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Algebro-Geometric Solutions of the Baxter–Szegő Difference Equation*

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Abstract: We derive theta function representations of algebro-geometric solutions of a discrete system governed by a transfer matrix associated with (an extension of) the trigonometric moment problem studied by Szegő and Baxter. We also derive a new hierarchy of coupled nonlinear difference equations satisfied by these algebro-geometric solutions.

1. Introduction

Let $\{\alpha(n)\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence of complex numbers subject to the condition

$$|\alpha(n)| < 1 \text{ for all } n \in \mathbb{N}, \tag{1.1}$$

and define the transfer matrix

$$T(z) = \begin{pmatrix} z & \alpha \\ \overline{\alpha} z & 1 \end{pmatrix}, \quad z \in \mathbb{T}, \tag{1.2}$$

with spectral parameter z on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Consider the system of difference equations

$$\Phi(z,n) = T(z,n)\Phi(z,n-1), \quad (z,n) \in \mathbb{T} \times \mathbb{N} \tag{1.3}$$

with initial condition $\Phi(z,0) = \binom{1}{1}$, $z \in \mathbb{T}$, where

$$\Phi(z,n) = \begin{pmatrix} \varphi(z,n) \\ \varphi^*(z,n) \end{pmatrix}, \quad (z,n) \in \mathbb{T} \times \mathbb{N}_0. \tag{1.4}$$

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(Here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.) Then $\varphi(\,\cdot\,, n)$ are monic polynomials of degree n and

$$\varphi^*(z, n) = z^n \overline{\varphi(1/z, n)}, \quad (z, n) \in \mathbb{T} \times \mathbb{N}_0, \tag{1.5}$$

the reversed *-polynomial of $\varphi(z, n)$, is of degree at most n. These polynomials were first introduced by Szegő in the 1920's in his work on the asymptotic distribution of eigenvalues of sections of Toeplitz forms [43, 44] (see also [33, Chs. 1–4], [45, Ch. XI]). Szegő's point of departure was the trigonometric moment problem and hence the theory of orthogonal polynomials on the unit circle: Given a probability measure $d\sigma$ supported on an infinite set on the unit circle, find monic polynomials of degree n in $z = e^{i\theta}$, $\theta \in [0, 2\pi]$, such that

$$\int_0^{2\pi} \gamma(n)^2 d\sigma(e^{i\theta})\, \overline{\varphi(e^{i\theta}, m)} \varphi(e^{i\theta}, n) = \delta_{m,n}, \quad m, n \in \mathbb{N}_0, \tag{1.6}$$

where

$$\gamma(n)^2 = \begin{cases} 1 & \text{for } n = 0, \\ \prod_{j=1}^{n} \left(1 - |\alpha(j)|^2\right)^{-1} & \text{for } n \in \mathbb{N}. \end{cases} \tag{1.7}$$

Here we chose to emphasize monic polynomials $\varphi(\,\cdot\,, n)$ in order to keep the factor γ out of the transfer matrix T. Szegő showed that the polynomials (1.4) satisfy the recurrence formula (1.3). Early work in this area includes contributions by Akhiezer [9, Ch. 5], Geronimus [25, 26], [27, Ch. I], Krein [34], Tomčuk [46], and Verblunsky [48, 49]. For a modern treatment of the theory of orthogonal polynomials on the unit circle and an exhaustive bibliography on the subject we refer to the forthcoming monumental two-volume treatise by Simon [41] (see also [42]). For fascinating connections between orthogonal polynomials and random matrix theory we refer, for instance, to Deift [18].

An extension of (1.3) was developed by Baxter in a series of papers on Toeplitz forms [10–13] in 1960–63. In these papers the transfer matrix T in (1.2) is replaced by the more general transfer matrix

$$U(z) = \begin{pmatrix} z & \alpha \\ \beta z & 1 \end{pmatrix} \tag{1.8}$$

with $\alpha = \alpha(n)$, $\beta = \beta(n)$, subject to the condition

$$\alpha(n)\beta(n) \neq 1 \text{ for all } n \in \mathbb{N}. \tag{1.9}$$

Studying the following extension of (1.3),

$$\Psi(z, n) = U(z, n)\Psi(z, n-1), \quad (z, n) \in \mathbb{T} \times \mathbb{N}, \tag{1.10}$$

Baxter was led to biorthogonal polynomials on the unit circle with respect to a complex-valued measure. In this paper we will primarily be concerned with Baxter's extension (1.10) of (1.3).

To simplify our notation in the following, shifts on the lattice are denoted using superscripts, that is, we write for complex-valued sequences f,

$$(S^{\pm} f)(n) = f^{\pm}(n) = f(n \pm 1), \quad n \in \mathbb{Z}, \ \{f(n)\}_{n \in \mathbb{Z}} \subset \mathbb{C}, \tag{1.11}$$

and apply the analogous convention to 2×2 matrices and their entries.

In the mid seventies, Ablowitz and Ladik, in a series of papers [3–6] (see also [1], [2, sect. 3.2.2], [7, Ch. 3]), used inverse scattering methods to analyze certain integrable differential-difference systems. One of their integrable variants of such systems, a discretization of the AKNS-ZS system, is of the type

$$-i\alpha_t - (\alpha^+ - 2\alpha + \alpha^-) + \alpha\beta(\alpha^+ + \alpha^-) = 0, \tag{1.12}$$

$$-i\beta_t + (\beta^+ - 2\beta + \beta^-) - \alpha\beta(\beta^+ + \beta^-) = 0 \tag{1.13}$$

with $\alpha = \alpha(n,t)$, $\beta = \beta(n,t)$. In particular, Ablowitz and Ladik [4] (see also [7, Ch. 3]) showed that in the focusing case, where $\beta = -\overline{\alpha}$, and in the defocusing case, where $\beta = \overline{\alpha}$ (cf. (1.2)), (1.12) and (1.13) yield the discrete analog of the nonlinear Schrödinger equation

$$-i\alpha_t + 2\alpha - (1 \pm |\alpha|^2)(\alpha^+ + \alpha^-) = 0. \tag{1.14}$$

For the closely related case of the discrete modified KdV equation, or the equation of the Schur flows, and the link to the trigonometric moment problem, we refer to [20] and [36].

Algebro-geometric solutions of the AL system (1.12), (1.13) have been studied by Ahmad and Chowdury [8], Bogolyubov, Prikarpatskii, and Samoilenko [15], Bogolyubov and Prikarpatskii [16], Geng, Dai, and Cewen [21], Vekslerchik [47], and especially, by Miller, Ercolani, Krichever, and Levermore [35] in an effort to analyze models describing oscillations in non-linear dispersive wave systems. In [35] the authors use the fact that the AL system (1.12), (1.13) arises as the compatibility requirements of the equations

$$\Phi = U\Phi^-, \quad \Phi_t^- = W\Phi^-. \tag{1.15}$$

Here U is precisely Baxter's matrix in (1.8) and W is defined as follows:

$$U(z) = \begin{pmatrix} z & \alpha \\ \beta z & 1 \end{pmatrix}, \quad W(z) = i\begin{pmatrix} z - 1 - \alpha\beta^- & \alpha - \alpha^- z^{-1} \\ \beta^- z - \beta & 1 + \alpha^-\beta - z^{-1} \end{pmatrix}. \tag{1.16}$$

Thus, the AL system (1.12), (1.13) is equivalent to the zero-curvature equations

$$U_t + UW - W^+U = 0. \tag{1.17}$$

Miller, Ercolani, Krichever, and Levermore [35] then performed a thorough analysis of the solutions $\Phi = \Phi(z,n,t)$ associated with the pair (U, W) and derived the theta function representations of α, β satisfying the AL system (1.12), (1.13). In the particular focusing and defocusing cases they also discussed periodic and quasi-periodic solutions α with respect to n and t.

Unaware of the paper [35], Geronimo and Johnson [23] studied the defocusing case (1.3) in the case where the coefficients α are random variables. They provide a detailed study of the corresponding Weyl–Titchmarsh functions, $m_\pm$, which satisfy the Riccati-type equation (for $z \in \mathbb{C}\backslash\mathbb{T}$, $n \in \mathbb{Z}$),

$$\alpha(n)m_\pm(z,n)m_\pm(z,n-1) - m_\pm(z,n-1) + zm_\pm(z,n) = z\overline{\alpha}(n) \tag{1.18}$$

(which should be compared to the identical equation (3.22) for the fundamental function ϕ in the defocusing case $\beta = \overline{\alpha}$). These functions take on the values $|m_+(z)| < 1$ and $|m_-(z)| > 1$ for $|z| < 1$ (cf. [22, 23]). Utilizing the fact that m_+ is a Schur function (i.e., analytic in the open unit disc with modulus less than one) and the close relation between such functions and the orthogonality measure $d\sigma_+$, they perform the transformation

$$\widehat{\Phi} = A\Phi, \quad A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}. \tag{1.19}$$

In this case U transforms into $\widehat{U}$ given by

$$\widehat{U}(z) = AU(z)A^{-1} = \frac{1}{2} \begin{pmatrix} (1-\overline{\alpha})z + 1 - \alpha & -i((1-\overline{\alpha})z - 1 + \alpha) \\ i((1+\overline{\alpha})z - 1 - \alpha) & (1+\overline{\alpha})z + 1 + \alpha \end{pmatrix}. \tag{1.20}$$

With this change of variables $m_\pm$ transform into

$$\widehat{m}_\pm(z,n) = i\,\frac{1 + m_\pm(z,n)}{1 - m_\pm(z,n)}, \quad z \in \mathbb{C}\backslash\mathbb{T}, \ n \in \mathbb{Z}. \tag{1.21}$$

The Schur property of m_+ (equivalently, the relation between Schur functions, Caratheodory functions, and positive measures on the unit circle [9, 40–42]) implies the standard representation,

$$\widehat{m}_+(z,n) = i \int_0^{2\pi} d\sigma_+(e^{i\theta}, n)\, \frac{e^{i\theta} + z}{e^{i\theta} - z}, \quad z \in \mathbb{C}\backslash\mathbb{T}, \ n \in \mathbb{Z}. \tag{1.22}$$

Under appropriate ergodicity assumptions on α and the hypothesis of a vanishing Lyapunov exponent on the prescribed spectral arcs on the unit circle $\mathbb{T}$, Geronimo and Johnson [23] showed that the m-functions associated with (1.3) are reflectionless, that is, $\widehat{m}_+$ is the analytic continuation of $\widehat{m}_-$ through the spectral arcs and vice versa, or equivalently, $\widehat{m}_\pm$ are the two branches of an analytic function $\widehat{m}$ on the hyperelliptic Riemann surface with branch points given by the end points of the spectral arcs on $\mathbb{T}$. They developed the corresponding spectral theory associated with (1.3) and the unitary operator it generates in $\ell^2(\mathbb{Z})$ (cf. [24]). This can be viewed as analogous to the case of real-valued finite-gap potentials for Schrödinger operators on $\mathbb{R}$ (cf., e.g., [14, 28]) and self-adjoint Jacobi operators on $\mathbb{Z}$ (cf., e.g., [17]). In particular, Geronimo and Johnson [23] prove the quasi-periodicity of the coefficients α in the defocusing case $\beta = \overline{\alpha}$. Connections with aspects of integrability, a zero-curvature or Lax formalism, and the theta function representation of α, are not discussed in [23]. The whole topic has been reconsidered in great detail and partially simplified in the upcoming two-volume monograph by Simon [41, Ch. 11] and aspects of integrability (Lax pairs, etc.) in the periodic defocusing case further explored by Nenciu [38].

The principal contribution of this paper to this circle of ideas is a short derivation of theta function formulas for algebro-geometric coefficients α, β associated with Baxter's finite difference system (1.10). Rather than considering solutions of a particular AL flow such as (1.12), (1.13), we will focus on a derivation of the coupled system of nonlinear difference equations satisfied by algebro-geometric solutions α, β of (1.10) (a new result) and its algebro-geometric solutions. In this sense our contribution represents the analog of determining algebro-geometric coefficients (generally, complex-valued) in one-dimensional Schrödinger and Jacobi operators and deriving the corresponding Its–Matveev-type theta function formulas. As a by-product in the special defocusing case $\beta = \overline{\alpha}$ with $|\alpha(n)| < 1$, $n \in \mathbb{Z}$, we recover the original result of Geronimo and Johnson [23] that α is quasi-periodic without the use of Fay's generalized Jacobi variety, double covers, etc.

In Sect. 2 we describe our zero-curvature formalism and the ensuing hierarchy of nonlinear difference equations for α, β. Our principal Sect. 3 then is devoted to a detailed derivation of the theta function formulas of all algebro-geometric quantities involved. Appendix A collects relevant material on hyperelliptic curves and their theta functions and introduces the terminology freely used in Sect. 3

2. Zero-Curvature Equations and Hyperelliptic Curves

In this section we introduce the basic zero-curvature setup for algebro-geometric solutions of (1.10). We follow the approach employed in [17, 28–31] in the analogous cases of stationary KdV, AKNS, and Toda solutions.

