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# Dynamical Yang-Baxter Equation and Quantum Vector Bundles★

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**Abstract:** We develop a categorical approach to the dynamical Yang-Baxter equation (DYBE) for arbitrary Hopf algebras. In particular, we introduce the notion of a dynamical extension of a monoidal category, which provides a natural environment for quantum dynamical R-matrices, dynamical twists, *etc*. In this context, we define dynamical associative algebras and show that such algebras give quantizations of vector bundles on coadjoint orbits. We build a dynamical twist for any pair of a reductive Lie algebra and its Levi subalgebra. Using this twist, we obtain an equivariant star product quantization of vector bundles on semisimple coadjoint orbits of reductive Lie groups.

## Contents



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## 1. Introduction

The quantum dynamical Yang-Baxter equation (DYBE) appeared in the mathematical physics literature, [GN, AF, Fad, F, ABB], in connection with integrable models of conformal field theories. The classical DYBE was first considered in [BDFh], rediscovered in [F], and systematically studied by Etingof, Schiffmann, and Varchenko in [EV1, ES2, S]. For a guide in the DYBE theory and an extended bibliography the reader is referred to the lecture course [ES1].

The theory of the DYBE over the Cartan subalgebra in a simple Lie algebra has been developed in detail. Classical dynamical r-matrices were classified in [EV1] and their explicit quantization built in [EV2, EV3, ESS]. Concerning the classical DYBE over an arbitrary (non-commutative) base, much is known about classification of its solutions and there are numerous explicit examples, [AM, ES2, Fh, S, Xu2]. At the same time, there is no generally accepted definition of quantum DYBE over a non-commutative Lie algebra or, say, over an arbitrary Hopf algebra. A generalization of the quantum DYBE for several particular cases was proposed in [Xu2] and [EE1]. Such a generalization was motivated by a relation between DYBE and the star product, [Xu1, Xu2]. An open question is an interpretation of the quantum DYBE of [Xu2] and [EE1] from a categorical point of view.

Another interesting question is a relation of DYBE to the equivariant quantization. It was observed by Lu, [Lu1], that the list of classical r-matrices over the Cartan subalgebra of a simple Lie algebra is in intriguing correspondence with the list of Poisson-Lie structures on its maximal coadjoint orbits. However, the precise relation between quantum dynamical R-matrices and the equivariant quantization has not been established.

The purpose of the present paper is to develop a theory of DYBE over an arbitrary Hopf algebra and relate it to equivariant quantization of vector bundles. Firstly, we generalize the classical dynamical Yang-Baxter equation for any Lie bialgebra $\mathfrak{h}$ extending the concept of base manifold, which is the dual space $\mathfrak{h}^*$ in the standard approach. Secondly, we build dynamical extensions of monoidal categories and define the quantum dynamical R-matrix over an arbitrary base. Our third result is a construction of dynamical twist for Levi subalgebras in a reductive Lie algebra. Finally, we introduce a notion of dynamical associative algebras as algebras in dynamical categories and relate them to equivariant quantization of vector bundles. As an application, we construct an equivariant star product quantization of vector bundles (including function algebras) on semisimple coadjoint orbits of reductive Lie groups.

It turns out that there is a general procedure of "dynamical extension", $\bar{\mathcal{O}}$, of every monoidal category $\mathcal{O}$ over a base $\mathcal{B}$, which is an $\mathcal{O}$-module category. This new category has the same objects as $\mathcal{O}$ but more morphisms. The objects are considered as functors from $\mathcal{B}$ to $\mathcal{B}$ by the tensor product action. Morphisms in $\bar{\mathcal{O}}$ are natural transformations between these functors. This category admits a tensor product making it a monoidal category with $\mathcal{O}$ being a subcategory. One can consider the standard notions as algebras, twists, and R-matrices relative to $\bar{\mathcal{O}}$. In terms of the original category $\mathcal{O}$, they satisfy "shifted" axioms, like shifted associativity, shifted cocycle condition, shifted or dynamical Yang-Baxter equation.

The construction of dynamical extension admits various formulations. One of them uses the so-called base algebras, which are commutative algebras in the Yetter-Drinfeld categories. From the algebraic point of view, a Yetter-Drinfeld category is a category of modules over the double $D(\mathcal{H})$ of a Hopf algebra $\mathcal{H}$. In the quasi-classical limit, the base algebras are function algebras on the so-called Poisson base manifolds. A Poisson base manifold $L$ is endowed with an action of the double $D(\mathfrak{h})$ of the Lie bialgebra $\mathfrak{h}$, the classical analog of $\mathcal{H}$. The Poisson structure on $L$ is induced by the canonical $r$-matrix of the double.

The category of $\mathcal{H}$-modules can be dynamically extended over the dual Hopf algebra $\mathcal{H}^*$. This approach is convenient for definition of dynamical associative algebras. A dynamical associative algebra is equipped with an equivariant family of binary operations (multiplications) depending on elements of $\mathcal{H}^*$. This family satisfies a "shifted" associativity condition. We show that the dynamical associative algebras give vector bundles on quantum spaces.

In this paper we consider vector bundles on coadjoint orbits. In the classical situation, the function algebra on a homogeneous space is a subalgebra in the function algebra on the group. In general, the quantized function algebra on a homogeneous space cannot be realized as a subalgebra in a quantized function algebra on the group. For example, in the case of semisimple coadjoint orbits, such a realization exists only for symmetric or bisymmetric orbits, [DGS1, DM1]. Nevertheless, a quantization of the function algebra on the group as a **dynamical associative algebra** contains quantum orbits as (associative) subalgebras. Moreover, a dynamical quantization on the group quantizes the algebra of sections of homogeneous vector bundles on orbits. Such quantizations are parameterized by group-like elements of $\mathcal{H}^*$.

A way of constructing (quantum) dynamical R-matrices and dynamical associative algebras is by twists in dynamical categories. We build such twists for Levi subalgebras in simple Lie algebras, using generalized Verma modules. This gives a construction of star product on the semisimple orbits.

The paper is organized as follows. In Sect. 2 we recall basic definitions concerning DYBE and the compatible star product of [Xu2].

Section 3 presents generalizations of DYBE using the concepts of base algebras and base manifolds.

Section 4 is devoted to various formulations of dynamical categories; therein we study dynamical associative algebras.

In Sect. 5 we study objects that are interesting for applications: dynamical twists and dynamical R-matrices. We consider various types of dynamical categories and give expressions of dynamical twists and R-matrices in terms of the original category.

In Sect. 6 we suggest a method of constructing dynamical twists. The method is based on a notion of dynamical adjoint functors. We build such functors using generalized Verma modules corresponding to Levi subalgebras in the (quantum) universal enveloping algebra of simple Lie algebras.

In Sect. 7 we study relations between quantization of vector bundles and dynamical associative algebras in a purely algebraic setting.

In Sect. 8 we give a detailed consideration to the dynamical associative algebra which is a quantized function algebra on a simple Lie group $G$. We relate this algebra to quantum vector bundles on coadjoint semisimple orbits of $G$.

Note that the equivariant star product on function algebras on coadjoint orbits was also constructed in the papers [AL] and [KMST] which appeared after the first version of this article. Our method of building dynamical twists is developed for a more general case in [EE2].

## 2. Dynamical r-Matrix and Compatible Star Product

*2.1. Classical dynamical Yang-Baxter equation.* In this section we recall basic definitions concerning the dynamical Yang-Baxter equation. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ its Lie subalgebra. The dual space $\mathfrak{h}^*$ is considered as an $\mathfrak{h}$-module with respect to the coadjoint action. Let $\{h_i\} \subset \mathfrak{h}$ be a basis and $\{\lambda^i\} \subset \mathfrak{h}^*$ its dual.

**Definition 2.1 ([F, EV1]).** *A classical dynamical r-matrix over the base $\mathfrak{h}$ is an $\mathfrak{h}$-equivariant meromorphic function $r\colon \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$ satisfying*

1. *the normal condition: the sum $r(\lambda) + r_{21}(\lambda)$ is $\mathfrak{g}$-invariant,*
2. *the classical dynamical Yang-Baxter equation (DYBE)*:

$$\sum_i \frac{\partial r_{23}}{\partial \lambda^i} h_i^{(1)} - \frac{\partial r_{13}}{\partial \lambda^i} h_i^{(2)} + \frac{\partial r_{12}}{\partial \lambda^i} h_i^{(3)} = [r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}]. \quad (1)$$

A constant dynamical $r$-matrix is a solution to the ordinary Yang-Baxter equation. It follows that the sum $r(\lambda) + r_{21}(\lambda)$ does not depend on $\lambda$, [ES2]. If it is identically zero, the r-matrix is called triangular.

*2.2. Quantum dynamical Yang-Baxter equation over an abelian base.* Suppose that $\mathfrak{h}$ is a commutative Lie algebra and $V$ is a semisimple $\mathfrak{h}$-module. Given a family $\Omega(\lambda)$, $\lambda \in \mathfrak{h}^*$, of linear operators on $V^{\otimes 3}$, let us denote by $\Omega(\lambda + th^{(1)})$ the family of operators on $V^{\otimes 3}$ acting by $v_1 \otimes v_2 \otimes v_3 \mapsto \Omega(\lambda + t\ \mathrm{wt}(v_1))(v_1 \otimes v_2 \otimes v_3)$, where $\mathrm{wt}(v)$ stands for the weight of $v \in V$ with respect to $\mathfrak{h}$ and $t$ is a formal parameter. The operators $\Omega(\lambda + th^{(i)})$, $i = 2, 3$, are defined similarly.

**Definition 2.2.** *Let $\mathfrak{h}$ be a commutative Lie subalgebra of a Lie algebra $\mathfrak{g}$. Let $R(\lambda)$ be an $\mathfrak{h}$-equivariant meromorphic function $\mathfrak{h}^* \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$ (we consider $\mathfrak{h}^*$ equipped with the coadjoint and $\mathcal{U}(\mathfrak{g})$ with adjoint action of $\mathfrak{h}$). Then $\mathcal{R}(\lambda)$ is called (universal) quantum dynamical R-matrix if it satisfies the quantum dynamical Yang-Baxter equation (QDYBE)*

$$\mathcal{R}_{12}(\lambda)\mathcal{R}_{13}(\lambda + th^{(2)})\mathcal{R}_{23}(\lambda) = \mathcal{R}_{23}(\lambda + th^{(1)})\mathcal{R}_{13}(\lambda)\mathcal{R}_{12}(\lambda + th^{(3)}). \qquad (2)$$

Assuming $\mathcal{R}(\lambda) = 1 \otimes 1 + t\, r(\lambda) + O(t^2)$, the element $r(\lambda)$ satisfies Eq. (1), i.e. Eq. (1) is the quasi-classical limit of Eq. (2). In this case $R(\lambda)$ is called quantization of $r(\lambda)$. The problem of quantizing classical DYBE has been solved for $\mathfrak{g}$ a complex semisimple Lie algebra and $\mathfrak{h}$ its reductive commutative subalgebra, [ESS]. As to the case of general $\mathfrak{h}$, there is no generally accepted concept of what should be taken as the quantum DYBE. In the next subsection we render a construction of [Xu2] suggesting a version of quantum DYBE as a quantization ansatz for triangular dynamical r-matrices. This will be the starting point for our study.

*2.3. Compatible star product.* Let $\mathfrak{g}$ be a complex Lie algebra and $G$ the corresponding connected Lie group. Let $\mathfrak{h}$ be a Lie subalgebra in $\mathfrak{g}$. Denote by $\vec{\xi}$ the left invariant vector field on $G$ induced by $\xi \in \mathfrak{g}$ via the right regular action. Let $\pi_{\mathfrak{h}^*}$ denote the Poisson-Lie bracket on $\mathfrak{h}^*$.

**Theorem 2.3 ([Xu2]).** *A smooth function $r\colon \mathfrak{h}^* \to \wedge^2\mathfrak{g}$ is a triangular dynamical r-matrix if and only if the bivector field*

$$\pi_{\mathfrak{h}^*} + \sum_i \frac{\partial}{\partial \lambda_i} \wedge \vec{h}_i + \vec{r}(\lambda) \qquad (3)$$

*is a Poisson structure on $\mathfrak{h}^* \times G$.*

Thus, the bivector field $\vec{r}(\lambda)$ on $G$ is a “part” of a special Poisson bracket on a bigger space, $\mathfrak{h}^* \times G$. Xu proposed to look at a star product on $\mathfrak{h}^* \times G$ of special form, as a quantization of (3). Let $\mathfrak{h}_t := \mathfrak{h}[[t]]$ be the Lie algebra over $\mathbb{C}[[t]]$ with the Lie bracket $[x, y]_t := t[x, y]$ for $x, y \in \mathfrak{h}$. The universal enveloping algebra $\mathcal{U}(\mathfrak{h}_t)$ can be considered as a deformation quantization of the polynomial algebra on $\mathfrak{h}^*$. It is known that this quantization can be presented as a star product on $\mathfrak{h}^*$ by the PBW map $S(\mathfrak{h})[[t]] \to \mathcal{U}(\mathfrak{h}_t)$, where elements of the symmetric algebra $S(\mathfrak{h})$ are identified with polynomial functions on $\mathfrak{h}^*$. We call this star product the **PBW star product**.

**Definition 2.4 ([Xu2]).** *A star product $*_t$ on $\mathfrak{h}^* \times G$ is called* **compatible** *if*

1. *when restricted to $C^\infty(\mathfrak{h}^*)$, it coincides with the PBW star product;*

*2. for $f \in C^\infty(G)$ and $g \in C^\infty(\mathfrak{h}^*)$,*

$$(f *_t g)(\lambda, x) := f(x)g(\lambda), \quad (g *_t f)(\lambda, x) := \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\partial^k g(\lambda)}{\partial\lambda^{i_1}\dots\partial\lambda^{i_k}} \vec{h}_{i_1}\dots\vec{h}_{i_k} f(x); \tag{4}$$

*3. for $f, g \in C^\infty(G)$,*

$$(f *_t g)(\lambda, x) := \vec{\mathcal{F}}(\lambda)(f, g)(x), \tag{5}$$

*where $\mathcal{F}(\lambda)$ is a smooth function $\mathcal{F}\colon \mathfrak{h}^* \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})[[h]]$ such that $\mathcal{F} = 1 \otimes 1 + \frac{t}{2} r(\lambda) + O(t^2)$.*

*For this star product to be associative, $\mathcal{F}$ should satisfy a certain condition called the shifted cocycle condition.*

Also, Xu proposed a generalization of the quantum DYBE (2) for an arbitrary Lie algebra $\mathfrak{h}$ in the form

$$\mathcal{R}_{12}(\lambda) *_t \mathcal{R}_{13}\big(\lambda + th^{(2)}\big) *_t \mathcal{R}_{23}(\lambda) = \mathcal{R}_{23}\big(\lambda + th^{(1)}\big) *_t \mathcal{R}_{13} *_t \mathcal{R}_{12}\big(\lambda + th^{(3)}\big), \tag{6}$$

where $\mathcal{R}$ is an equivariant function $\mathfrak{h}^* \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$, and the subscripts mark the tensor components in $\mathcal{U}^{\otimes 3}(\mathfrak{g})$. Notation $f(\lambda + th)$ for $f \in C^\infty(\mathfrak{h}^*)$ means

$$f(\lambda + th) := \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\partial^k f(\lambda)}{\partial\lambda^{i_1}\dots\partial\lambda^{i_k}}\, h_{i_1}\dots h_{i_k}. \tag{7}$$

Here $\{h_i\} \subset \mathfrak{h}$ and $\{\lambda^i\} \subset \mathfrak{h}^*$ are dual bases; the superscript of $h^{(i)}$, $i = 1, 2, 3$, in (6) means that $\mathfrak{h}$ is embedded in the $i^{\text{th}}$ component of $\mathcal{U}^{\otimes 3}(\mathfrak{g})$.

The compatible star product of [Xu2] is defined on smooth functions on $\mathfrak{h}^* \times G$. When restricted to polynomial functions on $\mathfrak{h}^*$, it gives the multiplication in the universal enveloping algebra $\mathcal{U}(\mathfrak{h})$. Formula (4) expresses the product of elements from $\mathcal{U}(\mathfrak{h})$ and $C^\infty(G)$ through the comultiplication in $\mathcal{U}(\mathfrak{h})$ and the action of $\mathcal{U}(\mathfrak{h})$ on $C^\infty(G)$. It seems natural to replace $\mathcal{U}(\mathfrak{h})$ with an arbitrary Hopf algebra $\mathcal{H}$ and $C^\infty(G)$ with a left $\mathcal{H}$-module $\mathcal{A}$. However, the bidifferential operator $\mathcal{F}(\lambda)$ in (5) may be a meromorphic or even a formal function in $\lambda \in \mathfrak{h}^*$. This requires to consider appropriate extensions of $\mathcal{U}(\mathfrak{h})$, which may no longer be Hopf algebras. On the other hand, there is a class of **admissible** algebras which are close, in a sense, to the Hopf ones. Those are commutative algebras in the so-called Yetter-Drinfeld category of $\mathcal{H}$-modules and $\mathcal{H}$-comodules, which are, roughly speaking, modules over the double of $\mathcal{H}$. We will define a **dynamical extension** of the monoidal category of $\mathcal{H}$-modules over an admissible algebra, where the notions of compatible star products, dynamical Yang-Baxter equations, *etc.*, acquire a natural algebraic formulation. Depending on a particular choice of admissible algebra, we come to different quasi-classical limits of quantum dynamical objects. Also, it appears useful (and often technically simpler) to consider a "dual" version of the dynamical extension, for example, a dynamical extension of the monoidal category of $\mathcal{H}^*$-comodules. In this way we obtain a "linearization" of the theory; in particular, smooth or meromorphic functions on $\mathfrak{h}^*$ become linear functions on $\mathcal{U}(\mathfrak{h})^*$. Moreover, it will be useful to introduce the notion of dynamical extension of an arbitrary monoidal category, defined without involving any Hopf algebra. Below we present all the formulations.

## 3. Generalizations of Dynamical Yang-Baxter Equations

*3.1. Base algebras.* In this subsection we define two objects of our primary concern: a base algebra $\mathcal{L}$ and a dynamical associative algebra over $\mathcal{L}$.

By $k$ we mean a commutative ring over a field of zero characteristic. The reader may think of it as $\mathbb{C}$ or $\mathbb{C}[[t]]$, the ring of formal series in $t$. Given a Hopf algebra $\mathcal{H}$ over $k$ we denote the multiplication, comultiplication, counit, and antipode by $m$, $\Delta$, $\varepsilon$, and $\gamma$. We use the standard Sweedler notation for the comultiplication in Hopf algebras: $\Delta(x) = x^{(1)} \otimes x^{(2)}$. In the same fashion we denote the $\mathcal{H}$-coaction on a right comodule $A$: $\delta(a) = a^{[0]} \otimes a^{(1)}$, where the square brackets label the $A$-component and the parentheses mark that belonging to $\mathcal{H}$. The Hopf algebra with the opposite multiplication will be denoted by $\mathcal{H}_{op}$ while with the opposite comultiplication by $\mathcal{H}^{op}$.

The Hopf algebra $\mathcal{H}$ is considered as a left module over itself with respect to the adjoint action

$$x \otimes a \mapsto x^{(1)} a \gamma(x^{(2)}); \tag{8}$$

then the multiplication in $\mathcal{H}$ is equivariant. It is a standard fact that for any left $\mathcal{H}$-module $A$ the map $\mathcal{H} \otimes A \to A \otimes \mathcal{H}$, $h \otimes a \mapsto h^{(1)} \triangleright a \otimes h^{(2)}$, is $\mathcal{H}$-equivariant.

Recall that an algebra and $\mathcal{H}$-module $\mathcal{A}$ is called a module algebra if the multiplication in $\mathcal{A}$ is $\mathcal{H}$-equivariant. An algebra and $\mathcal{H}$-comodule $\mathcal{A}$ is called a comodule algebra if the coaction $\mathcal{A} \to \mathcal{H} \otimes \mathcal{A}$ is a homomorphism of algebras.

**Definition 3.1 (Base algebras).** *A left $\mathcal{H}$-module and left $\mathcal{H}$-comodule algebra $\mathcal{L}$ is called* **base algebra** *over $\mathcal{H}$ if the coaction $\delta\colon \mathcal{L} \to \mathcal{H} \otimes \mathcal{L}$ satisfies the condition*

$$\left\{x^{(1)} \triangleright \ell\right\}^{(1)} x^{(2)} \otimes \left\{x^{(1)} \triangleright \ell\right\}^{[2]} = x^{(1)} \ell^{(1)} \otimes x^{(2)} \triangleright \ell^{[2]} \tag{9}$$

*for all $x \in \mathcal{H}$ and $\ell \in \mathcal{L}$, and the condition*

$$\ell_1 \ell_2 = \left(\ell_1^{(1)} \triangleright \ell_2\right) \ell_1^{[2]}, \tag{10}$$

*for all $\ell_1, \ell_2 \in \mathcal{L}$.*

The coaction $\delta$ defines a permutation $\tau_A : \mathcal{L} \otimes A \to A \otimes \mathcal{L}$ with every $\mathcal{H}$-module $A$:

$$\tau_A(\ell \otimes a) := \ell^{(1)} \triangleright a \otimes \ell^{[2]}, \quad \ell \otimes a \in \mathcal{L} \otimes A. \tag{11}$$

Condition (9) ensures that this permutation is $\mathcal{H}$-equivariant. Condition (10) means that the multiplication in $\mathcal{L}$ is $\tau_{\mathcal{L}}$-commutative.

*Remark 3.2.* A base algebra is a commutative algebra in the braided category of Yetter-Drinfeld modules. From the purely algebraic point of view, Yetter-Drinfeld modules are modules over the double Hopf algebra $D(\mathcal{H})$. The left $\mathcal{H}$-coaction induces a left $\mathcal{H}^*_{op}$-action. Together with the $\mathcal{H}$-action, the $\mathcal{H}^*_{op}$-action gives a $D(\mathcal{H})$-action. In our theory, an $\mathcal{H}$-base algebra plays the same role as the $\mathcal{U}(\mathfrak{h})$-module algebra of functions on $\mathfrak{h}^*$ in the theory of DYBE over a commutative base.

One can also introduce the dual notion of a base coalgebra as a comodule over $D(\mathcal{H})$. We will use $\mathcal{H}^*$, a dual to the Hopf algebra $\mathcal{H}$, as an example of such a base coalgebra.

*Example 3.3.* The algebra $\mathcal{H}$ itself is a base algebra over $\mathcal{H}$ with respect to the left adjoint action and the coproduct $\Delta$ considered as the left regular $\mathcal{H}$-coaction. Conditions (9) and (10) are checked directly.

*Example 3.4.* Suppose that $\mathcal{H}$ is the tensor product of two Hopf algebras, $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_1$. Then both $\mathcal{H}_0$ and $\mathcal{H}_1$ are natural base algebras over $\mathcal{H}$. The $\mathcal{H}$-action on $\mathcal{H}_i$ is the adjoint action restricted to $\mathcal{H}_i$. The $\mathcal{H}$-coaction on $\mathcal{H}_i$ is the coproduct coaction considered as a map with values in $\mathcal{H}_i \otimes \mathcal{H}_i \subset \mathcal{H} \otimes \mathcal{H}_i$.

*Example 3.5 (PBW star product).* Consider the algebra $C^\infty(\mathfrak{h}^*)[[t]]$ from Definition 2.4 equipped with the PBW star product. It is obviously a left $\mathcal{U}(\mathfrak{h})$-module algebra, and formula (7) defines a coaction $C^\infty(\mathfrak{h}^*)[[t]] \to \mathcal{U}(\mathfrak{h}) \otimes C^\infty(\mathfrak{h}^*)[[t]]$ (the completed tensor product). It is straightforward to check that $C^\infty(\mathfrak{h}^*)[[t]]$ is a base algebra over $\mathcal{U}(\mathfrak{h})$. The algebra $C^\infty(\mathfrak{h}^*)[[t]]$ is an extension of $\mathcal{U}(\mathfrak{h}_t)$, which is realized as the subalgebra in $\mathcal{U}(\mathfrak{h})[[t]]$ generated by $t\mathfrak{h}$. The algebra $\mathcal{U}(\mathfrak{h}_t)$ is a Hopf one, hence it is a base algebra over itself. At the same time, it is a base algebra over $\mathcal{U}(\mathfrak{h})[[t]]$. Indeed, it is invariant under the adjoint $\mathcal{U}(\mathfrak{h})$-action, and it is a left $\mathcal{U}(\mathfrak{h})$-comodule under the map $(\varphi_t \otimes \mathrm{id}) \circ \Delta$, where $\Delta$ is the coproduct in $\mathcal{U}(\mathfrak{h}_t)$ and $\varphi_t$ the natural embedding of $\mathcal{U}(\mathfrak{h}_t)$ in $\mathcal{U}(\mathfrak{h})[[t]]$.

