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Stable Bundles on Non-Kähler Elliptic Surfaces

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Abstract: In this paper, we study the moduli spaces $\mathcal{M}_{\delta,c_2}$ of stable rank-2 vector bundles on non-Kähler elliptic surfaces, thus giving a classification of these bundles; in the case of Hopf and Kodaira surfaces, these moduli spaces admit the structure of an algebraically completely integrable Hamiltonian system.

1. Introduction

Vector bundles on elliptic fibrations have been extensively studied over the past fifteen years; in fact, there is by now a well-understood theory for projective elliptic surfaces (see, for example, [D, F1, FMW]). However, not very much is known about the non-Kähler case. In this article, we partly remedy this problem by examining the stability properties of holomorphic rank-2 vector bundles on non-Kähler elliptic surfaces; their existence and classification are investigated in [BrMo1, BrMo2]. One of the motivations for the study of bundles on non-Kähler elliptic fibrations comes from recent developments in superstring theory, where six-dimensional non-Kähler manifolds occur in the context of $\mathcal{N} = 1$ supersymmetric heterotic and type II string compactifications with non-vanishing background H-field; in particular, all the non-Kähler examples appearing in the physics literature so far are non-Kähler principal elliptic fibrations (see [BBDG, CCFLMZ, GP] and the references therein). The techniques developed here and in [BrMo1, BrMo2] can also be used to study holomorphic vector bundles of arbitrary rank on higher dimensional non-Kähler elliptic and torus fibrations.

A minimal non-Kähler elliptic surface X is a Hopf-like surface that admits a holomorphic fibration $\pi : X \to B$, over a smooth connected compact curve B, whose smooth fibres are isomorphic to a fixed smooth elliptic curve T; the fibration π can have at most

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a finite number of singular fibres, in which case, they are isogeneous to multiples of T. More precisely, if the surface X does not have multiple fibres, then it is the quotient of a complex surface by an infinite cyclic group (see Example 3.1); multiple fibres can then be introduced by performing a finite number of logarithmic transformations on its relative Jacobian $J(X)$. To study bundles on X, a natural operation is restriction to the smooth fibres of π; this gives rise to an important invariant, called the *spectral curve* or *cover*, which is an effective divisor on $J(X)$ that encodes the holomorphic type of the bundle over each smooth fibre of π.

Consider the moduli space $\mathcal{M}_{\delta,c_2}$ of stable holomorphic rank-2 vector bundles on X with fixed determinant δ and second Chern class c_2. This moduli space can be identified, via the Kobayashi-Hitchin correspondence, with the moduli space of gauge-equivalence classes of Hermitian-Einstein connections in the fixed differentiable rank-2 vector bundle determined by δ and c_2 (see, for example, [Bh, LT]). In particular, if the determinant δ is the trivial line bundle $\mathcal{O}_X$, then there is a one-to-one correspondence between $\mathcal{M}_{\mathcal{O}_X,c_2}$ and the moduli space of $SU(2)$-instantons, that is, antiselfdual connections. Note that the determinant line bundle δ induces an involution i_δ of the relative Jacobian $J(X)$; furthermore, the spectral cover of any bundle in $\mathcal{M}_{\delta,c_2}$ is invariant with respect to this involution, thus descending to an effective divisor on the ruled surface $\mathbb{F}_\delta := J(X)/i_\delta$ called the *graph* of the bundle. We can then define a map

$$G : \mathcal{M}_{\delta,c_2} \to \mathrm{Div}(\mathbb{F}_\delta)$$

that associates to each stable vector bundle its graph in $\mathrm{Div}(\mathbb{F}_\delta)$, called the *graph map*. In [BH, Mo], the stability properties of vector bundles on Hopf surfaces were studied by analysing the image and the fibres of this map; in particular, it was shown [Mo] that the moduli spaces admit a natural Poisson structure with respect to which the graph map is a Lagrangian fibration whose generic fibre is an abelian variety: the map G admits an algebraically completely integrable system structure. In this paper, we adopt this approach to study stable vector bundles on arbitrary non-Kähler surfaces.

The article is organised as follows. We begin with a brief review of some existence and classification results for holomorphic vector bundles on non-Kähler elliptic surfaces that were proven in [BrMo1, BrMo2]. In the third section, we obtain explicit conditions for the stability of rank-2 vector bundles: we show that unfiltrable bundles are always stable and then classify the destabilising bundles of filtrable bundles. The moduli spaces $\mathcal{M}_{\delta,c_2}$ are studied in the last section. We first prove that these spaces are smooth on an open dense subset consisting of vector bundles that are regular on every fibre of π (on a smooth elliptic curve, a bundle of degree zero is said to be *regular* if its group of automorphisms is of the smallest possible dimension). However, for Hopf and Kodaira surfaces, the moduli spaces are also smooth at points that are not regular; in this case, the moduli are smooth complex manifolds of dimension $4c_2 - c_1^2(\delta)$. Then, we determine the image of the graph map; for simplicity, we focus our presentation on non-Kähler elliptic surfaces without multiple fibres, but similar results hold in the multiple fibre case. Furthermore, we give an explicit description of the fibres of the graph map, which follows immediately from the classification results of [BrMo2, Mo] and the stability conditions of the third section; in particular, the generic fibre at a graph $\mathcal{G} \in Div(\mathbb{F}_\delta)$ is isomorphic to a finite number of copies of a Prym variety associated to $\mathcal{G}$. We conclude by noting that for Kodaira surfaces the graph map is also an algebraically completely integrable Hamiltonian system, with respect to a given symplectic structure on $\mathcal{M}_{\delta,c_2}$.

2. Holomorphic Vector Bundles

Let $X \stackrel{\pi}{\to} B$ be a minimal non-Kähler elliptic surface, with B a smooth compact connected curve; it is well-known that $X \stackrel{\pi}{\to} B$ is a quasi-bundle over B, that is, all the smooth fibres are pairwise isomorphic and the singular fibres are multiples of elliptic curves [K, Br]. Let T be the general fibre of π, which is an elliptic curve, and denote its dual T^* (we fix a non-canonical identification $T^* := \mathrm{Pic}^0(T)$); in this case, the relative Jacobian of $X \stackrel{\pi}{\to} B$ is simply

$$J(X) = B \times T^* \stackrel{p_1}{\to} B$$

(see, for example, [K, BPV, Br]) and X is obtained from $J(X)$ by a finite number of logarithmic transformations [K, BPV, BrU]. In addition, if the fibration π has multiple fibres, then one can associate to X a principal T-bundle $\pi' : X' \to B'$ over an m-cyclic covering $\varepsilon : B' \to B$, where the integer m depends on the multiplicities of the singular fibres; note that the map ε induces natural m-cyclic coverings $J(X') \to J(X)$ and $\psi : X' \to X$.

To study bundles on X, one of our main tools is restriction to the smooth fibres of the fibration $\pi : X \to B$. It is important to point out that since X is non-Kähler, the restriction of *any* vector bundle on X to a smooth fibre of π *always* has trivial first Chern class [BrMo1]. Therefore, a vector bundle E on X is semistable on the generic fibre of π; in fact, its restriction to a fibre $\pi^{-1}(b)$ is unstable on at most an isolated set of points $b \in B$; these isolated points are called the *jumps* of the bundle. Furthermore, there exists a divisor S_E in the relative Jacobian of X, called the *spectral curve* or *cover* of the bundle, that encodes the isomorphism class of the bundle E over each smooth fibre of π; for a detailed description of this divisor, we refer the reader to [BrMo1]. We should note that, if the fibration π has multiple fibres, then the spectral cover S_E of E is actually defined as the projection in $J(X)$ of the spectral cover $S_{\psi^*E} \subset J(X')$ of ψ^*E, where $\psi : X' \to X$ is the m-cyclic covering defined above.

