Commun. Math. Phys. 256, 681–735 (2005)
Digital Object Identifier (DOI) 10.1007/s00220-004-1224-2

Communications in **Mathematical Physics**

Anomalous Universality in the Anisotropic Ashkin–Teller Model

A. Giuliani[1,⋆], **V. Mastropietro**[2]

[1] Dipartimento di Fisica, Università di Roma "La Sapienza", P.zzale A. Moro, 2, 00185 Roma, Italy, and INFN, sezione di Roma1, Roma, Italy. E-mail: alessandro.giuliani@roma1.infn.it
[2] Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy. E-mail: mastropi@mat.uniroma2.it

Received: 19 March 2004 / Accepted: 6 May 2004
Published online: 29 December 2004 –

Abstract: The Ashkin–Teller (AT) model is a generalization of Ising 2–d to a four states spin model; it can be written in the form of two Ising layers (in general with different couplings) interacting via a four–spin interaction. It was conjectured long ago (by Kadanoff and Wegner, Wu and Lin, Baxter and others) that AT has in general two critical points, and that universality holds, in the sense that the critical exponents are the same as in the Ising model, except when the couplings of the two Ising layers are equal (isotropic case). We obtain an explicit expression for the specific heat from which we prove this conjecture in the weakly interacting case and we locate precisely the critical points. We find the somewhat unexpected feature that, despite universality, holds for the specific heat, nevertheless nonuniversal critical indexes appear: for instance the distance between the critical points rescale with an anomalous exponent as we let the couplings of the two Ising layers coincide (isotropic limit); and so does the constant in front of the logarithm in the specific heat. Our result also explains how the crossover from universal to nonuniversal behaviour is realized.

1. Introduction

1.1. Historical introduction. Ashkin and Teller [AT] introduced their model as a generalization of the Ising model to a four component system; in each site of a bidimensional lattice there is a spin which can take four values, and only nearest neighbor spins interact. The model can be also considered a generalization of the four state Potts model to which it reduces for a suitable choice of the parameters.

A very convenient representation of the Ashkin Teller model is in terms of Ising spins [F]; one associates with each site of the square lattice two spin variables, $\sigma_{\mathbf{x}}^{(1)}$ and $\sigma_{\mathbf{x}}^{(2)}$; the partition function is given by $\Xi^{AT}_{\Lambda_M} = \sum_{\sigma^{(1)},\sigma^{(2)}} e^{-H_{\Lambda_M}}$, where

⋆ Partially supported by NSF Grant DMR 01–279–26

$$H_{\Lambda_M}(\sigma^{(1)},\sigma^{(2)}) = J^{(1)}H_I(\sigma^{(1)}) + J^{(2)}H_I(\sigma^{(2)}) + \lambda V(\sigma^{(1)},\sigma^{(2)}) = \sum_{\mathbf{x}\in\Lambda_M} H^{AT}_{\mathbf{x}} ,$$
$$H_I(\sigma^{(j)}) = -\sum_{\mathbf{x}\in\Lambda_M}[\sigma^{(j)}_{\mathbf{x}}\sigma^{(j)}_{\mathbf{x}+\hat e_1} + \sigma^{(j)}_{\mathbf{x}}\sigma^{(j)}_{\mathbf{x}+\hat e_0}] ,$$
$$V(\sigma^{(1)},\sigma^{(2)}) = -\sum_{\mathbf{x}\in\Lambda_M}[\sigma^{(2)}_{\mathbf{x}}\sigma^{(2)}_{\mathbf{x}+\hat e_0}\sigma^{(1)}_{\mathbf{x}}\sigma^{(1)}_{\mathbf{x}+\hat e_0} + \sigma^{(2)}_{\mathbf{x}}\sigma^{(2)}_{\mathbf{x}+\hat e_1}\sigma^{(1)}_{\mathbf{x}}\sigma^{(1)}_{\mathbf{x}+\hat e_1}] , \tag{1.1}$$

where H_I is the Ising model hamiltonian, $\hat e_1$, $\hat e_0$ are the unit vectors $\hat e_1 = (1,0)$, $\hat e_0 = (0,1)$ and Λ_M is a square subset of $\mathbb{Z}^2$ of side M. The free energy and the specific heat are given by

$$f = \lim_{M\to\infty}\frac{1}{M^2}\log \Xi^{AT}_{\Lambda_M} \quad , \quad C_v = \lim_{M\to\infty}\frac{1}{M^2}\sum_{\mathbf{x},\mathbf{y}\in\Lambda_M} < H^{AT}_{\mathbf{x}} H^{AT}_{\mathbf{y}} >_{\Lambda_M,T} , \tag{1.2}$$

where $<\cdot>_{\Lambda_M,T}$ denotes the truncated expectation w.r.t. the Gibbs distribution with the Hamiltonian (1.1). The case $J^{(1)} = J^{(2)}$ is called *isotropic*. For $\lambda = 0$ the model reduces to two independent Ising models and it has *two critical points* if $J^{(1)} \neq J^{(2)}$; it was conjectured by Kadanoff and Wegner [K, KW] and later on by Wu and Lin [WL] that the AT model has in general two critical points also when $\lambda \neq 0$, except when the model is isotropic.

The *isotropic* case was studied by Kadanoff [K] who, by scaling theory, conjectured a relation between the critical exponents of *isotropic* AT and those of the *Eight vertex* model, which had been solved by Baxter and has nonuniversal indexes. Further evidence for the validity of Kadanoff's prediction was given by [PB] (using second order renormalization group arguments) and by [LP, N] (by a heuristic mapping of both models into the massive Luttinger model describing one dimensional interacting fermions in the continuum). Indeed nonuniversal critical behaviour in the specific heat in the isotropic AT model, for small λ, has been rigorously established in [M1].

The *anisotropic* case is much less understood. As we said, it is believed that there are *two* critical points, contrary to what happens in the isotropic case. Baxter [Ba] conjectured that "presumably" *universality* holds at the critical points for $J^{(1)} \neq J^{(2)}$ (*i.e.* the critical indices are the same as in the Ising model), except when $J^{(1)} = J^{(2)}$ when the two critical points coincide and nonuniversal behaviour is found. Since the 1970's, the anisotropic AT model was studied by various approximate or numerical methods: Migdal–Kadanoff Renormalization Group [DR], Monte Carlo Renormalization group [Be], finite size scaling [Bad]; such results give evidence of the fact that, far away from the isotropic point, AT has two critical points and belongs to the same universality class of Ising; however they do not give information about the precise relative location of the critical points and the critical behaviour of the specific heat when $J^{(1)}$ is close to $J^{(2)}$. The problem of how the crossover from universal to nonuniversal behaviour is realized in the isotropic limit remained for years completely unsolved, even at a heuristic level.

We will study the anisotropic Ashkin–Teller model by writing the partition function and the specific heat as Grassmann integrals corresponding to a $d = 1 + 1$ *interacting* fermionic theory; this is possible because the Ising model can be reformulated as a *free fermions* model (see [SML, H, S or ID]). One can then take advantage from the theory of Grassmann integrals for weakly interacting $d = 1 + 1$ fermions, which is quite well developed, starting from [BG1] (see also [BG, GM or BM] for extensive reviews). Fermionic RG methods for classical spin models have been already applied in [PS] to the Ising model perturbed by a four spin interaction, proving a *universality* result for the specific heat; and in [M1] to prove a *nonuniversality* result for the 8 vertex or the isotropic AT model. By such techniques one can develop a perturbative expansion, convergent up to the critical points, uniformly in the parameters.

1.2. Main results. We find it convenient to introduce the variables $t^{(j)} = \tanh J^{(j)}$, $j = 1, 2$ and

$$t = \frac{t^{(1)} + t^{(2)}}{2} \quad , \quad u = \frac{t^{(1)} - t^{(2)}}{2}. \tag{1.3}$$

The parameter u measures the *anisotropy* of the system. We consider then the free energy or the specific heat as functions of t, u, λ.

If $\lambda = 0$, AT is exactly solvable, because the Hamiltonian (1.1) is the sum of two independent Ising model Hamiltonians. From the Ising model exact solution [O, SML, MW] one finds that f is analytic for all t, u except for

$$t = t_c^{\pm} = \sqrt{2} - 1 \pm |u|, \tag{1.4}$$

and for t close to $t_c^{\pm}$ the specific heat C_v has a logarithmic divergence: $C_v \simeq -C \log |t - t_c^{\pm}|$, where $C > 0$ and $\simeq$ means that the ratio of both sides tends to 1 as $t \to t_c^{\pm}$.

We consider the case in which λ is small with respect to $\sqrt{2} - 1$ and we distinguish two regimes.

1) If u is much bigger than λ (so that the unperturbed critical points are well separated) we find that the presence of λ just changes by a small amount the location of the critical points, *i.e.* we find that the critical points have the form $t_c^{\pm} = \sqrt{2} - 1 + O(\lambda) \pm |u|\big(1 + O(\lambda)\big)$; moreover the asymptotic behaviour of C_v at criticality remains essentially unchanged: $C_v \simeq -C \log |t - t_c^{\pm}|$.
2) When u is small compared to λ the interaction has a more dramatic effect. We find that the system has still only two critical points $t_c^{\pm}(\lambda, u)$; their center $(t_c^+ + t_c^-)/2$ is just shifted by $O(\lambda)$ from $\sqrt{2} - 1$, as in item (1); however their relative location scales, as $u \to 0$, with an "anomalous critical exponent" $\eta(\lambda)$, continuously varying with λ: more precisely we find that $t_c^+ - t_c^- = O(|u|^{1+\eta})$, where η is analytic in λ near $\lambda = 0$ and $\eta = -b\lambda + O\big(\lambda^2\big)$, $b > 0$. In particular the relative location of the critical points as a function of the anisotropy parameter u with λ fixed and small has a different qualitative behaviour, depending on the sign of λ, see Fig 1.

For $t \to t_c^{\pm}(\lambda, u)$ the specific heat C_v still has a logarithmic divergence but, for all $u \neq 0$, the constant in front of the log is $O(|u|^{\eta_c})$, where η_c is analytic in λ for small λ and $\eta_c = a\lambda + O\big(\lambda^2\big)$, $a \neq 0$. The logarithmic behaviour is found only in an extremely small region around the critical points; outside this region, C_v varies as $t \to t_c^{\pm}(\lambda, u)$ according to a power law behaviour with nonuniversal exponent. The conclusion is that, for all $u \neq 0$, there is universality for the specific heat (which diverges with the same exponent as in the Ising model); nevertheless *nonuniversal critical indexes* appear in the theory, in the difference between the critical points and in the constant in front of the logarithm in the specific heat. One can speak of *anomalous universality* as the specific heat diverges at criticality as in Ising, but the isotropic limit $u \to 0$ is reached with nonuniversal critical indices.

With the notations introduced above and calling D a sufficiently small $O(1)$ interval (*i.e.* with amplitude independent of λ) centered around $\sqrt{2} - 1$, we can express our main result as follows.

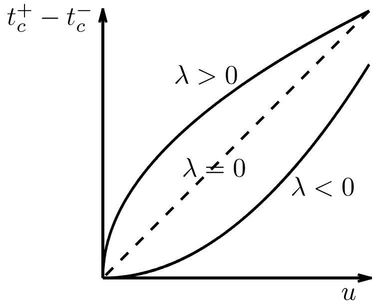

Fig. 1. The qualitative behavIour of $t_c^+(\lambda, u) - t_c^-(\lambda, u)$ as a function of u for two different values of λ (in arbitrary units). The graphs are (qualitative) plots of $2|u|^{1+\eta}$, with $\eta \simeq -b\lambda$, $b > 0$

Main Theorem. *There exists ε_1 such that, for $t \pm u \in D$, $j = 1, 2$, and $|\lambda| \leq \varepsilon_1$, one can define two functions $t_c^{\pm}(\lambda, u)$ with the following properties:*

$$t_c^{\pm}(\lambda, u) = \sqrt{2} - 1 + \nu^*(\lambda) \pm |u|^{1+\eta}\big(1 + F^{\pm}(\lambda, u)\big) , \tag{1.5}$$

where $|\nu^*(\lambda)| \leq c|\lambda|$, $|F^{\pm}(\lambda, u)| \leq c|\lambda|$, for some positive constant c and $\eta = \eta(\lambda)$ is an analytic function of λ s.t. $\eta(\lambda) = -b\lambda + O(\lambda^2)$, $b > 0$, and:

1) *the free energy $f(t, u, \lambda)$ and the specific heat $C_v(t, u, \lambda)$ in (1.2) are analytic in the region $t \pm u \in D$, $|\lambda| \leq \varepsilon_1$ and $t \neq t_c^{\pm}(\lambda, u)$;*
2) *in the same region of parameters, the specific heat can be written as:*

$$C_v = -C_1 \Delta^{2\eta_c} \log \frac{\left|t - t_c^-\right|\left|t - t_c^+\right|}{\Delta^2} + C_2 \frac{1 - \Delta^{2\eta_c}}{\eta_c} + C_3, \tag{1.6}$$

where $\Delta^2 \stackrel{def}{=} (t - \bar{t}_c)^2 + (u^2)^{1+\eta}$ and $\bar{t}_c \stackrel{def}{=} (t_c^+ + t_c^-)/2$; the exponent $\eta_c = \eta_c(\lambda) = a\lambda + O(\lambda^2)$, $a \neq 0$, is analytic in λ; the functions $C_j = C_j(\lambda, t, u)$, $j = 1, 2, 3$, are bounded above and below by $O(1)$ constants; finally $C_1 - C_2$ vanishes for $\lambda = u = 0$.

Remarks. 1) The key hypothesis for the validity of the Main Theorem is the smallness of λ. When $\lambda = 0$ the critical points correspond to $t \pm u = \sqrt{2} - 1$: hence for simplicity we restrict $t \pm u$ in a sufficiently small $O(1)$ interval around $\sqrt{2} - 1$. A possible explicit choice for D, convenient for our proof, could be $D = [\frac{3(\sqrt{2}-1)}{4}, \frac{5(\sqrt{2}-1)}{4}]$. Our technique would allow us to prove the above theorem, at the cost of a lengthier discussion, for any $t^{(1)}, t^{(2)} > 0$: of course in that case we should distinguish different regions of parameters and treat in a different way the cases of low or high temperature or the case of big anisotropy (*i.e.* the cases $t << \sqrt{2} - 1$ or $t >> \sqrt{2} - 1$ or $|u| >> 1$).

2) Equation (1.6) shows how the crossover from universal to nonuniversal behaviour is realized. When $u \neq 0$ only the first term in (1.6) can be singular in correspondence to the two critical points; it has a logarithmic singularity (as in the Ising model) with a constant $O(\Delta^{2\eta_c})$ in front. However the logarithmic term dominates the second one only if t varies inside an extremely small region $O(|u|^{1+\eta} e^{-a/|\lambda|})$, $a > 0$, around the critical points. Outside such a region the power law behaviour corresponding to the second addend in (1.6) dominates. When $u \to 0$ one recovers the power law decay found in [M1] for the isotropic case. See Fig 2.

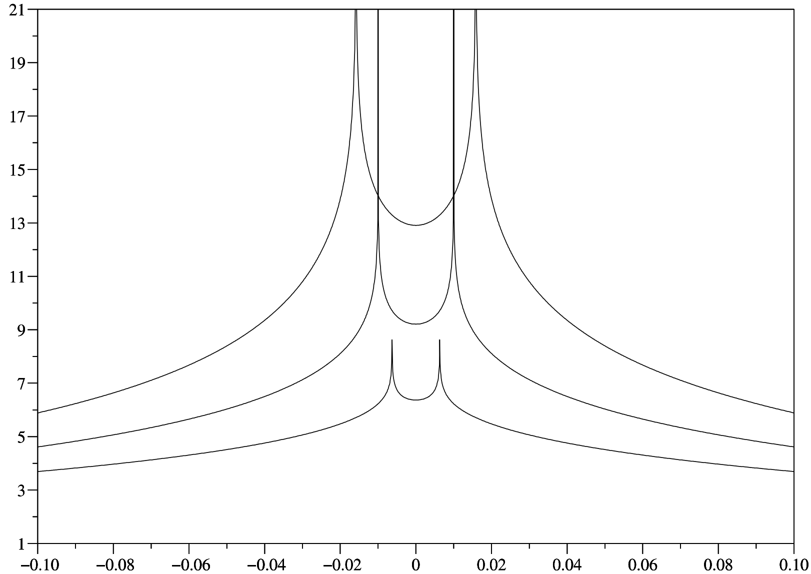

Fig. 2. The qualitative behaviour of C_v as a function of $t - \bar{t}_c$, where $\bar{t}_c = (t_c^+ + t_c^-)/2$. The three graphs are plots of (1.6), with $C_1 = C_2 = 1$, $C_3 = 0$, $u = 0.01$, $\eta = \eta_c = 0.1, 0, -0.1$ respectively; the central curve corresponds to $\lambda = 0$, the upper one to $\lambda > 0$ and the lower to $\lambda < 0$

3) By the result of item (1) of the Main Theorem, C_v is analytic in λ, t, u outside the critical line. This is not apparent from (1.6), because Δ is non-analytic in u at $u = 0$ (of course the bounded functions C_j are non-analytic in u also, in a suitable way compensating the non analyticity of Δ). We get to (1.6) by interpolating two different asymptotic behaviours of C_v in the regions $|t - \bar{t}_c| < 2|u|^{1+\eta}$ and $|t - \bar{t}_c| \geq 2|u|^{1+\eta}$ and the non analyticity of Δ is introduced "by hand" by our estimates and it is not intrinsic for C_v. Equation (1.6) is simply a convenient way to describe the crossover between different critical behaviours of C_v.
4) We do not study the free energy directly at $t = t_c^\pm(\lambda, u)$, therefore in order to show that $t = t_c^\pm(\lambda, u)$ is a critical point we must study some thermodynamic property like the specific heat by evaluating it at $t \neq t_c^\pm(\lambda, u)$ and $M = \infty$ and then verify that it has a singular behavior as $t \to t_c^\pm$. The case t precisely equal to $t_c^\pm$ cannot be discussed at the moment with our techniques, in spite of the uniformity of our bounds as $t \to t_c^\pm$. The reason is that we write the AT partition function as a sum of 16 different partition functions, differing for boundary terms. Our estimates on each single term are uniform up to the critical point; however, in order to show that the free energy computed with one of the 16 terms is the same as the complete free energy, we need to stay at $t \neq t_c^\pm$: in this case boundary terms are suppressed as $\sim e^{-\kappa M|t-t_c^\pm|}$, $\kappa > 0$, as $M \to \infty$. If we stay exactly at the critical point, cancellations between the 16 terms can be present (as it is well known already from the Ising model exact solution [MW]) and we do not have control on the behaviour of the free energy, as the infinite volume limit is approached.

1.3. Strategy of the proof. It is well known that the free energy and the specific heat of the Ising model can be expressed as a sum of *Pfaffians* [MW] which can be equivalently written, see [ID, S], as *Grassmann functional integrals*, see for instance App A of [M1] or §4 of [GM] for the basic definitions of Grassmann variables and Grassmann integration. The formal action of the Ising model in terms of Grassmann variables ψ, $\overline{\psi}$ has the form

$$\sum_{\mathbf{x}} \frac{t}{4}\Big[\psi_{\mathbf{x}}(\partial_1 - i\partial_0)\psi_{\mathbf{x}} + \overline{\psi}_{\mathbf{x}}(\partial_1 + i\partial_0)\overline{\psi}_{\mathbf{x}} - 2i\overline{\psi}_{\mathbf{x}}(\partial_1 + \partial_0)\psi_{\mathbf{x}}\Big] + i(\sqrt{2} - 1 - t)\overline{\psi}_{\mathbf{x}}\psi_{\mathbf{x}} \,, \tag{1.7}$$

where ∂_j are discrete derivatives. ψ and $\overline{\psi}$ are called *Majorana* fields, see [ID], because of an analogy with relativistic Majorana fermions. They are massive, because of the presence of the last term in (1.7); criticality corresponds to the massless case ($t = \sqrt{2} - 1$). If $\lambda = 0$ the free energy and specific heat can be written as the sum of Grassmann integrals describing *two* kinds of Majorana fields, with masses $m^{(1)} = t^{(1)} - \sqrt{2} + 1$ and $m^{(2)} = t^{(2)} - \sqrt{2} + 1$. The critical points are obtained by choosing one of the two fields massless (in the isotropic case $t^{(1)} = t^{(2)}$ and the two fields become massless together).

If $\lambda \neq 0$ again the free energy and the specific heat can be written as Grassmann integrals, but the Majorana fields are *interacting* with a short range potential. By performing a suitable change of variables, the partition function can be written, see §2 and §3, as a sum of terms $\Xi_{AT}^{\gamma_1,\gamma_2}$ (γ_1, γ_2 label different boundary conditions) of the form

$$\Xi_{AT}^{\gamma_1,\gamma_2} = \int P(d\psi) e^{-\mathcal{V}^{(1)}(\sqrt{Z_1}\psi)} \quad , \quad P(d\psi) = \mathcal{D}\psi\, e^{-Z_1(\psi^+, A\psi)} \,, \tag{1.8}$$

where $\psi = \{\psi^+_{\omega,\mathbf{x}}, \psi^-_{\omega,\mathbf{x}}\}_{\omega=\pm 1}$ are elements of a Grassmann algebra; $\mathcal{D}\psi$ is a symbol for the Grassmann integration; $\mathcal{V}^{(1)}$ is a short range interaction, sum of monomials in ψ of any degree, whose quartic term is weighted by a constant $\lambda_1 = O(\lambda)$; and $Z_1(\psi^+, A\psi)$ has the form:

$$Z_1 \sum_{\mathbf{x},\omega} \psi^+_{\omega,\mathbf{x}}(\partial_1 - i\omega\partial_0)\psi^-_{\omega,\mathbf{x}} - i\omega\sigma_1\psi^+_{\omega,\mathbf{x}}\psi^-_{-\omega,\mathbf{x}} + i\omega\mu_1\psi^\alpha_{\omega,\mathbf{x}}\psi^\alpha_{-\omega,-\mathbf{x}} - \beta_1\psi^\alpha_{\omega,\mathbf{x}}(\partial_1 - i\omega\partial_0)\psi^\alpha_{\omega,\mathbf{x}} \tag{1.9}$$

with $\sigma_1 = O(t - \sqrt{2} + 1) + O(\lambda)$, $\mu_1, \beta_1 = O(u)$ (in particular in the isotropic case the terms proportional to μ_1 and β_1 are absent). If $\lambda = 0$, $\sigma_1 = (m^{(1)} + m^{(2)})/2$ and $\mu_1 = (m^{(2)} - m^{(1)})/2$. $\psi^\pm$ are called *Dirac* fields, because of an analogy with relativistic Dirac fermions; they are combinations of the Majorana variables $\psi^{(j)}$, $\overline{\psi}^{(j)}$, $j = 1, 2$, associated with the two Ising layers in (1.1): hence the description in terms of Dirac variables mixes intrinsically the two Ising models and will be useful in a range of momentum scale in which the two layers appear to be essentially equal.

