Commun. Math. Phys. 254, 695–717 (2005)
Digital Object Identifier (DOI) 10.1007/s00220-004-1246-9

Communications in
**Mathematical
Physics**

(Non)regularity of Projections of Measures Invariant Under Geodesic Flow

Esa Järvenpää, Maarit Järvenpää*, Mika Leikas*

Department of Mathematics and Statistics, University of Jyväskylä, P. O. Box 35, 40014 Finland.
E-mail: esaj@maths.jyu.fi; amj@maths.jyu.fi; mileikas@maths.jyu.fi

Received: 20 December 2003 / Accepted: 25 June 2004
Published online: 11 January 2005 –

Abstract: We show that, unlike in the 2-dimensional case [LL], the Hausdorff dimension of a measure invariant under the geodesic flow is not necessarily preserved under the projection from the unit tangent bundle onto the base manifold if the base manifold is at least 3-dimensional. In the 2-dimensional case we reprove the preservation theorem due to Ledrappier and Lindenstrauss [LL] using the general projection formalism of Peres and Schlag [PS]. The novelty of our proof is that it illustrates the reason behind the failure of the preservation in the higher dimensional case. Finally, we show that the projected measure has fractional derivatives of order γ for all $\gamma < (\alpha - 2)/2$ provided that the invariant measure has finite α-energy for some $\alpha > 2$ and the base manifold has dimension 2.

1. Introduction

Several indications have been brought for and against the importance and relevance of fractality for different observed phenomena. In this context, there are two important aspects related to physical experiments. First of all, the number of degrees of freedom in realistic systems is usually huge, that is, the phase space is high dimensional. On the other hand, the number of measurements which can be reasonably taken in one experiment is relatively small. As a result, one obtains sharp information only on a few variables whilst the remaining ones must be treated in some averaging or effective manner. This may be interpreted by saying that a measurement is a projection which leads to the need to understand the mathematical theory of projections. Indeed, fractal features of projections have recently been the subject of intensive study. These include, for example, projections of SRB-measures of coupled map lattices [BKL, JJ] and those of measures invariant under the geodesic flow [LL].

In the theory of coupled map lattices projections play a crucial rôle in the very definition of SRB-measures (see [BS, BK1, BK2]). It has turned out that the projectional

* MJ and ML acknowledge the support of the Academy of Finland, project #48557.

properties of dimensions imply that the natural definition of the SRB-measure given by Bunimovich and Sinai [BS] and Bricmont and Kupiainen [BK1, BK2] has to be modified in order to obtain a physically acceptable concept (see [JJ, J]).

Dimensional properties of projections of sets and measures have been investigated for decades. The study of the behaviour of Hausdorff dimension under projection-type mappings dates back to the 1950's when Marstrand [Mar] proved a well-known theorem according to which the Hausdorff dimension of a planar set is preserved under typical orthogonal projections. In [K] Kaufman verified the same result using potential theoretical methods, and in [Mat1] Mattila generalized it to higher dimensions. For measures the analogous principle, discovered by Kaufman [K], Mattila [Mat2], Hu and Taylor [HT], and Falconer and Mattila [FM], can be stated in the following form: Let m and n be integers such that $0 < m < n$ and let μ_V be the image of a compactly supported Radon measure μ on $\mathbb{R}^n$ under the orthogonal projection onto an m-plane V. Then for almost all m-planes V we have

$$\dim_{\mathrm{H}} \mu_V = \dim_{\mathrm{H}} \mu \text{ provided that } \dim_{\mathrm{H}} \mu \leq m. \tag{1.1}$$

On the other hand, for almost all m-planes V,

$$\mu_V \ll \mathcal{L}^m \text{ provided that } \dim_{\mathrm{H}} \mu > m. \tag{1.2}$$

(Above $\dim_{\mathrm{H}}$ is the Hausdorff dimension, $\mathcal{L}^m$ is the m-dimensional Lebesgue measure, and the symbol $\ll$ denotes the absolute continuity.) In the case that μ has finite m-energy a substantially stronger form of (1.2) holds: we have for all typical m-planes that

$$\mu_V \ll \mathcal{L}^m \text{ with Radon–Nikodym derivative in } L^2. \tag{1.3}$$

Analogies of these results have been investigated for typical smooth mappings in the sense of prevalence and for infinite dimensional spaces in [SY, HK1, and HK2]. In [PS] Peres and Schlag extended (1.1), (1.2), and (1.3) to Sobolev dimensions of measures on compact metric spaces and parametrized families of transversal mappings in an elegant way. Their formalism has turned out to be a powerful tool when considering the uniqueness of SRB-measures of coupled map lattices [JJ]. For the purposes of the present paper, a significant difference between the earlier results and those of [PS] is that Peres and Schlag generalized (1.3) in terms of fractional derivatives by showing that if the original measure has finite $(m+\varepsilon)$-energy, then densities of typical projections onto m-dimensional spaces have fractional derivatives of order $\varepsilon/2$ in L^2. For more detailed information about a variety of related contributions, see [Mat4] and [PS].

In this paper we address the question of studying measures on Riemannian manifolds which are invariant under the geodesic flow. Although they are measures on the unit tangent bundle of the manifold, that is, on (a subset of) the phase space of the system, from the physical point of view it is important to try to describe their properties on the configuration space. After all, in many situations one is interested only in the positions of the particles and not their velocities. This leads to the study of the natural projection from the unit tangent bundle onto the base manifold. (For a discussion of connections to the Besicovitch-Kakeya problem, see [LL].) Even though the above mentioned results (1.1), (1.2), and (1.3) are genuinely "almost all"-results, meaning that they do not provide information about any specified projection, similar methods work for the natural projection from the unit tangent bundle onto the Riemannian surface. This interesting feature was discovered quite recently by Ledrappier and Lindenstrauss in [LL].

Theorem 1.1 (Ledrappier, Lindenstrauss). *Let M be a compact Riemannian surface, let μ be a Radon probability measure on the unit tangent bundle SM, and let $\Pi : SM \to M$ be the natural projection. Assuming that μ is invariant under the geodesic flow, the following properties hold for the image $\Pi_*\mu$ of μ under Π:*

1. *If* $\dim_H \mu \leq 2$, *then* $\dim_H \Pi_*\mu = \dim_H \mu$.
2. *If* $\dim_H \mu > 2$, *then* $\Pi_*\mu \ll \mathcal{L}^2$.

Analogously to (1.3), Ledrappier and Lindenstrauss proved that if μ has finite α-energy for $\alpha > 2$, then the Radon–Nikodym derivative is a L^2-function. They also addressed the question of whether this could be further generalized in terms of fractional derivatives. In addition to giving a positive answer to this question by employing the techniques from [PS], we consider another issue brought up in [LL] which is the validity of Theorem 1.1 for higher dimensional base manifolds. Quite surprisingly, it appears that the Hausdorff dimension is not necessarily preserved. Recalling the case of (1.1), (1.2), and (1.3), one might first think that the generalization from dimension 2 to higher dimensions is a question of finding correct methods. However, in Sect. 4 we give a new proof for Theorem 1.1 which explains why the preservation fails in higher dimensions.

This paper is organized as follows: In Sect. 2 we discuss the general projection formalism of Peres and Schlag [PS] which plays an important rôle in this work, whereas in Sect. 3 we recall the basic assumptions from [LL] and introduce our setting. The main part of Sect. 4 is devoted to proving that the parametrized family of mappings we are working with is transversal (Proposition 4.1). Then we apply the machinery of [PS] and a result from [JJL] to reprove Theorem 1.1, and explain why this does not work for higher dimensional base manifolds (Remark 4.6). The question concerning the fractional derivatives of the density of the projected measure will be dealt with in Sect. 5. We prove that if the α-energy of μ is finite for some $\alpha > 2$, then $\Pi_*\mu$ has fractional derivatives of order γ in L^2 for all $\gamma < (\alpha - 2)/2$ (Theorem 5.1). Finally, in the last section we give examples of higher dimensional manifolds and invariant measures on the unit tangent bundles whose Hausdorff dimensions decrease when projected onto the base manifolds. Remark 4.6 gives a base for constructing such examples.

2. General Projection Formalism of Peres and Schlag

In this section we recall the notation and results we need from [PS]. Given $\gamma \geq 0$, let $\|\nu\|_{2,\gamma}$ be the Sobolev norm of a finite Borel measure ν on $\mathbb{R}^n$, that is,

$$\|\nu\|_{2,\gamma} = \Big(\int |\hat{\nu}(\xi)|^2 |\xi|^{2\gamma} \, d\mathcal{L}^n(\xi) \Big)^{1/2},$$

where

$$\hat{\nu}(\xi) = \int e^{-i\xi\cdot x} d\nu(x)$$

is the Fourier transform of ν. The Sobolev dimension of ν is

$$\dim_S \nu = \sup\Big\{\alpha \in \mathbb{R} \mid \int |\hat{\nu}(\xi)|^2 (1 + |\xi|)^{\alpha - n} \, d\mathcal{L}^d(\xi) < \infty\Big\}.$$

Given $\alpha \geq 0$, the α-energy of a finite Borel measure ν on a compact metric space (Y, d) is denoted by $I_\alpha(\nu)$, that is,

$$I_\alpha(\nu) = \int_Y \int_Y d(x, y)^{-\alpha} d\nu(x) d\nu(y).$$

For the rest of this section, we restrict our consideration to the one dimensional parameter space.

Basic assumptions. Let (Y, d) be a compact metric space, let $J \subset \mathbb{R}$ be an open interval, and let $P : J \times Y \to \mathbb{R}$ be a continuous function. Assume that for any $l = 0, 1, \dots$ there is a constant $\widetilde{C}_l \geq 1$ such that

$$|\partial_t^l P(t, y)| \leq \widetilde{C}_l \tag{2.1}$$

for all $t \in J$ and $y \in Y$. Here ∂_t^l is the l^{th} partial derivative with respect to t.