2.1. Line bundles. The spectral cover of a line bundle L on X is a section Σ of $J(X)$ such that the restriction of L to any smooth fibre $\pi^{-1}(b)$ of π is isomorphic to the line bundle Σ_b of degree zero on $T \cong \pi^{-1}(b)$. Conversely, given any section Σ of $J(X)$, there exists at least one line bundle on X with spectral cover Σ [BrMo1]. Before giving a classification of line bundles on X, we fix some notation. Suppose that π has a multiple fibre mF over the point b in B; the line bundle associated to the divisor F of X is then such that $(\mathcal{O}_X(F))^m = \mathcal{O}_X(mF) = \pi^*\mathcal{O}_B(b)$. Let P_2 be the subgroup of $\mathrm{Pic}(X)$ generated by $\pi^*\mathrm{Pic}(B)$ and the $\mathcal{O}_X(T_i)'$, where $m_1T_1, \ldots, m_rT_r$ are the multiple fibres (if any) of X; we then have [BrMo1]:

Proposition 2.1. *Let Σ be a section of $J(X)$. Then, the set of all line bundles on X with spectral cover Σ is a principal homogeneous space over P_2.*

2.2. Rank-2 vector bundles. Consider a rank-2 vector bundle E on X; its *discriminant* is then defined as

$$\Delta(E) := \frac{1}{2}\left(c_2(E) - \frac{c_1(E)^2}{4}\right).$$

In this case, the spectral curve of E is a divisor S_E in $J(X)$ of the form

$$S_E := \left(\sum_{i=1}^{k} \{x_i\} \times T^* \right) + \overline{C},$$

where $\overline{C}$ is a bisection of $J(X)$ and $x_1, \cdots, x_k$ are points in B that correspond to the jumps of E. Let δ be the determinant line bundle of E. It then defines the following involution on the relative Jacobian $J(X) = B \times T^*$ of X:

$$\begin{aligned} i_\delta : J(X) &\to J(X), \\ (b, \lambda) &\mapsto (b, \delta_b \otimes \lambda^{-1}), \end{aligned}$$

where δ_b denotes the restriction of δ to the fibre $T_b = \pi^{-1}(b)$. By construction, the spectral curve S_E of E is invariant with respect to this involution; in particular, the pair of points lying on the bisection $\overline{C}$ over b is of the form $\{\lambda_b, \delta_b \otimes \lambda_b^{-1}\}$, where λ_b and $\delta_b \otimes \lambda_b^{-1}$ are the subline bundles of $E|_{\pi^{-1}(b)}$. Finally, note that the quotient of $J(X)$ by the involution i_δ is a ruled surface $\mathbb{F}_\delta := J(X)/i_\delta$ over B; let $\eta : J(X) \to \mathbb{F}_\delta$ be the canonical map. The spectral cover S_E of E then descends to a divisor on $\mathbb{F}_\delta$ of the form

$$\mathcal{G}_E := \sum_{i=1}^{k} f_i + A,$$

where f_i is the fibre of the ruled surface $\mathbb{F}_\delta$ over the point x_i and A is a section of the ruling such that $\eta^* A = \overline{C}$.

2.2.1. Bundles without jumps We begin with some properties of filtrable bundles without jumps. Let E be a rank-2 vector bundle on X with determinant δ, and spectral cover $(\Sigma_1 + \Sigma_2)$, where Σ_1 and Σ_2 are sections of $J(X)$; there exists a line bundle $\mathcal{D}$ on X associated to Σ_1 such that E is given by an extension

$$0 \to \mathcal{D} \to E \to \mathcal{D}^{-1} \otimes \delta \to 0. \tag{2.1}$$

Consequently,

$$\Delta(E) = -\frac{1}{8} \left(c_1(\delta) - 2c_1(\mathcal{D})\right)^2. \tag{2.2}$$

Given the above considerations, we have the following results.

Lemma 2.2. *If* $\Sigma_1 = \Sigma_2$*, then* $\Delta(E) = 0$*. Furthermore, the extension* (2.1) *either splits on every fibre of* π *or else it splits on at most a finite number of fibres.*

Proof. Note that $c_1(\mathcal{D}) = c_1(\mathcal{D}^{-1} \otimes \delta)$ because $\Sigma_1 = \Sigma_2$; referring to (2.2), we then have $\Delta(E) = 0$. Suppose that there exists at least one fibre T_{b_0} of π over which the extension is non-trivial; therefore, $h^1(T_{b_0}, \mathcal{D}^{-1} \otimes E) = 1$. But if the extension splits over the fibre T_b, then $h^1(T_b, \mathcal{D}^{-1} \otimes E) = 2$. The upper semi-continuity of the map $b \mapsto h^1(T_b, \mathcal{D}^{-1} \otimes E)$ thus implies that $h^1(T_b, \mathcal{D}^{-1} \otimes E) = 1$ for generic b. □

Lemma 2.3. *If* $\Sigma_1 \neq \Sigma_2$*, then* $|\Sigma_1 \cap \Sigma_2| = 4\Delta(E)$*. In addition, the extension* (2.1) *splits globally whenever* $\Delta(E) = 0$*.*

Proof. Since $\Sigma_1 \neq \Sigma_2$, the sheaf $\pi_*(\mathcal{D}^{-2} \otimes \delta)$ vanishes and the first direct image sheaf $R^1\pi_*(\mathcal{D}^{-2} \otimes \delta)$ is a skyscraper sheaf supported on the points of $\Sigma_1 \cap \Sigma_2$. Therefore, $c_1(R^1\pi_*(\mathcal{D}^{-2} \otimes \delta)) = |\Sigma_1 \cap \Sigma_2|$ and by Grothendieck-Riemann-Roch,

$$|\Sigma_1 \cap \Sigma_2| = -\frac{1}{2}\left(c_1(\delta) - 2c_1(\mathcal{D})\right)^2,$$

which is equal to $4\Delta(E)$ by (2.2). Consequently, if $\Delta(E) = 0$, then $\Sigma_1 \cap \Sigma_2 = \emptyset$; in this case, the extension (2.1) splits on every fibre of π and $R^1\pi_*(\mathcal{D}^2 \otimes \delta^{-1}) = 0$. Hence, the Leray spectral sequence gives $H^1(X, \mathcal{D}^2 \otimes \delta^{-1}) = 0$ and the extension splits globally. □

We have seen that we can associate to any rank-2 vector bundle on X a bisection in $J(X)$. Conversely, given any bisection of $J(X)$, there exists at least one rank-2 vector bundle on X associated to it; if the bisection is smooth, the bundles that correspond to it are classified as follows (see [BrMo2] for precise statements).

Theorem 2.4. *Fix a line bundle δ on X and its associated involution i_δ of $J(X)$. Let C be a smooth bisection of $J(X)$ that is invariant with respect to this involution; it is then a double cover of B of genus $4\Delta(2, c_1, c_2) + 2g - 1$. The set of all rank-2 vector bundles on X with spectral cover C and determinant δ is then parametrised by a finite number of copies of the Prym variety $Prym(C/B)$ associated to the double cover $C \to B$.*

2.2.2. Bundles with jumps. Consider a rank-2 vector bundle E on X with determinant δ that has a jump of multiplicity μ over the smooth fibre $T = \pi^{-1}(x_0)$. The restriction of E to the fibre T is then of the form $\lambda \oplus (\lambda^* \otimes \delta_{x_0})$, for some $\lambda \in \mathrm{Pic}^{-h}(T)$, $h > 0$; the integer h is called the *height* of the jump at T. Moreover, up to a multiple of the identity, there is a *unique* surjection $E|_T \to \lambda$, which defines a canonical elementary modification of E that we denote $\bar{E}$; this elementary modification is called *allowable* [F2]. Therefore, we can associate to E a finite sequence $\{\bar{E}_1, \bar{E}_2, \ldots, \bar{E}_l\}$ of allowable elementary modifications such that $\bar{E}_l$ is the only element of the sequence that does not have a jump at T.

