second statement follows.

3. Proof of Theorem 1

We can assume that $x_0 = 0$ without losing generality. We will use a compactly supported scaling function and wavelets of class C^n, where $n \ge s$ and we actually suppose that their supports are included in $B(0, M)$, with $M > 0$. Thus the support of the wavelet $\psi_{j,k}$ is included in the cube

$$\lambda_{j,k} = k2^{-j} + \left[\frac{-M}{2^j}, \frac{M}{2^j}\right]^d.$$

3.1. Proof of the embedding $\dot{X}_0^{s,-s,p,1} \subset T^p_{s-\frac{d}{p}}(0)$. Let $f \in \dot{X}_0^{s,-s,p,1}$; (20) can be rewritten

$$\sum_{|k|\le A2^j} |C_{j,k}|^p (1+|k|)^{-sp} \le C\varepsilon_j{}^p 2^{(-sp+d)j} \tag{31}$$

with $\varepsilon_j \in l^1$. We want to prove that there exists a polynomial P of degree less than or equal to $s-\frac{d}{p}$, $C>0$ and $R>0$ such that (11) holds (with $x_0=0$). Let

$$\Delta_j f(x) = \sum_{|k|\le A2^j} C_{j,k}\psi(2^j x - k). \tag{32}$$

The wavelets are compactly supported and, since $f \in L^p_{loc}$,

$$f(x) = \sum_k C_k \phi(x-k) + \sum_{j=0}^{+\infty} \Delta_j f(x), \tag{33}$$

where convergence takes place in L^p_{loc}. The function $\sum_k C_k\phi(x-k)$ belongs to $C^n(\mathbb{R}^d)$, where $n \ge s$ corresponds to the smoothness of the wavelets; thus it belongs to $T^p_{s-\frac{d}{p}}(0)$; therefore we can restrict our study to the function $\sum_{j=0}^{+\infty} \Delta_j f(x)$. Let us define the polynomial P.

- If $s < \frac{d}{p}$, we set $P=0$.
  In this case, for $\rho \le A$, we will have to bound

$$\left[\int_{|x|\le\rho} \left|\sum_{j=0}^{+\infty} \Delta_j f(x) - P(x)\right|^p dx\right]^{\frac{1}{p}} = \left[\int_{|x|\le\rho} \left|\sum_{j=0}^{+\infty} \Delta_j f(x)\right|^p dx\right]^{\frac{1}{p}}. \tag{34}$$

- If $s \ge \frac{d}{p}$, let $N = \left[s - \dfrac{d}{p}\right]$. In this case, we set

$$P(x) = \sum_{|\alpha|\le N} \sum_{j=0}^{+\infty} (\Delta_j^{(\alpha)} f)(0) \frac{x^\alpha}{\alpha!}. \tag{35}$$

We first have to check that the definition of P in the second case makes sense, i.e. that $\sum_{j=0}^{+\infty} \Delta_j^{(\alpha)} f(0)$ is finite for all α such that $|\alpha| \le N$. It follows from (32) that

$$|\Delta_j^{(\alpha)} f(0)| \le C 2^{j|\alpha|} \sum_{|k|\le M} |C_{j,k}|. \tag{36}$$

Since (31) implies that $2^{j|\alpha|} \sum_{|k|\le M} |C_{j,k}| \le C\varepsilon_j 2^{j(|\alpha|+\frac{d}{p}-s)}$, this yields

$$|\Delta_j^{(\alpha)} f(0)| \le 2M\varepsilon_j 2^{j(|\alpha|+\frac{d}{p}-s)}; \tag{37}$$

but $|\alpha| + \frac{d}{p} - s \leq 0$, and $\varepsilon_j \in l^1$; it follows that the series in (35) are convergent. In this case we have to bound

$$\left[\int_{|x|\leq\rho}\left|\sum_{j=0}^{+\infty}\Delta_j f(x) - P(x)\right|^p dx\right]^{\frac{1}{p}}$$
$$= \left[\int_{|x|\leq\rho}\left|\sum_{j=0}^{+\infty}\Delta_j f(x) - \sum_{|\alpha|\leq N}\sum_{j=0}^{+\infty}\Delta_j^{(\alpha)} f(0)\frac{x^\alpha}{\alpha!}\right|^p dx\right]^{\frac{1}{p}}. \tag{38}$$