We start by introducing the complex-valued sequences

$$\{\alpha(n)\}_{n\in\mathbb{Z}}, \{\beta(n)\}_{n\in\mathbb{Z}} \subset \mathbb{C}, \tag{2.1}$$

and define the recursion relations

$$f_0 = -2\alpha^+, \quad g_0 = 1, \quad h_0 = 2\beta, \tag{2.2}$$

$$g_{\ell+1} - g_{\ell+1}^- = \alpha h_\ell^- + \beta f_\ell, \quad \ell \in \mathbb{N}_0, \tag{2.3}$$

$$f_{\ell+1}^- = f_\ell - \alpha(g_{\ell+1} + g_{\ell+1}^-), \quad \ell \in \mathbb{N}_0, \tag{2.4}$$

$$h_{\ell+1} = h_\ell^- + \beta(g_{\ell+1} + g_{\ell+1}^-), \quad \ell \in \mathbb{N}_0. \tag{2.5}$$

Here shifts on the lattice are denoted using superscripts as introduced in (1.11).

In addition we get the relations

$$g_{\ell+1} - g_{\ell+1}^- = \alpha h_{\ell+1} + \beta f_{\ell+1}^-, \quad \ell \in \mathbb{N}_0, \tag{2.6}$$

which are derived as follows:

$$\begin{aligned} \alpha h_{\ell+1} + \beta f_{\ell+1}^- &= \alpha h_\ell^- + \alpha\beta(g_{\ell+1} + g_{\ell+1}^-) + \beta f_\ell - \alpha\beta(g_{\ell+1} + g_{\ell+1}^-) \\ &= \alpha h_\ell^- + \beta f_\ell = g_{\ell+1} - g_{\ell+1}^-, \quad \ell \in \mathbb{N}_0, \end{aligned} \tag{2.7}$$

using relations (2.4), (2.5), and (2.3).

Remark 2.1. One can compute the sequences $\{f_\ell\}$, $\{g_\ell\}$, and $\{h_\ell\}$ recursively as follows. Assume that f_ℓ, g_ℓ, and h_ℓ are known. Equation (2.3) is a first-order difference equation in $g_{\ell+1}$ that can be solved directly and yields a local lattice function. The coefficient $g_{\ell+1}$ is determined up to a new constant denoted by $c_{\ell+1} \in \mathbb{C}$. Relations (2.4) and (2.5) then determine $f_{\ell+1}$ and $h_{\ell+1}$, etc. The choice of the recursion relations (2.2)–(2.5) will be motivated in Remark 2.2 (iii).

Explicitly, one obtains

$$\begin{aligned} &f_0 = -2\alpha^+, \quad f_1 = 2\big((\alpha^+)^2\beta + \alpha^+\alpha^{++}\beta^+ - \alpha^{++}\big) + c_1(-2\alpha^+), \\ &g_0 = 1, \quad g_1 = -2\alpha^+\beta + c_1, \\ &h_0 = 2\beta, \quad h_1 = 2\big(-\alpha^+\beta^2 - \alpha\beta^-\beta + \beta^-\big) + c_1 2\beta, \ \text{etc.}, \end{aligned} \tag{2.8}$$

where $\{c_\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{C}$ denote certain summation constants.

Next, assuming $z \in \mathbb{C}$, we introduce the 2×2 matrix $U(z)$ by

$$U(z,n) = \begin{pmatrix} z & \alpha(n) \\ z\beta(n) & 1 \end{pmatrix}, \quad n \in \mathbb{Z}. \tag{2.9}$$

In addition, we introduce for each fixed $p \in \mathbb{N}$ the following 2×2 matrix $V_{p+1}(z)$,

$$V_{p+1}(z,n) = \begin{pmatrix} G_{p+1}^-(z,n) & -F_p^-(z,n) \\ H_{p+1}^-(z,n) & -G_{p+1}^-(z,n) \end{pmatrix}, \quad n \in \mathbb{Z}, \tag{2.10}$$

supposing $F_p(\,\cdot\,,n)$ and $G_{p+1}(\,\cdot\,,n)$, $H_{p+1}(\,\cdot\,,n)$ to be polynomials of degree p and $p+1$, respectively (cf., however, Remark 3.1), with respect to the spectral parameter $z \in \mathbb{C}$.

Postulating the stationary zero-curvature condition

$$U(z,n)V_{p+1}(z,n) - V_{p+1}^+(z,n)U(z,n) = 0, \quad p \in \mathbb{N}_0, \tag{2.11}$$

then yields the following fundamental relationships between the polynomials F_p, G_{p+1}, and H_{p+1}:

$$F_p - zF_p^- - \alpha\big(G_{p+1} + G_{p+1}^-\big) = 0, \tag{2.12}$$

$$z\beta\big(G_{p+1} + G_{p+1}^-\big) + H_{p+1}^- - zH_{p+1} = 0, \tag{2.13}$$

$$z\big(G_{p+1}^- - G_{p+1}\big) + \alpha H_{p+1}^- + z\beta F_p = 0, \tag{2.14}$$

$$G_{p+1} - G_{p+1}^- - \alpha H_{p+1} - z\beta F_p^- = 0. \tag{2.15}$$

Moreover, using relations (2.12)–(2.15) one shows that the quantity $G_{p+1}^2 - F_p H_{p+1}$ is a lattice constant and hence the expression

$$G_{p+1}(z,n)^2 - F_p(z,n)H_{p+1}(z,n) = R_{2p+2}(z) \tag{2.16}$$

is an n-independent polynomial of degree $2p+2$ with respect to z. (That $G_{p+1}^2 - F_p H_{p+1}$, $z \neq 0$, is a lattice constant also immediately follows from (2.11) taking determinants.)

In order to make the connection between the zero-curvature formalism and the recursion relation (2.2)–(2.5), we now introduce the polynomial ansatz with respect to the spectral parameter z,

$$F_p(z) = \sum_{\ell=0}^{p} f_{p-\ell} z^\ell, \quad G_{p+1}(z) = \sum_{\ell=0}^{p+1} g_{p+1-\ell} z^\ell, \quad H_{p+1}(z) = \sum_{\ell=0}^{p+1} h_{p+1-\ell} z^\ell. \tag{2.17}$$

The stationary zero-curvature condition (2.11) imposes further restrictions on the coefficients of V_{p+1} that we will now explore. Since $g_0 = 1$, the quantity R_{2p+2} in (2.16) is a monic polynomial of degree $2p+2$, that is,

$$R_{2p+2}(z) = \prod_{m=0}^{2p+1} (z - E_m), \quad \{E_m\}_{m=0,\dots,2p+1} \subset \mathbb{C}. \tag{2.18}$$

Next we assume $p \in \mathbb{N}$ to avoid cumbersome case distinctions concerning the trivial case $p = 0$. Insertion of (2.17) into (2.12)–(2.15) then yields the relations (2.2) (normalizing $g_0 = 1$) and the recursion relations (2.3), (2.4), and (2.5) for $\ell = 0, \dots, p-1$. In addition, one obtains the equations

$$f_p - \alpha(g_{p+1} + g_{p+1}^-) = 0, \tag{2.19}$$

$$\beta(g_{p+1} + g_{p+1}^-) + h_p^- - h_{p+1} = 0, \tag{2.20}$$

$$h_{p+1}^- = 0, \tag{2.21}$$

$$g_{p+1}^- - g_{p+1} + \alpha h_p^- + \beta f_p = 0, \tag{2.22}$$

$$\alpha h_{p+1} + g_{p+1}^- - g_{p+1} = 0. \tag{2.23}$$

Moreover, one infers the relations (cf. (2.6))

$$g_\ell - g_\ell^- = \alpha h_\ell + \beta f_\ell^-, \quad \ell = 0, \dots, p. \tag{2.24}$$

Combining (2.21) and (2.23), we first conclude that g_{p+1} is a lattice constant, that is,

$$g_{p+1} = g_{p+1}^+ \in \mathbb{C}. \tag{2.25}$$

In addition, using (2.20), (2.21), and (2.25) one obtains

$$0 = h_{p+1} = h_p^- + \beta(g_{p+1} + g_{p+1}^-) = h_p^- + 2g_{p+1}\beta, \tag{2.26}$$

and hence,

$$h_p = -2g_{p+1}\beta^+. \tag{2.27}$$

Equations (2.19) and (2.25) also yield

$$f_p = 2g_{p+1}\alpha, \tag{2.28}$$

in agreement with (2.22). Moreover, Eq. (2.25) is consistent with taking $z = 0$ in (2.16) which yields

$$g_{p+1}^2 = \prod_{m=0}^{2p+1} E_m. \tag{2.29}$$

Thus, the stationary zero-curvature condition (2.11) is equivalent to a coupled system of nonlinear difference equations for α and β which we write as

$$\text{s-SB}_p(\alpha, \beta) = \frac{1}{2}\begin{pmatrix} f_p(\alpha, \beta) - 2g_{p+1}\alpha \\ h_p^-(\alpha, \beta) + 2g_{p+1}\beta \end{pmatrix} = 0, \quad g_{p+1} = g_{p+1}^+, \tag{2.30}$$

in honor of the pioneering work by Szegő and Baxter in connection with the transfer matrices (1.2) and (1.8). Varying $p \in \mathbb{N}_0$ in (2.30) then defines the corresponding stationary SB hierarchy of nonlinear difference equations. The first few equations explicitly read

$$\text{s-SB}_0(\alpha, \beta) = \begin{pmatrix} -\alpha^+ - g_1\alpha \\ \beta^- + g_1\beta \end{pmatrix} = 0, \quad g_1 = g_1^+,$$

$$\text{s-SB}_1(\alpha,\beta) = \begin{pmatrix} \alpha^+\alpha^{++}\beta^+ + (\alpha^+)^2\beta - \alpha^{++} - c_1\alpha^+ - g_2\alpha \\ -\alpha^-\beta^{--}\beta^- - \alpha(\beta^-)^2 + \beta^{--} + c_1\beta^- + g_2\beta \end{pmatrix} = 0, \qquad (2.31)$$
$$g_2 = g_2^+, \text{ etc.}$$

By definition, the set of solutions of (2.30), with p ranging in $\mathbb{N}_0$, represents the class of algebro-geometric solutions associated with Baxter's finite difference system (1.10). The hierarchy of coupled nonlinear difference equations (2.30) is new.

Remark 2.2. (*i*) The scaling behavior $f_\ell(A\alpha, A^{-1}\beta) = Af_\ell(\alpha,\beta)$, $g_\ell(A\alpha, A^{-1}\beta) = g_\ell(\alpha,\beta)$, $h_\ell(A\alpha, A^{-1}\beta) = A^{-1}h_\ell(\alpha,\beta)$, $\ell \in \mathbb{N}_0$, $A \in \mathbb{C}\backslash\{0\}$, shows that the stationary SB hierarchy (2.30) has the scaling invariance,

$$(\alpha,\beta) \mapsto (A\alpha, A^{-1}\beta), \quad A \in \mathbb{C}\backslash\{0\}. \qquad (2.32)$$

In the special focusing and defocusing cases, where $\beta = -\overline{\alpha}$ and $\beta = \overline{\alpha}$, respectively, the scaling constant A in (2.32) is further restricted to

$$|A| = 1. \qquad (2.33)$$

(ii) In the defocusing case $\beta = \overline{\alpha}$, the compatibility requirement of the two equations in (2.30) requires the constraint $|g_{p+1}|^2 = 1$ and additional spectral theoretic considerations in connection with the trigonometric moment problem, assuming $|\alpha(n)| < 1$, $n \in \mathbb{Z}$, enforce $\{E_m\}_{m=0,\dots,2p+1} \subset \mathbb{T}$. The additional condition of periodicity of α then implies further constraints on $\{E_m\}_{m=0,\dots,2p+1}$ (cf. [41, Ch. 11]). The special case of real-valuedness of α also enforces additional constraints on $\{E_m\}_{m=0,\dots,2p+1}$.
(iii) The ansatz (2.17) inserted into the zero-curvature equation (2.11) yields the beginning of the recursion relations (2.2)–(2.5) as described in the paragraph following (2.18). Since this consideration is p-dependent, we decided to start Sect. 2 with the infinite recursion relations rather than the p-dependent zero-curvature equation. The "right" choice of the ansatz (2.17) (i.e., the appropriate degrees) for the polynomial entries in the matrix V_{p+1} was obtained upon a careful comparison with Eqs. (2.10)–(2.12) in [23], taking into account the transformation induced by (1.19), (1.20). We subsequently realized that the polynomial structure can also be inferred from Eqs. (4.14) in [35].

3. Theta Function Representations

In this our principal section, we present a detailed study of algebro-geometric solutions associated with (1.10) with special emphasis on theta function representations of α, β and related quantities. We employ the techniques discussed in [17] and [28] in connection with other integrable systems such as the KdV, AKNS, and Toda hierarchies.

Throughout this section we suppose

$$\{\alpha(n)\}_{n\in\mathbb{Z}}, \{\beta(n)\}_{n\in\mathbb{Z}} \subset \mathbb{C}, \quad \alpha(n)\beta(n) \neq 0, 1, \ n \in \mathbb{Z}, \qquad (3.1)$$

and assume (2.2)–(2.5), (2.11), (2.17). Moreover, we freely employ the formalism developed in Sect. 2, keeping $p \in \mathbb{N}_0$ fixed.