**Proposition 3.6.** *Suppose that $\mathcal{H}$ is a quasitriangular Hopf algebra, with the universal R-matrix $\mathcal{R}$. Let $\mathcal{L}$ be a quasi-commutative $\mathcal{H}$-module algebra, i.e. obeying $(\mathcal{R}_2 \triangleright \ell_2)(\mathcal{R}_1 \triangleright \ell_1) = \ell_1 \otimes \ell_2$ for all $\ell_1, \ell_2 \in \mathcal{L}$. Then $\mathcal{L}$ is an $\mathcal{H}$-base algebra, with the left $\mathcal{H}$-coaction*

$$\delta(\ell) := \mathcal{R}_2 \otimes \mathcal{R}_1 \triangleright \ell, \quad \ell \in \mathcal{L}. \tag{12}$$

*Proof.* The condition (10) is satisfied by construction. The equality $(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$ implies that the map (12) is an algebra homomorphism. The map (12) makes $\mathcal{L}$ a left $\mathcal{H}$-comodule, because of $(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$. The condition (9) holds by virtue of $\mathcal{R}\Delta(h) = \Delta^{op}(h)\mathcal{R}$ for every $h \in \mathcal{H}$. □

**Corollary 3.7.** *Within the hypothesis of Proposition 3.6, suppose that $\mathcal{R} \in \mathcal{H} \otimes \mathcal{K} \subset \mathcal{H} \otimes \mathcal{H}$, where $\mathcal{K}$ is a Hopf subalgebra in $\mathcal{H}$. Then $\mathcal{L}$ is endowed with a structure of the $\mathcal{K}$-base algebra, with the $\mathcal{K}$-coaction (12).*

*Proof.* The $\mathcal{H}$-coaction (12) is, in fact, an $\mathcal{K}$-coaction. Now the statement immediately follows from Proposition 3.6. □

Remark that an $\mathcal{R}$-commutative algebra $\mathcal{L}$ is commutative with respect to the element $\mathcal{R}_{21}^{-1}$, which is also a universal R-matrix for $\mathcal{H}$. Thus $\mathcal{L}$ has two $\mathcal{H}$-base algebra structures, and they are different in general. In particular, an $\mathcal{H}$-base algebra has two different $\mathfrak{D}(\mathcal{H})$-base algebra structures.

*Example 3.8 (The FRT algebras).* The FRT-dual Hopf algebra $\mathcal{H}^*$, [FRT], of a quasitriangular Hopf algebra $\mathcal{H}$ is a quasi-commutative $\mathcal{H} \otimes \mathcal{H}_{op}$-algebra. Therefore it has two structures of $\mathcal{H} \otimes \mathcal{H}_{op}$-base algebras.

*Example 3.9 (Reflection equation algebras).* Recall that a twist of a Hopf algebra $\mathcal{H}$ is a Hopf algebra with the same multiplication and the new comultiplication $\tilde{\Delta}(x) := \mathcal{F}^{-1}\Delta(x)\mathcal{F}$; the element $\mathcal{F}$ called a twisting cocycle satisfies certain conditions, see [Dr3]. For every quasitriangular Hopf algebra $\mathcal{H}$ with the R-matrix $\mathcal{R}$, there is a twist, $\mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}$, of its tensor square, [RS]. It is obtained by applying the twisting cocycle $\mathcal{R}_{23} \in (\mathcal{H}\otimes\mathcal{H})\otimes(\mathcal{H}\otimes\mathcal{H})$ to the comultiplication in $\mathcal{H} \otimes \mathcal{H}$. The twisted tensor square is a quasitriangular Hopf algebra with the R-matrix

$$\mathcal{R}' := \mathcal{R}^-_{14}\mathcal{R}^-_{13}\mathcal{R}^+_{24}\mathcal{R}^+_{23} \in (\mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}) \otimes (\mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}), \tag{13}$$

where $\mathcal{R}^+ := \mathcal{R}$ and $\mathcal{R}^- := \mathcal{R}_{21}^{-1}$. Recall that $\mathcal{H}$ is a Hopf subalgebra in $\mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}$ through the embedding $\Delta\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. Observe that the R-matrix (13) can be presented as $\mathcal{R}' = (\mathcal{R}_1^- \otimes \mathcal{R}_1^+) \otimes \Delta(\mathcal{R}_2^-)\Delta(\mathcal{R}_2^+)$. In other words, its right tensor component belongs to $\Delta(\mathcal{H}) \subset \mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}$. Applying the argument from Corollary 3.7 to $\mathcal{K} = \Delta(H)$, we come to the following proposition.

**Proposition 3.10.** *A quasi-commutative $\mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}$-module algebra is a base algebra over $\mathcal{H}$.*

The reflection equation algebra associated with a finite dimensional representation of $\mathcal{H}$, [KSkl, KS], is a quasi-commutative $\mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}$-algebra, [DM3]. As a corollary of Proposition 3.10, we obtain that the reflection equation algebra is an $\mathcal{H}$-base algebra.

More examples of base algebras are obtained by quantizing Poisson base algebras (see Subsect. 3.2.2), according to Theorem 3.23.

*3.2. Dynamical associative algebras.* Let $\mathcal{L}$ be a base algebra over a Hopf algebra $\mathcal{H}$.

**Definition 3.11.** *A left $\mathcal{H}$-module $\mathcal{A}$ is called a* **dynamical associative algebra** *over the base algebra $\mathcal{L}$ if it is equipped with an $\mathcal{H}$-equivariant bilinear map $\circledast\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{L}$ such that the following diagram is commutative:*

$$\begin{array}{ccccccc}
\mathcal{A}\otimes\mathcal{L}\otimes\mathcal{A} & \xrightarrow{\mathrm{id}\otimes\tau_{\mathcal{A}}} & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{L} & \xrightarrow{\circledast\otimes\mathrm{id}} & \mathcal{A}\otimes\mathcal{L}\otimes\mathcal{L} & \xrightarrow{\mathrm{id}\otimes\mathrm{m}} & \mathcal{A}\otimes\mathcal{L} \\
{\scriptstyle \circledast\otimes\mathrm{id}}\uparrow & & & & & & \| \\
\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xrightarrow{\mathrm{id}\otimes\circledast} & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{L} & \xrightarrow{\circledast\otimes\mathrm{id}} & \mathcal{A}\otimes\mathcal{L}\otimes\mathcal{L} & \xrightarrow{\mathrm{id}\otimes\mathrm{m}} & \mathcal{A}\otimes\mathcal{L}
\end{array} \tag{14}$$

*Here* m *stands for the multiplication in $\mathcal{L}$ and the permutation $\tau_{\mathcal{A}}$ is defined by (11).*

An example of dynamical associative algebra is the function algebra on a group $G$ twisted by the dynamical twist from [Xu2]. It defines the compatible star product in the sense of Definition 2.4; it turns out that the multiplication $\circledast$ in a dynamical associative algebra over an arbitrary base can be extended to an ordinary associative multiplication in a bigger algebra, according to the following proposition.

**Proposition 3.12.** *Let $\mathcal{A}$ be a left $\mathcal{H}$-module equipped with an equivariant map $\circledast\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{L}$. Then $\mathcal{A}$ is a dynamical associative algebra with respect to $\circledast$ if and only if the operation*

$$(\mathcal{A}\otimes\mathcal{L})\otimes(\mathcal{A}\otimes\mathcal{L}) \xrightarrow{\tau_{\mathcal{A}}} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{L}\otimes\mathcal{L} \xrightarrow{\circledast\otimes\mathrm{m}} \mathcal{A}\otimes\mathcal{L}\otimes\mathcal{L} \xrightarrow{\mathrm{m}} \mathcal{A}\otimes\mathcal{L}$$

*makes $\mathcal{A}\otimes\mathcal{L}$ an associative $\mathcal{H}$-module algebra, denoted further by $\mathcal{A}\circledast\mathcal{L}$.*

*Proof.* The proof can be conducted by a straightforward verification. Below we give another proof using our categorical approach to dynamical associative algebras, see Example 4.21. □

*3.3 Infinitesimal analogs of base algebras and dynamical associative algebras.* In the present subsection, we introduce quasi-classical analogs of base algebras and dynamical associative algebras.

*3.2.1. Poisson-Lie manifolds.* Let us recall some basic facts about Poisson-Lie manifolds.

Throughout the text an $\mathfrak{g}$-manifold means a manifold equipped with a left $\mathfrak{g}$-action on functions. This corresponds to a right action on the manifold of a Lie group $G$ relative to $\mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie bialgebra, i.e. a Lie algebra equipped with a cobracket map $\mu\colon \mathfrak{g} \to \wedge^2\mathfrak{g}$. The cobracket defines on the dual space $\mathfrak{g}^*$ a Lie algebra structure compatible with the Lie algebra structure on $\mathfrak{g}$ in the sense of [Dr1]. Recall from [Dr2] that $\mu$ induces a Poisson structure on the Lie group $G$ such that the multiplication map $G \times G \to G$ is a Poisson map (the manifold $G \times G$ is equipped with the standard Poisson structure of Cartesian product of two Poisson manifolds). A right $G$-manifold $P$ is called a Poisson-Lie manifold if the action $P \times G \to P$ is Poisson. The right $G$-action on $P$ induces a left action of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ on the function algebra $\mathcal{A}(P)$. For an element $x \in \mathcal{U}(\mathfrak{g})$, let $\vec{x}_P$ (or simply $\vec{x}$, if $P$ is clear from the context) denote the corresponding differential operator on $P$. For the bidifferential operator on $P$ generated by a bivector field $\pi$, we use the notation $\pi(a,b) := (m \circ \pi)(a \otimes b)$, $a, b \in \mathcal{A}(P)$, where $m$ is the multiplication in $\mathcal{A}(P)$.

The following fact is well known and can be checked directly.

**Proposition 3.13.** *Let $\mathfrak{g}$ be a Lie bialgebra with cobracket $\mu$, $G$ the corresponding connected simply connected Poisson-Lie group, and $P$ a right $G$-manifold equipped with a Poisson bracket $\pi$. Then $P$ is a Poisson-Lie $G$-manifold if and only if for any $x \in \mathfrak{g}$ and $a, b \in \mathcal{A}(P)$*

$$\vec{x}\pi(a,b) - \pi(\vec{x}a, b) - \pi(a, \vec{x}b) = \overrightarrow{\mu(x)}(a,b). \tag{15}$$

Any Lie bialgebra structure on $\mathfrak{g}$ can be quantized to a $\mathbb{C}[[t]]$-Hopf algebra $\mathcal{U}_t(\mathfrak{g})$ (quantum group), see [EK]. If $\mathcal{A}_t(P)$ is a $\mathcal{U}_t(\mathfrak{g})$-equivariant quantization of $\mathcal{A}(P)$, then the quasi-classical limit of $\mathcal{A}_t(P)$ gives a Poisson-Lie bracket on $P$.

An important particular case of Lie bialgebras is a coboundary one, with the cobracket $\mu(x) := [x \otimes 1 + 1 \otimes x, r]$, where the element $r \in \wedge^2\mathfrak{g}$ satisfies the modified classical Yang-Baxter equation

$$[\![r, r]\!] := [r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] = \varphi \in \wedge^3(\mathfrak{g})^{\mathfrak{g}}. \tag{16}$$

Formula (15) then reads

$$[\vec{x} \otimes 1 + 1 \otimes \vec{x}, \pi - \vec{r}] = 0. \tag{17}$$

In other words, a Poisson-Lie bracket differs from $\vec{r}$ by an invariant bivector $f := \pi - \vec{r}$ such that $[\![f, f]\!]$ is equal to $-\vec{\varphi}$ from (16). Here the operation $f \mapsto [\![f, f]\!]$ is defined by (16) for the Lie algebra of vector fields; this operation is proportional to the Schouten bracket. Note that the Poisson-Lie bracket on a Poisson-Lie $\mathfrak{g}$-manifold $P$ is the infinitesimal object for the $\mathcal{U}_t(\mathfrak{g})$-equivariant quantization of the function algebra on $P$, where $\mathcal{U}_t(\mathfrak{g})$ is the corresponding quantum group. Such brackets are classified in [DGS1, Kar, D2, DO] for homogeneous manifolds $G/H$, where $G$ is a simple Lie group and $H$ its reductive Lie subgroup of maximal rank.

*3.2.2. Poisson base algebras and Poisson base manifolds.* Let $D(\mathfrak{h})$ denote the double of a Lie bialgebra $\mathfrak{h}$, [Dr1]. As a linear space, $D(\mathfrak{h})$ is the direct sum $\mathfrak{h} + \mathfrak{h}^*_{op}$, where $\mathfrak{h}^*$ is the dual Lie algebra. The double $D(\mathfrak{h})$ is endowed with a non-degenerate symmetric bilinear form induced by the natural pairing between $\mathfrak{h}$ and $\mathfrak{h}^*_{op}$. There is a unique extension of the Lie algebra structure from $\mathfrak{h}$ and $\mathfrak{h}^*_{op}$ to a Lie algebra on $D(\mathfrak{h})$ such that this form is ad-invariant. The double is a coboundary Lie bialgebra with the r-matrix $r := \sum_i \eta^i \wedge h^i = \frac{1}{2}\sum_i(\eta^i \otimes h^i - h^i \otimes \eta^i)$, where $\{h^i\}$ is a basis in $\mathfrak{h}$ and $\{\eta^i\}$ is the dual basis in $\mathfrak{h}^*_{op}$. The canonical element $\theta := \frac{1}{2}\sum_i(\eta^i \otimes h^i + h^i \otimes \eta^i)$ is ad-invariant. The pair $(r, \theta)$ makes $D(\mathfrak{h})$ a quasitriangular Lie bialgebra.

**Definition 3.14.** *A commutative $D(\mathfrak{h})$-algebra $\mathcal{L}_0$ is called a* **Poisson base algebra** *over $\mathfrak{h}$, or simply an $\mathfrak{h}$-base algebra, if $\theta$ induces the zero bidifferential operator on $\mathcal{L}_0$.*

*When a Poisson base algebra $\mathcal{L}_0$ over $\mathfrak{h}$ appears as the function algebra on a manifold L, i.e. $\mathcal{L}_0 := \mathcal{A}(L)$, we call L a* **Poisson base manifold** *over $\mathfrak{h}$, or simply an $\mathfrak{h}$-base manifold.*

**Proposition 3.15.** *An $\mathfrak{h}$-base manifold L is a Poisson-Lie $D(\mathfrak{h})$-manifold with respect to the bracket*

$$\varpi := \sum_i \vec{\eta^i} \wedge \vec{h^i}, \tag{18}$$

*which is automatically equal to the bivector field $\sum_i \vec{\eta^i} \otimes \vec{h^i}$.*

*Proof.* The element $\sum_i \eta^i \wedge h^i \in \wedge^2 D(\mathfrak{h})$ satisfies the modified Yang-Baxter equation (16) with $\varphi := [\theta_{12}, \theta_{23}]$. Since $\theta$ yields the zero bivector field on $L$, the three-vector field induced by $[\theta_{12}, \theta_{23}]$ is zero, too. This implies the following two assertions. Firstly, the bivector $\varpi$ defines a Poisson structure on $L$. Secondly, for any $D(\mathfrak{h})$-invariant Poisson bracket $f$ the bracket $f + \varpi$ hence $\varpi$, is automatically a Poisson-Lie one. □

The following are examples of Poisson base manifolds. According to Theorem 3.23 below, they can be quantized to $\mathcal{U}_t(\mathfrak{h})$-base algebras, where $\mathcal{U}_t(\mathfrak{h})$ is the quantized universal enveloping algebra of $\mathfrak{h}$.

*Example 3.16 (Group spaces $H^*$ and $H$).* Let $H$ be the Lie subgroup in the double $D(H)$ corresponding to the Lie subalgebra $\mathfrak{h} \subset D(\mathfrak{h})$. We will show that the left coset space $H\backslash D(H)$ is an $\mathfrak{h}$-base manifold. Note that the manifold $H\backslash D(H)$ is locally isomorphic to the Lie group $H^*$ corresponding to the Lie algebra $\mathfrak{h}^*$. The algebra of functions on $H\backslash D(H)$ is realized as a subalgebra of functions $f \in \mathcal{A}\big(D(H)\big)$ obeying $f(hx) = f(x)$ for $h \in H$. This subalgebra is invariant under the right regular action of $D(H)$ on itself. The element $\theta$ is $D(\mathfrak{h})$-invariant, hence the bivector $\theta^{l,l} - \theta^{r,r}$, where the superscripts $l, r$ denote the left- and right-invariant field extensions, gives the zero operator on $\mathcal{A}\big(D(H)\big)$. Therefore the bivector $\theta^{l,l}$ gives the zero operator on the left $H$-invariant functions, where it equals $\theta^{l,l} - \theta^{r,r}$. Thus, $H\backslash D(H)$ and therefore $H^*$ are Poisson $\mathfrak{h}$-base manifolds. In this example, the Poisson bracket on $H^*$ is the Drinfeld-Sklyanin bracket on $D(H)$ projected to $H^*$.

Similarly to $H\backslash D(H)$, one can consider the coset space $H^*\backslash D(H)$, which is locally isomorphic to the group space $H$. So $H$ is a Poisson $\mathfrak{h}$-base manifold as well.

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By the function algebra $\mathcal{A}(P)$ we understand, depending on a particular type of the manifold $P$, the algebra of polynomial, analytical, meromorphic, or smooth functions.

*Example 3.17 (Coset spaces $K\backslash H$).* Let us generalize Example 3.16. Suppose that $\mathfrak{k}$ is a Lie sub-bialgebra in $\mathfrak{h}$. Then the linear sum $\mathfrak{k} + \mathfrak{h}^*_{op}$ is a Lie sub-bialgebra in $D(\mathfrak{h})$. Let $K$ be the Lie subgroup in $H$ corresponding to $\mathfrak{k}$. Using the same arguments as in Example 3.16, one can prove that the coset space $K\backslash H$ is a Poisson $\mathfrak{h}$-base manifold. Indeed, let $K \cdot H^*$ denote the connected subgroup in $\mathfrak{D}(H)$ whose Lie algebra is $\mathfrak{k}+\mathfrak{h}^*_{op}$. The coset space $(K \cdot H^*)\backslash D(H)$ is locally isomorphic to $K\backslash H$ as a smooth manifold. Consider the functions on the group $D(H)$ that are invariant under $K \cdot H^*$ as functions on $(K \cdot H^*)\backslash D(H)$. The rest of the construction is exactly the same as in the previous example. Namely, one can check that the projection of the Drinfeld-Sklyanin bracket from $D(H)$ makes $K\backslash H$ a Poisson $\mathfrak{h}$-base manifold.

It follows that the quotient spaces of the standard Drinfeld-Jimbo simple Poisson-Lie group $H$ by the Levi and parabolic subgroups are Poisson $\mathfrak{h}$-base manifolds.

Obviously, the same construction works for the dual Lie bialgebra $\mathfrak{h}^*_{op}$ and its sub-bialgebras; the corresponding coset spaces will be $\mathfrak{h}$-base manifolds.

*Example 3.18 (Group $H$, the quasitriangular case).* Suppose that $\mathfrak{h}$ is a quasi-triangular Lie bialgebra, i.e. $\mathfrak{h}$ is endowed with an r-matrix $r$ and a symmetric invariant element $\omega \in \mathfrak{h}\otimes\mathfrak{h}$ such that $r$ satisfies (16) with $\varphi := [\omega_{12}, \omega_{23}]$. We can treat $r$ and $\omega$ as linear maps from $\mathfrak{h}^*_{op}$ to $\mathfrak{h}$ via pairing with the first tensor factor. Consider the Lie group $H$ corresponding to $\mathfrak{h}$ as a right $H$-manifold via the action $x \mapsto y^{-1}xy$, $x, y \in H$. This action generates the action of $\mathfrak{h}$ on the function algebra $\mathcal{A}(H)$ by vector fields $\vec{h} := h^l - h^r$, $h \in \mathfrak{h}$. Here the superscripts $l, r$ stand for the left- and right- $H$-invariant vector fields generated, respectively, by the right and the left regular actions of $H$ on itself. The group $H$ is also a right $\mathfrak{h}^*_{op}$-manifold. Namely, the element $\eta \in \mathfrak{h}^*_{op}$ acts on functions from $\mathcal{A}(H)$ by the vector field $\vec{\eta} := r(\eta)^l - r(\eta)^r + \omega(\eta)^l + \omega(\eta)^r$. We have

$$\begin{aligned} 2\vec{\theta} &= (r^{l,l} - r^{r,l} - r^{l,r} + r^{r,r}) + (\omega^{l,l} - \omega^{r,l} + \omega^{l,r} - \omega^{r,r}) \\ &\quad -(r^{l,l} - r^{l,r} - r^{r,l} + r^{r,r}) + (\omega^{l,l} - \omega^{l,r} + \omega^{r,l} - \omega^{r,r}) \\ &= 2(\omega^{l,l} - \omega^{r,r}), \end{aligned} \tag{19}$$

which vanishes on functions, because $\omega$ is invariant. These actions of $\mathfrak{h}$ and $\mathfrak{h}^*_{op}$ define an action of the double $D(\mathfrak{h})$, thus the group space $H$ is an $\mathfrak{h}$-base manifold. In this example, $\varpi$ is the reflection equation Poisson bracket, [Sem]. Quantization of this bracket is an RE algebra, cf. Example 3.9.

*3.2.3. Poisson dynamical algebras.* In this subsection, we define a Poisson dynamical bracket as an infinitesimal object for the deformation quantization of a commutative algebra $\mathcal{A}$ to a dynamical associative algebra, in the sense of Definition 3.11. We assume $\mathcal{A} := \mathcal{A}(P)$, a function algebra on a manifold $P$.

Given a linear space $X$, by Alt we denote a linear endomorphism of $X^{\otimes 3}$ acting by

$$\mathrm{Alt}\colon x_1 \otimes x_2 \otimes x_3 \mapsto x_1 \otimes x_2 \otimes x_3 - x_2 \otimes x_1 \otimes x_3 + x_2 \otimes x_3 \otimes x_1, \quad x_i \in X.$$

**Definition 3.19.** *Let $\mathfrak{h}$ be a Lie bialgebra with the cobracket $\mu$, $L$ an $\mathfrak{h}$-base manifold, and $P$ an $\mathfrak{h}$-manifold. Let $T(P)$ denote the tangent bundle to $P$. A function $\pi\colon L \to \wedge^2 T(P)$ is called a* **Poisson dynamical bracket** *on $P$ (or on $\mathcal{A}(P)$) over the base manifold $L$ (or over the Poisson $\mathfrak{h}$-base algebra $\mathcal{A}(L)$) if*

*1. for any $h \in \mathfrak{h}$ and $a, b \in \mathcal{A}(P)$*

$$\vec{h}_L \pi(\lambda)(a,b) + \vec{h}_P\big(\pi(\lambda)(a,b)\big) - \pi(\lambda)(\vec{h}_P a, b) - \pi(\lambda)(a, \vec{h}_P b)$$
$$= \overrightarrow{\mu(h)}_P(a,b), \tag{20}$$

*2. $\pi$ satisfies the equation*

$$\sum_i \mathrm{Alt}\big(\vec{h^i}_P \otimes \vec{\eta^i}_L \pi(\lambda)\big) = [\![\pi(\lambda), \pi(\lambda)]\!]. \tag{21}$$

In this definition the expression $\pi(a, b)$ is a function on $P \times L$. The vector fields $\vec{h}_L$ and $\vec{h}_P$ are generated by the actions of $\mathfrak{h}$ on $L$ and $P$, respectively; $\overrightarrow{\mu(h)}_P$ is a bivector field induced on $P$ by the Lie cobracket $\mu(h) \in \wedge^2\mathfrak{h}$. The vector field $\vec{\eta}^i_L$ is induced by the actions of $\mathfrak{h}^*_{op}$ on $L$ (recall that $L$ is a $D(\mathfrak{h})$-manifold).

When $P$ is endowed with a Poisson dynamical bracket over a base $L$, we say that $\mathcal{A}(P)$ is a **Poisson dynamical algebra**. While a Poisson base manifold is a classical analog of a base algebra, a Poisson dynamical algebra is a classical analog of dynamical associative algebra. The Poisson dynamical bracket may be viewed as a map $\pi : \mathcal{A}(P) \wedge \mathcal{A}(P) \to \mathcal{A}(P) \otimes \mathcal{A}(L)$.

The following proposition is a generalization of Theorem 2.3.

**Proposition 3.20.** *Let $\mathfrak{h}$ be a Lie bialgebra with cobracket $\mu$, $L$ an $\mathfrak{h}$-base and $P$ an $\mathfrak{h}$-manifold. A function $\pi : L \to \wedge^2 T(P)$ is a Poisson dynamical bracket on $P$ over the base $L$ if and only if the bivector*

$$\sum_i \vec{\eta^i}_L \wedge \vec{h^i}_L + 2\sum_i \vec{\eta^i}_L \wedge \vec{h^i}_P + \pi \tag{22}$$

*is a Poisson-Lie bracket on the $\mathfrak{h}$-manifold $P \times L$.*

*Proof.* This statement is proven by a direct computation. It can be considered as an infinitesimal analog of Theorem 3.12. □

If the base manifold $L$ has $\mathfrak{h}$-stable points, then a Poisson dynamical bracket on $P$ can be restricted to the coset space $P/H$, where it becomes an ordinary Poisson bracket. This is formalized by the following proposition.

**Proposition 3.21.** *Let $\lambda_0 \in L$ be a stable point under the action of $\mathfrak{h}$. Then $\pi(\lambda_0)$ restricts to a Poisson bracket on the subalgebra of $\mathfrak{h}$-invariants in $\mathcal{A}(P)$.*

*Proof.* By the equivariance condition (20), the function $\pi(\lambda_0)(f, g)$ is $\mathfrak{h}$-invariant when $f, g \in \mathcal{A}(P)$ are $\mathfrak{h}$-invariant. The Schouten bracket of $\pi(\lambda_0)$ with itself vanishes on $\mathfrak{h}$-invariant elements from $\mathcal{A}(P)$, as follows from (21). □

Proposition 3.21 gives rise to a quantization method for the class of Poisson structures coming from Poisson dynamical structures. This method is developed in Sect. 7 and uses dynamical associative algebras, which are quantizations of Poisson dynamical algebras.