In the following, J denotes the integer defined by

$$2^{-J} \leq \rho < 2^{-J+1} \leq A. \tag{39}$$

3.1.1. The case $s \geq \frac{d}{p}$ We will estimate the contributions of the $\Delta_j f$ in (38) separately for $j \leq J$ and $j \geq J$.

Let us first consider the case $j \geq J$. The corresponding term in (38) is bounded by $R_J^1 + R_J^2$ where

$$R_J^1 = \left[\int_{|x]\leq\rho}\left|\sum_{j=J}^{+\infty}\sum_{|\alpha|\leq N}\Delta_j^{(\alpha)} f(0)\frac{x^\alpha}{\alpha!}\right|^p dx\right]^{\frac{1}{p}} \text{ and } R_J^2 = \left[\int_{|x|\leq\rho}\left|\sum_{j=J}^{+\infty}\Delta_j f(x)\right|^p dx\right]^{\frac{1}{p}}. \tag{40}$$

We can bound R_J^1 by

$$R_J^1 \leq \sum_{|\alpha|\leq N}\left[\int_{|x|\leq\rho}\left|\sum_{j=J}^{+\infty}\Delta_j^{(\alpha)} f(0)\right|^p \frac{|x|^{|\alpha|p}}{(|\alpha|!)^p}dx\right]^{\frac{1}{p}}. \tag{41}$$

Using (37), we get

$$\left|\sum_{j=J}^{+\infty}\Delta_j^{(\alpha)} f(0)\right| \leq C\sum_{j=J}^{+\infty}\varepsilon_j 2^{j(|\alpha|+\frac{d}{p}-s)} \leq C2^{J(|\alpha|+\frac{d}{p}-s)}\sum_{j=J}^{+\infty}\varepsilon_j.$$

Using (39), it follows that

$$\left|\sum_{j=J}^{+\infty}\Delta_j^{(\alpha)} f(0)\right| \leq C\rho^{s-\frac{d}{p}-|\alpha|}\sum_{j=J}^{+\infty}\varepsilon_j.$$

This yields

$$R_J^1 \leq C\sum_{|\alpha|\leq N}\rho^{s-\frac{d}{p}-|\alpha|}\sum_{j=J}^{+\infty}\varepsilon_j\left[\int_{|x|\leq\rho}|x|^{|\alpha|p}dx\right]^{\frac{1}{p}}$$
$$\leq C\rho^{s-\frac{d}{p}-|\alpha|}\rho^{|\alpha|+\frac{d}{p}}\sum_{j=J}^{+\infty}\varepsilon_j \leq C\rho^s\sum_{j=J}^{+\infty}\varepsilon_j. \tag{42}$$

Furthermore,

$$R_J^2 = \left[\int_{|x|\leq\rho} \left| \sum_{j=J}^{+\infty} \Delta_j f(x) \right|^p dx \right]^{\frac{1}{p}} \leq \sum_{j=J}^{+\infty} \left[\int_{|x|\leq\rho} \left| \Delta_j f(x) \right|^p dx \right]^{\frac{1}{p}}. \tag{43}$$