Returning to (2.18) we now introduce the hyperelliptic curve $\mathcal{K}_p$ with nonsingular affine part defined by

$$\mathcal{K}_p\colon \mathcal{F}_p(z,y) = y^2 - R_{2p+2}(z) = 0, \qquad (3.2)$$

$$R_{2p+2}(z) = \prod_{m=0}^{2p+1} (z - E_m), \quad \{E_m\}_{m=0,\dots,2p+1} \subset \mathbb{C}\backslash\{0\}, \tag{3.3}$$

$$E_m \neq E_{m'} \text{ for } m \neq m', \ m, m' = 0, \dots, 2p + 1. \tag{3.4}$$

Equations (3.1)–(3.4) are assumed for the remainder of this section. We compactify $\mathcal{K}_p$ by adding two points P_{∞_+} and P_{∞_-}, $P_{\infty_+} \neq P_{\infty_-}$, at infinity, still denoting its projective closure by $\mathcal{K}_p$. Finite points P on $\mathcal{K}_p$ are denoted by $P = (z, y)$, where $y(P)$ denotes the meromorphic function on $\mathcal{K}_p$ satisfying $\mathcal{F}_p(z, y) = 0$. The complex structure on $\mathcal{K}_p$ is then defined in a standard manner and $\mathcal{K}_p$ has topological genus p. Moreover, we use the involution

$$*\colon \mathcal{K}_p \to \mathcal{K}_p, \quad P = (z, y) \mapsto P^* = (z, -y), \ P_{\infty_\pm} \mapsto {P_{\infty_\pm}}^* = P_{\infty_\mp}. \tag{3.5}$$

For further properties and notation concerning hyperelliptic curves we refer to Appendix A.

Remark 3.1. The assumption $\alpha(n) \neq 0$, $\beta(n) \neq 0$, $n \in \mathbb{Z}$, in (3.1) is not an essential one. It has the advantage of guaranteeing that for all $n \in \mathbb{Z}$, $F_p(\,\cdot\,, n)$ and $H_{p+1}(\,\cdot\,, n)$ are polynomials of degree p and $p + 1$, respectively. If $\alpha^+(n_0) = 0$ (resp., $\beta(n_0) = 0$) for some $n_0 \in \mathbb{Z}$, then $F_p(\,\cdot\,, n_0)$ has at most degree $p - 1$ (resp., $H_{p+1}(\,\cdot\,, n_0)$ has at most degree p). The latter n-dependence of the degree of the polynomials F_p and H_{p+1} enforces numerous case distinctions in connection with our fundamental function ϕ in (3.14) below. For simplicity we will in almost all situations avoid these cumbersome case distinctions and hence assume $\alpha(n) \neq 0$, $\beta(n) \neq 0$, $n \in \mathbb{Z}$ throughout this section. (For an exception see Remark 3.6.) In the extreme case that $\alpha \equiv 0$ (i.e., $\alpha(n) = 0$ for all $n \in \mathbb{Z}$), then $F_p \equiv 0$ and hence the curve $\mathcal{K}_p$ becomes singular as $R_{2p+2}(z) = G_{p+1}(z, n)^2$, $z \in \mathbb{C}$, by (2.16), and thus the branch points of $\mathcal{K}_p$ necessarily occur in pairs. (In addition, $G_{p+1}(z, n)$ then becomes independent of $n \in \mathbb{Z}$.) The same argument applies to $\beta \equiv 0$ since then $H_{p+1} \equiv 0$. For this reason the trivial cases $\alpha \equiv 0$ and $\beta \equiv 0$ in (3.1) are excluded in the remainder of this paper. Finally, in order to avoid numerous case distinctions in connection with the trivial case $p = 0$, we shall assume $p \geq 1$ for the remainder of this section (with the exception of Example 3.15).

In the following, the zeros of the polynomials $F_p(\,\cdot\,, n)$ and $H_{p+1}(\,\cdot\,, n)$ (cf. (2.17)) will play a special role. We denote them by $\{\mu_j(n)\}_{j=1,\dots,p}$ and $\{\nu_\ell(n)\}_{\ell=0,\dots,p}$ and hence write

$$F_p(z) = -2\alpha^+ \prod_{j=1}^{p} (z - \mu_j), \quad H_{p+1}(z) = 2\beta \prod_{\ell=0}^{p} (z - \nu_\ell). \tag{3.6}$$

In addition, we lift these zeros to $\mathcal{K}_p$ by introducing

$$\hat{\mu}_j(n) = (\mu_j(n), G_{p+1}(\mu_j(n), n)) \in \mathcal{K}_p, \quad j = 1, \dots, p, \ n \in \mathbb{Z}, \tag{3.7}$$

$$\hat{\nu}_\ell(n) = (\nu_\ell(n), -G_{p+1}(\nu_\ell(n), n)) \in \mathcal{K}_p, \quad \ell = 0, \dots, p, \ n \in \mathbb{Z}. \tag{3.8}$$

We recall that $h_{p+1} = 0$ (cf. (2.21)). Hence we may choose

$$\nu_0(n) = 0, \quad n \in \mathbb{Z}. \tag{3.9}$$

Define

$$P_{0,\pm} = (0, \pm G_{p+1}(0)) = (0, \pm g_{p+1}), \tag{3.10}$$

where

$$y(P_{0,\pm}) = \pm g_{p+1}, \quad g_{p+1}^2 = \prod_{m=0}^{2p+1} E_m. \tag{3.11}$$

We emphasize that $P_{0,\pm}$ and $P_{\infty_\pm}$ are not necessarily on the same sheet of $\mathcal{K}_p$. The actual sheet on which $P_{0,\pm}$ lie depends on the sign of g_{p+1}. Thus, one obtains

$$\hat{\nu}_0 = P_{0,-}. \tag{3.12}$$

The branch of $y(\,\cdot\,)$ near $P_{\infty_\pm}$ is fixed according to

$$\lim_{\substack{|z(P)|\to\infty \\ P\to P_{\infty_\pm}}} \frac{y(P)}{G_{p+1}(z(P))} = \lim_{\substack{|z(P)|\to\infty \\ P\to P_{\infty_\pm}}} \frac{y(P)}{z(P)^{p+1}} = \mp 1. \tag{3.13}$$

Next, we introduce the fundamental meromorphic function ϕ on $\mathcal{K}_p$ by

$$\phi(P,n) = \frac{y + G_{p+1}(z,n)}{F_p(z,n)} = \frac{-H_{p+1}(z,n)}{y - G_{p+1}(z,n)}, \quad P = (z,y) \in \mathcal{K}_p,\ n \in \mathbb{Z} \tag{3.14}$$

with divisor $(\phi(\,\cdot\,,n))$ (cf. the notation for divisors introduced in (A.20) and (A.21)) given by

$$(\phi(\,\cdot\,,n)) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n)}. \tag{3.15}$$

Here we abbreviated (cf. (A.20), (A.21))

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\},\ \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_p\} \in \operatorname{Sym}^p \mathcal{K}_p. \tag{3.16}$$

Remark 3.2. It may be worth emphasizing the important dual role played by $z = \infty$ and $z = 0$ in this context. In the special defocusing case $\beta = \overline{\alpha}$, where all the zeros E_m, $m = 0, \dots, 2p+1$, of R_{2p+2} lie on the unit circle $\mathbb{T}$, the points 0 and ∞ play a symmetric role with respect to $\mathbb{T}$ and hence the fact that 0 and ∞ acquire equal importance as demonstrated in (3.15), is not too surprising. However, in the general case discussed in this paper, where the zeros E_m of R_{2p+2} are in arbitrary positions away from $z = 0$ and only restricted by being pairwise distinct (cf. (3.3), (3.4)), it may at first sight be surprising that 0 still plays a distinguished role together with ∞. The latter is a consequence of

$$H_{p+1}(0) = h_{p+1} = 0 \tag{3.17}$$

according to (2.17) and (2.21), and hence of

$$y(P_{0,\pm})^2 = R_{2p+2}(0) = \prod_{m=0}^{2p+1} E_m = g_{p+1}^2 \tag{3.18}$$

by (2.16), (2.17), (3.2), and (3.11).

The stationary Baker–Akhiezer vector $\Psi(P, n, n_0)$ is defined on $\mathcal{K}_p$ by

$$\Psi(P, n, n_0) = \begin{pmatrix} \psi_1(P, n, n_0) \\ \psi_2(P, n, n_0) \end{pmatrix}, \tag{3.19}$$

$$\psi_1(P, n, n_0) = \begin{cases} \prod_{m=n_0+1}^{n} (z + \alpha(m)\phi^-(P, m)), & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod_{m=n+1}^{n_0} (z + \alpha(m)\phi^-(P, m))^{-1}, & n \leq n_0 - 1, \end{cases} \tag{3.20}$$

$$\psi_2(P, n, n_0) = \phi(P, n_0) \begin{cases} \prod_{m=n_0+1}^{n} (z\beta(m)\phi^-(P, m)^{-1} + 1), & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod_{m=n+1}^{n_0} (z\beta(m)\phi^-(P, m)^{-1} + 1)^{-1}, & n \leq n_0 - 1, \end{cases}$$

$$P \in \mathcal{K}_p, \ (n, n_0) \in \mathbb{Z}^2. \tag{3.21}$$

(Equations (3.19)–(3.21) were obtained by trial and error with a view toward (3.26)–(3.28) and in analogy to the corresponding formulas in connection with the stationary Toda hierarchy, cf. Eq. (3.23) in [17].)

Clearly $\Psi(\,\cdot\,, n, n_0)$ is meromorphic on $\mathcal{K}_p$ since $\phi(\,\cdot\,, m)$ is meromorphic on $\mathcal{K}_p$. Fundamental properties of ϕ and Ψ are summarized next.

Lemma 3.3. *Suppose $\alpha, \beta \subset \mathbb{C}$ satisfy* (3.1) *and the p^{th} stationary SB system* (2.30). *Moreover, assume* (3.2)–(3.4) *and let $P = (z, y) \in \mathcal{K}_p \backslash \{P_{\infty_+}, P_{\infty_-}\}$, $(n, n_0) \in \mathbb{Z}^2$. Then ϕ satisfies the Riccati-type equation*

$$\alpha\phi(P)\phi^-(P) - \phi^-(P) + z\phi(P) = z\beta \tag{3.22}$$

as well as

$$\phi(P)\phi(P^*) = \frac{H_{p+1}(z)}{F_p(z)}, \tag{3.23}$$

$$\phi(P) + \phi(P^*) = 2\frac{G_{p+1}(z)}{F_p(z)}, \tag{3.24}$$

$$\phi(P) - \phi(P^*) = 2\frac{y}{F_p(z)}. \tag{3.25}$$

The vector Ψ fulfills

$$\psi_2(P, n, n_0)/\psi_1(P, n, n_0) = \phi(P, n), \tag{3.26}$$

$$\Psi(P, n, n_0) = U(z, n)\Psi^-(P, n, n_0), \tag{3.27}$$

$$- y\Psi^-(P, n, n_0) = V_{p+1}(z, n)\Psi^-(P, n, n_0), \tag{3.28}$$

$$\psi_1(P, n, n_0)\psi_1(P^*, n, n_0) = z^{n-n_0}\frac{F_p(z, n)}{F_p(z, n_0)}\Pi(n, n_0), \tag{3.29}$$

$$\psi_2(P, n, n_0)\psi_2(P^*, n, n_0) = z^{n-n_0}\frac{H_{p+1}(z, n)}{F_p(z, n_0)}\Pi(n, n_0), \tag{3.30}$$

$$\psi_1(P, n, n_0)\psi_2(P^*, n, n_0) + \psi_1(P^*, n, n_0)\psi_2(P, n, n_0) = 2z^{n-n_0}\frac{G_{p+1}(z, n)}{F_p(z, n_0)}\Pi(n, n_0), \tag{3.31}$$

$$\psi_1(P,n,n_0)\psi_2(P^*,n,n_0) - \psi_1(P^*,n,n_0)\psi_2(P,n,n_0)$$
$$= -2z^{n-n_0}\frac{y}{F_p(z,n_0)}\Pi(n,n_0), \tag{3.32}$$

where

$$\Pi(n,n_0) = \begin{cases} \prod_{m=n_0+1}^{n}(1-\alpha(m)\beta(m)), & n \geq n_0+1, \\ 1, & n = n_0, \\ \prod_{m=n+1}^{n_0}(1-\alpha(m)\beta(m))^{-1}, & n \leq n_0 - 1. \end{cases} \tag{3.33}$$

Proof. Using $y^2 = G_{p+1}^2 - F_p H_{p+1}$ (cf. (2.16), (3.2)) and (3.14), the left-hand side of (3.22) can be rewritten as follows:

$$\alpha\phi\phi^- - \phi^- + z\phi - z\beta = (F_p F_p^-)^{-1}\big[\alpha\big(G_{p+1}^2 - F_p H_{p+1} + y(G_{p+1} + G_{p+1}^-)$$
$$+ G_{p+1}G_{p+1}^-\big) - (y + G_{p+1}^-)F_p + z(y + G_{p+1})F_p^- - z\beta F_p F_p^-\big]. \tag{3.34}$$