For all $t \in J$ and $x, y \in Y$ with $x \neq y$, define

$$T_t(x, y) = \frac{P(t, x) - P(t, y)}{d(x, y)}. \tag{2.2}$$

We assume that the following form of transversality holds: there is a constant C_T such that for all $t \in J$ and for all $x, y \in Y$ with $x \neq y$ the condition $|T_t(x, y)| \leq C_T$ implies that

$$|\partial_t T_t(x, y)| \geq C_T. \tag{2.3}$$

In addition, the function T_t is assumed to be regular in the following sense: for all $l = 0, 1, \dots$ there exists a constant C_l such that

$$|\partial_t^l T_t(x, y)| \leq C_l \tag{2.4}$$

for all $t \in J$ and $x, y \in Y$ with $x \neq y$.

In the following theorem from [PS], which serves as a significant tool in Proposition 4.3, we use the notation $P_t(\cdot) = P(t, \cdot)$. Moreover, we denote by $f_*\mu$ the image of a measure μ on X under a mapping $f : X \to Z$ defined as $f_*\mu(A) = \mu(f^{-1}(A))$ for all $A \subset Z$.

Theorem 2.1. *Suppose that the assumptions* (2.1)*,* (2.3)*, and* (2.4) *are satisfied. Let $\alpha > 0$ and let ν be a finite Borel measure on Y such that $I_\alpha(\nu) < \infty$. Then there is a constant C_γ such that*

$$\int_J \|(P_t)_*\nu\|_{2,\gamma}^2 d\mathcal{L}^1(t) \leq C_\gamma I_\alpha(\nu) \tag{2.5}$$

provided that $0 < 1 + 2\gamma \leq \alpha$. Moreover, for any $\sigma \in (0, \min\{\alpha, 1\}]$ we have

$$\dim_{\mathrm{H}}\{t \in J \mid \dim_{\mathrm{S}}(P_t)_*\nu \leq \sigma\} \leq 1 + \sigma - \alpha. \tag{2.6}$$

Proof. See [PS, Theorem 2.8]. □

We complete this section by stating a technical lemma which plays an important rôle in relating our setting to that of [PS].

Lemma 2.2. *For all* $t \in (0,1)$*, let* ν_t *be a compactly supported Radon measure on* $\mathbb{R}$*. Suppose that* μ *is a Radon measure on* $\mathbb{R} \times (0,1)$ *such that for all Borel functions* $g : \mathbb{R} \times (0,1) \to \mathbb{R}$,

$$\int g(x,t)\, d\mu(x,t) = \iint g(x,t)\, d\nu_t(x)\, d\mathcal{L}^1(t). \tag{2.7}$$

Assume that there is $\alpha > 0$ *such that* $\dim_{\mathrm{H}} \nu_t \geq \alpha$ *for* $\mathcal{L}^1$*-almost all* $t \in (0,1)$*. Then* $\dim_{\mathrm{H}} \mu \geq 1 + \alpha$.

Proof. The proof of [JJL, Lemma 3.4] goes through in our setting. One simply needs to replace in the proof of [JJL, Lemma 3.4] the assumption according to which $I_\alpha(\nu_t) < \infty$ for all t by the weaker one of Lemma 2.2. □

3. Notation

In this section, we define a transversal mapping appropriate to the setting of Sect. 2. Our notation is similar to that in [LL]. Assume that M is a smooth compact 2-dimensional Riemannian manifold. Denoting by SM the unit tangent bundle, let μ be a Radon probability measure on SM which is invariant under the geodesic flow, and let $\Pi : SM \to M$ be the natural projection.

Since, in general, the measure μ is too complicated to handle, we have to divide it into small pieces. The fact that μ is invariant under the geodesic flow implies that locally a suitable restriction of μ is roughly of the form $\nu \times \mathcal{L}^1$, where ν is a measure on a two dimensional square. We will proceed by showing that the projection of this restriction of μ is in a certain sense of the form $\nu_t \times \mathcal{L}^1$ (see Lemma 3.2) where ν_t is a projection of ν onto one dimensional space. In this way one obtains a family of projections parametrized by t and this family will turn out to be transversal (see Proposition 4.1). We continue by formalizing this idea.

Taking $p_1, p_2 \in M$ sufficiently close to each other, we denote by γ_{p_1,p_2} the unique shortest geodesic, parametrized by the Riemannian arc length, which connects p_1 and p_2, that is,

$$\gamma_{p_1,p_2}(0) = p_1 \text{ and } \gamma_{p_1,p_2}(d_M(p_1,p_2)) = p_2. \tag{3.1}$$

Here d_M is the distance induced by the Riemannian metric.

Basic assumptions. Let $I = [0,1]$. We choose an open set $U \subset M$ and a chart $\Phi : U \to \mathbb{R}^2$ with the following properties:

(1) $I^2 \subset \Phi(U)$.
(2) Defining

$$\mathcal{C}_1 = \Phi^{-1}(I \times \{0\}) \text{ and } \mathcal{C}_2 = \Phi^{-1}(I \times \{1\})$$

and picking any $c_1 \in \mathcal{C}_1$ and $c_2 \in \mathcal{C}_2$, there exists a unique geodesic γ_{c_1,c_2} connecting c_1 and c_2 such that its image $\Phi(\gamma_{c_1,c_2}(t)) = (x_1(t), x_2(t))$ satisfies

$$|x_1'(t)| \leq C|x_2'(t)|$$

for some $C > 0$ for all $t \in [0, d_M(c_1,c_2)]$. Thus the tangents of the (images of) geodesics are uniformly bounded away from being horizontal. Further, U is assumed to be so small that geodesics are close to straight lines. (We use scaled normal coordinates around a fixed point $m \in U$ with $\Phi(m) = (1/2, 1/2)$.)

(3) Denoting by $\Gamma_1 : [0, t_1] \to M$ and $\Gamma_2 : [0, t_2] \to M$ the unique geodesics connecting the left-hand side end points of $\mathcal{C}_1$ and $\mathcal{C}_2$, and their right-hand side end points, respectively, we assume that $\Phi(\Gamma_1) \subset \Phi(U)$ and $\Phi(\Gamma_2) \subset \Phi(U)$.

As in [LL], we define a smooth map $\Psi : I^2 \times \mathbb{R} \to SM$ as follows:

$$\Psi(y_1, y_2, t) = (\gamma_{p_1,p_2}(t), \gamma'_{p_1,p_2}(t)), \tag{3.2}$$

where $p_1 = \Phi^{-1}(y_1, 0)$ and $p_2 = \Phi^{-1}(y_2, 1)$ (see Fig. 1). Set

$$D = \{(y_1, y_2, t) \mid (y_1, y_2) \in I^2, 0 \le t \le d_M(p_1, p_2)\}.$$

Then $\Psi : D \to \Psi(D)$ is a diffeomorphism by the uniqueness of geodesics (see (2)).

Next we analyze how the preimages of the projection Π behave on $I^2 \subset \Phi(U)$ keeping in mind that we will project the restriction of μ. Any $(x_1, x_2) \in I^2$ is a projection of some $v \in SM$ if there is (an image under Φ of) a geodesic starting from $a \in I \times \{0\}$, ending at $b \in I \times \{1\}$, and going through (x_1, x_2). Note that by the uniqueness of geodesics, for each $a = (a_1, 0)$ the corresponding $b = (b_1, 1)$ is unique (if it exists). Thus a pair of points (a_1, b_1) defines uniquely a point $v \in SM$ which is projected onto (x_1, x_2). Since the pair (a_1, b_1) contains also the information about the distance $d_M(a, x)$, we may suppress the "time" coordinate and define a function $a_1 \mapsto b_1$ such that all points on the graph of this function are mapped onto (x_1, x_2) under the projection Π (see Figs. 2 and 3). Letting x_1 vary and keeping x_2 fixed, we obtain a family of graphs filling I^2 (see Fig. 4). The fact that all points in the same graph are mapped onto the same point under Π implies that these graphs define a projection $P_{x_2} : I^2 \to \mathbb{R}$ associated with Π. Note that x_2 will play the rôle of the parameter and x_1 determines the domain of the associated projection.

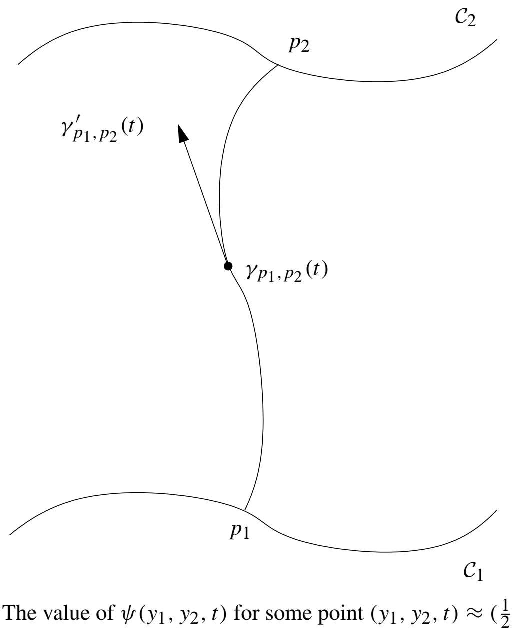

Fig. 1. The value of $\psi(y_1, y_2, t)$ for some point $(y_1, y_2, t) \approx (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