*3.4. Quantization of Poisson base algebras and Poisson dynamical algebras.* Let $\mathfrak{h}$ be a Lie bialgebra and $\mathcal{U}_t(\mathfrak{h})$ the corresponding quantization of $\mathcal{U}(\mathfrak{h})$. Suppose $\mathcal{L}_0$ is a Poisson $\mathfrak{h}$-base algebra. By Proposition 3.15, $\mathcal{L}_0$ is endowed with a Poisson bracket induced by the tensor $\sum_i \eta^i \otimes h^i \in D(\mathfrak{h})^{\otimes 2}$, where $\{h^i\} \subset \mathfrak{h}$ and $\{\eta^i\} \subset \mathfrak{h}^*_{op}$ are dual bases.

**Definition 3.22.** *A quantization of the Poisson* $\mathfrak{h}$*-base algebra* $\mathcal{L}_0$ *is a base algebra* $\mathcal{L}_t$ *over* $\mathcal{U}_t(\mathfrak{h})$ *that is a* $\mathcal{U}_t(\mathfrak{h})$*-equivariant deformation quantization of* $\mathcal{L}_0$ *with the multiplication*

$$a *_t b = ab + O(t), \quad a *_t b - b *_t a = t \sum_i (\vec{\eta}^i a)(\vec{h}^i b) + O(t^2) \tag{23}$$

*and the coaction* $\delta\colon \mathcal{L}_t \to \mathcal{U}_t(\mathfrak{h}) \otimes \mathcal{L}_t$*,*

$$\delta(a) = 1 \otimes a + t \sum_i h^i \otimes (\vec{\eta}^i a) + O(t^2), \tag{24}$$

*where* $a, b \in \mathcal{L}_t$*.*

When $\mathcal{L}_0 = \mathcal{A}(L)$, the function algebra on an $\mathfrak{h}$-base manifold, one may require in the definition that $\mathcal{L}_t$ is a star product. Then $D(\mathfrak{h})$ acts on $\mathcal{L}_0$ by vector fields.

**Theorem 3.23.** *Any Poisson base algebra can be quantized.*

*Proof.* Let $\theta = \frac{1}{2}\sum_i (h^i \otimes \eta^i + \eta^i \otimes h^i) \in D(\mathfrak{h}) \otimes D(\mathfrak{h})$ be the canonical symmetric invariant of the double Lie algebra $D(\mathfrak{h})$. Consider the quasi-Hopf algebra $\mathcal{U}\big(D(\mathfrak{h})\big)[[t]]$ with the R-matrix $e^{t\theta}$ and the associator $\Phi_t$, which is expressed through $t\theta_{12}$ and $t\theta_{23}$, [Dr3]. Since $\theta$ vanishes on $\mathcal{L}_0$, so do $e^{t\theta}$ and $\Phi_t$. Therefore $\mathcal{L}_0[[t]]$ is a commutative algebra not only in the classical monoidal category of $\mathcal{U}\big(D(\mathfrak{h})\big)[[t]]$-modules, but also in the category with the associator $\Phi_t$, i.e. $\mathcal{L}_0$ is $e^{t\theta}$-commutative and $\Phi_t$-associative.

According to [EK], there exists a twist $J_t$ converting the quasi-Hopf algebra $\mathcal{U}\big(D(\mathfrak{h})\big)[[t]]$ into a Hopf one, $\mathcal{U}_t\big(D(\mathfrak{h})\big)$. This Hopf algebra contains the quantized enveloping algebras $\mathcal{U}_t(\mathfrak{h})$ and $\mathcal{U}_t(\mathfrak{h}^*_{op})$ as Hopf subalgebras. The Hopf algebra $\mathcal{U}_t\big(D(\mathfrak{h})\big)$ is quasitriangular, with the universal R-matrix $\mathcal{R}_t = (J_t)^{-1}_{21} e^{t\theta} J_t$ lying in $\mathcal{U}_t(\mathfrak{h}^*_{op}) \otimes \mathcal{U}_t(\mathfrak{h}) \subset \mathcal{U}_t\big(D(\mathfrak{h})\big) \otimes \mathcal{U}_t\big(D(\mathfrak{h})\big)$.

Applying the twist $J_t$ to the algebra $\mathcal{L}_0[[t]]$, we obtain a quasi-commutative algebra $\mathcal{L}_t$ in the category of $\mathcal{U}_t\big(D(\mathfrak{h})\big)$-modules. We introduce on $\mathcal{L}_t$ a structure of the $\mathcal{U}_t(\mathfrak{h})$-comodule algebra by setting

$$\delta(\ell) := (\mathcal{R}_t)_2 \otimes (\mathcal{R}_t)_1 \triangleright \ell, \quad \ell \in \mathcal{L}_t. \tag{25}$$

Together with the $\mathcal{U}_t(\mathfrak{h})$-action restricted from $\mathcal{U}_t\big(D(\mathfrak{h})\big)$, the coaction (25) makes $\mathcal{L}_t$ a $\mathcal{U}_t(\mathfrak{h})$-base algebra. This follows from Corollary 3.7, where one should set $\mathcal{H} = \mathcal{U}_t\big(D(\mathfrak{h})\big)$ and $\mathcal{K} = \mathcal{U}_t(\mathfrak{h})$. $\square$

**Definition 3.24.** *Let* $L$ *be an* $\mathfrak{h}$*-base manifold. Let* $P$ *be an* $\mathfrak{h}$*-manifold and* $\pi$ *a Poisson dynamical bracket on* $P$ *over* $L$*. A quantization of Poisson dynamical* $\mathfrak{h}$*-algebra* $\mathcal{A}(P)$ *is a pair* $\big(\mathcal{L}_t, \mathcal{A}_t(P)\big)$*, where a)* $\mathcal{L}_t$ *is a quantization of the Poisson base algebra* $\mathcal{A}(L)$ *in the sense of Definition 3.22 and b)* $\mathcal{A}_t(P)$ *is a flat* $\mathbb{C}[[t]]$*-module and a dynamical associative* $\mathcal{U}_t(\mathfrak{h})$*-algebra over* $\mathcal{L}_t$ *such that* $\mathcal{A}_t(P)/t\mathcal{A}_t(P) = \mathcal{A}(P)$ *and* $a \circledast b - b \circledast a = t\pi(a, b) + O(t^2)$*.*

**Conjecture 3.25.** *Any Poisson dynamical algebra can be quantized.*

In Subsect. 7.2, we develop a method of quantizing vector bundles on the coset space $P/H$, using dynamical associative algebras. By duality, the construction of Sect. 7.2 can be formulated in terms of base algebras rather than coalgebras. Namely, let $\mathcal{A}$ be a dynamical associative algebra over an $\mathcal{H}$-base algebra $\mathcal{L}$ and let $\chi$ be an $\mathcal{H}$-invariant character of $\mathcal{L}$. Then the composition map $\mathcal{A} \otimes \mathcal{A} \xrightarrow{\circledast} \mathcal{A} \otimes \mathcal{L} \xrightarrow{\mathrm{id}\otimes\chi} \mathcal{A}$ yields an associative multiplication on the subspace of $\mathcal{H}$-invariant elements of $\mathcal{A}$. Thus invariant characters of base algebras are important for our approach to quantization (see also [DM1]).

In the deformation situation, the infinitesimal analogs of $\mathcal{U}_t(\mathfrak{h})$-invariant characters of the base algebra $\mathcal{L}_t$ are $\mathfrak{h}$-stable points on the $\mathfrak{h}$-base manifold $L$. By Proposition 3.21, each $\mathfrak{h}$-stable point defines a Poisson structure on $P/H$. It is natural to quantize this Poisson structure by the corresponding invariant character applying it to the dynamical associative quantization of $\mathcal{A}(P)$. The question is whether every $\mathfrak{h}$-stable point can be quantized to a $\mathcal{U}_t(\mathfrak{h})$-invariant character of $\mathcal{L}_t$. The answer to this question is affirmative.

**Proposition 3.26.** *Let $\mathcal{L}_t$ be the quantization of the function algebra on a base manifold $L$ built in Theorem 3.23. Then every $\mathfrak{h}$-stable point $\lambda_0$ on $L$ defines an $\mathcal{U}_t(\mathfrak{h})$-invariant character of $\mathcal{L}_t$ by $\chi^{\lambda_0}(f) = f(\lambda_0)$ for $f \in \mathcal{L}_t$.*

*Proof.* As follows from the explicit form of the twist $J_t$ constructed in [EK], it reduces to $1 \otimes 1$ at every $\mathfrak{h}$-stable point $\lambda_0 \in L$. It follows from the proof of Theorem 3.23 that the star product in $\mathcal{L}_t$ satisfies $(f * g)(\lambda_0) = (fg)(\lambda_0) = f(\lambda_0)g(\lambda_0)$ for any pair of functions $f, g \in \mathcal{A}(L)$. $\square$

*3.5. Dynamical Yang-Baxter equations.* In this subsection we give definitions of the classical and quantum dynamical Yang-Baxter equations over an arbitrary base algebra.

**Definition 3.27.** *Let $\mathfrak{g}$ be a Lie bialgebra and $\mathfrak{h} \subset \mathfrak{g}$ its sub-bialgebra; let $\mu$ denote the Lie cobracket on $\mathfrak{h}$. Let $L$ be a Poisson $\mathfrak{h}$-base manifold. A function $\bar{r}\colon L \to \mathfrak{g} \otimes \mathfrak{g}$ is called a* **classical dynamical r-matrix** *over base $L$ if*

*1. for any $h \in \mathfrak{h}$*

$$\vec{h}_L \bar{r}(\lambda) + [h \otimes 1 + 1 \otimes h, \bar{r}(\lambda)] = \mu(h), \tag{26}$$

*2. the sum $\bar{r}(\lambda) + \bar{r}_{21}(\lambda)$ is $\mathfrak{g}$-invariant,*
*3. $\bar{r}$ satisfies the equation*

$$\sum_i \mathrm{Alt}\big(h^i \otimes \vec{\eta^i}_L \bar{r}(\lambda)\big) = [\![\bar{r}(\lambda), \bar{r}(\lambda)]\!]. \tag{27}$$

We call (27) the classical dynamical Yang-Baxter equation over the base $L$. The skew part of $\bar{r}$ satisfies the "modified" version of the dynamical Yang-Baxter equation with non-zero right-hand side being an invariant element from $\wedge^3\mathfrak{g}$. We call it skew dynamical r-matrix. Condition (26) means quasi-equivariance of $\bar{r}(\lambda)$ with respect to the action of $\mathfrak{h}$. In fact, the symmetric part $\bar{\omega} = \frac{1}{2}(\bar{r} + \bar{r}_{21})$ is constant on every $D(\mathfrak{h})$-orbit in $L$, i.e. $\vec{x}_L \bar{\omega} = 0$ for any $x \in D(\mathfrak{h})$.

Consider the opposite Lie bialgebra $\mathfrak{h}_{op}$ equipped with the opposite bracket and the same cobracket. Endowed with the opposite bracket, the manifold $L$ becomes a Poisson base manifold for $\mathfrak{h}_{op}$. The Cartesian product $L \times L$ is a Poisson base manifold with respect to the Lie bialgebra $\mathfrak{h} \oplus \mathfrak{h}_{op}$. Let $G$ be the Lie group corresponding to $\mathfrak{g}$. The following proposition characterizes the dynamical r-matrices.

**Proposition 3.28.** *A function $\bar r\colon L \to \mathfrak{g} \wedge \mathfrak{g}$ is a skew dynamical r-matrix over the Poisson $\mathfrak{h}$-base manifold $L$ if and only if $M := G \times L \times L$ is equipped with the $\mathfrak{h} \oplus \mathfrak{h}_{op}$-Poisson Lie structure such that the projection $M \to L \times L$ is a Poisson map and*

$$\{f, a\} := \sum_i (\vec{\eta}^i_L f)\big(h^{i\,l} + h^{i\,r}\big)(a), \tag{28}$$

$$\{a, b\} := \big(\bar r^{l,l}(\lambda') - \bar r^{r,r}(\lambda'')\big)(a, b) \tag{29}$$

*for $f \in \mathcal{A}(L \times L)$, $a, b \in \mathcal{A}(G)$, and $(\lambda', \lambda'') \in L \times L$.*

*Proof.* Straightforward. $\square$

Suppose that $\mathfrak{g}$ is a quasitriangular Lie bialgebra with an r-matrix $r_{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}$. Let $\omega_{\mathfrak{g}} = \frac{1}{2}(r_{\mathfrak{g}} + r_{\mathfrak{g}}^{21})$ denote the symmetric part of $r_{\mathfrak{g}}$. Assume that $L$ is $D(\mathfrak{h})$-transitive, i.e. $D(\mathfrak{h})$-invariants in $\mathcal{A}(L)$ are scalars.

**Proposition 3.29.** *A function $\bar r\colon L \to \mathfrak{g} \otimes \mathfrak{g}$ subject to $\frac{1}{2}(\bar r + \bar r) = \omega_{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}$ is a dynamical r-matrix if and only if $M := L \times G$ is equipped with an $H$-invariant Poisson structure, such that the projection $M \to L$ is a Poisson map, and*

$$\{f, a\} := \sum_i (\vec{\eta}^i_L f)\big(h^{i\,l} a\big), \quad \{a, b\} := (\bar r^{l,l} - r_{\mathfrak{g}}^{r,r})(a, b) \tag{30}$$

*for $f \in \mathcal{A}(L)$, $a, b \in \mathcal{A}(G)$,*

*Proof.* Straightforward. $\square$

Proposition 3.28 implies that the bivector field $\bar r^{l,l}(\lambda') - \bar r^{r,r}(\lambda'')$ makes $\mathcal{A}(G)$ a Poisson dynamical algebra over the $\mathfrak{h} \oplus \mathfrak{h}_{op}$-base manifold $L \times L$. By Proposition 3.29, the bivector field $\bar r^{l,l}(\lambda) - r_{\mathfrak{g}}^{r,r}$ makes $\mathcal{A}(G)$ a Poisson dynamical algebra over the $\mathfrak{h}$-base manifold $L$.

**Proposition 3.30.** *Let $\bar r\colon L \to \mathfrak{g} \otimes \mathfrak{g}$ be a classical dynamical r-matrix on an $\mathfrak{h}$-base manifold L. Let $r_{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}$ be a constant r-matrix whose symmetric part coincides with the symmetric part of $\bar r$. Suppose that $\lambda_0 \in L$ is an $\mathfrak{h}$-stable point. Then the bivector field $\bar r^{l,l}(\lambda_0) - r_{\mathfrak{g}}^{r,r}$ yields a $\mathfrak{g}$-Poisson-Lie structure on the coset space $G/H$.*

*Proof.* Applying Proposition 3.21 to the Poisson dynamical bracket $\bar r^{l,l}(\lambda) - r_{\mathfrak{g}}^{r,r}$ on $G$, we obtain a Poisson structure on the subalgebra in $\mathcal{A}(G)$ that consists of invariants under the action of $\mathfrak{h}$ by the left-invariant vector fields. This algebra is canonically identified with the algebra of functions on the coset space $G/H$. Obviously, this Poisson structure is a Poisson-Lie one, with respect to the right $\mathfrak{g}$-action on $\mathcal{A}(G/H)$ induced by the left $G$-action on $G/H$. $\square$

*Remark 3.31.* Let $\mathfrak{h} = \mathfrak{g}$ be quasitriangular, with the classical r-matrix $r_{\mathfrak{g}}$ whose symmetric part is equal to the symmetric part of $\bar r$. Then $\bar r(\lambda) - r_{\mathfrak{g}}$ is the dynamical r-matrix of [FhMrsh].

We complete this subsection with a definition of the quantum dynamical Yang-Baxter equation over an arbitrary base algebra. This definition naturally follows from our categorical point of view presented in Sect. 5.

The quantum DYBE will be defined for any triple $(\mathcal{U}, \mathcal{H}, \mathcal{L})$, where $\mathcal{H}$ is a Hopf subalgebra in a Hopf algebra $\mathcal{U}$ and $\mathcal{L}$ is an $\mathcal{H}$-base algebra.

**Definition 3.32.** *An element $\bar{\mathcal{R}} = \bar{\mathcal{R}}_1 \otimes \bar{\mathcal{R}}_2 \otimes \bar{\mathcal{R}}_3 \in \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{L}$ is called a* **universal quantum dynamical R-matrix** *of $\mathcal{U}$ over the $\mathcal{H}$-base algebra $\mathcal{L}$ if it satisfies the equivariance condition*

$$h^{(2)}\bar{\mathcal{R}}_1 \otimes h^{(1)}\bar{\mathcal{R}}_2 \otimes h^{(3)} \triangleright \bar{\mathcal{R}}_3 = \bar{\mathcal{R}}_1 h^{(1)} \otimes \bar{\mathcal{R}}_2 h^{(2)} \otimes \bar{\mathcal{R}}_3, \quad h \in \mathcal{H}, \tag{31}$$

*and the quantum dynamical Yang-Baxter equation*

$$\bar{\mathcal{R}}_{12}\ {}^{(2)}\bar{\mathcal{R}}_{13}\ \bar{\mathcal{R}}_{23} = {}^{(1)}\bar{\mathcal{R}}_{23}\ \bar{\mathcal{R}}_{13}\ {}^{(3)}\bar{\mathcal{R}}_{12}, \tag{32}$$

*in $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{L}$.*

Here the notation ${}^{(i)}\bar{\mathcal{R}}$ means the following. Applying the coaction to the $\mathcal{L}$-component of $\mathcal{R}$, we get the element ${}^{(3)}\bar{\mathcal{R}} := \bar{\mathcal{R}}_1 \otimes \bar{\mathcal{R}}_2 \otimes \bar{\mathcal{R}}_3^{(1)} \otimes \bar{\mathcal{R}}_3^{[2]}$. The other two are obtained from this by permutations, namely ${}^{(2)}\bar{\mathcal{R}} := \bar{\mathcal{R}}_1 \otimes \bar{\mathcal{R}}_3^{(1)} \otimes \bar{\mathcal{R}}_2 \otimes \bar{\mathcal{R}}_3^{[2]}$ and ${}^{(1)}\bar{\mathcal{R}} := \bar{\mathcal{R}}_3^{(1)} \otimes \bar{\mathcal{R}}_1 \otimes \bar{\mathcal{R}}_2 \otimes \bar{\mathcal{R}}_3^{[2]}$.

Equation (32) specializes to (6) for $\mathcal{H} = \mathcal{U}(\mathfrak{h}), \mathcal{U} = \mathcal{U}(\mathfrak{g})$, and $\mathcal{L}$ being the extension of $\mathcal{U}(\mathfrak{h}_t)$ to the PBW star product on functions on $\mathfrak{h}^*$, cf. Example 3.5. Also, Eq. (32) coincides with the conventional dynamical Yang-Baxter equation (2) for $\mathfrak{h}$ a commutative Lie subalgebra in $\mathfrak{g}$ and $\mathcal{L}$ being the algebra of functions on $\mathfrak{h}^*$, [EV2].

Suppose that $\mathcal{U}$ and $\mathcal{H}$ are quantizations of the universal enveloping algebras $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{h})$ and $\mathcal{L}$ is a quantization of the function algebra on a Poisson $\mathfrak{h}$-base manifold $L$. Suppose that the universal dynamical R-matrix has the form $\mathcal{R} = 1 \otimes 1 \otimes 1 + t\bar{r} + O(t^2)$. Then $\bar{r}$ is a function on $L$ with values in $\mathfrak{g} \otimes \mathfrak{g}$. It satisfies Eq. (26) and (27), which are the consequences of Eqs. (31) and (32).

*Remark 3.33.* The definitions of the classical and quantum dynamical R-matrix given above admit further generalization. The reader is referred to [DM5], where the classical dynamical r-matrices are studied in connection with Lie bialgebroids. The present definitions are conditioned by our specific approach confined to the strict monoidal categories (i.e. with trivial associator). If one considers general monoidal categories, as in Example 5.5, Eq. (32) would involve an associator. In the quasi-classical limit, it will give the dynamical r-matrix of the Alekseev-Meinrenken type, [AM].

## 4. Dynamical Categories

*4.1. Base algebra in a monoidal category.* A dynamical associative algebra $\mathcal{A}$ from Definition 3.11 may serve as a model for further generalizations. It turns out that there is a monoidal category where $\mathcal{A}$ is an associative algebra. Such categories can be built for all Hopf algebras and they include dynamical categories of Etingof-Varchenko introduced for commutative cocommutative Hopf algebras in [EV3]. Such notions as dynamical twist and the dynamical Yang-Baxter equation can be naturally formulated and generalized within the dynamical categories, which are the subject of our further study.

Let $(\hat{\mathcal{O}}, \otimes)$ be a monoidal category. We will work, for simplicity, with only strict monoidal categories, i.e. having the trivial associator; all the constructions can be carried over to the general case in a straightforward way. Let $Z(\hat{\mathcal{O}})$ be the center of $\hat{\mathcal{O}}$, see [Kas]. The center is a braided monoidal category consisting of pairs $(A, \tau)$, where $A$ is an object of $\hat{\mathcal{O}}$ and $\tau$ the collection of permutations $\tau_X\colon A \otimes X \to X \otimes A$, satisfying natural conditions.

**Definition 4.1.** *A* **base algebra** *in the category $\hat{\mathcal{O}}$ is commutative algebra from a $Z(\hat{\mathcal{O}})$.*

In other words, a base algebra is an algebra in $\hat{\mathcal{O}}$ and a collection of morphisms $\tau_A \in \mathrm{Hom}_{\hat{\mathcal{O}}}(\mathcal{L} \otimes A, A \otimes \mathcal{L})$ and $A \in \mathrm{Ob}\ \hat{\mathcal{O}}$, such that the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{L}\otimes B & \xrightarrow{\tau_B} & B\otimes\mathcal{L} \\ {\scriptstyle \mathrm{id}_{\mathcal{L}}\otimes\psi}\downarrow & & \downarrow{\scriptstyle \psi\otimes\mathrm{id}_{\mathcal{L}}} \\ \mathcal{L}\otimes A & \xrightarrow{\tau_A} & A\otimes\mathcal{L} \end{array} \tag{33}$$

$$\begin{array}{ccc} \mathcal{L}\otimes A\otimes B & \xrightarrow{\tau_{A\otimes B}} & A\otimes B\otimes\mathcal{L} \\ & {\scriptstyle \tau_A}\searrow \quad \swarrow{\scriptstyle \tau_B} & \\ & A\otimes\mathcal{L}\otimes B & \end{array} \tag{34}$$

$$\begin{array}{ccccc} \mathcal{L}\otimes\mathcal{L}\otimes A & \xrightarrow{\tau_A} & \mathcal{L}\otimes A\otimes\mathcal{L} & \xrightarrow{\tau_A} & A\otimes\mathcal{L}\otimes\mathcal{L} \\ {\scriptstyle \mathrm{m}_{\mathcal{L}}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{m}_{\mathcal{L}}} \\ \mathcal{L}\otimes A & & \xrightarrow{\tau_A} & & A\otimes\mathcal{L} \end{array} \tag{35}$$

$$\begin{array}{ccc} \mathcal{L}\otimes\mathcal{L} & \xrightarrow{\tau_{\mathcal{L}}} & \mathcal{L}\otimes\mathcal{L} \\ & {\scriptstyle \mathrm{m}_{\mathcal{L}}}\searrow \quad \swarrow{\scriptstyle \mathrm{m}_{\mathcal{L}}} & \\ & \mathcal{L} & \end{array} \tag{36}$$

for all $A, B \in \mathrm{Ob}\ \hat{\mathcal{O}}$, $\psi \in \mathrm{Hom}_{\hat{\mathcal{O}}}(B, A)$.

*Example 4.2.* The unit object $1_{\hat{\mathcal{O}}}$ is the simplest example of a base algebra. The algebra structure and permutation are defined by the canonical isomorphisms $1_{\hat{\mathcal{O}}} \otimes A \simeq A \simeq A \otimes 1_{\hat{\mathcal{O}}}$ for all $A \in \mathrm{Ob}\ \hat{\mathcal{O}}$.

*Example 4.3.* When the category $\hat{\mathcal{O}}$ is braided with braiding $\sigma$, any commutative algebra $\mathcal{L}$ in this category has two natural base algebra structures, with respect to the $\tau = \sigma$ and $\tau = \sigma^{-1}$.

*Example 4.4.* Let $\mathcal{H}$ be a Hopf algebra and $\hat{\mathcal{O}}$ the monoidal category of left $\mathcal{H}$-modules. Any $\mathcal{H}$-base algebra in the sense of Definition 3.1 is a base algebra in the category $\hat{\mathcal{O}}$. Indeed, for a left $\mathcal{H}$-module $A$ we define the permutation $\tau_A\colon \mathcal{L} \otimes A \to A \otimes \mathcal{L}$ by

$$\ell \otimes a \to \ell^{(1)} \triangleright a \otimes \ell^{[2]}, \quad a \in A,\ \ell \in \mathcal{L}. \tag{37}$$

The permutation (37) is $\mathcal{H}$-equivariant, as follows from (9), hence the condition (33) is satisfied. Conditions (34) and (35) hold because $\mathcal{L}$ is an $\mathcal{H}$-comodule algebra. Equation (36) follows from (10).