Insertion of (2.12) and (2.15) into (3.34) then shows that the right-hand side of (3.34) vanishes. This proves (3.22). Equations (3.23)–(3.25) are clear from (3.14) and $y^2 = G_{p+1}^2 - F_p H_{p+1}$. Relation (3.26) is proven inductively as follows. Since it holds for $n = n_0$ by (3.20), (3.21), we assume that

$$\psi_2(P,m,n_0)/\psi_1(P,m,n_0) = \phi(P,m), \quad m = n_0, \ldots, n-1. \tag{3.35}$$

Then combining (3.20), (3.21), and (3.35), one obtains

$$\frac{\psi_2(P,n,n_0)}{\psi_1(P,n,n_0)} = \phi^-(P,n)\frac{z\beta(n)\phi^-(P,n)^{-1} + 1}{z + \alpha(n)\phi^-(P,n)}, \tag{3.36}$$

and hence

$$\alpha(n)\frac{\psi_2(P,n,n_0)}{\psi_1(P,n,n_0)}\phi^-(P,n) - \phi^-(P,n) + z\frac{\psi_2(P,n,n_0)}{\psi_1(P,n,n_0)} - z\beta(n) = 0. \tag{3.37}$$

Comparison with (3.22) (cf. also (3.34)) then proves (3.26) for all $n \geq n_0$. The case $n \leq n_0 - 1$ is proven analogously. By (3.20) and (3.21) one infers

$$\psi_1(P,n,n_0) = [z + \alpha(n)\phi^-(P,n)]\psi_1^-(P,n,n_0)$$
$$= z\psi_1^-(P,n,n_0) + \alpha(n)\psi_2^-(P,n,n_0), \tag{3.38}$$
$$\psi_2(P,n,n_0) = [z\beta(n)\phi^-(P,n)^{-1} + 1]\psi_2^-(P,n,n_0)$$
$$= z\beta(n)\psi_1^-(P,n,n_0) + \psi_2^-(P,n,n_0), \tag{3.39}$$

by (3.26). This proves (3.27). An application of (3.14) implies

$$V_{p+1}\Psi^- = \begin{pmatrix} G_{p+1}^-\psi_1^- - F_p^-\psi_2^- \\ H_{p+1}^-\psi_1^- - G_{p+1}^-\psi_2^- \end{pmatrix} = \begin{pmatrix} (G_{p+1}^- - F_p^-\phi^-)\psi_1^- \\ (H_{p+1}^-(\phi^-)^{-1} - G_{p+1}^-)\psi_2^- \end{pmatrix} = -y\Psi^-, \tag{3.40}$$

and hence (3.28). Combining (2.12), (2.14), (3.14), (3.20), (3.23), (3.24), and $y^2 = G_{p+1}^2 - F_p H_{p+1}$ yields (assuming $n \geq n_0 + 1$ for simplicity)

$$\psi_1(P)\psi_1(P^*) = (z + \alpha\phi^-(P))(z + \alpha\phi^-(P^*))\psi_1^-(P)\psi_1^-(P^*)$$

$$
\begin{aligned}
&= (F_p^-)^{-1}\big(z^2F_p^- + 2z\alpha G_{p+1}^- + \alpha^2 H_{p+1}^-\big)\psi_1^-(P)\psi_1^-(P^*) \\
&= (F_p^-)^{-1}\big(z^2F_p^- + 2z\alpha G_{p+1}^- - z\alpha\beta F_p + z\alpha(G_{p+1} - G_{p+1}^-)\big) \\
&\quad \times \psi_1^-(P)\psi_1^-(P^*) \\
&= (F_p^-)^{-1}\big(z^2F_p^- - z\alpha\beta F_p + z\alpha(G_{p+1} + G_{p+1}^-)\big)\psi_1^-(P)\psi_1^-(P^*) \\
&= (F_p^-)^{-1}\big(zF_p - z\alpha\beta F_p\big)\psi_1^-(P)\psi_1^-(P^*) \\
&= z(1-\alpha\beta)\big(F_p/F_p^-\big)\psi_1^-(P)\psi_1^-(P^*)
\end{aligned} \tag{3.41}
$$

and thus (3.29). Combining (3.23) and (3.29) proves (3.30). Similarly, combining (3.24) (resp. (3.25)) and (3.29) proves (3.31) (resp. (3.32)). □

We note that the Riccati-type equation (3.22) for ϕ coincides with that of $m_\pm$ in (1.18) in the defocusing case $\beta = \overline{\alpha}$.

Next, we derive trace formulas for α and β in terms of the zeros μ_j and ν_j of F_p and H_{p+1}, respectively. For simplicity we just record the simplest case below.

Lemma 3.4. *Suppose that* $\alpha, \beta \subset \mathbb{C}$ *satisfy* (3.1) *and the* p^{th} *stationary SB system* (2.30). *Then,*

$$
\frac{\alpha}{\alpha^+} = \frac{(-1)^{p+1}}{g_{p+1}}\prod_{j=1}^{p}\mu_j, \quad \frac{\beta^+}{\beta} = \frac{(-1)^{p+1}}{g_{p+1}}\prod_{\ell=1}^{p}\nu_\ell. \tag{3.42}
$$

Proof. Combining (2.17), $f_p = 2g_{p+1}\alpha$, and (3.6) yields

$$
2g_{p+1}\alpha = f_p = f_0(-1)^p\prod_{j=1}^{p}\mu_j = -2\alpha^+(-1)^p\prod_{j=1}^{p}\mu_j. \tag{3.43}
$$

Using $h_p = -2g_{p+1}\beta^+$, the case β^+/β is analogous. □

The following result describes the asymptotic behavior of ϕ and ψ_j, $j = 1, 2$, as $P \to P_{\infty_\pm}$ and $P \to P_{0,\pm}$.

Lemma 3.5. *Suppose that* $\alpha, \beta \subset \mathbb{C}$ *satisfy* (3.1) *and the* p^{th} *SB system* (2.30). *In addition, assume* (3.2)–(3.4) *and let* $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}\}$ *and* $(n, n_0) \in \mathbb{Z}^2$. *Then,*

$$
\phi(P) \underset{\zeta\to 0}{=} \begin{cases} \beta + (1-\alpha\beta)\beta^-\zeta + O(\zeta^2) & \text{as } P \to P_{\infty_+}, \\ -(\alpha^+)^{-1}\zeta^{-1} + (1-\alpha^+\beta^+)\alpha^{++}(\alpha^+)^{-2} + O(\zeta) & \text{as } P \to P_{\infty_-}, \end{cases}
\quad \zeta = 1/z, \tag{3.44}
$$

$$
\phi(P) \underset{\zeta\to 0}{=} \begin{cases} (\alpha)^{-1} - (1-\alpha\beta)\alpha^-(\alpha)^{-2}\zeta + O(\zeta^2) & \text{as } P \to P_{0,+}, \\ -\beta^+\zeta - (1-\alpha^+\beta^+)\beta^{++}\zeta^2 + O(\zeta^3) & \text{as } P \to P_{0,-}, \end{cases} \quad \zeta = z. \tag{3.45}
$$

Moreover, for $n > n_0$, $\psi_1(\,\cdot\,, n, n_0)$ *has a pole of order* $n - n_0$ *at* P_{∞_+}, *and a zero of order* $n - n_0$ *at* $P_{0,-}$. *For* $n < n_0$, $\psi_1(\,\cdot\,, n, n_0)$ *has a zero of order* $n_0 - n$ *at* P_{∞_+}, *and a pole of* *order $n_0 - n$ at $P_{0,-}$. Generically, $\psi_1(\,\cdot\,, n, n_0)$ has simple poles at $\hat{\mu}_1(n_0), \dots, \hat{\mu}_p(n_0)$ and simple zeros at $\hat{\mu}_1(n), \dots, \hat{\mu}_p(n)$. Moreover,*

$$\psi_1(P, n, n_0) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{n_0-n}(1+O(\zeta)) & \text{as } P \to P_{\infty_+}, \\ \Pi(n,n_0)(\alpha^+(n)/\alpha^+(n_0)) + O(\zeta) & \text{as } P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \tag{3.46}$$

$$\psi_1(P, n, n_0) \underset{\zeta\to 0}{=} \begin{cases} (\alpha(n)/\alpha(n_0)) + O(\zeta), & \text{as } P \to P_{0,+}, \\ \Pi(n,n_0)\zeta^{n-n_0}(1+O(\zeta)) & \text{as } P \to P_{0,-}, \end{cases} \quad \zeta = z. \tag{3.47}$$

Similarly, for $n > n_0$, $\psi_2(\,\cdot\,, n, n_0)$ has a pole of order $n - n_0$ at P_{∞_+}, a simple pole at P_{∞_-}, and a zero of order $n - n_0 + 1$ at $P_{0,-}$. For $n < n_0$, $\psi_2(\,\cdot\,, n, n_0)$ has a zero of order $n_0 - n$ at P_{∞_+}, a pole of order $n_0 - n - 1$ at $P_{0,-}$, and a simple pole at P_{∞_-}. Generically, $\psi_2(\,\cdot\,, n, n_0)$ has simple poles at $\hat{\mu}_1(n_0), \dots, \hat{\mu}_p(n_0)$ and simple zeros at $\hat{\nu}_1(n), \dots, \hat{\nu}_p(n)$. Moreover,

$$\psi_2(P, n, n_0) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{n_0-n}\beta(n)(1+O(\zeta)) & \text{as } P \to P_{\infty_+}, \\ -\Pi(n,n_0)(\alpha^+(n_0))^{-1}\zeta^{-1}(1+O(\zeta)) & \text{as } P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \tag{3.48}$$

$$\psi_2(P, n, n_0) \underset{\zeta\to 0}{=} \begin{cases} (\alpha(n_0))^{-1} + O(\zeta), & \text{as } P \to P_{0,+}, \\ -\Pi(n,n_0)\beta^+(n)\zeta^{n+1-n_0}(1+O(\zeta)) & \text{as } P \to P_{0,-}, \end{cases} \quad \zeta = z. \tag{3.49}$$

Finally, the divisor $(\psi_j(\,\cdot\,, n, n_0))$ of the meromorphic functions $\psi_j(\,\cdot\,, n, n_0)$, $j = 1, 2$, is given by

$$(\psi_1(\,\cdot\,, n, n_0)) = \mathcal{D}_{\underline{\hat{\mu}}(n)} - \mathcal{D}_{\underline{\hat{\mu}}(n_0)} + (n - n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_+}}), \tag{3.50}$$

$$(\psi_2(\,\cdot\,, n, n_0)) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n_0)} + (n - n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_+}}). \tag{3.51}$$

Proof. Inserting the ansatz

$$\phi(P, n) \underset{z\to\infty}{=} \phi_{-1}(n)z + \phi_0(n) + \phi_1(n)z^{-1} + O(z^{-2}) \tag{3.52}$$

and

$$\phi(P, n) \underset{z\to 0}{=} \phi_0(n) + \phi_1(n)z + O(z^2) \tag{3.53}$$

into the Riccati-type equation (3.22) produces (3.44) and (3.45).

Since

$$z + \alpha\phi^-(P) = \begin{cases} z + O(1) & \text{as } P \to P_{\infty_+}, \\ (1-\alpha\beta)\alpha^+(\alpha)^{-1} + O(z^{-1}) & \text{as } P \to P_{\infty_-}, \end{cases} \tag{3.54}$$

and

$$z + \alpha\phi^-(P) = \begin{cases} \alpha(\alpha^-)^{-1} + O(z) & \text{as } P \to P_{0,+}, \\ (1-\alpha\beta)z + O(z^2) & \text{as } P \to P_{0,-}, \end{cases} \tag{3.55}$$

by (3.44), and (3.45), (3.20) shows that $\psi_1(\,\cdot\,,n,n_0)$ has a pole (resp. zero) at least of order $n-n_0$ at $P=P_{\infty_+}$ and a zero (resp. pole) at least of order $n-n_0$ at $P=P_{0,-}$ for $n>n_0$ (resp. $n<n_0$). Moreover, using (2.14) and (3.14) one infers for $n>n_0+1$,

$$\begin{aligned}
\psi_1(P,n,n_0) &= \prod_{m=n_0+1}^{n} \big(z+\alpha(m)\phi^-(P,m)\big)\\
&= \prod_{m=n_0+1}^{n} (y-G_{p+1}^-(z,m))^{-1}\big(z(y-G_{p+1}^-(z,m))-\alpha(m)H_{p+1}^-(z,m)\big)\\
&= \prod_{m=n_0+1}^{n} (y-G_{p+1}^-(z,m))^{-1}\big(z(y-G_{p+1}(z,m))+z\beta(m)F_p(z,m)\big)\\
&= z^{n-n_0}\bigg(\prod_{m=n_0+1}^{n} (y-G_{p+1}^-(z,m))^{-1}\bigg)\\
&\quad\times\bigg(\prod_{m=n_0+1}^{n} \big(y-G_{p+1}(z,m)+\beta(m)F_p(z,m)\big)\bigg)\\
&= z^{n-n_0}(y-G_{p+1}(z,n_0))^{-1}\big(y-G_{p+1}(z,n)+\beta(n)F_p(z,n)\big)\\
&\quad\times\prod_{m=n_0+1}^{n-1}\bigg(1+\beta(m)\frac{F_p(z,m)}{y-G_{p+1}(z,m)}\bigg)\\
&= z^{n-n_0}(y-G_{p+1}(z,n_0))^{-1}\big(y-G_{p+1}(z,n)+\beta(n)F_p(z,n)\big)\\
&\quad\times\prod_{m=n_0+1}^{n-1}\bigg(1-\beta(m)\frac{y+G_{p+1}(z,m)}{H_{p+1}(z,m)}\bigg) \qquad (3.56)\\
&= \prod_{m=n_0+1}^{n}\bigg(z+\alpha(m)\frac{y+G_{p+1}^-(z,m)}{F_p^-(z,m)}\bigg)\\
&= \prod_{m=n_0+1}^{n} F_p^-(z,m)^{-1}\big(F_p(z,m)+\alpha(m)(y-G_{p+1}(z,m))\big)\\
&= F_p(z,n_0)^{-1}\big(F_p(z,n)+\alpha(n)(y-G_{p+1}(z,n))\big)\\
&\quad\times\prod_{m=n_0+1}^{n-1}\bigg(1+\alpha(m)\frac{y-G_{p+1}(z,m)}{F_p(z,m)}\bigg). \qquad (3.57)
\end{aligned}$$

(With the usual convention that a product over an empty index set equals one, (3.56) and (3.57) also extend to the case $n=n_0+1$.) Analogous considerations hold for $n\le n_0-1$. From these facts, and from (3.20), (3.44), and (3.45), the properties (3.46), (3.47), and (3.50) of ψ_1 can be read off.