One can compute $\Xi_{AT}^{\gamma_1,\gamma_2}$ by expanding $e^{-\mathcal{V}^{(1)}(\sqrt{Z_1}\psi)}$ in Taylor series and integrating term by term the Grassmann monomials; since the propagators of $P(d\psi)$ (*i.e.* the elements of A^{-1}, see (1.8), (1.9)) diverge for $\mathbf{k} = \mathbf{0}$ and $\sigma_1 \pm \mu_1 = 0$ in the infinite volume limit $M \to \infty$, the series can converge uniformly in M only in a region outside $|\sigma_1 \pm \mu_1| \leq c$, for some c, *i.e.* in the thermodynamic limit it can converge only far from the critical points.

Since we are interested in the critical behaviour of the system, we set up a more complicated procedure to evaluate the partition function, based on the (Wilsonian) Renormalization Group (RG). The first step is to decompose the integration $P(d\psi)$ as a product of independent integrations: $P(d\psi) = \prod_{h=-\infty}^{1} P(d\psi^{(h)})$, where the momentum space propagator corresponding to $P(d\psi^{(h)})$ is not singular, but $O(\gamma^{-h})$, for $M \to \infty$, γ being a fixed *scaling parameter* larger than 1. This decomposition is realized by slicing in a smooth way the momentum space, so that $\psi^{(h)}$, if $h \le 0$, depends only on the momenta between γ^{h-1} and γ^{h+1}. We compute the Grassmann integrals defining the partition function by iteratively integrating the fields $\psi^{(1)}, \psi^{(0)}, \ldots$, see §4. After each integration step we rewrite the partition function in a way similar to (1.8), with the quadratic form $Z_1(\psi^+, A\psi)$ replaced by $Z_h(\psi^+, A^{(h)}\psi)$, which has the same structure of (1.9), with Z_h, σ_h, μ_h replacing Z_1, σ_1, μ_1; the structure of $Z_h(\psi^+, A^{(h)}\psi)$ is preserved because of symmetry properties, guaranteeing that many other possible quadratic "local" terms are indeed vanishing, or *irrelevant* in a RG sense. The interaction $\mathcal{V}^{(1)}$ is replaced by an *effective action* $\mathcal{V}^{(h)}$, $h \le 0$, given by a sum of monomials of ψ of arbitrary order, with kernels decaying in real space on scale γ^{-h}; in particular the quartic term is weighted by a coupling constant λ_h and the kernels of $\mathcal{V}^{(h)}$ are *analytic functions* of $\{\lambda_h, \ldots, \lambda_1\}$, if λ_k are small enough, $k \ge h$, and $|\sigma_k|\gamma^{-k}, |\mu_k|\gamma^{-k} \le 1$ (say – the constant 1 could be replaced by any other constant $O(1)$).

In this way the problem of finding good bounds for $\log \Xi_{\Lambda_M}^{AT}$ is reformulated into the problem of controlling the size of $\lambda_h, \sigma_h, \mu_h$, $h \le 0$, under the RG iterations. We use a crucial property, called *vanishing of Beta function*, to prove that actually, if λ is small enough, $|\lambda_h| \le 2|\lambda_1|$ (recall that $\lambda_1 = O(\lambda)$). The possibility of controlling the flow of λ_h is the main reason for describing the system in terms of Dirac variables. For σ_h, μ_h, Z_h, we find that, under RG iterations, they evolve as: $\sigma_h \simeq \sigma_1\gamma^{b_2\lambda h}$, $\mu_h \simeq \mu_1\gamma^{-b_2\lambda h}$, $Z_h \simeq \gamma^{-b_1\lambda^2 h}$. Note in particular that Z_h grows exponentially with an exponent $O(\lambda^2)$; this is connected with the presence of "critical indexes" in the correlation functions, which means that their long distance behaviour is qualitatively changed by the interaction.

We perform the iterative integration described above up to a scale h_1^* such that $(|\sigma_{h_1^*}| + |\mu_{h_1^*}|)\gamma^{-h_1^*} = O(1)$, in such a way that $(|\sigma_h| + |\mu_h|)\gamma^{-h} \le O(1)$, for all $h \ge h_1^*$ and convergence of the kernels of the effective potential can be guaranteed by our estimates. In the range of scales $h \ge h_1^*$ the flow of the effective coupling constant λ_h is essentially the same as for the isotropic AT model [M1] (since $|\mu_h|\gamma^{-h}$ is small, the iteration "does not see" the anisotropy and the system seems to behave as if there was just one critical point) and nonuniversal critical indexes are generated (they appear in the flows of σ_h, μ_h and Z_h), following the same mechanism of the isotropic case.

We note that after the integration of $\psi^{(1)}, \ldots, \psi^{(h_1^*+1)}$, we can still reformulate the problem in terms of the original Majorana fermions $\psi^{(1,\le h_1^*)}$, $\psi^{(2,\le h_1^*)}$ associated with the two Ising models in (1.1). On scale h_1^* their masses are deeply changed w.r.t. $t^{(1)} - \sqrt{2} + 1$ and $t^{(2)} - \sqrt{2} + 1$: they are given by $m_{h_1^*}^{(1)} = |\sigma_{h_1^*}| + |\mu_{h_1^*}|$ and $m_{h_1^*}^{(2)} = |\sigma_{h_1^*}| - |\mu_{h_1^*}|$. Note that the condition $|\sigma_{h_1^*}| + |\mu_{h_1^*}| = O(\gamma^{h_1^*})$ implies that the field $\psi^{(1,\le h_1^*)}$ is massive on scale h_1^* (so that the Ising layer with $j = 1$ is "far from criticality" on the same scale). This implies that we can integrate (without any multiscale decomposition) the massive Majorana field $\psi^{(1,\le h_1^*)}$, obtaining an effective theory of a single Majorana field with mass $|\sigma_{h_1^*}| - |\mu_{h_1^*}|$, which can be arbitrarily small. The integration of the scales $\le h_1^*$, see §6, is done again by a multiscale decomposition similar to the one just described; an important feature is however that there are no more quartic marginal terms, because the anticommutativity of Grassmann variables forbids local quartic monomials of a single Majorana fermion. The problem is essentially equivalent to the study of a single perturbed Ising model with "upper" cutoff on momentum space $O(\gamma^{h_1^*})$ and mass $|\sigma_{h_1^*}| - |\mu_{h_1^*}|$. The flow of the effective mass and of Z_h is non-anomalous in this regime: in particular the mass of the Majorana field is just shifted by $O(\lambda\gamma^{h_1^*})$ from $|\sigma_{h_1^*}| - |\mu_{h_1^*}|$. Criticality is found when the effective mass on scale $-\infty$ is vanishing; the values of t, u for which this happens are found by solving a non-trivial implicit function problem.

Finally, see §7, we define a similar expansion for the specific heat and we compute its asymptotic behaviour arbitrarily near the critical points.

Technically it is an interesting feature of this problem that there are two regimes in which the system must be described in terms of different fields: the first one in which the natural variables are Dirac Grassmann variables, and the second one in which they are Majorana; note that the scale separating the two regimes is dynamically generated by the RG iterations (and of course its precise location is not crucial and h_1^* can be modified in $h_1^* + n$, $n \in \mathbb{Z}$, without qualitatively affecting the bounds).

2. Fermionic Representation

2.1. The partition function $\Xi_I^{(j)} = \sum_{\sigma^{(j)}} \exp\{-J^{(j)} H_I(\sigma^{(j)})\}$ of the Ising model can be written as a Grassmann integral; this is a classical result, mainly due to [LMS], [Ka, H, MW, S]. In Appendix A1, starting from a formula obtained in [MW], we prove that

$$\Xi_I^{(j)} = (-1)^{M^2} \frac{(2\cosh J^{(j)})^{M^2}}{2} \times \sum_{\varepsilon,\varepsilon'=\pm} \int \prod_{\mathbf{x}\in\Lambda_M} dH_{\mathbf{x}}^{(j)} d\overline{H}_{\mathbf{x}}^{(j)} dV_{\mathbf{x}}^{(j)} d\overline{V}_{\mathbf{x}}^{(j)} (-1)^{\delta_\gamma} e^{S_\gamma^{(j)}(t^{(j)})}, \quad (2.1)$$

where $j = 1, 2$ denotes the lattice, $\gamma = (\varepsilon, \varepsilon')$ and δ_γ is $\delta_{+,+} = 1$, $\delta_{+,-} = \delta_{-,+} = \delta_{-,-} = 2$ and, if $t^{(j)} = \tanh J^{(j)}$,

$$\begin{aligned} S_\gamma^{(j)}(t^{(j)}) = t^{(j)} \sum_{\mathbf{x}\in\Lambda_M} \Big[\overline{H}_{\mathbf{x}}^{(j)} H_{\mathbf{x}+\hat{e}_1}^{(j)} + \overline{V}_{\mathbf{x}}^{(j)} V_{\mathbf{x}+\hat{e}_0}^{(j)}\Big] \\ + \sum_{\mathbf{x}\in\Lambda_M} \Big[\overline{H}_{\mathbf{x}}^{(j)} H_{\mathbf{x}}^{(j)} + \overline{V}_{\mathbf{x}}^{(j)} V_{\mathbf{x}}^{(j)} + \overline{V}_{\mathbf{x}}^{(j)} \overline{H}_{\mathbf{x}}^{(j)} \\ + V_{\mathbf{x}}^{(j)} \overline{H}_{\mathbf{x}}^{(j)} + H_{\mathbf{x}}^{(j)} \overline{V}_{\mathbf{x}}^{(j)} + V_{\mathbf{x}}^{(j)} H_{\mathbf{x}}^{(j)}\Big], \end{aligned} \quad (2.2)$$

where $H_{\mathbf{x}}^{(j)}, \overline{H}_{\mathbf{x}}^{(j)}, V_{\mathbf{x}}^{(j)}, \overline{V}_{\mathbf{x}}^{(j)}$ are *Grassmann variables* verifying different boundary conditions depending on the label $\gamma = (\varepsilon, \varepsilon')$ which is not affixed explicitly, to simplify the notations, *i.e.*

$$\begin{aligned} \overline{H}_{\mathbf{x}+M\hat{e}_0}^{(j)} = \varepsilon \overline{H}_{\mathbf{x}}^{(j)} \quad &, \quad \overline{H}_{\mathbf{x}+M\hat{e}_1}^{(j)} = \varepsilon' \overline{H}_{\mathbf{x}}^{(j)} \\ H_{\mathbf{x}+M\hat{e}_0}^{(j)} = \varepsilon H_{\mathbf{x}}^{(j)} \quad &, \quad H_{\mathbf{x}+\hat{e}_1}^{(j)} = \varepsilon' H_{\mathbf{x}}^{(j)} \quad , \quad \varepsilon, \varepsilon' = \pm \end{aligned} \quad (2.3)$$

and identical definitions are set for the variables $V^{(j)}, \overline{V}^{(j)}$; we shall say that $\overline{H}^{(j)}$, $H^{(j)}$, $\overline{V}^{(j)}$, $V^{(j)}$ satisfy ε–periodic (ε'–periodic) boundary conditions in the vertical (horizontal) direction.

2.2. By expanding in power series $\exp\{-\lambda V\}$, we see that the partition function of the model (1.1) is

$$
\begin{aligned}
\Xi^{AT}_{\Lambda_M} &= \sum_{\sigma^{(1)},\,\sigma^{(2)}} e^{-J^{(1)}H_I(\sigma^{(1)})} e^{-J^{(2)}H_I(\sigma^{(2)})} e^{-\lambda V(\sigma^{(1)},\,\sigma^{(2)})} \\
&= (\cosh\lambda)^{2M^2} \sum_{\sigma^{(1)},\,\sigma^{(2)}} e^{-J^{(1)}H_I(\sigma^{(1)})-J^{(2)}H_I(\sigma^{(2)})} \\
&\quad\cdot \prod_{\mathbf{x}\in\Lambda_M} \left(1+\hat\lambda \sigma^{(1)}_{\mathbf{x}}\sigma^{(1)}_{\mathbf{x}+\hat e_1}\sigma^{(2)}_{\mathbf{x}}\sigma^{(2)}_{\mathbf{x}+\hat e_1}\right)\left(1+\hat\lambda \sigma^{(1)}_{\mathbf{x}}\sigma^{(1)}_{\mathbf{x}+\hat e_0}\sigma^{(2)}_{\mathbf{x}}\sigma^{(2)}_{\mathbf{x}+\hat e_0}\right)\,, \qquad (2.4)
\end{aligned}
$$

where $\hat\lambda = \tanh\lambda$. The r.h.s. of (2.4) can be rewritten as:

$$
\Xi^{AT}_{\Lambda_M} = \left[\prod_{\substack{\mathbf{x}\in\Lambda_M\\ i=0,1}} \left(1+\hat\lambda \frac{\partial^2}{\partial J^{(1)}_{\mathbf{x},\mathbf{x}+\hat e_i}\,\partial J^{(2)}_{\mathbf{x},\mathbf{x}+\hat e_i}}\right)\right] \Xi^{(1)}_I(\{J^{(1)}_{\mathbf{x},\mathbf{x}'}\})\,\Xi^{(2)}_I(\{J^{(2)}_{\mathbf{x},\mathbf{x}'}\})\Big|_{\{J^{(j)}_{\mathbf{x},\mathbf{x}'}\}=\{J^{(j)}\}}, \qquad (2.5)
$$

where $\Xi^{(j)}_I(\{J^{(j)}_{\mathbf{x},\mathbf{x}'}\})$ is the partition function of an Ising model in which the couplings are allowed to depend on the bonds (the coupling associated to the n.n. bond $(\mathbf{x},\mathbf{x}')$ on the lattice j is called $J^{(j)}_{\mathbf{x},\mathbf{x}'}$). Using for $\Xi^{(1)}_I(\{J^{(1)}_{\mathbf{x},\mathbf{x}'}\})$ an expression similar to (2.1), we find that we can express Ξ_{AT} as a sum of sixteen partition functions labeled by $\gamma_1,\gamma_2 = (\varepsilon_1,\varepsilon'_1),(\varepsilon_2,\varepsilon'_2)$ (corresponding to choosing each ε_j and ε'_j as $\pm$):

$$
\Xi^{AT}_{\Lambda_M} = \frac{1}{4}(\cosh\lambda)^{2M^2}\sum_{\gamma_1,\gamma_2}(-1)^{\delta_{\gamma_1}+\delta_{\gamma_2}}\,\Xi^{\gamma_1,\gamma_2}_{AT}\,, \qquad (2.6)
$$

each of which is given by a functional integral

$$
\begin{aligned}
\Xi^{\gamma_1,\gamma_2}_{AT} &= \left[4(1+\hat\lambda t^{(1)}t^{(2)})\right]^{M^2}\prod_{j=1}^{2}(\cosh J^{(j)})^{M^2}(-1)^{M^2} \\
&\quad\cdot\int\prod_{\mathbf{x}\in\Lambda_M}^{j=1,2} dH^{(j)}_{\mathbf{x}}\,d\overline{H}^{(j)}_{\mathbf{x}}\,dV^{(j)}_{\mathbf{x}}\,d\overline{V}^{(j)}_{\mathbf{x}}\; e^{S^{(1)}_{\gamma_1}(t^{(1)}_\lambda)+S^{(2)}_{\gamma_2}(t^{(2)}_\lambda)+V_\lambda}\,, \qquad (2.7)
\end{aligned}
$$

where, if we define

$$
\lambda^{(j)} = \frac{\hat\lambda\left[t(1-t^2+u^2)+(-1)^j u(1+t^2-u^2)\right]}{1+\hat\lambda(t^2-u^2)}\,, \qquad (2.8)
$$

we have that $t^{(j)}_\lambda$, $j=1,2$, is given by $t^{(j)}_\lambda = t^{(j)}+\lambda^{(j)}$ and V_λ by:

$$
\begin{aligned}
V_\lambda &= \sum_{\mathbf{x}\in\Lambda_M}\tilde\lambda\left(\overline{H}^{(1)}_{\mathbf{x}}H^{(1)}_{\mathbf{x}+\hat e_1}\overline{H}^{(2)}_{\mathbf{x}}H^{(2)}_{\mathbf{x}+\hat e_1}+\overline{V}^{(1)}_{\mathbf{x}}V^{(1)}_{\mathbf{x}+\hat e_0}\overline{V}^{(2)}_{\mathbf{x}}V^{(2)}_{\mathbf{x}+\hat e_0}\right)\,, \\
\tilde\lambda &= \frac{\lambda^{(1)}\lambda^{(2)}}{\hat\lambda(t^2-u^2)}. \qquad (2.9)
\end{aligned}
$$

2.3. From now on, we shall study in detail only the partition function $\Xi^-_{AT} \stackrel{def}{=} \Xi^{(-,-),(-,-)}_{AT}$, *i.e.* the partition function in which all Grassmannian variables verify antiperiodic boundary conditions (see (2.3)). We shall see in §5.5 below that, if (λ, t, u) *does not belong* to the *critical surface*, which is a suitable 2–dimensional subset of $[-\varepsilon_1, \varepsilon_1] \times D \times [-\frac{|D|}{2}, \frac{|D|}{2}]$ which we will explicitly determine in §5.6, the partition function $\Xi^{\gamma_1,\gamma_2}_{AT}$ divided by $\Xi^{(1)\gamma_1}_I \Xi^{(2)\gamma_2}_I$ is exponentially insensitive to boundary conditions as $M \to \infty$.

As in [M1] we find it convenient to perform the following change of variables, $\alpha = \pm$, $\omega = \pm 1$:

$$\frac{1}{\sqrt{2}} \sum_{j=1,2} (-i\alpha)^{j-1} \big(\overline{H}^{(j)}_{\mathbf{x}} + i\omega H^{(j)}_{\mathbf{x}}\big) = e^{i\omega\pi/4} \big(\psi^\alpha_{\omega,\mathbf{x}} - \chi^\alpha_{\omega,\mathbf{x}}\big),$$
$$\frac{1}{\sqrt{2}} \sum_{j=1,2} (-i\alpha)^{j-1} \big(\overline{V}^{(j)}_{\mathbf{x}} + i\omega V^{(j)}_{\mathbf{x}}\big) = \psi^\alpha_{\omega,\mathbf{x}} + \chi^\alpha_{\omega,\mathbf{x}} \,. \tag{2.10}$$

Let $\mathbf{k} \in D_{-,-}$, where $D_{-,-}$ is the set of $\mathbf{k}$'s such that $k = 2\pi/M(n_1 + 1/2)$ and $k_0 = 2\pi/M(n_0 + 1/2)$, where $-[M/2] \le n_0, n_1 \le [(M-1)/2]$, $n_0, n_1 \in \mathbb{Z}$. The Fourier transform of the Grassmannian fields $\phi^\alpha_{\omega,\mathbf{x}}$, $\phi = \psi, \chi$, is given by $\hat\phi^\alpha_{\omega,\mathbf{k}} \stackrel{def}{=} \sum_{\mathbf{x}\in\Lambda_M} e^{-i\alpha\mathbf{k}\mathbf{x}} \phi^\alpha_{\omega,\mathbf{x}}$.

With the above definitions, it is straightforward algebra to verify that the final expression is:

$$\Xi^-_{AT} = e^{-EM^2} \int P(d\psi) P(d\chi) e^{Q(\psi,\chi)+V(\psi,\chi)} \,, \tag{2.11}$$

where E is a suitable constant; $Q(\psi, \chi)$ collects the quadratic terms of the form $\psi^{\alpha_1}_{\omega_1,\mathbf{x}_1} \chi^{\alpha_2}_{\omega_2,\mathbf{x}_2}$; $V(\psi, \chi)$ is the quartic interaction (it is equal to V_λ, see (2.9), in terms of the $\psi^\pm_\omega$, $\chi^\pm_\omega$ variables); $P(d\phi)$, $\phi = \psi, \chi$, is

$$P(d\phi) = \mathcal{N}_\phi^{-1} \prod_{\mathbf{k}\in D_{-,-}} \prod_{\omega=\pm1} d\phi^+_{\omega,\mathbf{k}} d\phi^-_{\omega,\mathbf{k}} \exp\Big\{-\frac{t_\lambda}{4M^2} \sum_{\mathbf{k}\in D_{-,-}} \Phi^{+,\mathbf{T}}_{\mathbf{k}} A_\phi(\mathbf{k}) \Phi_{\mathbf{k}}\Big\} \,,$$
$$A_\phi(\mathbf{k}) = \begin{pmatrix} i\sin k + \sin k_0 & -i\sigma_\phi(\mathbf{k}) & -\frac{\mu}{2}(i\sin k + \sin k_0) & i\mu(\mathbf{k}) \\ i\sigma_\phi(\mathbf{k}) & i\sin k - \sin k_0 & -i\mu(\mathbf{k}) & -\frac{\mu}{2}(i\sin k - \sin k_0) \\ -\frac{\mu}{2}(i\sin k + \sin k_0) & i\mu(\mathbf{k}) & i\sin k + \sin k_0 & -i\sigma_\phi(\mathbf{k}) \\ -i\mu(\mathbf{k}) & -\frac{\mu}{2}(i\sin k - \sin k_0) & i\sigma_\phi(\mathbf{k}) & i\sin k - \sin k_0 \end{pmatrix}, \tag{2.12}$$

where

$$\Phi^{+,\mathbf{T}}{}_{\mathbf{k}} = (\hat\phi^+_{1,\mathbf{k}}, \hat\phi^+_{-1,\mathbf{k}}, \hat\phi^-_{1,-\mathbf{k}}, \hat\phi^-_{-1,-\mathbf{k}}) \quad , \quad \Phi^{\mathbf{T}}{}_{\mathbf{k}} = (\hat\phi^-_{1,\mathbf{k}}, \hat\phi^-_{-1,\mathbf{k}}, \hat\phi^+_{1,-\mathbf{k}}, \hat\phi^+_{-1,-\mathbf{k}}), \tag{2.13}$$

$\mathcal{N}_\phi$ is chosen in such a way that $\int P(d\phi) = 1$ and, if we define $t_\lambda \stackrel{def}{=} (t^{(1)}_\lambda + t^{(2)}_\lambda)/2$, $u_\lambda \stackrel{def}{=} (t^{(1)}_\lambda - t^{(2)}_\lambda)/2$, for $\phi = \psi, \chi$ we have:

$$\sigma_\phi(\mathbf{k}) = 2\Big(1 + \frac{\pm\sqrt{2}+1}{t_\lambda}\Big) + \cos k_0 + \cos k - 2,$$
$$\mu(\mathbf{k}) = -(u_\lambda/t_\lambda)(\cos k + \cos k_0). \tag{2.14}$$

In the first of (2.14) the $-$ $(+)$ sign corresponds to $\phi = \psi$ $(\phi = \chi)$. The parameter μ in (2.12) is given by $\mu \stackrel{def}{=} \mu(\mathbf{0})$.