*Example 4.5.* Let $\hat{\mathcal{O}}$ be the category of semisimple modules over a commutative finite dimensional Lie algebra $\mathfrak{h}$. Take $\mathcal{L}$ to be the algebra of functions on $\mathfrak{h}^*$, which is a trivial $\mathfrak{h}$-module. Let $A$ be a semisimple $\mathfrak{h}$-module. The permutation $\tau_A$ between $\mathcal{L}$ and $A$ is defined by $f(x) \otimes a \mapsto a \otimes f\big(x + \alpha(a)\big)$, where $f \in \mathcal{L}$, $a \in A$, and $\alpha(a)$ is the weight of $a$.

*4.2. Dynamical categories over base algebras.* Let $(\hat{\mathcal{O}}, \otimes)$ be a monoidal category and $\mathcal{O}$ be a monoidal subcategory in $\hat{\mathcal{O}}$. Given a base algebra $(\mathcal{L}, \tau)$ in $\hat{\mathcal{O}}$, let us construct a new monoidal category $\bar{\mathcal{O}}_{\mathcal{L}}$. Objects in $\bar{\mathcal{O}}_{\mathcal{L}}$ are the same as in $\mathcal{O}$. For two objects $A$ and $B$ in $\bar{\mathcal{O}}_{\mathcal{L}}$, morphisms $\mathrm{Hom}_{\bar{\mathcal{O}}_{\mathcal{L}}}(A, B)$ are $\hat{\mathcal{O}}$-morphisms $\mathrm{Hom}_{\hat{\mathcal{O}}}(A, B \otimes \mathcal{L})$. Since the algebra $\mathcal{L}$ is unital, every morphism $\phi \in \mathrm{Hom}_{\mathcal{O}}(A, B)$ naturally becomes a morphism from $\mathrm{Hom}_{\bar{\mathcal{O}}_{\mathcal{L}}}(A, B)$ through the composition $A \xrightarrow{\phi} B \otimes 1_{\hat{\mathcal{O}}} \to B \otimes \mathcal{L}$.

The composition of two morphisms $A \xrightarrow{\phi} B$ and $B \xrightarrow{\psi} C$ in $\bar{\mathcal{O}}_{\mathcal{L}}$ is defined as the composition

$$A \xrightarrow{\phi} B \otimes \mathcal{L} \xrightarrow{\psi} C \otimes \mathcal{L} \otimes \mathcal{L} \xrightarrow{\mathrm{m}_{\mathcal{L}}} C \otimes \mathcal{L}, \tag{38}$$

in $\hat{\mathcal{O}}$, where the rightmost arrow is the multiplication in $\mathcal{L}$. It is easy to see that the composition is associative. The identity morphism $\mathrm{id}_A$ for $A \in \mathrm{Ob}\ \mathcal{O}_{\mathcal{L}}$ is the composition $A \to A \otimes 1_{\hat{\mathcal{O}}} \to A \otimes \mathcal{L}$, where the first arrow is the canonical isomorphism and the second one is the natural inclusion $1_{\hat{\mathcal{O}}} \to \mathcal{L}$ via the unit of $\mathcal{L}$. Thus $\bar{\mathcal{O}}_{\mathcal{L}}$ is a category.

Let us introduce a monoidal structure $\bar{\otimes}$ in $\bar{\mathcal{O}}_{\mathcal{L}}$ setting it on objects as in $\mathcal{O}$; on the morphisms it is defined by the composition

$$A \otimes C \xrightarrow{\phi \otimes \psi} B \otimes \mathcal{L} \otimes D \otimes \mathcal{L} \xrightarrow{\tau_D} B \otimes D \otimes \mathcal{L} \otimes \mathcal{L} \xrightarrow{\mathrm{m}_{\mathcal{L}}} B \otimes D \otimes \mathcal{L}, \tag{39}$$

for $\phi \in \mathrm{Hom}_{\bar{\mathcal{O}}_{\mathcal{L}}}(A, B)$ and $\psi \in \mathrm{Hom}_{\bar{\mathcal{O}}_{\mathcal{L}}}(C, D)$.

**Proposition 4.6.** *The tensor product $\bar{\otimes}$ defined by (39) makes $\bar{\mathcal{O}}_{\mathcal{L}}$ a monoidal category.*

*Proof.* The unit object $1_{\mathcal{O}}$ is obviously the neutral element for $\bar{\otimes}$. Let us prove associativity of $\bar{\otimes}$. Using compatibility (35) of $\tau$ with the multiplication $\mathrm{m}_{\mathcal{L}}$ and associativity (34) we find that the diagram

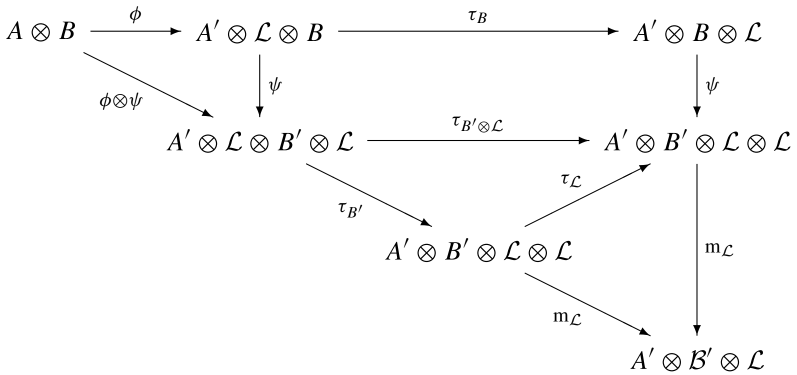
$$\begin{array}{ccccc}
\mathcal{L} \otimes A \otimes \mathcal{L} \otimes B & \xrightarrow{\tau_A} & A \otimes \mathcal{L} \otimes \mathcal{L} \otimes B & \xrightarrow{\mathrm{m}_{\mathcal{L}}} & A \otimes \mathcal{L} \otimes B \\
\downarrow{\scriptstyle \tau_B} & & & & \downarrow{\scriptstyle \tau_B} \\
\mathcal{L} \otimes A \otimes B \otimes \mathcal{L} & \xrightarrow{\tau_{A \otimes B}} & A \otimes B \otimes \mathcal{L} \otimes \mathcal{L} & \xrightarrow{\mathrm{m}_{\mathcal{L}}} & A \otimes B \otimes \mathcal{L}
\end{array}$$

is commutative for all $A, B \in \mathrm{Ob}\ \mathcal{O}$. From this one can readily deduce associativity of $\bar{\otimes}$.

Now we will prove functoriality of $\bar{\otimes}$. It is equivalent to the four conditions:

$$(\mathrm{id}\bar{\otimes}\phi)\bar{\circ}(\mathrm{id}\bar{\otimes}\psi) = \mathrm{id}\bar{\otimes}(\phi\bar{\circ}\psi), \tag{40}$$
$$(\phi\bar{\otimes}\mathrm{id})\bar{\circ}(\mathrm{id}\bar{\otimes}\psi) = \phi\bar{\otimes}\psi, \tag{41}$$
$$(\phi\bar{\otimes}\mathrm{id})\bar{\circ}(\psi\bar{\otimes}\mathrm{id}) = (\phi\bar{\circ}\psi)\bar{\otimes}\mathrm{id}, \tag{42}$$
$$(\mathrm{id}\bar{\otimes}\psi)\bar{\circ}(\phi\bar{\otimes}\mathrm{id}) = \phi\bar{\otimes}\psi \tag{43}$$

for any pair of morphisms $\phi$, $\psi$. Observe that $\phi\bar{\otimes}\mathrm{id}_B = (\mathrm{id}_{A'} \otimes \tau_B) \circ (\phi \otimes \mathrm{id}_B)$ and $\mathrm{id}_B\bar{\otimes}\phi = \mathrm{id}_B \otimes \phi$ for any morphism $A \xrightarrow{\phi} A'$ and any object $B$. This immediately leads to (40) and (41). Condition (42) follows from (35). Let us prove condition (43) assuming $\phi \in \mathrm{Hom}_{\bar{\mathcal{O}}_{\mathcal{L}}}(A, A')$ and $\psi \in \mathrm{Hom}_{\bar{\mathcal{O}}_{\mathcal{L}}}(B, B')$. It suffices to show that the following diagram is commutative (the identity maps are suppressed):




(44)

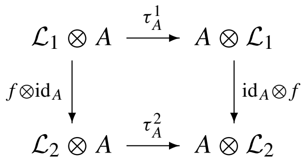
Commutativity of the rectangle follows from (33); the two lower triangles are commutative by virtue of (34 ) and (36). □

The category $\bar{\mathcal{O}}_{\mathcal{L}}$ naturally includes $\mathcal{O}$ as a monoidal subcategory. We call $\bar{\mathcal{O}}_{\mathcal{L}}$ the **dynamical extension** of $\mathcal{O}$ over the base algebra $\mathcal{L}$.

*Example 4.7.* The simplest example is $\mathcal{L} = 1_{\hat{\mathcal{O}}}$ and $\mathcal{O} = \hat{\mathcal{O}}$; then the category $\bar{\mathcal{O}}_{\mathcal{L}}$ is canonically isomorphic to $\mathcal{O}$.

*Example 4.8.* Let $\mathcal{H}$ be a Hopf algebra and $\hat{\mathcal{O}}$ the category of left $\mathcal{H}$-modules. As was mentioned in Example 4.4, any $\mathcal{H}$-base algebra, including $\mathcal{H}$ itself, is a base algebra in $\hat{\mathcal{O}}$. Let $\mathcal{M}_{\mathcal{H}}$ be the subcategory of locally finite $\mathcal{H}$-modules (a module is called locally finite if every one of its elements lies in a finite dimensional submodule). Its dynamical extension over a base algebra $\mathcal{L}$ is denoted further by $\bar{\mathcal{M}}_{\mathcal{H};\mathcal{L}}$, or simply $\bar{\mathcal{M}}_{\mathcal{H}}$ for $\mathcal{L} = \mathcal{H}$.

*4.3. Morphisms of base algebras.* By a morphism of base algebras $(\mathcal{L}_1, \tau^1) \to (\mathcal{L}_2, \tau^2)$ in a category $\hat{\mathcal{O}}$ we mean a morphism of $\hat{\mathcal{O}}$-algebras $\mathcal{L}_1 \xrightarrow{f} \mathcal{L}_2$ such that the diagram




is commutative for all $A \in \mathrm{Ob}\,\hat{\mathcal{O}}$.

*Example 4.9.* Let $\mathcal{H}$ be a Hopf algebra and $\hat{\mathcal{O}}$ the category of left $\mathcal{H}$-modules. A homomorphism of two $\mathcal{H}$-base algebras can be defined as a homomorphism of $\mathcal{H}$-algebras and $\mathcal{H}$-comodules. Then it is a morphism of base algebras in $\hat{\mathcal{O}}$, cf. Example 4.4.

*Example 4.10.* Any invariant character $\chi$ of $\mathcal{L}$ defines a homomorphism of base algebras $\mathcal{L} \to \mathcal{H}$ by the formula $\ell \mapsto \ell^{(1)}\chi\big(\ell^{[2]}\big)$. Indeed, this is an algebra and comodule map because $\mathcal{L}$ is an $\mathcal{H}$-comodule algebra. This map is equivariant for invariant $\chi$, by virtue of (9).

Recall that a functor from one monoidal category to another is called strong monoidal if it is unital (relates the units) and commutes with tensor products. We conclude this subsection with an obvious proposition.

**Proposition 4.11.** *A morphism of base algebras* $(\mathcal{L}_1, \tau^1) \to (\mathcal{L}_2, \tau^2)$ *induces a strong monoidal functor* $\bar{\mathcal{O}}_{\mathcal{L}_1} \to \bar{\mathcal{O}}_{\mathcal{L}_2}$.

*4.4. Category* $\bar{\mathcal{M}}^{\mathcal{H}^*}$. The dynamical extension of a monoidal category can be defined using a notion of base coalgebra instead of base algebra. We will present such a formulation for the case when the monoidal category $\hat{\mathcal{O}}$ is a category of $\mathcal{H}$-modules and the base coalgebra is a restricted dual to $\mathcal{H}$.

Let $\mathcal{H}^*$ denote the Hopf algebra formed by matrix elements of finite dimensional representations of $\mathcal{H}$ (we assume that the supply of such elements is big enough to induce a non-degenerate pairing between $\mathcal{H}^*$ and $\mathcal{H}$). We equip $\mathcal{H}^*$ with the structure of a left $\mathcal{H}$-module with respect to the action

$$x \otimes \lambda \mapsto x^{(2)} \triangleright \lambda \triangleleft \gamma(x^{(1)}), \quad x \in \mathcal{H}, \quad \lambda \in \mathcal{H}^*, \tag{45}$$

expressed through the coregular left and right actions, $\triangleright$ and $\triangleleft$, of $\mathcal{H}$ on $\mathcal{H}^*$.

Let $\hat{O}$ be the category of left $\mathcal{H}$-modules. We can consider the category of locally finite right $\mathcal{H}^*$-comodules as a subcategory in $\hat{\mathcal{O}}$, since every right $\mathcal{H}^*$-comodule is a natural left $\mathcal{H}$-module. We denote this category by $\mathcal{M}^{\mathcal{H}^*}$.

The following statement introduces a permutation between $\mathcal{H}^*$ and other $\mathcal{H}^*$-comodules.

**Proposition 4.12.** *For any* $A \in \mathrm{Ob}\, \mathcal{M}^{\mathcal{H}^*}$ *the map* $\tau^A \colon \mathcal{H}^* \otimes A \to A \otimes \mathcal{H}^*$ *defined as*

$$\tau^A(\lambda \otimes a) := a^{[0]} \otimes \lambda a^{(1)} \tag{46}$$

*is an isomorphism of* $\mathcal{H}$*-modules.*

*Proof.* First of all observe that $\tau^A$ is invertible and its inverse is

$$(\tau^A)^{-1}(a \otimes \lambda) = \lambda\gamma^{-1}(a^{(1)}) \otimes a^{[0]}, \quad \lambda \in \mathcal{H}^*,\ a \in A.$$

Further, for all $x, y \in \mathcal{H}$ we have

$$\begin{aligned} \langle \tau^A\big(x \triangleright (\lambda \otimes a)\big), \mathrm{id} \otimes y\rangle &= \langle \tau^A\big(x^{(1)} \triangleright \lambda \otimes a^{[0]}\big), \mathrm{id} \otimes y\rangle\langle a^{(1)}, x^{(2)}\rangle \\ &= a^{[0]} \otimes \langle (x^{(1)} \triangleright \lambda)a^{(1)}, y\rangle\langle a^{(2)}, x^{(2)}\rangle \\ &= a^{[0]} \otimes \langle x^{(1)} \triangleright \lambda, y^{(1)}\rangle\langle a^{(1)}, y^{(2)}x^{(2)}\rangle. \end{aligned} \tag{47}$$

On the other hand,

$$\begin{aligned}\langle x \triangleright \tau^A(\lambda \otimes a), \mathrm{id} \otimes y\rangle &= a^{[0]} \otimes \langle a^{(1)}, x^{(1)}\rangle \langle \lambda a^{(2)}, \gamma(x^{(2)}) y x^{(3)}\rangle \\ &= a^{[0]} \otimes \langle a^{(1)}, x^{(1)}\rangle \langle \lambda, \gamma(x^{(3)}) y^{(1)} x^{(4)}\rangle \langle a^{(2)}, \gamma(x^{(2)}) y^{(2)} x^{(5)}\rangle \\ &= a^{[0]} \otimes \langle a^{(1)}, y^{(2)} x^{(3)}\rangle \langle \lambda, \gamma(x^{(1)}) y^{(1)} x^{(2)}\rangle \end{aligned} \tag{48}$$

for all $x, y \in \mathcal{H}$, $\lambda \in \mathcal{H}^*$, and $a \in A$. The resulting expression in (48) is easily brought to (47). □

Let us define the dynamical extension, $\bar{\mathcal{M}}^{\mathcal{H}^*}$, of the category $\mathcal{M}^{\mathcal{H}^*}$. The objects in $\bar{\mathcal{M}}^{\mathcal{H}^*}$ are locally finite right $\mathcal{H}^*$- comodules. The set of morphisms $\mathrm{Hom}_{\bar{\mathcal{M}}^{\mathcal{H}^*}}(A, B)$ consists of $\mathcal{H}$-equivariant maps from $\mathcal{H}^* \otimes A$ to $B$. The composition $\phi \bar{\circ} \psi$ of morphisms $\phi \in \mathrm{Hom}(A, A')$ and $\phi \in \mathrm{Hom}(A', A'')$ is defined as the composition map

$$\mathcal{H}^* \otimes A \xrightarrow{\Delta \otimes \mathrm{id}_A} \mathcal{H}^* \otimes \mathcal{H}^* \otimes A \xrightarrow{\mathrm{id}_{\mathcal{H}^*} \otimes \psi} \mathcal{H}^* \otimes A' \xrightarrow{\quad \phi \quad} A'' \tag{49}$$

This operation is apparently associative and $\varepsilon \otimes \mathrm{id}_A$ is the identity in $\mathrm{Hom}_{\bar{\mathcal{M}}^{\mathcal{H}^*}}(A, A)$; here $\varepsilon$ is the counit in $\mathcal{H}^*$.

Now we introduce a monoidal structure on $\bar{\mathcal{M}}^{\mathcal{H}^*}$. We put the tensor product of objects from $\bar{\mathcal{M}}^{\mathcal{H}^*}$ as in $\mathcal{M}^{\mathcal{H}^*}$. The tensor product $\phi \bar{\otimes} \psi$ of $\phi \in \mathrm{Hom}_{\bar{\mathcal{M}}^{\mathcal{H}^*}}(A, A')$ and $\psi \in \mathrm{Hom}_{\bar{\mathcal{M}}^{\mathcal{H}^*}}(B, B')$ is defined as the composition

$$\mathcal{H}^* \otimes A \otimes B \xrightarrow{\Delta} \mathcal{H}^* \otimes \mathcal{H}^* \otimes A \otimes B \xrightarrow{\tau^A} \mathcal{H}^* \otimes A \otimes \mathcal{H}^* \otimes B \xrightarrow{\phi \otimes \psi} A' \otimes B'. \tag{50}$$

One can check that, indeed, the operation $\bar{\otimes}$ makes $\bar{\mathcal{M}}^{\mathcal{H}^*}$ a monodial category.

*4.5. Comparison of categories $\bar{\mathcal{M}}^{\mathcal{H}^*}$ and $\bar{\mathcal{M}}^{\mathcal{H}^*}_{\mathcal{H}}$.* Since $\mathcal{M}^{\mathcal{H}^*}$ is a subcategory in the category of $\mathcal{H}$-modules, it can be extended to the dynamical category $\bar{\mathcal{M}}^{\mathcal{H}^*}_{\mathcal{H}}$ over the base algebra $\mathcal{L} = \mathcal{H}$ along the lines of Subsect. 4.2. Our next goal is to compare the categories $\bar{\mathcal{M}}^{\mathcal{H}^*}_{\mathcal{H}}$ and $\bar{\mathcal{M}}^{\mathcal{H}^*}$. Since they have the same supply of objects, we will study relations between their morphisms.

Introduce a pairing between $\mathcal{H}^*$ and $\mathcal{H}$ by the formula

$$(h, x) := \langle \gamma^{-1}(h), x\rangle, \tag{51}$$

where $\langle ., .\rangle$ is the canonical Hopf pairing $\mathcal{H}^* \otimes \mathcal{H} \to k$. It is $\mathcal{H}$-invariant, since $\mathcal{H}$ is considered as the adjoint $\mathcal{H}$-module (8) and $\mathcal{H}^*$ is an $\mathcal{H}$-module by (45).

**Lemma 4.13.** *For any right $\mathcal{H}^*$-comodule $A \in \mathcal{M}^{\mathcal{H}^*}$ the diagram*

$$\begin{array}{ccc} \mathcal{H}^* \otimes A \otimes \mathcal{H} & \xrightarrow{\tau^A} & A \otimes \mathcal{H}^* \otimes \mathcal{H} \\ \downarrow{\scriptstyle \tau_A^{-1}} & & \downarrow{\scriptstyle (.,.)} \\ \mathcal{H}^* \otimes \mathcal{H} \otimes A & \xrightarrow{(.,.)} & A \end{array} \tag{52}$$

*is commutative.*

*Proof.* Straightforward. □

To any equivariant map $\phi\colon A \to B \otimes \mathcal{H}$ we put into correspondence an equivariant map $\phi'\colon \mathcal{H}^* \otimes A \to B$ being the composition

$$\mathcal{H}^* \otimes A \xrightarrow{\phi} \mathcal{H}^* \otimes B \otimes \mathcal{H} \xrightarrow{\tau^B} B \otimes \mathcal{H}^* \otimes \mathcal{H} \xrightarrow{(.,.)} B. \tag{53}$$

Clearly, this correspondence induces a natural embedding $\mathrm{Hom}\ \bar{\mathcal{M}}^{\mathcal{H}^*}_{\mathcal{H}} \to \mathrm{Hom}\ \bar{\mathcal{M}}^{\mathcal{H}^*}$. Note that this embedding is not an isomorphism, in general.

**Proposition 4.14.** *The correspondence* $\mathrm{Hom}\ \bar{\mathcal{M}}^{\mathcal{H}^*}_{\mathcal{H}} \to \mathrm{Hom}\ \bar{\mathcal{M}}^{\mathcal{H}^*}$, $\phi \mapsto \phi'$, *given by (53), induces a strong monoidal functor* $\bar{\mathcal{M}}^{\mathcal{H}^*}_{\mathcal{H}} \to \bar{\mathcal{M}}^{\mathcal{H}^*}$.

The proof of this proposition uses the diagram technique, the properties of permutations $\{\tau_A\}$ and $\{\tau^A\}$, and relies on Lemma 4.13. The details are left to the reader.

*4.6. Dynamical extension of a monoidal category over a module category.* Let $\mathcal{O}$ be a monoidal category and $\mathcal{B}$ its left module category, see [O]. For example, $\mathcal{B}$ is a monoidal category and $\mathcal{O}$ its monoidal subcategory. We denote the tensor product in $\mathcal{O}$ and action of $\mathcal{O}$ on $\mathcal{B}$ by the same symbol $\otimes$. For simplicity, all monoidal categories are assumed to be strict (with trivial associativity); the same is assumed for actions on module categories.

Let us define a **dynamical extension**, $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$, of $\mathcal{O}$ over $\mathcal{B}$ in the following way. The collection of objects in $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$ coincides with that of $\mathcal{O}$. An object $A$ of $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$ is treated as a functor from $\mathcal{B}$ to $\mathcal{B}$, namely $X \xrightarrow{A} A \otimes X$ for all $X \in \mathrm{Ob}\ \mathcal{B}$. Morphisms of $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$ are natural transformations of the functors. Namely, $\phi \in \mathrm{Hom}_{\bar{\mathcal{O}}_{\triangleright\mathcal{B}}}(A, B)$ is a collection $\{\phi_X\}$ of morphisms $\phi_X \in \mathrm{Hom}_{\mathcal{B}}(A \otimes X, B \otimes X)$ such that

$$\phi_X \circ (\mathrm{id}_A \otimes \xi) = (\mathrm{id}_B \otimes \xi) \circ \phi_{X'} \tag{54}$$

for any $\xi \in \mathrm{Hom}_{\mathcal{B}}(X', X)$. The composition of morphisms in $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$ is "pointwise", $(\phi\bar{\circ}\psi)_X = \phi_X \circ \psi_X$. Obviously, the condition (54) holds for $\bar{\circ}$. Clearly, $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$ defined in this way is a category. It includes $\mathcal{O}$ as a subcategory. Indeed, any morphism $\phi$ from $\mathcal{O}$ gives rise to the family $\{\phi \otimes \mathrm{id}_X\}$, which is a morphism in $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$.

**Proposition 4.15.** $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$ *is a monoidal category with respect to the tensor product on the objects as in* $\mathrm{Ob}\ \mathcal{O}$ *and defined on the morphisms by*

$$(\phi\bar{\otimes}\psi)_X := (\mathrm{id}_C \otimes \psi_X) \circ (\phi_{B\otimes X}) = (\phi_{D\otimes X}) \circ (\mathrm{id}_A \otimes \psi_X), \tag{55}$$

*for* $\phi \in \mathrm{Hom}_{\bar{\mathcal{O}}_{\triangleright\mathcal{B}}}(A, C)$, *and* $\psi \in \mathrm{Hom}_{\bar{\mathcal{O}}_{\triangleright\mathcal{B}}}(B, D)$.