For each j and k given, there are at most $(2M)^d$ wavelets $\psi_{j,k'}$ non-vanishing on the support of $\psi_{j,k}$. We can split the wavelets at scale j into $(2M)^d$ sets P_i such that the wavelets in the same P_i have disjoint supports. The convexity of the function x^p (if $p \geq 1$) yields

$$\begin{aligned} \int_{|x|\leq\rho} \left| \Delta_j f(x) \right|^p dx &\leq C \sum_{i\in I} \left(\int_{|x|\leq\rho} \left| \sum_{k\in P_i} C_{j,k} \psi(2^j x - k) \right|^p dx \right) \\ &\leq C \sum_{i\in I} \sum_{k\in P_i} |C_{j,k}|^p \int |\psi(2^j x - k)|^p dx \\ &\leq C \sum_{k\in\Omega(j,\rho)} 2^{-dj} |C_{j,k}|^p, \end{aligned} \tag{44}$$

where the sum on k actually bears on the indices belonging to the set

$$\Omega(j,\rho) = \{k : \lambda_{j,k} \bigcap B(0,\rho) \neq \emptyset\}.$$

If $k \in \Omega(j,\rho)$, then $|k2^{-j}| \leq \rho + \dfrac{2M}{2^j}$. Since $2^{-j} \leq 2^{-J} \leq \rho$, it follows that $|k2^{-j}| \leq \rho(1+M)$; this yields $2^{-j} \leq \dfrac{\rho(1+M)}{|k|}$ if $|k| \geq 1$, so that, in all cases, $2^{-j} \leq \dfrac{C\rho}{|k|+1}$.

Since we can assume that $s \geq 0$ it follows that $2^{-spj} \leq \dfrac{\rho^{sp}}{(|k|+1)^{sp}}$. Therefore, it follows from (44) that

$$\int_{|x|\leq\rho} |\Delta_j f(x)|^p dx \leq C \sum_{k\in\Omega(j,\rho)} |C_{j,k}|^p 2^{(sp-d)j} \frac{\rho^{sp}}{(|k|+1)^{sp}}. \tag{45}$$

Thus, using (31), $\int_{|x|\leq\rho} |\Delta_j f(x)|^p dx \leq C\varepsilon_j^p \rho^{sp}$; and (43) implies that

$$R_J^2 \leq C\rho^s \sum_{j=J}^{+\infty} \varepsilon_j. \tag{46}$$

Our estimates for R_J^1 and R_J^2 therefore yield

$$\left[\int_{|x]\leq\rho} \left| \sum_{j=J}^{+\infty} \left(\Delta_j f(x) - \sum_{|\alpha|\leq N} \Delta_j^{(\alpha)} f(0) \frac{x^\alpha}{\alpha!} \right) \right|^p dx \right]^{\frac{1}{p}} \leq C\rho^s \sum_{j=J}^{+\infty} \varepsilon_j. \tag{47}$$

Let us consider now the case $j \leq J$. We have to estimate

$$S_J = \left[\int_{|x| \leq \rho} \left| \sum_{j=0}^{J} \Delta_j f(x) - \sum_{|\alpha| \leq N} \sum_{j=0}^{J} \Delta_j^{(\alpha)} f(0) \frac{x^\alpha}{\alpha!} \right|^p dx \right]^{\frac{1}{p}}$$

$$\leq \sum_{j=0}^{J} \left[\int_{|x| \leq \rho} \left| \Delta_j f(x) - \sum_{|\alpha| \leq N} \Delta_j^{(\alpha)} f(0) \frac{x^\alpha}{\alpha!} \right|^p dx \right]^{\frac{1}{p}}. \tag{48}$$

Therefore, using the mean value theorem,

$$S_J \leq \sum_{j=0}^{J} \left[\int_{|x| \leq \rho} |x|^{(N+1)p} \sup_{|x| \leq \rho, |\alpha| = N+1} \left| \Delta_j^\alpha f(x) \right|^p dx \right]^{\frac{1}{p}}. \tag{49}$$

The wavelets $\psi_{j,k}$ which bring a non-vanishing contribution to (49) satisfy $\lambda_{j,k} \bigcap B(0, \rho) \neq \emptyset$. Since $j \leq J$, we have $\rho \leq 2^{-J+1} \leq 2.2^{-j}$ and

$$\left| k 2^{-j} \right| \leq \rho + M 2^{-j} \leq 2^{-j}(1 + 2M).$$

Thus $|k| \leq 1 + 2M$, and therefore

$$\forall t \in B(0, \rho) : \Delta_j^\alpha f(t) = \sum_{|k| \leq 2M+1} 2^{j(N+1)} C_{j,k} \psi^{(\alpha)}(2^j t - k).$$

