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Regularity of Solutions to Vorticity Navier–Stokes System on $\mathbb{R}^2$

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Abstract: The Cauchy problem for the Navier–Stokes system for vorticity on plane is considered. If the Fourier transform of the initial data decays as a power at infinity, then at any positive time the Fourier transform of the solution decays exponentially, i.e. the solution is analytic.

1. Introduction. Main Results

We consider the Cauchy problem for the Navier–Stokes system on $\mathbb{R}^2$ in its vorticity formulation:

$$\frac{\partial\omega(x,t)}{\partial t} + u_1(x,t)\frac{\partial\omega(x,t)}{\partial x_1} + u_2(x,t)\frac{\partial\omega(x,t)}{\partial x_2} = \nu\Delta\omega(x,t) + f(x,t), \quad (1)$$

$$\omega(x,t) = \frac{\partial u_2(x,t)}{\partial x_1} - \frac{\partial u_1(x,t)}{\partial x_2}, \quad (2)$$

$$\lim_{|x|\to\infty} |u(x,t)| = 0, \quad t \geqslant 0, \quad (3)$$

$$\omega(x,0) = \omega_0(x). \quad (4)$$

Here the spatial variable x belongs to the Euclidean space $\mathbb{R}^2$ with the inner product $\langle\cdot,\cdot\rangle$, $\omega : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$ is the vorticity of the velocity field $u : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^2$ which is assumed to be divergence-free (i.e. $\langle\nabla, u\rangle = 0$), $f : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$ is the vorticity of the external forcing, $\nu > 0$ is the viscosity parameter and $\omega_0 : \mathbb{R}^2 \to \mathbb{R}$ is the initial data.

The theory of existence and uniqueness of solutions for the 2-dimensional Navier–Stokes system was developed by Leray and Ladyzhenskaya, see, e.g. the survey [8]. The following existence and uniqueness theorem for the vorticity system (1) – (4) was proved in [10].

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Theorem 1. *Suppose $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and all second derivatives of ω_0 are uniformly Hölder in $\mathbb{R}^2$ with some exponent $\lambda > 0$. Let $T > 0$ be such that $f \in L^1(Q_T) \cap L^\infty(Q_T)$, where $Q_T = \mathbb{R}^2 \times [0, T]$ and f is locally Hölder with the same exponent λ with respect to spatial variables for all $t \in [0, T]$. Then there exists a bounded classical solution ω to the Cauchy problem (1)—(4) on $[0, T]$. All the derivatives of the solution arising in the statement of the Cauchy problem are bounded and continuous on Q_T.*

This solution is unique in the class of functions which are bounded for every $t \geqslant 0$,

$$\sup_{t\in[0,T]} \left[\|\omega(\cdot,t)\|_{L_\infty(\mathbb{R}^2)} + \|\omega(\cdot,t)\|_{L_1(\mathbb{R}^2)}\right] \leqslant \|\omega_0\|_{L^\infty(\mathbb{R}^2)} + \|\omega_0\|_{L^1(\mathbb{R}^2)} + T\|f\|_{L^\infty(Q_T)} + \|f\|_{L^1(Q_T)},$$

and for every t the following representation (the Biot-Savart law) holds true:

$$u(\cdot,t) = K * \omega(\cdot,t), \tag{5}$$

where $*$ means convolution and $K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}$, $x^\perp = (-x_2, x_1)$ for $x = (x_1, x_2) \in \mathbb{R}^2$.

This result allows to consider the uniquely determined global (i.e. defined on $\mathbb{R}_+$) solution ω.

The problem (1) – (4) was also studied in e.g. [1, 2, 6, 7] where some existence-uniqueness theorems were obtained as well as some regularity properties of solutions.

In this note we are concerned with the study of regularity of solution $\omega(x,t)$ in terms of its Fourier transform under the conditions of Theorem 1.

The Gevrey class regularity of solutions to the Navier-Stokes system on 2-dimensional torus (the periodic case) was obtained in [4]. It is shown in a recent paper [3] how the techniques of [4] can be used to prove analyticity of solutions in the 3-dimensional situation under some modest regularity assumptions on solutions to a mollified Navier–Stokes system. This approach can be also adapted to the 2-dimensional non-periodic case under study in this paper. However we prove analyticity for this case assuming only minimal regularity properties of the initial data and the forcing.