It is convenient to split the $\sqrt{2}-1$ appearing in the definition of $\sigma_\psi(\mathbf{k})$ as:

$$\sqrt{2}-1 = (\sqrt{2}-1+\frac{\nu}{2}) - \frac{\nu}{2} \stackrel{def}{=} t_\psi - \frac{\nu}{2} \,, \tag{2.15}$$

where ν is a parameter to be properly chosen later as a function of λ, in such a way that the average location of the critical points will be given by $t_\lambda = t_\psi$; in other words ν has the role of a *counterterm* fixing the middle point of the critical temperatures. The splitting (2.15) induces the following splitting of $P(d\psi)$:

$$P(d\psi) = P_\sigma(d\psi) e^{-\nu F_\nu(\psi)} \quad , \quad F_\nu(\psi) \stackrel{def}{=} \frac{1}{2M^2} \sum_{\mathbf{k},\omega} (-i\omega) \hat{\psi}^+_{\omega,\mathbf{k}} \hat{\psi}^-_{-\omega,\mathbf{k}} \,, \tag{2.16}$$

where $P_\sigma(d\psi)$ is given by (2.12) with $\phi = \psi$ and $\sigma \stackrel{def}{=} 2(1 - t_\psi/t_\lambda)$ replacing $\sigma_\psi(\mathbf{0})$.

2.4. Integration of the χ variables. The propagators $< \phi^\sigma_{\mathbf{x},\omega} \phi^{\sigma'}_{\mathbf{y},\omega'} >$ of the fermionic integration $P(d\phi)$ verify the following bound, for some $A, \kappa > 0$:

$$| < \phi^\sigma_{\mathbf{x},\omega} \phi^{\sigma'}_{\mathbf{y},\omega'} > | \le A e^{-\kappa \bar{m}_\phi |\mathbf{x}-\mathbf{y}|} \,, \tag{2.17}$$

where $\bar{m}_\phi$ is the minimum between $|m^{(1)}_\phi|$ and $|m^{(2)}_\phi|$ and, for $j = 1, 2$, $m^{(j)}_\phi$ is given by $m^{(j)}_\phi \stackrel{def}{=} 2(t^{(j)}_\lambda - t_\phi)/t_\lambda$, $j = 1, 2$. Note that both $m^{(1)}_\chi$ and $m^{(2)}_\chi$ are $O(1)$. This suggests to integrate first the χ variables.

After the integration of the χ variables we shall rewrite (2.11) as

$$\Xi^-_{AT} = e^{-M^2 E_1} \int P_{Z_1,\sigma_1,\mu_1,C_1}(d\psi) e^{-\mathcal{V}^{(1)}(\sqrt{Z_1}\psi)} \,, \qquad \mathcal{V}^{(1)}(0) = 0 \,, \tag{2.18}$$

where $C_1(\mathbf{k}) = 1$, $Z_1 = t_\psi$, $\sigma_1 = \sigma/(1-\frac{\sigma}{2})$, $\mu_1 = \mu/(1-\frac{\sigma}{2})$ and $P_{Z_1,\sigma_1,\mu_1,C_1}(d\psi)$ is the exponential of a quadratic form:

$$\begin{aligned}
P_{Z_1,\sigma_1,\mu_1,C_1}(d\psi) &= \mathcal{N}_1^{-1} \prod_{\mathbf{k}\in D_{-,-}}^{\omega=\pm1} d\psi^+_{\omega,\mathbf{k}} d\psi^-_{\omega,\mathbf{k}} \\
&\quad \times \exp\Big[-\frac{1}{4M^2} \sum_{\mathbf{k}\in D_{-,-}} Z_1 C_1(\mathbf{k}) \Psi^{+,T}_{\mathbf{k}} A^{(1)}_\psi(\mathbf{k}) \Psi_{\mathbf{k}}\Big] \,, \\
A^{(1)}_\psi(\mathbf{k}) &= \begin{pmatrix} M^{(1)}(\mathbf{k}) & N^{(1)}(\mathbf{k}) \\ N^{(1)}(\mathbf{k}) & M^{(1)}(\mathbf{k}) \end{pmatrix} , \\
M^{(1)}(\mathbf{k}) &= \begin{pmatrix} i\sin k + \sin k_0 + a^+_1(\mathbf{k}) & -i\,(\sigma_1 + c_1(\mathbf{k})) \\ i\,(\sigma_1 + c_1(\mathbf{k})) & i\sin k - \sin k_0 + a^-_1(\mathbf{k}) \end{pmatrix} , \\
N^{(1)}(\mathbf{k}) &= \begin{pmatrix} b^+_1(\mathbf{k}) & i\,(\mu_1 + d_1(\mathbf{k})) \\ -i\,(\mu_1 + d_1(\mathbf{k})) & b^-_1(\mathbf{k}) \end{pmatrix} ,
\end{aligned} \tag{2.19}$$

where $\mathcal{N}_1$ is chosen in such a way that $\int P_{Z_1,\sigma_1,\mu_1,C_1}(d\psi) = 1$. Moreover $\mathcal{V}^{(1)}$ is the *interaction*, which can be expressed as a sum of monomials in ψ of arbitrary order:

$$\mathcal{V}^{(1)}(\psi) = \sum_{n=1}^{\infty} \sum_{\substack{\mathbf{k}_1,\dots,\mathbf{k}_{2n}\\ \underline{\alpha},\underline{\omega}}} \prod_{i=1}^{2n} \hat{\psi}^{\alpha_i(\le 1)}_{\omega_i,\mathbf{k}_i} \widehat{W}^{(1)}_{2n,\underline{\alpha},\underline{\omega}}(\mathbf{k}_1,\dots,\mathbf{k}_{2n-1})\delta\Big(\sum_{i=1}^{2n}\alpha_i\mathbf{k}_i\Big) \tag{2.20}$$

and $\delta(\mathbf{k}) = \sum_{\mathbf{n}\in\mathbb{Z}^2}\delta_{\mathbf{k},2\pi\mathbf{n}}$. The constant E_1 in (2.18), the functions $a_1^{\pm}, b_1^{\pm}, c_1, d_1$ in (2.19) and the kernels $\widehat{W}^{(1)}_{2n,\underline{\alpha},\underline{\omega}}$ in (2.20) have the properties described in the following theorem, proved in Appendix A2. Note that from now on we will consider all functions appearing in the theory as functions of λ, σ_1, μ_1 (of course t and u can be analytically and elementarily expressed in terms of λ, σ_1, μ_1). We shall also assume $|\sigma_1|, |\mu_1|$ bounded by some $O(1)$ constant. Note that if $t \pm u$ belong to a sufficiently small interval D centered around $\sqrt{2}-1$, as assumed in the hypothesis of the Main Theorem in §1, then of course $|\sigma_1|, |\mu_1| \le c_1$ for a suitable constant c_1 (in particular, if D is chosen as in Remark (1) following the Main Theorem, we find $|\sigma_1| \le 1 + O(\varepsilon_1)$ and $|\mu_1| \le 2 + O(\varepsilon_1)$).

Theorem 2.1. *Assume that $|\sigma_1|, |\mu_1| \le c_1$ for some constant $c_1 > 0$. There exists a constant ε_1 such that, if $|\lambda|, |\nu| \le \varepsilon_1$, then Ξ^-_{AT} can be written as in (2.18), (2.19), (2.20), where:*

1) E_1 is an $O(1)$ constant;
2) $a_1^{\pm}(\mathbf{k}), b_1^{\pm}(\mathbf{k})$ are analytic odd functions of $\mathbf{k}$ and $c_1(\mathbf{k}), d_1(\mathbf{k})$ real analytic even functions of $\mathbf{k}$; in a neighborhood of $\mathbf{k} = \mathbf{0}$, $a_1^{\pm}(\mathbf{k}) = O(\sigma_1\mathbf{k}) + O(\mathbf{k}^3)$, $b_1^{\pm}(\mathbf{k}) = O(\mu_1\mathbf{k}) + O(\mathbf{k}^3)$, $c_1(\mathbf{k}) = O(\mathbf{k}^2)$ and $d_1(\mathbf{k}) = O(\mu_1\mathbf{k}^2)$;
3) the determinant $|\det A_\psi(\mathbf{k})|$ can be bounded above and below by some constant times $\big[(\sigma_1-\mu_1)^2 + |c(\mathbf{k})|\big]\big[(\sigma_1+\mu_1)^2 + |c(\mathbf{k})|\big]$ and $c(\mathbf{k}) = \cos k_0 + \cos k - 2$;
4) $\widehat{W}^{(1)}_{2n,\underline{\alpha},\underline{\omega}}$ are analytic functions of $\mathbf{k}_i, \lambda, \nu, \sigma_1, \mu_1$, $i = 1,\dots,2n$ and, for some constant C,

$$|\widehat{W}^{(1)}_{2n,\underline{\alpha},\underline{\omega}}(\mathbf{k}_1,\dots,\mathbf{k}_{2n-1})| \le M^2 C^n |\lambda|^{\max\{1,n/2\}}\ ; \tag{2.21}$$

4) –a) the terms in (2.21) with $n = 2$ can be written as

$$L_1 \sum_{\mathbf{k}_1,\dots,\mathbf{k}_4} \hat{\psi}^+_{1,\mathbf{k}_1}\hat{\psi}^+_{-1,\mathbf{k}_2}\hat{\psi}^-_{-1,\mathbf{k}_3}\hat{\psi}^-_{1,\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)$$
$$+ \sum_{\mathbf{k}_1,\dots,\mathbf{k}_4}\sum_{\underline{\alpha},\underline{\omega}} \widetilde{W}_{4,\underline{\alpha},\underline{\omega}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\hat{\psi}^{\alpha_1}_{\omega_1,\mathbf{k}_1}\hat{\psi}^{\alpha_2}_{\omega_2,\mathbf{k}_2}\hat{\psi}^{\alpha_3}_{\omega_3,\mathbf{k}_3}\hat{\psi}^{\alpha_4}_{\omega_4,\mathbf{k}_4}\delta\Big(\sum_{i=1}^{4}\alpha_i\mathbf{k}_i\Big)\ , \tag{2.22}$$

where L_1 is real and $\widetilde{W}_{4,\underline{\alpha},\underline{\omega}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ vanishes at $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3 = \big(\frac{\pi}{M}, \frac{\pi}{M}\big)$;
4) –b) the term in (2.21) with $n = 1$ can be written as:

$$\frac{1}{4}\sum_{\omega,\alpha=\pm}\sum_{\mathbf{k}}\Big[S_1(-i\omega)\hat{\psi}^+_{\omega,\mathbf{k}}\hat{\psi}^-_{-\omega,\mathbf{k}} + M_1(i\omega)\hat{\psi}^\alpha_{\omega,\mathbf{k}}\hat{\psi}^\alpha_{-\omega,-\mathbf{k}}$$
$$+F_1(i\sin k + \omega\sin k_0)\hat{\psi}^\alpha_{\omega,\mathbf{k}}\hat{\psi}^\alpha_{\omega,-\mathbf{k}}$$
$$+G_1(i\sin k + \omega\sin k_0)\hat{\psi}^+_{\omega,\mathbf{k}}\hat{\psi}^-_{\omega,\mathbf{k}}\Big]$$
$$+\sum_{\mathbf{k}}\sum_{\underline{\alpha},\underline{\omega}}\widetilde{W}_{2,\underline{\alpha},\underline{\omega}}(\mathbf{k})\hat{\psi}^{\alpha_1}_{\omega_1,\mathbf{k}}\hat{\psi}^{\alpha_2}_{\omega_2,-\alpha_1\alpha_2\mathbf{k}}, \tag{2.23}$$

where $\widetilde{W}_{2,\underline{\alpha},\underline{\omega}}(\mathbf{k})$ *is* $O(\mathbf{k}^2)$ *in a neighborhood of* $\mathbf{k}=\mathbf{0}$*;* S_1, M_1, F_1, G_1 *are real analytic functions of* $\lambda, \sigma_1, \mu_1, \nu$ *s.t.* $F_1 = O(\lambda\mu_1)$ *and*

$$L_1 = l_1 + O(\lambda\sigma_1) + O(\lambda\mu_1) \quad , \quad S_1 = s_1 + \gamma n_1 + O(\lambda\sigma_1^2) + O(\lambda\mu_1^2),$$
$$M_1 = m_1 + O(\lambda\mu_1\sigma_1) + O(\lambda\mu_1^3) \quad , \quad G_1 = z_1 + O(\lambda\sigma_1) + O(\lambda\mu_1)|, \tag{2.24}$$

with $s_1 = \sigma_1 f_1$*,* $m_1 = \mu_1 f_2$ *and* l_1, n_1, f_1, f_2, z_1 *independent of* σ_1, μ_1*; moreover* $l_1 = \tilde{\lambda}/Z_1^2 + O(\lambda^2)$*,* $f_1, f_2 = O(\lambda)$*,* $\gamma n_1 = \nu/Z_1 + c_1^\nu\lambda + O(\lambda^2)$*, for some* c_1^ν *independent of* λ*, and* $z_1 = O(\lambda^2)$*.*

Remark. The meaning of Theorem 2.1 is that after the integration of the χ fields we are left with a fermionic integration similar to (2.12) up to corrections which are at least $O(\mathbf{k}^2)$, and an effective interaction containing terms with any number of fields.

A priori many bilinear terms with kernel $O(1)$ or $O(\mathbf{k})$ with respect to $\mathbf{k}$ near $\mathbf{k}=\mathbf{0}$ could be generated by the χ–integration besides the ones originally present in (2.12); however *symmetry considerations restrict drastically the number of possible bilinear terms* $O(1)$ or $O(\mathbf{k})$. Only one new term of the form $\sum_{\mathbf{k}}(i\sin k + \omega\sin k_0)\hat\psi^\alpha_{\omega,\mathbf{k}}\hat\psi^\alpha_{\omega,-\mathbf{k}}$ appears, which is "dimensionally" *marginal* in a RG sense; however it is weighted by a constant $O(\lambda\mu_1)$ and this will improve its "dimension", so that it will result to be *irrelevant*, see §3.2 below.

3. Integration of the ψ Variables: First Regime

3.1. Multiscale analysis. From the bound on det $A_\psi^{(1)}(\mathbf{k})$ described in Theorem 2.1, we see that the ψ fields have a mass given by $\min\{|\sigma_1-\mu_1|, |\sigma_1+\mu_1|\}$, which can be arbitrarily small; their integration in the infrared region (small $\mathbf{k}$) needs a multiscale analysis. We introduce a *scaling parameter* $\gamma > 1$ which will be used to define a geometrically growing sequence of length scales $1, \gamma, \gamma^2, \ldots$, *i.e.* of geometrically decreasing momentum scales γ^h, $h = 0, -1, -2, \ldots$ Correspondingly we introduce C^∞ compact support functions $f_h(\mathbf{k})$ $h \le 1$, with the following properties: if $|\mathbf{k}| \stackrel{def}{=} \sqrt{\sin^2 k + \sin^2 k_0}$, when $h \le 0$, $f_h(\mathbf{k}) = 0$ for $|\mathbf{k}| < \gamma^{h-2}$ or $|\mathbf{k}| > \gamma^h$, and $f_h(\mathbf{k}) = 1$, if $|\mathbf{k}| = \gamma^{h-1}$; $f_1(\mathbf{k}) = 0$ for $|\mathbf{k}| \le \gamma^{-1}$ and $f_1(\mathbf{k}) = 1$ for $|\mathbf{k}| \ge 1$; furthermore:

$$1 = \sum_{h=h_M}^{1} f_h(\mathbf{k}) \quad , \quad \text{where :} \qquad h_M = \min\{h : \gamma^h > \sqrt{2}\sin\frac{\pi}{M}\}\ , \tag{3.1}$$

and $\sqrt{2}\sin(\pi/M)$ is the smallest momentum allowed by the antiperiodic boundary conditions, *i.e.* $\sqrt{2}\sin(\pi/M) = \min_{\mathbf{k}\in D_{-,-}} |\mathbf{k}|$.

The purpose is to perform the integration of (2.19) over the fermion fields in an iterative way. After each iteration we shall be left with a "simpler" Grassmannian integration to perform: if $h = 1, 0, -1, \ldots, h_M$, we shall write

$$\Xi^-_{AT} = \int P_{Z_h,\sigma_h,\mu_h,C_h}(d\psi^{(\le h)})\, e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) - M^2 E_h}\ , \quad \mathcal{V}^{(h)}(0) = 0\ , \tag{3.2}$$

where the quantities Z_h, σ_h, μ_h, C_h, $P_{Z_h,\sigma_h,\mu_h,C_h}(d\psi^{(\le h)})$, $\mathcal{V}^{(h)}$ and E_h have to be defined recursively and the result of the last iteration will be $\Xi^-_{AT} = e^{-M^2 E_{-1+h_M}}$, *i.e.* the value of the partition function.

$P_{Z_h,\sigma_h,\mu_h,C_h}(d\psi^{(\le h)})$ is defined by (2.19) in which we replace $Z_1, \sigma_1, \mu_1, a_1^\omega, b_1^\omega, c_1, d_1, C_1(\mathbf{k})$ with $Z_h, \sigma_h, \mu_h, a_h^\omega, b_h^\omega, c_h, d_h, C_h(\mathbf{k})$, where $C_h(\mathbf{k})^{-1} = \sum_{j=h_M}^h f_j(\mathbf{k})$. Moreover

$$\mathcal{V}^{(h)}(\psi) = \sum_{n=1}^\infty \frac{1}{M^{2n}} \sum_{\substack{\mathbf{k}_1,\dots,\mathbf{k}_{2n-1},\\ \underline{\alpha},\underline{\omega}}} \prod_{i=1}^{2n} \hat\psi^{\alpha_i(\le h)}_{\omega_i,\mathbf{k}_i} \widehat{W}^{(h)}_{2n,\underline{\alpha},\underline{\omega}}(\mathbf{k}_1,\dots,\mathbf{k}_{2n-1})\delta(\sum_{i=1}^{2n}\alpha_i\mathbf{k}_i) \stackrel{def}{=}$$

$$\stackrel{def}{=} \sum_{n=1}^\infty \sum_{\substack{\mathbf{x}_1,\dots,\mathbf{x}_{2n},\\ \underline{\sigma},\underline{j},\underline{\omega},\underline{\alpha}}} \prod_{i=1}^{2n} \partial_{j_i}^{\sigma_i} \psi^{\alpha_i(\le h)}_{\omega_i,\mathbf{x}_i} W^{(h)}_{2n,\underline{\sigma},\underline{j},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1,\dots,\mathbf{x}_{2n}) , \qquad (3.3)$$

where in the last line $j_i = 0, 1$, $\sigma_i \ge 0$ and ∂_j is the forward discrete derivative in the $\hat{e}_j$ direction.

Note that the field $\psi^{(\le h)}$, whose propagator is given by the inverse of $Z_h C_h(\mathbf{k}) A_\psi^{(h)}$, has the same support of $C_h^{-1}(\mathbf{k})$, that is on a strip of width γ^h around the singularity $\mathbf{k} = \mathbf{0}$. The field $\psi^{(\le 1)}$ coincides with the field ψ of the previous section, so that (2.18) is the same as (3.2) with $h = 1$.

It is crucial for the following to think $\widehat{W}^{(h)}_{2n,\underline{\alpha},\underline{\omega}}$, $h \le 1$, as functions of the variables $\sigma_k(\mathbf{k}), \mu_k(\mathbf{k})$, $k = h, h+1, \dots, 0, 1$, $\mathbf{k} \in D_{-,-}$. The iterative construction below will inductively imply that the dependence on these variables is well defined (note that for $h = 1$ we can think of the kernels of $\mathcal{V}^{(1)}$ as functions of σ_1, μ_1, see Theorem 2.1).

3.2. The localization operator. We now begin to describe the iterative construction leading to (3.2). The first step consists in defining a *localization* operator $\mathcal{L}$ acting on the kernels of $\mathcal{V}^{(h)}$, in terms of which we shall rewrite $\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$, where $\mathcal{R} = 1 - \mathcal{L}$. The iterative integration procedure will use such splitting, see §3.3 below.

$\mathcal{L}$ will be non-zero only if acting on a kernel $\widehat{W}^{(h)}_{2n,\underline{\alpha},\underline{\omega}}$ with $n = 1, 2$. In this case $\mathcal{L}$ will be the combination of four different operators: $\mathcal{L}_j$, $j = 0, 1$, whose effect on a function of $\mathbf{k}$ will be essentially to extract the term of order j from its Taylor series in $\mathbf{k}$; and $\mathcal{P}_j$, $j = 0, 1$, whose effect on a functional of the sequence $\sigma_h(\mathbf{k}), \mu_h(\mathbf{k}), \dots, \sigma_1, \mu_1$ will be essentially to extract the term of order j from its power series in $\sigma_h(\mathbf{k}), \mu_h(\mathbf{k}), \dots, \sigma_1, \mu_1$.

The action of $\mathcal{L}_j$, $j = 0, 1$, on the kernels $\widehat{W}^{(h)}_{2n,\underline{\alpha},\underline{\omega}}(\mathbf{k}_1,\dots,\mathbf{k}_{2n})$ is defined as follows:

1) If $n = 1$,

$$\mathcal{L}_0 \widehat{W}^{(h)}_{2,\underline{\alpha},\underline{\omega}}(\mathbf{k}, \alpha_1\alpha_2\mathbf{k}) = \frac{1}{4}\sum_{\eta,\eta'=\pm1} \widehat{W}^{(h)}_{2,\underline{\alpha},\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'}, \alpha_1\alpha_2\bar{\mathbf{k}}_{\eta\eta'}),$$

$$\mathcal{L}_1 \widehat{W}^{(h)}_{2,\underline{\alpha},\underline{\omega}}(\mathbf{k}, \alpha_1\alpha_2\mathbf{k}) = \frac{1}{4}\sum_{\eta,\eta'=\pm1} \widehat{W}^{(h)}_{2,\underline{\alpha},\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'}, \alpha_1\alpha_2\bar{\mathbf{k}}_{\eta\eta'})\big[\eta\frac{\sin k}{\sin\frac{\pi}{M}} + \eta'\frac{\sin k_0}{\sin\frac{\pi}{M}}\big] , \qquad (3.4)$$

where $\bar{\mathbf{k}}_{\eta\eta'} = \left(\eta\frac{\pi}{M}, \eta'\frac{\pi}{M}\right)$ are the smallest momenta allowed by the antiperiodic boundary conditions.

2) If $n = 2$, $\mathcal{L}_1 \widehat{W}^{(h)}_{4,\underline{\alpha},\underline{\omega}} = 0$ and

$$\mathcal{L}_0 \widehat{W}^{(h)}_{4,\underline{\alpha},\underline{\omega}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) \stackrel{def}{=} \widehat{W}^{(h)}_{4,\underline{\alpha},\underline{\omega}}(\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++}) . \qquad (3.5)$$

3) If $n > 2$, $\mathcal{L}_0 \widehat{W}_{2n,\underline{\alpha},\underline{\omega}} = \mathcal{L}_1 \widehat{W}_{2n,\underline{\alpha},\underline{\omega}} = 0$.