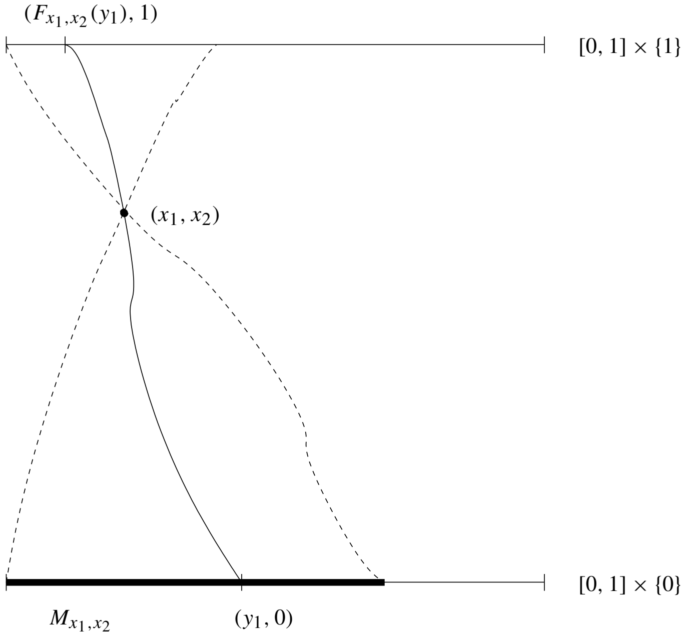

Fig. 2. Definitions of the domain M_{x_1,x_2} and the function F_{x_1,x_2}

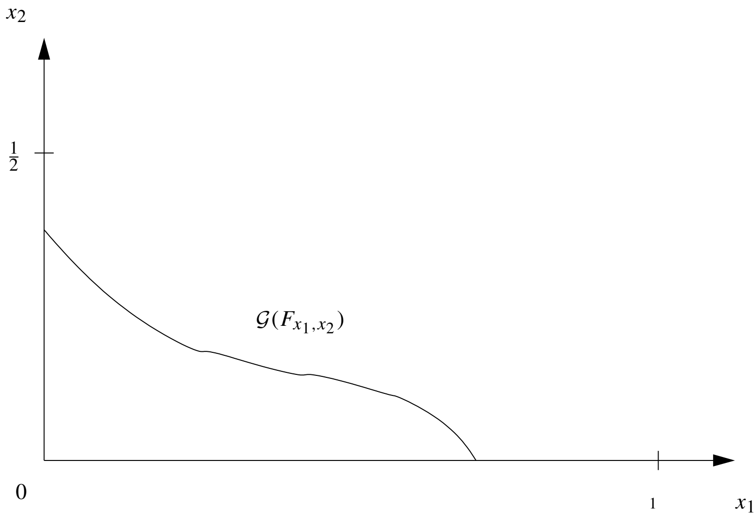

Fig. 3. The graph of F_{x_1,x_2} in the situation of Fig. 2

For the purpose of making the above idea rigorous, denote by E the subset of $\mathbb{R}^2$ restricted by the curves $I \times \{0\}$, $I \times \{1\}$, $\Phi(\Gamma_1)$, and $\Phi(\Gamma_2)$. Given any $(x_1, x_2) \in E$, let

$$M_{x_1,x_2} = \{y_1 \in I \mid \text{there is } y_2 \in I \text{ such that the geodesic } \gamma_{\Phi^{-1}(y_1,0),\Phi^{-1}(y_2,1)} \text{ goes through } \Phi^{-1}(x_1, x_2)\}. \quad (3.3)$$

Note that, by (2), for all $y_1 \in M_{x_1,x_2}$ the point $y_2 \in I$ in (3.3) is unique (provided $x_2 > 0$). Moreover, $M_{x_1,x_2} \neq \emptyset$ for all $(x_1, x_2) \in E$. For all $(x_1, x_2) \in E$, we define a function $F_{x_1,x_2} : M_{x_1,x_2} \to I$ by $F_{x_1,x_2}(y_1) = y_2$ where y_2 is as in (3.3) (see Fig. 2 and 3). (If $x_2 = 0$, we consider the vertical line segment I above y_1 recalling that the important object is the graph of F_{x_1,x_2}.)

Lemma 3.1. *The mapping F_{x_1,x_2} has the following properties:*

1. *If (x_1, x_2), $(\tilde{x}_1, x_2) \in E$ such that $\tilde{x}_1 > x_1$, we have $F_{\tilde{x}_1,x_2}(y_1) > F_{x_1,x_2}(y_1)$ for all $y_1 \in M_{x_1,x_2} \cap M_{\tilde{x}_1,x_2}$.*
2. *Given (x_1, x_2), $(\tilde{x}_1, x_2) \in E$ with $\tilde{x}_1 \to x_1$, we have $F_{\tilde{x}_1,x_2}(y_1) \to F_{x_1,x_2}(y_1)$ for all $y_1 \in M_{x_1,x_2} \cap M_{\tilde{x}_1,x_2}$.*
3. *For all y_1, $y_2 \in I$ and $x_2 \in I$ there exists x_1 such that $(x_1, x_2) \in E$ and $F_{x_1,x_2}(y_1) = y_2$.*

Proof. The claims follow directly from the definitions by (2). □

Now we are ready to define the family of projections associated with Π. All the points belonging to the same graph $\mathcal{G}(F_{x_1,x_2})$ should be mapped onto the same point. To choose this point, we fix a line L_{x_2} which is roughly perpendicular to the graphs and define the image of the points in $\mathcal{G}(F_{x_1,x_2})$ to be the intersection point of this graph and the line L_{x_2}. Since near the corners of I^2 there is no intersection point (see Fig. 4) we have to replace I^2 by a smaller square $\widetilde{I}^2$ with the same centre as I^2.

To be more precise, given $t \in I$, let L_t be the line in $\mathbb{R}^2$ which goes through $(1/2, 1/2)$ and is orthogonal to the line segment going through the points in $\partial(I^2) \cap \mathcal{G}(F_{1/2,t})$ (see Fig. 4). (Here the boundary of a set A is denoted by ∂A.) Note that our assumptions guarantee that $\{(1/2, t) \mid t \in I\} \subset E$, and furthermore, the set $\partial(I^2) \cap \mathcal{G}(F_{1/2,t})$ contains

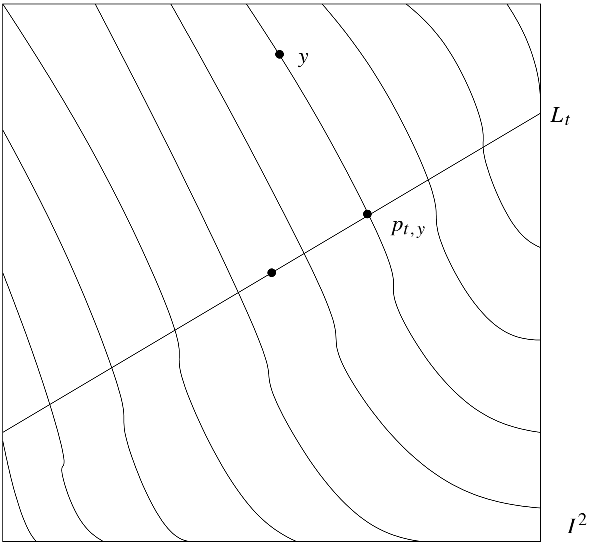

Fig. 4. The foliation of I^2 and the value of P_t for some $t \approx \frac{1}{3}$

exactly two points. We may choose $\widetilde{I^2} \subset I^2$ such that for all $t \in I$ and $(y_1, y_2) \in \widetilde{I^2}$ the intersection $L_t \cap \mathcal{G}(F_{x,t})$ is a singleton for $x \in I$ with $F_{x,t}(y_1) = y_2$ (see Lemma 3.1 (3)). This enables us to define a function $P : I \times \widetilde{I^2} \to \mathbb{R}$ by

$$P(t, y) = p_{t,y}, \tag{3.4}$$

where $y = (y_1, y_2) \in \widetilde{I^2}$, $p_{t,y}$ is the unique point in $L_t \cap \mathcal{G}(F_{x,t})$, and the point x is determined by $F_{x,t}(y_1) = y_2$. Here L_t is identified with $\mathbb{R}$ such that the origin is at $(1/2, 1/2)$. Later we will use the abbreviation $P_t(\cdot)$ for the map $P(t, \cdot)$.

Invariant measure under geodesic flow. Similarly as in [LL], we restrict our consideration to the normalized restriction measure $\widetilde{\mu} = \mu(\widetilde{U})^{-1}\mu|_{\widetilde{U}}$, where $\widetilde{U} = \Psi(\widetilde{D})$ and

$$\widetilde{D} = \{(y_1, y_2, t) \mid (y_1, y_2) \in \widetilde{I^2}, 0 \le t \le d_M(\Phi^{-1}(y_1, 0), \Phi^{-1}(y_2, 1))\}.$$

(Here $\mu|_{\widetilde{U}}(A) = \mu(\widetilde{U} \cap A)$ for all $A \subset SM$.) Since μ is invariant under the geodesic flow, there is a measure ν on $\widetilde{I^2}$ such that $\Psi_*(\nu \times \mathcal{L}^1) = \widetilde{\mu}$. We call a measure locally invariant if it is of this form for some ν.

Next we will represent the measure $\Pi_*\widetilde{\mu}$ in a form which allows us to apply the general projection formalism of Sect. 2. Observe that the preimage of a point $(x_1, x_2) \in E$ under $\Phi \circ \Pi \circ \Psi$ is a curve on $\widetilde{D}$ whose projection onto $\widetilde{I^2}$ is $\mathcal{G}(F_{x_1,x_2})$. Since the distance from $(y_1, 0)$ to (x_1, x_2) depends on y_1, the "time" coordinate of this preimage on $\widetilde{D}$ is not constant. Hence we have to first rescale "time" and then use the map P. For this purpose, let $V = \Phi \circ \Pi(\widetilde{U})$. We define, for given $t \in I$ and $\omega \in (\Phi \circ \Pi \circ \Psi)^{-1}\{(x, t) \mid (x, t) \in V\}$,

$$B_1(\omega_1, \omega_2, \omega_3) = (\omega_1, \omega_2, t). \tag{3.5}$$

Now $B_1 : \widetilde{D} \to \widetilde{I^2} \times I$ is a diffeomorphism since geodesics are not horizontal. Setting $\widetilde{P}(\omega_1, \omega_2, t) = (P_t(\omega_1, \omega_2), t)$ for all $(\omega_1, \omega_2, t) \in \widetilde{I^2} \times I$, we find for all $(x, t) \in V$ a unique point $\tilde{x} \in \mathbb{R}$ such that $\widetilde{P} \circ B_1((\Phi \circ \Pi \circ \Psi)^{-1}\{(x, t)\}) = (\tilde{x}, t)$. Defining

$$B_2(x, t) = (\tilde{x}, t) \tag{3.6}$$

and using the fact that

$$B_1((\Phi \circ \Pi \circ \Psi)^{-1}\{(x, t)\}) = \{(y_1, y_2, t) \mid y_2 = F_{x,t}(y_1)\},$$

we get a diffeomorphism $B_2 : V \to B_2(V)$.