Our results and techniques are parallel to those of [9] where the 2-dimensional periodic case was studied. The results are stated in this section and their proofs are given in Sect. 2.

By Fourier transform of a function f with respect to the spatial variable we mean the function

$$\hat{f}(k) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\langle k,x\rangle} f(x)dx.$$

For the properties of Fourier transform and its inverse see [12].

To state the main theorem we need the following notation for an arbitrary function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$|f|_{\gamma,\alpha} = \sup_k \frac{|f(k)|}{(1 \wedge |k|^{-\gamma})e^{-\alpha|k|}}, \quad \alpha \geqslant 0, \gamma > 0,$$
$$|f|_\gamma = |f|_{\gamma,0}, \quad \gamma > 0.$$

Thus, if $|f|_\gamma$ is finite then $f(k)$ decays as a power at infinity and if $|f|_{\gamma,\alpha}$ is finite then $f(k)$ decays exponentially.

Theorem 2. *Let the initial data ω_0 and the forcing f satisfy the conditions of Theorem 1. Suppose that $|\widehat{\omega}_0|_\gamma < \infty$ for some $\gamma > 0$ and $|\hat{f}(\cdot, t)|_{\gamma,\alpha} \leqslant C_f$ for some $\alpha > 0$, $C_f > 0$, all $t > 0$ and the same γ. Then there exist nondecreasing and positive for $t > 0$ functions $\beta(t)$ and $D(t)$ such that the solution ω of the Cauchy problem* (1)–(4) *satisfies inequality*

$$|\widehat{\omega}(\cdot, t)|_{\gamma,\beta(t)} \leqslant D(t).$$

There exist positive constants B and T such that $\beta(t) = Bt$ for $t \in [0, T]$ and $\beta(t) \equiv BT$ for $t \geqslant T$. The function $D(t)$ may be chosen to be linear for $t \geqslant T$. If the external forcing is absent then $D(t)$ may be chosen to be constant for $t \geqslant T$.

Remark 1. If $\gamma > 4$ in this theorem then the conditions of Theorem 1 are satisfied automatically. This remark is also applicable to the auxiliary Theorems 3—5 below.

Remark 2. Theorem 2 means that if the Fourier transform of the initial data decays as a negative power when $|k| \to \infty$, then for any positive time the Fourier transform of the solution decays exponentially at infinity, i.e. the solution is analytic.

Remark 3. In the case of unforced system analyticity of the solution for $t > 0$ was proved in [11]. For unforced system with nondecaying initial velocity C^∞-smoothness of solutions has been obtained, see [5] and references therein.

The proof of Theorem 2 is based on the study of the following system describing the evolution of the Fourier transform of vorticity:

$$\frac{\partial \hat{\omega}(k,\ t)}{\partial t} = -\nu |k|^2 \hat{\omega}(k,\ t) + \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{\omega}(l,\ t)\hat{\omega}(k-l,\ t)\frac{\langle k,\ l^\perp \rangle}{|l|^2} dl + \hat{f}(k, t). \quad (6)$$

The proof will be conducted in several steps. First, we shall obtain the following result (the demonstration is given in Sect. 2) on invariance of the set of functions decaying as a negative power at infinity.

Theorem 3. *Let the initial data ω_0 and forcing f satisfy the conditions of Theorem 1. Suppose that $|\widehat{\omega}_0|_\gamma < \infty$ for some $\gamma > 0$ and $|\hat{f}(\cdot, t)|_\gamma \leqslant C_f$ for some constant $C_f > 0$ and all $t > 0$. Then there exists a function $D(t)$ such that the solution ω of the Cauchy problem* (1)–(4) *satisfies $|\widehat{\omega}(\cdot, t)|_\gamma \leqslant D(t)$ for all $t \geqslant 0$. The function $D(t)$ may be chosen to grow linearly and if the forcing is absent then $D(t)$ may be chosen to be constant.*

Then, using the same method and appropriate changes of variables we shall prove Theorems 4 and 5 which immediately imply Theorem 2. Sketches of the proofs are given in Sect. 2.