The action of $\mathcal{P}_j$, $j = 0, 1$, on the kernels $\widehat{W}_{2n,\underline{\alpha},\underline{\omega}}$, thought of as functionals of the sequence $\sigma_h(\mathbf{k}), \mu_h(\mathbf{k}), \ldots, \sigma_1, \mu_1$ is defined as follows:

$$\mathcal{P}_0 \widehat{W}_{2n,\underline{\alpha},\underline{\omega}} \stackrel{def}{=} \widehat{W}_{2n,\underline{\alpha},\underline{\omega}}\Big|_{\underline{\sigma}^{(h)}=\underline{\mu}^{(h)}=0},$$

$$\mathcal{P}_1 \widehat{W}_{2n,\underline{\alpha},\underline{\omega}} \stackrel{def}{=} \sum_{k\ge h,\mathbf{k}} \left[\sigma_k(\mathbf{k}) \frac{\partial \widehat{W}_{2n,\underline{\alpha},\underline{\omega}}}{\partial \sigma_k(\mathbf{k})}\Big|_{\underline{\sigma}^{(h)}=\underline{\mu}^{(h)}=0} + \mu_k(\mathbf{k}) \frac{\partial \widehat{W}_{2n,\underline{\alpha},\underline{\omega}}}{\partial \mu_k(\mathbf{k})}\Big|_{\underline{\sigma}^{(h)}=\underline{\mu}^{(h)}=0}\right]. \tag{3.6}$$

Given $\mathcal{L}_j, \mathcal{P}_j$, $j = 0, 1$ as above, we define the action of $\mathcal{L}$ on the kernels $\widehat{W}_{2n,\underline{\alpha},\underline{\omega}}$ as follows:

1) If $n = 1$, then

$$\mathcal{L}\widehat{W}_{2,\underline{\alpha},\underline{\omega}} \stackrel{def}{=} \begin{cases} \mathcal{L}_0(\mathcal{P}_0 + \mathcal{P}_1)\widehat{W}_{2,\underline{\alpha},\underline{\omega}} & \text{if } \omega_1 + \omega_2 = 0 \text{ and } \alpha_1 + \alpha_2 = 0, \\ \mathcal{L}_0\mathcal{P}_1 \widehat{W}_{2,\underline{\alpha},\underline{\omega}} & \text{if } \omega_1 + \omega_2 = 0 \text{ and } \alpha_1 + \alpha_2 \neq 0, \\ \mathcal{L}_1\mathcal{P}_0 \widehat{W}_{2,\underline{\alpha},\underline{\omega}} & \text{if } \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 = 0, \\ 0 & \text{if } \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 \neq 0. \end{cases}$$

2) If $n = 2$, then $\mathcal{L}\widehat{W}_{4,\underline{\alpha},\underline{\omega}} \stackrel{def}{=} \mathcal{L}_0\mathcal{P}_0 \widehat{W}_{4,\underline{\alpha},\underline{\omega}}$.
3) If $n > 2$, then $\mathcal{L}\widehat{W}_{2n,\underline{\alpha},\underline{\omega}} = 0$.

Finally, the effect of $\mathcal{L}$ on $\mathcal{V}^{(h)}$ is, by definition, to replace on the r.h.s. of (3.3) $\widehat{W}_{2n,\underline{\alpha},\underline{\omega}}$ with $\mathcal{L}\widehat{W}_{2n,\underline{\alpha},\underline{\omega}}$. Note that $\mathcal{L}^2\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)}$.

Using the previous definitions we get the following result, proven in Appendix A2.2. We use the notation $\underline{\sigma}^{(h)} = \{\sigma_k(\mathbf{k})\}_{\mathbf{k}\in D_{-,-}}^{k=h,\ldots,1}$ and $\underline{\mu}^{(h)} = \{\mu_k(\mathbf{k})\}_{\mathbf{k}\in D_{-,-}}^{k=h,\ldots,1}$.

Lemma 3.1. *Let the action of $\mathcal{L}$ on $\mathcal{V}^{(h)}$ be defined as above. Then*

$$\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h)}) = (s_h + \gamma^h n_h)F_\sigma^{(\le h)} + m_h F_\mu^{(\le h)} + l_h F_\lambda^{(\le h)} + z_h F_\zeta^{(\le h)}\ , \tag{3.7}$$

where s_h, n_h, m_h, l_h and z_h are real constants and s_h is linear in $\underline{\sigma}^{(h)}$ and independent of $\underline{\mu}^{(h)}$; m_h is linear in $\underline{\mu}^{(h)}$ and independent of $\underline{\sigma}^{(h)}$; n_h, l_h, z_h are independent of $\underline{\sigma}^{(h)}, \underline{\mu}^{(h)}$; moreover, if $D_h \stackrel{def}{=} D_{-,-} \cap \{\mathbf{k} : C_h^{-1}(\mathbf{k}) > 0\}$,

$$F_\sigma^{(\le h)}(\psi^{(\le h)}) = \frac{1}{2M^2}\sum_{\mathbf{k}\in D_h}\sum_{\omega=\pm1}(-i\omega)\widehat{\psi}^{+(\le h)}_{\omega,\mathbf{k}}\widehat{\psi}^{-(\le h)}_{-\omega,\mathbf{k}} \stackrel{def}{=} \frac{1}{M^2}\sum_{\mathbf{k}\in D_h}\widehat{F}_\sigma^{(\le h)}(\mathbf{k})\ ,$$

$$F_\mu^{(\le h)}(\psi^{(\le h)}) = \frac{1}{4M^2}\sum_{\mathbf{k}\in D_h}\sum_{\alpha,\omega=\pm1} i\omega\widehat{\psi}^{\alpha(\le h)}_{\omega,\mathbf{k}}\widehat{\psi}^{\alpha(\le h)}_{-\omega,-\mathbf{k}} \stackrel{def}{=} \frac{1}{M^2}\sum_{\mathbf{k}\in D_h}\widehat{F}_\mu^{(\le h)}(\mathbf{k})\ ,$$

$$F_\lambda^{(\le h)}(\psi^{(\le h)}) = \frac{1}{M^8}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_4\in D_h}\widehat{\psi}^{+(\le h)}_{1,\mathbf{k}_1}\widehat{\psi}^{+(\le h)}_{-1,\mathbf{k}_2}\widehat{\psi}^{-(\le h)}_{-1,\mathbf{k}_3}\widehat{\psi}^{-(\le h)}_{1,\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\ ,$$

$$F_\zeta^{(\le h)}(\psi^{(\le h)}) = \frac{1}{2M^2}\sum_{\mathbf{k}\in D_h}\sum_{\omega=\pm1}(i\sin k + \omega\sin k_0) \times\widehat{\psi}^{+(\le h)}_{\omega,\mathbf{k}}\widehat{\psi}^{-(\le h)}_{\omega,\mathbf{k}} \stackrel{def}{=} \frac{1}{M^2}\sum_{\mathbf{k}\in D_h}\widehat{F}_\zeta^{(\le h)}(\mathbf{k}), \tag{3.8}$$

where $\delta(\mathbf{k}) = M^2\sum_{\mathbf{n}\in\mathbb{Z}^2}\delta_{\mathbf{k},2\pi\mathbf{n}}$.

Remark. The application of $\mathcal{L}$ to the kernels of the effective potential generates the sum in (3.7), *i.e.* a linear combination of the Grassmannian monomials in (3.8) which, in the renormalization group language, are called "*relevant*" (the first two) or "*marginal*" operators (the two others).

We now consider the operator $\mathcal{R}\stackrel{def}{=}1-\mathcal{L}$. The following result holds, see Appendix A2 for the proof. We use the notation $\mathcal{R}_1 = 1-\mathcal{L}_0$, $\mathcal{R}_2 = 1-\mathcal{L}_0-\mathcal{L}_1$, $\mathcal{S}_1 = 1-\mathcal{P}_0$, $\mathcal{S}_2 = 1-\mathcal{P}_0-\mathcal{P}_1$.

Lemma 3.2. *The action of* $\mathcal{R}$ *on* $\widehat{W}_{2n,\underline{\alpha},\underline{\omega}}$ *for* $n=1,2$ *is the following:*

1) If $n=1$*, then*

$$\mathcal{R}\widehat{W}_{2,\underline{\alpha},\underline{\omega}} = \begin{cases} [\mathcal{S}_2+\mathcal{R}_2(\mathcal{P}_0+\mathcal{P}_1)]\widehat{W}_{2,\underline{\alpha},\underline{\omega}} & if\, \omega_1+\omega_2=0,\\ [\mathcal{R}_1\mathcal{S}_1+\mathcal{R}_2\mathcal{P}_0]\widehat{W}_{2,\underline{\alpha},\underline{\omega}} & if\, \omega_1+\omega_2\neq 0 \; and\; \alpha_1+\alpha_2=0,\\ \mathcal{R}_1\mathcal{S}_1\widehat{W}_{2,\underline{\alpha},\underline{\omega}} & if\, \omega_1+\omega_2\neq 0 \; and\; \alpha_1+\alpha_2\neq 0.\end{cases}$$

2) If $n=2$*, then* $\mathcal{R}\widehat{W}_{4,\underline{\alpha},\underline{\omega}} = [\mathcal{S}_1+\mathcal{R}_1\mathcal{P}_0]\widehat{W}_{4,\underline{\alpha},\underline{\omega}}$.

Remark. The effect of $\mathcal{R}_j$, $j=1,2$ on $\widehat{W}^{(h)}_{2n,\underline{\alpha},\underline{\omega}}$ consists in extracting the rest of a Taylor series in $\mathbf{k}$ of order j. The effect of $\mathcal{S}_j$, $j=1,2$ on $\widehat{W}^{(h)}_{2n,\underline{\alpha},\underline{\omega}}$ consists in extracting the rest of a power series in $(\underline{\sigma}^{(h)},\underline{\mu}^{(h)})$ of order j. The definitions are given in such a way that $\mathcal{R}\widehat{W}_{2n,\underline{\alpha},\underline{\omega}}$ is at least quadratic in $\mathbf{k}$, $\underline{\sigma}^{(h)}$, $\underline{\mu}^{(h)}$ if $n=1$ and at least linear in $\mathbf{k}$, $\underline{\sigma}^{(h)}$, $\underline{\mu}^{(h)}$ when $n=2$. This will give dimensional gain factors in the bounds for $\mathcal{R}\widehat{W}^{(h)}_{2n,\underline{\alpha},\underline{\omega}}$ w.r.t. the bounds for $\widehat{W}^{(h)}_{2n,\underline{\alpha},\underline{\omega}}$, $n=1,2$, as we shall see in detail in Appendix A4.

3.3. Renormalization. Once the above definitions are given we can describe our integration procedure for $h\le 0$.

We start from (3.2) and we rewrite it as

$$\int P_{Z_h,\sigma_h,\mu_h,C_h}(d\psi^{(\le h)})\, e^{-\mathcal{L}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})-\mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})-M^2E_h}\ , \tag{3.9}$$

with $\mathcal{L}\mathcal{V}^{(h)}$ as in (3.7). Then we include the quadratic part of $\mathcal{L}\mathcal{V}^{(h)}$ (except the term proportional to n_h) in the fermionic integration, so obtaining

$$\int P_{\widehat{Z}_{h-1},\sigma_{h-1},\mu_{h-1},C_h}(d\psi^{(\le h)}) \\ \times e^{-l_hF_\lambda(\sqrt{Z_h}\psi^{(\le h)})-\gamma^h n_h F_\sigma(\sqrt{Z_h}\psi^{(\le h)})-\mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})-M^2E_h}\ , \tag{3.10}$$

where $\widehat{Z}_{h-1}(\mathbf{k})\stackrel{def}{=}Z_h(1+z_hC_h^{-1}(\mathbf{k}))$ and

$$\sigma_{h-1}(\mathbf{k})\stackrel{def}{=}\frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})}(\sigma_h(\mathbf{k})+s_hC_h^{-1}(\mathbf{k}))\quad ,\quad \mu_{h-1}(\mathbf{k})\stackrel{def}{=}\frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})}(\mu_h(\mathbf{k})+m_hC_h^{-1}(\mathbf{k})),$$
$$a^\omega_{h-1}(\mathbf{k})\stackrel{def}{=}\frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})}a^\omega_h(\mathbf{k})\quad ,\quad b^\omega_{h-1}(\mathbf{k})\stackrel{def}{=}\frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})}b^\omega_h(\mathbf{k}),$$
$$c_{h-1}(\mathbf{k})\stackrel{def}{=}\frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})}c_h(\mathbf{k})\quad ,\quad d_{h-1}(\mathbf{k})\stackrel{def}{=}\frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})}d_h(\mathbf{k})\ . \tag{3.11}$$

The integration in (3.10) differs from the one in (3.2) and (3.9): $P_{\widehat{Z}_{h-1},\sigma_{h-1},\mu_{h-1},C_h}$ is defined by (2.19) with Z_1 and $A_\psi^{(1)}$ replaced by $\widehat{Z}_{h-1}(\mathbf{k})$ and $A_\psi^{(h-1)}$.

Now we can perform the integration of the $\psi^{(h)}$ field. It is convenient to rescale the fields:

$$\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}) \stackrel{def}{=} \lambda_h F_\lambda(\sqrt{Z_{h-1}}\psi^{(\le h)}) \\ +\gamma^h \nu_h F_\sigma(\sqrt{Z_{h-1}}\psi^{(\le h)}) + \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) , \tag{3.12}$$

where $\lambda_h = \left(\frac{Z_h}{Z_{h-1}}\right)^2 l_h$, $\nu_h = \frac{Z_h}{Z_{h-1}} n_h$ and $\mathcal{R}\mathcal{V}^{(h)} = (1-\mathcal{L})\mathcal{V}^{(h)}$ is the irrelevant part of $\mathcal{V}^{(h)}$, and rewrite (3.10) as

$$e^{-M^2(t_h+E_h)} \int P_{Z_{h-1},\sigma_{h-1},\mu_{h-1},C_{h-1}}(d\psi^{(\le h-1)}) \\ \times \int P_{Z_{h-1},\sigma_{h-1},\mu_{h-1},\widetilde{f}_h^{-1}}(d\psi^{(h)})\, e^{-\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})}, \tag{3.13}$$

where we used the decomposition $\psi^{(\le h)} = \psi^{(\le h-1)} + \psi^{(h)}$ (and $\psi^{(\le h-1)}$, $\psi^{(h)}$ are independent) and $\widetilde{f}_h(\mathbf{k})$ is defined by the relation $C_h^{-1}(\mathbf{k})\widehat{Z}_{h-1}^{-1}(\mathbf{k}) = C_{h-1}^{-1}(\mathbf{k})Z_{h-1}^{-1} + \widetilde{f}_h(\mathbf{k})Z_{h-1}^{-1}$, namely:

$$\widetilde{f}_h(\mathbf{k}) \stackrel{def}{=} Z_{h-1}\Big[\frac{C_h^{-1}(\mathbf{k})}{\widehat{Z}_{h-1}(\mathbf{k})} - \frac{C_{h-1}^{-1}(\mathbf{k})}{Z_{h-1}}\Big] = f_h(\mathbf{k})\Big[1 + \frac{z_h f_{h+1}(\mathbf{k})}{1+z_h f_h(\mathbf{k})}\Big] . \tag{3.14}$$

Note that $\widetilde{f}_h(\mathbf{k})$ has the same support as $f_h(\mathbf{k})$. Moreover $P_{Z_{h-1},\sigma_{h-1},\mu_{h-1},\widetilde{f}_h^{-1}}(d\psi^{(h)})$ is defined in the same way as $P_{\widehat{Z}_{h-1},\sigma_{h-1},\mu_{h-1},C_h}(d\psi^{(h)})$, with $\widehat{Z}_{h-1}(\mathbf{k})$ resp. C_h replaced by Z_{h-1}, resp. $\widetilde{f}_h^{-1}$. The *single scale* propagator is

$$\int P_{Z_{h-1},\sigma_{h-1},\mu_{h-1},\widetilde{f}_h^{-1}}(d\psi^{(h)})\, \psi^{\alpha(h)}_{\mathbf{x},\omega} \psi^{\alpha'(h)}_{\mathbf{y},\omega'} \\ = \frac{1}{Z_{h-1}} g^{(h)}_{\underline{a},\underline{a}'}(\mathbf{x}-\mathbf{y}) \quad , \quad \underline{a} = (\alpha,\omega) \quad , \quad \underline{a}' = (\alpha',\omega') , \tag{3.15}$$

where

$$g^{(h)}_{\underline{a},\underline{a}'}(\mathbf{x}-\mathbf{y}) = \frac{1}{2M^2}\sum_{\mathbf{k}} e^{i\alpha\alpha'\mathbf{k}(\mathbf{x}-\mathbf{y})}\, \widetilde{f}_h(\mathbf{k})[A_\psi^{(h-1)}(\mathbf{k})]^{-1}_{j(\underline{a}),j'(\underline{a}')} \tag{3.16}$$

with $j(-,1) = j'(+,1) = 1$, $j(-,-1) = j'(+,-1) = 2$, $j(+,1) = j'(-,1) = 3$ and $j(+,-1) = j'(-,-1) = 4$. One finds that $g^{(h)}_{\underline{a},\underline{a}'}(\mathbf{x}) = g^{(1,h)}_{\omega,\omega'}(\mathbf{x}) - \alpha\alpha' g^{(2,h)}_{\omega,\omega'}(\mathbf{x})$, where $g^{(j,h)}_{\omega,\omega'}(\mathbf{x})$, $j = 1, 2$ are defined in Appendix A3, see (A3.1).

The long distance behaviour of the propagator is given by the following lemma, proved in Appendix A3.

Lemma 3.3. *Let* $\sigma_h \stackrel{def}{=} \sigma_h(\mathbf{0})$ *and* $\mu_h \stackrel{def}{=} \mu_h(\mathbf{0})$ *and assume* $|\lambda| \le \varepsilon_1$ *for a small constant* ε_1. *Suppose that for* $h > \bar{h}$,

$$|z_h| \le \frac{1}{2} \quad , \quad |s_h| \le \frac{1}{2}|\sigma_h| \quad , \quad |m_h| \le \frac{1}{2}|\mu_h| , \tag{3.17}$$

that there exists c s.t.

$$e^{-c|\lambda|} \leq \left|\frac{\sigma_h}{\sigma_{h-1}}\right| \leq e^{c|\lambda|} \quad , \quad e^{-c|\lambda|} \leq \left|\frac{\mu_h}{\mu_{h-1}}\right| \leq e^{c|\lambda|} \quad ,$$
$$e^{-c|\lambda|^2} \leq \left|\frac{Z_h}{Z_{h-1}}\right| \leq e^{c|\lambda|^2} , \tag{3.18}$$

and that, for some constant C_1,

$$\frac{|\sigma_{\bar{h}}|}{\gamma^{\bar{h}}} \leq C_1 \quad , \quad \frac{|\mu_{\bar{h}}|}{\gamma^{\bar{h}}} \leq C_1 \; ; \tag{3.19}$$

then, for all $h \geq \bar{h}$, *given the positive integers* N, n_0, n_1 *and putting* $n = n_0 + n_1$, *there exists a constant* $C_{N,n}$ *s.t.*

$$|\partial_{x_0}^{n_0}\partial_x^{n_1} g^{(h)}_{\underline{a},\underline{a}'}(\mathbf{x}-\mathbf{y})| \leq C_{N,n}\frac{\gamma^{(1+n)h}}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N} \quad , \quad \textit{where} \quad \mathbf{d}(\mathbf{x})$$
$$= \frac{M}{\pi}\big(\sin\frac{\pi x}{M}, \sin\frac{\pi x_0}{M}\big) \; . \tag{3.20}$$

Furthermore, if $\mathcal{P}_0$, $\mathcal{P}_1$ *are defined as in (3.6) and* $\mathcal{S}_1$, $\mathcal{S}_2$ *are defined as in Lemma 3.2, we have that* $\mathcal{P}_j g^{(h)}_{\underline{a},\underline{a}'}$, $j = 0, 1$ *and* $\mathcal{S}_j g^{(h)}_{\underline{a},\underline{a}'}$, $j = 1, 2$, *satisfy the same bound (3.20), times a factor* $\big(\frac{|\sigma_h|+|\mu_h|}{\gamma^h}\big)^j$. *The bounds for* $\mathcal{P}_0 g^{(h)}_{\underline{a},\underline{a}'}$ *and* $\mathcal{P}_1 g^{(h)}_{\underline{a},\underline{a}'}$ *hold even without hypothesis (3.19).*

After the integration of the field on scale h we are left with an integral involving the fields $\psi^{(\leq h-1)}$ and the new effective interaction $\mathcal{V}^{(h-1)}$, defined as

$$e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)})-\tilde{E}_h M^2} = \int P_{Z_{h-1},\sigma_{h-1},\mu_{h-1},\widetilde{f}_h}(d\psi^{(h)})e^{-\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)})} . \tag{3.21}$$

It is easy to see that $\mathcal{V}^{(h-1)}$ is of the form (3.3) and that $E_{h-1} = E_h + t_h + \tilde{E}_h$. It is sufficient to use the well known identity

$$M^2\tilde{E}_h + \mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)}) = \sum_{n\geq 1}\frac{1}{n!}(-1)^{n+1}\mathcal{E}_h^T(\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}); n), \tag{3.22}$$

where $\mathcal{E}_h^T(X(\psi^{(h)}); n)$ is the truncated expectation of order n w.r.t. the propagator $Z_{h-1}^{-1}g^{(h)}_{\underline{a},\underline{a}'}$, defined as

$$\mathcal{E}_h^T(X(\psi^{(h)}); n) = \frac{\partial}{\partial\lambda^n}\log\int P_{Z_{h-1},\sigma_{h-1},\mu_{h-1},\widetilde{f}_h}(d\psi^{(h)})e^{\lambda X(\psi^{(h)})}\Big|_{\lambda=0} . \tag{3.23}$$

Note that the above procedure allows us to write the *running coupling constants* $\vec{v}_{h-1} = (\lambda_{h-1}, \nu_{h-1})$, $h \leq 1$, in terms of $\vec{v}_k$, $h \leq k \leq 1$, namely $\vec{v}_{h-1} = \beta_h(\vec{v}_h, \ldots, \vec{v}_1)$, where β_h is the so–called *Beta function*.

3.4. Analticity of the effective potential. We have expressed the effective potential $\mathcal{V}^{(h)}$ in terms of the *running coupling constants* $\lambda_k, \nu_k, k \geq h$, and of the *renormalization constants* $Z_k, \mu_k(\mathbf{k}), \sigma_k(\mathbf{k}), k \geq h$.

In Appendix A4 we will prove the following result.

Theorem 3.4. *Let* $\sigma_h \stackrel{def}{=} \sigma_h(\mathbf{0})$ *and* $\mu_h \stackrel{def}{=} \mu_h(\mathbf{0})$ *and assume* $|\lambda| \leq \varepsilon_1$ *for a small constant* ε_1. *Suppose that for* $h > \bar{h}$ *the hypothesis (3.17), (3.18) and (3.19) hold. If, for some constant c,*

$$\max_{h>\bar{h}}\{|\lambda_h|, |\nu_h|\} \leq c|\lambda| \; , \tag{3.24}$$

then there exists $C > 0$ *s.t. the kernels in (3.3) satisfy*

$$\int d\mathbf{x}_1 \cdots d\mathbf{x}_{2n} |W^{(\bar{h})}_{2n,\underline{\sigma},\underline{j},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1, \dots, \mathbf{x}_{2n})| \leq M^2 \gamma^{-\bar{h} D_k(n)} \, (C\,|\lambda|)^{max(1,n-1)}, \tag{3.25}$$

where $D_k(n) = -2 + n + k$ *and* $k = \sum_{i=1}^{2n} \sigma_i$.