Let us now assume that π has a multiple fibre m_0T_0. One can then associate to X an elliptic quasi-bundle $\pi' : X' \to B'$, over an m_0-cyclic covering $\varepsilon : B' \to B$, such that $T_0' := \psi^{-1}(T_0) \subset X'$ is a smooth fibre of π', where $\psi : X' \to X$ is the m_0-cyclic covering induced by ε. Given this, we say that E has a jump over T_0 if and only if the restriction of ψ^*E to the fibre T_0' is unstable. Naturally, the height and multiplicity of the jump of E over T_0 are defined as the height and multiplicity of the jump of ψ^*E over T_0'. We can now define the following important invariants.

Definition 2.5. *Let T be a smooth fibre of π. Suppose that the vector bundle E has a jump over T and consider the corresponding sequence of allowable elementary modifications $\{\bar{E}_1, \bar{E}_2, \ldots, \bar{E}_l\}$. The integer l is called the length of the jump at T. The jumping sequence of T is defined as the set of integers $\{h_0, h_1, \ldots, h_{l-1}\}$, where $h_0 = h$ is the height of E and h_i is the height of $\bar{E}_i$, for $0 < i \leq l - 1$.*

If the vector bundle E has a jump over a multiple fibre m_0T_0 of π, we define the length and jumping sequence of T_0 to be the length and jumping sequence of the jump of ψ^*E over the smooth fibre $T_0' = \psi^{-1}(T_0)$ of ψ, where $\psi : X' \to X$ is the m_0-cyclic covering defined above.

Note that if a vector bundle E jumps over a smooth fibre T of π, with multiplicity μ and jumping sequence $\{h_0, \ldots, h_{l-1}\}$, then $\mu = \sum_{i=1}^{l-1} h_i$. For a detailed description of jumps, we refer the reader to [Mo, BrMo2]; moreover, the basic properties of elementary modifications can be found, for example, in [F2]. We finish this section by stating the following existence result [BrMo2]: if X does not have multiple fibres and S is a spectral cover in $J(X)$ (that may have vertical components), then one can associate to S at least one rank-2 vector bundle on X.

3. Stable Rank Two Bundles

3.1. Degree and stability. The degree of a vector bundle can be defined on any compact complex manifold M. Let $d = \dim_{\mathbb{C}} M$. A theorem of Gauduchon's [G] states that any hermitian metric on M is conformally equivalent to a metric, called a *Gauduchon metric*, whose associated (1,1) form ω satisfies $\partial\bar{\partial}\omega^{d-1} = 0$. Suppose that M is endowed with such a metric and let L be a holomorphic line bundle on M. The *degree of L with respect to ω* is defined [Bh], up to a constant factor, by

$$\deg L := \int_M F \wedge \omega^{d-1},$$

where F is the curvature of a hermitian connection on L, compatible with $\bar{\partial}_L$. Any two such forms F differ by a $\partial\bar{\partial}$-exact form. Since $\partial\bar{\partial}\omega^{d-1} = 0$, the degree is independent of the choice of connection and is therefore well defined. This notion of degree is an extension of the Kähler case. If M is Kähler, we get the usual topological degree defined on Kähler manifolds; but in general, this degree is not a topological invariant, for it can take values in a continuum (see below).

Having defined the degree of holomorphic line bundles, we define the *degree* of a torsion-free coherent sheaf $\mathcal{E}$ on M by

$$\deg(\mathcal{E}) := \deg(\det \mathcal{E}),$$

where $\det \mathcal{E}$ is the determinant line bundle of $\mathcal{E}$, and the *slope of $\mathcal{E}$* by

$$\mu(\mathcal{E}) := \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E}).$$

The notion of stability then exists for any compact complex manifold:

A torsion-free coherent sheaf $\mathcal{E}$ on M is **stable** *if and only if for every coherent subsheaf $\mathcal{S} \subset \mathcal{E}$ with $0 < rk(\mathcal{S}) < rk(\mathcal{E})$, we have $\mu(\mathcal{S}) < \mu(\mathcal{E})$.*

Remark. With this definition of stability, many of the properties from the Kähler case hold. In particular, all line bundles are stable; for rank two vector bundles on a surface, it is sufficient to verify stability with respect to line bundles. Finally, if a vector bundle E is stable, then $H^0(M, End(E)) = \mathbb{C}$.

Example 3.1. Let $X \xrightarrow{\pi} B$ be a non-Kähler principal elliptic bundle over a curve B of genus g and with fibre T. The surface X is then isomorphic to a quotient of the form

$$X = \Theta^*/\langle\tau\rangle,$$

where Θ is a line bundle on B with positive Chern class d, Θ^* is the complement of the zero section in the total space of Θ, and $\langle\tau\rangle$ is the multiplicative cyclic group generated by a fixed complex number $\tau \in \mathbb{C}$, with $|\tau| > 1$. In this case, the degree of torsion line bundles can be computed explicitly (for details, see [T]). Every line bundle $L \in \mathrm{Pic}^\tau(X)$

decomposes uniquely as $L = H \otimes L_\alpha$, for $H \in \cup_{i=0}^{d-1}\text{Pic}(B)$ and $\alpha \in \mathbb{C}^*$. Taking into account this decomposition, the degree of L is given by

$$\deg L = c_1(H) - \frac{d}{\ln|\tau|}\ln|\alpha|.$$

In particular, $\deg(\pi^* H) = \deg H$, for all $H \in \text{Pic}(B)$.

We end this example by observing that if X has a multiple fibre $m_0 T_0$, then we have $\deg(\mathcal{O}_X(T_0)) = 1/m_0$.

3.2. Stable vector bundles. Let $X \xrightarrow{\pi} B$ be a non-Kähler elliptic surface with multiple fibres $m_1 T_1, \ldots, m_r T_r$ (if any); the canonical bundle of X is then $K_X = \pi^* K_B \otimes \mathcal{O}_X\left(\sum_{i=1}^r (m_i - 1)T_i\right)$, giving $\omega_{X/B} = \mathcal{O}_X\left(\sum_{i=1}^r (m_i - 1)T_i\right)$ as the dualising sheaf of π. Note that $\deg \omega_{X/B} = r - \sum_{i=1}^r 1/m_i \geq 0$ (see Example 3.1). Fix a rank-2 vector bundle E on X and let δ be its determinant line bundle; there exists a sufficient condition on the spectral cover of E that ensures its stability:

Proposition 3.2. *Suppose that the spectral cover of E includes an irreducible bisection $\overline{C}$ of $J(X)$. Then E is stable.*

Proof. Suppose that there exists a line bundle $\mathcal{D}$ on X that maps into E. After possibly tensoring $\mathcal{D}$ by the pullback of a suitable line bundle on B, the rank-2 bundle E is then given as an extension

$$0 \to \mathcal{D} \to E \to \mathcal{D}^{-1} \otimes \delta \otimes I_Z \to 0, \tag{3.1}$$

where $Z \subset X$ is a locally complete intersection of codimension 2. In fact, Z is the set of points $\{x_1, \ldots, x_k\}$ corresponding to the fibres $\pi^{-1}(x_i)$ over which E is unstable. Let Σ_1 and Σ_2 be the sections of $J(X)$ determined by the line bundles $\mathcal{D}$ and $\mathcal{D}^{-1} \otimes \delta$, respectively. The extension (3.1) then implies $\overline{C} = \Sigma_1 + \Sigma_2$. □

Consequently, the spectral covers of unstable bundles include bisections of the form $\overline{C} = (\Sigma_1 + \Sigma_2)$, where Σ_1 and Σ_2 are sections of the Jacobian surface.