Since $\psi_2=\psi_1\phi$ and ϕ behaves near $P_{\infty_\pm}$ and $P_{0,\pm}$ as described in (3.44) and (3.45), the corresponding statements for ψ_2 near $P_{\infty_\pm}$ and $P_{0,\pm}$ follow. Finally, using again $\psi_2=\psi_1\phi$ and (3.15) then shows that $\psi_2(P,n,n_0)$ has zeros at $\hat\nu_1(n),\dots,\hat\nu_p(n)$. These results are summarized in (3.51). Formulas (3.48), (3.49) then again follow from $\psi_2=\psi_1\phi$ and from (3.44)–(3.47). ◻

Next, we briefly consider the asymptotic behavior of ϕ in the case where the conditions $\alpha(n)\beta(n) \neq 0$ are violated for some $n \in \mathbb{Z}$.

Remark 3.6. First we note that if $\alpha^+ \neq 0$, then by (3.44) no pole $\hat{\mu}_j$ of ϕ hits the point P_{∞_-}. Similarly, by (3.45), $P_{0,-} = \hat{\nu}_0$ is a zero of ϕ. The case $\beta(n) = 0$ for some $n \in \mathbb{Z}$ poses no difficulty and (3.44) and (3.45) extend continuously in this case. The case $\alpha(n) = 0$ for some $n \in \mathbb{Z}$ is more involved and causes higher order poles in (3.44) and (3.45). An explicit calculation yields ($\zeta = 1/z$)

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} O(1) & \text{as } P \to P_{\infty_+} \\ -(\alpha^{++})^{-1}\zeta^{-2} + O(\zeta^{-1}) & \text{as } P \to P_{\infty_-} \end{cases} \quad \text{if } \alpha^+ = 0, \alpha^{++} \neq 0. \tag{3.58}$$

Thus, if $\alpha^+ = 0$, $\alpha^{++} \neq 0$, one of the poles $\hat{\mu}_j$ of ϕ hits the point P_{∞_-}. However, still no pole of ϕ hits P_{∞_+}. Similarly, using

$$\begin{aligned} &f_p = 2g_{p+1}\alpha, \quad f_{p-1} = 2g_{p+1}(\alpha^- - \alpha^2\beta^+ - \alpha^-\alpha\beta) + 2C\alpha, \\ &g_p = -2g_{p+1}\alpha\beta^+ + g_{p+1}^{-1}2(2p+1)c_1, \\ &h_p = -2g_{p+1}\beta^+, \quad h_{p-1} = 2g_{p+1}(-\beta^{++} + \alpha\beta\beta^+ + \alpha^-\beta^2) - 2C\beta, \end{aligned} \tag{3.59}$$

one derives ($\zeta = z$)

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} (\alpha^-)^{-1}\zeta^{-1} + O(1) & \text{as } P \to P_{0,+} \\ O(\zeta) & \text{as } P \to P_{0,-} \end{cases} \quad \text{if } \alpha = 0, \alpha^- \neq 0. \tag{3.60}$$

Thus, if $\alpha = 0$, $\alpha^- \neq 0$, one of the poles $\hat{\mu}_j$ of ϕ hits the point $P_{0,+}$. In addition, $P_{0,-}$ remains a zero of ϕ.

Since nonspecial divisors will play a fundamental role in this section, we now take a closer look at them.

Lemma 3.7. *Suppose that* $\alpha, \beta \subset \mathbb{C}$ *satisfy* (3.1) *and the* p^{th} *stationary SB system* (2.30). *Moreover, assume* (3.2)–(3.4) *and let* $n \in \mathbb{Z}$. *Let* $\mathcal{D}_{\underline{\hat{\mu}}}$, $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}}$, $\underline{\hat{\nu}} = \{\underline{\hat{\nu}}_1, \dots, \underline{\hat{\nu}}_p\}$, *be the pole and zero divisors of degree* p, *respectively, associated with* α, β *and* ϕ *defined according to* (3.7), (3.8), *that is,*

$$\hat{\mu}_j(n) = (\mu_j(n), G_{p+1}(\mu_j(n), n)), \quad j = 1, \dots, p, \; n \in \mathbb{Z}, \tag{3.61}$$

$$\hat{\nu}_j(n) = (\nu_j(n), -G_{p+1}(\nu_j(n), n)), \quad j = 1, \dots, p, \; n \in \mathbb{Z}. \tag{3.62}$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ *are nonspecial for all* $n \in \mathbb{Z}$.

Proof. We provide a detailed proof in the case of $\mathcal{D}_{\underline{\hat{\mu}}(n)}$. By Theorem A.2, $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is special if and only if $\{\hat{\mu}_1(n), \dots, \hat{\mu}_p(n)\}$ contains at least one pair of the type $\{\hat{\mu}(n), \hat{\mu}(n)^*\}$. Hence $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is certainly nonspecial as long as the projections $\mu_j(n)$ of $\hat{\mu}_j(n)$ are mutually distinct, $\mu_j(n) \neq \mu_k(n)$ for $j \neq k$. On the other hand, if two or more projections coincide for some $n_0 \in \mathbb{Z}$, for instance,

$$\mu_{j_1}(n_0) = \cdots = \mu_{j_N}(n_0) = \mu_0, \quad N \in \{2, \dots, p\}, \tag{3.63}$$

then $G_{p+1}(\mu_0, n_0) \neq 0$ as long as $\mu_0 \notin \{E_0, \dots, E_{2p+1}\}$. This fact immediately follows from (2.16) since $F_p(\mu_0, n_0) = 0$ but $R_{2p+2}(\mu_0) \neq 0$ by hypothesis. In particular, $\hat{\mu}_{j_1}(n_0), \dots, \hat{\mu}_{j_N}(n_0)$ all meet on the same sheet since

$$\hat{\mu}_{j_r}(n_0) = (\mu_0, G_{p+1}(\mu_0, n_0)), \quad r = 1, \dots, N, \tag{3.64}$$

and hence no special divisor can arise in this manner. It remains to study the case where two or more projections collide at a branch point, say at $(E_{m_0}, 0)$ for some $n_0 \in \mathbb{Z}$. In this case one concludes $F_p(z, n_0) \underset{z \to E_{m_0}}{=} O\big((z - E_{m_0})^2\big)$ and

$$G_{p+1}(E_{m_0}, n_0) = 0, \tag{3.65}$$

using again (2.16) and $F_p(E_{m_0}, n_0) = R_{2p+2}(E_{m_0}) = 0$. Since $G_{p+1}(\,\cdot\,, n_0)$ is a polynomial (of degree $p+1$), (3.65) implies $G_{p+1}(z, n_0) \underset{z \to E_{m_0}}{=} O((z - E_{m_0}))$. Thus, using (2.16) once more, one obtains the contradiction,

$$\begin{aligned} O\big((z - E_{m_0})^2\big) &\underset{z \to E_{m_0}}{=} R_{2p+2}(z) \\ &\underset{z \to E_{m_0}}{=} (z - E_{m_0}) \Bigg(\prod_{\substack{m=1 \\ m \neq m_0}}^{2p+1} \big(E_{m_0} - E_m\big) + O(z - E_{m_0}) \Bigg). \end{aligned} \tag{3.66}$$

Consequently, at most one $\hat{\mu}_j(n)$ can hit a branch point at a time and again no special divisor arises. Finally, by (3.44), $\hat{\mu}_j(n)$ never reaches the point P_{∞_+}. Hence if some $\hat{\mu}_j(n)$ tend to infinity, they all necessarily converge to P_{∞_-}. Again no special divisor can arise in this manner.

The proof for $\mathcal{D}_{\hat{\underline{\nu}}(n)}$ is completely analogous (replacing F_p by H_{p+1} and noticing that by (3.44), ϕ has no zeros near $P_{\infty_\pm}$), thereby completing the proof. $\square$

Remark 3.8. For simplicity we assumed $\alpha(n) \neq 0$, $\beta(n) \neq 0$, $n \in \mathbb{Z}$, in Lemma 3.7. However, the asymptotic behavior in (3.58) (resp., (3.60)) shows that no special divisors can be created at infinity (resp., zero) and hence the results of Lemma 3.7 extend by continuity to the situation considered in Remark 3.6. In particular, it extends to the case where $\beta(n_0) = 0$ for some $n_0 \in \mathbb{Z}$. The case $\alpha(n_0) = 0$ for some $n_0 \in \mathbb{Z}$ is more involved and requires more and more case distinctions as is clear from Remark 3.6, but the pattern persists.

Next we turn to the representation of ϕ, Ψ, α, and β in terms of the Riemann theta function associated with $\mathcal{K}_p$. We freely use the notation established in Appendix A, assuming $\mathcal{K}_p$ to be nonsingular as in (3.2)–(3.4). To avoid the trivial case $p = 0$ (considered separately in Example 3.15), we assume $p \in \mathbb{N}$ for the remainder of this argument.

We choose a fixed base point $Q_0 \in \mathcal{K}_p \backslash \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, in fact, we will choose a branch point for convenience, $Q_0 \in \mathcal{B}(\mathcal{K}_p)$. Moreover we denote by $\omega^{(3)}_{P_1, P_2}$ a normal differential of the third kind (cf. (A.11), (A.12)) with simple poles at P_1 and P_2 with residues 1 and -1, respectively. Explicitly, one computes for $\omega^{(3)}_{P_{0,-}, P_{\infty_-}}$ and $\omega^{(3)}_{P_{0,-}, P_{\infty_+}}$ the following expressions:

$$\omega^{(3)}_{P_{0,-}, P_{\infty_\pm}} = \frac{y + y_{0,-}}{z} \frac{dz}{2y} \mp \frac{1}{2y} \prod_{j=1}^{p} (z - \lambda_{\pm,j}) dz, \quad P_{0,-} = (0, y_{0,-}) = (0, -g_{p+1}), \tag{3.67}$$

where $\{\lambda_{\pm,j}\}_{j=1,\dots,p}$ are uniquely determined by the normalization

$$\int_{a_j} \omega^{(3)}_{P_{0,-},P_{\infty\pm}} = 0, \quad j = 1,\dots,p. \tag{3.68}$$

The explicit formula (3.67) then implies the following asymptotic expansions (using the local coordinate $\zeta = z$ near $P_{0,\pm}$ and $\zeta = 1/z$ near $P_{\infty\pm}$),

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty-}} \underset{\zeta\to 0}{=} \begin{Bmatrix} 0 \\ \ln(\zeta) \end{Bmatrix} + \omega_0^{0,\pm}(P_{0,-},P_{\infty-}) + O(\zeta) \text{ as } P \to P_{0,\pm}, \tag{3.69}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty-}} \underset{\zeta\to 0}{=} \begin{Bmatrix} 0 \\ -\ln(\zeta) \end{Bmatrix} + \omega_0^{\infty\pm}(P_{0,-},P_{\infty-}) + O(\zeta) \text{ as } P \to P_{\infty\pm}, \tag{3.70}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty+}} \underset{\zeta\to 0}{=} \begin{Bmatrix} 0 \\ \ln(\zeta) \end{Bmatrix} + \omega_0^{0,\pm}(P_{0,-},P_{\infty+}) + O(\zeta) \text{ as } P \to P_{0,\pm}, \tag{3.71}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty+}} \underset{\zeta\to 0}{=} \begin{Bmatrix} -\ln(\zeta) \\ 0 \end{Bmatrix} + \omega_0^{\infty\pm}(P_{0,-},P_{\infty+}) + O(\zeta) \text{ as } P \to P_{\infty\pm}. \tag{3.72}$$

Here $Q_0 \in \mathcal{B}(\mathcal{K}_p)$ is a fixed base point and we agree to choose the same path of integration from Q_0 to P in all Abelian integrals in this section.