Thus, using (31),

$$\begin{aligned} \sup_{|t| \leq \rho} \left| \Delta_j^\alpha f(t) \right|^p &\leq C 2^{jp(N+1)} \left[\sum_{|k| \leq 2M+1} |C_{j,k}| \right]^p \\ &\leq C 2^{jp(N+1)} \sum_{|k| \leq 2M+1} |C_{j,k}|^p \\ &\leq C 2^{jp(N+1)} (4M+3)^d \varepsilon_j^p 2^{j(d-sp)}, \end{aligned} \tag{50}$$

which, together with (49) yields

$$S_J \leq C \sum_{j=0}^{J} \rho^{N+1+\frac{d}{p}} 2^{(N+1)j} \varepsilon_j 2^{(\frac{d}{p}-s)j} \leq C \rho^{N+1+\frac{d}{p}} \sum_{j=0}^{J} \varepsilon_j 2^{(-s+N+1+\frac{d}{p})j}.$$

Since $2^{-J} \leq \rho \leq 2^{-J+1}$, and $N + 1 - s + \frac{d}{p} \geq 0$, we have

$$S_J \leq C \rho^{N+1+\frac{d}{p}} \sum_{j=0}^{J} \varepsilon_j 2^{(-s+N+1+\frac{d}{p})J} \leq C \rho^s \sum_{j=0}^{J} \varepsilon_j. \tag{51}$$

Using (51) and (47), we obtain

$$\left[\int_{|x| \leq \rho} \left| \sum_{j=0}^{+\infty} \Delta_j f(x) - P(x) \right|^p dx \right]^{\frac{1}{p}} \leq C \rho^s.$$

Thus $\dot{X}_0^{s,-s,p,1} \subset T^p_{s-\frac{d}{p}}(0)$ if $s \geq \frac{d}{p}$.

3.1.2. The case $s < \frac{d}{p}$ We have to estimate (34). As before, we split the sum into two terms, depending whether $j \geq J$ or $j < J$. If $j \geq J$, we can use the bound obtained above for R_J^2 since it was obtained under the sole assumption $s \geq 0$. Therefore

$$\left[\int_{|x|\leq\rho}\left|\sum_{j=J}^{+\infty}\Delta_j f(x)\right|^p dx\right]^{\frac{1}{p}} \leq C\rho^s \sum_{j=J}^{+\infty}\varepsilon_j. \tag{52}$$

Now, we want to bound the sum in (34) restricted to $j \leq J$. Let

$$S_J = \left[\int_{|x|\leq\rho}\left|\sum_{j=0}^{J}\Delta_j f(x)\right|^p dt\right]^{\frac{1}{p}} \leq \sum_{j=0}^{J}\left[\int_{|x|\leq\rho}\sup_{|t|\leq\rho}\left|\Delta_j f(t)\right|^p dx\right]^{\frac{1}{p}}. \tag{53}$$

As in the previous case ($s \geq \frac{d}{p}$), we have a finite number of non vanishing wavelets in the integral and this yields

$$\forall t \in B(0,\rho), \quad \Delta_j f(t) = \sum_{|k|\leq 2M+1} C_{j,k}\psi(2^j t - k).$$

Thus, using (31),

$$\sup_{|t|\leq\rho}\left|\Delta_j f(t)\right|^p \leq C(4M+1)^d \varepsilon_j^p 2^{j(d-sp)}.$$

Coming back to (53), it follows that

$$S_J \leq C\sum_{j=0}^{J}\rho^{\frac{d}{p}}\varepsilon_j 2^{(\frac{d}{p}-s)j};$$

since $s < \frac{d}{p}$, and since $2^{-J} \leq \rho \leq 2^{-J+1}$, then

$$S_J \leq C\sum_{j=0}^{J}\rho^{\frac{d}{p}}\varepsilon_j 2^{(\frac{d}{p}-s)J} \leq C\rho^s\sum_{j=0}^{J}\varepsilon_j. \tag{54}$$