*Proof.* Let us check that the family $\{(\phi\bar{\otimes}\psi)_X\}$ defines a morphism of functors, $A \otimes B \to C \otimes D$. First of all, observe that condition (54) is satisfied. We will show that operation (55) is functorial. Take $\{\alpha_X\} \in \mathrm{Hom}_{\bar{\mathcal{O}}_{\triangleright\mathcal{B}}}(A', A)$ and $\{\beta_X\} \in \mathrm{Hom}_{\bar{\mathcal{O}}_{\triangleright\mathcal{B}}}(B', B)$. We have for $(\phi\bar{\circ}\alpha)\bar{\otimes}(\psi\bar{\circ}\beta)$:

$$\begin{aligned}
\big(\mathrm{id}_C \otimes (\psi_X \circ \beta_X)\big) \circ (\phi_{B'\otimes X} \circ \alpha_{B'\otimes X}) &= (\mathrm{id}_C \otimes \psi_X) \circ (\mathrm{id}_C \otimes \beta_X) \circ \phi_{B'\otimes X} \circ \alpha_{B'\otimes X}\\
&= (\mathrm{id}_C \otimes \psi_X) \circ \phi_{B\otimes X} \circ (\mathrm{id}_C \otimes \beta_X) \circ \alpha_{B'\otimes X}\\
&= (\phi\bar{\otimes}\psi)_X \circ (\alpha\bar{\otimes}\beta)_X
\end{aligned}$$

for all $X \in \mathrm{Ob}\ \mathcal{B}$. In transition to the middle line we used the condition (54), in order to permute the morphisms $\mathrm{id}_C \otimes \beta_X$ and $\phi_{B'\otimes X}$. To prove associativity, we take $\zeta \in \mathrm{Hom}_{\bar{\mathcal{O}}_{\triangleright\mathcal{B}}}(A, U)$, $\phi \in \mathrm{Hom}_{\bar{\mathcal{O}}_{\triangleright\mathcal{B}}}(B, V)$, $\psi \in \mathrm{Hom}_{\bar{\mathcal{O}}_{\triangleright\mathcal{B}}}(C, W)$ and find that $\big(\zeta\bar{\otimes}(\phi\bar{\otimes}\psi)\big)_X$ and $\big((\zeta\bar{\otimes}\phi)\bar{\otimes}\psi\big)_X$ are equal to the same composition map

$$A \otimes B \otimes C \otimes X \xrightarrow{\zeta_{B\otimes C\otimes X}} U \otimes B \otimes C \otimes X \xrightarrow{\phi_{C\otimes X}} U \otimes V \otimes C \otimes X \xrightarrow{\psi_X} U \otimes V \otimes W \otimes X.$$

This completes the proof. □

Note that $\mathcal{O}$ is a monoidal subcategory in $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$.

**Definition 4.16.** *The category $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$ is called a* **dynamical extension** *of $\mathcal{O}$ over $\mathcal{B}$.*

*Remark 4.17.* Similarly to $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$, one can define a dynamical extension, ${}_{\mathcal{B}\triangleleft}\bar{\mathcal{O}}$, of a monoidal category $\mathcal{O}$ over its **right** module category $\mathcal{B}$. Thus, the set $\mathrm{Hom}_{{}_{\mathcal{B}\triangleleft}\bar{\mathcal{O}}}(A, B)$ is formed by families $\{{}_X\psi\}$ from $\mathrm{Hom}_{\mathcal{B}}(X \otimes A \to X \otimes B)$ subject to the natural condition analogous to (54). The composition $\bar{\circ}$ is defined as the composition of functor morphisms, similarly to the $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$ case. Formula (55) for tensor products of morphisms is changed to

$$ {}_X(\phi\bar{\otimes}\psi) := ({}_X\phi \otimes \mathrm{id}_D) \circ {}_{X\otimes A}\psi, \ \phi \in \mathrm{Hom}_{{}_{\mathcal{B}\triangleleft}\bar{\mathcal{O}}}(A, C), \quad \psi \in \mathrm{Hom}_{{}_{\mathcal{B}\triangleleft}\bar{\mathcal{O}}}(B, D). \quad (56)$$

*4.7. Comparison of categories $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$ and ${}_{\mathcal{B}\triangleleft}\bar{\mathcal{O}}$ with $\bar{\mathcal{M}}_{\mathcal{H};\mathcal{L}}$ and $\bar{\mathcal{M}}^{\mathcal{H}^*}$.* Let $\mathcal{L}$ be a base algebra over a Hopf algebra $\mathcal{H}$. Let $\mathcal{B}$ be the category of left $\mathcal{L}$-modules, and $\mathcal{O}$ the category $\mathcal{M}_{\mathcal{H}}$ of locally finite left $\mathcal{H}$-modules. Then $\mathcal{B}$ is a left $\mathcal{O}$-module category. The tensor product of $A \in \mathrm{Ob}\ \mathcal{O}$ and $X \in \mathrm{Ob}\ \mathcal{B}$ is an $\mathcal{L}$-module by

$$\ell \blacktriangleright (a \otimes x) = \ell^{(1)} \triangleright a \otimes \ell^{[2]} \blacktriangleright x, \quad \ell \in \mathcal{L},\ a \in A,\ x \in X, \quad (57)$$

where $\blacktriangleright$ denotes the action of $\mathcal{L}$ and $\triangleright$ the action of $\mathcal{H}$.

Consider the dynamical extension $\bar{\mathcal{M}}_{\mathcal{H};\mathcal{L}}$ of $\mathcal{M}_{\mathcal{H}}$ over the base algebra $\mathcal{L}$ as in Example 4.8. Let $\psi$ be a morphism from $\mathrm{Hom}_{\bar{\mathcal{M}}_{\mathcal{H};\mathcal{L}}}(A, B)$. Consider the family of maps $\psi_X : A \otimes X \to B \otimes X$, $X \in \mathcal{B}$, defined by the composition

$$A \otimes X \xrightarrow{\psi\otimes\mathrm{id}_X} B \otimes \mathcal{L} \otimes X \xrightarrow{\mathrm{id}_B\otimes\blacktriangleright} B \otimes X. \quad (58)$$

The maps $\psi_X$ defined by (58) are $\mathcal{L}$-equivariant, due to quasi-commutativity of $\mathcal{L}$. The following proposition is immediate.

**Proposition 4.18.** *The correspondence of morphisms $\psi \mapsto \{\psi_X\}$ induces a strong monoidal functor $\bar{\mathcal{M}}_{\mathcal{H};\mathcal{L}} \to \bar{\mathcal{O}}_{\triangleright\mathcal{B}}$ identical on objects.*

Now take $\mathcal{O}$ to be the category $\mathcal{M}^{\mathcal{H}^*}$ of right locally finite $\mathcal{H}^*$-comodules also considered as left $\mathcal{H}$-modules. Put $\mathcal{B}$ the category of locally finite $\mathcal{H}$-modules.

**Proposition 4.19.** *There exists a strong monoidal functor $\bar{\mathcal{M}}^{\mathcal{H}^*} \to {}_{\mathcal{B}\triangleleft}\bar{\mathcal{O}}$.*

*Proof.* We will give a sketch of the proof. Categories ${}_{\mathcal{B}\triangleleft}\bar{\mathcal{O}}$ and $\bar{\mathcal{M}}^{\mathcal{H}^*}$ have the same collection of objects, and the functor in question is set to be identical on objects. Let us define it on morphisms. Let $f : \mathcal{H}^* \otimes A \to B$ be a morphism in $\bar{\mathcal{M}}^{\mathcal{H}^*}$. For every finite dimensional $\mathcal{H}$-module $X$ there is a natural map $X^* \otimes X \to \mathcal{H}^*$, where $X^* \otimes X$ is considered as the (left) dual module to the space of right endomorphisms (over $k$) of $X$. Hence, $f$ defines a collection of $\mathcal{H}$-equivariant maps $X^* \otimes X \otimes A \to B$, or, equivalently, a collection $\{f_X\}$ of $\mathcal{H}$-equivariant maps $X \otimes A \to X \otimes B$. This family extends to all locally finite $\mathcal{H}$-modules $X$. Thus we have built an embedding of morphisms Hom $\bar{\mathcal{M}}^{\mathcal{H}^*} \to$ Hom ${}_{\mathcal{B}}\!\!\!\not{\;}\bar{\mathcal{O}}$, $f \mapsto \{f_X\}$. It remains to check that the above correspondence is functorial and respects the composition and the tensor product of morphisms. We leave the details to the reader. □

*Remark 4.20.* The functor from Proposition 4.19 is an isomorphism when $\mathcal{H}^*$ decomposes into the direct sum of $X^* \otimes X$, where $X$ runs over simple $\mathcal{H}$-modules.

*4.8. Dynamical associative algebras.* Dynamical associative algebra as an algebra in a monoidal (dynamical) category is defined in the standard way. Below we give examples of dynamical associative algebras in the categories $\bar{\mathcal{M}}^{\mathcal{H}^*}$ and $\bar{\mathcal{M}}_{\mathcal{H};\mathcal{L}}$.

*Example 4.21 (Dynamical algebras in $\bar{\mathcal{M}}_{\mathcal{H};\mathcal{L}}$).* Let us consider the dynamical extension $\bar{\mathcal{M}}_{\mathcal{H};\mathcal{L}}$ of the category $\mathcal{M}_{\mathcal{H}}$ over a $\mathcal{H}$-base algebra $\mathcal{L}$. An algebra $\mathcal{A}$ in $\bar{\mathcal{M}}_{\mathcal{H};\mathcal{L}}$ is an object equipped with a morphism $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ obeying the associativity axiom. In terms of $\mathcal{M}_{\mathcal{H}}$, this is equivalent to Definition 3.11. Namely, the multiplication in $\mathcal{A}$ is an $\mathcal{H}$-equivariant map $\circledast\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{L}$, which is shifted associative in the sense of (14).

Now let us prove Proposition 3.12. This is a corollary of the following general fact. Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ be a monoidal category whose objects are vector spaces over $k$ and morphisms are linear maps. Suppose there is an object $\mathcal{A} \in \mathrm{Ob}\,\mathcal{C}$, a morphism $\iota\colon 1_{\mathcal{C}} \to \mathcal{A}$, and an operation $\mathrm{Hom}_{\mathcal{C}}(X, \mathcal{A}) \otimes_k \mathrm{Hom}_{\mathcal{C}}(Y, \mathcal{A}) \xrightarrow{\circledast} \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, \mathcal{A})$ for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$. We say that $\circledast$ is a) natural if $(\phi \circ \alpha) \circledast (\psi \circ \beta) = (\phi \circledast \psi) \circ (\alpha \otimes \beta)$, b) associative if $(\phi \circledast \psi) \circledast \vartheta = \phi \circledast (\psi \circledast \vartheta)$, and c) unital if $\phi \circledast (\iota \circ \chi) = \phi \otimes \chi$, $(\iota \circ \chi) \circledast \phi = \chi \otimes \phi$ for all morphisms $\chi$ with target in $1_{\mathcal{C}}$. The multiplication m in $\mathcal{A}$ and the operation $\circledast$ are related by $\mathrm{m} = \mathrm{id}_{\mathcal{A}} \circledast \mathrm{id}_{\mathcal{A}}$, $\phi \circledast \psi = \mathrm{m} \circ (\phi \otimes \psi)$. Now let $\mathcal{A}$ be an algebra in $\bar{\mathcal{M}}_{\mathcal{H};\mathcal{L}}$. The unit morphism $\iota\colon k \to \mathcal{A}$ in $\bar{\mathcal{M}}_{\mathcal{H};\mathcal{L}}$ gives the unit map $k \to \mathcal{A} \otimes \mathcal{L}$ in $\mathcal{M}_{\mathcal{H}}$. The multiplication $\circledast$ in $\mathcal{A}$ defines a natural associative unital operation on morphisms from Hom $\bar{\mathcal{M}}_{\mathcal{H};\mathcal{L}}$ with target in $\mathcal{A}$. Hence it defines a natural associative unital operation on morphisms from Hom $\mathcal{M}_{\mathcal{H}}$ with target in $\mathcal{A} \otimes \mathcal{L}$.

*Example 4.22 (Dynamical algebras in $\bar{\mathcal{M}}^{\mathcal{H}^*}$).* Let us describe dynamical associative algebras in the category $\bar{\mathcal{M}}^{\mathcal{H}^*}$. The multiplication in an algebra $\mathcal{A} \in \mathrm{Ob}\,\bar{\mathcal{M}}^{\mathcal{H}^*}$ is an $\mathcal{H}$-equivariant map $\Lambda : \mathcal{H}^* \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. Associativity, in terms of $\mathcal{M}^{\mathcal{H}^*}$, is formalized by the following commutative diagram:

$$\begin{array}{ccccccc} \mathcal{H}^* \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\Delta} & \mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\Lambda} & \mathcal{H}^* \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\Lambda} & \mathcal{A} \\ & & {\scriptstyle\tau^{\mathcal{A}}}\downarrow & & & & \| \\ & & \mathcal{H}^* \otimes \mathcal{A} \otimes \mathcal{H}^* \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\Lambda} & \mathcal{H}^* \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\Lambda} & \mathcal{A} \end{array} \quad (59)$$

This diagram is a "partial dualization" of the diagram (14). The algebra $\mathcal{A}$ is unital if there is an element $1 \in \mathcal{A}$ such that $\Lambda(\lambda, 1, a) = \Lambda(\lambda, a, 1) = \varepsilon(\lambda)a$, for all $a \in \mathcal{A}$, $\lambda \in \mathcal{H}^*$.

The map $\Lambda$ defines a family of bilinear operations $\overset{\lambda}{*}$ depending on elements $\lambda \in \mathcal{H}^*$. In terms of $\overset{\lambda}{*}$, the "shifted" associativity (59) reads (summation implicit)

$$(a \overset{\lambda^{(2)}}{*} b) \overset{\lambda^{(1)}}{*} c = a^{[0]} \overset{\lambda^{(1)}}{*} (b \overset{\lambda^{(2)}a^{(1)}}{*} c). \quad (60)$$

I. Kantor proposed to consider the multiplication map $\mathcal{H}^* \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{\Lambda} \mathcal{A}$ as a ternary operation $\lambda \otimes a \otimes b \mapsto (\lambda ab)$ which is associative in the sense

$$\big(\lambda^{(1)}(\lambda^{(2)}ab)c\big) = \big(\lambda^{(1)}a'(\lambda^{(2)''}bc)\big).$$

Here $a' \otimes \lambda'' = \tau^{\mathcal{A}}(\lambda \otimes a)$, the permutation (46).

## 5. Categorical Approach to Quantum DYBE

*5.1. Dynamical twisting cocycles.* In this subsection we study transformations of dynamical categories. Recall that a functor $\tilde{\mathcal{C}} \xrightarrow{\Upsilon} \mathcal{C}$ between two monoidal categories is called monoidal if there is a functor isomorphism $F$ between $\Upsilon(A) \otimes \Upsilon(B)$ and $\Upsilon(A\tilde{\otimes}B)$. This implies a family of isomorphisms,

$$\Upsilon(A) \otimes \Upsilon(B) \xrightarrow{F_{A,B}} \Upsilon(A\tilde{\otimes}B)$$

fulfilling the cocycle conditions (for simplicity, we assume the trivial associator)

$$F_{A\tilde{\otimes}B,C} \circ \big(F_{A,B} \otimes \mathrm{id}_C\big) \;=\; F_{A,B\tilde{\otimes}C} \circ \big(\mathrm{id}_A \otimes F_{B,C}\big), \tag{61}$$

$$F_{A,1} = \mathrm{id}_A = F_{1,A}, \tag{62}$$

where 1 is the unit of $\mathcal{C}$. We are mostly interested in the situation when Ob $\tilde{\mathcal{C}}$ = Ob $\mathcal{C}$ and $\Upsilon$ is identical on objects.

Suppose that $F$ is a cocycle in $\mathcal{C}$, i.e. a family of invertible morphisms $F_{A,B} \in \mathrm{Aut}_{\mathcal{C}}(A \otimes B)$ fulfilling the conditions (61) and (62). Then it is possible to define a new monoidal structure on $\mathcal{C}$. It is the same on objects and defined by

$$\phi\tilde{\otimes}\psi := F \circ (\phi \otimes \psi) \circ F^{-1} \tag{63}$$

on morphisms. This new monoidal category $\tilde{\mathcal{C}}$ coincides with the old one if $F$ respects morphisms of $\mathcal{C}$, i.e.

$$\phi \otimes \psi = F \circ (\phi \otimes \psi) \circ F^{-1} \tag{64}$$

for all $f, g \in \mathrm{Hom}\, \mathcal{C}$.

*Remark 5.1.* One can define the category $\tilde{\mathcal{C}}$ using an arbitrary family $F_{A,B} \in \mathrm{Aut}_{\mathcal{C}}(A, B)$ of morphisms, which is not necessarily a cocycle. Then $\tilde{\mathcal{C}}$ will not be strictly monoidal, but rather with the associator $\Phi_{A,B,C} = F_{A,BC}F_{B,C}F^{-1}_{A,B}F^{-1}_{AB,C}$, which satisfies the pentagon identity in $\tilde{\mathcal{C}}$. The identity functor $\tilde{\mathcal{C}} \to \mathcal{C}$ yields an isomorphism of monoidal categories.

**Definition 5.2 (Dynamical twist).** *Let $\bar{\mathcal{O}}$ be a dynamical extension of a monoidal category $\mathcal{O}$.* **Dynamical twist** *is a cocycle in $\bar{\mathcal{O}}$ that respects morphisms from $\mathcal{O}$.*

A dynamical twist is identical on $\mathcal{O}$, therefore $\mathcal{O}$ remains a subcategory in the twisted category $\tilde{\bar{\mathcal{O}}}$.

One of the applications of twist is transformation of algebras. Any cocycle $F$ in a category $\mathcal{C}$ makes a $\mathcal{C}$-algebra with the multiplication m into a $\tilde{\mathcal{C}}$-algebra, with the multiplication $\mathrm{m} \circ F^{-1}$. Let us apply this to the specific situation of dynamical twist and build an $\bar{\mathcal{O}}$-algebra out of $\mathcal{O}$-algebra.

**Proposition 5.3.** *Let $F$ be a dynamical twist in $\bar{\mathcal{O}}$. Let $\mathcal{A}$ be an algebra in $\mathcal{O}$ with multiplication* m. *Then the multiplication* $\mathrm{m} \circ F$ *makes $\mathcal{A}$ a dynamical associative algebra, i.e. an algebra in $\bar{\mathcal{O}}$.*

*Proof.* It follows from (63) that dynamical twist preserves $\mathcal{O}$ as a monoidal subcategory in $\tilde{\bar{\mathcal{O}}}$. Therefore $\mathcal{A}$ turns out to be an algebra in $\tilde{\bar{\mathcal{O}}}$ as well. The family $\{F^{-1}_{A,B}\}$ is a cocycle in $\tilde{\bar{\mathcal{O}}}$; the corresponding twist of $\tilde{\bar{\mathcal{O}}}$ gives $\bar{\mathcal{O}}$. Applying this inverse twist to the algebra $\mathcal{A}$ we obtain an $\bar{\mathcal{O}}$-algebra with the multiplication $\mathrm{m} \circ F$. □

Below we specialize the cocycle equations (61) and (62) for various types of dynamical categories.

*Example 5.4 (Dynamical twist in $\mathcal{O}_{\triangleright\mathcal{B}}$).* Let us express a cocycle in dynamical category $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$ in terms of $\mathcal{O}$ and $\mathcal{B}$. A cocycle in $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$ is a collection $(F_{V,W})_X$ from $\mathrm{Aut}_{\mathcal{B}}(V \otimes W \otimes X)$, $V, W \in \mathrm{Ob}\,\mathcal{O}$, $X \in \mathrm{Ob}\,\mathcal{B}$, satisfying conditions

$$(F_{V\otimes W,U})_X \circ (F_{V,W})_{U\otimes X} \quad = \quad (F_{V,W\otimes U})_X \circ (F_{W,U}), \tag{65}$$

$$(F_{V,1_{\bar{\mathcal{O}}}})_X = \mathrm{id}_{V\otimes X} = (F_{1_{\bar{\mathcal{O}}},V})_X. \tag{66}$$

*Example 5.5 (Drinfeld associator as a twist in ${}_{\mathcal{O}\triangleleft}\mathcal{O}$).* Let $\mathfrak{g}$ be a complex simple Lie algebra. In [EE1], Enriquez and Etingof proposed a quantization of the Alekseev-Meinrenken dynamical r-matrix [AM] using the Drinfeld associator $\Phi \in \mathcal{U}^{\otimes 3}(\mathfrak{g})[[t]]$. This quantization can be interpreted as a twist in the category ${}_{\mathcal{O}\triangleleft}\bar{\mathcal{O}}$, where $\mathcal{O}$ is the category of free $\mathbb{C}[[t]]$-modules of finite rank with $\mathcal{U}(\mathfrak{g})[[t]]$-action. Indeed, let us put ${}_X(F_{A,B}) := \Phi_{X,A,B}$. Then the pentagon identity on $\Phi$ takes the form

$$\Phi_{A,B,C} \circ {}_X(F_{A\otimes B,C}) \circ {}_X(F_{A,B}) = {}_X(F_{A,B\otimes C}) \circ {}_{X\otimes A}(F_{B,C}).$$

The twisted dynamical category is not strictly monoidal, cf. Remark 5.1. It is equipped with the associator $\{\Phi_{A,B,C}\}$.

*Example 5.6 (Dynamical twist in $\bar{\mathcal{O}}_{\mathcal{L}}$).* Consider a cocycle in $\bar{\mathcal{O}}_{\mathcal{L}}$, the dynamical extension of a category $\mathcal{O}$ over a base algebra $(\mathcal{L}, \tau)$. In terms of $\mathcal{O}$, condition (61) reads

$$\mathrm{m}_{\mathcal{L}\otimes\mathcal{L}} \circ F_{V\otimes W,U} \circ (\mathrm{id}_{V\otimes W} \otimes \tau_U) \circ F_{V,W} = \mathrm{m}_{\mathcal{L}\otimes\mathcal{L}} \circ F_{V,W\otimes U} \circ F_{W,U}, \tag{67}$$

where $F_{V,W} \in \mathrm{Hom}_{\mathcal{O}}(V \otimes W, V \otimes W \otimes \mathcal{L})$ (the id-automorphisms are dropped from the formulas).

*Example 5.7 (Dynamical twist in $\bar{\mathcal{M}}^{\mathcal{H}^*}$).* Let us specialize the notion of cocycle for the category $\bar{\mathcal{M}}^{\mathcal{H}^*}$. A morphism $\mathcal{H}^* \otimes A \xrightarrow{f} B$ in the category $\mathcal{M}^{\mathcal{H}^*}$ can be thought of as a family of maps $f^\lambda\colon A \to B$ parameterized by elements $\lambda \in \mathcal{H}^*$. Let $\Omega^\lambda$ be a family of linear operators on the tensor product $\otimes_{l=1}^m V_l$ of $\mathcal{H}^*$-comodules $V_l$, $l = 1, \ldots, m$. By ${}^{V_i}\Omega^\lambda$, or simply by ${}^i\Omega^\lambda$, we denote the family of linear operators on $\otimes_{l=1}^m V_l$ defined by

$${}^i\Omega^\lambda(v_1 \otimes \ldots \otimes v_m) := \Omega^{\lambda v_i^{(1)}}(v_1 \otimes \ldots \otimes v_i^{[0]} \otimes \ldots \otimes v_m),$$

where $v_i^{[0]} \otimes v_i^{(1)}$ denotes the right $\mathcal{H}^*$-coaction $\delta(v_i)$ (as always, the summation is implicit). The collection of morphisms $F^\lambda_{V,W} \in \mathrm{Hom}_{\mathcal{M}^{\mathcal{H}^*}}(\mathcal{H}^* \otimes V \otimes W, V \otimes W)$ satisfies condition (61) and (62) in $\bar{\mathcal{M}}^{\mathcal{H}^*}$ if and only if

$$F^{\lambda^{(1)}}_{V\otimes W,U} F^{\lambda^{(2)}}_{V,W} \quad = \quad F^{\lambda^{(1)}}_{V,W\otimes U}\ {}^V F^{\lambda^{(2)}}_{W,U}, \tag{68}$$

$$F^\lambda_{V,k} = \mathrm{id}_V = F^\lambda_{k,V}. \tag{69}$$

Here $\lambda^{(1)} \otimes \lambda^{(2)}$ stands for $\Delta_{\mathcal{H}^*}(\lambda)$.

*Example 5.8 (Universal cocycle).* Assume that $\mathcal{H}$ is a Hopf subalgebra of another Hopf algebra, $\mathcal{U}$. Consider the category $\mathcal{M}_\mathcal{U}$ as a subcategory of the category $\mathcal{M}_\mathcal{H}$. Let $\mathcal{L}$ be a base algebra over $\mathcal{H}$. Suppose there is an invertible element $\bar{\mathcal{F}} = \bar{\mathcal{F}}_1 \otimes \bar{\mathcal{F}}_2 \otimes \bar{\mathcal{F}}_3 \in \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{L}$ that satisfies the condition

$$h^{(1)}\bar{\mathcal{F}}_1 \otimes h^{(2)}\bar{\mathcal{F}}_2 \otimes h^{(3)} \triangleright \bar{\mathcal{F}}_3 = \bar{\mathcal{F}}_1 h^{(1)} \otimes \bar{\mathcal{F}}_2 h^{(2)} \otimes \bar{\mathcal{F}}_3 \tag{70}$$

for all $h \in \mathcal{H}$, and the conditions

$$(\Delta \otimes \mathrm{id})(\bar{\mathcal{F}})\ {}^{(3)}\bar{\mathcal{F}}_{12} \quad = \quad (\mathrm{id} \otimes \Delta)(\bar{\mathcal{F}})(\bar{\mathcal{F}}_{23}), \tag{71}$$

$$(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\bar{\mathcal{F}}) = 1 \otimes 1 \otimes 1 = (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\bar{\mathcal{F}}), \tag{72}$$

where Equation (71) is in $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{L}$. Here the notation ${}^{(3)}\bar{\mathcal{F}}$ means $\delta(\bar{\mathcal{F}})$, where $\delta$ is the coaction $\mathcal{L} \to \mathcal{H} \otimes \mathcal{L}$; the $\mathcal{H}$-component is embedded to the third tensor factor in $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{L}$. The element $\bar{\mathcal{F}}$ defines a cocycle in $\bar{\mathcal{M}}_{\mathcal{U};\mathcal{L}}$, namely $F_{V,W} := \rho_V(\bar{\mathcal{F}}_1) \otimes \rho_W(\bar{\mathcal{F}}_2) \otimes \bar{\mathcal{F}}_3$ for $\mathcal{U}$-modules $V$ and $W$. This cocycle clearly respects morphisms in $\mathcal{M}_\mathcal{U}$, hence it is a dynamical twist. The element $\bar{\mathcal{F}}$ may be called a **universal dynamical twist**, by the analogy with the universal R-matrix. Equation (71) leads to the shifted cocycle condition of [Xu2] for $\mathcal{H}$ being a universal enveloping algebra.