Lemma 3.9. *With* $\omega_0^{\infty_\sigma}(P_{0,-},P_{\infty\pm})$ *and* $\omega_0^{0,\sigma'}(P_{0,-},P_{\infty\pm})$, $\sigma,\sigma' \in \{+,-\}$, *defined as in* (3.69)–(3.72) *one has*

$$\begin{aligned} \exp\big[&\omega_0^{0,-}(P_{0,-},P_{\infty\pm}) - \omega_0^{\infty+}(P_{0,-},P_{\infty\pm}) \\ &- \omega_0^{\infty-}(P_{0,-},P_{\infty\pm}) + \omega_0^{0,+}(P_{0,-},P_{\infty\pm})\big] = 1. \end{aligned} \tag{3.73}$$

Proof. Pick $Q_{1,\pm} = (z_1, \pm y_1) \in \mathcal{K}_p\backslash\{P_{\infty\pm}\}$ in a neighborhood of $P_{\infty\pm}$ and $Q_{2,\pm} = (z_2, \pm y_2) \in \mathcal{K}_p\backslash\{P_{0,\pm}\}$ in a neighborhood of $P_{0,\pm}$. Without loss of generality we may assume that $P_{\infty+}$ and $P_{0,+}$ lie on the same sheet. Then by (3.67),

$$\begin{aligned} &\int_{Q_0}^{Q_{2,-}} \omega^{(3)}_{P_{0,-},P_{\infty-}} - \int_{Q_0}^{Q_{1,+}} \omega^{(3)}_{P_{0,-},P_{\infty-}} - \int_{Q_0}^{Q_{1,-}} \omega^{(3)}_{P_{0,-},P_{\infty-}} + \int_{Q_0}^{Q_{2,+}} \omega^{(3)}_{P_{0,-},P_{\infty-}} \\ &\quad = \int_{Q_0}^{Q_{2,+}} \frac{dz}{z} - \int_{Q_0}^{Q_{1,+}} \frac{dz}{z} = \ln(z_2) - \ln(z_1) + 2\pi i k, \end{aligned} \tag{3.74}$$

for some $k \in \mathbb{Z}$. On the other hand, by (3.69)–(3.72) one obtains

$$\begin{aligned} &\int_{Q_0}^{Q_{2,-}} \omega^{(3)}_{P_{0,-},P_{\infty-}} - \int_{Q_0}^{Q_{1,+}} \omega^{(3)}_{P_{0,-},P_{\infty-}} - \int_{Q_0}^{Q_{1,-}} \omega^{(3)}_{P_{0,-},P_{\infty-}} + \int_{Q_0}^{Q_{2,+}} \omega^{(3)}_{P_{0,-},P_{\infty-}} \\ &\quad = \ln(z_2) + \ln(1/z_1) + \omega_0^{0,-}(P_{0,-},P_{\infty-}) - \omega_0^{\infty+}(P_{0,-},P_{\infty-}) - \omega_0^{\infty-}(P_{0,-},P_{\infty-}) \\ &\qquad + \omega_0^{0,+}(P_{0,-},P_{\infty-}) + O(z_2) + O(1/z_1), \end{aligned} \tag{3.75}$$

and hence the part of (3.73) concerning $\omega^{(3)}_{P_{0,-},P_{\infty-}}$ follows. The corresponding result for $\omega^{(3)}_{P_{0,-},P_{\infty+}}$ is proved analogously. □

In the following it is convenient to use the abbreviation

$$\underline{z}(P,\underline{Q}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \quad P \in \mathcal{K}_p,\ \underline{Q} = \{Q_1,\dots,Q_p\} \in \operatorname{Sym}^p \mathcal{K}_p. \tag{3.76}$$

For subsequent purposes we state the following result.

Lemma 3.10. *The following relations hold:*

$$\underline{z}(P_{\infty_+},\underline{\hat{\mu}}^+) = \underline{z}(P_{\infty_-},\underline{\hat{\nu}}) = \underline{z}(P_{0,-},\underline{\hat{\mu}}) = \underline{z}(P_{0,+},\underline{\hat{\nu}}^+), \tag{3.77}$$

$$\underline{z}(P_{\infty_+},\underline{\hat{\nu}}^+) = \underline{z}(P_{0,-},\underline{\hat{\nu}}), \quad \underline{z}(P_{0,+},\underline{\hat{\mu}}^+) = \underline{z}(P_{\infty_-},\underline{\hat{\mu}}). \tag{3.78}$$

Proof. We indicate the proof of some of the relations to be used in (3.92) and (3.93). Suppose $\underline{\hat{\lambda}}$ stands for either $\underline{\hat{\mu}}$ or $\underline{\hat{\nu}}$. Then,

$$\begin{aligned}
\underline{z}(P_{0,+},\underline{\hat{\lambda}}^+) &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{0,+}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}^+})\\
&= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{0,+}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}}) + \underline{A}_{P_{0,-}}(P_{\infty_+})\\
&= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty_-}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}})\\
&= \underline{z}(P_{\infty_-},\underline{\hat{\lambda}}),
\end{aligned} \tag{3.79}$$

$$\begin{aligned}
\underline{z}(P_{\infty_+},\underline{\hat{\lambda}}^+) &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty_+}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}^+})\\
&= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty_+}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}}) + \underline{A}_{P_{0,-}}(P_{\infty_+})\\
&= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{0,-}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}})\\
&= \underline{z}(P_{0,-},\underline{\hat{\lambda}}), \ \text{etc.}
\end{aligned} \tag{3.80}$$

□

The theta function representations of ϕ, ψ_j, $j=1,2$, and α, β then read as follows.

Theorem 3.11. *Suppose that* $\alpha, \beta \subset \mathbb{C}$ *satisfy* (3.1) *and the* p^{th} *SB system* (2.30). *In addition, assume* (3.2)–(3.4) *and let* $P \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ *and* $(n,n_0) \in \mathbb{Z}^2$. *Then for each* $n \in \mathbb{Z}$, $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ *are nonspecial. Moreover,*[1]

$$\phi(P,n) = C(n)\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))}\exp\left(\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}}\right), \tag{3.81}$$

$$\psi_1(P,n,n_0) = C(n,n_0)\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0)))}\exp\left((n-n_0)\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_+}}\right), \tag{3.82}$$

$$\begin{aligned}
\psi_2(P,n,n_0) &= C(n)C(n,n_0)\\
&\quad\times\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0)))}\exp\left(\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}}\right.\\
&\quad\left.+(n-n_0)\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_+}}\right),
\end{aligned} \tag{3.83}$$

[1] To avoid multi-valued expressions in formulas such as (3.81)–(3.83), etc., we always agree to choose the same path of integration connecting Q_0 and P and refer to Remark A.4 for additional tacitly assumed conventions.

where

$$C(n) = (-1)^{n-n_0} \exp\big[(n-n_0)(\omega_0^{0,-}(P_{0,-}, P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-}, P_{\infty_-}))\big] \\ \times \frac{1}{\alpha(n_0)} \exp\big[-\omega_0^{0,+}(P_{0,-}, P_{\infty_-})\big] \frac{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(n_0)))}, \tag{3.84}$$

$$C(n, n_0) = \exp\big[-(n-n_0)\omega_0^{\infty_+}(P_{0,-}, P_{\infty_+})\big] \frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n)))}. \tag{3.85}$$

The Abel map linearizes the auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) + \underline{A}_{P_{0,-}}(P_{\infty_+})(n-n_0), \tag{3.86}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n_0)}) + \underline{A}_{P_{0,-}}(P_{\infty_+})(n-n_0). \tag{3.87}$$

Finally, α, β are of the form

$$\alpha(n) = \alpha(n_0)(-1)^{n-n_0} \exp\big[-(n-n_0)(\omega_0^{0,-}(P_{0,-}, P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-}, P_{\infty_-}))\big] \\ \times \frac{\theta(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(n_0)))\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n_0)))\theta(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(n)))}, \tag{3.88}$$

$$\beta(n) = \beta(n_0)(-1)^{n-n_0} \exp\big[(n-n_0)(\omega_0^{0,-}(P_{0,-}, P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-}, P_{\infty_-}))\big] \\ \times \frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n_0)))\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\nu}}(n_0)))\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n)))}, \tag{3.89}$$

$$\alpha(n)\beta(n) = \exp\big[\omega_0^{\infty_+}(P_{0,-}, P_{\infty_-}) - \omega_0^{0,+}(P_{0,-}, P_{\infty_-})\big] \\ \times \frac{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n)))\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(n)))\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n)))}. \tag{3.90}$$

Proof. While Eq. (3.86) is clear from (3.50), Eq. (3.87) follows by combining (3.15) and (3.51). By Lemma 3.7, $\mathcal{D}_{\underline{\hat{\mu}}}$ and $\mathcal{D}_{\underline{\hat{\nu}}}$ are nonspecial. By (3.15), Theorem A.3, and Remark A.4, $\phi(P,n)\exp\big(-\int_{Q_0}^P \omega_{P_{0,-},P_{\infty_-}}^{(3)}\big)$ must be of the type

$$\phi(P,n)\exp\left(-\int_{Q_0}^P \omega_{P_{0,-},P_{\infty_-}}^{(3)}\right) = C(n)\frac{\theta(\underline{z}(P, \underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P, \underline{\hat{\mu}}(n)))} \tag{3.91}$$

for some constant $C(n)$. A comparison of (3.91) and the asymptotic relations (3.44) and (3.45) then yields with the help of (3.69), (3.70) and (3.77), (3.78) below the following expressions for α and β:

$$(\alpha^+)^{-1} = C^+ e^{\omega_0^{0,+}(P_{0,-}, P_{\infty_-})} \frac{\theta(\underline{z}(P_{0,+}, \underline{\hat{\nu}}^+))}{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}^+))} \\ = C^+ e^{\omega_0^{0,+}(P_{0,-}, P_{\infty_-})} \frac{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\nu}}))}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}))} \\ = -C e^{\omega_0^{\infty_-}(P_{0,-}, P_{\infty_-})} \frac{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\nu}}))}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}))}. \tag{3.92}$$

Similarly one obtains

$$\begin{aligned}\beta^+ &= C^+ e^{\omega_0^{\infty_+}(P_{0,-},P_{\infty_-})} \frac{\theta(\underline{z}(P_{\infty_+}, \hat{\underline{\nu}}^+))}{\theta(\underline{z}(P_{\infty_+}, \hat{\underline{\mu}}^+))} \\ &= C^+ e^{\omega_0^{\infty_+}(P_{0,-},P_{\infty_-})} \frac{\theta(\underline{z}(P_{0,-}, \hat{\underline{\nu}}))}{\theta(\underline{z}(P_{0,-}, \hat{\underline{\mu}}))} \\ &= -C e^{\omega_0^{0,-}(P_{0,-},P_{\infty_-})} \frac{\theta(\underline{z}(P_{0,-}, \hat{\underline{\nu}}))}{\theta(\underline{z}(P_{0,-}, \hat{\underline{\mu}}))}. \end{aligned} \tag{3.93}$$

Here we used (3.86) and (3.87), more precisely,

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}^+}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}}) + \underline{A}_{P_{0,-}}(P_{\infty_+}), \quad \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\nu}}^+}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\nu}}}) + \underline{A}_{P_{0,-}}(P_{\infty_+}), \tag{3.94}$$

(3.76), and relations of the type (3.77) and (3.78). Thus, one concludes

$$C(n+1) = -\exp\big[\omega_0^{0,-}(P_{0,-}, P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-}, P_{\infty_-})\big]C(n), \quad n \in \mathbb{Z} \tag{3.95}$$

and

$$C(n+1) = -\exp\big[\omega_0^{\infty_-}(P_{0,-}, P_{\infty_-}) - \omega_0^{0,+}(P_{0,-}, P_{\infty_-})\big]C(n), \quad n \in \mathbb{Z}, \tag{3.96}$$

which is consistent with (3.73). The first-order difference equation (3.95) then implies

$$C(n) = (-1)^{(n-n_0)} \exp\big[(n-n_0)(\omega_0^{0,-}(P_{0,-}, P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-}, P_{\infty_-}))\big]C(n_0), \quad n, n_0 \in \mathbb{Z}. \tag{3.97}$$

Thus one infers (3.88) and (3.89). Moreover, (3.97) and taking $n = n_0$ in the first line in (3.92) yield (3.84). Dividing the first line in (3.93) by the first line in (3.92) then proves (3.90).