Coming back to (34), it follows from (52) and (54) that

$$\left(\int_{|x|\leq\rho}\left|\sum_{j=0}^{+\infty}\Delta_j f(x)\right|^p dx\right)^{\frac{1}{p}} \leq C\rho^s. \tag{55}$$

Since $\sum_k C_k\phi(x-k)$ belongs to $T^p_{s-\frac{d}{p}}(0)$, we can conclude that

$$\left(\int_{|x|\leq\rho}|f(x)|^p dx\right)^{\frac{1}{p}} \leq C\rho^s. \tag{56}$$

Thus, if $s < \frac{d}{p}$, and if f belongs to $X_0^{s,-s,p,1}$, then f belongs to $T^p_{s-\frac{d}{p}}(0)$, which completes the proof of Part 1 in Theorem 1.

3.2. Proof of Part 2 of Theorem 1. If $f \in T^p_{s-\frac{d}{p}}(0)$, then (11) holds and there exists a polynomial P of degree less than or equal to $s-\frac{d}{p}$ and constants C and R such that

$$\forall \rho \le R, \quad \int_{|x-x_0|\le\rho} |f(x)-P(x-x_0)|^p dx \le C\rho^{sp}.$$

Since f belongs to $T^p_{s-\frac{d}{p}}(0)$, it belongs to L^p_{loc}. We want to prove that (21) holds. We rewrite (21) on the orthonormal wavelet basis $\tilde{\psi}_{j,k} = 2^{\frac{dj}{2}}\psi_{j,k}$ of $L^2(\mathbb{R}^d)$:

$$f(x) = \sum_j \sum_k \tilde{C}_{j,k}\tilde{\psi}_{j,k} \quad \text{with} \quad \tilde{\psi}_{j,k}(x) = 2^{\frac{dj}{2}}\psi(2^j x - k) \quad \text{and} \quad \tilde{C}_{j,k} = 2^{-\frac{dj}{2}}C_{j,k}.$$

We want to prove that, for an $A > 0$,

$$2^{(sp-d+\frac{dp}{2})j} \sum_{|k|\le A2^j} |\tilde{C}_{j,k}|^p (1+|k|)^{-sp} \le Cj. \tag{57}$$

We pick $A = R$ and we define $a \in \mathbb{Z}$ by the condition $2^a \le R < 2^{a+1}$. For $l \in \{1, ..., j+a+1\}$, let

$$A_{l,j} = \sum_{2^{l-1}<|k|\le 2^l} |\tilde{C}_{j,k}|^p (1+|k|)^{-sp}; \tag{58}$$

(57) can be rewritten

$$2^{(sp-d+\frac{dp}{2})j} \left[|\tilde{C}_{j,0}|^p + \sum_{l=1}^{j+a+1} A_{l,j} \right] \le Cj. \tag{59}$$

Let us now derive a bound for $A_{l,j}$. The basis $\tilde{\psi}_{j,k}$ is orthonormal in L^2 so that

$$A_{l,j} = \left\langle \sum_{j'\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \tilde{C}_{j',k}\tilde{\psi}_{j',k} \;\middle|\; \sum_{2^{l-1}<|k|\le 2^l} |\tilde{C}_{j,k}|^{p-1} sgn(\tilde{C}_{j,k})(1+|k|)^{-sp}\tilde{\psi}_{j,k} \right\rangle, \tag{60}$$

where $< .|. >$ denotes the scalar product in L^2 and sgn the function such that for any $a \neq 0$, $sgn(a) = \frac{a}{|a|}$, and $sgn(0) = 0$.