Lemma 3.2. *The following properties hold:*

(1) $(\Phi \circ \Pi)_*\widetilde{\mu} = (B_2^{-1} \circ \widetilde{P} \circ B_1)_*(\nu \times \mathcal{L}^1)$.
(2) For all non-negative Borel functions $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\int f(x, t) d(\widetilde{P}_*(\nu \times \mathcal{L}^1))(x, t) = \iint f(x, t) d((P_t)_*\nu)(x)\, d\mathcal{L}^1(t).$$

(3) For all non-negative Borel functions $g : \mathbb{R}^3 \to \mathbb{R}$*,*

$$\int g\, d((B_1)_*(\nu \times \mathcal{L}^1)) = \int g |\det DB_1^{-1}|\, d(\nu \times \mathcal{L}^1),$$

where $\det DB_1^{-1}$ *is the determinant of the derivative of* B_1^{-1}*. Furthermore, there is a constant* $C > 0$ *such that*

$$C^{-1} \leq |\det DB_1^{-1}| \leq C.$$

(4) There exists a constant $C > 0$ *such that for all Borel sets* $A \subset \mathbb{R}^2$*,*

$$\frac{1}{C}\widetilde{P}_*(\nu \times \mathcal{L}^1)(A) \leq (\widetilde{P} \circ B_1)_*(\nu \times \mathcal{L}^1)(A) \leq C\widetilde{P}_*(\nu \times \mathcal{L}^1)(A).$$

(5) There is a constant $C > 0$ *such that*

$$C^{-1} \leq |\det DB_2^{-1}| \leq C.$$

Proof. Clearly, (1) follows from the definitions, and (2) is a straightforward consequence of Fubini's theorem. Noting that B_1 can be written in the form $B_1(x_1, x_2, t) = (x_1, x_2, b(x_1, x_2, t))$, Fubini's theorem gives the equality in (3). Our basic assumption (2) guarantees the existence of a constant C such that $C^{-1} \leq |\det D(B_1^{-1})| \leq C$ concluding the proof of (3). Finally, applying (3) gives (4), and (5) follows similarly as (3). □

4. Transversality and Preservation of Hausdorff Dimension in Two Dimensional Manifolds

In this section we discuss connections between [LL] and [PS]. In particular, we give a new proof of Theorem 1.1 which explains why the corresponding result fails if the dimension of the base manifold is more than 2 (see Remark 4.6). The machinery developed in this section leads us to prove in Sect. 5 that the Radon–Nikodym derivative $\frac{d\Pi_*\mu}{d\mathcal{L}^2}$ has fractional derivatives in the Sobolev sense. An essential step is to prove that the function T_t, defined as in (2.2) in terms of the function P given in (3.4), has the crucial property of being transversal.

Proposition 4.1. *Let P be as in* (3.4)*. Then* (2.1) *is satisfied. Furthermore, defining for all* $t \in I$ *and* $x \neq y \in \widetilde{I^2}$*,*

$$T_t(x, y) = \frac{P(t, x) - P(t, y)}{|x - y|},$$

properties (2.3) *and* (2.4) *hold.*

Proof. Observing that (2.1) and (2.4) follow directly from the definitions, it suffices to prove that the transversality condition (2.3) is satisfied.

The idea of the proof of transversality is most easily explained if we assume that the manifold is a flat torus. Then the geodesics are straight lines and the graphs $\mathcal{G}(F_{x,t})$ are parallel straight lines with slopes given by the equation $\tan\alpha = (1 - t)/t$ (see Fig. 5 and (4.1) where now $a = b$). Moreover, $P(t, \cdot)$ is an orthogonal projection and $T_t(x, y)$ reduces to $P(t, v)$, where $v = (x - y)/|x - y|$. By (4.2), the change of the parameter

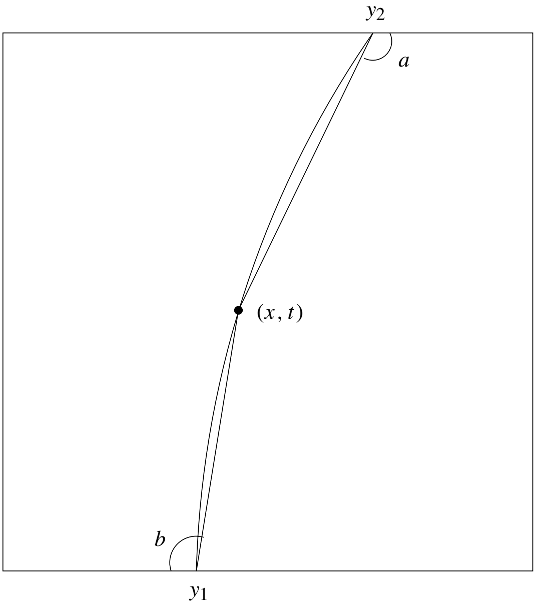

Fig. 5. The notation for determining the slope of the graph $F_{x,t}$ at a point (y_1, y_2)

t means a comparable change of the slope of L_t. Therefore the proof of transversality reduces to the case where L_t is spanned by $(\cos t, \sin t)$, $v = (0, 1)$, and $t = 0$. Then $\partial_t P(t, v)|_{t=0} = \partial_t \sin t|_{t=0} = 1$. To check that this simple idea works also in the general case involves several careful estimates which we make below.

Given (x, t), let α be the slope of the graph of $F_{x,t}$ at a point (y_1, y_2). Using the notation introduced in Fig. 5, one may deduce the formula

$$\tan \alpha = \frac{(1-t)\sin^2 b(x,t)}{t \sin^2 a(x,t)} \tag{4.1}$$

from elementary geometrical arguments. Note that the basic assumption (2) in Sect. 3 guarantees that both the angles a and b are bounded away from 0 and π and are close to each other. Combining this with Eq. (4.1), in turn, implies the existence of a positive constant C_1 such that

$$\left|\frac{d\alpha}{dt}\right| \geq C_1 \tag{4.2}$$

for all t.

Letting $\varepsilon > 0$, consider $x \neq y$ such that

$$|P(t, x) - P(t, y)| \leq \varepsilon|x - y|. \tag{4.3}$$

We will show that, choosing ε small enough, we have for all small h,

$$|P(t+h, x) - P(t+h, y) - (P(t, x) - P(t, y))| \geq \varepsilon|x - y|h. \tag{4.4}$$

This clearly gives the transversality condition (2.3).

Note that our assumptions guarantee the existence of a constant C_F (independent of x, y, and t) such that

$$\max_{K\|L_t}\{|z_1 - z_2| \mid z_1 \in \mathcal{G}(F_{x,t}) \cap K,\ z_2 \in \mathcal{G}(F_{y,t}) \cap K\} \\ \leq C_F \min_{K\|L_t}\{|z_1 - z_2| \mid z_1 \in \mathcal{G}(F_{x,t}) \cap K,\ z_2 \in \mathcal{G}(F_{y,t}) \cap K\}, \tag{4.5}$$

where both the maximum and the minimum are taken over all lines K that are parallel to L_t (denoted by the symbol $K \parallel L_t$). Using the notation shown in Fig. 6, we have

$$|x - a| \leq \varepsilon C_F |x - y|, \\ |a - b| \geq C_2 |x - y|\, h, \\ \big||c - d| - |e - f|\big| \leq C_3 \varepsilon |x - y|\, h, \tag{4.6}$$

where both C_2 and C_3 are constants that do not depend on x, y, and t. In fact, the first inequality in (4.6) is a consequence of (4.5) and (4.3). Choosing $\varepsilon < 1/(2C_F)$, the second inequality follows from the first one and the fact that there is a constant C such that $|a - b| \geq C|a - y|\, h$ (see (4.2)). For the last one, observe first that, since the geodesics are close to lines in V and depend smoothly on the initial data, there is a constant C (independent of x, y, and t) such that

$$\big||c - d| - |e - f|\big| \leq C\big||w_1 - w_2| - |w_3 - w_4|\big|, \tag{4.7}$$

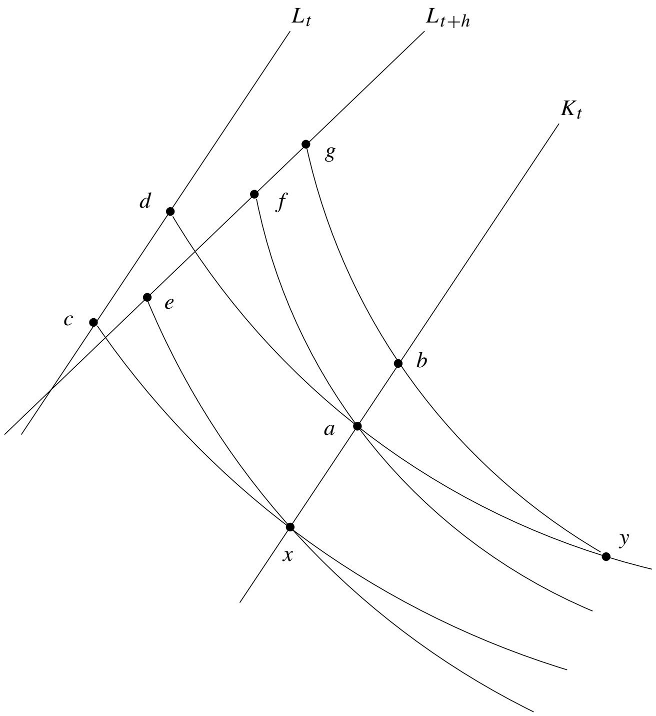

Fig. 6. Above the line K_t goes through x and is parallel line to L_t, $\{a\} = K_t \cap \mathcal{G}(F_{\cdot,t})$, $\{b\} = K_t \cap \mathcal{G}(F_{\cdot,t+h})$, $c = P(t, x)$, $d = P(t, y)$, $e = P(t + h, x)$, $f = P(t + h, a)$, and $g = P(t + h, y)$

where w_1, w_2, w_3, and w_4 are as in Fig. 7. Using the fact that the closer to each other the geodesics are, the more they look like parallel curves in V, we get

$$\big||w_1 - w_2| - |w_3 - w_4|\big| \leq \widetilde{C}|w_1 - w_2|\,h \leq \widehat{C}|x - a|\,h.$$

(Here $\widetilde{C}$ and $\widehat{C}$ are constants that are independent of x, y, and t.) This, in turn, combined with (4.7) and the first inequality in (4.6), completes the proof of the last inequality of (4.6).