Theorem 4. *Let the initial data ω_0 and forcing f satisfy the conditions of Theorem 1. Suppose that $|\widehat{\omega}_0|_{\gamma,\alpha} < \infty$ for some $\gamma, \alpha > 0$ and $|\hat{f}(\cdot, t)|_{\gamma,\alpha} \leqslant C_f$ for some constant $C_f > 0$, all $t > 0$ and the same γ, α.*

Then there exists a function $D(t)$ such that the solution ω of the Cauchy problem (1)—(4) *satisfies $|\widehat{\omega}(\cdot, t)|_{\gamma,\alpha} \leqslant D(t)$ for all $t \geqslant 0$. The function $D(t)$ may be chosen to grow linearly and if the forcing is absent then $D(t)$ may be chosen to be constant.*

Theorem 5. *Let the initial data* ω_0 *and forcing* f *satisfy the conditions of Theorem 1. Suppose that* $|\widehat{\omega}_0|_\gamma < \infty$ *for some* $\gamma > 0$ *and* $|\hat{f}(\cdot, t)|_{\gamma,\alpha} \leqslant C_f$ *for some constant* $C_f > 0$*, all* $t > 0$*, the same* γ *and some* $\alpha > 0$*. Then there exist a time* $T > 0$ *and a nondecreasing function* $D(t)$ *such that for* $t \in [0, T]$ *the solution* ω *of the Cauchy problem* (1)–(4) *satisfies* $|\widehat{\omega}(\cdot, t)|_{\gamma,t\alpha} \leqslant D(t)$*.*

2. Proofs

The following estimate of the nonlinear term in (6) plays the key role in the proof of Theorems 3—5.

Lemma 1. *Let* $D_v = |v|_\gamma \vee \|v\|_{L^2}$ *for a function* $v : \mathbb{R}^2 \to \mathbb{R}$ *and* $\gamma > 0$*. There exists a constant* $\mathcal{Q} = \mathcal{Q}(\gamma)$ *such that*

$$\left|\frac{1}{2\pi}\int_{\mathbb{R}^2} v(l)w(k-l)\frac{\langle k, l^\perp\rangle}{|l|^2}dl\right| \leqslant \begin{cases} \mathcal{Q}D_vD_w|k|^{1-\gamma}\sqrt{1+\ln|k|}, & |k| \geqslant 1 \\ \mathcal{Q}D_vD_w, & |k| < 1. \end{cases} \tag{7}$$

Proof. We shall prove

$$\left|\frac{1}{2\pi}\int_{\mathbb{R}^2} v(l)w(k-l)\frac{\langle k, l^\perp\rangle}{|l|^2}dl\right| \leqslant \begin{cases} \tilde{\mathcal{Q}}D_vD_w|k|^{1-\gamma}\sqrt{1+\ln|k|}, & |k| \geqslant 2 \\ \tilde{\mathcal{Q}}D_vD_w, & |k| < 2, \end{cases} \tag{8}$$

with some $\tilde{\mathcal{Q}}$ which will immediately imply (7).

First, consider the case $|k| \geqslant 2$. Denote J the integral we are interested in and split the domain of integration into four parts:

$$J = J_1 + J_2 + J_3 + J_4 = \int_{|l|\leqslant 1} + \int_{1<|l|\leqslant|k|/2} + \int_{|k|/2<|l|\leqslant 2|k|} + \int_{2|k|<|l|}. \tag{9}$$

In the first term $|k-l| \geqslant |k| - 1$ and $|w(k-l)| \leqslant D_w(|k|-1)^{-\gamma}$ for $|k| \geqslant 2$. Using inequality

$$\frac{|\langle k, l^\perp\rangle|}{|l|^2} \leqslant \frac{|k|}{|l|} \tag{10}$$

and the conditions of the lemma we obtain

$$J_1 \leqslant \frac{1}{2\pi}D_vD_w|k|(|k|-1)^{-\gamma}\int_{|l|\leqslant 1}\frac{1}{|l|}dl \leqslant 2^\gamma D_vD_w|k|^{1-\gamma}. \tag{11}$$