Moreover $|\tilde{E}_{\bar{h}+1}| + |t_{\bar{h}+1}| \leq c|\lambda|\gamma^{2\bar{h}}$ *and the kernels of* $\mathcal{L}\mathcal{V}^{(\bar{h})}$ *satisfy*

$$|s_{\bar{h}}| \leq C|\lambda||\sigma_{\bar{h}}| \quad , \quad |m_{\bar{h}}| \leq C|\lambda||\mu_{\bar{h}}| \tag{3.26}$$

and

$$|n_{\bar{h}}| \leq C|\lambda| \quad , \quad |z_{\bar{h}}| \leq C|\lambda|^2 \quad , \quad |l_{\bar{h}}| \leq C|\lambda|^2 \; . \tag{3.27}$$

The bounds (3.26) hold even if (3.19) does not hold. The bounds (3.27) hold even if (3.19) and the first two of (3.18) do not hold.

Remarks. 1) The above result immediately implies analyticity of the effective potential of scale h in the running coupling constants $\lambda_k, \nu_k, k \geq h$, under the assumptions (3.17), (3.18), (3.19) and (3.24).

2) The assumptions (3.18) and (3.24) will be proved in §4 and Appendix A5 below, solving the *flow equations* for $\vec{v}_h = (\lambda_h, \nu_h)$ and Z_h, σ_h, μ_h, given by $\vec{v}_{h-1} = \beta_h(\vec{v}_h, \dots, \vec{v}_1)$, $Z_{h-1} = Z_h(1 + z_h)$ and (3.11). They will be proved to be true up to $h = -\infty$.

4. The Flow of the Running Coupling Constants

The convergence of the expansion for the effective potential is proved by Theorem 3.1 under the hypothesis that the running coupling constants are small, see (3.24), and that the bounds (3.17), (3.18) and (3.19) are satisfied. We now want to show that, choosing λ small enough and ν as a suitable function of λ, such hypotheses are indeed verified. In order to prove this, we will solve the flow equations for the renormalization constants (following from (3.11) and the preceding line):

$$\frac{Z_{h-1}}{Z_h} = 1 + z_h \quad , \quad \frac{\sigma_{h-1}}{\sigma_h} = 1 + \frac{s_h/\sigma_h - z_h}{1 + z_h} \quad , \quad \frac{\mu_{h-1}}{\mu_h} = 1 + \frac{m_h/\mu_h - z_h}{1 + z_h}, \tag{4.1}$$

together with those for the running coupling constants:

$$\begin{aligned} \lambda_{h-1} &= \lambda_h + \beta^h_\lambda(\lambda_h, \nu_h; \dots; \lambda_1, \nu_1), \\ \nu_{h-1} &= \gamma \nu_h + \beta^h_\nu(\lambda_h, \nu_h; \dots; \lambda_1, \nu_1) \; . \end{aligned} \tag{4.2}$$

The functions $\beta^h_\lambda, \beta^h_\nu$ are called the λ and ν components of the Beta function, see the comment after (3.23), and, by construction, are *independent* of σ_k, μ_k, so that their convergence follow just from (3.24) and the last of (3.18), *i.e.* without assuming (3.19), see Theorem 3.1. While for a general kernel we will apply Theorem 3.1 just up to a finite scale h^*_1 (in order to insure the validity of (3.19) with $\bar{h} = h^*_1$), we will inductively study the flow generated by (4.2) up to scale $-\infty$, and we shall prove that it is bounded for all scales. The main result on the flows of λ_h and ν_h, proven in Appendix A5, is the following.

Theorem 4.1. *If λ is small enough, there exists an analytic function $\nu^*(\lambda)$ independent of t, u such that the running coupling constants $\{\lambda_h, \nu_h\}_{h\le 1}$ with $\nu_1 = \nu^*(\lambda)$ verify $|\nu_h| \le c|\lambda|\gamma^{(\vartheta/2)h}$ and $|\lambda_h| \le c|\lambda|$. Moreover the kernels z_h, s_h and m_h satisfy (3.17) and the solutions of the flow equations (4.1) satisfy (3.18).*

Once ν_1 is conveniently chosen as in Theorem 4.1, one can study in more detail the flows of the renormalization constants. In Appendix A5 we prove the following.

Lemma 4.2. *If λ is small enough and ν_1 is chosen as in Theorem 4.1, the solution of (4.1) can be written as:*

$$Z_h = \gamma^{\eta_z(h-1)+F^h_\zeta} \quad , \quad \mu_h = \mu_1\gamma^{\eta_\mu(h-1)+F^h_\mu} \quad , \quad \sigma_h = \sigma_1\gamma^{\eta_\sigma(h-1)+F^h_\sigma}, \quad (4.3)$$

where η_z, η_μ, η_z and $F^h_\zeta, F^h_\mu, F^h_\sigma$ are $O(\lambda)$ functions, independent of σ_1, μ_1.

Moreover $\eta_\sigma - \eta_\mu = -b\lambda + O(|\lambda|^2), b > 0$.

4.1. The scale h^*_1. The integration described in §3 is iterated up to a scale h^*_1 defined in the following way:

$$h^*_1 \stackrel{def}{=} \begin{cases} \min\left\{1, \left[\log_\gamma |\sigma_1|^{\frac{1}{1-\eta_\sigma}}\right]\right\} & if\, |\sigma_1|^{\frac{1}{1-\eta_\sigma}} > 2|\mu_1|^{\frac{1}{1-\eta_\mu}}, \\ \min\left\{1, \left[\log_\gamma |u|^{\frac{1}{1-\eta_\mu}}\right]\right\} & if\, |\sigma_1|^{\frac{1}{1-\eta_\sigma}} \le 2|\mu_1|^{\frac{1}{1-\eta_\mu}}. \end{cases} \quad (4.4)$$

From (4.4) it follows that

$$C_2\gamma^{h^*_1} \le |\sigma_{h^*_1}| + |\mu_{h^*_1}| \le C_1\gamma^{h^*_1}\ , \quad (4.5)$$

with C_1, C_2 independent of λ, μ_1, σ_1.

This is obvious in the case $h^*_1 = 1$. If $h^*_1 < 1$ and $|\sigma_1|^{\frac{1}{1-\eta_\sigma}} > 2|\mu_1|^{\frac{1}{1-\eta_\mu}}$, then $\gamma^{h^*_1-1} = c_\sigma|\sigma_1|^{\frac{1}{1-\eta_\sigma}}$, with $1 \le c_\sigma < \gamma$, so that, using the third of (4.3), we see that $C_2\gamma^{h^*_1} \le |\sigma_{h^*_1}| \le C'_1 g^{h^*_1}$, for some $C'_1, C_2 = O(1)$. Furthermore, using also the second of (4.3), we find

$$\frac{|\mu_{h^*_1}|}{|\sigma_{h^*_1}|} = c_\sigma^{\eta_\mu-\eta_\sigma}|\mu_1||\sigma_1|^{-\frac{1-\eta_\mu}{1-\eta_\sigma}}\gamma^{F^{h^*_1}_\mu - F^{h^*_1}_\sigma} < 1 \quad (4.6)$$

and (4.5) follows.

If $h_1^* < 1$ and $|\sigma_1|^{\frac{1}{1-\eta_\sigma}} \leq 2|\mu_1|^{\frac{1}{1-\eta_\mu}}$, then $\gamma^{h_1^*-1} = c_u|u|^{\frac{1}{1-\eta_\mu}}$, with $1 \leq c_\mu < \gamma$, so that, using the second of (4.3) and $|\mu_1| = O(|u|)$, we see that $C_2\gamma^{h_1^*} \leq |\mu_{h_1^*}| \leq C_1'\gamma^{h_1^*}$. Furthermore, using the third (4.3), we find

$$\frac{|\sigma_{h_1^*}|}{|\mu_{h_1^*}|} = c_u^{\eta_\sigma-\eta_\mu}|\sigma_1||u|^{-\frac{1-\eta_\sigma}{1-\eta_\mu}}\gamma^{F_\sigma^{h_1^*}-F_\mu^{h_1^*}} < C_1'' , \tag{4.7}$$

for some $C_1'' = O(1)$, and (4.5) again follows.

Remark. The specific value of h_1^* is not crucial: if we change h_1^* in $h_1^* + n$, $n \in \mathbb{Z}$, the constants C_1, C_2 in (4.5) are replaced by different $O(1)$ constants and the estimates below are not qualitatively modified. Of course, the specific values of C_1, C_2 (then, the specific value of h_1^*) can affect the convergence radius of the pertubative series in λ. The optimal value of h_1^* should be chosen by maximizing the corresponding convergence radius. Since here we are not interested in optimal estimates, we find the choice in (4.4) convenient.

Note also that h_1^* is a non-analytic function of (λ, t, u) (in particular for small u we have $\gamma^{h_1^*} \sim |u|^{1+O(\lambda)}$). As a consequence, the asymptotic expression for the specific heat near the critical points (that we shall obtain in the next section) will contain non-analytic functions of u (in fact it will contain terms depending on h_1^*). However, as explained in Remark (3) after the Main Theorem, this does not imply that C_v is non analytic: it is clear that in this case the non analyticity is introduced "by hand" by our specific choice of h_1^*.

From the results of Theorem 4.1 and Lemma 4.2, together with (4.4) and (4.5), it follows that the assumptions of Theorem 3.4 are satisfied for any $\bar{h} \geq h_1^*$. The integration of the scales $\leq h_1^*$ must be performed in a different way, as discussed in next the section.

5. Integration of the ψ Variables: Second Regime

5.1. Integration of the $\psi^{(1)}$ field. If h_1^* is fixed as in §4.1, we can use Theorem 3.4 up to the scale $\bar{h} = h_1^* + 1$.

Once all the scales $> h_1^*$ are integrated out, it is more convenient to describe the system in terms of the fields $\psi_\omega^{(1)}, \psi_\omega^{(2)}$, $\omega = \pm 1$, defined through the following change of variables:

$$\hat{\psi}_{\omega,\mathbf{k}}^{\alpha(\leq h_1^*)} = \frac{1}{\sqrt{2}}(\hat{\psi}_{\omega,-\alpha\mathbf{k}}^{(1,\leq h_1^*)} - i\alpha\hat{\psi}_{\omega,-\alpha\mathbf{k}}^{(2,\leq h_1^*)}) , \quad \psi_{\omega,\mathbf{x}}^{(j)} = \frac{1}{M^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}}\hat{\psi}_{\omega,\mathbf{k}}^{(j)} . \tag{5.1}$$

If we perform this change of variables, we find $P_{Z_{h_1^*},\sigma_{h_1^*},\mu_{h_1^*},C_{h_1^*}} = \prod_{j=1}^2 P^{(j)}_{Z_{h_1^*},m^{(j)}_{h_1^*},C_{h_1^*}}$ where, if $\Psi^{(j,\le h_1^*),T}_{\mathbf{k}} \stackrel{def}{=} (\psi^{(j,\le h_1^*)}_{1,\mathbf{k}},\ \psi^{(j,\le h_1^*)}_{-1,\mathbf{k}})$,

$$P^{(j)}_{Z_{h_1^*},m^{(j)}_{h_1^*},C_{h_1^*}}(d\psi^{(j,\le h_1^*)}) \stackrel{def}{=}$$

$$\stackrel{def}{=} \frac{1}{N^{(j)}_{h_1^*}} \prod_{\mathbf{k},\omega} d\psi^{(j,\le h_1^*)}_{\omega,\mathbf{k}} \exp\Big\{ - \frac{Z_{h_1^*}}{4M^2} \sum_{\mathbf{k}\in D_{h_1^*}} C_{h_1^*}(\mathbf{k}) \Psi^{(j,\le h_1^*),T}_{\mathbf{k}} A^{(h_1^*)}_j(\mathbf{k}) \Psi^{(j,\le h_1^*)}_{-\mathbf{k}} \Big\}$$

$$A^{(h_1^*)}_j(\mathbf{k}) \stackrel{def}{=} \begin{pmatrix} (-i\sin k - \sin k_0) + a^{+(j)}_{h_1^*}(\mathbf{k}) & -i\big(m^{(j)}_{h_1^*}(\mathbf{k}) + c^{(j)}_{h_1^*}(\mathbf{k})\big) \\ i\big(m^{(j)}_{h_1^*}(\mathbf{k}) + c^{(j)}_{h_1^*}(\mathbf{k})\big) & (-i\sin k + \sin k_0) + a^{-(j)}_{h_1^*}(\mathbf{k}) \end{pmatrix} \quad (5.2)$$

and $a^{\omega(j)}_{h_1^*}$, $m^{(j)}_{h_1^*}$, $c^{(j)}_{h_1^*}$ are given by (A3.2) with $h = h^* + 1$.

The propagators $g^{(j,\le h_1^*)}_{\omega_1,\omega_2}$ associated with the fermionic integration (5.2) are given by (A3.1) with $h = h_1^* + 1$. Note that, by (4.5), $\max\{|m^{(1)}_{h_1^*}|, |m^{(2)}_{h_1^*}|\} = |\sigma_{h_1^*}| + |\mu_{h_1^*}| = O(\gamma^{h_1^*})$ (see (A3.2) for the definition of $m^{(1)}_{h_1^*}$, $m^{(2)}_{h_1^*}$). From now on, for definiteness we shall suppose that $\max\{|m^{(1)}_{h_1^*}|, |m^{(2)}_{h_1^*}|\} = |m^{(1)}_{h_1^*}|$. Then, it is easy to realize that the propagator $g^{(1,\le h_1^*)}_{\omega_1,\omega_2}$ is bounded as follows:

$$|\partial^{n_0}_{x_0}\partial^{n_1}_{x} g^{(1,\le h_1^*)}_{\omega_1,\omega_2}(\mathbf{x})| \le C_{N,n} \frac{\gamma^{(1+n)h_1^*}}{1 + (\gamma^{h_1^*}|\mathbf{d}(\mathbf{x})|)^N} \quad , \quad n = n_0 + n_1 \ , \quad (5.3)$$

namely $g^{(1,\le h_1^*)}_{\omega_1,\omega_2}$ satisfies the same bound as the single scale propagator on scale $h = h_1^*$. This suggests to integrate out $\psi^{(1,\le h_1^*)}$, without any other scale decomposition. We find the following result:

Lemma 5.1. *If $|\lambda| \le \varepsilon_1$, $|\sigma_1|, |\mu_1| \le c_1$ (c_1, ε_1 being the same as in Theorem 2.1) and ν_1 is fixed as in Theorem 4.1, we can rewrite the partition function as*

$$\Xi^-_{AT} = \int P^{(2)}_{Z_{h_1^*},\widehat{m}^{(2)}_{h_1^*},C_{h_1^*}}(d\psi^{(2,\le h_1^*)}) e^{-\overline{\mathcal{V}}^{(h_1^*)}(\sqrt{Z_{h_1^*}}\psi^{(2,\le h_1^*)}) - M^2 \overline{E}_{h_1^*}} \ , \quad (5.4)$$

where: $\widehat{m}^{(2)}_{h_1^*}(\mathbf{k}) = m^{(2)}_{h_1^*}(\mathbf{k}) - \gamma^{h_1^*}\pi_{h_1^*} C^{-1}_{h_1^*}(\mathbf{k})$, with $\pi_{h_1^*}$ a free parameter, s.t. $|\pi_{h_1^*}| \le c|\lambda|$; $|\overline{E}_{h_1^*} - E_{h_1^*}| \le c|\lambda|\gamma^{2h_1^*}$; and

$$\begin{aligned} &\overline{\mathcal{V}}^{(h_1^*)}(\psi^{(2)}) - \gamma^{h_1^*}\pi_{h_1^*} F^{(2,\le h_1^*)}_\sigma(\psi^{(2\le h_1^*)}) \\ &= \sum_{n=1}^\infty \sum_{\underline{\omega}} \prod_{i=1}^{2n} \hat{\psi}^{(2)}_{\omega_i,\mathbf{k}_i} \overline{W}^{(h_1^*)}_{2n,\underline{\omega}}(\mathbf{k}_1, \dots, \mathbf{k}_{2n-1}) \delta\Big(\sum_{i=1}^{2n} \mathbf{k}_i\Big) \\ &= \sum_{n=1}^\infty \sum_{\underline{\sigma},\underline{j},\underline{\omega}} \prod_{i=1}^{2n} \partial^{\sigma_i}_{j_i} \psi^{(2)}_{\omega_i,\mathbf{x}_i} \overline{W}^{(h_1^*)}_{2n,\underline{\sigma},\underline{j},\underline{\omega}}(\mathbf{x}_1, \dots, \mathbf{x}_{2n}) \ , \end{aligned} \quad (5.5)$$

with $F_\sigma^{(2,\le h)}$ *given by the first of (3.8) with* $\hat\psi^{(2,\le h)}_{\omega,\mathbf{k}}\hat\psi^{(2,\le h)}_{\omega',-\mathbf{k}}$ *replacing* $\hat\psi^{+(\le h)}_{\omega,\mathbf{k}}\hat\psi^{-(\le h)}_{\omega',\mathbf{k}}$*; and* $\overline{W}^{(h_1^*)}_{2n,\underline{\sigma},\underline{j},\underline{\omega}}$ *satisfying the same bound (3.25) as* $W^{(\bar h)}_{2n,\underline{\sigma},\underline{j},\underline{\alpha},\underline{\omega}}$ *with* $\bar h = h_1^*$.

In order to prove the lemma it is sufficient to consider (3.2) with $h = h_1^*$ and rewrite $P_{Z_{h_1^*},\sigma_{h_1^*},\mu_{h_1^*},C_{h_1^*}}$ as the product $\prod_{j=1}^2 P^{(j)}_{Z_{h_1^*},m^{(j)}_{h_1^*},C_{h_1^*}}$. Then the integration over the $\psi^{(1,\le h_1^*)}$ field is done as the integration of the χ's in Appendix A2, recalling the bound (5.3).

Finally we rewrite $m^{(2)}_{h_1^*}(\mathbf{k})$ as $\widehat{m}^{(2)}_{h_1^*}(\mathbf{k}) + \gamma^{h_1^*}\pi_{h_1^*}C^{-1}_{h_1^*}(\mathbf{k})$, where $\pi_{h_1^*}$ is a parameter to be suitably fixed below as a function of λ, σ_1, μ_1.

5.2. The localization operator. The integration of the r.h.s. of (5.4) is done in an iterative way similar to the one described in §3. If $h = h_1^*, h_1^* - 1, \ldots,$ we shall write:

$$\Xi^-_{AT} = \int P^{(2)}_{Z_h,\widehat{m}^{(2)}_h,C_h}(d\psi^{(2,\le h)})e^{-\overline{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(2,\le h)})-M^2E_h}\ , \tag{5.6}$$

where $\overline{\mathcal{V}}^{(h)}$ is given by an expansion similar to (5.5), with h replacing h_1^* and $Z_h, \widehat{m}^{(2)}_h$ are defined recursively in the following way. We first introduce a *localization operator* $\mathcal{L}$. As in §(3.2), we define $\mathcal{L}$ as a combination of four operators $\mathcal{L}_j$ and $\overline{\mathcal{P}}_j$, $j = 0, 1$. $\mathcal{L}_j$ are defined as in (3.4) and (3.5), while $\overline{\mathcal{P}}_0$ and $\overline{\mathcal{P}}_1$, in analogy with (3.6), are defined as the operators extracting from a functional of $\widehat{m}^{(2)}_h(\mathbf{k})$, $h \le h_1^*$, the contributions independent and linear in $\widehat{m}^{(2)}_h(\mathbf{k})$. Note that inductively the kernels $\overline{W}^{(h)}_{2n,\underline{\omega}}$ can be thought of as functionals of $\widehat{m}_k(\mathbf{k})$, $h \le k \le h_1^*$. Given $\mathcal{L}_j, \overline{\mathcal{P}}_j$, $j = 0, 1$ as above, we define the action of $\mathcal{L}$ on the kernels $\overline{W}^{(h)}_{2n,\underline{\omega}}$ as follows.

1) If $n = 1$, then

$$\mathcal{L}\overline{W}^{(h)}_{2,\underline{\omega}} \stackrel{def}{=} \begin{cases} \mathcal{L}_0(\overline{\mathcal{P}}_0 + \overline{\mathcal{P}}_1)\overline{W}^{(h)}_{2,\underline{\omega}} & if\, \omega_1 + \omega_2 = 0, \\ \mathcal{L}_1\overline{\mathcal{P}}_0\overline{W}^{(h)}_{2,\underline{\omega}} & if\, \omega_1 + \omega_2 \ne 0. \end{cases}$$

2) If $n > 2$, then $\mathcal{L}\overline{W}^{(h)}_{2n,\underline{\omega}} = 0$.

It is easy to prove the analogue of Lemma 3.1:

$$\mathcal{L}\overline{\mathcal{V}}^{(h)} = (s_h + \gamma^h p_h)F_\sigma^{(2,\le h)} + z_h F_\zeta^{(2,\le h)}\ , \tag{5.7}$$

where s_h, p_h and z_h are real constants and s_h is linear in $\widehat{m}^{(2)}_k(\mathbf{k})$, $h \le k \le h_1^*$; p_h and z_h are independent of $\widehat{m}^{(2)}_k(\mathbf{k})$. Furthermore $F_\sigma^{(2,\le h)}$ and $F_\zeta^{(2,\le h)}$ are given by the first and the last of (3.8) with $\hat\psi^{(2,\le h)}_{\omega,\mathbf{k}}\hat\psi^{(2,\le h)}_{\omega',-\mathbf{k}}$ replacing $\hat\psi^{+(\le h)}_{\omega,\mathbf{k}}\hat\psi^{-(\le h)}_{\omega',\mathbf{k}}$.

Remark. Note that the action of $\mathcal{L}$ on the quartic terms is trivial. The reason for such a choice is that in the present case no quartic local term can appear, because of the Pauli principle: $\psi^{(2,h)}_{1,\mathbf{x}}\psi^{(2,h)}_{1,\mathbf{x}}\psi^{(2,h)}_{-1,\mathbf{x}}\psi^{(2,h)}_{-1,\mathbf{x}} = 0$, so that $\mathcal{L}_0\overline{W}_{4,\underline{\omega}} = 0$.

Using the symmetry properties exposed in Appendix A2.2, we can prove the analogue of Lemma 3.2: if $n = 1$, then

$$\mathcal{R}\overline{W}_{2,\underline{\omega}} = \begin{cases} [\overline{\mathcal{S}}_2 + \mathcal{R}_2(\overline{\mathcal{P}}_0 + \overline{\mathcal{P}}_1)]\overline{W}_{2,\underline{\omega}} & if\, \omega_1 + \omega_2 = 0, \\ [\mathcal{R}_1\overline{\mathcal{S}}_1 + \mathcal{R}_2\overline{\mathcal{P}}_0]\overline{W}_{2,\underline{\alpha},\underline{\omega}} & if\, \omega_1 + \omega_2 \neq 0, \end{cases} \tag{5.8}$$

where $\overline{\mathcal{S}}_1 = 1 - \overline{\mathcal{P}}_0$ and $\overline{\mathcal{S}}_2 = 1 - \overline{\mathcal{P}}_0 - \overline{\mathcal{P}}_1$; if $n = 2$, then $\overline{W}_{4,\underline{\omega}} = \mathcal{R}_1\overline{W}_{4,\underline{\omega}}$.