Finally, after noting that for small h we have $|f - g| \geq (1/(2C_F))|a - b|$ by (4.5), we deduce from (4.6)

$$\big||c - d| - |e - g|\big| = |f - g| - \big||c - d| - |e - f|\big| \geq C_3\varepsilon|x - y|\,h$$

for $\varepsilon < \min\{1/(2C_F), C_2/(4C_F C_3)\}$. Hence (4.4) follows. □

As a corollary of Proposition 4.1, one obtains quite easily a new proof for Theorem 1.1. This is achieved by means of Proposition 4.3. Recall that the Hausdorff dimension of a finite Borel measure μ on a Riemannian manifold X is defined using lower local dimensions, $\underline{\dim}_{\mathrm{loc}}$, as follows:

$$\dim_{\mathrm{H}} \mu = \mu\text{-}\operatorname*{ess\,inf}_{x \in X} \underline{\dim}_{\mathrm{loc}}\, \mu(x),$$

where

$$\underline{\dim}_{\mathrm{loc}}\, \mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.$$

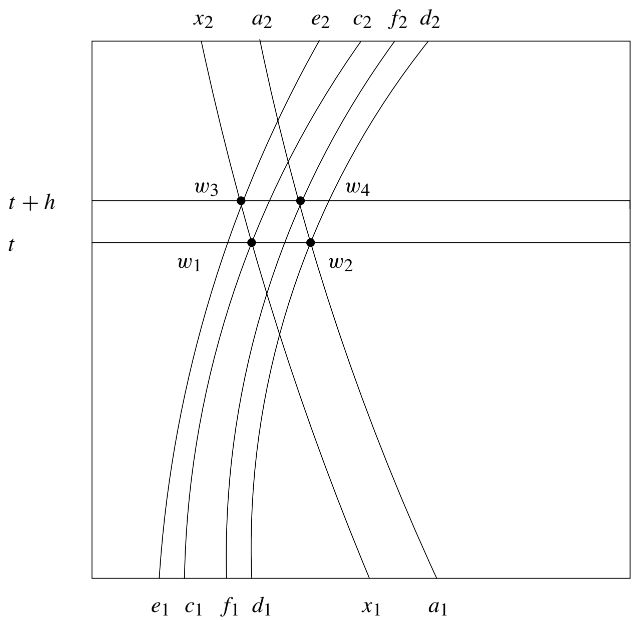

Fig. 7. The setting for the proof of the last inequality in (4.6). The notation corresponds to Fig. 6 in a natural way

Here $B(x, r)$ is the open ball with centre at x and radius $r > 0$. The following equality relates Hausdorff dimension of measures to that of sets:

$$\dim_H \mu = \inf\{\dim_H A \mid A \text{ is a Borel set with } \mu(A) > 0\}$$

(see [F, Proposition 10.2]).

Remark 4.2. It follows from [Mat4, Proposition 5.1] and [Mat3, Theorem 8.7] that $\dim_H \mu \geq \dim_S \mu$ provided that $\dim_S \mu < \dim X$.

Proposition 4.3. *With the notation introduced in Sect. 3, we have:*

1. *Assuming that* $\dim_H \nu \leq 1$, *we have* $\dim_H (P_t)_* \nu = \dim_H \nu$ *for* $\mathcal{L}^1$*-almost all* $t \in (0, 1)$.
2. *Assuming that* $\dim_H \nu > 1$, *we have* $(P_t)_* \nu \ll \mathcal{L}^1$ *for* $\mathcal{L}^1$*-almost all* $t \in (0, 1)$.

Proof. To verify (1), let $\beta < \dim_H \nu$. Defining $\nu_i = \nu|_{A_i}$ for all $i = 1, 2, \ldots$, where

$$A_i = \{x \in \mathbb{R}^2 \mid \nu(B(x, r)) \leq i r^\beta \text{ for all } r > 0\},$$

one easily checks that $I_\alpha(\nu_i) < \infty$ for all $\alpha < \beta$, and $\nu_i(B) \to \nu(B)$ for all $B \subset \mathbb{R}^2$. Given $\sigma < \alpha$, we get from inequality (2.6) in Theorem 2.1 using Remark 4.2 that for $\mathcal{L}^1$-almost all $t \in (0, 1)$,

$$\dim_H (P_t)_* \nu_i \geq \dim_S (P_t)_* \nu_i > \sigma \tag{4.8}$$

for all i. This, in turn, implies that $\dim_H (P_t)_* \nu \geq \sigma$ for $\mathcal{L}^1$-almost all $t \in (0, 1)$. Finally, taking a sequence $\sigma_j \to \dim_H \nu$, gives (1), since P_t does not increase dimension as a Lipschitz function.

For (2), we consider $1 < \beta < \dim_H \nu$ and proceed as above to find a sequence (ν_i) of measures with $I_\beta(\nu_i) < \infty$ such that $\nu_i(B) \to \nu(B)$ for all $B \subset \mathbb{R}^2$. Now inequality (2.5) in Theorem 2.1 implies that for $\mathcal{L}^1$-almost all $t \in (0, 1)$ one has $((P_t)_* \nu_i)^\wedge \in L^2$ for all i, and therefore $(P_t)_* \nu_i \ll \mathcal{L}^1$ for all i. This gives (2). □

We continue by explaining how Theorem 1.1 follows from Proposition 4.3. For this purpose we need two intermediate steps:

Corollary 4.4. *Using the same notation as in Sect. 3, we have:*

1. *If* $\dim_H \widetilde{\mu} \leq 2$, *then* $\dim_H \widetilde{P}_*(\nu \times \mathcal{L}^1) = \dim_H \widetilde{\mu}$.
2. *If* $\dim_H \widetilde{\mu} > 2$, *then* $\widetilde{P}_*(\nu \times \mathcal{L}^1) \ll \mathcal{L}^2$.

Proof. Note that $\dim_H \widetilde{\mu} = \dim_H \nu + 1$ (see [H] or [Mat3, Theorem 8.10]). To prove (1), Proposition 4.3 (1) gives $\dim_H (P_t)_* \nu = \dim_H \nu$ for $\mathcal{L}^1$-almost all $t \in \mathbb{R}$. From Lemma 2.2 and Lemma 3.2 (2), we deduce that $\dim_H \widetilde{P}_*(\nu \times \mathcal{L}^1) \geq \dim_H \nu + 1 = \dim_H \widetilde{\mu}$. The fact that $\widetilde{P}$ is a Lipschitz mapping yields (1).

For (2), let $A \subset \mathbb{R}^2$ be a Borel set with $\mathcal{L}^2(A) = 0$. Setting $A_t = \{x \in \mathbb{R} \mid (x, t) \in A\}$ for all $t \in \mathbb{R}$, and using Fubini's theorem and Proposition 4.3 (2), we get $(P_t)_* \nu(A_t) = 0$ for $\mathcal{L}^1$-almost all $t \in \mathbb{R}$. Combining this with Lemma 3.2 (2) concludes the proof. □

Corollary 4.5. *Using the notation given in Sect. 3, we have:*

1. *If* $\dim_H \widetilde{\mu} \leq 2$, *then* $\dim_H (\Phi \circ \Pi)_* \widetilde{\mu} = \dim_H \widetilde{\mu}$.
2. *If* $\dim_H \widetilde{\mu} > 2$, *then* $(\Phi \circ \Pi)_* \widetilde{\mu} \ll \mathcal{L}^2$.

Proof. Corollary 4.4, Lemma 3.2 (4), and the fact that B_2^{-1} is a bi-Lipschitz mapping (see Lemma 3.2 (5)) combine to give the equality $\dim_{\mathrm{H}}(B_2^{-1} \circ \widetilde{P} \circ B_1)_*(\nu \times \mathcal{L}^1) = \dim_{\mathrm{H}} \widetilde{\mu}$ provided that $\dim_{\mathrm{H}} \widetilde{\mu} \leq 2$, and furthermore, $(B_2^{-1} \circ \widetilde{P} \circ B_1)_*(\nu \times \mathcal{L}^1) \ll \mathcal{L}^2$ under the assumption $\dim_{\mathrm{H}} \widetilde{\mu} > 2$. This in turn gives the claim by Lemma 3.2 (1). □

Since Φ is bi-Lipschitz mapping, Theorem 1.1 follows immediately from Corollary 4.5 by representing the original measure μ as a finite sum of measures $\widetilde{\mu}_i$ having the same properties as the measure $\widetilde{\mu}$ above.

Remark 4.6. In Sect. 6 we construct examples which show that Theorem 1.1 fails for higher dimensional base manifolds. The reason for the failure, which may be deduced from the above methods, is as follows: The local invariance produces a parametrized family of projections onto $(n-1)$-dimensional planes in $2(n-1)$-dimensional space. The parameter is given by the time coordinate, and therefore the family is one dimensional. Since the dimension of the space of $(n-1)$-planes in $2(n-1)$ dimensional space is greater than 1, if $n \geq 3$, the transversality condition cannot hold.