For the second term in (9) we have $|k-l| \geqslant |k|/2$ and $|w(k-l)| \leqslant 2^\gamma D_w/|k|^\gamma$. The conditions of the lemma, inequality (10) and the Cauchy-Schwartz inequality imply

$$\begin{aligned} J_2 &\leqslant \frac{2^\gamma}{2\pi}|k|^{1-\gamma}D_w\int_{1<l\leqslant|k|/2}\frac{|v(l)|}{|l|}dl \leqslant \frac{2^\gamma}{2\pi}|k|^{1-\gamma}D_w\|v\|_2\left(\int_{1<l\leqslant|k|/2}\frac{dl}{|l|^2}\right)^{1/2} \\ &\leqslant \frac{2^\gamma}{\sqrt{2\pi}}D_vD_w|k|^{1-\gamma}[\ln(|k|/2)]^{1/2}. \end{aligned} \tag{12}$$

For the third term in (9) we have $|v(l)| \leqslant 2^\gamma D_v |k|^{-\gamma}$ and $\frac{|\langle k, l^\perp \rangle|}{|l|^2} \leqslant 2$. The Cauchy-Schwartz inequality gives

$$\begin{aligned} J_3 &\leqslant \frac{2^{1+\gamma}}{2\pi} D_v |k|^{-\gamma} \int_{|k|/2<|l|\leqslant 2|k|} |w(k-l)| dl \\ &\leqslant \frac{2^{1+\gamma}}{2\pi} D_v |k|^{-\gamma} \|w\|_2 \left(\int_{|k|/2<|l|\leqslant 2|k|} dl \right)^{1/2} \\ &\leqslant \frac{2^{1+\gamma}}{(2\pi)^{1/2}} \left(\frac{15}{8} \right)^{1/2} D_v D_w |k|^{1-\gamma}. \end{aligned} \tag{13}$$

The last term can be estimated by means of the Cauchy-Schwartz inequality and (10):

$$\begin{aligned} J_4 &\leqslant \frac{1}{2\pi} |k| \|w\|_2 \left(\int_{2|k|<l} \frac{|v(l)|^2}{|l|^2} dl \right)^{1/2} \\ &\leqslant \frac{1}{(2\pi)^{1/2}} |k| \|w\|_2 D_v \left(\int_{2|k|}^\infty \frac{1}{r^{1+2\gamma}} dr \right)^{1/2} \\ &\leqslant \frac{1}{(2\pi)^{1/2}} \frac{1}{2^{\gamma+1/2} \gamma^{1/2}} |k|^{1-\gamma} D_v D_w. \end{aligned} \tag{14}$$

Now (8) for $|k| \geqslant 2$ follows from (9) and (11)–(14).

In the case of $|k| < 2$ let us split the domain of integration in two parts:

$$J = J_1 + J_2 = \int_{|l|\leqslant 4} + \int_{|l|>4}.$$

Then

$$J_1 \leqslant \frac{1}{2\pi} D_v D_w \int_{|l|\leqslant 4} \frac{dl}{|l|} = 4 D_v D_w.$$

If $|l| > 4$ and $|k| < 2$ then $|k-l| > |l|/2$ and

$$\begin{aligned} J_2 &\leqslant \frac{1}{2\pi} |k| \|w\|_2 \left(\int_{2|k|<l} \frac{|v(l)|^2}{|l|^2} dl \right)^{1/2} \leqslant \frac{2}{(2\pi)^{1/2}} \|w\|_2 D_v \left(\int_4^\infty \frac{1}{r^{1+2\gamma}} dr \right)^{1/2} \\ &\leqslant \frac{2}{(2\pi)^{1/2}} \frac{1}{4^\gamma (2\gamma)^{1/2}} D_v D_w. \end{aligned}$$

These estimates for J_1 and J_2 imply inequality (8) for $|k| < 2$. □

Proof of Theorem 3. First, let us prove the following local theorem of existence and uniqueness of the solution with finite norm $|\cdot|_\gamma$.