### 5.2. *Quantum dynamical R-matrix.*

*5.2.1. Dynamical Yang-Baxter equation.* Let us consider the Yang-Baxter equation in dynamical categories. Let $\mathcal{C}$ be a braided monoidal category with braiding $\sigma$. The braiding is a collection, $\{\sigma_{A,B}\}$, of morphisms $A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$ for $A, B \in \mathrm{Ob}\,\mathcal{C}$ obeying conditions

$$\sigma_{A,B} \circ \sigma_{A,C} \circ \sigma_{B,C} = \sigma_{B,C} \circ \sigma_{A,C} \circ \sigma_{A,B}, \tag{73}$$

$$\sigma_{A \otimes B,C} = \sigma_{A,C} \circ \sigma_{B,C}, \qquad \sigma_{C,A\otimes B} = \sigma_{C,B} \circ \sigma_{C,A} \tag{74}$$

and respecting morphisms, i.e. $(f \otimes g) \circ \sigma = \sigma \circ (g \otimes f)$ for all $f, g \in \mathrm{Hom}\,\mathcal{C}$ (in fact, (73) follows from (74) and functoriality of $\sigma$). Condition (73) is called the Yang-Baxter equation, conditions (74) are, in fact, the hexagon identities. If $\sigma$ fulfills (73) and (74) but is not functorial (does not respect morphisms), we call it **pre-braiding**. This is the case when $\sigma$ is a braiding in a subcategory $\mathcal{C}'$ of $\mathcal{C}$ such that $\mathrm{Ob}\,\mathcal{C}' = \mathrm{Ob}\,\mathcal{C}$, e.g., when $\mathcal{C}$ is a dynamical extension of a $\mathcal{C}'$. Then $\mathcal{C}$ has more morphisms than $\mathcal{C}'$, and they are not respected by $\sigma$, in general.

Given a pre-braiding $\sigma$ in $\mathcal{C}$, it is possible to restrict it to a braiding in a subcategory $\mathcal{C}_\sigma$ defined as follows. The objects in $\mathcal{C}_\sigma$ are those of $\mathcal{C}$. A morphism $f \in \mathrm{Hom}_\mathcal{C}(A, B)$ is a morphism in $\mathrm{Hom}_{\mathcal{C}_\sigma}(A, B)$ if and only if

$$\sigma_{B,C} \circ (f \otimes \mathrm{id}_C) = (\mathrm{id}_C \otimes f) \circ \sigma_{A,C}, \quad (f \otimes \mathrm{id}_C) \circ \sigma_{C,A} = \sigma_{C,B} \circ (\mathrm{id}_C \otimes f) \tag{75}$$

for all $C \in \mathrm{Ob}\,\mathcal{C}$.

**Proposition 5.9.** *$\mathcal{C}_\sigma$ is a braided category with braiding $\sigma$.*

For instance, the dynamical extension $\bar{\mathcal{O}}_\mathcal{L}$ of a braided category $(\mathcal{O}, \sigma)$ over a commutative algebra $\mathcal{L}$ in $\mathcal{O}$ (cf. Example 4.3) is braided if and only if $\sigma_{A,\mathcal{L}} \circ \sigma_{\mathcal{L},A} = \mathrm{id}_{\mathcal{L}\otimes A}$ for all $A \in \mathrm{Ob}\,\mathcal{O}$, e.g. when $\mathcal{O}$ is a symmetric category.

*Proof.* It follows from (73) and (74) that $\sigma$ lies in $\mathcal{C}_\sigma$. Condition (74) guarantees that $\mathcal{C}_\sigma$ is a monoidal category. Therefore, $\sigma$ is a pre-braiding in $\mathcal{C}_\sigma$ and respects morphisms in it by construction; hence $\sigma$ is a braiding in $\mathcal{C}_\sigma$. □

**Proposition 5.10.** *Let $\sigma$ be a pre-braiding in $\mathcal{C}$ and let $F$ be a cocycle in $\mathcal{C}$ respecting morphisms from $\mathcal{C}_\sigma$. Then the family*

$$\bar{\sigma}_{A,B} := F^{-1}_{B,A} \circ \sigma_{A,B} \circ F_{A,B} \tag{76}$$

*satisfies the Yang-Baxter equation (73).*

*Proof.* Define $\Omega_{A,B,C} := F_{A\otimes B,C} \circ (F_{A,B} \otimes \mathrm{id}_C) = F_{A,B\otimes C} \circ (\mathrm{id}_A \otimes F_{B,C})$ for all $A, B, C \in \mathrm{Ob}\,\mathcal{C}$. Since $F$ respects morphisms from $\mathcal{C}_\sigma$, we have $\Omega^{-1}_{A,C,B} \circ (\mathrm{id}_A \otimes \sigma_{B,C}) \circ \Omega_{A,B,C} = \mathrm{id}_A \otimes \bar{\sigma}_{B,C}$ and $\Omega^{-1}_{B,A,C} \circ (\sigma_{A,B} \otimes \mathrm{id}_C) \circ \Omega_{A,B,C} = \bar{\sigma}_{A,B} \otimes \mathrm{id}_C$ for all $A, B, C$. Multiplying Eq. (73) by $\Omega^{-1}_{C,B,A}$ from the left and by $\Omega_{A,B,C}$ from the right, we prove the statement. □

Applied to dynamical twists, Proposition 5.10 yields the following corollary.

**Corollary 5.11.** *Let $\mathcal{O}$ be a braided category with the braiding $\sigma$. Let $\bar{\mathcal{O}}$ be a dynamical extension of $\mathcal{O}$ and $F$ a dynamical twist in $\bar{\mathcal{O}}$. The collection of morphisms (76) for $A, B \in \bar{\mathcal{O}}$ satisfies the Yang-Baxter equation in $\bar{\mathcal{O}}$.*

In general, a twist destroys the hexagon identities in the twisted category $\tilde{\mathcal{C}}$. However, it yields a pre-braiding in an equivalent category to $\tilde{\mathcal{C}}$, which is constructed in Subsect. 5.2.2. There is another way to fix the situation when $\mathcal{C} = \bar{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$, the dynamical extension of the category of $\mathcal{H}$-modules over a base algebra $\mathcal{L}$. There exists a realization of $\bar{\mathcal{M}}_{\mathcal{H},\mathcal{L}}$ as a category of modules over a certain bialgebroid, [DM5]. A dynamical twist gives rise to a bialgebroid twist, which transforms the braiding in the category of modules over the bialgebroid.

We call a solution of (73) in a dynamical category a **dynamical R-matrix**.

Below we specialize this definition of dynamical R-matrix to various types of dynamical categories.

*Example 5.12 (Dynamical R-matrix in $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$).* The dynamical R-matrix in the category $\bar{\mathcal{O}}_{\triangleright\mathcal{B}}$ is defined by

$$(\sigma_{A,B})_X \circ (\sigma_{A,C})_{B\otimes X} \circ (\sigma_{B,C})_X = (\sigma_{B,C})_{A\otimes X} \circ (\sigma_{A,C})_X \circ (\sigma_{A,B})_{C\otimes X}, \tag{77}$$

where $\sigma$ is a collection of invertible morphisms $(\sigma_{A,B})_X \in \mathrm{Aut}_{\mathcal{B}}(A \otimes B \otimes X)$.

*Example 5.13 (Dynamical R-matrix in $\bar{\mathcal{O}}_{\mathcal{L}}$).* Consider the category $\bar{\mathcal{O}}_{\mathcal{L}}$, a dynamical extension of a monoidal category $\mathcal{O}$ over a base algebra $(\mathcal{L}, \tau)$, cf. Subsect. 4.2. Let m be the multiplication in the algebra $\mathcal{L}$ and $\mathrm{m}^3$ denote the three-fold product $\mathrm{m}\circ(\mathrm{m}\otimes\mathrm{id}_{\mathcal{L}})$. In terms of $\mathcal{O}$ and $(\mathcal{L}, \tau)$, Eq. (73) reads

$$\mathrm{m}^3 \circ \sigma_{A,B} \circ \tau_B \circ \sigma_{A,C} \circ \sigma_{B,C} = \mathrm{m}^3 \circ \tau_A \circ \sigma_{B,C} \circ \sigma_{A,C} \circ \tau_C \circ \sigma_{A,B}, \tag{78}$$

where $\sigma_{A,B} \in \mathrm{Hom}_{\hat{\mathcal{O}}}(A \otimes B, A \otimes B \otimes \mathcal{L})$.

*Example 5.14 (Dynamical R-matrix in $\bar{\mathcal{M}}^{\mathcal{H}^*}$).* We use the notation of Example 5.7. A collection of morphisms $\{\sigma^{\lambda}_{A,B}\}$ from Hom $\mathcal{M}^{\mathcal{H}^*}$ fulfills Eq. (73) in $\bar{\mathcal{M}}^{\mathcal{H}^*}$ if and only if

$$ {}^{A}\sigma^{\lambda^{(1)}}_{B,C}\ \sigma^{\lambda^{(2)}}_{A,C}\ {}^{C}\sigma^{\lambda^{(3)}}_{A,B} = \sigma^{\lambda^{(1)}}_{A,B}\ {}^{B}\sigma^{\lambda^{(2)}}_{A,C}\ \sigma^{\lambda^{(3)}}_{B,C}. \tag{79}$$

*Example 5.15 (Universal dynamical R-matrix).* Consider the situation of Example 5.8 assuming that $\mathcal{H}$ is a Hopf subalgebra in another Hopf algebra, $\mathcal{U}$, and $\mathcal{L}$ is a $\mathcal{H}$-base algebra. Definition 3.32 of Subsect. 3.5 introduces a universal quantum dynamical R-matrix of $\mathcal{U}$ over the base $\mathcal{L}$. For any pair $V$ and $W$ of $\mathcal{U}$-modules considered as modules over $\mathcal{H}$, it gives $\sigma_{V,W} := P_{V,W}R_{V,W}$, where $P$ is the usual flip and $R_{V,W} = (\rho_V \otimes \rho_W)(\bar{R})$ is the image of $\bar{R}$ in $\mathrm{End}(V) \otimes \mathrm{End}(W) \otimes \mathcal{L}$.

**Proposition 5.16.** *Suppose the Hopf algebra $\mathcal{U}$ is quasitriangular and let $\mathcal{R}$ be its universal R-matrix. Let $\mathcal{L}$ be a base algebra over $\mathcal{H} \subset \mathcal{U}$ and $\bar{\mathcal{F}} \in \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{L}$ a universal dynamical twist. Then the element $\bar{\mathcal{R}} := \bar{\mathcal{F}}_{21}^{-1}\mathcal{R}\bar{\mathcal{F}}$ is a universal dynamical R-matrix.*

*Proof.* This statement can be checked directly. Another way to verify it is to consider representations of $\mathcal{U}$. Then the statement follows from Proposition (76). □

*5.2.2. Dynamical (pre-) braiding.* Let $\mathcal{C}$ be a monoidal category. Let $F$ be a cocycle in $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be the twisted category defined in Subsect. 5.1. Suppose $\sigma$ is a pre-braiding in $\mathcal{C}$. As was mentioned above, the hexagon identities (74) are destroyed in $\tilde{\mathcal{C}}$. We are going to construct an equivalent monoidal category $F(\mathcal{C})$ where the twist of $\sigma$ will be a pre-braiding.

We consider formal sequences (words) $\mathbf{A} := (A_1, A_2, \dots, A_n)$, $n > 0$, of objects from $\mathcal{C}$. For two words $\mathbf{A}$ and $\mathbf{B}$, let $\mathbf{A} \bullet \mathbf{B}$ denote the concatenation $(A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_m)$.

Let $\alpha(\mathbf{A})$ denote the tensor product $A_1 \otimes \ldots \otimes A_n \in \mathrm{Ob}\,\mathcal{C}$. By induction on the length of words, let us introduce an isomorphism $\Omega_{\mathbf{A}}$ of $\alpha(\mathbf{A}) \in \mathrm{Ob}\,\mathcal{C}$ setting $\Omega_{\mathbf{A}} := \mathrm{id}_A$ for $\mathbf{A} = A \in \mathrm{Ob}\,\mathcal{C}$ and

$$\Omega_{\mathbf{A}\bullet\mathbf{B}} := F_{\alpha(\mathbf{A}),\alpha(\mathbf{B})}(\Omega_{\mathbf{A}} \otimes \Omega_{\mathbf{B}}). \tag{80}$$

One can check, using the cocycle condition (61), that $\Omega_{\mathbf{A}}$ does not depend on a particular partition of $\mathbf{A}$ into two concatenated words. Using the family $\{\Omega_{\mathbf{A}}\}$, define a transformation $F(f)$ of morphisms $\alpha(\mathbf{A}) \xrightarrow{f} \alpha(\mathbf{B})$ in $\mathcal{C}$ setting

$$F(f) := \Omega_{\mathbf{B}} f \Omega_{\mathbf{A}}{}^{-1}. \tag{81}$$

Let us construct the category $F(\mathcal{C})$. The objects of $F(\mathcal{C})$ are finite formal sequences of objects from $\mathcal{C}$. The set of morphisms $\mathrm{Hom}_{F(\mathcal{C})}(\mathbf{A}, \mathbf{B})$ consists of $F(f)$, where $f$ is a morphism from $\mathrm{Hom}_{\mathcal{C}}\big(\alpha(\mathbf{A}), \alpha(\mathbf{B})\big)$.

We define the tensor product of objects $\mathbf{A}$ and $\mathbf{B}$ of $F(\mathcal{C})$ as the concatenation $\mathbf{A} \bullet \mathbf{B}$. The empty word plays the role of the unit object.

Let us define the tensor product of morphisms in $F(\mathcal{C})$. Let $F(f)\colon \mathbf{A} \to \mathbf{A}'$ and $F(g)\colon \mathbf{B} \to \mathbf{B}'$ be two morphisms. Then we put

$$F(f) \bullet F(g) := F(f \otimes g)\colon \mathbf{A} \bullet \mathbf{B} \to \mathbf{A}' \bullet \mathbf{B}'. \tag{82}$$

The category $F(\mathcal{C})$ is equivalent to $\mathcal{C}$. Indeed, the correspondence $\mathbf{A} \mapsto \alpha(\mathbf{A})$, $F(f) \mapsto f$ gives a strong monoidal functor $\alpha\colon F(\mathcal{C}) \to \mathcal{C}$. Consider also the functor $\beta\colon \mathcal{C} \to$

$F(\mathcal{C})$ defined on objects by $\beta(A) = A$, the word of length $n = 1$, and on morphisms by $\beta(f) = f$. This functor is monoidal. Indeed, one can interpret $\Omega_{\mathbf{A}}$ as a morphism in $F(\mathcal{C})$, namely,

$$\Omega_{\mathbf{A}}^{-1} = F(\mathrm{id}_{A_1} \otimes \ldots \otimes \mathrm{id}_{A_n}) \in \mathrm{Hom}_{F(\mathcal{C})}(A_1 \bullet \ldots \bullet A_n, A_1 \otimes \ldots \otimes A_n).$$

So we obtain the transformation of the tensor products $\beta(A) \bullet \beta(B) \xrightarrow{\Omega^{-1}_{(A,B)}} \beta(A \otimes B)$. The functors $\alpha$ and $\beta$ give the equivalence of categories $\mathcal{C}$ and $F(\mathcal{C})$.

**Proposition 5.17.** *Let $\sigma$ be a pre-braiding in $\mathcal{C}$. Then the collection $\sigma_{\mathbf{A},\mathbf{B}} := F(\sigma_{\alpha(\mathbf{A}),\alpha(\mathbf{B})})$ is a pre-braiding in $F(\mathcal{C})$.*

*Proof.* Apply $F$ to Eq. (73) and (74) and use the definition (82). □

For example, let us specialize $\sigma_{\mathbf{A},\mathbf{B}}$ for $\mathbf{A} = A$ and $\mathbf{B} = B$. In this case, we have $\Omega_{\mathbf{A},\mathbf{B}} = F_{A,B}$. Applying formula (82), we obtain $\sigma_{\mathbf{A},\mathbf{B}} = F_{B,A}\sigma_{A,B}{F^{-1}}_{A,B}$ for $\mathbf{A} = A$ and $\mathbf{B} = B$.

## 6. A Construction of Dynamical Twisting Cocycles

*6.1. Associative operations on morphisms and twists.* Let $\mathcal{C}$ be a monoidal category and $\mathcal{C}'$ a subcategory in $\mathcal{C}$. We are going to show that cocycles in $\mathcal{C}'$ (see Subsect. 5.1) are in one-to-one correspondence with natural associative operations on morphisms $\mathrm{Hom}_{\mathcal{C}}(A, V)$, where $A \in \mathrm{Ob}\,\mathcal{C}$ and $V \in \mathrm{Ob}\,\mathcal{C}'$. First of all observe that a cocycle $F$ in $\mathcal{C}'$ defines such an operation by the formula $\phi \circledast \psi := F \circ (\phi \otimes \psi)$. The converse is also true.

**Lemma 6.1.** *Suppose there is an associative operation*

$$\mathrm{Hom}_{\mathcal{C}}(A, V) \otimes \mathrm{Hom}_{\mathcal{C}}(B, W) \xrightarrow{\circledast} \mathrm{Hom}_{\mathcal{C}}(A \otimes B, V \otimes W)$$

*for all $A, B \in \mathrm{Ob}\,\mathcal{C}$ and $V, W \in \mathrm{Ob}\,\mathcal{C}'$ that is natural with respect to its $\mathcal{C}$-arguments:*

$$(\phi \circ \alpha) \circledast (\psi \circ \beta) = (\phi \circledast \psi) \circ (\alpha \otimes \beta), \tag{83}$$

*whenever $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, V)$, $\psi \in \mathrm{Hom}_{\mathcal{C}}(B, W)$, $\alpha, \beta \in \mathrm{Hom}\,\mathcal{C}$. Suppose it is unital, i.e.*

$$\phi \circledast \chi = \phi \otimes \chi, \quad \chi \circledast \phi = \chi \otimes \phi$$

*for any morphism $\phi$ and any $\chi \in \mathrm{Hom}_{\mathcal{C}}(B, 1_{\mathcal{C}})$. Then the family*

$$F_{V,W} := \mathrm{id}_V \circledast \mathrm{id}_W \in \mathrm{End}_{\mathcal{C}}(V \otimes W) \tag{84}$$

*is a cocycle in $\mathcal{C}'$. This cocycle respects morphisms from a subcategory $\mathcal{C}''$ in $\mathcal{C}'$ if and only if the operation $\circledast$ is natural with respect to $\mathcal{C}''$-arguments, i.e.*

$$(\zeta \circ \phi) \circledast (\eta \circ \psi) = (\zeta \otimes \eta) \circ (\phi \circledast \psi)$$

*whenever $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, V)$, $\psi \in \mathrm{Hom}_{\mathcal{C}}(B, W)$, $\zeta, \eta \in \mathrm{Hom}\,\mathcal{C}''$.*

*Proof.* By the definition (84), the expression $F_{U\otimes V,W} \circ (F_{U,V} \otimes \mathrm{id}_W)$ is equal to

$$\big(\mathrm{id}_{U\otimes V} \circledast \mathrm{id}_W\big) \circ \big((\mathrm{id}_U \circledast \mathrm{id}_V) \otimes \mathrm{id}_W\big) = \mathrm{id}_U \circledast \mathrm{id}_V \circledast \mathrm{id}_W. \tag{85}$$

Here we have used condition (83). Similarly, the expression $F_{U,V\otimes W} \circ (\mathrm{id}_U \otimes F_{V,W})$ is brought to the right-hand side of (85). Thus $F_{V,W}$ satisfies the cocycle condition. □

*6.2. Dynamical adjoint functors.* In this subsection we formulate the notion of dynamical adjoint functor, which appears to be very useful in constructing dynamical twists. Let $\mathcal{O}$ be a monoidal category and $\mathcal{O}'$ its monoidal subcategory; the embedding functor $\mathcal{O}' \to \mathcal{O}$ is denoted by R. Let $\mathcal{B}$ and $\mathcal{B}'$ be right module categories over $\mathcal{O}$ and $\mathcal{O}'$, respectively.

**Definition 6.2.** *A functor $\mathcal{B} \xrightarrow{\mathrm{M}} \mathcal{B}'$ is called* **dynamical adjoint** *to* R *if there is an isomorphism of the following three-functors from $\mathcal{B} \times \mathcal{B} \times \mathcal{O}'$ to the category of linear spaces:*

$$Y \times X \times V \to \mathrm{Hom}_{\mathcal{B}}\big(Y, X \otimes \mathrm{R}(V)\big) \simeq Y \times X \times V \to \mathrm{Hom}_{\mathcal{B}'}\big(\mathrm{M}(Y), \mathrm{M}(X) \otimes V\big). \quad (86)$$

Given a pair of dynamical adjoint functors, we introduce an operation $\circledast$ on morphisms from Ob $_{\mathcal{B}\lhd}\bar{\mathcal{O}}$ to its subcategory Ob $_{\mathcal{B}\lhd}\bar{\mathcal{O}}'$ in the following way. A pair $\{{}_X\phi\} \in \mathrm{Hom}_{\mathcal{B}\lhd\bar{\mathcal{O}}}\big(A, \mathrm{R}(V)\big)$ and $\{{}_X\psi\} \in \mathrm{Hom}_{\mathcal{B}\lhd\bar{\mathcal{O}}}\big(B, \mathrm{R}(W)\big)$ defines a family of $\mathcal{B}'$-morphisms $\mathrm{M}(X \otimes A \otimes B) \to \mathrm{M}(X) \otimes V \otimes W$, for all $X \in \mathcal{B}$, via the composition

$$\mathrm{M}(X \otimes A \otimes B) \xrightarrow{(X\otimes A)\tilde{\psi}} \mathrm{M}(X \otimes A) \otimes W \xrightarrow{X\tilde{\phi}\otimes \mathrm{id}_W} \mathrm{M}(X) \otimes V \otimes W. \quad (87)$$

By the tilde we denote the image of a morphism from Hom $_{\mathcal{B}\lhd}\bar{\mathcal{O}}$ under the correspondence (86). By condition (86), the composition (87) yields a morphism,

$$X \otimes A \otimes B \xrightarrow{X(\phi\circledast\psi)} X \otimes \mathrm{R}(V \otimes W), \quad (88)$$

in the category $\mathcal{B}$. Functoriality with respect to the first argument in (86) implies that the family $\{{}_X(\phi \circledast \psi)\}$ is in fact an $_{\mathcal{B}\lhd}\bar{\mathcal{O}}$-morphism. The associativity of the operation $\circledast$ follows from the associativity of composition of morphisms in the category $\mathcal{B}'$. Thus we obtain the following result.

**Proposition 6.3.** *A pair of dynamical adjoint functors defines, by formula (88), an associative operation $\phi \otimes \psi \to \phi \circledast \psi$ that satisfies the conditions of Lemma 6.1 for $\mathcal{C} = {}_{\mathcal{B}\lhd}\bar{\mathcal{O}}$ and $\mathcal{C}' = {}_{\mathcal{B}\lhd}\bar{\mathcal{O}}'$. It is $\mathcal{O}'$-functorial and thus yields a dynamical twist of $\mathcal{O}'$.*

In the next subsection, using Lemma 6.1 and Proposition 6.3, we construct a dynamical cocycle in the category of $\mathfrak{g}$-modules considered as a subcategory of $\mathfrak{l}$-modules, where $\mathfrak{l}$ is an arbitrary Levi subalgebra in $\mathfrak{g}$.

*6.3. Generalized Verma modules.* Let $\mathfrak{g}$ be a complex reductive Lie algebra with the Cartan subalgebra $\mathfrak{h}$ and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ a polarization with respect to $\mathfrak{h}$.

We fix a Levi subalgebra $\mathfrak{l}$, which is, by definition, the centralizer of an element in $\mathfrak{h}$. The algebra $\mathfrak{l}$ is reductive, so it is decomposed into the direct sum of its center and the semisimple part, $\mathfrak{l} = \mathfrak{c} \oplus \mathfrak{l}_0$, where $\mathfrak{l}_0 = [\mathfrak{l}, \mathfrak{l}]$. Also, there exists a decomposition

$$\mathfrak{g} = \mathfrak{n}_{\mathfrak{l}}^- \oplus \mathfrak{l} \oplus \mathfrak{n}_{\mathfrak{l}}^+, \quad (89)$$

where $\mathfrak{n}_{\mathfrak{l}}^{\pm}$ are subalgebras in $\mathfrak{n}^{\pm}$. Let $\mathfrak{p}^{\pm}$ denote the parabolic subalgebras $\mathfrak{l} \oplus \mathfrak{n}_{\mathfrak{l}}^{\pm}$.