By (3.50) and Theorem A.3, $\psi_1(P, n, n_0)$ must be of the type (3.82). A comparison of (3.20), (3.44), and (3.82) as $P \to P_{\infty_+}$ ($\zeta = 1/z$) then yields

$$\psi_1(P, n, n_0) \underset{\zeta \to 0}{=} \zeta^{n_0 - n}(1 + O(\zeta)) \tag{3.98}$$

and

$$\begin{aligned}\psi_1(P, n, n_0) &\underset{\zeta \to 0}{=} C(n, n_0) \frac{\theta(\underline{z}(P_{\infty_+}, \hat{\underline{\mu}}(n)))}{\theta(\underline{z}(P_{\infty_+}, \hat{\underline{\mu}}(n_0)))} \\ &\quad \times \exp\big[(n-n_0)\omega_0^{\infty_+}(P_{0,-}, P_{\infty_+})\big]\zeta^{n_0-n}(1 + O(\zeta)),\end{aligned} \tag{3.99}$$

proving (3.85). Equation (3.83) is clear from (3.26), (3.81), and (3.82). □

Remark 3.12. (i) By (3.86), (3.87), the arguments of all theta functions in (3.81)–(3.83) (3.85), and (3.88)–(3.90) are linear with respect to n.
(ii) Using relations of the type (3.77), (3.78) and

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\nu}}}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}}) + \underline{A}_{P_{0,-}}(P_{\infty_-}), \tag{3.100}$$

one can rewrite formulas (3.81)–(3.90) in terms of $\hat{\underline{\mu}}$ (or $\hat{\underline{\nu}}$) only.

(iii) For simplicity we assumed $\alpha(n) \neq 0$, $\beta(n) \neq 0$, $n \in \mathbb{Z}$, in Theorem 3.11. Since by (3.44) and (3.45) no $\hat{\mu}_j$ and no $\hat{\nu}_\ell$ hits $P_{0,+}$ or P_{∞_+}, the expressions (3.88) and (3.89) for α and β are consistent with this assumption.

(iv) Generally, α and β will not be quasi-periodic with respect to $n \in \mathbb{Z}$. Only under certain restrictions on the distribution of $\{E_m\}_{m=0,\dots,2p+1}$, such as the (de)focusing cases discussed in Corollary 3.13 next, one can expect to uniformly bound the exponential terms in (3.88) and (3.89) and prove quasi-periodicity of α and β.

The special defocusing and focusing cases are briefly considered next.

Corollary 3.13. *Suppose that $\alpha, \beta \subset \mathbb{C}$ satisfy* (3.1) *and the p^{th} SB system* (2.30) *and assume* (3.2)–(3.4). *Moreover, assume either the defocusing case, where $\beta(n) = \overline{\alpha(n)}$, or the focusing case, where $\beta(n) = -\overline{\alpha(n)}$, $n \in \mathbb{Z}$. In either case, α is quasi-periodic with respect to $n \in \mathbb{Z}$.*

Proof. We start by noting that the ratio of theta functions in (3.88) and (3.89) is bounded as n varies in $\mathbb{Z}$ since by (3.15) (see also (3.44) and (3.45)) $P_{0,+}$ is never hit by any $\hat{\nu}_\ell(n)$ and P_{∞_+} is never hit by any $\hat{\mu}_j(n)$. Thus, α (and of course β) is quasi-periodic if and only if the exponential term in (3.88) is bounded (i.e., unimodular). Assume the defocusing case $\beta = \overline{\alpha}$. Then, writing

$$\alpha(n) = b(n)e^{nc}, \quad \beta(n) = \tilde{b}(n)e^{-nc}, \ n \in \mathbb{Z}, \quad b, \tilde{b} \in \ell^\infty(\mathbb{Z}) \tag{3.101}$$

(cf. (3.88), (3.89)), $\beta = \overline{\alpha}$ implies

$$\beta(n) = \tilde{b}(n)e^{-n\mathrm{Re}(c) - in\mathrm{Im}(c)} = \overline{\alpha}(n) = \overline{b(n)}e^{n\mathrm{Re}(c) - in\mathrm{Im}(c)}, \tag{3.102}$$

and hence $\mathrm{Re}(c) = 0$. The analogous argument applies in the focusing case. □

Remark 3.14. (i) The additional (de)focusing assumption $\beta = \pm\overline{\alpha}$ in Corollary 3.13, implies strong restrictions on the possible location of the branch points $(E_m, 0)$, $m = 0, \dots, 2p+1$. In particular, in analogy to the Ablowitz–Ladik model discussed in [35], one expects all $(E_m, 0)$ to occur in pairs which are reflection symmetric with respect to the unit circle $\mathbb{T}$ in $\mathbb{C}$. In the defocusing case, $\beta = \overline{\alpha}$ with $|\alpha(n)| < 1, n \in \mathbb{Z}$, all branch points are seen to lie on $\mathbb{T}$ as discussed in [23] and [41]. For $|\alpha| > 1$ one expects them to bifurcate off the unit circle $\mathbb{T}$.

(ii) In analogy to the defocusing case of the nonlinear Schrödinger equation (cf. [28, Ch. 3]), the isospectral manifold of algebro-geometric solutions of (1.3) can be identified with a $(p+1)$-dimensional real torus $\mathbb{T}^{p+1}$ as discussed in detail in [41, Ch. 11]. This isospectral torus is of dimension $p+1$ (rather than p, given the p divisors $\hat{\mu}_j(n_0)$, $j = 1, \dots, p$) due to the additional scaling invariance discussed in (2.32), (2.33) involving an arbitary constant multiple of absolute value equal to one.

(iii) By Remark 3.8, no special divisors arise if $\beta(n_1) = \pm\overline{\alpha(n_1)} = 0$ for some $n_1 \in \mathbb{Z}$, and hence Corollary 3.13 extends to this case as long as $\beta(n_0) = \pm\overline{\alpha(n_0)} \neq 0$ in (3.89).

(iv) In the special defocusing case $\beta = \overline{\alpha}$, with $|\alpha(n)| < 1, n \in \mathbb{Z}$, Corollary 3.13 recovers the original result of Geronimo and Johnson [23] that α is quasi-periodic without the use of Fay's generalized Jacobi variety, double covers, etc.

(v) After submitting our paper we became aware of a first draft by Peherstorfer [39] who considers Caratheodory functions $\widehat{m}_{\pm}$ of the form (1.22) related to algebro-geometric Schur functions $m_{\pm}$, the branches of ϕ in the defocusing case $\beta = \overline{\alpha}$ (cf. (1.18), (1.21), (3.14), and (3.22)), and related theta function representations. We thank Peter Yuditskii and Franz Peherstorfer for providing us with a copy of this manuscript.

Finally, we briefly consider the case $p = 0$ excluded in Theorem 3.11.

Example 3.15. Let $p = 0$, $P = (z, y) \in \mathcal{K}_0 \backslash \{P_{0,+}, P_{0,-}, P_{\infty_+}, P_{\infty_-}\}$, and $(n, n_0) \in \mathbb{Z}^2$. Then,

$$\mathcal{K}_0 \colon \mathcal{F}_0(z, y) = y^2 - R_2(z) = y^2 - (z - E_0)(z - E_1) = 0,$$
$$E_0, E_1 \in \mathbb{C}\backslash\{0\},\ \ E_0 \neq E_1, \quad g_1^2 = E_0 E_1, \quad g_1 = y(P_{0,+}), \quad c_1 = -(E_0 + E_1)/2,$$
$$\alpha(n) = \alpha(n_0)(-g_1)^{n-n_0}, \quad \beta(n) = \beta(n_0)(-g_1)^{n_0-n},$$
$$\text{s-SB}_0(\alpha, \beta) = \begin{pmatrix} -\alpha^+ - g_1\alpha \\ \beta^- + g_1\beta \end{pmatrix} = 0, \quad \alpha(n)\beta(n) = [1 - (c_1/g_1)]/2,$$
$$\phi(P) = \frac{y + z - 2\alpha^+\beta + c_1}{-2\alpha^+} = \frac{-2\beta z}{y - z + 2\alpha^+\beta - c_1}.$$

One verifies that $E_0 \neq E_1$ is equivalent to $\alpha\beta \in \mathbb{C}\backslash\{0, 1\}$. For a Borg-type theorem related to this example in the special defocusing case $\beta = \overline{\alpha}$ with $|\alpha(n)| < 1$, $n \in \mathbb{Z}$, we refer to [32].

Appendix A. Hyperelliptic Curves and Their Theta Functions

We give a brief summary of some of the fundamental properties and notations needed from the theory of hyperelliptic curves. More details can be found in some of the standard textbooks [19] and [37], as well as monographs dedicated to integrable systems such as [14, ch. 2], [28, App. A, B].

Fix $g \in \mathbb{N}$. The hyperelliptic curve $\mathcal{K}_g$ of genus g used in Sect. 3 is defined by

$$\mathcal{K}_g : \mathcal{F}_g(z, y) = y^2 - R_{2g+2}(z) = 0, \quad R_{2g+2}(z) = \prod_{m=0}^{2g+1} (z - E_m), \tag{A.1}$$

$$\{E_m\}_{m=0,\dots,2g+1} \subset \mathbb{C}, \quad E_m \neq E_{m'} \text{ for } m \neq m',\ m, m' = 0, \dots, 2g+1. \tag{A.2}$$

The curve (A.1) is compactified by adding the points P_{∞_+} and P_{∞_-}, $P_{\infty_+} \neq P_{\infty_-}$, at infinity. One then introduces an appropriate set of $g+1$ nonintersecting cuts $\mathcal{C}_j$ joining $E_{m(j)}$ and $E_{m'(j)}$. We denote

$$\mathcal{C} = \bigcup_{j \in \{1,\dots,g+1\}} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k. \tag{A.3}$$

Define the cut plane $\Pi = \mathbb{C}\backslash\mathcal{C}$, and introduce the holomorphic function

$$R_{2g+2}(\,\cdot\,)^{1/2} \colon \Pi \to \mathbb{C}, \quad z \mapsto \left(\prod_{m=0}^{2g+1} (z - E_m)\right)^{1/2} \tag{A.4}$$

on Π with an appropriate choice of the square root branch in (A.4). Define

$$\mathcal{M}_g = \{(z, \sigma R_{2g+2}(z)^{1/2}) \mid z \in \mathbb{C},\ \sigma \in \{\pm 1\}\} \cup \{P_{\infty_+}, P_{\infty_-}\} \tag{A.5}$$

by extending $R_{2g+2}(\,\cdot\,)^{1/2}$ to $\mathcal{C}$. The hyperelliptic curve $\mathcal{K}_g$ is then the set $\mathcal{M}_g$ with its natural complex structure obtained upon gluing the two sheets of $\mathcal{M}_g$ crosswise along the cuts. The set of branch points $\mathcal{B}(\mathcal{K}_g)$ of $\mathcal{K}_g$ is given by

$$\mathcal{B}(\mathcal{K}_g) = \{(E_m, 0)\}_{m=0,\dots,2g+1} \tag{A.6}$$

and finite points P on $\mathcal{K}_g$ are denoted by $P = (z, y)$, where $y(P)$ denotes the meromorphic function on $\mathcal{K}_g$ satisfying $\mathcal{F}_g(z, y) = y^2 - R_{2g+2}(z) = 0$. Local coordinates near $P_0 = (z_0, y_0) \in \mathcal{K}_g \backslash (\mathcal{B}(\mathcal{K}_g) \cup \{P_{\infty_+}, P_{\infty_-}\})$ are given by $\zeta_{P_0} = z - z_0$, near $P_{\infty_\pm}$ by $\zeta_{P_{\infty_\pm}} = 1/z$, and near branch points $(E_{m_0}, 0) \in \mathcal{B}(\mathcal{K}_g)$ by $\zeta_{(E_{m_0},0)} = (z - E_{m_0})^{1/2}$. The Riemann surface $\mathcal{K}_g$ defined in this manner has topological genus g.

One verifies that dz/y is a holomorphic differential on $\mathcal{K}_g$ with zeros of order $g-1$ at $P_{\infty_\pm}$ and that

$$\eta_j = \frac{z^{j-1}dz}{y}, \quad j = 1, \dots, g \tag{A.7}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_g$. Introducing the invertible matrix C in $\mathbb{C}^g$,

$$\begin{aligned} &C = (C_{j,k})_{j,k=1,\dots,g}, \quad C_{j,k} = \int_{a_k} \eta_j, \\ &\underline{c}(k) = (c_1(k), \dots, c_n(k)), \quad c_j(k) = C^{-1}_{j,k}, \quad j, k = 1, \dots, g, \end{aligned} \tag{A.8}$$

the corresponding basis of normalized holomorphic differentials ω_j, $j = 1, \dots, g$ on $\mathcal{K}_g$ is given by

$$\omega_j = \sum_{\ell=1}^{g} c_j(\ell)\eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j, k = 1, \dots, g. \tag{A.9}$$

Here $\{a_j, b_j\}_{j=1,\dots,g}$ is a homology basis for $\mathcal{K}_g$ with intersection matrix of the cycles satisfying

$$a_j \circ b_k = \delta_{j,k},\ a_j \circ a_k = 0,\ b_j \circ b_k = 0, \quad j, k = 1, \dots, g. \tag{A.10}$$

Associated with the homology basis $\{a_j, b_j\}_{j=1,\dots,g}$ we also recall the canonical dissection of $\mathcal{K}_g$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_g$ of the fundamental polygon $\partial\widehat{\mathcal{K}}_g$ given by $\partial\widehat{\mathcal{K}}_g = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g^{-1} b_g^{-1}$. Let $\mathcal{M}(\mathcal{K}_g)$ and $\mathcal{M}^1(\mathcal{K}_g)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on $\mathcal{K}_g$. The residue of a meromorphic differential $\nu \in \mathcal{M}^1(\mathcal{K}_g)$ at a point $Q \in \mathcal{K}_g$ is defined by $\operatorname{res}_Q(\nu) = \frac{1}{2\pi i} \int_{\gamma_Q} \nu$, where γ_Q is a counterclockwise oriented, smooth, simple, closed contour encircling Q but no other pole of ν. Holomorphic differentials are also called Abelian differentials of the first kind. Abelian differentials of the second kind, $\omega^{(2)} \in \mathcal{M}^1(\mathcal{K}_g)$, are characterized by the property that all their residues vanish. Any meromorphic differential $\omega^{(3)}$ on $\mathcal{K}_g$ not of the first or second kind is said to be of the third kind. A differential of the third kind $\omega^{(3)} \in \mathcal{M}^1(\mathcal{K}_g)$ is usually normalized by the vanishing of its a-periods, that is,

$$\int_{a_j} \omega^{(3)} = 0, \quad j = 1, \dots, g. \tag{A.11}$$

A normal differential of the third kind $\omega^{(3)}_{P_1,P_2}$ associated with two points $P_1, P_2 \in \widehat{\mathcal{K}}_g$, $P_1 \neq P_2$ by definition has simple poles at P_j with residues $(-1)^{j+1}$, $j = 1, 2$ and vanishing a-periods. If $\omega^{(3)}_{P,Q}$ is a normal differential of the third kind associated with $P, Q \in \widehat{\mathcal{K}}_g$, holomorphic on $\mathcal{K}_g \backslash \{P, Q\}$, then

$$\frac{1}{2\pi i} \int_{b_j} \omega^{(3)}_{P,Q} = \int_Q^P \omega_j, \quad j = 1, \dots, g, \tag{A.12}$$

where the path from Q to P lies in $\widehat{\mathcal{K}}_g$ (i.e., does not touch any of the cycles a_j, b_j).