Lemma 2. *Under the conditions of Theorem 3 there exist $c, \tau > 0$ depending only on viscosity ν and $|\cdot|_\gamma$-norms of the initial data and the forcing such that on the time interval $[0, \tau]$ the solution $\hat{\omega}$ of* (6) *satisfies $|\hat{\omega}(\cdot, t)|_\gamma \leqslant c$ and it is continuous in time with respect to $|\cdot|_\gamma$.*

Proof. Consider the integral form of (6):

$$\hat{\omega}(k,t) = e^{-\nu|k|^2 t}\hat{\omega}(k,0) + \frac{1}{2\pi}\int_0^t e^{-\nu|k|^2(t-s)}\int\limits_{\mathbb{R}^2}\hat{\omega}(l,\ s)\hat{\omega}(k-l,\ s)\frac{\langle k,\ l^{\perp}\rangle}{|l|^2}dlds$$
$$+\int_0^t e^{-\nu|k|^2(t-s)}\hat{f}(k,s)ds, \tag{15}$$

and the following approximation scheme. Let $\hat{\omega}_1(k,t) = \hat{\omega}_0(k)$ for all $k \in \mathbb{R}^2$, $t \in \mathbb{R}_+$, and for $n \geqslant 1$,

$$\hat{\omega}_{n+1}(k,t) = e^{-\nu|k|^2 t}\hat{\omega}_0(k)$$
$$+\frac{1}{2\pi}\int_0^t e^{-\nu|k|^2(t-s)}\int\limits_{\mathbb{R}^2}\hat{\omega}_n(l,\ s)\hat{\omega}_n(k-l,\ s)\frac{\langle k,\ l^{\perp}\rangle}{|l|^2}dlds$$
$$+\int_0^t e^{-\nu|k|^2(t-s)}\hat{f}(k,s)ds. \tag{16}$$

Let $|\omega_n(k,s)| \leqslant C_n(1 \wedge |k|^{-\gamma})$ for $s \in [0,t]$ some t and all $k \in \mathbb{R}^2$. Then for $|k| < 1$ Lemma 1 implies that

$$|\hat{\omega}_{n+1}(k,t)| \leqslant e^{-\nu t|k|^2}|\omega_0(k)| + \mathcal{Q}C_n^2\int_0^t e^{-\nu(t-s)|k|^2}ds + \int_0^t e^{-\nu(t-s)|k|^2}f(k,t)ds$$
$$\leqslant C_0 + \mathcal{Q}C_n^2 t + C_f t, \tag{17}$$

where constants $\mathcal{Q}$ and C_f are defined in the statements of Lemma 1 and Theorem 3 respectively and $C_0 = |\hat{\omega}_0|_\gamma$.

If $|k| \geqslant 1$, then Lemma 1 implies

$$|\hat{\omega}_{n+1}(k,t)| \leqslant e^{-\nu t|k|^2}|\omega_0(k)| + \mathcal{Q}C_n^2|k|^{-\gamma}\int_0^t e^{-\nu(t-s)|k|^2}|k|\sqrt{1+\ln|k|}ds$$
$$+\int_0^t e^{-\nu(t-s)|k|^2}|f(k,t)|ds = I_1 + I_2 + I_3. \tag{18}$$

Clearly, $I_1 \leqslant C_0|k|^{-\gamma}$, $I_2 \leqslant tC_f|k|^{-\gamma}$, and I_2 may be estimated using the Hõlder inequality:

$$I_2 \leqslant \mathcal{Q}C_n^2|k|^{-\gamma}\left(\int_0^t |k|^p e^{-\frac{p\nu(t-s)|k|^2}{3}}ds\right)^{\frac{1}{p}}\left(\int_0^t (1+\ln|k|)^{\frac{q}{2}}e^{-\frac{q\nu(t-s)|k|^2}{3}}ds\right)^{\frac{1}{q}}$$
$$\times\left(\int_0^t e^{-\frac{r\nu(t-s)|k|^2}{3}}ds\right)^{\frac{1}{r}} = \mathcal{Q}C_n^2|k|^{-\gamma}J_1J_2J_3$$

for $p,q,r > 1$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$. If we choose $p \geqslant 2$, then J_1 will be bounded by some constant, independent of t. Under arbitrary choice of $q > 0$ the same holds for J_2. In J_3 a bounded function is integrated. Hence,

$$I_2 \leqslant KC_n^2 t^{1/r}$$

for a constant $K = K(\nu, r) > \mathcal{Q}$. Here $r > 2$ because of the imposed restrictions on p and q.