5.3. Renormalization for $h \le h_1^*$. If $\mathcal{L}$ and $\mathcal{R} = 1 - \mathcal{L}$ are defined as in the previous subsection, we can rewrite (5.6) as:

$$\int P^{(2)}_{Z_h,\widehat{m}_h^{(2)},C_h}(d\psi^{(2,\le h)})e^{-\mathcal{L}\overline{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(2,\le h)})-\mathcal{R}\overline{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(2,\le h)})-M^2E_h}\,. \tag{5.9}$$

Furthermore, using (5.7) and defining:

$$\widehat{Z}_{h-1}(\mathbf{k})\stackrel{def}{=}Z_h(1+C_h^{-1}(\mathbf{k})z_h)\quad,\quad \widehat{m}^{(2)}_{h-1}(\mathbf{k})\stackrel{def}{=}\frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})}\left(\widehat{m}^{(2)}_h(\mathbf{k})+C_h^{-1}(\mathbf{k})s_h\right), \tag{5.10}$$

we see that (5.9) is equal to

$$\int P^{(2)}_{\widehat{Z}_{h-1},\widehat{m}^{(2)}_{h-1},C_h}(d\psi^{(2,\le h)})e^{-\gamma^h p_h F_\sigma^{(2,\le h)}(\sqrt{Z_h}\psi^{(2),\le h})-\mathcal{R}\overline{\mathcal{V}}^h(\sqrt{Z_h}\psi^{(2),\le h})-M^2(E_h+t_h)}. \tag{5.11}$$

Again, we rescale the potential:

$$\widetilde{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})\stackrel{def}{=}\gamma^h\pi_h F_\sigma^{(2,\le h)}(\sqrt{Z_{h-1}}\psi^{(2,\le h)})+\mathcal{R}\overline{\mathcal{V}}^h(\sqrt{Z_h}\psi^{(2,\le h)}), \tag{5.12}$$

where $Z_{h-1} = \widehat{Z}_{h-1}(\mathbf{0})$ and $\pi_h = (Z_h/Z_{h-1})p_h$; we define $\widetilde{f}_h^{-1}$ as in (3.14), we perform the single scale integration and we define the new effective potential as

$$e^{-\overline{\mathcal{V}}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(2,\le h-1)})-M^2\tilde{E}_h}\stackrel{def}{=}\int P^{(2)}_{Z_{h-1},\widehat{m}^{(2)}_{h-1},\widetilde{f}_h^{-1}}(d\psi^{(2,h)})e^{-\widetilde{\mathcal{V}}^h(\sqrt{Z_h}\psi^{(2,\le h)})} \tag{5.13}$$

Finally we pose $E_{h-1} = E_h + t_h + \tilde{E}_h$. Note that the above procedure allows us to write the π_h in terms of π_k, $h \le k \le h_1^*$, namely $\pi_{h-1} = \gamma^h\pi_h + \beta_\pi^h(\pi_h, \dots, \pi_{h_1^*})$, where β_π^h is the *Beta function.*

Proceeding as in §3 we can inductively show that $\overline{\mathcal{V}}^{(h)}$ has the structure of (5.5), with h replacing h_1^* and that the kernels of $\overline{\mathcal{V}}^{(h)}$ are bounded as follows.

Lemma 5.2. *Let the hypothesis of Lemma 5.1 be satisfied and suppose that, for $\bar{h} < h \le h_1^*$ and some constants $c, \vartheta > 0$,*

$$e^{-c|\lambda|}\le\frac{\widehat{m}_h^{(2)}}{\widehat{m}_{h-1}^{(2)}}\le e^{c|\lambda|}\quad,\quad e^{-c|\lambda|^2}\le\frac{Z_h}{Z_{h-1}}\le e^{c|\lambda|^2}\quad,$$
$$|\pi_h|\le c|\lambda|\quad,\quad |\widehat{m}_{\bar{h}}^{(2)}|\le\gamma^{\bar{h}}. \tag{5.14}$$

Then the partition function can be rewritten as in 5.6 and there exists $C > 0$ *s.t. the kernels of* $\overline{\mathcal{V}}^{(h)}$ *satisfy:*

$$\int d\mathbf{x}_1 \cdots d\mathbf{x}_{2n} |\overline{W}^{(\bar{h})}_{2n,\underline{\sigma},\underline{j},\underline{\omega}}(\mathbf{x}_1, \dots, \mathbf{x}_{2n})| \leq M^2 \gamma^{-\bar{h} D_k(n)} (C\,|\lambda|)^{max(1,n-1)}, \tag{5.15}$$

where $D_k(n) = -2 + n + k$ *and* $k = \sum_{i=1}^{2n} \sigma_i$. *Finally* $|E_{\bar{h}+1}| + |t_{\bar{h}+1}| \leq c|\lambda|\gamma^{2\bar{h}}$.

The proof of Lemma 5.2 is essentially identical to the proof of Theorem 3.4 and we do not repeat it here.

It is possible to fix $\pi_{h_1^*}$ so that the first three assumptions in (5.14) are valid for any $h \leq h_1^*$. More precisely, the following result holds, see Appendix A6.

Lemma 5.3. *If* $|\lambda| \leq \varepsilon_1$, $|\sigma_1|, |\mu_1| \leq c_1$ *and* ν_1 *is fixed as in Theorem 4.1, there exists* $\pi^*_{h_1^*}(\lambda, \sigma_1, \mu_1)$ *such that, if we fix* $\pi_{h_1^*} = \pi^*_{h_1^*}(\lambda, \sigma_1, \mu_1)$*, for* $h \leq h_1^*$ *we have:*

$$|\pi_h| \leq c|\lambda|\gamma^{(\vartheta/2)(h-h_1^*)} \quad , \quad \widehat{m}^{(2)}_h = \widehat{m}^{(2)}_{h_1^*}\gamma^{F^h_m} \quad , \quad Z_h = Z_{h_1^*}\gamma^{\overline{F}^h_\zeta} , \tag{5.16}$$

where F^h_m *and* $\overline{F}^h_\zeta$ *are* $O(\lambda)$. *Moreover:*

$$\left|\pi^*_{h_1^*}(\lambda, \sigma_1, \mu_1) - \pi^*_{h_1^*}(\lambda, \sigma_1', \mu_1')\right| \leq c|\lambda| \left(\gamma^{(\eta_\sigma - 1)h_1^*}|\sigma_1 - \sigma_1'| + \gamma^{(\eta_\mu - 1)h_1^*}|\mu_1 - \mu_1'|\right). \tag{5.17}$$

5.4. The integration of the scales $\leq h_2^*$. In order to insure that the last assumption in (5.14) holds, we iterate the preceding construction up to the scale h_2^* defined as the scale s.t. $|\widehat{m}^{(2)}_k| \leq \gamma^{k-1}$ for any $h_2^* \leq k \leq h_1^*$ and $|\widehat{m}^{(2)}_{h_2^*-1}| > \gamma^{h_2^*-2}$.

Once we have integrated all the fields $\psi^{(>h_2^*)}$, we can integrate $\psi^{(2,\leq h_2^*)}$ without any further multiscale decomposition. Note in fact that by definition the propagator satisfies the same bound (5.3) with h_2^* replacing h_1^*. Then, if we define

$$e^{-M^2 \tilde{E}_{\leq h_2^*}} \stackrel{def}{=} \int P_{Z_{h_2^*-1}, \widehat{m}^{(2)}_{h_2^*-1}, C_{h_2^*}} e^{-\widetilde{\mathcal{V}}^{(h_2^*)}(\sqrt{Z_{h_2^*-1}}\psi^{(2,\leq h_2^*)})} , \tag{5.18}$$

we find that $|\tilde{E}_{\leq h_2^*}| \leq c|\lambda|\gamma^{2h_2^*}$ (the proof is a repetition of the estimates on the single scale integration).

Combining this bound with the results of Theorem 3.4, Lemma 5.1, Lemma 5.2 and Lemma 5.3, together with the results of §4 we finally find that the free energy associated to Ξ^-_{AT} is given by the following *finite* sum, uniformly convergent with the size of Λ_M:

$$\lim_{M\to\infty} \frac{1}{M^2} \log \Xi^-_{AT} = E_{\leq h_2^*} + (\overline{E}_{h_1^*} - E_{h_1^*}) + \sum_{h=h_2^*+1}^{1} (\tilde{E}_h + t_h) , \tag{5.19}$$

where $E_{\leq h_2^*} = \lim_{M\to\infty} \tilde{E}_{\leq h_2^*}$ and it is easy to see that $E_{\leq h_2^*}$, for any finite h_2^*, exists and satisfies the same bound of $\tilde{E}_{h_2^*}$.

5.5. Keeping h_2^* finite. From the discussion of the previous subsection, it follows that, for any finite h_2^*, (5.19) is an analytic function of λ, t, u, for $|\lambda|$ sufficiently small, uniformly in h_2^* (this is an elementary consequence of Vitali's convergence theorem). Moreover, repeating the discussion of Appendix G in [M1], it can be proved that, for any $\gamma^{h_2^*} > 0$ (here $\gamma^{h_2^*}$ plays the role of $|t - t_c|$ in Appendix G of [M1]), the limit (5.19) coincides with $\lim_{M\to\infty} 1/M^2 \log \Xi_{AT}^{\gamma_1,\gamma_2}$ for any choice γ_1, γ_2 of boundary conditions; hence this limit coincides with $-2\log\cosh\lambda$ plus the free energy in (1.2), see also (2.6). We can state the result as follows.

Lemma 5.4. *There exists $\varepsilon_1 > 0$ such that, if $|\lambda| \le \varepsilon_1$ and $t \pm u \in D$ (the same as in the Main Theorem), the free energy f defined in (1.2) is real analytic in λ, t, u, except possibly for the choices of λ, t, u such that $\gamma^{h_2^*} = 0$.*

We shall see in §6 below that the specific heat is logarithmically divergent as $\gamma^{h_2^*} \to 0$. So the critical point is really given by the condition $\gamma^{h_2^*} = 0$. We shall explicitly solve the equation for the critical point in the next subsection.

5.6. The critical points. In the present subsection we check that, if $t \pm u \in D$, D being a suitable interval centered around $\sqrt{2} - 1$, see the Main Theorem, there are precisely two critical points of the form (1.5). More precisely, keeping in mind that the equation for the critical point is simply $\gamma^{h_2^*} = 0$ (see the end of the previous subsection), we prove the following.

Lemma 5.5. *Let $|\lambda| \le \varepsilon_1$, $t \pm u \in D$ and $\pi_{h_1^*}$ be fixed as in Lemma 5.3. Then $\gamma^{h_2^*} = 0$ only if $(\lambda, t, u) = (\lambda, t_c^{\pm}(\lambda, u), u)$, where $t_c^{\pm}(\lambda, u)$ is given by (1.5).*

Proof. From the definition of h_2^* given above, see §5.4, it follows that h_2^* satisfies the following equation:

$$\gamma^{h_2^*-1} = c_m \gamma^{F_m^{h_2^*}} \left| |\sigma_{h_1^*}| - |\mu_{h_1^*}| - \alpha_\sigma \gamma^{h_1^*} \pi_{h_1^*} \right| , \tag{5.20}$$

for some $1 \le c_m < \gamma$ and $\alpha_\sigma = \operatorname{sign} \sigma_1$. Then, the equation $\gamma^{h_2^*} = 0$ can be rewritten as:

$$|\sigma_{h_1^*}| - |\mu_{h_1^*}| - \alpha_\sigma \gamma^{h_1^*} \pi_{h_1^*} = 0 . \tag{5.21}$$

First note that the result of Lemma 5.5 is trivial when $h_1^* = 1$. If $h_1^* < 1$, (5.21) cannot be solved when $|\sigma_1|^{\frac{1}{1-\eta_\sigma}} > 2|\mu_1|^{\frac{1}{1-\eta_\mu}}$. In fact,

$$\begin{aligned} &|\sigma_1|\gamma^{\eta_\sigma(h_1^*-1)+F_\sigma^{h_1^*}} - |\mu_1|\gamma^{\eta_\mu(h_1^*-1)+F_\mu^{h_1^*}} - \alpha_\sigma \gamma^{h_1^*}\pi_{h_1^*} \\ &\quad = |\sigma_1|^{1+\frac{\eta_\sigma}{1-\eta_\sigma}} c_1 - \Big(|\mu_1||\sigma_1|^{-\frac{1-\eta_\mu}{1-\eta_\sigma}}\Big)|\sigma_1|^{\frac{1-\eta_\mu}{1-\eta_\sigma}-\frac{\eta_\mu}{1-\eta_\sigma}} c_1' - \alpha_\sigma \gamma^{h_1^*}\pi_{h_1^*} \ge \frac{\gamma^{h_1^*-1}}{3\gamma}, \end{aligned} \tag{5.22}$$

where c_1, c_1' are constants $= 1 + O(\lambda)$, $\pi_{h_1^*} = O(\lambda)$ and $\gamma^{h_1^*-1} = c_\sigma |\sigma_1|^{\frac{1}{1-\eta_\sigma}}$, with $1 \le c_\sigma < \gamma$. Now, if $|\mu_1| > 0$, the r.h.s. of (5.22) equation is strictly positive.

So, let us consider the case $h_1^* < 1$ and $|\sigma_1|^{\frac{1}{1-\eta_\sigma}} \le 2|\mu_1|^{\frac{1}{1-\eta_\mu}}$ (s.t. $\gamma^{h_1^*} = c_u \log_\gamma |u|^{\frac{1}{1-\eta_\mu}}$, with $1 \le c_u \le \gamma$). In this case (5.21) can be easily solved to find:

$$|\sigma_1| = |\mu_1||u|^{\frac{\eta_\mu - \eta_\sigma}{1-\eta_\mu}} c_u^{\eta_\mu-\eta_\sigma} \gamma^{F_\mu^{h_1^*} - F_\sigma^{h_1^*}} + |u|^{\frac{1-\eta_\sigma}{1-\eta_\mu}} c_u^{1-\eta_\sigma} \alpha_\sigma \gamma^{1-F_\sigma^{h_1^*}} \pi_{h_1^*} . \qquad (5.23)$$

Note that $c_u^{\eta_\mu-\eta_\sigma} \gamma^{F_\mu^{h_1^*} - F_\sigma^{h_1^*}} = 1 + O(\lambda)$ is just a function of u, (it does not depend on t), because of our definition of h_1^*. Moreover $\pi_{h_1^*}$ is a smooth function of t: if we call $\pi_{h_1^*}(t, u)$, resp. $\pi_{h_1^*}(t', u)$, the correction corresponding to the initial data $\sigma_1(t, u)$, $\mu_1(t, u)$, resp. $\sigma_1(t', u)$, $\mu_1(t', u)$, we have

$$|\pi_{h_1^*}(t, u) - \pi_{h_1^*}(t', u)| \le c|\lambda||u|^{\frac{\eta_\sigma - 1}{1-\eta_\mu}} |t - t'| , \qquad (5.24)$$

where we used (5.17) and the bounds $|\sigma_1 - \sigma_1'| \le c|t - t'|$ and $|\mu_1 - \mu_1'| \le c|u||t - t'|$, following from the definitions of (σ_1, μ_1) in terms of (σ, μ) and of (t, u), see §2.

Using the same definitions we also realize that (5.23) can be rewritten as

$$t = \Big[\sqrt{2} - 1 + \frac{\nu(\lambda)}{2} \pm |u|^{1+\eta}\Big(1 + \lambda f(t, u)\Big)\Big] \frac{1 + \hat\lambda(t^2 - u^2)}{1 + \hat\lambda} , \qquad (5.25)$$

where

$$1 + \eta \stackrel{def}{=} \frac{1 - \eta_\sigma}{1 - \eta_\mu} , \qquad (5.26)$$

and the crucial property is that $\eta = -b\lambda + O(\lambda^2)$, $b > 0$, see Lemma 4.2 and Appendix A5. We also recall that both η and ν are functions of λ and are independent of t, u. Moreover $f(t, u)$ is a suitable bounded function s.t. $|f(t, u) - f(t', u)| \le c|u|^{-(1+\eta)}|t - t'|$, as it follows from the Lipshitz property of $\pi_{h_1^*}$ (5.24). The r.h.s. of (5.25) is Lipshitz in t with constant $O(\lambda)$, so that (5.25) can be inverted w.r.t. t by contractions and, for both choices of the sign, we find a unique solution

$$t = t_c^\pm(\lambda, u) = \sqrt{2} - 1 + \nu^*(\lambda) \pm |u|^{1+\eta}\big(1 + F^\pm(\lambda, u)\big) , \qquad (5.27)$$

with $|F^\pm(\lambda, u)| \le c\big|\lambda|$, for some c. □

5.7. Computation of h_2^*. Let us now solve (5.20) in the general case of $\gamma^{h_2^*} \ge 0$. Calling $\varepsilon \stackrel{def}{=} \gamma^{h_2^* - h_1^* - F_m^{h_2^*}}/c_m$, we find:

$$\begin{aligned}\varepsilon &= \left| |\sigma_1|\gamma^{(\eta_\sigma-1)(h_1^*-1)+F_\sigma^{h_1^*}} - |\mu_1|\gamma^{(\eta_\mu-1)(h_1^*-1)+F_\mu^{h_1^*}} - \alpha_\sigma\gamma\pi_{h_1^*} \right| \\ &= \gamma^{(\eta_\sigma-1)(h_1^*-1)+F_\sigma^{h_1^*}} \left| |\sigma_1| - |\mu_1|\gamma^{(\eta_\mu-\eta_\sigma)(h_1^*-1)+F_\mu^{h_1^*}-F_\sigma^{h_1^*}} - \alpha_\sigma\gamma^{1+(1-\eta_\sigma)(h_1^*-1)-F_\sigma^{h_1^*}}\pi_{h_1^*} \right| .\end{aligned} \qquad (5.28)$$

If $|\sigma_1|^{1/(1-\eta_\sigma)} \le 2|\mu_1|^{1/(1-\eta_\mu)}$, we use $\gamma^{h_1^*-1} = c_u|u|^{1/(1-\eta_\mu)}$ and, from the second row of (5.27), we find: $\varepsilon = C\Big||\sigma_1| - |\sigma_{1,c}^{\alpha_\sigma}|\Big| |u|^{-(1+\eta)}$, where $\sigma_{1,c}^\pm = \sigma_1(\lambda, t_c^\pm, u)$ and

$C = C(\lambda, t, u)$ is bounded above and below by $O(1)$ constants; defining Δ as in (1.6), we can rewrite:

$$\varepsilon = C \frac{\Big| |\sigma_1| - |\sigma_{1,c}^{\alpha_\sigma}| \Big|}{|u|^{1+\eta}} = C' \frac{\Big| \sigma_1^2 - (\sigma_{1,c}^{\alpha_\sigma})^2 \Big|}{\Delta |u|^{1+\eta}} = C'' \frac{|t - t_c^+| \cdot |t - t_c^-|}{\Delta^2} \ , \tag{5.29}$$

where $C' = C'(\lambda, t, u)$ and $C'' = C''(\lambda, t, u)$ are bounded above and below by $O(1)$ constants.

In the opposite case ($|\sigma_1|^{1/(1-\eta_s)} > 2|\mu_1|^{1/(1-\eta_\mu)}$), we use $\gamma^{h_1^*-1} = c_\sigma |\sigma_1|^{1/(1-\eta_\sigma)}$ and, from the first row of (5.27), we find $\varepsilon = \tilde{C}(1 - |\mu_1||\sigma_1|^{-1/(1+\eta)} - \alpha_\sigma \gamma \pi_{h_1^*}) = \bar{C}$, where $\tilde{C}$ and $\bar{C}$ are bounded above and below by $O(1)$ constants. Since in this region of parameters $|t - t_c^\pm|\Delta^{-1}$ is also bounded above and below by $O(1)$ constants, we can in both cases write

$$\varepsilon = C_\varepsilon(\lambda, t, u) \frac{|t - t_c^+| \cdot |t - t_c^-|}{\Delta^2} \quad , \quad C_{1,\varepsilon} \le C_\varepsilon(\lambda, t, u) \le C_{2,\varepsilon} \tag{5.30}$$

and $C_{j,\varepsilon}$, $j = 1, 2$, are suitable positive $O(1)$ constants.

6. The Specific Heat

Consider the specific heat defined in (1.2). The correlation function $< H_{\mathbf{x}}^{AT} H_{\mathbf{y}}^{AT} >_{\Lambda_M, T}$ can be conveniently written as

$$\begin{aligned} < H_{\mathbf{x}}^{AT} H_{\mathbf{y}}^{AT} >_{\Lambda, T} \ &= \ \frac{\partial^2}{\partial \phi_{\mathbf{x}} \partial \phi_{\mathbf{y}}} \log \Xi_{AT}(\phi) \Big|_{\phi = 0} \quad , \\ \Xi_{AT}(\phi) &\stackrel{def}{=} \sum_{\sigma^{(1)}, \sigma^{(2)}} e^{-\sum_{\mathbf{x} \in \Lambda} (1 + \phi_{\mathbf{x}}) H_{\mathbf{x}}^{AT}} , \end{aligned} \tag{6.1}$$

where $\phi_{\mathbf{x}}$ is a real commuting auxiliary field (with periodic boundary conditions).