5. Fractional Derivatives

In this section we answer the question concerning the fractional derivatives of the density of the projected measure $\Pi_*\mu$ addressed in [LL]. The main theorem of this section is as follows:

Theorem 5.1. *Let M be a compact smooth Riemannian surface and let $\Pi : SM \to M$ be the natural projection from the unit tangent bundle SM onto the base manifold M. Assume that μ is a Radon probability measure on SM such that μ is invariant under the geodesic flow and $I_\alpha(\mu) < \infty$ for some $\alpha > 2$. Then for all $\gamma < (\alpha - 2)/2$ the projected measure $\Pi_*\mu$ has fractional derivatives of order γ in L^2, that is, $\|\Pi_*\mu\|_{2,\gamma} < \infty$.*

Below the proof of Theorem 5.1 is divided into a sequence of lemmas. Observe that Theorem 2.1 combined with Proposition 4.1 implies the existence of fractional derivatives for almost all horizontal slices of $\Pi_*\mu$, which are, in fact, diffeomorphic images of the measures $(P_t)_*\nu$. However, since this approach does not give the desired result for the measure $\Pi_*\mu$, we modify the methods of [PS] in a more effective way.

The idea of the proof is roughly as follows: In order to estimate Sobolev norms, we will first use the Littlewood–Paley decomposition to separate different frequencies (see Lemma 5.5). When estimating the Sobolev norm of the projection of μ with an appropriate energy of ν, one is essentially forced to deduce that the measure of parameters t for which $|P_t(q) - P_t(q')|$ is small is less than some power of $|q - q'|$ (see (5.5)). This estimate will be divided into several steps (see Lemmas 5.3, 5.4, and 5.6), where we will use effectively two properties of ψ given by the Littlewood–Paley decomposition. First of all, ψ decays faster than any power guaranteeing the desired behaviour of the integral over domains where the argument of ψ is not too small. Secondly, after using the first property several times, we are reduced to a domain where the argument is small. Then we will use the fact that ψ has vanishing moments of all orders, and so the integral over this domain may be calculated over its complement. Finally, the fast decay of ψ will be applied again.

Using the same notation as in the previous sections, we begin with a small technical lemma.

Lemma 5.2. *Let $\alpha > 1$. Assume that $\mu = F_*(\nu \times \mathcal{L}^1|_K)$, where $K \subset \mathbb{R}$ is a compact set and F is a diffeomorphism such that $C^{-1} \le |\det DF| \le C$ for some $C > 0$. Then*

$$I_\alpha(\mu) < \infty \iff I_{\alpha-1}(\nu) < \infty.$$

Proof. The claim follows from straightforward calculations. □

The next lemma shows that for fixed $q \neq q' \in \widetilde{I}^2$ the mapping $a \mapsto T_a(q, q')$ is small only in neighbourhoods of finitely many zeroes.

Lemma 5.3. *For any $q \neq q' \in \widetilde{I}^2$ there exist $a_1, \ldots, a_N \in I$ such that*

$$\{a \in I \mid |T_a(q, q')| \le d\} \subset \bigcup_{i=1}^{N} B(a_i, C_T^{-1} d)$$

for all $d < C_T$. Moreover, the mapping $a \mapsto T_a(q, q')$ is a diffeomorphism on $B(a_i, C_1^{-1} C_T)$ for all $i = 1, \ldots, N$, and $N \le C_1/C_T + 2$. (Here C_T is as in (2.3) *and C_1 as in* (2.4).*)*

Proof. Let $a_1, \ldots, a_{N-2}$ be the zeroes of the function $a \mapsto T_a(q, q')$, and let $a_{N-1} = 0$ and $a_N = 1$. Then all the claims follow from (2.3) and (2.4). □

We continue by defining mappings $H_{q,q'}$ and by studying their basic properties which will be needed in the proof of Lemma 5.6.

Lemma 5.4. *Given $q \neq q' \in \widetilde{I}^2$, let $r = |q - q'|$. Define $H_{q,q'} : I^2 \to \mathbb{R}^2$ by*

$$H_{q,q'}(a, b) = (T_a(q, q') + r^{-1}(P_a(q') - P_b(q')), r^{-1}(a - b)).$$

Let $a_1, \ldots, a_N \in I$ be as in Lemma 5.3. For any $i = 1, \ldots, N$, set

$$O_i = \{(a, b) \in \big(B(a_i, C_1^{-1} C_T) \cap (0, 1)\big) \times (0, 1) \mid |T_a(q, q')| < C_T \text{ and } |a - b| < (2\widetilde{C}_2)^{-1} C_T r\},$$

where $\widetilde{C}_2$, C_T, and C_1 are as in (2.1), (2.3), *and* (2.4), *respectively. Then the restriction of $H_{q,q'}$ to the set O_i is a diffeomorphism onto $H_{q,q'}(O_i)$. Furthermore, there are constants c and $c(l)$ for all $l \in \mathbb{N}$ which are independent of q and q' such that*

$$\|DH_{q,q'}^{-1}\| < c, \ |\partial^\eta H_{q,q'}^{-1}| < c(|\eta|), \text{ and } |\partial^\eta \det DH_{q,q'}^{-1}| < c(|\eta|) \tag{5.1}$$

for all indices $\eta = (\eta_1, \eta_2) \in \mathbb{N}^2$. Here $|\eta| = \eta_1 + \eta_2$ and $\partial^\eta = \partial_a^{\eta_1} \partial_b^{\eta_2}$.

Proof. By (2.1) and (2.3) we have for all $(a, b) \in O_i$,

$$\begin{aligned} |\det DH_{q,q'}(a, b)| &= r^{-1} |\partial_a T_a(q, q') - r^{-1}(\partial_b P_b(q') - \partial_a P_a(q'))| \\ &\ge (2r)^{-1} C_T. \end{aligned} \tag{5.2}$$

For the first claim it is therefore sufficient to show that the restriction of $H_{q,q'}$ to O_i is an injection. This, in turn, follows from two easy observations: If $(a, b), (a', b') \in O_i$ with $a - b \neq a' - b'$, then clearly $H_{q,q'}(a, b) \neq H_{q,q'}(a', b')$. On the other hand, $H_{q,q'}$ is strictly monotone on the line segments $\{(a, b) \in O_i \mid b - a = d\}$, where $d \in \mathbb{R}$, since

$$|\partial_a T_a(q, q') - r^{-1}(\partial_a P_{a+d}(q') - \partial_a P_a(q'))| \ge 2^{-1} C_T.$$

For (5.1) note that

$$DH^{-1}_{q,q'}(y) = (\det DH_{q,q'}(H^{-1}_{q,q'}(y)))^{-1}\begin{pmatrix} -r^{-1} & r^{-1}\partial_b P_b(q') \\ -r^{-1} & \partial_a T_a(q,q') + r^{-1}\partial_a P_a(q') \end{pmatrix}$$
$$=: (\det DH_{q,q'}(H^{-1}_{q,q'}(y)))^{-1}A,$$

where $(a,b) = H^{-1}_{q,q'}(y)$. Combining this with inequality (5.2), (2.1), and (2.4), gives $\|DH^{-1}_{q,q'}\| < c$. Using similar arguments and the fact that for all $l \in \mathbb{N}$ there exists a constant $C(l)$ such that $|\partial^\eta A_{ij}| < r^{-1}C(|\eta|)$ for all η and i, j, the second claim in (5.1) follows by induction. Finally, the last estimate is a consequence of the previous one. □

In the following lemma which is from [PS] we denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of smooth functions such that all of their derivatives decay faster than any power.

Lemma 5.5. *There exists $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\psi} > 0$, spt $\hat{\psi} \subset \{\xi \in \mathbb{R}^n \mid 1 \le |\xi| \le 4\}$, and $\sum_{j=-\infty}^{\infty} \hat{\psi}(2^{-j}\xi) = 1$ for all $\xi \ne 0$. Furthermore, for any finite Radon measure ν on $\mathbb{R}^n$ and any $\gamma \in \mathbb{R}$ there exists a constant C such that*

$$\frac{1}{C}\|\nu\|^2_{2,\gamma} \le \sum_{j=-\infty}^{\infty} 2^{2j\gamma}\int_{\mathbb{R}^n}(\psi_{2^{-j}} * \nu)(x)\,d\nu(x) \le C\|\nu\|^2_{2,\gamma},$$

where $\psi_{2^{-j}}(x) = 2^{jn}\psi(2^j x)$. (Above $*$ is the convolution.)

Proof. See [PS, Lemma 4.1]. □

Next we prove a lemma which is a modification of [PS, Lemma 7.10] tailored for our purposes.

Lemma 5.6. *Assume that ρ is a smooth non-negative real valued function which is supported inside the open unit square $(0,1)^2$. Let ψ be as in Lemma 5.5. Then for all $q, q' \in \widetilde{I}^2$ with $q \ne q'$, $j \in \mathbb{Z}$, and $k \in \mathbb{N} \setminus \{0\}$ we have*

$$\Big|\int_{\mathbb{R}}\int_{\mathbb{R}} \rho(a,b)\psi(2^j(P_a(q) - P_b(q'), a-b))\,d\mathcal{L}^1(a)\,d\mathcal{L}^1(b)\Big|$$
$$\le C\min\{(1+2^j|q-q'|)^{-k}, (1+2^j)^{-1}\},$$

where the constant C does not depend on q, q', and j.

Proof. Observing that it is enough to study positive integers j, and using the fast decay of ψ, we have

$$\Big|\int_{\mathbb{R}} \rho(a,b)\psi(2^j(P_a(q) - P_b(q')), 2^j(a-b))\,d\mathcal{L}^1(a)\Big|$$
$$\le c2^{-j} + c'\int_{t>2^{-j}}(2^j t)^{-2}\,d\mathcal{L}^1(t) \le C(1+2^j)^{-1}.$$

As indicated by the above calculation, the difficult part to handle is the domain where a is roughly equal to b and $|P_a(q) - P_b(q')|$ is much smaller than $|q-q'|$. We will first estimate the "easy" parts using the fast decay of ψ, and finally, we will use the fact that ψ has vanishing moments of all orders to replace the “difficult” domain by its complement.