Let $X$ be a finite dimensional semisimple representation of $\mathfrak{l}$. We consider $X$ as a left $\mathcal{U}(\mathfrak{l})$-module. Being extended by the trivial action of $\mathfrak{n}_{\mathfrak{l}}^+$, this representation can be considered as a left $\mathcal{U}(\mathfrak{p}^+)$-module. We denote by $M_X$ the generalized Verma module

$M_X := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}^+)} X$. It is a left $\mathcal{U}(\mathfrak{g})$-module, and the natural map $\mathcal{U}(\mathfrak{n}_\mathfrak{l}^-) \otimes_\mathbb{C} X \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}^+)} X$ is an isomorphism of vector spaces.

Let us consider the dual representation $X^*$ as a left $\mathcal{U}(\mathfrak{l})$-module with the action

$$(u\varphi)(x) = \varphi(\gamma(u)x), \tag{90}$$

where $\varphi \in X^*$, $x \in X$, $u \in \mathfrak{l}$, and $\gamma$ denotes the antipode in $\mathcal{U}(\mathfrak{g})$. Analogously to $M_X$, we define the generalized Verma module $M^-_{X^*} := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}^-)} X^*$ naturally isomorphic as a vector space to $\mathcal{U}(\mathfrak{n}_\mathfrak{l}^+) \otimes_\mathbb{C} X^*$.

There exists the following equivariant pairing between $M^-_{X^*}$ and $M_X$. Let $u_1 \otimes \varphi \in \mathcal{U}(\mathfrak{n}_\mathfrak{l}^+) \otimes_\mathbb{C} X^*$, $u_2 \otimes x \in \mathcal{U}(\mathfrak{n}_\mathfrak{l}^-) \otimes_\mathbb{C} X$. We put $\langle u_1 \otimes \varphi, u_2 \otimes x\rangle = \varphi\big(s(\gamma(u_1)u_2)x\big)$, where $s$ is the projection $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{l})$ along the direct sum decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{l}) \oplus (\mathfrak{n}_\mathfrak{l}^- \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n}_\mathfrak{l}^+).$$

It is obvious that this pairing defines the $\mathcal{U}(\mathfrak{g})$-equivariant map

$$M^-_{X^*} \to M^*_X, \tag{91}$$

where $M^*_X$ denotes the restricted dual $\mathcal{U}(\mathfrak{g})$-module to $M_X$, which is defined as follows. It is clear that $M_X = \oplus_\mu M_X[\mu]$, where $M_X[\mu]$ is the finite dimensional subspace of weight $\mu \in \mathfrak{h}^*$. We put $M^*_X := \oplus_\mu (M_X[\mu])^*$ with the $\mathcal{U}(\mathfrak{g})$-action similar to (90). It is known that map (91) is an isomorphism for representations $X$ satisfying conditions of Proposition 6.4 below.

Since $\mathcal{U}(\mathfrak{l}) = \mathcal{U}(\mathfrak{l}_0) \otimes \mathcal{U}(\mathfrak{c})$, where $\mathfrak{l}_0$ is the semisimple part of $\mathfrak{l}$ and $\mathfrak{c}$ its center, a $\mathcal{U}(\mathfrak{l})$-module $X$ is irreducible if and only if it can be presented as the tensor product of two representations:

$$X = X_0 \otimes \mathbb{C}_\lambda. \tag{92}$$

Here $X_0$ is an irreducible representation of $\mathfrak{l}_0$, and $\mathbb{C}_\lambda$ is a one dimensional representation of $\mathfrak{c}$ defined by a character $\lambda \in \mathfrak{c}^*$; both $X_0$ and $\mathbb{C}_\lambda$ are lifted to $\mathcal{U}(\mathfrak{l})$-modules in the natural way. It is clear that representation (92) is unique. We call the element $\lambda$ from (92) the **character** of $X$.

Let $\alpha_i$, $i = 1, \dots, \dim \mathfrak{c}$, be the simple roots with respect to $\mathfrak{h}$ that are not roots of $\mathfrak{l}$, and $e_{\pm\alpha_i}$ the corresponding root vectors such that $(e_{\alpha_i}, e_{-\alpha_i}) = 1$ for the Killing form $(.,.)$ in $\mathfrak{g}$. Put $h_i := [e_{\alpha_i}, e_{-\alpha_i}]$, $i = 1, \dots, \dim \mathfrak{c}$. Denote by $\mathcal{Y}$ the union of hyperplanes in $\mathfrak{c}^*$ consisting of $\lambda \in \mathfrak{c}^*$ having at least one coordinate $\lambda(h_i)$ integer.

**Proposition 6.4 ([J]).** *Let $X$ be a semisimple representation of $\mathfrak{l}$. If the characters of its irreducible components do not belong to $\mathcal{Y}$, then the map (91) is an isomorphism.*

We call an $\mathfrak{l}$-module $X$ generic if it satisfies this proposition.

*6.4. Dynamical twist via generalized Verma modules.* In this subsection we construct a dynamical cocycle for the case when the Hopf algebra $\mathcal{H}$ is a (quantum) universal enveloping algebra of a Levi subalgebra $\mathfrak{l}$ in a reductive Lie algebra $\mathfrak{g}$. Our method is a generalization to noncommutative and non-cocommutative Hopf algebras of the construction of Etingof and Varchenko, [EV3]. For simplicity we consider only classical universal enveloping algebras $\mathcal{U} = \mathcal{U}(\mathfrak{g})$, $\mathcal{H} = \mathcal{U}(\mathfrak{l})$. The construction carries over to the quantum groups in a straightforward way. Recall that $\mathcal{M}_{\mathcal{U}(\mathfrak{l})}$ and $\mathcal{M}_{\mathcal{U}(\mathfrak{g})}$ denote the categories of locally finite semisimple modules over $\mathcal{U}(\mathfrak{l})$ and $\mathcal{U}(\mathfrak{g})$, respectively.

**Lemma 6.5.** *For all $Y \in \mathcal{M}_{\mathcal{U}(\mathfrak{l})}$, $V \in \mathcal{M}_{\mathcal{U}(\mathfrak{g})}$, and generic $X \in \mathcal{M}_{\mathcal{U}(\mathfrak{l})}$,*

$$\mathrm{Hom}_{\mathfrak{g}}(M_Y, M_X \otimes V) \simeq \mathrm{Hom}_{\mathfrak{l}}(Y, X \otimes V). \tag{93}$$

*Proof.* Since, by Proposition 6.4, the module $M_X^*$ is isomorphic to $M_{X^*}^-$ for generic $X$, we have

$$\mathrm{Hom}_{\mathfrak{g}}(M_Y, M_X \otimes V) \simeq \mathrm{Hom}_{\mathfrak{g}}(M_Y \otimes M_X^*, V) \simeq \mathrm{Hom}_{\mathfrak{g}}(M_Y \otimes M_{X^*}^-, V), \tag{94}$$

where $M_X^*$ is the restricted dual to $M_X$. Since $M_Y \otimes M_{X^*}^- \simeq \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(Y \otimes X^*)$ as a $\mathfrak{g}$-module, we can apply the Frobenius reciprocity and obtain

$$\mathrm{Hom}_{\mathfrak{g}}(M_Y \otimes M_{X^*}^-, V) \simeq \mathrm{Hom}_{\mathfrak{l}}(Y \otimes X^*, V) \simeq \mathrm{Hom}_{\mathfrak{l}}(Y, X \otimes V). \tag{95}$$

Combining (94) and (95) we prove the lemma. □

Set, in terms of Definition 6.2, $\mathcal{O}$ to be the full subcategory in $\mathcal{M}_{\mathcal{U}(\mathfrak{l})}$ of modules whose characters belong to the weight lattice of $\mathfrak{g}$ relative to $\mathfrak{h}$. This category contains $\mathcal{M}_{\mathcal{U}(\mathfrak{g})}$ as a subcategory, which we put to be $\mathcal{O}'$. Let $\mathcal{B}$ be the full subcategory in $\mathcal{M}_{\mathcal{U}(\mathfrak{l})}$ of modules whose characters do not belong to $\mathcal{Y}$; it is a module category over $\mathcal{O}$. Let $\mathcal{B}'$ be the category of all $\mathcal{U}(\mathfrak{g})$-modules. Put $\mathrm{R}\colon \mathcal{O}' \to \mathcal{O}$ to be the restriction functor making an $\mathcal{U}(\mathfrak{g})$-module a module over $\mathcal{U}(\mathfrak{l})$. We define the adjoint functor M as follows. For $X \in \mathrm{Ob}\mathcal{M}_{\mathcal{U}(\mathfrak{l})}$ we put $\mathrm{M}(X) = M_X$, the generalized Verma module corresponding to $X$. It is clear that any morphism $X \to Y$ of $\mathcal{U}(\mathfrak{l})$-modules naturally corresponds to a morphism $M_X \to M_Y$ in the category $\mathcal{B}'$.

**Corollary 6.6.** *The functor $X \overset{\mathrm{M}}{\to} M_X$ is dynamical adjoint to the restriction functor $\mathcal{M}_{\mathcal{U}(\mathfrak{g})} \overset{\mathrm{R}}{\to} \mathcal{O}$.*

*Proof.* All we have to check is that correspondence (93) is natural with respect to $Y$, $V$, and generic $X$. This holds because the Frobenius reciprocity gives a natural isomorphism between adjoint functors for generic $X$. □

Let us consider the category $\mathcal{M}^{\mathcal{U}^*(\mathfrak{g})}$ of locally finite semisimple right $\mathcal{U}^*(\mathfrak{g})$-comodules. Note that $\mathcal{M}^{\mathcal{U}^*(\mathfrak{g})}$ is naturally isomorphic to the category $\mathcal{M}_{\mathcal{U}(\mathfrak{g})}$ of locally finite semisimple left $\mathcal{U}(\mathfrak{g})$-modules and hence to a subcategory of locally finite semisimple left $\mathcal{U}(\mathfrak{l})$-modules. We call the **dynamical extension of** $\mathcal{M}^{\mathcal{U}^*(\mathfrak{g})}$ **within** $\bar{\mathcal{M}}^{\mathcal{U}^*(\mathfrak{l})}$ the full subcategory in $\bar{\mathcal{M}}^{\mathcal{U}^*(\mathfrak{l})}$ whose objects belong to $\mathcal{M}^{\mathcal{U}^*(\mathfrak{g})}$.

Let us consider in more detail the structure of $\mathcal{U}^*(\mathfrak{l})$. First of all, $\mathcal{U}^*(\mathfrak{l})$ can be interpreted as the algebra of polynomial functions on the connected simply connected Lie group $\hat{H}$ corresponding to the Lie algebra $\mathfrak{l}$. That is, $\mathcal{U}^*(\mathfrak{l})$ is generated over $\mathbb{C}$ by matrix elements of all finite dimensional semisimple representations of $\mathfrak{l}$. The group $\hat{H}$ is presented as the Cartesian product $\hat{H}_0 \times \mathfrak{c}$ of the semisimple subgroup $\hat{H}_0$ and $\mathfrak{c}$ viewed as an abelian group.

It is well known that $\mathcal{U}^*(\mathfrak{l}) = \oplus_V \mathrm{End}^*_{\mathbb{C}}(V)$, where $V$ runs over the irreducible $\mathcal{U}(\mathfrak{l})$-modules. Each irreducible representation of $\mathfrak{l}$ has the form $V = V_0 \otimes \mathbb{C}_\mu$, where $V_0$ is a module over the semisimple part $\mathfrak{l}_0$ of $\mathfrak{l}$ and $\mathbb{C}_\mu$ is a one dimensional representation of the center $\mathfrak{c}$. Let $e^\mu\colon \mathfrak{c} \to \mathbb{C}^\times$ be the matrix element of $\mathbb{C}_\mu$ and $e_{ij}^{V_0}$ the matrix elements of $V_0$. The elements $\{e_{ij}^{V_0} e^\mu\}$ form a basis in the vector space $\mathcal{U}^*(\mathfrak{l})$. We call an element $\lambda \in \mathcal{U}^*(\mathfrak{l})$ generic if the decomposition of $\lambda$ via this basis contains no $e^\mu$, for $\mu \in \mathcal{Y}$.

**Theorem 6.7.** *Let $\mathfrak{l}$ be a Levi subalgebra in a reductive Lie algebra $\mathfrak{g}$, $\mathcal{U}(\mathfrak{l})$ the corresponding Hopf subalgebra in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. There exists an $\mathcal{U}(\mathfrak{l})$-equivariant map*

$$\bar{\mathcal{F}}\colon \mathcal{U}^*(\mathfrak{l}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$$

*such that for generic $\lambda \in \mathcal{U}^*(\mathfrak{l})$ the family $F^{\lambda}_{V,W} := (\rho_V \otimes \rho_W)\big(\bar{\mathcal{F}}(\lambda)\big)$, $V, W \in$ Ob $\mathcal{M}^{\mathcal{U}^*(\mathfrak{g})}$, is a dynamical twist (68–69) in the dynamical extension of the category $\mathcal{M}^{\mathcal{U}^*(\mathfrak{g})}$ within $\bar{\mathcal{M}}^{\mathcal{U}^*(\mathfrak{l})}$.*

*Proof.* By Corollary 6.6 and Proposition 6.3, there exists a dynamical twist in the category ${}_{\mathcal{B}}\bar{\mathcal{O}}$, which is a collection of morphisms $\chi(F_{V,W}) \in \mathrm{End}_{\mathfrak{l}}(X \otimes V \otimes W)$, where $V$ and $W$ are $\mathfrak{g}$-modules and $X$ is a generic $\mathfrak{l}$-module. Using the natural filtration in generalized Verma modules, one can prove that morphisms $\{\chi(F_{V,W})\}$ are invertible in $\bar{\mathcal{M}}^{\mathcal{U}^*(\mathfrak{l})}$. The morphisms $\chi(F_{V,W})$ define a collection of $\mathfrak{l}$-equivariant maps $X^* \otimes X \to \mathrm{End}_k(V \otimes W)$, which gives rise to a collection of maps $X^* \otimes X \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ for generic $X$, since the dynamical twist is natural with respect to the arguments $V$ and $W$. This collection determines an $\mathfrak{l}$-equivariant map $\bar{\mathcal{F}}\colon \mathcal{U}^*(\mathfrak{l}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ defined for generic of elements $\mathcal{U}^*(\mathfrak{l})$. Indeed, by Remark 4.20 the dynamical category ${}_{\mathcal{B}}\bar{\mathcal{O}}$ is isomorphic to $\bar{\mathcal{M}}^{\mathcal{U}^*(\mathfrak{l})}$. Under this isomorphism, the dynamical twist $\{\chi(F_{V,W})\}$ goes over to the map $\lambda \mapsto \bar{\mathcal{F}}(\lambda)$ for $\lambda \in X^* \otimes X$ and $X$ generic, which reduces to the twisting cocycle (68–69) in representations. □

## 7. Dynamical Associative Algebras and Quantum Vector Bundles

*7.1. Classical vector bundles.* Let $H$ be a Lie group and $P$ be a principal $H$-bundle. Denote by $\mathcal{A} = \mathcal{A}(P)$ the algebra of functions on $P$. Let $V$ be a finite dimensional left $H$-module. An associated vector bundle $V(M)$ on $M = P/H$ with the fiber $V$ is defined as the coset space $(P \times V)/H$ by the action $(p, v) \mapsto (ph, h^{-1}v)$, $(p, v) \in (P \times V)$, $h \in H$. The global sections of $V(M)$ are identified with the space $\big(\mathcal{A}(P) \otimes V\big)^H \simeq \mathrm{Hom}_H\big(V^*, \mathcal{A}(P)\big)$. Let us denote by $\mathcal{A}^V$ the space of global sections of $V(M)$. When $V = k$, the trivial $H$-module, the space $\mathcal{A}^k$ is canonically identified with the subalgebra in $\mathcal{A}$ of $H$-invariant functions; in other words, $\mathcal{A}^k = \mathcal{A}(M)$. The tensor product of vector bundles corresponds to the tensor product of sections, which is induced by multiplication in $\mathcal{A}$: given $s_V \in \mathcal{A}^V$ and $s_W \in \mathcal{A}^W$ the section $s_W \otimes s_V \in \mathcal{A}^{V \otimes W}$ is

$$(s_W \otimes s_V)(w \otimes v) := s_W(w) s_V(v), \quad w \otimes v \in W^* \otimes V^* \simeq (V \otimes W)^*.$$

In particular, the tensor product of sections makes the space $\mathcal{A}^V$ a two-sided module over $\mathcal{A}^k$.

*7.2. Quantum vector bundles.* Fix a Hopf algebra $\mathcal{H}$ over the ground ring $k$ and consider a dynamical associative algebra $\mathcal{A}$ in the category $\bar{\mathcal{M}}^{\mathcal{H}^*}$, cf. Example 4.22. We are going to introduce associated vector bundles over the "non-commutative coset space" corresponding to the action of $\mathcal{H}$ on $\mathcal{A}$.

**Definition 7.1.** *Let $V$ be a right $\mathcal{H}$-module.* **The associated vector bundle** $\mathcal{A}^V$ *with fiber $V$ is the space of all $\mathcal{H}$-equivariant maps (sections) $s_V\colon V^* \to \mathcal{A}$.*

Observe that the restriction of the dynamical multiplication in $\mathcal{A}$ to a group-like element $\lambda$ of $\mathcal{H}^*$ defines a bilinear operation $\overset{\lambda}{*}: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, which is $\mathcal{H}$-equivariant, since group-like elements are invariant in $\mathcal{H}^*$ under the coadjoint action (45). Now we can define a product of sections. Let $V$ and $W$ be two $\mathcal{H}$-modules. Take $s_V: V^* \to \mathcal{A}$ and $s_W: W^* \to \mathcal{A}$ to be sections of $\mathcal{A}^V$ and $\mathcal{A}^W$. Fix a group-like element $\lambda \in \mathcal{H}^*$. The map $s_W \overset{\lambda}{*} s_V: (V \otimes W)^* \simeq W^* \otimes V^* \to \mathcal{A}$,

$$(s_W \overset{\lambda}{*} s_V)(w \otimes v) := s_W(w) \overset{\lambda}{*} s_V(v), \quad w \otimes v \in W^* \otimes V^*, \tag{96}$$

is a section of the bundle $\mathcal{A}^{V \otimes W}$. The subspace of $\mathcal{H}$-invariants $\mathcal{A}^k \subset \mathcal{A}$ is obviously closed under $\overset{\lambda}{*}$ for every group-like element $\lambda \in \mathcal{H}^*$.

**Theorem 7.2.** *For any group-like element $\lambda \in \mathcal{H}^*$ and any finite dimensional $\mathcal{H}$-module $V$ the multiplication $\overset{\lambda}{*}$ provides $\mathcal{A}^k$ with the structure of an associative algebra, $\mathcal{A}^k_\lambda$, and makes the space $\mathcal{A}^V$ a left $\mathcal{A}^k_\lambda$-module. If $V = k_\alpha$, i.e. is the 1-dimensional representation of $\mathcal{H}$ defined by the character $\alpha$, then the line bundle $\mathcal{A}^V$ is also a right $\mathcal{A}^k_{\lambda\alpha^{-1}}$-module with respect to $\overset{\lambda}{*}$. For any $a \in \mathcal{A}^k_\lambda$, $s_V \in \mathcal{A}^V$, and $s_W \in \mathcal{A}^W$*

$$a \overset{\lambda}{*} (s_V \overset{\lambda}{*} s_W) = (a \overset{\lambda}{*} s_V) \overset{\lambda}{*} s_W. \tag{97}$$

*Proof.* Sections of the line bundle $\mathcal{A}^{k_\alpha}$ may be treated as elements $a \in \mathcal{A}$ such that $h \triangleright a = \alpha^{-1}(h)a$ for all $h \in \mathcal{H}$ (the inverse is understood in the sense of the algebra $\mathcal{H}^*$). For $a \in A^{k_\alpha}$ and $b, c \in \mathcal{A}$, the formula (60) turns into

$$(a \overset{\lambda}{*} b) \overset{\lambda}{*} c = a \overset{\lambda}{*} (b \overset{\lambda\alpha^{-1}}{*} c) \tag{98}$$

under the assumption that $\lambda \in \mathcal{H}^*$ is group-like. Setting $\alpha = 1$ (the unit of $\mathcal{H}^*$) in (98), we find that $\overset{\lambda}{*}$ is associative when restricted to $\mathcal{A}^k$ and makes it an associative algebra, $\mathcal{A}^k_\lambda$. Also we see that $\mathcal{A}$ is a left $\mathcal{A}^k_\lambda$-module. This induces the structure of a left $\mathcal{A}^k_\lambda$-module on $\mathcal{A}^V$ for every $\mathcal{H}$-module $V$, by formula (98). Assuming $b, c \in \mathcal{A}^k$ in (98) we obtain a right $\mathcal{A}^k_{\lambda\alpha^{-1}}$-module structure on the space $\mathcal{A}^{k_\alpha}$. □

## 8. Vector Bundles on Semisimple Coadjoint Orbits

The problem of equivariant quantization of function algebras on semisimple coadjoint orbits of simple Lie groups was studied, e.g., in [DGS1, DolJ, DM1, DM2, DM4, DS]. Quantization of vector bundles on semisimple orbits as modules over the quantized function algebras was considered in [D1, GLS]. In this section we use dynamical associative algebras for quantization of the entire "algebra" of sections of all vector bundles on semisimple coadjoint orbits.

*8.1. Dynamical quantization of the function algebra on a group.* Let $\mathfrak{g}$ be a simple Lie algebra and $G$ the corresponding connected Lie group. We will apply the previous considerations to the problem of equivariant quantization of vector bundles on semisimple orbits in $\mathfrak{g}^*$ with respect to the coadjoint action of $G$. Denote by $\mathcal{A}(G)$ the algebra of polynomial functions on $G$. The group $G$ acts on itself by the left and the right regular actions. These actions induce two left commuting actions of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{A}(G)$ via the differential operators $\rho_1(x)$ and $\rho_2(x)$, $x \in \mathcal{U}(\mathfrak{g})$, respectively. Here $\rho_1(x)$ ($\rho_2(x)$) is the differential operator on $G$ that is the right (left) invariant extension of $\gamma(x)$ $(x)$, where $\gamma$ denotes the antipode in $\mathcal{U}(\mathfrak{g})$.

Given a representation $\pi : G \to \mathrm{End}(V)$ assign to each $f \in \mathrm{End}(V)^*$ the function $f \circ \pi$ on $G$. Identifying $\mathrm{End}(V)^*$ with $\mathrm{End}(V) = V \otimes V^*$ via the trace pairing, these assignments give the well known isomorphism

$$\oplus_E E \otimes E^* \to \mathcal{A}(G), \tag{99}$$

where $E$ runs over all irreducible representations of $G$. Then the $\rho_1$-action can be treated as an action on the $E$-factor while the $\rho_2$-action – on the $E^*$-factor of each term $E \otimes E^*$ in the direct sum (99).

As a manifold, a semisimple orbit is the quotient $M = G/H$, where $H$ is a Levi subgroup with the Lie algebra $\mathfrak{l} \subset \mathfrak{g}$. Recall the decomposition $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{c}$, where $\mathfrak{l}_0$ is the semisimple part and $\mathfrak{c}$ the center of $\mathfrak{l}$. The manifold $G$ may be considered as a principal $H$-bundle on $M$. Any equivariant vector bundle on $M$ is associated to the bundle $G$ via a representation $V$ of $H$. We denote this vector bundle by $V(M)$. It has the vector space $V$ as the fiber.

The global sections of the bundle $V(M)$ are identified with the space $\big(\mathcal{A}(G) \otimes V\big)^{\mathfrak{l}} = \mathrm{Hom}_{\mathfrak{l}}\big(V^*, \mathcal{A}(G)\big)$, where $\mathcal{A}(G)$ is considered as a $\mathcal{U}(\mathfrak{g})$-module with respect to the $\rho_2$-action. Since $M$ is an affine variety, one can identify the vector bundle $V(M)$ with its global sections and consider it as a $\mathcal{U}(\mathfrak{g})$-module with respect to the $\rho_1$-action. In particular, suppose $V = \mathbb{C}_\lambda$ is the one dimensional representation of $\mathfrak{l}$ induced by the character $\lambda \in \mathfrak{c}^*$. Let us consider $\lambda$ as an element of $\mathfrak{h}^*$ via the embedding $\mathfrak{c}^* \subset \mathfrak{h}^*$ along the decomposition $\mathfrak{h} = \mathfrak{c} \oplus \mathfrak{c}^\perp$. Due to isomorphism (99), the assignment $\mathrm{Hom}_{\mathfrak{l}}\big(\mathbb{C}^*_\lambda, \mathcal{A}(G)\big) \ni \varphi \mapsto \varphi(1) \in \mathcal{A}(G)$ gives a natural isomorphism of $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{l})$-modules

$$\mathrm{Hom}_{\mathfrak{l}}\big(\mathbb{C}^*_\lambda, \mathcal{A}(G)\big) \to \mathcal{A}(G)[-\lambda], \tag{100}$$

where $\mathcal{A}(G)[-\lambda]$ is a subspace of $\mathfrak{l}_0$-invariant elements of $\mathcal{A}(G)$ of weight $-\lambda$ with respect to the $\rho_2$-action. It is obvious that $\mathcal{A}(G)[-\lambda]$ is a $\mathcal{U}(\mathfrak{g})$-module with respect to the $\rho_1$-action and it is naturally isomorphic to $\oplus_E E \otimes E^*[-\lambda]$, where $E^*[-\lambda]$ denotes the subspace of $\mathfrak{l}_0$-invariant elements of $E^*$ of weight $-\lambda$. It is clear that the isomorphism (100) is actually non-zero only if $\lambda$ is an integer weight. In this case the map (100) identifies $\mathcal{A}(G)[-\lambda]$ with the space of global sections of the line bundle $\mathbb{C}_\lambda(M)$. In particular, the function algebra on $M$ is naturally isomorphic to the $\mathcal{U}(\mathfrak{g})$-module algebra $\oplus_E E \otimes E^*[0] \subset \mathcal{A}(G)$.