Repeating the construction of §2, we see that $\Xi_{AT}(\phi)$ admit a Grassmannian representation similar to the one of Ξ_{AT}, and in particular, if $\mathbf{x} \neq \mathbf{y}$:

$$\begin{aligned} \frac{\partial^2}{\partial \phi_{\mathbf{x}} \partial \phi_{\mathbf{y}}} \log \Xi_{AT}(\phi) \Big|_{\phi=0} &= \frac{\partial^2}{\partial \phi_{\mathbf{x}} \partial \phi_{\mathbf{y}}} \log \sum_{\gamma_1, \gamma_2} (-1)^{\delta_{\gamma_1} + \delta_{\gamma_2}} \widehat{\Xi}_{AT}^{\gamma_1, \gamma_2}(\phi) \Big|_{\phi=0}, \\ \widehat{\Xi}_{AT}^{\gamma_1, \gamma_2}(\phi) = \int \prod_{\mathbf{x} \in \Lambda_M}^{j=1,2} & dH_{\mathbf{x}}^{(j)} d\overline{H}_{\mathbf{x}}^{(j)} dV_{\mathbf{x}}^{(j)} d\overline{V}_{\mathbf{x}}^{(j)} \, e^{S_{\gamma_1}^{(1)}(t^{(1)}) + S_{\gamma_2}^{(2)}(t^{(2)}) + V_\lambda + \mathcal{B}(\phi)}, \end{aligned} \tag{6.2}$$

where δ_γ, $S^{(j)}(t^{(j)})$ and V_λ where defined in §2 (see (2.2) and previous lines, and (2.9)), the apex γ_1, γ_2 attached to $\widehat{\Xi}_{AT}$ refers to the boundary conditions assigned to the Grassmannian fields, as in §2 and finally $\mathcal{B}(\phi)$ is defined as:

$$\begin{aligned} \mathcal{B}(\phi) = \sum_{\mathbf{x} \in \Lambda} \phi_{\mathbf{x}} \Big\{ & a^{(1)} \big(\overline{H}_{\mathbf{x}}^{(1)} H_{\mathbf{x}+\hat{e}_1}^{(1)} + \overline{V}_{\mathbf{x}}^{(1)} V_{\mathbf{x}+\hat{e}_0}^{(1)} \big) + a^{(2)} \big(\overline{H}_{\mathbf{x}}^{(2)} H_{\mathbf{x}+\hat{e}_1}^{(2)} + \overline{V}_{\mathbf{x}}^{(2)} V_{\mathbf{x}+\hat{e}_0}^{(2)} \big) \\ & + \lambda \widetilde{a} \big(\overline{H}_{\mathbf{x}}^{(1)} H_{\mathbf{x}+\hat{e}_1}^{(1)} \overline{H}_{\mathbf{x}}^{(2)} H_{\mathbf{x}+\hat{e}_1}^{(2)} + \overline{V}_{\mathbf{x}}^{(1)} V_{\mathbf{x}+\hat{e}_0}^{(1)} \overline{V}_{\mathbf{x}}^{(2)} V_{\mathbf{x}+\hat{e}_0}^{(2)} \big) \Big\} \stackrel{def}{=} \sum_{\mathbf{x} \in \Lambda} \phi_{\mathbf{x}} A_{\mathbf{x}} \ , \end{aligned} \tag{6.3}$$

where $a^{(1)}$, $a^{(2)}$ and $\widetilde{a}$ are $O(1)$ constants, with $a^{(1)} - a^{(2)} = O(u)$. Using (6.2) and (6.3) we can rewrite:

$$< H^{AT}_{\mathbf{x}} H^{AT}_{\mathbf{y}} >_{\Lambda,T} = \frac{1}{4}(\cosh J)^{2M^2} \sum_{\gamma_1,\gamma_2} (-1)^{\delta_{\gamma_1}+\delta_{\gamma_2}} \frac{\Xi^{\gamma_1,\gamma_2}_{AT}}{\Xi_{AT}} < A_{\mathbf{x}} A_{\mathbf{y}} >^{\gamma_1,\gamma_2}_{\Lambda_M,T}, \quad (6.4)$$

where $< \cdot >^{\gamma_1,\gamma_2}_{\Lambda_M,T}$ is the average w.r.t. the boundary conditions γ_1, γ_2. Proceeding as in Appendix G of [M1] one can show that, if $\gamma^{h_2^*} > 0$, $< A_{\mathbf{x}} A_{\mathbf{y}} >^{\gamma_1,\gamma_2}_{\Lambda_M,T}$ is exponentially insensitive to boundary conditions and $\sum_{\gamma_1,\gamma_2} (-1)^{\delta_{\gamma_1}+\delta_{\gamma_2}} \Xi^{\gamma_1,\gamma_2}_{AT} / \Xi_{AT}$ is an $O(1)$ constant. Then from now on we will study only $\Xi^{-}_{AT}(\phi) \stackrel{def}{=} \widehat{\Xi}^{(-,-),(-,-)}_{AT}(\phi)$ and $< A_{\mathbf{x}} A_{\mathbf{y}} >^{(-,-),(-,-)}_{\Lambda_M,T}$.

As in §2 we integrate out the χ fields and, proceeding as in Appendix A2.1, we find:

$$\Xi^{-}_{AT}(\phi) = \int P_{Z_1,\sigma_1,\mu_1,C_1}(d\psi) e^{\mathcal{V}^{(1)}+\mathcal{B}^{(1)}}, \quad (6.5)$$

where

$$\mathcal{B}^{(1)}(\psi,\phi) = \sum_{m,n=1}^{\infty} \sum_{\substack{\mathbf{x}_1\cdots\mathbf{x}_m \\ \mathbf{y}_1\cdots\mathbf{y}_{2n}}}^{\underline{\sigma},\underline{j},\underline{\alpha},\underline{\omega}} B^{(1)}_{m,2n;\underline{\sigma},\underline{j},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_m;\mathbf{y}_1,\ldots,\mathbf{y}_{2n}) \Big[\prod_{i=1}^{m} \phi_{\mathbf{x}_i}\Big]\Big[\prod_{i=1}^{2n} \partial^{\sigma_i}_{j_i} \psi^{\alpha_i}_{\mathbf{y}_i,\omega_i}\Big]. \quad (6.6)$$

We proceed as for the partition function, namely as described in §3 above. We introduce the scale decomposition described in §3 and we perform iteratively the integration of the single scale fields, starting from the field of scale 1. After the integration of the fields $\psi^{(1)},\ldots,\psi^{(h+1)}$, $h_1^* < h \le 0$, we are left with

$$\Xi^{-}_{AT}(\phi) = e^{-M^2 E_h + S^{(h+1)}(\phi)} \int P_{Z_h,\sigma_h,\mu_h,C_h}(d\psi^{\le h}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) + \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi)}, \quad (6.7)$$

where $P_{Z_h,\sigma_h,\mu_h m_h,C_h}(d\psi^{(\le h)})$ and $\mathcal{V}^{(h)}$ are the same as in §3, $S^{(h+1)}(\phi)$ denotes the sum of the contributions dependent on ϕ but independent of ψ, and finally $\mathcal{B}^{(h)}(\psi^{(\le h)},\phi)$ denotes the sum over all terms containing at least one ϕ field and two ψ fields. $S^{(h+1)}$ and $\mathcal{B}^{(h)}$ can be represented as

$$S^{(h+1)}(\phi) = \sum_{m=1}^{\infty} \sum_{\mathbf{x}_1\cdots\mathbf{x}_m} S^{(h+1)}_m(\mathbf{x}_1,\ldots,\mathbf{x}_m) \prod_{i=1}^{m} \phi_{\mathbf{x}_i}$$

$$\mathcal{B}^{(h)}(\psi^{(\le h)},\phi) = \sum_{m,n=1}^{\infty} \sum_{\substack{\mathbf{x}_1\cdots\mathbf{x}_m \\ \mathbf{y}_1\cdots\mathbf{y}_{2n}}}^{\underline{\sigma},\underline{j},\underline{\alpha},\underline{\omega}} B^{(h)}_{m,2n;\underline{\sigma},\underline{j},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_m;\mathbf{y}_1,\ldots,\mathbf{y}_{2n}) \Big[\prod_{i=1}^{m} \phi_{\mathbf{x}_i}\Big] \times \Big[\prod_{i=1}^{2n} \partial^{\sigma_i} \psi^{(\le h)\alpha_i}_{\mathbf{y}_i,\omega_i}\Big]. \quad (6.8)$$

Since the field ϕ is equivalent, regarding dimensional bounds, to two ψ fields (see Theorem 6.1 below for a more precise statement), the only terms in the expansion for $\mathcal{B}^{(h)}$ which are not irrelevant are those with $m = n = 1$, $\sigma_1 = \sigma_2 = 0$ and they are marginal. Hence we extend the definition of the localization operator $\mathcal{L}$, so that its action on $\mathcal{B}^{(h)}(\psi^{(\le h)},\phi)$ is defined by its action on the kernels $\widehat{B}^{(h)}_{m,2n;\underline{\alpha},\underline{\omega}}(\mathbf{q}_1,\ldots,\mathbf{q}_m;\mathbf{k}_1,\ldots,\mathbf{k}_{2n})$:

1) if $m = n = 1$ and $\alpha_1 + \alpha_2 = \omega_1 + \omega_2 = 0$, then $\mathcal{L}\widehat{B}^{(h)}_{1,2;\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{q}_1; \mathbf{k}_1, \mathbf{k}_2) \stackrel{def}{=} \mathcal{P}_0 \widehat{B}^{(h)}_{1,2;\underline{\alpha},\underline{\omega}}(\mathbf{k}_+; \mathbf{k}_+, \mathbf{k}_+)$, where $\mathcal{P}_0$ is defined as in (3.6);
2) in all other cases $\mathcal{L}\widehat{B}^{(h)}_{m,2n;\underline{\alpha},\underline{\omega}} = 0$.

Using the symmetry considerations of Appendix B together with the remark that $\phi_{\mathbf{x}}$ is invariant under *Complex conjugation*, *Hole–particle* and (1) $\longleftrightarrow$ (2), while under *Parity* $\phi_{\mathbf{x}} \to \phi_{-\mathbf{x}}$ and under *Rotation* $\phi_{(x,x_0)} \to \phi_{(-x_0,-x)}$, we easily realize that $\mathcal{L}\mathcal{B}^{(h)}$ has necessarily the following form:

$$\mathcal{L}\mathcal{B}^{(h)}(\psi^{(\leq h)}, \phi) = \frac{\overline{Z}_h}{Z_h} \sum_{\mathbf{x},\omega} \frac{(-i\omega)}{2} \phi_{\mathbf{x}} \psi^{(\leq h)+}_{\omega,\mathbf{x}} \psi^{(\leq h)-}_{-\omega,\mathbf{x}} \ , \tag{6.9}$$

where $\overline{Z}_h$ is real and $\overline{Z}_1 = a^{(1)}|_{\sigma=\mu=0} = a^{(2)}|_{\sigma=\mu=0}$.

Note that apriori a term $\sum_{\mathbf{x},\omega,\alpha} \phi_{\mathbf{x}} \psi^{(\leq h)\alpha}_{\omega,\mathbf{x}} \psi^{(\leq h)\alpha}_{-\omega,\mathbf{x}}$ is allowed by symmetry but, using (1) $\longleftrightarrow$ (2) symmetry, one sees that its kernel is proportional to μ_k, $k \geq h$. So, with our definition of localization, such a term contributes to $\mathcal{R}\mathcal{B}^{(h)}$.

Now that the action of $\mathcal{L}$ on $\mathcal{B}$ is defined, we can describe the single scale integration, for $h > h_1^*$. The integral in the r.h.s. of (6.7) can be rewritten as:

$$e^{-M^2 t_h} \int P_{Z_{h-1},\sigma_{h-1},\mu_{h-1},C_{h-1}}(d\psi^{\leq h-1}) \cdot \int P_{Z_{h-1},\sigma_{h-1},\mu_{h-1},\widetilde{f}_h^{-1}}(d\psi^{(h)}) e^{-\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}) + \widehat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)},\phi)}, \tag{6.10}$$

where $\widehat{\mathcal{V}}^{(h)}$ was defined in (3.12) and

$$\widehat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}, \phi) \stackrel{def}{=} \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}, \phi) \ . \tag{6.11}$$

Finally we define

$$e^{-\widetilde{E}_h M^2 + \widetilde{S}^{(h)}(\phi) - \mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)}) + \mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)},\phi)} \stackrel{def}{=} \int P_{Z_{h-1},\sigma_{h-1},\mu_{h-1},\widetilde{f}_h^{-1}}(d\psi^{(h)}) e^{-\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}) + \widehat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)},\phi)} \ , \tag{6.12}$$

and

$$E_{h-1} \stackrel{def}{=} E_h + t_h + \widetilde{E}_h \quad , \quad S^{(h)}(\phi) \stackrel{def}{=} S^{(h+1)}(\phi) + \widetilde{S}^{(h)}(\phi) \ . \tag{6.13}$$

With the definitions above, it is easy to verify that $\overline{Z}_{h-1}$ satisfies the equation $\overline{Z}_{h-1} = \overline{Z}_h(1 + \overline{z}_h)$, where $\overline{z}_h = \overline{b}\lambda_h + O(\lambda^2)$, for some $\overline{b} \neq 0$. Then, for some $c > 0$, $\overline{Z}_1 e^{-c|\lambda|h} \leq \overline{Z}_h \leq \overline{Z}_1 e^{c|\lambda|h}$. The analogue of Theorem 3.1 for the kernels of $\mathcal{B}^{(h)}$ holds:

Theorem 6.1. *Suppose that the hypothesis of Lemma 5.1 is satisfied. Then, for $h_1^* \leq \bar{h} \leq 1$ and a suitable constant C, the kernels of $\mathcal{B}^{(h)}$ satisfy*

$$\int d\mathbf{x}_1 \cdots d\mathbf{x}_{2n} |B^{(\bar{h})}_{2n,m;\underline{\sigma},\underline{j},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1, \ldots, \mathbf{x}_m; \mathbf{y}_1, \ldots, \mathbf{y}_{2n})| \leq M^2 \gamma^{-\bar{h}(D_k(n)+m)} (C|\lambda|)^{max(1,n-1)} \ , \tag{6.14}$$

where $D_k(n) = -2 + n + k$ and $k = \sum_{i=1}^{2n} \sigma_i$.

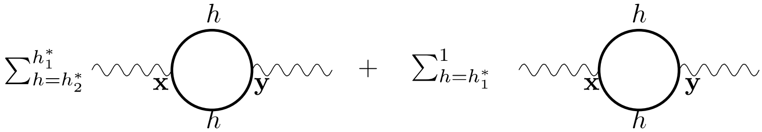

Fig. 3. The lowest order diagrams contributing to $< H^{AT}_{\mathbf{x}} H^{AT}_{\mathbf{y}} >_{\Lambda_M,T}$. The wavy lines ending in the points labeled $\mathbf{x}$ and $\mathbf{y}$ represent the fields $\phi_{\mathbf{x}}$ and $\phi_{\mathbf{y}}$ respectively. The solid lines labeled by h and going from $\mathbf{x}$ to $\mathbf{y}$ represent the propagators $g^{(h)}(\mathbf{x}-\mathbf{y})$. The sums are over the scale indices and, even if not explicitly written, over the indexes $\underline{\alpha}$, $\underline{\omega}$ (and the propagators depend on these indexes)

Note that, consistently with our definition of localization, the dimension of $B^{(h)}_{2,1;(0,0),(+,-),(\omega,-\omega)}$ is $D_0(1)+1=0$.

Again, proceeding as in §4, we can study the flow of $\overline{Z}_h$ up to $h=-\infty$ and prove that $\overline{Z}_h = \overline{Z}_1 \gamma^{\overline{\eta}(h-1)+F^h_{\overline{z}}}$, where $\overline{\eta}$ is a non-trivial analytic function of λ (its linear part is non-vanishing) and $F^h_{\overline{z}}$ is a suitable $O(\lambda)$ function (independent of σ_1,μ_1). We recall that $\overline{Z}_1 = O(1)$.

We proceed as above up to the scale h_1^*. Once the scale h_1^* is reached we pass to the $\psi^{(1)}$, $\psi^{(2)}$ variables, we integrate out (say) the $\psi^{(1)}$ fields and we get

$$\int P^{(2)}_{Z_{h_1^*},\widehat{m}^{(2)}_{h_1^*},C_{h_1^*}}(d\psi^{(2)(\le h_1^*)})e^{-\overline{\mathcal{V}}^{(h_1^*)}(\sqrt{Z_{h_1^*}}\psi^{(2,\le h_1^*)})+\overline{B}^{(h_1^*)}(\sqrt{Z_{h_1^*}}\psi^{(2,\le h_1^*)})}, \tag{6.15}$$

with $\mathcal{L}\overline{B}^{h_1^*}(\sqrt{Z_{h_1^*}}\psi^{(2),\le h_1^*}) = \overline{Z}_{h_1^*}\sum_{\mathbf{x}} i\phi_{\mathbf{x}}\psi^{(2,\le h_1^*)}_{1,\mathbf{x}}\psi^{(2,\le h_1^*)}_{-1,\mathbf{x}}$.

The scales $h_2^* \le h \le h_1^*$ are integrated as in §5 and one finds that the flow of $\overline{Z}_h$ in this regime is trivial, *i.e.* if $h_2^* \le h \le h_1^*$, $\overline{Z}_h = \overline{Z}_{h_1^*}\gamma^{F^h_z}$, with $F^h_z = O(\lambda)$.

The result is that the correlation function $< H^{AT}_{\mathbf{x}} H^{AT}_{\mathbf{y}} >_{\Lambda_M,T}$ is given by a convergent power series in λ, uniformly in Λ_M. Then, the leading behaviour of the specific heat is given by the sum over $\mathbf{x}$ and $\mathbf{y}$ of the lowest order contributions to $< H^{AT}_{\mathbf{x}} H^{AT}_{\mathbf{y}} >_{\Lambda_M,T}$, namely by the diagrams in Fig. 3. Absolute convergence of the power series of $< H^{AT}_{\mathbf{x}} H^{AT}_{\mathbf{y}} >_{\Lambda_M,T}$ implies that the rest is a small correction.

The conclusion is that C_v, for λ small and $|t-\sqrt{2}+1|$, $|u| \le (\sqrt{2}-1)/4$, is given by:

$$\begin{aligned} C_v = \frac{1}{|\Lambda|}\sum_{\mathbf{x},\mathbf{y}\in\Lambda_M}\sum_{\omega_1,\omega_2=\pm1}\sum_{h,h'=h_2^*}^{1}\frac{(Z^{(1)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}} \\ \times\Big[G^{(h)}_{(+,\omega_1),(+,\omega_2)}(\mathbf{x}-\mathbf{y})G^{(h')}_{(-,-\omega_2),(-,-\omega_1)}(\mathbf{y}-\mathbf{x}) \\ +G^{(h)}_{(+,\omega_1),(-,-\omega_2)}(\mathbf{x}-\mathbf{y})G^{(h')}_{(-,-\omega_1),(+,\omega_2)}(\mathbf{x}-\mathbf{y})\Big] \\ +\frac{1}{|\Lambda|}\sum_{\mathbf{x},\mathbf{y}\in\Lambda_M}\sum_{h_2^*}^{1}\left(\frac{\overline{Z}_h}{Z_h}\right)^2\Omega^{(h)}_{\Lambda_M}(\mathbf{x}-\mathbf{y}), \end{aligned} \tag{6.16}$$

where $h\vee h' = \max\{h,h'\}$ and $G^{(h)}_{(\alpha_1,\omega_1),(\alpha_2,\omega_2)}(\mathbf{x})$ must be interpreted as

$$G^{(h)}_{(\alpha_1\omega_1),(\alpha_2,\omega_2)}(\mathbf{x}) = \begin{cases} g^{(h)}_{(\alpha_1\omega_1),(\alpha_2,\omega_2)}(\mathbf{x}) & if\, h > h_1^*, \\ g^{(1,\le h_1^*)}_{\omega_1,\omega_2}(\mathbf{x}) + g^{(2,h_1^*)}_{\omega_1,\omega_2}(\mathbf{x}) & if\, h = h_1^*, \\ g^{(2,h)}_{\omega_1,\omega_2}(\mathbf{x}) & if\, h_2^* < h < h_1^*, \\ g^{(2,\le h_2^*)}_{\omega_1,\omega_2}(\mathbf{x}) & if\, h = h_2^*. \end{cases}$$

Moreover, if $N, n_0, n_1 \ge 0$ and $n = n_0 + n_1$, $|\partial_x^{n_0}\partial_{x_0}\Omega^{(h)}_{\Lambda_M}(\mathbf{x})| \le C_{N,n}|\lambda|\frac{\gamma^{(2+n)h}}{1+(\gamma^h|\mathbf{d}(\mathbf{x})|)^N}$. Now, calling η_c the exponent associated to $\overline{Z}_h/Z_h$, from (6.16) we find:

$$C_v = -C_1\gamma^{2\eta_c h_1^*}\log_\gamma \gamma^{h_1^*-h_2^*}\big(1+\Omega^{(1)}_{h_1^*,h_2^*}(\lambda)\big) + C_2\frac{1-\gamma^{2\eta_c(h_1^*-1)}}{2\eta_c}\big(1+\Omega^{(2)}_{h_1^*}(\lambda)\big)\,, \tag{6.17}$$

where $|\Omega^{(1)}_{h_1^*,h_2^*}(\lambda)|, |\Omega^{(2)}_{h_1^*}(\lambda)| \le c|\lambda|$, for some c. Note that, defining Δ as in (1.6), $\gamma^{(1-\eta_\sigma)h_1^*}\Delta^{-1}$ is bounded above and below by $O(1)$ constants. Then, using (5.30), (1.6) follows.

Appendix A1. Proof of (2.1)

We start from Eq. (V.2.12) in [MW], expressing the partition function of the Ising model with periodic boundary condition on a lattice with an even number of sites as a combination of the Pfaffians of four matrices with different boundary conditions, defined by (V.2.10) and (V.2.11) in [MW]. In the general case (*i.e.* M^2 not necessarily even), the (V.2.12) of [MW] becomes:

$$Z_I = \sum_\sigma e^{-\beta J H_I(\sigma)} = (-1)^{M^2}\frac{1}{2}(2\cosh\beta J)^{M^2}\Big(-\mathrm{Pf}\,\overline{A}_1 + \mathrm{Pf}\,\overline{A}_2 + \mathrm{Pf}\,\overline{A}_3 + \mathrm{Pf}\,\overline{A}_4\Big), \tag{A1.1}$$

where $\overline{A}_i$ are matrices with elements $(\overline{A}_i)_{\mathbf{x},j;\mathbf{y},k}$, with $\mathbf{x},\mathbf{y}\in\Lambda_M$, $j,k = 1,\dots,6$, given by:

$$(\overline{A}_i)_{\mathbf{x};\mathbf{x}} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \tag{A1.2}$$

and $\big((\overline{A}_i)_{\mathbf{x};\mathbf{x}+\hat{e}_1}\big)_{i,j} = t\delta_{i,1}\delta_{j,2}$, $\big((\overline{A}_i)_{\mathbf{x};\mathbf{x}+\hat{e}_0}\big)_{i,j} = t\delta_{i,2}\delta_{j,1}$, $(\overline{A}_i)_{\mathbf{x};\mathbf{x}+\hat{e}_1} = -(\overline{A}_i^{\,T})_{\mathbf{x}+\hat{e}_1;\mathbf{x}}$, $(\overline{A}_i)_{\mathbf{x};\mathbf{x}+\hat{e}_0} = -(\overline{A}_i^{\,T})_{\mathbf{x}+\hat{e}_0;\mathbf{x}}$; moreover

$$\begin{aligned} (\overline{A}_i)_{(M,x_0);(1,x_0)} &= -(\overline{A}_i^{\,T})_{(1,x_0);(M,x_0)} = (-1)^{[\frac{i-1}{2}]}(\overline{A}_i)_{(1,x_0);(2,x_0)}, \\ (\overline{A}_i)_{(x,M);(x,1)} &= -(\overline{A}_i^{\,T})_{(x,1);(x,M)} = (-1)^{i-1}(\overline{A}_i)_{(x,1);(x,2)}\,, \end{aligned} \tag{A1.3}$$

where $[\frac{i-1}{2}]$ is the bigger integer $\le \frac{i-1}{2}$; in all the other cases the matrices $(\overline{A}_i)_{\mathbf{x},\mathbf{y}}$ are identically zero.