For the other upper bound, fix $k, j \in \mathbb{N}$ such that $k \geq 1$. Setting $r = |q - q'|$, we may assume that $2^j r > 1$. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $[-1, 1]^2$, and $\phi \equiv 0$ on $\mathbb{R}^2 \setminus [-2, 2]^2$. Letting $H_{q,q'} : I^2 \to \mathbb{R}^2$ be as in Lemma 5.4, one obtains

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} \rho(a, b)\psi(2^j(P_a(q) - P_b(q'), a - b))\, d\mathcal{L}^1(a)\, d\mathcal{L}^1(b) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(a, b)\psi(2^j r H_{q,q'}(a, b))\phi(C_T^{-1} H_{q,q'}(a, b))\, d\mathcal{L}^1(a)\, d\mathcal{L}^1(b) \\
&\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(a, b)\psi(2^j r H_{q,q'}(a, b))(1 - \phi(C_T^{-1} H_{q,q'}(a, b)))\, d\mathcal{L}^1(a)\, d\mathcal{L}^1(b) \\
&=: A_1 + A_2.
\end{aligned}$$

Since the integrand of A_2 is non-zero only if $|H_{q,q'}| > C_T$, the fact that the support $\operatorname{spt} \rho$ of ρ is inside $(0, 1)^2$ and $\psi \in \mathcal{S}(\mathbb{R}^2)$ implies

$$|A_2| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(a, b)(C_T 2^j r)^{-k}\, d\mathcal{L}^1(a)\, d\mathcal{L}^1(b) \leq C(1 + 2^j r)^{-k}.$$

We continue by estimating A_1. Picking $a_1, \ldots, a_N$ as in Lemma 5.3, we find d_2, $d_3 < \min\{C_T, C_1^{-1} C_T\}$ such that

$$\{a \in (0, 1) \mid |T_a(q, q')| \leq d_3\} \subset \bigcup_{i=1}^{N} B(a_i, d_2/2) \tag{5.3}$$

and

$$\bigcup_{i=1}^{N} B(a_i, d_2) \cap (0, 1) \subset \{a \in (0, 1) \mid |T_a(q, q')| \leq C_T/4\}. \tag{5.4}$$

Let $d_1 < \min\{(2\widetilde{C}_2)^{-1} C_T, (4\widetilde{C}_1)^{-1} C_T\}$. For all $i = 0, \ldots, N$, there exists a smooth function $\chi_i : \mathbb{R} \to [0, 1]$ with the following properties:

(1) $\operatorname{spt} \chi_0 \subset B(0, d_1)$.
(2) $\operatorname{spt} \chi_i \subset B(a_i, d_2)$ for all $i = 1, \ldots, N$.
(3) Letting O_i be as in Lemma 5.4, we have

$$\chi_0(r^{-1}(a - b))\chi_i(a) = 0$$

for all $i = 1, \ldots, N$ and $(a, b) \in (0, 1)^2 \setminus O_i$.
(4) For all $(a, b) \in \operatorname{spt} \rho$ with $|T_a(q, q')| \leq d_3$ and $r^{-1}|a - b| \leq (8\widetilde{C}_1)^{-1} d_3$ we have

$$\sum_{i=1}^{N} \chi_0(r^{-1}(a - b))\chi_i(a) = 1.$$

(5) For all $l \in N$ there is a constant c_l such that

$$\sup_{0 \leq i \leq N} \|\partial^l \chi_i\|_\infty \leq c_l.$$

(Note that above property (3) follows from (1), (2), and (5.4), and (5.3) makes the choice of property (4) possible.) Combining (2.1), (5.4), and properties (2) and (3) leads to

$$\chi_0(r^{-1}(a-b))\chi_i(a) = \chi_0(r^{-1}(a-b))\chi_i(a)\phi(C_T^{-1}H_{q,q'}(a,b))$$

for all $(a,b) \in \mathbb{R}^2$ and $i = 1,\dots,N$. (This follows from the fact that $\phi(C_T^{-1}H_{q,q'}(a,b)) = 1$ if the left-hand side in the above equality is non-zero.) Therefore

$$\begin{aligned} A_1 &= \sum_{i=1}^{N}\int_{\mathbb{R}}\int_{\mathbb{R}} \rho(a,b)\chi_0(r^{-1}(a-b))\chi_i(a)\psi(2^j r H_{q,q'}(a,b))\, d\mathcal{L}^1(a)d\mathcal{L}^1(b) \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} \rho(a,b)\Big(1-\sum_{i=1}^{N}\chi_0(r^{-1}(a-b))\chi_i(a)\Big)\psi(2^j r H_{q,q'}(a,b)) \\ &\qquad\qquad \times \phi(C_T^{-1}H_{q,q'}(a,b))\, d\mathcal{L}^1(a)\, d\mathcal{L}^1(b) =: \sum_{i=1}^{N} D_i + D. \end{aligned}$$

From (4) we deduce that on the support of the integrand of D we have $|H_{q,q'}(a,b)| \geq (8\widetilde{C}_1)^{-1}d_3$, and so, similarly as before, we get

$$|D| \leq C(1+2^j r)^{-k}.$$

Since $N \leq C_1/C_T + 2$ by Lemma 5.3, it suffices to show that each D_i has an upper bound of the desired form. Fixing $1 \leq i \leq N$ and applying (3) and Lemma 5.4 gives

$$\begin{aligned} D_i &= \int_{O_i} \rho(a,b)\chi_0(r^{-1}(a-b))\chi_i(a)\psi(2^j r H_{q,q'}(a,b))\, d\mathcal{L}^2(a,b) \\ &= \int_{H_{q,q'}(O_i)} \rho(H_{q,q'}^{-1}(u,v))\chi_0(v)\chi_i((H_{q,q'}^{-1})_1(u,v))\psi(2^j r(u,v)) \\ &\qquad\qquad \times |\det(DH_{q,q'}^{-1}(u,v))|\, d\mathcal{L}^2(u,v), \end{aligned}$$

where $(H_{q,q'}^{-1})_1(u,v)$ is the first coordinate of $H_{q,q'}^{-1}(u,v)$. Since the integrand of D_i is zero outside O_i by (3), we may modify $H_{q,q'}$ in such a way that it becomes a diffeomorphism on $\mathbb{R}^2$, and all the bounds given in Lemma 5.4 remain unchanged. Defining for all $(u,v) \in \mathbb{R}^2$,

$$G(u,v) := \rho(H_{q,q'}^{-1}(u,v))\chi_0(v)\chi_i((H_{q,q'}^{-1})_1(u,v))|\det(DH_{q,q'}^{-1}(u,v))|,$$

and choosing $0 < \varepsilon < 1$ such that $(k+2)(1-\varepsilon) > k$, we rewrite D_i as

$$\begin{aligned} D_i &= \int_{|y|<(2^j r)^{\varepsilon-1}} G(y)\psi(2^j r y)\, d\mathcal{L}^2(y) + \int_{|y|>(2^j r)^{\varepsilon-1}} G(y)\psi(2^j r y)\, d\mathcal{L}^2(y) \\ &=: J_1 + J_2. \end{aligned}$$

From (5.1) we obtain that

$$|J_2| \leq c\int_{t>(2^j r)^{\varepsilon-1}} (2^j r t)^{-\frac{k}{\varepsilon}-1} t\, d\mathcal{L}^1(t) \leq C(1+2^j r)^{-k-1},$$

and therefore, it remains to estimate J_1.

Note that ψ has vanishing moments of all orders since $\partial^\eta \hat{\psi}(0) = 0$ for all η. Using the Taylor expansion for the function G, we calculate

$$\begin{aligned} J_1 &= -\sum_{|\eta|<k} \int_{|y|>(2^j r)^{\varepsilon-1}} (\eta!)^{-1} \partial^\eta G(0) y^\eta \psi(2^j r y)\, d\mathcal{L}^2(y) \\ &\quad + \sum_{|\eta|=k} \int_{|y|<(2^j r)^{\varepsilon-1}} (\eta!)^{-1} \partial^\eta G(t(y)y) \psi(2^j r y) y^\eta \, d\mathcal{L}^2(y) \\ &=: -\sum_{|\eta|<k} K_\eta + K. \end{aligned}$$

Here $y^\eta = y_1^{\eta_1} y_2^{\eta_2}$, $\eta! = \eta_1!\eta_2!$, and $t(y) \in [0, 1]$. Finally,

$$\begin{aligned} |K| &\le c \int_{|y|<(2^j r)^{\varepsilon-1}} \sup_{|\eta|=k} \|\partial^\eta G\|_\infty |y|^k \, d\mathcal{L}^2(y) \\ &\le c \sup_{|\eta|=k} \|\partial^\eta G\|_\infty (2^j r)^{-(1-\varepsilon)(k+2)} \le C \sup_{|\eta|=k} \|\partial^\eta G\|_\infty (1 + 2^j r)^{-k} \end{aligned}$$

and

$$\begin{aligned} |K_\eta| &\le c \|\partial^\eta G\|_\infty \int_{|y|>(2^j r)^{\varepsilon-1}} |y|^{|\eta|} |2^j r y|^{-|\eta|-1-\frac{k}{\varepsilon}} \, d\mathcal{L}^2(y) \\ &\le C \|\partial^\eta G\|_\infty (1 + 2^j r)^{-k-1}. \end{aligned}$$

Thus the claim follows from Lemma 5.4. □

As an immediate consequence of Lemma 5.6 we obtain the following result.

Corollary 5.7. *Let ρ and ψ be as in Lemma 5.6, and let $q, q' \in \widetilde{I}^2$ with $q \ne q'$. Then for any $k, n \in \mathbb{N} \setminus \{0\}$ we have*

$$\begin{aligned} &\Big| \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(a, b) \psi(2^j (P_a(q) - P_b(q'), a - b))\, d\mathcal{L}^1(a)\, d\mathcal{L}^1(b) \Big| \\ &\qquad \le C (1 + 2^j |q - q'|)^{-\frac{k}{n}} (1 + 2^j)^{-\frac{n-1}{n}}, \end{aligned}$$

where C does not depend on q, q' or j.