Proposition 3.3. *Suppose that the spectral cover of E is given by*

$$\left(\sum_{i=1}^k \{x_i\} \times T^*\right) + (\Sigma_1 + \Sigma_2).$$

Then, there exist line bundles K_1 and K_2 on X (corresponding to the sections Σ_1 and Σ_2, respectively) such that the set of all line bundles that map non-trivially to E is given by

$$\left\{K_j \otimes \pi^* H \otimes \mathcal{O}_X\left(\sum_{i=1}^r a_i T_i\right) \;:\; H \in \text{Pic}^{\leq 0}(B) \text{ and } a_i \leq 0\right\}.$$

Also, E is stable if and only if $\deg K_1$ *and* $\deg K_2$ *are both smaller than* $\deg \delta/2$. *Note that if $\Sigma_1 = \Sigma_2$, then $K_1 = K_2$.*

The line bundles K_1 and K_2 are called the **destabilising line bundles of** *E.*

Proof. Let $\mathcal{D}$ be a line bundle that corresponds to the section Σ_1 and suppose that there exists a non-trivial map $\mathcal{D} \to E$. We begin by assuming that E is regular on the generic fibre of π. In this case, the direct image sheaf $\pi_*(\mathcal{D}^{-1} \otimes E)$ is a line bundle on B, say L, of positive degree. Set $K = \mathcal{D} \otimes (\pi^* L^{-1})^{-1}$; then, K restricts to Σ_{1b} over the smooth fibres $\pi^{-1}(b)$ of π and $\pi_*(K^{-1} \otimes E) \cong \mathcal{O}_B$. However, any line bundle $\mathcal{D}'$ on X corresponding to Σ_1 can be written as $K \otimes \pi^* H \otimes \mathcal{O}_X\left(\sum_{i=1}^r a_i T_i\right)$, for some $H \in \mathrm{Pic}(B)$ and integers $0 \le a_i \le m_i - 1$. Moreover, one can easily show that $\pi_*\left(\mathcal{F} \otimes \mathcal{O}_X\left(\sum_{i=1}^r a_i T_i\right)\right) = \pi_*(\mathcal{F})$, for any locally free sheaf $\mathcal{F}$ on X. Consequently, if $\mathcal{D}'$ also maps into E, then the line bundle $\pi_*(\mathcal{D}'^{-1} \otimes E) \cong H^{-1}$ has a non-trivial section, implying that $H \in \mathrm{Pic}^{\le 0}(B)$. Note that $\pi_*(K'^{-1} \otimes E) \cong \mathcal{O}_B$ for any line bundle K' of the form $K \otimes \mathcal{O}_X\left(\sum_{i=1}^r b_i T_i\right)$, where $0 \le b_i \le m_i - 1$. But, the destabilising bundle K_1 is the line bundle associated to Σ_1 of maximal degree that maps into E; we therefore set $K_1 = K \otimes \mathcal{O}_X\left(\sum_{i=1}^r (m_i - 1) T_i\right) = K \otimes \omega_{X/B}$. Clearly, any line bundle corresponding to Σ_1 that maps into E can be written as $K_1 \otimes \pi^* H \otimes \mathcal{O}_X\left(\sum_{i=1}^r a_i T_i\right)$, for $H \in \mathrm{Pic}^{\le 0}(B)$ and integers $a_i \le 0$.

We now assume that E is not regular on the generic fibre of π. The direct image sheaf $\pi_*(\mathcal{D}^{-1} \otimes E)$ is then a rank-2 vector bundle on B, say $\mathcal{F}$; it must have a subline bundle L such that $\mathcal{F}/L$ is torsion free. If we set $K_1 = \mathcal{D} \otimes (\pi^* L^{-1})^{-1} \otimes \omega_{X/B}$, then $\pi_*({K_1}^{-1} \otimes E)$ has a nowhere vanishing section and, as above, any line bundle induced by Σ_1 that maps into E is of the required form. □

In fact, the destabilising line bundles of filtrable bundles without jumps can be described explicitly as follows:

Proposition 3.4. *Let E be a holomorphic rank-2 vector bundle on X with invariants $\det(E) = \delta$, $c_2(E) = c_2$, and spectral cover $(\Sigma_1 + \Sigma_2)$, where Σ_1 and Σ_2 are sections of $J(X)$. Let K_1 be the destabilising line bundle of E induced by Σ_1; there is an extension*

$$0 \to K_1 \to E \to {K_1}^{-1} \otimes \delta \to 0. \tag{3.2}$$

- *(i) If $\Sigma_1 = \Sigma_2$ and the extension is trivial on every fibre of π, then there exists a line bundle H_- on B of non-positive degree d_0 such that ${K_1}^2 = \delta \otimes \pi^*(H_-) \otimes \omega_{X/B}$.*
- *(ii) If $\Sigma_1 = \Sigma_2$ and the extension splits only on a finite number $n \ge 0$ of fibres of π, then ${K_1}^2 = \delta \otimes \pi^*(H_+) \otimes \omega_{X/B}$, where H_+ is a line bundle of degree n on B that is trivial whenever $n = 0$.*
- *(iii) If $\Sigma_1 \ne \Sigma_2$ and the extension is non-trivial on a finite number $n \le 4\Delta(E)$ of fibres, then the second destabilising bundle of E is $K_2 = {K_1}^{-1} \otimes \delta \otimes \pi^*(H_-) \otimes \omega_{X/B}$, where H_- is a line bundle of non-positive degree $-n$ on B that is trivial for $n = 0$.*

Proof. Let us first assume that the extension (3.2) splits on every fibre of π; note that if $\Sigma_1 \ne \Sigma_2$, then the extension in fact splits globally. If the extension splits globally, then $\pi_*(K_1 \otimes \delta^{-1} \otimes E)$ has a nowhere vanishing global section, implying that the second destabilising bundle of E is $K_2 = {K_1}^{-1} \otimes \delta \otimes \omega_{X/B}$. Otherwise, every subline bundle of $\pi_*(K_1 \otimes \delta^{-1} \otimes E)$ has negative degree; let H_- be its subline bundle of maximal degree $d_0 < 0$. Then, ${K_1}^{-1} \otimes \delta \otimes \pi^* H_- \otimes \omega_{X/B}$ is the destabilising line bundle of E so that it is isomorphic to K_1. This proves (i) and (iii) for $n = 0$.