Finally we have

$$|\hat{\omega}_{n+1}(k,t)||k|^{\gamma} \leqslant C_0 + Kt^{1/r} + C_f t.$$

So, $|\hat{\omega}_{n+1}(k,t)| \leqslant C_{n+1}(1 \wedge |k|^{-\gamma})$,

$$C_{n+1} \leqslant C_0 + KC_n^2 t^{1/r} + C_f t. \tag{19}$$

Let us show that the sequence (C_n) is bounded for sufficiently small $t > 0$. For small t the quadratic equation

$$Kt^{1/r}x^2 - x + C_0 + C_f t = 0$$

has two real roots. It is easily verified that if

$$c = \frac{1 - \sqrt{1 - 4Kt^{1/r}(C_0 + C_f t)}}{2Kt^{1/r}}$$

is the smallest root then the segment $[0, c]$ is mapped into itself under the map $x \mapsto Kt^{1/r}x^2 + C_0 + C_f t$. Besides that, inequality $\sqrt{1+x} < 1 + x/2$ which is true for $|x| < 1$ implies that $0 < C_0 < c$. Hence, $C_n \leqslant c$ for all n.

Now let us estimate the difference between two successive approximations obtained according to (16):

$$\begin{aligned}
&|\hat{\omega}_{n+1}(k,t) - \hat{\omega}_n(k,t)| \\
&\quad \leqslant \frac{1}{2\pi}\int_0^t e^{-\nu|k|^2(t-s)} \int_{\mathbb{R}^2} |\hat{\omega}_n(l,\ s)||\hat{\omega}_n(k-l,\ s) - \hat{\omega}_{n-1}(k-l,\ s)|\frac{|\langle k,\ l^{\perp}\rangle|}{|l|^2} dlds \\
&\qquad + \frac{1}{2\pi}\int_0^t e^{-\nu|k|^2(t-s)} \int_{\mathbb{R}^2} |\hat{\omega}_n(l,\ s) - \hat{\omega}_{n-1}(l,\ s)||\hat{\omega}_{n-1}(k-l,\ s)|\frac{|\langle k,\ l^{\perp}\rangle|}{|l|^2} dlds.
\end{aligned} \tag{20}$$

Let $|\hat{\omega}_{n+1}(k,t) - \hat{\omega}_n(k,t)| \leqslant \Delta_n(1 \wedge |k|^{-\gamma})$. Then estimates involving Lemma 1 analogous to those derived above show that one may choose

$$\Delta_{n+1} \leqslant 2Kt^{1/r}c\Delta_n.$$

So, for some $\tau > 0$ and $t < \tau$ the series $\sum_{n=1}^{\infty}\Delta_n$ is convergent. Hence $\hat{\omega}_n$ is a Cauchy sequence with respect to the norm $|f(\cdot,\cdot)|_{\gamma} = \sup_{k,t}\frac{f(k,t)}{1\wedge|k|^{-\gamma}}$ and converges to some limiting function. Passing to the limit in (16), we have that this limiting function is a solution of (6) and hence coincides with $\hat{\omega}$. So, $|\hat{\omega}|_{\gamma} \leqslant \limsup|\hat{\omega}_n|_{\gamma} \leqslant c$. It is easy to verify along the same lines that the family of functions $(\hat{\omega}_n)$ is equicontinuous on $[0, \tau]$ with respect to $|\cdot|_{\gamma}$. Consequently, the limiting function $\hat{\omega}$ is continuous with respect to this norm. Lemma 2 is proved. □

Coming back to the proof of Theorem 3, let us denote $\hat{\omega}^{(1)} = \Re\hat{\omega}$, $\hat{\omega}^{(2)} = \Im\hat{\omega}$ and rewrite the system (6) as

$$\frac{\partial \hat{\omega}^{(1)}(k,\ t)}{\partial t} = -\nu|k|^2\hat{\omega}^{(1)}(k,\ t) + \frac{1}{2\pi}\int\limits_{\mathbb{R}^2}[\hat{\omega}^{(1)}(l,\ t)\hat{\omega}^{(1)}(k-l,\ t) - \hat{\omega}^{(2)}(l,\ t)\hat{\omega}^{(2)}(k-l,\ t)] \times \frac{\langle k,\ l^{\perp}\rangle}{|l|^2}dl + f^{(1)}(k,t), \tag{21}$$

$$\frac{\partial \hat{\omega}^{(2)}(k,\ t)}{\partial t} = -\nu|k|^2\hat{\omega}^{(2)}(k,\ t) + \frac{1}{2\pi}\int\limits_{\mathbb{R}^2}[\hat{\omega}^{(1)}(l,\ t)\hat{\omega}^{(2)}(k-l,\ t) + \hat{\omega}^{(1)}(l,\ t)\hat{\omega}^{(2)}(k-l,\ t)] \times \frac{\langle k,\ l^{\perp}\rangle}{|l|^2}dl + f^{(2)}(k,t), \tag{22}$$