Applying the dynamical twist $\bar{\mathcal{F}}$ constructed in Theorem 6.7 to the $\mathcal{U}(\mathfrak{g})$-module algebra $\mathcal{A}(G)$ with respect to the $\rho_2$-action, we obtain a dynamical associative $\mathcal{U}(\mathfrak{l})$-algebra in the category $\bar{\mathcal{O}}^{\mathcal{U}^*(\mathfrak{l})}$. This algebra is equal to $\mathcal{A}(G)$ as a $\mathcal{U}(\mathfrak{g})$-module (with respect to $\rho_1$-action) and has the family of multiplications parameterized by generic $\lambda \in \mathcal{U}^*(\mathfrak{l})$ and defined as $\bar{m}_\lambda = m \circ \bar{\mathcal{F}}^\lambda$, where $m$ is the original multiplication in the algebra $\mathcal{A}(G)$. Applying Theorem 7.2 to this dynamical associative algebra, we obtain a quantization of vector bundles on $G/H$. Obviously, this quantization is equivariant with respect to the $\rho_1$-action of $\mathcal{U}(\mathfrak{g})$.

Let us consider the dynamical twist $\bar{\mathcal{F}}(\lambda)$ restricted to $\mathcal{U}^*(\mathfrak{c})$. Applied to the subalgebra $\mathcal{A}(G)^{\mathfrak{l}_0} \subset \mathcal{A}(G)$ of $\mathfrak{l}_0$-invariant functions on $G$ with respect to the $\rho_2$-action, this restriction makes $\mathcal{A}(G)^{\mathfrak{l}_0}$ a dynamical associative $\mathcal{U}(\mathfrak{l})$-algebra over the base $\mathcal{U}^*(\mathfrak{c})$. As a $\mathcal{U}(\mathfrak{g})$-module, it is formed by sections of all linear bundles on $M$. Let us describe this dynamical algebra in more detail.

Let $M_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}^+)} \mathbb{C}_\lambda$ be the Verma module corresponding to the one dimensional representation of $\mathcal{U}(\mathfrak{l})$ associated to $\lambda \in \mathfrak{c}^*$. Let $\mathrm{Hom}^0(M_\lambda, M_\mu)$ denote the subspace of locally finite elements in $\mathrm{Hom}_{\mathbb{C}}(M_\lambda, M_\mu)$ with respect to the adjoint action of $\mathcal{U}(\mathfrak{g})$. In fact, $\mathrm{Hom}^0(M_\lambda, M_\mu)$ is not zero only when $\mu - \lambda$ is an integer weight. Recall that for a $\mathcal{U}(\mathfrak{g})$-module $E$, we denote by $E[\lambda]$, $\lambda \in \mathfrak{c}^*$, the subspace of $\mathfrak{l}_0$-invariant elements of weight $\lambda$.

**Proposition 8.1.** *Let $V$ be a finite dimensional representation of $\mathcal{U}(\mathfrak{g})$. Then, there is a natural morphism of $\mathcal{U}(\mathfrak{g})$-modules: $V \otimes V^*[\lambda - \mu] \to \mathrm{Hom}^0(M_\lambda, M_\mu)$, $\lambda, \mu \in \mathfrak{c}^*$, $\mu$ is generic. When $V$ is irreducible, this morphism is embedding. These embeddings give rise to the natural isomorphism*

$$j_{\lambda,\mu} : \oplus_E E \otimes E^*[\lambda - \mu] \to \mathrm{Hom}^0(M_\lambda, M_\mu) \tag{101}$$

*of $\mathcal{U}(\mathfrak{g})$-modules, where $E$ runs over all finite dimensional irreducible representations of $\mathcal{U}(\mathfrak{g})$.*

*Proof.* It is enough to prove the first part of the proposition and show that the multiplicity of $V$ in $\mathrm{Hom}^0(M_\lambda, M_\mu)$ is equal to $\dim V^*[\lambda - \mu]$. Applying the Frobenius reciprocity, one proves that for generic $\mu \in \mathfrak{c}^*$ the space $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(M_\lambda, M_\mu \otimes V^*)$ is naturally isomorphic to $V^*[\lambda - \mu]$; the proof is the same as in [ES1]. But $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(M_\lambda, M_\mu \otimes V^*) \cong \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}\big(V, \mathrm{Hom}^0(M_\lambda, M_\mu)\big)$, which proves the proposition. □

Compositions $M_\nu \to M_\mu \to M_\lambda$ generate the map $\mathrm{Hom}^0(M_\mu, M_\lambda) \otimes \mathrm{Hom}^0(M_\nu, M_\mu) \to \mathrm{Hom}^0\big(M_\nu, M_\lambda\big)$, $\lambda, \mu, \nu \in \mathfrak{c}^*$. Due to isomorphisms (101) and (99), this map defines the morphism of $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{l})$-modules

$$\mathcal{A}(G)[\mu - \lambda] \otimes \mathcal{A}(G)[\nu - \mu] \to \mathcal{A}(G)[\nu - \lambda]. \tag{102}$$

Since $\mathcal{A}(G)[\beta] = 0$ unless $\beta$ is a positive integer weight, this morphism is defined for generic $\lambda \in \mathfrak{c}^*$, i.e. for $\lambda \notin \mathcal{Y}$, where $\mathcal{Y}$ is from Proposition 6.4. Indeed, if $\lambda \notin \mathcal{Y}$, then also $\mu, \nu \notin \mathcal{Y}$ when the differences $\mu - \lambda$ and $\nu - \mu$ are integer weights.

Fixing a generic $\lambda$ in (102) and varying $\mu$ and $\nu$, we obtain from (102) a morphism

$$\mathcal{A}(G)^{\mathfrak{l}_0} \otimes \mathcal{A}(G)^{\mathfrak{l}_0} \to \mathcal{A}(G)^{\mathfrak{l}_0}. \tag{103}$$

These morphisms form a family of multiplications parameterized by elements $e^\lambda \in \mathcal{U}^*(\mathfrak{c})$ (or $\lambda \in \mathfrak{c}^*$) for generic $\lambda$. Since the elements $e^\lambda$ form a basis of $\mathcal{U}^*(\mathfrak{c})$ over $\mathbb{C}$, this family extends by linearity to all generic elements of $\mathcal{U}^*(\mathfrak{c})$. One can check that this family makes $\mathcal{A}(G)^{\mathfrak{l}_0}$ a dynamical associative $\mathcal{U}(\mathfrak{l})$-algebra over the base $\mathcal{U}^*(\mathfrak{c})$. Comparing the construction of this multiplication and the construction of twist from Theorem 6.7, we come to the following.

**Proposition 8.2.** *For generic $\lambda \in \mathfrak{c}^*$, the dynamical associative multiplication (103) has the form $\bar{m}_\lambda = m \circ \bar{\mathcal{F}}(\lambda)$, where $\bar{\mathcal{F}}$ is the dynamical twist over the base $\mathcal{U}^*(\mathfrak{l})$ from Theorem 6.7.*

*8.2. Deformation quantization of the Kirillov brackets and vector bundles on coadjoint orbits.* Dynamical twist from Theorem 6.7 applied to $\mathcal{A}(G)$ gives a dynamical algebra, which, by Theorem 7.2, defines quantization of vector bundles on $M = G/H$ as left modules over the quantized algebra of functions on $M$. This quantization of vector bundles is obviously $\mathfrak{g}$-equivariant. Restricted to the function algebra on $M$, it gives quantization of the Kirillov brackets in the following way.

Let $t$ be an independent variable. Denote by $\mathfrak{g}_t$ the Lie algebra over $\mathbb{C}[[t]]$ with bracket $[x, y]_t := t[x, y]$ for $x, y \in \mathfrak{g}$, where $[.,.]$ is the original bracket in $\mathfrak{g}$. Then there is an algebra morphism $\varphi_t : \mathcal{U}(\mathfrak{g}_t) \to \mathcal{U}(\mathfrak{g})[[t]]$ induced by the correspondence $x \mapsto tx$ for $x \in \mathfrak{g}$. As was shown in [DGS2], the equivariant quantization of the Kirillov Poisson bracket corresponding to the semisimple orbit passing through $\lambda \in \mathfrak{c}^* \subset \mathfrak{g}^*$ is identified with the image of $\mathcal{U}(\mathfrak{g}_t)$ by the composition map $\mathcal{U}(\mathfrak{g}_t) \to \mathcal{U}(\mathfrak{g})[[t]] \to \mathrm{Hom}^0(M_{\lambda/t}, M_{\lambda/t})$, where the first arrow is $\varphi_t$ and the second one is the representaion map.

Using this fact, one can show that the multiplication $\bar{m}_{\lambda,t} := \bar{m}_{\lambda/t}$ from Proposition 8.2 being restricted to $\mathcal{A}(M) = \mathcal{A}(G)[0]$ gives a $\mathcal{U}(\mathfrak{g})$-equivariant deformation quantization of the Kirillov Poisson bracket on $M$ realized as a coadjoint orbit passing through $\lambda$. Since the multiplication $\bar{m}_{\lambda,t}$ depends, in fact, on $\lambda/t$, we do not need $\lambda$ to be generic in the deformation quantization. Note that also for any formal path $\lambda(t) = \lambda_0 + t\lambda_1 + \ldots \in \mathfrak{c}^*[[t]]$ the multiplication $\bar{m}_{\lambda(t),t}$ gives a $\mathcal{U}(\mathfrak{g})$-equivariant deformation quantization on the orbit passing through $\lambda_0$, with the appropriate Kirillov bracket.

*Remark 8.3.* Any equivariant deformation quantization of the Kirillov bracket on the orbit can be obtained in this way, and different paths in $\mathfrak{c}^*$ give non-equivalent quantizations, [D1].

For $\lambda \in \mathfrak{c}^*$, consider $\bar{m}_\lambda = m \circ \bar{\mathcal{F}}(\lambda)$, where $\bar{\mathcal{F}}$ is the dynamical twist from Theorem 6.7 and $m$ the classical multiplication, as a map $\mathcal{A}(G)^{\otimes 2} \to \mathcal{A}(G)$. Recall that $\mathfrak{c}^*$ can be naturally identified with a subspace in $\mathfrak{g}^*$. Let us call an element $\lambda_0 \in \mathfrak{c}^*$ regular if its stabilizer in $\mathfrak{g}$ (under the coadjoint action) is the Levi subalgebra $\mathfrak{l}$. In a similar way, one can prove

**Proposition 8.4.** *For any regular $\lambda_0 \in \mathfrak{c}^*$ and for any formal path $\lambda(t) = \lambda_0 + t\lambda_1 + \ldots \in \mathfrak{c}^*[[t]]$, the multiplication $\bar{m}_{\lambda(t),t} := \bar{m}_{\lambda(t)/t}$ gives a $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{l})$-equivariant map $\mathcal{A}(G)^{\otimes 2} \to \mathcal{A}(G)[[t]]$ that coincides modulo $t$ with the original multiplication in $\mathcal{A}(G)$.*

Now, we fix regular $\lambda_0 \in \mathfrak{c}^*$ and consider $M = G/H$ as the semisimple orbit passing through $\lambda_0$. Recall that the space of global sections of the vector bundle $V(M)$ corresponding to an $H$-representation $V$ is identified with the space $\big(\mathcal{A}(G) \otimes V\big)^{\mathfrak{l}} = \mathrm{Hom}_{\mathfrak{l}}\big(V^*, \mathcal{A}(G)\big)$. We identify $V(M)$ with its global sections. Applying Theorem 7.2 to the dynamical algebra obtained from $\mathcal{A}(G)$ by the dynamical twist from Theorem 6.7 and using Proposition 8.4, we obtain

**Theorem 8.5.** *Let $\lambda_0$ be a regular element from $\mathfrak{c}^*$ and $\lambda(t) = \lambda_0 + t\lambda_1 + \ldots$ a formal path in $\mathfrak{c}^*$. Then the dynamical multiplication $\bar{m}_{\lambda(t)/t}$ defines a $\mathcal{U}(\mathfrak{g})$-equivariant multiplication $\overset{\lambda}{*}$ on the global sections of equivariant vector bundles on $M$. This multiplication is a deformation of the usual tensor product of the sections and satisfies the following properties:*

*1) Restricted to $\mathcal{A}(M)$, the operation $\overset{\lambda(t)}{*}$ defines a deformation quantization $\mathcal{A}_{\lambda(t)}(M)$ of the function algebra $\mathcal{A}(M)$ corresponding to the Kirillov bracket on the orbit passing through $\lambda_0$;*

*2) Let $s_V$ and $s_W$ be global section of vector bundles $V(M)$ and $W(M)$, and $a$ is a function on $M$. Then*

$$a \overset{\lambda}{*} (s_V \overset{\lambda}{*} s_W) = (a \overset{\lambda}{*} s_V) \overset{\lambda}{*} s_W.$$

*In particular, any vector bundle $V(M)$ is a left $\mathcal{A}_{\lambda(t)}(M)$-module with respect to the action map $a \otimes s \mapsto a \overset{\lambda(t)}{*} s$, where $a \in \mathcal{A}_{\lambda(t)}(M)$ and $s \in V(M)$.*

*3) The line bundle $\mathbb{C}_\alpha(M)$, where $\alpha \in \mathfrak{c}^*$ is a positive integer weight, is also a right module over the algebra $\mathcal{A}_{\lambda(t)-t\alpha}(M)$ with respect to the action map $s \otimes b \mapsto s \overset{\lambda(t)}{*} b$, where $s \in \mathbb{C}_\alpha(M)$, $b \in \mathcal{A}_{\lambda(t)-t\alpha}(M)$.*

*8.3. The quantum group case.* Let $\mathcal{U}_q(\mathfrak{g})$ be the Drinfeld-Jimbo quantum group corresponding to $\mathfrak{g}$ and $\mathcal{U}_q(\mathfrak{l})$ be considered as its quantum subgroup corresponding to the Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$. Let $\mathcal{A}_q(G)$ denote the dual algebra to $\mathcal{U}_q(\mathfrak{g})$ consisting of matrix elements of finite dimensional representations of $\mathcal{U}_q(\mathfrak{g})$. The algebra $\mathcal{A}_q(G)$ is a quantization of the classical algebra $\mathcal{A}(G)$, it is equivariant under the left and the right regular actions of $\mathcal{U}_q(\mathfrak{g})$, which we replace by two left actions, $\rho_1$ and $\rho_2$, as above.

Let $\bar{\mathcal{F}}_q$ be the dynamical twist constructed in Theorem 6.7 with the help of generalized Verma modules over $\mathcal{U}_q(\mathfrak{g})$. Applying this twist to the algebra $\mathcal{A}_q(G)$, we obtain a dynamical algebra $\left(\mathcal{A}(G), \bar{m}_{q,\lambda}\right)$ in the category $\bar{\mathcal{M}}^{\mathcal{U}_q^*(\mathfrak{l})}$. This algebra is equal to $\mathcal{A}_q(G)$ as a $\mathcal{U}_q(\mathfrak{g})$-module (with respect to $\rho_1$-action) and has the family of multiplications $\bar{m}_{q,\lambda}$ parameterized by generic $\lambda \in \mathcal{U}_q^*(\mathfrak{l})$. They are defined by $\bar{m}_{q,\lambda} := m_q \circ \bar{\mathcal{F}}_{q,\lambda}$, where $m_q$ is the original multiplication in $\mathcal{A}_q(G)$. It is obvious that $\mathcal{A}_{q,\lambda}(G)$ is a $\mathcal{U}_q(\mathfrak{g})$-module algebra with respect to the $\rho_1$-action.

One can show that replacing simultaneously $\lambda$ by $\lambda/t$ and $q$ by $q^t$ we obtain the family of multiplications $\bar{m}_{q^t,\lambda,t} := \bar{m}_{q^t,\lambda/t}$ that gives a $\mathcal{U}_{q^t}(\mathfrak{g})$-equivariant deformation quantization of $\mathcal{A}(M)$. Also, there exists a $q$-analog of Theorem 8.5 which reduces to Theorem 8.5 when $q = 1$.

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## References

[ABB] Avan, J., Babelon, O., Billey, E.: The Gervais-Neveu-Felder equation and the quantum Calogero-Moser systems. Commun. Math. Phys. **178**, 281–299 (1996)

[AL] Alekseev, A., Lachowska, A.: *Invariant $*$-product on coadjoint orbits and the Shapovalov pairing*. http://arxiv.org/abs/math.QA/0308100, 2003

[AM] Alekseev, A., Meinrenken, E.: The non-commutative Weil algebra. Invent. Math. **135**, 135–172 (2000)

[AF] Alekseev, A., Faddeev, L.: $T^*(G)_t$: a toy model of conformal field theory. Commun. Math. Phys. **141**, 413–422 (1991)

[BDFh] Balog, J., Dabrowski, L., Fehér, L.: Classical $r$-matrix and exchange algebra in WZNW and Toda field theories. Phys. Lett. B **244**, # 2, 227–234 (1990)

[Dr1] Drinfeld, V.: Quantum Groups. In Proc. Int. Congress of Mathematicians, Berkeley, 1986, ed. A. V. Gleason, Providence, RI: AMS, (1987), pp. 798–820

[Dr2] Drinfeld, V.: Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations. Sov. Math. Dokl. **27**, 68–71 (1983)

[Dr3] Drinfeld, V.: *Quasi-Hopf algebras*. Leningrad Math.J. **1**, 1419–1457 (1990)

[Dr4] Drinfeld, V.: On Poisson homogeneous spaces of Poisson-Lie groups. Theor. Math. Phys. **95**, 226-227 (1993)

[DolJ] Dolan, B. P., Jahn, O.: Fuzzy Complex Grassmannian Spaces and their Star Products. Int. J. Mod. Phys. **A 18**, 1935-1958 (2003)

[D1] Donin, J.: $\mathcal{U}_h(\mathfrak{g})$-invariant quantization of coadjoint orbits and vector bundles over them. J. Geom. Phys., **38**, #1, 54–80 (2001)

[D2] Donin, J.: Quantum $G$-manifolds. Contemp. Math. **315**, 47–60 (2002)

[DGS1] Donin, J., Gurevich, D., Shnider, S.: Double quantization in some orbits in the coadjoint representiations of simple Lie groups. Commun. Math. Phys. **204**, 39–60 (1999)

[DGS2] Donin, J., Gurevich, D., Shnider, S.: Quantization of function algebras on semisimple orbits in $\mathfrak{g}^*$. http://arxiv.org/abs/q-alg/9607008, 1996

[DM1] Donin, J., Mudrov, A.: Method of quantum characters in equivariant quantization. Commun. Math. Phys. **234**, 533–555 (2003)

[DM2] Donin, J., Mudrov, A.: Explicit equivariant quantization on coadjoint orbits of GL(n). Lett. Math. Phys. **62**, 17–32 (2002)

[DM3] Donin, J., Mudrov, A.: Reflection Equation, Twist, and Equivariant Quantization. Isr. J. Math. **136**, 11–28 (2003)

[DM4] Donin, J., Mudrov, A.: Quantum coadjoint orbits of GL(n) and generalized Verma modules. http://arxiv.org/abs/math.QA/0212318, 2002

[DM5] Donin, J., Mudrov, A.: Quantum groupoids associated with dynamical categories. http://arxiv.org/abs/math.QA/0311116, 2003

[DO] Donin, J., Ostapenko, V.: Equivariant quantization on quotients of simple Lie groups by reductive subgroups of maximal rank. Czech. J. of Phys. **52**, # 11, 1213–1218 (2002)

[DS] Donin, J., Shnider, S.: Quantum symmetric spaces. J. Pure & Applied Alg. **100**, 103–116 (1995)

[EE1] Enriquez, B., Etingof, P.: Quantization of Alekseev-Meinrenken dynamical r-matrices. Amer. Math. Soc. Transl. **210**, # 2, 81–98 (2003)

[EE2] Enriquez, B., Etingof, P.: Quantization of classical dynamical r-matrices withnonabelian base. To appear in Commun. Math. Phys., DOI 10.1007/S00220-004-1243-2, 2004

[EK] Etingof, P., Kazhdan, D.: Quantization of Lie bialgebras. Selecta Math. **2**, # 1, 1–41 (1996)

[ES1] Etingof, P., Schiffmann, O.: Lectures on the dynamical Yang-Baxter equation. Quantum Groups and Lie theory. London Math. Soc. Lecture Note **290**, 89–129 (2001)

[ES2] Etingof, P., Schiffmann, O.: On the moduli space of classical dynamical r-matrices. Math. Res. Lett. **9**, 157–170 (2001)

[ESS] Etingof, P., Schiffmann, O., Schedler, T.: Explicit quantization of dynamical r-matrices for finite dimensional semisimple Lie algebras. J.AMS **13**, 595–609 (2000)

[EV1] Etingof, P., Varchenko, A.: Geometry and classification of solutions of the classical dynamical Yang-Baxter equation. Commun. Math. Phys. **192**, 77–120 (1998)

[EV2] Etingof, P., Varchenko, A.: Solutions of the quantum dynamical Yang-Baxter equation and dynamical quantum groups. Commun. Math. Phys. **196**, 591–640 (1998)

[EV3] Etingof, P., Varchenko, A.: Exchange dynamical quantum groups. Commun. Math. Phys. **205**, 19–52 (1999)

[F] Felder, G.: Conformal field theories and integrable models associated to elliptic curves. In: Proc. ICM Zurich, Basel: Birkhäuser, 1994, pp. 1247–1255

[Fh] Fehér, L.: Dynamical $r$-matrices and Poisson-Lie symmetries of the chiral WZNW model. Talkgiven at the workshop on Integrable Theories, Solitens and Duality, IFT Sao Paulo, July 2002, http://arxiv.org/abs/hep-th/0212006, 2002

[FhMrsh] Fehér, L., Marshall, I.: On a Poisson-Lie analogue of the classical dynamical Yang-Baxter equation for self dual Lie algebras. Lett. Math. Phys. **62**, 51–62 (2002)

[Fad] Faddeev, L.: On the exchange matrix of the WZNW model. Commun. Math. Phys. **132**, 131–138 (1990)

[FRT] Faddeev, L., Reshetikhin, N., Takhtajan, L.: Quantization of Lie groups and Lie algebras. Leningrad Math. J. **1**, 193–226 (1990)

[GLS] Gurevich, D., Leclercq, R., Saponov, P.: *q-Index on braided spheres*. http://arxic.org/abs/math.QA/0207268, 2002

[GN] Gervais, J.-L., Neveu, A.: Novel triangle relation and absence of tachyons in Liouville string theory. Nucl. Phys. B **238**, 125–141 (1984)

[J] Jantzen, J.-C.: Kontravariante Formen auf induzierten Darstellungen halbeinfacher aa Lie-Algebren. Math. Ann. **226**, 53–65 (1977)

[Kas] Kassel, Ch.: *Quantum groups*. NY Springer, 1995.

[Kar] Karolinskii, E.: *A classification of Poisson homogeneous spaces of complex reductive Poisson-Lie groups*. In: Poisson Geometry, J. Grabouski, P. Urbauski (eds.), Banach Center Publ. **51**, Warsaw: Banach Center, 2000, pp. 103–108

[KMST] Karolinsky, E., Muzykin, K., Stolin, A., Tarasov, V.: *Dynamical Yang-Baxter equations, quasi-Poisson homogeneous spaces, and quantization*. http://arxiv.org/abs/math.QA/0309203, 2003
[KSkl] Kulish, P. P., Sklyanin, E. K.: Algebraic structure related to the reflection equation. J. Phys. A **25**, 5963–5976 (1992)
[KS] Kulish, P.P., Sasaki, R.: Covariance properties of reflection equation algebras. Prog. Theor. Phys. **89**, #3, 741–761 (1993)
[Lu1] Lu, J.-H.: Classical dynamical r-matrices and homogeneous Poisson structures on $G/H$ and $K/T$. Commun. Math. Phys. **212**, 337–370 (2000)
[Lu2] Lu, J.H.: Hopf algebroids and quantum gouppoids. Int. J. Math. **7**, 47–70 (1996)
[O] Ostrik, V.: Module categories, weak Hopf algebras, and modular invariants. Transform. Groups **8**, # 2, 177–206 (2003)
[RS] Reshetikhin, N., Semenov-Tian-Shansky, M.: Quantum $R$-matrices and factorization problem. J. Geom. Phys. **5**, 533–550 (1988)
[S] Schiffmann, O.: On classification of dynamical r-matrices. Math. Res. Lett. **5**, 13–30 (1998)
[Sem] Semenov-Tian-Shansky, M.: Poisson-Lie Groups, Quantum Duality Principle, and the Quantum Double. Contemp. Math. **175**, 219–248 (1994)
[Xu1] Xu, P.: Triangular dynamical r-matrices and quantization. Adv. Math. **166**, 1–49 (2002)
[Xu2] Xu, P.: Quantum Dynamical Yang-Baxter Equation Over a Nonabelian basis. Commun. Math. Phys. **226**, 475–495 (2002)

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