Next, we suppose that $\Sigma_1 = \Sigma_2$ and that the extension is non-trivial on the generic fibre of π. Note that the restriction of ${K_1}^2 \otimes \delta^{-1}$ is trivial on every fibre of π, implying that ${K_1}^2 \otimes \delta^{-1} = \pi^*(H_+) \otimes \mathcal{O}_X\left(\sum_{i=1}^r a_i T_i\right)$ for a line bundle H_+ on B and integers

$0 \leq a_i \leq m_i - 1$; tensoring the exact sequence (3.2) by ${K_1}^{-1}$ and pushing down to B, we obtain a new exact sequence:

$$0 \to {H_+}^{-1} \to \mathcal{O}_B \to R^1\pi_*({K_1}^{-1} \otimes E) \to {H_+}^{-1} \to 0. \tag{3.3}$$

Suppose that the extension (3.2) splits over n fibres of π (counting multiplicity). Referring to (3.3), the first direct image sheaf $R^1\pi_*({K_1}^{-1} \otimes E)$ is then given by the extension

$$0 \to \mathcal{S} \to R^1\pi_*({K_1}^{-1} \otimes E) \to {H_+}^{-1} \to 0,$$

where $\mathcal{S}$ is a skyscraper sheaf supported on the n points (counting multiplicity) corresponding to these fibres. By Grothendieck-Riemann-Roch,

$$\deg(R^1\pi_*({K_1}^{-1} \otimes E)) = -\frac{1}{2}c_1^2({K_1}^2 \otimes \delta^{-1}) = -\frac{1}{2}c_1^2(\pi^*(H_+)) = 0.$$

The degree of the line bundle H_+ is thus n; clearly, $H_+ = \mathcal{O}_B$ if $n = 0$. Note that by construction, $K_1^{-1} \otimes \delta \otimes \pi^*(H_+) \otimes \mathcal{O}_X \left(\sum_{i=1}^r a_i T_i\right)$ is the destabilising line bundle of E; however, $\pi_* \left(K_1 \otimes \delta^{-1} \otimes \pi^*(H_+)^{-1} \otimes \mathcal{O}_X \left(\sum_{i=1}^r b_i T_i\right) \otimes E\right) = \mathcal{O}_B$, for all integers $0 \leq b_i \leq m_i - 1$. Therefore, $a_i = m_i - 1$ for all $i = 1, \ldots, r$, proving (ii).

Finally, let us assume that $\Sigma_1 \neq \Sigma_2$. If the extension also splits over $m \leq 4\Delta(E)$ fibres of π (counting multiplicity) corresponding to points in $\Sigma_1 \cap \Sigma_2$, then the rank of $R^1\pi_*({K_1}^{-1} \otimes E)$ jumps at these m points; in fact, the first direct image sheaf is given by the extension

$$0 \to \mathcal{O}_B \to R^1\pi_*({K_1}^{-1} \otimes E) \to R^1\pi_*({K_1}^{-2} \otimes \delta) \to 0.$$

Dualising, we get $R^1\pi_*({K_1}^{-1} \otimes E)^* = H_-$, for $H_- \in \mathrm{Pic}(B)$. Let $n := 4\Delta(E) - m$; since the skyscraper sheaf $R^1\pi_*({K_1}^{-2} \otimes \delta)$ is supported on $4\Delta(E)$ points (see the proof of Lemma 2.3), the line bundle H_- has degree $-n$. Furthermore, by relative Serre duality, $\pi_*(K_1 \otimes \delta^{-1} \otimes E) = R^1\pi_*({K_1}^{-1} \otimes E)^* = H_-$; therefore, the second destabilising line bundle of E is $K_2 = {K_1}^{-1} \otimes \delta \otimes \pi^*(H_-) \otimes \omega_{X/B}$ and we are done. □

Recall that, for surfaces X with multiple fibres, the spectral cover of a vector bundle E on X was defined in Sect. 2 in terms of the vector bundle ψ^*E on an m-cyclic covering $\psi : X' \to X$, where X' is an elliptic fibre bundle over an m-cyclic covering $B' \to B$. Keeping this in mind, we now state the main result of the section.

Theorem 3.5. *Consider a filtrable rank-2 vector bundle E on X with determinant δ that has k jumps of lengths $l_1, \ldots, l_k$, respectively; furthermore, suppose that j of them occur over multiple fibres $m_{i_1}T_{i_1}, \ldots, m_{i_j}T_{i_j}$, respectively, for some integer $0 \leq j \leq k$. We set*

$$\nu := \sum_{s=1}^{j} l_s/m_{i_s} + \sum_{t=j+1}^{k} l_t.$$

Let K be one of the destabilising bundles of E. There is an extension

$$0 \to \psi^*K \to \overline{\psi^*E} \to \psi^*(K \otimes \delta^{-1}) \to 0,$$

where $\overline{\psi^*E}$ denotes the vector bundle on X' obtained by performing successive elementary modifications to eliminate the jumps of ψ^*E.

(i) If $\Sigma_1 = \Sigma_2$ and the extension is trivial on every fibre of π', then E is stable if and only if $\nu > d_0 + \deg \omega_{X/B}$, where d_0 is the integer of Proposition 3.4 (i).
(ii) Suppose that $\Sigma_1 = \Sigma_2$ and that the extension splits on only a finite number mn of fibres, then E is stable if and only if $\nu > n + \deg \omega_{X/B}$.
(iii) If $\Sigma_1 \neq \Sigma_2$ and the extension is non-trivial on a finite number mn of fibres of π', then E is stable if and only if $\deg K \in \big(\deg \delta/2 - \nu - n + \deg \omega_{X/B}, \deg \delta/2\big)$.

Proof. Note that any elementary modification of $\psi^* E$ has the same destabilising bundles as $\psi^* E$ (which are the pullbacks to X' of the destabilising bundles of E). Furthermore, the elementary modification $\overline{\psi^* E}$ has determinant $\psi^*\delta \otimes \mathcal{O}_{X'}(-D)$, where $D := \sum_{s=1}^{j} l_s T_{i_s} + (\sum_{t=j+1}^{k} l_t)T$. Applying Proposition 3.4 to the bundle $\overline{\psi^* E}$, we obtain the theorem. □

4. Moduli Spaces

Let X be a non-Kähler elliptic surface and consider a pair (c_1, c_2) in $NS(X) \times \mathbb{Z}$. For a fixed line bundle δ on X with $c_1(\delta) = c_1$, let $\mathcal{M}_{\delta,c_2}$ be the moduli space of stable holomorphic rank-2 vector bundles with invariants $\det(E) = \delta$ and $c_2(E) = c_2$. We define the following positive rational number:

$$m(2, c_1) := -\frac{1}{4} \max \left\{ \sum_1^n \left(\frac{c_1}{2} - \mu_i\right)^2, \mu_1, \ldots, \mu_r \in NS(X), \sum_1^n \mu_i = c_1 \right\}.$$

Note that, for any $c_1 \in NS(X)$, one can choose a line bundle δ on X such that

$$c_1(\delta) \in c_1 + 2NS(X) \text{ and } m(2, c_1) = -\frac{1}{2}\left(\frac{c_1(\delta)}{2}\right)^2; \tag{4.1}$$

moreover, if there exist line bundles a and δ' on X such that $\delta = a^2\delta'$, then there is a natural isomorphism between the moduli spaces $\mathcal{M}_{\delta,c_2}$ and $\mathcal{M}_{\delta',c_2}$, defined by $E \mapsto a \otimes E$. Therefore, if δ' is any other line bundle with Chern class in $c_1 + 2NS(X)$, it induces a moduli space that is isomorphic to $\mathcal{M}_{\delta,c_2}$. However, the advantage of using such a δ is that its Chern class has maximal self-intersection $-8m(2, c_1)$. Hence, we restrict our study to moduli spaces $\mathcal{M}_{\delta,c_2}$ of stable bundles whose determinant δ satisfies (4.1).