Since we assumed that $f \in T^p_{s-\frac{d}{p}}(0)$, there exists a polynomial P of degree less than $s-\frac{d}{p}$ (P vanishes if $s < \frac{d}{p}$), such that (11) holds. Since the $\tilde{\psi}_{j,k}$ form an orthonormal basis of $L^2(\mathbb{R}^d)$ with regularity C^n such that $n \ge N$, all the moments of ψ of order less than or equal to N of ψ vanish, so that, $\forall j, k$, $\int P(x)\tilde{\psi}_{j,k}(x)dx = 0$. Thus, if we define $g_{l,j}$ by

$$g_{l,j} = \sum_{2^{l-1}<|k|\le 2^l} |\tilde{C}_{j,k}|^{p-1} sgn(\tilde{C}_{j,k})(1+|k|)^{-sp}\tilde{\psi}_{j,k}, \tag{61}$$

then (60) can be rewritten

$$A_{l,j} = \int_{\mathcal{C}_{l,j}} (f(x) - P(x)) g_{l,j}(x) dx, \tag{62}$$

where $\mathcal{C}_{l,j}$ denotes the support of $g_{l,j}$.

Remark that if $2^{l-1} > M$, $\mathcal{C}_{l,j}$ is included in a ring of center 0, of inner diameter $2.\frac{2^{l-1}-M}{2^j}$ and outer diameter $2.\frac{2^l+M}{2^j}$. If $2^{l-1} \le M$, $\mathcal{C}_{l,j}$ is included in the ball of center 0 and diameter $2.\frac{2^l+M}{2^j}$.

If q denotes the conjugate exponent of p, Hölder's inequality yields

$$A_{l,j} \le \left|\left|(f-P).\chi_{\mathcal{C}_{l,j}}\right|\right|_{L^p} \cdot \left|\left|g_{l,j}\right|\right|_{L^q}. \tag{63}$$

Let us study $\left|\left|g_{l,j}\right|\right|_{L^q}$. The wavelet characterization of L^q (see Chapter 6.2 of [23]) yields

$$A_{l,j} \le C \parallel (f-P).\chi_{\mathcal{C}_{l,j}} \parallel_{L^p} \left[\sum_{2^{l-1}<|k|\le 2^l} |\tilde{C}_{j,k}|^{q(p-1)} (1+|k|)^{-sp-sq} 2^{\frac{djq}{2}-dj} \right]^{\frac{1}{q}}.$$

Since $|k| \ge 2^{l-1}$, it follows that $(1+|k|)^{-sp-sq} \le C(1+|k|)^{-sp}(1+2^l)^{-sq}$, which yields

$$\begin{aligned} A_{l,j} &\le C \parallel (f-P).\chi_{\mathcal{C}_{l,j}} \parallel_{L^p} (1+2^l)^{-s} \left[\sum_{2^{l-1}<|k|\le 2^l} |\tilde{C}_{j,k}|^p (1+|k|)^{-sp} \right]^{\frac{1}{q}} 2^{\frac{dj}{2}-\frac{dj}{q}} \\ &= C \parallel (f-P)\chi_{\mathcal{C}_{l,j}} \parallel_{L^p} (1+2^l)^{-s} 2^{\frac{dj}{2}-\frac{dj}{q}} A_{l,j}^{\frac{1}{q}}. \end{aligned}$$

Thus

$$A_{l,j} \le C \parallel (f-P).\chi_{\mathcal{C}_{l,j}} \parallel_{L^p}^p (1+2^l)^{-sp}.2^{\frac{djp}{2}-\frac{djp}{q}}, \tag{64}$$

or, equivalently

$$2^{(sp+\frac{dp}{2}-d)j} A_{l,j} \le C \parallel (f-P).\chi_{\mathcal{C}_{l,j}} \parallel_{L^p}^p 2^{spj}(1+2^l)^{-sp}. \tag{65}$$

Because of (58), in order to obtain an upper-bound of (59), we add up these estimates for $l \in \{1, ..., j+a+1\}$. We get

$$\begin{aligned} &2^{(sp-d+\frac{dp}{2})j} \sum_{l=1}^{j+a+1} \sum_{2^{l-1}<|k|\le 2^l} |\tilde{C}_{j,k}|^p (1+|k|)^{-sp} \\ &\le C \sum_{l=1}^{j+a+1} \parallel (f-P).\chi_{\mathcal{C}_{l,j}} \parallel_{L^p}^p 2^{spj}(1+2^l)^{-sp} \\ &= \sum_{l=1}^{j+a+1} 2^{spj}(1+2^l)^{-sp} \int_{\mathcal{C}_{l,j}} |f(t)-P(t)|^p dt. \end{aligned}$$