Theorem 1 implies that there exists a nondecreasing linear function $E(t)$ such that

$$\|\omega(\cdot,t)\|_{L^1} \leqslant E(t), t \geqslant 0, \tag{23}$$

$$\|\omega(\cdot,t)\|_{L^\infty} \leqslant E(t), t \geqslant 0, \tag{24}$$

$$|\hat{\omega}_0|_\gamma < E(0). \tag{25}$$

Then

$$\|\hat{\omega}(\cdot,t)\|_{L^2} = \|\omega(\cdot,t)\|_{L^2} \leqslant \sqrt{\|\omega(\cdot,t)\|_{L^1}\|\omega(\cdot,t)\|_{L^\infty}} \leqslant E(t), \tag{26}$$

$$\|\hat{\omega}(\cdot,t)\|_{L^\infty} \leqslant \frac{1}{2\pi}\|\omega\|_{L^1} \leqslant E(t). \tag{27}$$

Results from [6] imply that if there is no forcing term then function E may be chosen constant.

Denoting $D(t) = D_{K_0}(t) = E(t)K_0^\gamma$ for $K_0 > 1$, we get $|\hat{\omega}(k,t)| \leqslant D_{K_0}(t)(1 \wedge |k|^{-\gamma})$ for all $t \geqslant 0$ and $|k| \leqslant K_0$.

In order to show that this inequality is fulfilled also for all the other values of k let us assume that, on the contrary, $|\hat{\omega}(\cdot,t)|_\gamma > D_{K_0}(t)$ at some time $t \in (0,\tau]$. Let t_1 be the infimum of such times. Since $|\hat{\omega}(\cdot,t)|_\gamma$ depends on t continuously, we have $|\hat{\omega}(\cdot,t)|_\gamma \leqslant D_{K_0}(t)$ when $t \in [0,t_1]$.

If $t \leqslant t_1$, $|k| > K_0$, $i \in \{1,2\}$ $|\hat{\omega}^{(i)}(k,t)| \geqslant D_{K_0}(t)|k|^{-\gamma}/2$, then $\frac{d\hat{\omega}^{(i)}(k,t)}{dt} = -\operatorname{sgn}\hat{\omega}^{(i)}(k,t)$. Indeed (for definiteness we suppose $\hat{\omega}^{(i)}(k,t) > 0$ without loss of generality), Lemma 1 and (26) imply

$$\frac{d\hat{\omega}^{(i)}(k,t)}{dt} < -\nu|k|^{2-\gamma}\frac{D_{K_0}(t)}{2} + 2D_{K_0}^2(t)Q|k|^{1-\gamma}\sqrt{1+\ln|k|} + C_f|k|^{-\gamma},$$

i.e. the derivative is negative for sufficiently large K_0 and $|k| > K_0$. This contradicts the assumption made, because for all t in the time interval under consideration $|\hat{\omega}^{(i)}(k,t)| \leqslant D_{K_0}(t)/\sqrt{2}$ and $|\hat{\omega}(\cdot,t)|_\gamma \leqslant D_{K_0}(t)$ for $t \in [0,\tau]$.