4.1. Existence and dimension. A necessary condition for the existence of holomorphic rank-2 vector bundles is $\Delta(2, c_1, c_2) := 1/2\left(c_2 - c_1^2/4\right) \geq 0$ [BaL, Br]. Also, a theorem of Bănică - Le Potier's [BaL] states that there exists a filtrable holomorphic rank-2 vector bundle with Chern classes c_1 and c_2 if and only if $\Delta(2, c_1, c_2) \geq m(2, c_1)$. Given our choice of line bundle δ, any element E of $\mathcal{M}_{\delta,c_2}$ has discriminant

$$\Delta(E) = m(2, c_1) + \frac{1}{2}c_2 \geq 0.$$

Consequently, $c_2 \geq -2m(2, c_1)$; moreover, if $c_2 < 0$, then E unfiltrable. However, if the vector bundle E is unfiltrable, then its spectral cover contains an irreducible bisection; it is then stable by Proposition 3.2. Therefore, if a rank-2 vector bundle has second Chern class $-2m(2, c_1) \leq c_2 < 0$, then it is stable.

Assume that the moduli space $\mathcal{M}_{\delta,c_2}$ is non-empty. Consider one of its elements E; there is a natural splitting of the endomorphism bundle $\mathrm{End}(E) = \mathcal{O}_X \oplus \mathrm{ad}(E)$, where $\mathrm{ad}(E)$ is the kernel of the trace map. By deformation theory, the moduli space has expected dimension $h^1(X; \mathrm{ad}(E)) - h^2(X; \mathrm{ad}(E))$ at E. Since the vector bundle E is assumed to be stable, we have $h^0(X; \mathrm{ad}(E)) = 0$ and the expected dimension of the moduli space is equal to $-\chi(E) = 8\Delta(2, c_1, c_2) - 3\chi(\mathcal{O}_X) = 8\Delta(2, c_1, c_2)$.

4.2. Smoothness. Let us first assume that X is an elliptic fibre bundle over a curve of genus less than 2. Recall that a vector bundle E on a complex manifold X is said to be *good* if and only if $h^2(X; \mathrm{ad}(E)) = 0$, or equivalently, if $h^0(X; \mathrm{ad}(E) \otimes K_X) = 0$ (by Serre duality); furthermore, the moduli space $\mathcal{M}_{\delta,c_2}$ is smooth at E if and only if the vector bundle E is good. Given this, one easily proves the following:

Proposition 4.1. *Let X be a non-Kähler elliptic fibre bundle over a curve B of genus less than 2, that is, X is a Hopf surface or a primary Kodaira surface. The moduli spaces $\mathcal{M}_{\delta,c_2}$ are then smooth of dimension* $8\Delta(2, c_1, c_2)$.

Proof. It is sufficient to prove that every stable bundle on X with Chern classes c_1 and c_2 is good. In this case, the canonical bundle of the surface is $K_X = \pi^*(K_B)$. Since the genus of B is ≤ 1, the canonical bundle is given by $\mathcal{O}_X(-D)$, where D is an effective divisor. There is an inclusion $K_X = \mathcal{O}_X(-D) \subset \mathcal{O}_X$, which in turn induces an inclusion on the space of global sections $H^0(X; \mathrm{ad}\, E \otimes K_X) \subset H^0(X; \mathrm{ad}(E))$. However, the stability of E implies that $h^0(X; \mathrm{ad}(E)) = 0$ and we are done. □

For an arbitrary non-Kähler elliptic surface $X \xrightarrow{\pi} B$, we consider the elements of the moduli space that are regular, that is, vector bundles that are regular on every fibre of π. Note that for such a bundle E, the direct image sheaves $\pi_*(\mathrm{End}(E))$ and $R^1\pi_*(\mathrm{End}(E))$ are dual locally free sheaves of rank two; therefore, by Grothendieck-Riemann-Roch, we have $c_1(\pi_*(\mathrm{End}(E))) = 2ch_2(E)$. Given the natural splitting $\pi_*(\mathrm{End}(E)) = \mathcal{O}_B \oplus \pi_*(\mathrm{ad}(E))$, we conclude that

$$\deg(\pi_*(\mathrm{ad}(E))) = 2ch_2(E).$$

The Leray spectral sequence gives us $h^0(X; \mathrm{ad}(E) \otimes K_X) = h^0(B; \pi_*(\mathrm{ad}(E)) \otimes K_B)$; hence, if the degree of $\pi_*(\mathrm{ad}(E)) \otimes K_B$ is negative, we have $h^0(X; \mathrm{ad}(E) \otimes K_X) = 0$, leading us to the following:

Proposition 4.2. *Let X be a non-Kähler elliptic surface over a base curve B of genus g. Then, if $c_2 - c_1^2/2 > g - 1$, the moduli space $\mathcal{M}_{\delta,c_2}$ is smooth on the open dense subset of regular bundles.* □

Remark. We can also give a sufficient condition for smoothness of the moduli space at points that do not correspond to regular bundles. Consider a stable vector bundle E that is not regular over the fibres of π lying over the points $x_1, \ldots, x_s$ in B. In this case, $\pi_*(\mathrm{End}(E))$ is again a rank-2 vector bundle, but $R^1\pi_*(\mathrm{End}(E))$ is the sum of a rank 2 vector bundle with a skyscraper sheaf supported on the points $x_1, \ldots, x_s$, with multiplicities $\gamma_1, \ldots, \gamma_s$, respectively. Let $\gamma = \sum_i \gamma_i$. Then, one easily verifies that a sufficient condition for smoothness of the moduli space $\mathcal{M}_{\delta,c_2}$ at E is given by

$$c_2 - \frac{c_1^2}{2} > g - 1 + \frac{\gamma}{4}.$$

Note that γ depends not only on the spectral cover of E, but also on the geometry of its jumps.

4.3. The image of the graph map. Fix any pair $(c_1, c_2) \in NS(X) \times \mathbb{Z}$ such that $\Delta(2, c_1, c_2) \geq 0$ and let δ be a line bundle on X such that $m(2, c_1) = -\frac{1}{2}(c_1(\delta)/2)^2$. Referring to Sect. 2.2, this line bundle determines an involution i_δ of the Jacobian surface and there is an associated ruled surface $\mathbb{F}_\delta := J(X)/i_\delta$; the quotient map is denoted $\eta : J(X) \to \mathbb{F}_\delta$. Furthermore, to any rank-2 vector bundle E on X with determinant δ and second Chern class c_2, there corresponds a graph in $\mathbb{F}_\delta$. It was shown in [BrMo1] that these graphs are elements of linear systems in $\mathbb{F}_\delta$ of the form $|\eta_*(B_0) + \mathfrak{b} f|$, where B_0 is the zero section of $J(X)$, $\mathfrak{b}$ is the pullback to X of a line bundle on B of degree c_2, and f is a fibre of the ruled surface.

Let $\mathbb{P}_{\delta,c_2}$ be the set of divisors in $\mathbb{F}_\delta$ of the form $\sum_{i=1}^k f_i + A$, where A is a section and the f_i's are fibres of the ruled surface, that are numerically equivalent to $\eta_*(B_0) + c_2 f$. We have a well-defined map

$$G : \mathcal{M}_{\delta,c_2} \longrightarrow \mathbb{P}_{\delta,c_2}$$

that associates to each vector bundle its graph, called the *graph map*. Let us then describe the image of this map; we begin by noting that it is surjective on the open dense subset of graphs in $\mathbb{P}_{\delta,c_2}$ that correspond to irreducible bisections in $J(X)$. When considering the remaining graphs, we restrict ourselves, for simplicity, to the case where X has no multiple fibres; however, similar results hold if X does have multiple fibres.

Proposition 4.3. *Let $X \xrightarrow{\pi} B$ be a non-Kähler elliptic fibre bundle. Choose an element $c_1 \in NS(X)$ such that $m(2, c_1) = 0$; in this case, $\mathbb{F}_\delta = B \times \mathbb{P}^1$ and the elements of $\mathbb{P}_{\delta,c_2}$ are of the form*

$$\sum_{i=1}^{k}(\{b_i\} \times \mathbb{P}^1) + Gr(F),$$

where $b_1, \ldots, b_k$ are points in B and $Gr(F)$ is the graph of a rational map $F : B \to \mathbb{P}^1$ of degree $c_2 - k$. We then have the following.