Proof of Theorem 5.1. Assume that μ is a Radon probability measure on SM such that μ is invariant under the geodesic flow and $I_\alpha(\mu) < \infty$ for $\alpha > 2$. Let $\gamma < (\alpha - 2)/2$.

By Lemma 3.2 we may restrict our consideration to the measures $\mu_\delta = \widetilde{P}_*(\nu \times \rho \mathcal{L}^1)$ where $\delta > 0$ and ρ is a smooth function such that $\operatorname{spt} \rho \subset (0, 1)$ and $\rho(t) = 1$ for all $\delta < t < 1 - \delta$. Letting $n, k \in \mathbb{N}$ such that $\alpha > 2 + 2\gamma + 1/n$ and $k > n(1 + 2\gamma + 1/n)$, and using Lemma 5.5 and Corollary 5.7 for positive j, we have

$$\int_{\mathbb{R}^2} |\widehat{\mu_\delta}(\xi)|^2 |\xi|^{2\gamma} \, d\mathcal{L}^2(\xi) \leq C \sum_{j=-\infty}^{\infty} 2^{2j\gamma} \int_{\mathbb{R}^2} (\psi_{2^{-j}} * \mu_\delta)(x) \, d\mu_\delta(x)$$

$$\leq C \sum_{j=-\infty}^{\infty} 2^{2j\gamma+2j} \Big| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi(2^j(x-y)) \times d\widetilde{P}_*(\nu \times \rho\mathcal{L}^1)(x) d\widetilde{P}_*(\nu \times \rho\mathcal{L}^1)(y) \Big|$$

$$= C \sum_{j=-\infty}^{\infty} 2^{2j\gamma+2j} \Big| \int_{\widetilde{I}^2 \times \mathbb{R}} \int_{\widetilde{I}^2 \times \mathbb{R}} \rho(a)\rho(b)\psi(2^j(P_a(q) - P_b(q'), a-b)) \times d\nu(q) d\mathcal{L}^1(a) d\nu(q') d\mathcal{L}^1(b) \Big| \tag{5.5}$$

$$\leq C \int_{\widetilde{I}^2} \int_{\widetilde{I}^2} \sum_{j=-\infty}^{\infty} 2^{2j\gamma+2j} (1+2^j)^{-\frac{n-1}{n}} (1 + 2^j |q-q'|)^{-\frac{k}{n}} d\nu(q) d\nu(q')$$

$$\leq C \int_{\widetilde{I}^2} \int_{\widetilde{I}^2} |q-q'|^{-(1+2\gamma+1/n)} d\nu(q) d\nu(q') = C I_{1+2\gamma+1/n}(\nu).$$

Here the last inequality follows by picking a positive integer j_0 such that $2^{-j_0-1} \leq r < 2^{-j_0}$ and by dividing the sum into 3 parts: $j < 0$, $0 \leq j \leq j_0$, and $j > j_0$. Using the choice of n and applying Lemma 5.2 gives the claim. □

6. Non-Preservation of Hausdorff Dimension in Higher Dimensional Manifolds

In this section we construct examples of (locally) invariant measures whose Hausdorff dimensions decrease under the projection onto the base manifold. Because of Remark 4.6 the following setting is natural for such examples.

Example 6.1. For any $n \geq 3$ there exist an n-dimensional compact smooth Riemannian manifold M and a measure μ on the unit tangent bundle SM such that it is locally invariant and its Hausdorff dimension decreases under the projection $\Pi : SM \to M$.

In fact, let M be the flat n-dimensional torus $[-1, 2]^n$ and let $I^n = [0, 1]^n \subset M$. Using the notation of Sect. 3, we set

$$\mathcal{C}_1 = I^{n-1} \times \{0\} \text{ and } \mathcal{C}_2 = I^{n-1} \times \{1\},$$

and define a diffeomorphism $\Psi : D \to \Psi(D)$ by

$$\Psi(x, y, t) = (\gamma_{p,q}(t), \gamma'_{p,q}(t)),$$

where $p = (x_1, \ldots, x_{n-1}, 0) \in \mathcal{C}_1$, $q = (y_1, \ldots, y_{n-1}, 1) \in \mathcal{C}_2$, $\gamma_{p,q}$ is the unique shortest geodesic parametrized by the Riemannian arc length which connects p and q, and

$$D = \{(x, y, t) \mid x, y \in I^{n-1}, 0 \leq t \leq d_M(p, q)\}.$$

Taking any measure ν such that

$$\operatorname{spt} \nu \subset \{(x, y) \in I^{n-1} \times I^{n-1} \mid x_{n-1} = y_{n-1} = 0\}$$

and defining $\mu = \Psi_*(\nu \times \mathcal{L}^1)$, we have $\dim_H \Pi_*\mu \le n-1$ since $\Pi_*\mu$ is supported by the $(n-1)$-dimensional plane $\{(m_1,\dots,m_n) \in I^n \mid m_{n-1} = 0\}$. Furthermore, μ is locally invariant, and $\dim_H \mu = \dim_H \nu + 1$. Choosing ν such that $\dim_H \nu > n-2$ gives $\dim_H \Pi_*\mu < \dim_H \mu$.

Remark 6.2. (a) Example 6.1 is easily modified to verify the existence of a globally invariant measure whose Hausdorff dimension is not preserved when projected onto the base manifold. To see this, take $\nu = \mathcal{L}^{2(n-2)}$ and replace I^n by M in Example 6.1. Then it is a straightforward calculation to show that $\dim_H(\Pi \circ \psi)_*(\mathcal{L}^{2(n-2)} \times \mathcal{L}^1) = n-1$. Clearly, $\psi_*(\mathcal{L}^{2(n-2)} \times \mathcal{L}^1) = \mathcal{L}^{2n-3}$ is globally invariant under the geodesic flow.

(b) In the case $n = 3$ Example 6.1 may be reduced to the 2-dimensional case. Therefore we may apply the results of Sect. 4 to deduce that

$$\dim_H \Pi_*\mu = \begin{cases} \dim_H \mu, & \text{if } \dim_H \nu \le 1 \\ 2, & \text{if } \dim_H \nu > 1. \end{cases}$$

Acknowledgement. We thank the referee for valuable comments clarifying the exposition.

References

[BKL] Bonetto, F., Kupiainen, A., Lebowitz, J.L.: *Absolute continuity of projected SRB measures of coupled Arnold cat map lattices*. To appear in Ergodic Theory Dynam. Systems, http://arxiv.org/abs/nlin/0310009, 2003
[BK1] Bricmont, J., Kupiainen, A.: Coupled analytic maps. Nonlinearity **8**, 379–396 (1995)
[BK2] Bricmont, J., Kupiainen, A.: High temperature expansions and dynamical systems. Commun. Math. Phys. **178**, 703–732 (1996)
[BS] Bunimovich, L.A., Sinai, Ya. G.: Spacetime chaos in coupled map lattices. Nonlinearity **1**, 491–516 (1988)
[F] Falconer, K.: *Techniques in Fractal Geometry*. Chichester: John Wiley & Sons, Ltd., 1997
[FM] Falconer, K., Mattila, P.: The packing dimension of projections and sections of measures. Math. Proc. Cambridge Philos. Soc. **119**, 695–713 (1996)
[H] Haase, H.: On the dimension of product measures. Mathematika **37**, 316–323 (1990)
[HT] Hu, X., Taylor, J.: Fractal properties of products and projections of measures in $\mathbb{R}^n$. Math. Proc. Cambridge Philos. Soc. **115**, 527–544 (1994)
[HK1] Hunt, B.R., Kaloshin, Yu., V.: How projections affect the dimension spectrum of fractal measures?. Nonlinearity **10**, 1031–1046 (1997)
[HK2] Hunt, B.R., Kaloshin, Yu., V.: Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces. Nonlinearity **12**, 1263–1275 (1999)
[J] Järvenpää, E.: *SRB-measures for coupled map lattices*. To appear in Lecture Notes in Phys., Springer
[JJ] Järvenpää, E., Järvenpää, M.: On the definition of SRB-measures for coupled map lattices. Commun. Math. Phys. **220**, 1–12 (2001)
[JJL] Järvenpää, E., Järvenpää, M., Llorente, M.: Local dimensions of sliced measures and stability of packing dimensions of sections of sets. Adv. Math. **183**, 127–154 (2004)
[K] Kaufman, R.: On Hausdorff dimension of projections. Mathematika **15**, 153–155 (1968)
[LL] Ledrappier, F., Lindenstrauss, E.: On the projections of measures invariant under the geodesic flow. Int. Math. Res. Not. **9**, 511–526 (2003)
[Mar] Marstrand, M.: Some fundamental geometrical properties of plane sets of fractional dimension. Proc. London Math. Soc. **4**, 257–302 (1954)
[Mat1] Mattila, P.: Hausdorff dimension, orthogonal projections and intersections with planes. Ann. Acad. Sci. Fenn. Math. **1**, 227–244 (1975)
[Mat2] Mattila, P.: Orthogonal projections, Riesz capacities and Minkowski content. Indiana Univ. Math. J. **39**, 185–198 (1990)
[Mat3] Mattila, P.: *Geometry of Sets and Measures in Euclidean Spaces: Fractals and rectifiability*. Cambridge: Cambridge University Press, 1995
[Mat4] Mattila, P.: Hausdorff dimension, projections, and Fourier transform. Publ. Mat. **48**, 3–48 (2004)

[PS] Peres, Y., Schlag, W.: Smoothness of projections, Bernoulli convolutions, and the dimensions of exceptions. Duke Math. J. **102**, 193–251 (2000)

[SY] Sauer, T. D., Yorke, J. A.: Are dimensions of a set and its image equal for typical smooth functions?. Ergodic Theory Dynam. Systems **17**, 941–956 (1997)

Communicated by A. Kupiainen