Using Lemma 2 we always can continue the solution continuously in time on a time interval of positive length. On this interval we can apply again the estimate $|\hat{\omega}(\cdot,t)|_\gamma \leqslant D(t)$. Iterating this procedure we obtain this estimate for all $t \in \mathbb{R}_+$. The theorem is proved. □

Proof of Theorem 4. Consider the function

$$\hat{v}(k,t) = \hat{\omega}(k,t)e^{\alpha|k|}.$$

It suffices to show that $|v(\cdot,t)|_\gamma \leqslant D(t)$. To this end we rewrite (6) as

$$\frac{\partial \hat{v}(k,\ t)}{\partial t} = -\nu|k|^2\hat{v}(k,\ t) + \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\alpha(|l|+|k-l|-|k|)}\hat{v}(l,\ t)\hat{v}(k-l,\ t)\frac{\langle k,\ l^\perp\rangle}{|l|^2}dl + \hat{f}(k,t)e^{-\alpha|k|}.$$

Since $|k| < |l| + |k-l|$, we have $e^{-\alpha(|l|+|k-l|-|k|)} < 1$, and Theorem 4 may be proved by the literal repetition of the proof of Theorem 3. □

Proof of Theorem 5. Consider the function

$$\hat{v}(k,t) = \hat{\omega}(k,t)e^{t\alpha|k|}.$$

It suffices to show that $|v(\cdot,t)|_\gamma \leqslant D(t)$. We rewrite (6) as

$$\frac{\partial \hat{v}(k,\ t)}{\partial t} = -\nu|k|^2\hat{v}(k,\ t) + t\alpha|k|\hat{v}(k,\ t) + \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-t\alpha(|l|+|k-l|-|k|)}\hat{v}(l,\ t)\hat{v}(k-l,\ t)\frac{\langle k,\ l^\perp\rangle}{|l|^2}dl + \hat{f}(k,t)e^{-t\alpha|k|}.$$

Since $e^{-t\alpha(|l|+|k-l|-|k|)} < 1$ and for sufficiently small t the term $t\alpha|k|\hat{v}(k,\ t)$ is small compared to $\nu|k|^2\hat{v}(k,\ t)$ for $|k| > K_0$ and sufficiently large K_0, the proof of Theorem 5 may be obtained by an obvious modification of the proof of Theorem 3. □

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References

1. Ben-Artzi, M.: Global solutions of two-dimensional Navier–Stokes and Euler equations. Arch. Ration. Mech. Anal. **128**(4), 329–358 (1994)
2. Biagioni, H.A., Gramchev, T.: On the 2D Navier-Stokes equation with singular initial data and forcing term. Mat. Contemp. **10**, 1–20 (1996)
3. Constantin, P.: Near identity transformations for the Navier–Stokes equations. In: Friedlander S. (ed.) et al. *Handbook of mathematical fluid dynamics*. Vol. II. Amsterdam: North-Holland, 2003, pp. 117–141
4. Foias, C., Temam, R.: Gevrey class regularity for the solutions of the Navier–Stokes equations. J. Funct. Anal. **87**(2), 359–369 (1989)
5. Giga, Y., Matsui, S., Sawada, O.: Global existence of two-dimensional Navier-Stokes flow with nondecaying initial velocity. J. Math. Fluid Mech. **3**(3), 302–315 (2001)
6. Giga, Y. Miyakawa, T., Osada, H.: Two-dimensional Navier-Stokes flow with measures as initial vorticity. Arch. Ration. Mech. Anal. **104**(3), 223–250 (1988)
7. Kato, T.: The Navier-Stokes equation for an incompressible fluid in $\mathbb{R}^2$ with a measure as the initial vorticity. Differ. Integral Eq. **7**(3–4), 949–966 (1994)

8. Ladyzhenskaya, O.A.: The sixth Millenium problem: Navier–Stokes equations, existence and smoothness. Uspekhi Mat. Nauk **58**(2(350)), 45–78 (2003)
9. Mattingly, J.C., Sinai, Ya.G.: An elementary proof of the existence and uniqueness theorem for the Navier–Stokes equations. Commun. Contemp. Math. **1**(4), 497–516 (1999)
10. McGrath F.J.: Nonstationary plane flow of viscous and ideal fluids. Arch. Ration. Mech. Anal. **27**, 329–348 (1967)
11. Oliver, M., Titi E.S.: Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in $\mathbf{R}^n$. J. Funct. Anal. **172**(1), 1–18 (2000)
12. Reed, M., Simon, B.: *Methods of modern mathematical physics. II: Fourier analysis self- adjointness*. New York - San Francisco - London: Academic Press, a subsidiary of Harcourt Brace Jovanovich Publishers. XV, 1975

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