We shall always assume (without loss of generality) that all poles of differentials of the second and third kind on $\mathcal{K}_g$ lie on $\widehat{\mathcal{K}}_g$ (i.e., not on $\partial\widehat{\mathcal{K}}_n$).

Define the matrix $\tau = (\tau_{j,\ell})_{j,\ell=1,\dots,g}$ by

$$\tau_{j,\ell} = \int_{b_\ell} \omega_j, \quad j, \ell = 1, \dots, g. \tag{A.13}$$

Then $\operatorname{Im}(\tau) > 0$ and $\tau_{j,\ell} = \tau_{\ell,j}$, $j, \ell = 1, \dots, g$. Associated with τ one introduces the period lattice

$$L_g = \{\underline{z} \in \mathbb{C}^g \mid \underline{z} = \underline{m} + \underline{n}\tau,\ \underline{m}, \underline{n} \in \mathbb{Z}^g\} \tag{A.14}$$

and the Riemann theta function associated with $\mathcal{K}_g$ and the given homology basis $\{a_j, b_j\}_{j=1,\dots,g}$,

$$\theta(\underline{z}) = \sum_{n \in \mathbb{Z}^g} \exp\big(2\pi i(\underline{n}, \underline{z}) + \pi i(\underline{n}, \underline{n}\tau)\big), \quad \underline{z} \in \mathbb{C}^g, \tag{A.15}$$

where $(\underline{u}, \underline{v}) = \overline{\underline{u}}\,\underline{v}^\top = \sum_{j=1}^g \overline{u_j}\, v_j$ denotes the scalar product in $\mathbb{C}^g$. It has the fundamental properties

$$\theta(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_g) = \theta(\underline{z}), \tag{A.16}$$

$$\theta(\underline{z} + \underline{m} + \underline{n}\tau) = \exp\big(-2\pi i(\underline{n}, \underline{z}) - \pi i(\underline{n}, \underline{n}\tau)\big)\theta(\underline{z}), \quad \underline{m}, \underline{n} \in \mathbb{Z}^g. \tag{A.17}$$

Next, fix a base point $Q_0 \in \mathcal{K}_g \backslash \{P_{0,\pm}, P_{\infty_\pm}\}$, denote by $J(\mathcal{K}_g) = \mathbb{C}^g / L_g$ the Jacobi variety of $\mathcal{K}_g$, and define the Abel map $\underline{A}_{Q_0}$ by

$$\underline{A}_{Q_0} \colon \mathcal{K}_g \to J(\mathcal{K}_g), \quad \underline{A}_{Q_0}(P) = \left(\int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_g\right) \pmod{L_g}, \quad P \in \mathcal{K}_g. \tag{A.18}$$

Similarly, we introduce

$$\underline{\alpha}_{Q_0} \colon \operatorname{Div}(\mathcal{K}_g) \to J(\mathcal{K}_g), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_g} \mathcal{D}(P)\underline{A}_{Q_0}(P), \tag{A.19}$$

where $\operatorname{Div}(\mathcal{K}_g)$ denotes the set of divisors on $\mathcal{K}_g$. Here $\mathcal{D}\colon \mathcal{K}_g \to \mathbb{Z}$ is called a divisor on $\mathcal{K}_g$ if $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_g$. (In the main body of this paper we will choose Q_0 to be one of the branch points, i.e., $Q_0 \in \mathcal{B}(\mathcal{K}_g)$, and for simplicity we will always choose the same path of integration from Q_0 to P in all Abelian integrals.)

In connection with divisors on $\mathcal{K}_g$ we shall employ the following (additive) notation,

$$\begin{aligned} &\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}, \\ &\underline{Q} = \{Q_1, \dots, Q_m\} \in \operatorname{Sym}^m \mathcal{K}_g, \quad Q_0 \in \mathcal{K}_g,\ m \in \mathbb{N}, \end{aligned} \tag{A.20}$$

where for any $Q \in \mathcal{K}_g$,

$$\mathcal{D}_Q\colon \mathcal{K}_g \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_g \backslash \{Q\}, \end{cases} \tag{A.21}$$

and $\operatorname{Sym}^n \mathcal{K}_g$ denotes the n^{th} symmetric product of $\mathcal{K}_g$. In particular, $\operatorname{Sym}^m \mathcal{K}_g$ can be identified with the set of nonnegative divisors $0 \le \mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$ of degree m.

For $f \in \mathcal{M}(\mathcal{K}_g)\backslash\{0\}$, $\omega \in \mathcal{M}^1(\mathcal{K}_g)\backslash\{0\}$ the divisors of f and ω are denoted by (f) and (ω), respectively. Two divisors $\mathcal{D}$, $\mathcal{E} \in \operatorname{Div}(\mathcal{K}_g)$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(\mathcal{K}_g)\backslash\{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}] = \{\mathcal{E} \in \operatorname{Div}(\mathcal{K}_g) \mid \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0,\ \deg((\omega)) = 2(g-1),\ f \in \mathcal{M}(\mathcal{K}_g)\backslash\{0\},\ \omega \in \mathcal{M}^1(\mathcal{K}_g)\backslash\{0\}, \tag{A.22}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D}) = \sum_{P \in \mathcal{K}_g} \mathcal{D}(P)$. (f) is called a principal divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_g) \mid f = 0 \text{ or } (f) \ge \mathcal{D}\}, \quad r(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}), \tag{A.23}$$

$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_g) \mid \omega = 0 \text{ or } (\omega) \ge \mathcal{D}\}, \quad i(\mathcal{D}) = \dim \mathcal{L}^1(\mathcal{D}) \tag{A.24}$$

with $i(\mathcal{D})$ the index of speciality of $\mathcal{D}$, one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$. Moreover, we recall the following fundamental facts.

Theorem A.1. *Let* $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$, $\omega \in \mathcal{M}^1(\mathcal{K}_g) \setminus \{0\}$. *Then*

$$i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad g \in \mathbb{N}_0. \tag{A.25}$$

The Riemann–Roch theorem reads

$$r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - g + 1, \quad g \in \mathbb{N}_0. \tag{A.26}$$

By Abel's theorem, $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$, $g \in \mathbb{N}$, *is principal if and only if*

$$\deg(\mathcal{D}) = 0 \text{ and } \underline{\alpha}_{Q_0}(\mathcal{D}) = \underline{0}. \tag{A.27}$$

Finally, assume $g \in \mathbb{N}$. *Then* $\underline{\alpha}_{Q_0} : \operatorname{Div}(\mathcal{K}_g) \to J(\mathcal{K}_g)$ *is surjective (Jacobi's inversion theorem).*

Theorem A.2. *Let* $\mathcal{D}_{\underline{Q}} \in \operatorname{Sym}^g \mathcal{K}_g$, $\underline{Q} = \{Q_1, \dots, Q_g\}$. *Then* $1 \le i(\mathcal{D}_{\underline{Q}}) = s \le g/2$ *if and only if there are* s *pairs of the type* $(P, P^*) \in \{Q_1, \dots, Q_g\}$ *(this includes, of course, branch points for which* $P = P^*$*).*

Denote by $\underline{\Xi}_{Q_0} = (\Xi_{Q_{0,1}}, \dots, \Xi_{Q_{0,g}})$ the vector of Riemann constants,

$$\Xi_{Q_{0,j}} = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^{g} \int_{a_\ell} \omega_\ell(P) \int_{Q_0}^{P} \omega_j, \quad j = 1, \dots, g. \tag{A.28}$$

Theorem A.3. *Let $\underline{Q} = \{Q_1, \dots, Q_g\} \in \operatorname{Sym}^g \mathcal{K}_g$ and assume $\mathcal{D}_{\underline{Q}}$ to be nonspecial, that is, $i(\mathcal{D}_{\underline{Q}}) = 0$. Then*

$$\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \alpha_{Q_0}(\mathcal{D}_Q)) = 0 \text{ if and only if } P \in \{Q_1, \dots, Q_g\}. \tag{A.29}$$

Remark A.4. In Sect. 3 we dealt with theta function expressions of the type

$$\phi(P) = \frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_1))}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_2))} \exp\left(\int_{Q_0}^{P} \omega^{(3)}_{Q_1,Q_2}\right), \quad P \in \mathcal{K}_g, \tag{A.30}$$

where $\mathcal{D}_j \in \operatorname{Sym}^g(\mathcal{K}_g)$, $j = 1, 2$, are nonspecial positive divisors of degree g, $Q_j \in \mathcal{K}_g \setminus \{P_{\infty_+}, P_{\infty_-}\}$, $j = 1, 2$, and $\omega^{(3)}_{Q_1,Q_2}$ is a normal differential of the third kind. In particular, one has

$$\int_{a_j} \omega^{(3)}_{Q_1,Q_2} = 0, \quad j = 1, \dots, g. \tag{A.31}$$

Even though we agree to always choose identical paths of integration from Q_0 to P in all Abelian integrals (A.30), this is not sufficient to render ϕ single-valued on $\mathcal{K}_g$. To achieve single-valuedness one needs to replace $\mathcal{K}_g$ by its simply connected canonical dissection $\widehat{\mathcal{K}}_g$ and then replace the Abel maps $\underline{A}_{Q_0}$, $\underline{\alpha}_{Q_0}$ in (A.30) with the corresponding Abel maps $\widehat{\underline{A}}_{Q_0}$, $\hat{\underline{\alpha}}_{Q_0}$ on $\widehat{\mathcal{K}}_g$. In particular, one regards a_j, b_j as curves (being a part of $\partial\widehat{\mathcal{K}}_g$) and not as homology classes $[a_j]$, $[b_j]$ in $H_1(\mathcal{K}_g, \mathbb{Z})$. Similarly, one then replaces $\underline{\Xi}_{Q_0}$ by $\widehat{\underline{\Xi}}_{Q_0}$ (replacing $\underline{A}_{Q_0}$ by $\widehat{\underline{A}}_{Q_0}$ in (A.28), etc.). Moreover, to render ϕ single-valued on $\widehat{\mathcal{K}}_g$ one needs to assume in addition that

$$\hat{\underline{\alpha}}_{Q_0}(\mathcal{D}_1) - \hat{\underline{\alpha}}_{Q_0}(\mathcal{D}_2) = 0 \tag{A.32}$$

(as opposed to merely $\underline{\alpha}_{Q_0}(\mathcal{D}_1) - \underline{\alpha}_{Q_0}(\mathcal{D}_2) = 0 \pmod{L_g}$). These statements easily follow from (A.12) and (A.17). In fact, by (A.17),

$$\hat{\underline{\alpha}}_{Q_0}(\mathcal{D}_1 + \mathcal{D}_{Q_1}) - \hat{\underline{\alpha}}_{Q_0}(\mathcal{D}_2 + \mathcal{D}_{Q_2}) \in \mathbb{Z}^g, \tag{A.33}$$

suffices to guarantee single-valuedness of ϕ on $\widehat{\mathcal{K}}_g$. Without the replacement of $\underline{A}_{Q_0}$ and $\underline{\alpha}_{Q_0}$ by $\widehat{\underline{A}}_{Q_0}$ and $\hat{\underline{\alpha}}_{Q_0}$ in (A.30) and without the assumption (A.32) (or (A.33)), ϕ is a multiplicative (multi-valued) function on $\mathcal{K}_g$, and then most effectively discussed by introducing the notion of characters on $\mathcal{K}_g$ (cf. [19, Sect. III.9]). For simplicity, we decided to avoid the latter possibility and throughout this paper will tacitly always assume (A.32) without particularly emphasizing this convention each time it is used.

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