By definition of $T^p_{s-\frac{d}{p}}(0)$, $\int_{\mathcal{B}_{l,j}} |f(t)-P(t)|^p dt \le C\left(\frac{2^l}{2^j}\right)^{sp}$. It follows that, since s is positive,

$$\sum_{l=1}^{j+a+1} 2^{spj}(1+2^l)^{-sp}\int_{\mathcal{C}_{l,j}} |f(t)-P(t)|^p dt \le \sum_{l=1}^{j+a+1} 2^{spj}(1+2^l)^{-sp}(2^{l-j})^{sp} \le C(j+a+1). \qquad (66)$$

In order to obtain the upper-bound on (59), we just need to evaluate

$$|\tilde{C}_{j,0}| = \left|\int 2^{dj/2} f(x)\psi(2^j x)dx\right| = \left|\int 2^{dj/2}(f(x)-P(x))\psi(2^j x)dx\right|,$$

which, by Hölder's inequality, is bounded by

$$2^{dj/2}\left[\int_{|x|\le M2^{-j}} |f(x)-P(x)|^p\right]^{1/p} \| \psi(2^j\cdot) \|_q^{1/q} \le 2^{dj/2}[(M2^{-j})^{sp}]^{1/p}2^{-dj/q},$$

therefore $|\tilde{C}_{j,0}| \le C2^{-spj}2^{dj(1-p/2)}$. Thus (59) is finite.

4. Proof of Theorem 2

First, note that the independence of $i_p(x_0)$ from the wavelet basis chosen follows from the same result for the spaces $X^{s,s',p,q}_{x_0}$.

Lemma 5. *The following properties hold for any function $f \in L^p$:*

1. $\Sigma^p_j(s, A)$ *is an increasing function of* A,
2. *if* $B > A$, *then* $\Sigma^p_j(s, B) = \Sigma^p_j(s, A) + O(1)$,
3. *if* $s \le 0$, $\forall A$, $\Sigma^p_j(s, A) = O(1)$.

Proof. The first part is obvious. If $f \in L^p$, then $f \in B^{0,\infty}_p$ (see [23]), so that

$$\exists C,\ \forall j, \quad \sum_k 2^{-dj}|C_{j,k}|^p \le C;$$

but

$$\Sigma^p_j(s, B) = \Sigma^p_j(s, A) + 2^{-dj}\sum_{A\le|\lambda-x_0|\le B} |C_{j,k}|^p(2^{-j}+|\lambda-x_0|)^{-sp},$$

and the last term is bounded by $C2^{-dj}\sum_k |C_{j,k}|^p = O(1)$; therefore, the second part holds.

The third part is also a direct consequence of the embedding $L^p \hookrightarrow B^{0,\infty}_p$.

We can assume that the wavelets are compactly supported in $B(0, M)$ for an $M > 0$, and that $f \in L^p_{loc}$. Let $A > 0$ and $B > A$; on $\tilde{B} = B(0, B+2M)$, f coincides with a function $g \in L^p$ and, because of our choice of the radius of $\tilde{B}$, $\forall j \ge 0$, $\Sigma^p_j(s, B)$

coincide for f and g and $\Sigma_j^p(s, A)$ also coincide for f and g. Therefore Part 3 of Lemma 5 implies that $\forall A$, $i_p(x_0)$ defined by (23) is positive. Furthermore, since $i_p(x_0)$ is determined by the values of s such that

$$\exists \varepsilon > 0, \exists j_n : \quad \Sigma_j^p(s, A) \geq 2^{\varepsilon j_n}. \tag{67}$$

Part 2 of 5 implies that (67) holds for A if and only if it holds for B. Therefore, $i_p(x_0)$ is positive and independent of the value of A. Hence the first part of Theorem 2 holds.