(i) For $c_2 = 0$, the moduli space $\mathcal{M}_{\delta,0}$ consists of isomorphism classes of bundles of the form $L \otimes \pi^* E'$, where L is a line bundle on X and E' is a stable rank-2 vector bundle on B.

(ii) Let $\mathcal{S}$ be the set of points λ_0 in T^* such that the degree of any line bundle on X corresponding to the section $B \times \{\lambda_0\}$ in $J(X)$ is congruent to $\deg \delta/2$ *modulo $\mathbb{Z}$. If I is the projection of $\mathcal{S}$ onto $\mathbb{P}^1 = T^*/i_\delta$, then we denote $B \times I$ the set of graphs*

$$\left\{(\{b\} \times \mathbb{P}^1) + (B \times \{\bar{\lambda}\}) \;\middle|\; b \in B \text{ and } \bar{\lambda} \in I\right\}.$$

For $c_2 = 1$, the image of the graph map G is $\mathbb{P}_{\delta,1} \setminus (B \times I)$.

(iii) For $c_2 \geq 2$, the graph map is surjective.

Proof. Consider a graph $\mathcal{G} = \sum_{i=1}^k(\{b_i\} \times \mathbb{P}^1) + Gr(F)$, where $b_1, \ldots, b_k$ are points in B and $Gr(F)$ is the graph of a rational map $F : B \to \mathbb{P}^1$ of degree $c_2 - k$; we denote $\overline{C}$ the bisection of $J(X)$ determined by $Gr(F)$. Referring to Sect. 2, we can construct rank-2 vector bundles on X with graph $\mathcal{G}$; therefore, we only have to determine whether or not at least one of them is stable. Let us fix a bundle E with graph $\mathcal{G}$ and discuss its stability.

Suppose that $c_2 = 0$. Then, $\Delta(E) = 0$ and the map F is constant; moreover, the bisection $\overline{C}$ is reducible and E is a filtrable bundle without jumps. Set $\overline{C} = (B \times \{\lambda_1\}) +$

$(B \times \{\lambda_2\})$, with $\lambda_1, \lambda_2 \subset \mathrm{Pic}^0(T)$, and let K_1 be the destabilising bundle corresponding to λ_1. Referring to Proposition 3.4, if $\lambda_1 \neq \lambda_2$, then the second destabilising bundle of E is $K_2 = K_1{}^{-1} \otimes \delta$; therefore, $\deg K_1 + \deg K_2 = \deg \delta$ and at least one of the destabilising bundles has degree greater or equal to $\deg \delta/2$. Hence, every bundle with $\lambda_1 \neq \lambda_2$ is unstable. Similarly, if $\lambda_1 = \lambda_2$ and the bundle E is a split extension only on a finite number of fibres of π, then $\deg K_1 \geq \deg \delta/2$ implying that E is unstable. Finally, if $\lambda_1 = \lambda_2$ and the bundle E splits on every fibre of π, then there exists a rank-2 vector bundle E' on B such that $E = K_1 \otimes \pi^* E'$; therefore, the bundle E is stable if and only if E' is stable, proving (i).

Now, assume that $c_2 \geq 1$. Recall that the vector bundle E may be unstable only if the bisection $\overline{C}$ is reducible; therefore, suppose that $\overline{C} = \Sigma_1 + \Sigma_2$ for some sections $\Sigma_1, \Sigma_2 \subset J(X)$. If $\Sigma_1 = \Sigma_2$, then the bundle has at least one jump (otherwise, $\Delta(E) = 0$ (see Lemma 2.2), contradicting the fact that $\Delta(E) = c_2 \geq 1$); in this case, E is always stable by Theorem 3.5. If $\Sigma_1 \neq \Sigma_2$, then a stable bundle E can be constructed as follows. Choose a line bundle K corresponding to Σ_1; after possibly tensoring K by an element of P_2, one can assume that $\deg K \in (\deg \delta/2 - k - 2(c_2 - k), \deg \delta/2)$, unless $c_2 = 1$ and the degree of K is congruent to $\deg \delta/2$ modulo $\mathbb{Z}$. Then, consider a regular extension of $K^{-1} \otimes \delta(kT)$ by K and perform k elementary modifications (using a line bundle of degree 1 on T) to introduce the jumps. Note that K is one of the destabilising bundles of E; referring to Theorem 3.5, E is then stable. Finally, if $c_2 = 1$ and $\Sigma_1 \neq \Sigma_2$, then a bundle with graph $\mathcal{G}$ is stable if and only if the degrees of its destabilising bundles are in the interval $(\deg \delta/2 - 1, \deg \delta/2)$; if all bundles corresponding to Σ_1 have degree congruent to $\deg \delta/2$ modulo $\mathbb{Z}$, then this is never possible. □

Proposition 4.4. *Let $X \xrightarrow{\pi} B$ be a non-Kähler elliptic fibre bundle. Choose an element $c_1 \in NS(X)$ such that $m(2, c_1) > 0$, so that we may have $c_2 < 0$. Then, the graph map is surjective whenever the moduli spaces are non-empty, except in the following case. Suppose that $c_2 = 0$ and $m(2, c_1) = 1/4$. Furthermore, let J be the set of sections A in $\mathbb{P}_{\delta,0}$ such that $\eta^* A = \Sigma_1 + \Sigma_2$ is a reducible bisection of $J(X)$ and the degree of any line bundle on X associated to Σ_1 is congruent to* $\deg \delta/2$ *modulo $\mathbb{Z}$. In this case, the image of the graph map is $\mathbb{P}_{\delta,0} \backslash J$.*

Proof. Consider a graph $\mathcal{G} = \sum_{i=1}^{k} f_i + A$ and set $\overline{C} = \eta^* A$. We know that there exist bundles corresponding to this graph; let us then discuss the stability of a bundle E that has graph $\mathcal{G}$. If $c_2 < 0$, then $\Delta(E) < m(2, c_1)$: the bisection $\overline{C}$ is irreducible and the bundle is stable. If $c_2 = 0$, then $\Delta(E) = m(2, c_1) > 0$. There are now two possibilities. The first is $k \neq 0$, implying that $A^2 < 4m(2, c_1)$; therefore, the bisection $\overline{C} = \eta^* A$ is irreducible and the bundle is stable. The second is $k = 0$ and the bisection is reducible; suppose that $\overline{C} = \Sigma_1 + \Sigma_2$, for some sections $\Sigma_1, \Sigma_2 \subset J(X)$. Note that $\Sigma_1 \neq \Sigma_2$; otherwise, $k = 0$ would imply that $\Delta(E) = 0$, which is a contradiction. The vector bundle E is then an extension of $K^{-1} \otimes \delta$ by K, where K is the destabilising bundle of E corresponding to Σ_1, that can be assumed to be regular on every fibre of π. Hence, E is stable if and only if $\deg K \in (\deg \delta/2 - 4m(2, c_1), \deg \delta/2)$, as stated in Theorem 3.5. Clearly, if $m(2, c_1) = 1/4$ and the degree of every line bundle corresponding to Σ_1 is congruent to $\deg \delta/2$ modulo $\mathbb{Z}$, then E is unstable. Finally, by arguments similar to those used to prove Proposition 4.3, the graph map is surjective whenever $c_2 \geq 1$. □

4.4. Fibre of the graph map. If we consider graphs without vertical components, the description of most fibres of the graph map is then straightforward.