Let us now prove (24). By definition of $u_f^p(x_0)$, $\forall u_0 < u_f^p(x_0)$, f belongs to $T_{u_0}^p(x_0)$; thus, using Theorem 1, f satisfies (21), and there exists a constant $A > 0$ such that

$$\forall j \geq 0, \quad 2^{u_0 p j} \left[\sum_{|k-2^j x_0| \leq A 2^j} |C_{j,k}|^p (1 + |k - 2^j x_0|)^{-(u_0 p + d)} \right] \leq Cj, \tag{68}$$

which can be rewritten

$$\Sigma_j^p \left(u_0 + \frac{d}{p} \right) \leq Cj, \tag{69}$$

so that

$$\liminf_{j \to \infty} \frac{\log \left(\Sigma_j^p (u_0 + \frac{d}{p})^{\frac{1}{p}} \right)}{-j \log 2} \geq 0. \tag{70}$$

Coming back to (23), we see that

$$u_0 \leq i_p(x_0) - \frac{d}{p}. \tag{71}$$

Since this is true $\forall u_0 < u_f^p(x_0)$, it follows that (24) holds.

Let us now prove (25). Suppose first that $i_p(x_0) = 0$; since necessarily $u_f^p(x_0) \geq -\frac{d}{p}$, (24) implies that $u_f^p(x_0) = -\frac{d}{p} = i_p(x_0) - \frac{d}{p}$, and (25) holds.

Suppose now that $i_p(x_0) \neq 0$ and that f belongs to $B_p^{\delta,p}$ for a $\delta > 0$. We can assume without loss of generality that $x_0 = 0$ and $A = 1$. We want to prove that, if $i_0 < i_p(0)$, then f belongs to $T^p_{i_0 - \frac{d}{p}}(0)$. Using Part 1 of Theorem 1 it is sufficient to prove that f belongs to $\dot{X}_0^{i_0, -i_0, p, 1}$, i.e. that

$$2^{(i_0 p - d) j} \sum_{|k| \leq 2^j} |C_{j,k}|^p (1 + |k|)^{-i_0 p} \leq C \varepsilon_j^p \qquad \text{with } \varepsilon_j \in l^1. \tag{72}$$

Let $i_0 < i_p(0)$ be given. The hypotheses are the following:

- By definition of $i_p(0)$, $\forall s < i_p(0)$, $\forall \varepsilon > 0$,

$$2^{(sp-d)j} \sum_k |C_{j,k}|^p (1 + |k|)^{-sp} \leq C(s, \epsilon) 2^{\epsilon j}. \tag{73}$$

– $f \in B_p^{\delta,p}$ for a $\delta > 0$, so that

$$2^{(\delta p-d)j} \sum_k |C_{j,k}|^p \in l^1. \tag{74}$$

We pick an s satisfying $i_0 < s < i_p(0)$. Let $\theta \in (0, 1)$ which will be fixed later. First, we estimate the sum on the left hand side of (72) for $|k| \geq 2^{\theta j}$; it is bounded by

$$2^{(i_0 p-d)j} \sum_k |C_{j,k}|^p (1+2^{\theta j})^{-i_0 p},$$

which, using (74) is bounded by $2^{i_0 p(1-\theta)j} 2^{-\delta p j}$. Therefore, if we pick θ close enough to 1, this term decays exponentially.

Having thus fixed the value of θ, we now estimate the sum on the right hand side of (72) for $|k| < 2^{\theta j}$; it is equal to

$$2^{(i_0 p-d)j} \sum_{|k|<2^{\theta j}} |C_{j,k}|^p (1+|k|)^{-sp} (1+|k|)^{(s-i_0)p},$$

which, using (73) is bounded by

$$C(s,\epsilon) 2^{\epsilon j} 2^{-pj(1-\theta)(s-i_0)},$$

which also decays exponentially if ϵ is picked small enough.

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