Proposition 4.5. *Let X be a non-Kähler elliptic surface over a curve B of genus g. Fix a pair (c_1, c_2) in $NS(X) \times \mathbb{Z}$ and let δ be a line bundle on X, with $c_1(\delta) = c_1$, such that $\mathcal{M}_{\delta,c_2}$ is non-empty. Consider an element A of $\mathbb{P}_{\delta,c_2}$ that does not contain vertical components and let $\overline{C} = \eta^* A$ be the corresponding bisection in $J(X)$.*

(i) Suppose that $\overline{C}$ is a smooth bisection of $J(X)$. The fibre of the graph map G at A is then isomorphic to a finite number of copies of the Prym variety $Prym(\overline{C}/B)$ (see Theorem 2.4).

(ii) If the bisection $\overline{C} = \Sigma_1 + \Sigma_2$ is reducible, then the components of $G^{-1}(A)$ are parametrised by the set of line bundles on X associated to Σ_1 that satisfy the conditions of Theorem 3.5. In particular, the component given by the line bundle K consists of extensions of $K^{-1} \otimes \delta$ by K that are regular on at least one fibre $\pi^{-1}(b)$, where $\Sigma_{1,b} = \Sigma_{2,b}$, if $\deg K$ is not congruent to $\deg \delta/2$ modulo $\mathbb{Z}$, or that are regular on at least two such fibres, otherwise. □

For graphs with vertical components, the fibre of the graph map can be described by examining how jumps can be added to vector bundles, that is, by classifying elementary modifications. This is done in detail in [Mo] for vector bundles on Hopf surfaces. For the sake of completion, we briefly state how this translates to bundles on an arbitrary non-Kähler elliptic surface X.

Let E be a stable rank-2 vector bundle on X with $\det E = \delta$, $c_2(E) = c_2$, and a jump of length l over the smooth fibre $T = \pi^{-1}(x_0)$. This jump can be removed by performing l successive allowable elementary modifications, thus obtaining a bundle with determinant $\delta(-lT)$; note that this procedure is canonical. But, adding a jump to E implies several choices: a jumping sequence $\{h_0, \ldots, h_{l-1}\}$, a line bundle N on T for each distinct integer of the jumping sequence, and surjections to N that preserve stability. These choices are parametrised by a fibration that we now describe. Let $\mathcal{G}$ be a graph that contains a vertical component over x_0 of multiplicity μ and $\{h_0, \ldots, h_{l-1}\}$ be a jumping sequence such that $\sum_{i=0}^{l-1} h_i = \mu$. We set

$$\mathcal{E}^j \mathcal{J}^{c_2,l}_{\mathcal{G},\{h_0,\ldots,h_{l-1}\}} = \left\{ E \in \mathcal{M}_{\delta(jT),c_2} \;\middle|\; \begin{array}{c} G(E) = \mathcal{G} \text{ and } E \text{ has a jump} \\ \text{of length } l \text{ at } x_0 \text{ with jumping} \\ \text{sequence } \{h_0, \ldots, h_{l-1}\} \end{array} \right\}.$$

Associating to a bundle E its allowable elementary modification $\bar{E}$ therefore defines a natural map

$$\begin{aligned} \Psi : \mathcal{E}^{j+1} \mathcal{J}^{c_2,l+1}_{\mathcal{G},\{h_0,h_1,\ldots,h_l\}} &\longrightarrow \mathcal{E}^j \mathcal{J}^{c_2-h_0,l}_{G(\bar{E}),\{h_1,\ldots,h_l\}} \\ E &\longmapsto \bar{E}. \end{aligned}$$

Proposition 4.6. *The fibre of the natural projection Ψ at W is given by:*

(i) $Aut_{SL(2,\mathbb{C})}(W|_T)$, if $c_2 > h_0$ and $h_0 = h_1$,
(ii) $Pic^{-h_0}(T) \times Aut_{SL(2,\mathbb{C})}(W|_T)$, if $c_2 > h_0 = 1$ and $l = 0$,
(iii) $Pic^{-c_2}(T)$, if $c_2 = h_0 = 1$ and $l = 0$. □

4.5. Integrable systems. A Poisson structure on a surface X is given by a global section of its anticanonical bundle K_X^{-1} [Bo]. Suppose that $X \xrightarrow{\pi} B$ is a non-Kähler elliptic surface that may have multiple fibres $T_1, \ldots, T_r$ of multiplicities $m_1, \ldots, m_r$, respectively.

The anticanonical bundle of X is then $\pi^* K_B^{-1} \otimes \left(\bigotimes_{i=1}^r \mathcal{O}_X(1-m_i)\right)$, implying that X admits a Poisson structure if the genus of the base curve is smaller or equal to 1 and if π does not have multiple fibres. From now on, we suppose that X is a non-Kähler elliptic surface without multiple fibres over a curve B of genus $g = 0$ or 1, that is, a Hopf surface or a primary Kodaira surface. Let us fix a Poisson structure $s \in H^0(X, K_X^{-1})$ on X. A Poisson structure $\theta = \theta_s \in H^0(\mathcal{M}, \otimes^2 T\mathcal{M})$ on the moduli space $\mathcal{M} := \mathcal{M}_{c_2,\delta}$ is then defined as follows: for any bundle $E \in \mathcal{M}$, $\theta(E) : T_E^*\mathcal{M} \times T_E^*\mathcal{M} \longrightarrow \mathbb{C}$ is the composition

$$\begin{aligned}\theta(E) : H^1(X, \mathrm{ad}(E) \otimes K_X) \times H^1(X, \mathrm{ad}(E) \otimes K_X) \xrightarrow{\circ} \\ H^2(X, \mathrm{End}(E) \otimes K_X^2) \xrightarrow{s} H^2(X, \mathrm{End}(E) \otimes K_X) \xrightarrow{\mathrm{Tr}} \mathbb{C},\end{aligned}$$

where the first map is the cup-product of two cohomology classes, the second is multiplication by s, and the third is the trace map.

If the base curve B is elliptic, the canonical bundle of X is trivial and the Poisson structure s is non-degenerate; in this case, θ has maximal rank everywhere, that is, θ is symplectic. If the base curve is instead rational, the Poisson structure s is now degenerate; we denote its divisor $D := (s)$. Then, at any point $E \in \mathcal{M}$,

$$\mathrm{rk}\,\theta(E) = 4\dim_{\mathbb{C}} \mathcal{M} - \dim H^0(D, \mathrm{ad}(E|_D)).$$

Suppose that the locally free sheaf $\mathcal{O}_B(2)$ on $B \cong \mathbb{P}^1$ is given by the divisor $x_1 + x_2$, for some points $x_1, x_2 \in B$; then, $D = T_1 + T_2$, where $T_i = \pi^{-1}(x_i)$ for $i = 1, 2$. We now see that the rank of the Poisson structure is generically $4\dim_{\mathbb{C}} \mathcal{M} - 2$ and "drops" at the points of $\mathcal{M}$ corresponding to bundles that are not regular over the fibres T_1 and T_2 (for details, see [Mo]).

Referring to Sects. 4.2 and 4.4, the moduli space $\mathcal{M}$ has dimension $8\Delta(2, c_1, c_2)$ and the generic fibres of the graph map $G : \mathcal{M} \longrightarrow \mathbb{P}_{\delta,c_2}$ consist of Prym varieties of dimension $4\Delta(2, c_1, c_2) + g - 1$ (see Proposition 4.5). Also, one can show as in [Mo] that the component functions $H_1, \ldots, H_N$ of the graph map are in involution with respect to the Poisson structure, that is, $\{H_i, H_j\} = 0$ for all i, j. Consequently, the graph map G is an algebraically completely integrable Hamiltonian system.

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