Given a $(2n) \times (2n)$ antisymmetric matrix A, it is well–known that $\mathrm{Pf}\, A = (-1)^n \int d\psi_1 \cdots d\psi_{2n} \cdot \cdot \exp\{\frac{1}{2}\sum_{i,j} \psi_i A_{ij} \psi_j\}$, where $\psi_1, \ldots, \psi_{2n}$ are Grassmannian variables. Then, we can rewrite (A1.1) as:

$$\frac{1}{2}(2\cosh\beta J)^{M^2} \sum_{\gamma} (-1)^{\delta_\gamma} \int \prod_{\mathbf{x}\in\Lambda_M} d\overline{H}^\gamma_{\mathbf{x}} dH^\gamma_{\mathbf{x}} d\overline{V}^\gamma_{\mathbf{x}} dV^\gamma_{\mathbf{x}} d\overline{T}^\gamma_{\mathbf{x}} dT^\gamma_{\mathbf{x}} e^{S^\gamma(t;H,V,T)}, \tag{A1.4}$$

where: $\gamma = (\varepsilon, \varepsilon')$; $\varepsilon, \varepsilon' = \pm 1$; δ_γ is defined after (2.1); $\overline{H}^\gamma_{\mathbf{x}}, H^\gamma_{\mathbf{x}}, \overline{V}^\gamma_{\mathbf{x}}, V^\gamma_{\mathbf{x}}$ are Grassmannian variables with ε–periodic resp. ε'–periodic boundary conditions in the vertical, resp. horizontal, direction, see (2.3) and following lines. Furthermore:

$$\begin{aligned} S^\gamma(t; H, V, T) &= t\sum_{\mathbf{x}} \left[\overline{H}^\gamma_{\mathbf{x}} H^\gamma_{\mathbf{x}+\hat{e}_1} + \overline{V}^\gamma_{\mathbf{x}} V^\gamma_{\mathbf{x}+\hat{e}_0}\right] \\ &+ \sum_{\mathbf{x}} \left[\overline{V}^\gamma_{\mathbf{x}}\overline{H}^\gamma_{\mathbf{x}} + \overline{H}^\gamma_{\mathbf{x}} T^\gamma_{\mathbf{x}} + V^\gamma_{\mathbf{x}} H^\gamma_{\mathbf{x}} + H^\gamma_{\mathbf{x}}\overline{T}^\gamma_{\mathbf{x}} + T^\gamma_{\mathbf{x}}\overline{V}^\gamma_{\mathbf{x}} + \overline{T}^\gamma_{\mathbf{x}} V^\gamma_{\mathbf{x}} + \overline{T}^\gamma_{\mathbf{x}} T^\gamma_{\mathbf{x}}\right]. \end{aligned} \tag{A1.5}$$

The T–fields appear only in the diagonal elements and they can be easily integrated out:

$$\begin{aligned} &\prod_{\mathbf{x}\in\Lambda_M} \int d\overline{T}^\gamma_{\mathbf{x}} dT^\gamma_{\mathbf{x}} \exp\left\{\overline{H}^\gamma_{\mathbf{x}} T^\gamma_{\mathbf{x}} + H^\gamma_{\mathbf{x}}\overline{T}^\gamma_{\mathbf{x}} + T^\gamma_{\mathbf{x}}\overline{V}^\gamma_{\mathbf{x}} + \overline{T}^\gamma_{\mathbf{x}} V^\gamma_{\mathbf{x}} + \overline{T}^\gamma_{\mathbf{x}} T^\gamma_{\mathbf{x}}\right\} \\ &\quad = \prod_{\mathbf{x}\in\Lambda_M} (-1 - \overline{H}^\gamma_{\mathbf{x}} H^\gamma_{\mathbf{x}} - \overline{V}^\gamma_{\mathbf{x}} V^\gamma_{\mathbf{x}} - V^\gamma_{\mathbf{x}}\overline{H}^\gamma_{\mathbf{x}} - V^\gamma_{\mathbf{x}}\overline{H}^\gamma_{\mathbf{x}}) \\ &\quad = (-1)^M \exp\sum_{\mathbf{x}\in\Lambda_M} \left[\overline{H}^\gamma_{\mathbf{x}} H^\gamma_{\mathbf{x}} + \overline{V}^\gamma_{\mathbf{x}} V^\gamma_{\mathbf{x}} + V^\gamma_{\mathbf{x}}\overline{H}^\gamma_{\mathbf{x}} + H^\gamma_{\mathbf{x}}\overline{V}^\gamma_{\mathbf{x}}\right], \end{aligned} \tag{A1.6}$$

where in the last identity we used that $\left[\overline{H}^\gamma_{\mathbf{x}} H^\gamma_{\mathbf{x}} + \overline{V}^\gamma_{\mathbf{x}} V^\gamma_{\mathbf{x}} + V^\gamma_{\mathbf{x}}\overline{H}^\gamma_{\mathbf{x}} + H^\gamma_{\mathbf{x}}\overline{V}^\gamma_{\mathbf{x}}\right]^2 = 0$. Substituting (A1.6) into (A1.4) we find (2.1).

Appendix A2. Integration of the Heavy Fermions. Symmetry Properties

A2.1. Integration of the χ fields. Calling $\overline{\mathcal{V}}(\psi, \chi) = Q(\psi, \chi) - \nu F_\sigma(\psi) + V(\psi, \chi)$, we obtain

$$-\widetilde{E}_1 M^2 - Q^{(1)}(\psi) - \mathcal{V}^{(1)}(\psi) = \log\int P(d\chi) e^{\overline{\mathcal{V}}(\psi,\chi)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{E}^T_\chi(\overline{\mathcal{V}}(\psi, \chi); n), \tag{A2.1}$$

where $\widetilde{E}_1$ is a constant and $\mathcal{V}^{(1)}$ is at least quadratic in ψ and vanishing when $\lambda = \nu = 0$. $Q^{(1)}$ is the rest (quadratic in ψ). Given s set of labels P_{v_i}, $i = 1, \ldots, s$ and $\widetilde{\chi}(P_{v_i}) \stackrel{def}{=} \prod_{f\in P_{v_i}} \chi^{\alpha(f)}_{\omega(f),\mathbf{x}(f)}$, the truncated expectation $\mathcal{E}^T_\chi(\widetilde{\chi}(P_{v_1}), \ldots, \widetilde{\chi}(P_{v_s}))$ can be written as

$$\mathcal{E}^T_\chi(\widetilde{\chi}(P_{v_1}), \ldots, \widetilde{\chi}(P_{v_s})) = \sum_T \alpha_T \prod_{\ell\in T} g_\chi(f^1_\ell, f^2_\ell) \int dP_T(\mathbf{t}) \mathrm{Pf}\, G^T(\mathbf{t}), \tag{A2.2}$$

where T is a set of lines forming an *anchored tree* between the cluster of poins $P_{v_1}, \ldots, P_{v_s}$ *i.e.* T is a set of lines which becomes a tree if one identifies all the points in the same clusters; $\mathbf{t} = \{t_{i,i'} \in [0, 1], 1 \le i, i' \le s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^s$ of unit norm; α_T is a sign (irrelevant for the subsequent bounds); f^1_ℓ, f^2_ℓ are the field labels associated to the points connected by ℓ; if $\underline{a}(f) = (\alpha(f), \omega(f))$, the propagator $g_\chi(f, f')$ is equal to

$$g_\chi(f, f') = g^\chi_{\underline{a}(f),\underline{a}(f')}(\mathbf{x}(f) - \mathbf{x}(f')) \stackrel{def}{=} < \chi^{\alpha(f)}_{\omega(f),\mathbf{x}(f)} \chi^{\alpha(f')}_{\omega(f'),\mathbf{x}(f')} >; \tag{A2.3}$$

if $2n = \sum_{i=1}^s |P_{v_i}|$, then $G^T(\mathbf{t})$ is a $(2n - 2s + 2) \times (2n - 2s + 2)$ antisymmetrix matrix, whose elements are given by $G^T_{f,f'} = t_{i(f),i(f')} g_\chi(f, f')$, where: $f, f' \notin F_T$ and $F_T \stackrel{def}{=} \cup_{\ell \in T} \{f^1_\ell, f^2_\ell\}$; $i(f)$ is s.t. $f \in P_{i(f)}$; finally Pf G^T is the *Pfaffian* of G^T. If $s = 1$ the sum over T is empty, but we can still use the above equation by interpreting the r.h.s. as 1 if P_{v_1} is empty, and $\det G(P_1)$ otherwise.

Sketch of the proof of (A2.2). Equation (A2.2) is a trivial generalization of the well–known formula expressing truncated fermionic expectations in terms of sums of determinants [Le]. The only difference here is that the propagators $< \chi^\alpha_{\omega_1,\mathbf{x}_1} \chi^\alpha_{\omega_2,\mathbf{x}_2} >$ are not vanishing, so that Pfaffians appear instead of determinants. The proof can be done along the same lines of Appendix A3 of [GM]. The only difference here is that the identity known as the *Berezin integral*, see (A3.15) of [GM], that is the starting point to get to (A2.2), must be replaced by the (more general) identity:

$$\mathcal{E}_\chi \Big(\prod_{j=1}^s \widetilde{\chi}(P_j) \Big) = \text{Pf}\, G = (-1)^n \int \mathcal{D}\chi \, \exp \Big[\frac{1}{2} (\chi, G\chi) \Big] , \tag{A2.4}$$

where: the expectation $\mathcal{E}_\chi$ is w.r.t. $P(d\chi)$; if $2m = \sum_{j=1}^s |P_j|$, G is the $2m \times 2m$ antisymmetric matrix with entries $G_{f,f'} = g^\chi_{\underline{a}(f),\underline{a}(f')}(\mathbf{x}(f) - \mathbf{x}(f'))$; and

$$\mathcal{D}\chi = \prod_{j=1}^n \prod_{f \in P_j} d\chi^{\alpha(f)}_{\mathbf{x}(f),\omega(f)} \qquad (\chi, G\chi) = \sum_{f,f' \in \cup_i P_i} \chi^{\alpha(f)}_{\mathbf{x}(f),\omega(f)} G_{f,f'} \chi^{\alpha(f')}_{\mathbf{x}(f'),\omega(f')} . \tag{A2.5}$$

Starting from (A2.4), the proof in Appendix A3 of [GM] can be repeated step by step in the present case, to find finally the analogue of (A.3.55) of [GM]. Then, using again that $\int \mathcal{D}\chi \exp(\chi, G\chi)/2$ is, unless for a sign, the Pfaffian of G, we find (A2.2). □

We now use the well–known property $|\text{Pf}\, G^T| = \sqrt{|\det G^T|}$ and we can bound $\det G^T$ by the Gram–Hadamard (GH) inequality. Let $\mathcal{H} \stackrel{def}{=} \mathbb{R}^s \otimes \mathcal{H}_0$, where $\mathcal{H}_0$ is the Hilbert space of complex four dimensional vectors $F(\mathbf{k}) = (F_1(\mathbf{k}), \ldots, F_4(\mathbf{k}))$, $F_i(\mathbf{k})$ being a function on the set $\mathcal{D}_{-,-}$, with scalar product $< F, G >= \sum_{i=1}^4 1/M^2 \sum_{\mathbf{k}} F^*_i(\mathbf{k}) G_i(\mathbf{k})$. We can write the elements of G^T as inner products of vectors of $\mathcal{H}$:

$$G_{f,f'} = t_{i(f),i(f')} g_\chi(f, f') =< \mathbf{u}_{i(f)} \otimes A_f, \mathbf{u}_{i(f')} \otimes B_{f'} > , \tag{A2.6}$$

where $\mathbf{u}_i \in \mathbb{R}^s$, $i = 1, \ldots, s$, are vectors such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$, and, if $\hat{g}^\chi_{\underline{a},\underline{a}'}(\mathbf{k})$ is the Fourier transform of $g^\chi_{\underline{a},\underline{a}'}(\mathbf{x}-\mathbf{y})$, $A_f(\mathbf{k})$ and $B_{f'}(\mathbf{k})$ are given by

$$A_f(\mathbf{k}) = e^{-i\mathbf{k}\mathbf{x}(f)}\Big(\hat{g}^\chi_{\underline{a}(f),(-,1)}(\mathbf{k}), \hat{g}^\chi_{\underline{a}(f),(-,-1)}(\mathbf{k}), \hat{g}^\chi_{\underline{a}(f),(+,1)}(\mathbf{k}), \hat{g}^\chi_{\underline{a}(f),(+,-1)}(\mathbf{k})\Big),$$
$$B_{f'}(\mathbf{k}) = e^{-i\mathbf{k}\mathbf{x}(f')}\begin{cases}(1,0,0,0), & \text{if } \underline{a}(f') = (-,1),\\ (0,1,0,0), & \text{if } \underline{a}(f') = (-,-1),\\ (0,0,1,0), & \text{if } \underline{a}(f') = (+,1),\\ (0,0,0,1), & \text{if } \underline{a}(f') = (+,-1).\end{cases} \tag{A2.7}$$

With these definitions and remembering (2.17), it is now clear that $|Pf G^T| \le C^{n-s+1}$, for some constant C. Then, applying (A2.2) and the previous bound we find the second of (2.21).

We now turn to the construction of $P_{Z_1,\sigma_1,\mu_1,C_1}$, in order to prove (2.19).

We define $e^{-t_1 M^2} P_{Z_1,\sigma_1,\mu_1,C_1}(d\psi) \stackrel{def}{=} P_\sigma(d\psi) e^{-Q^{(1)}(\psi)}$, where t_1 is a normalization constant. In order to write $P_{Z_1,\sigma_1,\mu_1,C_1}(d\psi)$ as an exponential of a quadratic form, it is sufficient to calculate the correlations

$$\begin{aligned}< \psi^{\alpha_1}_{\omega_1,\mathbf{k}} \psi^{\alpha_2}_{\omega_2,-\alpha_1\alpha_2\mathbf{k}} >_1 &\stackrel{def}{=} \int P_{Z_1,\sigma_1,\mu_1,C_1}(d\psi)\psi^{\alpha_1}_{\omega_1,\mathbf{k}} \psi^{\alpha_2}_{\omega_2,-\alpha_1\alpha_2\mathbf{k}}\\ &= e^{-t_1 M^2}\int P_\sigma(d\psi)P(d\chi)e^{Q(\chi,\psi)}\psi^{\alpha_1}_{\omega_1,\mathbf{k}} \psi^{\alpha_2}_{\omega_2,-\alpha_1\alpha_2\mathbf{k}}.\end{aligned} \tag{A2.8}$$

It is easy to realize that the measure $\sim P_\sigma(d\psi)P(d\chi)e^{Q(\chi,\psi)}$ factorizes into the product of two measures generated by the fields $\psi^{(j)}_{\omega,\mathbf{x}}$, $j = 1, 2$, defined by $\psi^\alpha_{\omega,\mathbf{x}} = (\psi^{(1)}_{\omega,\mathbf{x}} + i(-1)^\alpha \psi^{(2)}_{\omega,\mathbf{x}})/\sqrt{2}$. In fact, using this change of variables, one finds that

$$\begin{aligned}P_\sigma(d\psi)P(d\chi)e^{Q(\chi,\psi)} &= \prod_{j=1,2} P^{(j)}(d\psi^{(j)}, d\chi^{(j)})\\ &= \prod_{j=1,2}\frac{1}{\mathcal{N}^{(j)}}\exp\{-\frac{t^{(j)}_\lambda}{4M^2}\sum_{\mathbf{k}}\xi^{(j),T}_{\mathbf{k}} C^{(j)}_{\mathbf{k}} \xi^{(j)}_{-\mathbf{k}}\},\end{aligned} \tag{A2.9}$$

for two suitable matrices $C^{(j)}_{\mathbf{k}}$, whose determinants $B^{(j)}(\mathbf{k}) \stackrel{def}{=} \det C^{(j)}_{\mathbf{k}}$ are equal to

$$B^{(j)}(\mathbf{k}) = \frac{16}{(t^{(j)}_\lambda)^4}\big\{2t^{(j)}_\lambda[1-(t^{(j)}_\lambda)^2](2-\cos k - \cos k_0) + (t^{(j)}_\lambda - t_\psi)^2(t^{(j)}_\lambda - t_\chi)^2\big\}. \tag{A2.10}$$

From the explicit expression of $C^{(j)}_{\mathbf{k}}$ one finds

$$\begin{aligned}&< \psi^{(j)}_{-\mathbf{k}}\psi^{(j)}_{\mathbf{k}} >_1 = \frac{4M^2}{t^{(j)}_\lambda}\frac{c^{(j)}_{1,1}(\mathbf{k})}{B^{(j)}(\mathbf{k})}, \quad < \overline{\psi}^{(j)}_{-\mathbf{k}}\psi^{(j)}_{\mathbf{k}} >_1 = \frac{4M^2}{t^{(j)}_\lambda}\frac{c^{(j)}_{-1,1}(\mathbf{k})}{B^{(j)}(\mathbf{k})},\\ &< \overline{\psi}^{(j)}_{-\mathbf{k}}\overline{\psi}^{(j)}_{\mathbf{k}} >_1 = \frac{4M^2}{t^{(j)}_\lambda}\frac{c^{(j)}_{-1,-1}(\mathbf{k})}{B^{(j)}(\mathbf{k})},\end{aligned} \tag{A2.11}$$

where, if $\omega = \pm 1$, recalling that $t_\psi = \sqrt{2} - 1 + \nu/2$ and defining $t_\chi = -\sqrt{2} - 1$,

$$\begin{aligned}
c^{(j)}_{\omega,\omega}(\mathbf{k}) &\stackrel{def}{=} \frac{4}{(t^{(j)}_\lambda)^2}\big\{2t^{(j)}_\lambda t_\chi(-i\sin k\cos k_0 + \omega\sin k_0\cos k)\\
&\quad+[(t^{(j)}_\lambda)^2 + t^2_\chi](i\sin k - \omega\sin k_0)\big\},\\
c^{(j)}_{\omega,-\omega}(\mathbf{k}) &\stackrel{def}{=} -i\omega\frac{4}{(t^{(j)}_\lambda)^2}\big\{-t^{(j)}_\lambda(3t_\chi + t_\psi)\cos k\cos k_0\\
&\quad+[(t^{(j)}_\lambda)^2 + 2t_\chi t_\psi + t^2_\chi](\cos k + \cos k_0)\\
&\quad-\big(t^{(j)}_\lambda(t_\psi + t_\chi) + 2\frac{t_\psi t^2_\chi}{t^{(j)}_\lambda}\big)\big\}\,. \qquad \text{(A2.12)}
\end{aligned}$$

It is clear that, for any monomial $F(\psi^{(j)})$, $\int P(d\psi^{(j)}, d\chi^{(j)})F(\psi^{(j)}) = \int P^{(j)}(d\psi^{(j)}) F(\psi^{(j)})$, with

$$\begin{aligned}
P^{(j)}(d\psi^{(j)}) &\stackrel{def}{=} \frac{1}{N_j}\prod_{\mathbf{k}} d\psi^{(j)}_{\mathbf{k}} d\overline{\psi}^{(j)}_{\mathbf{k}}\\
&\cdot\exp\Big\{-\frac{t^{(j)}_\lambda B^{(j)}(\mathbf{k})}{4M^2\det c^{(j)}_{\mathbf{k}}}(\psi^{(j)}_{\mathbf{k}}, \overline{\psi}^{(j)}_{\mathbf{k}})\begin{pmatrix} c^{(j)}_{-1,-1}(\mathbf{k}) & -c^{(j)}_{1,-1}(\mathbf{k})\\ -c^{(j)}_{-1,1}(\mathbf{k}) & c^{(j)}_{1,1}(\mathbf{k})\end{pmatrix}\begin{pmatrix}\psi^{(j)}_{-\mathbf{k}}\\ \overline{\psi}^{(j)}_{-\mathbf{k}}\end{pmatrix}\Big\}, \qquad \text{(A2.13)}
\end{aligned}$$

where $\det c^{(j)}_{\mathbf{k}} = c^{(j)}_{1,1}(\mathbf{k})c^{(j)}_{-1,-1}(\mathbf{k}) - c^{(j)}_{1,-1}(\mathbf{k})c^{(j)}_{-1,1}(\mathbf{k})$. If we now use the identity $t^{(j)}_\lambda = t_\psi(2 + (-1)^j\mu)/(2-\sigma)$ and rewrite the measure $P^{(1)}(d\psi^{(1)})P^{(2)}(d\psi^{(2)})$ in terms of $\psi^\pm_{\omega,\mathbf{k}}$ we find:

$$\begin{aligned}
P^{(1)}(d\psi^{(1)})P^{(2)}(d\psi^{(2)}) &= \frac{1}{\mathcal{N}^{(1)}}\prod_{\mathbf{k},\omega} d\psi^+_{\omega,\mathbf{k}} d\psi^-_{\omega,\mathbf{k}}\exp\{-\frac{Z_1C_1(\mathbf{k})}{4M^2}\Psi^{+,T}_{\mathbf{k}}A^{(1)}_\psi\Psi^-_{\mathbf{k}}\}\\
&= P_{Z_1,\sigma_1,\mu_1,C_1}(d\psi)\,, \qquad \text{(A2.14)}
\end{aligned}$$

with $C_1(\mathbf{k})$, Z_1, σ_1 and μ_1 defined as after (2.18), and $A^{(1)}_\psi(\mathbf{k})$ as in (2.19), with

$$\begin{aligned}
M^{(1)}(\mathbf{k}) &= \frac{2}{2-\sigma}\begin{pmatrix} -c^+_{-1,-1}(\mathbf{k}) & c^+_{-1,1}(\mathbf{k})\\ c^+_{1,-1}(\mathbf{k}) & -c^+_{1,1}(\mathbf{k})\end{pmatrix},\\
N^{(1)}(\mathbf{k}) &= \frac{2}{2-\sigma}\begin{pmatrix} -c^-_{-1,-1}(\mathbf{k}) & c^-_{-1,1}(\mathbf{k})\\ c^-_{1,-1}(\mathbf{k}) & -c^-_{1,1}(\mathbf{k})\end{pmatrix}, \qquad \text{(A2.15)}
\end{aligned}$$

where $c^\alpha_{\omega_1,\omega_2}(\mathbf{k}) \stackrel{def}{=} [(1-\mu/2)B^{(1)}(\mathbf{k})c^{(1)}_{\omega_1,\omega_2}(\mathbf{k})/\det c^{(1)}_{\mathbf{k}} + \alpha(1+\mu/2)B^{(2)}(\mathbf{k})c^{(2)}_{\omega_1,\omega_2}(\mathbf{k})/\det c^{(2)}_{\mathbf{k}}]/2$. It is easy to verify that $A^{(1)}_\psi(\mathbf{k})$ has the form (2.19). In fact, computing the functions in (A2.15), one finds that, for $\mathbf{k}$, σ_1 and μ_1 small,

$$\begin{aligned}
M^{(1)}(\mathbf{k}) &= \begin{pmatrix} \big(1+\frac{\sigma_1}{2}\big)(i\sin k + \sin k_0) + O(\mathbf{k}^3) & -i\sigma_1 + O(\mathbf{k}^2)\\ i\sigma_1 + O(\mathbf{k}^2) & \big(1+\frac{\sigma_1}{2}\big)(i\sin k - \sin k_0) + O(\mathbf{k}^3)\end{pmatrix},\\
N^{(1)}(\mathbf{k}) &= \begin{pmatrix} -\frac{\mu_1}{2}(i\sin k + \sin k_0) + O(\mathbf{k}^3) & i\mu_1 + O(\mu_1\mathbf{k}^2)\\ -i\mu_1 + O(\mu_1\mathbf{k}^2) & -\frac{\mu_1}{2}(i\sin k - \sin k_0) + O(\mathbf{k}^3)\end{pmatrix}, \qquad \text{(A2.16)}
\end{aligned}$$

where the higher order terms in $\mathbf{k}$, σ_1 and μ_1 contribute to the corrections $a_1^{\pm}(\mathbf{k})$, $b_1^{\pm}(\mathbf{k})$, $c_1(\mathbf{k})$ and $d_1(\mathbf{k})$. They have the reality and parity properties described after (2.19) and it is apparent that $a_1^{\pm}(\mathbf{k}) = O(\sigma_1\mathbf{k}) + O(\mathbf{k}^3)$, $b_1^{\pm}(\mathbf{k}) = O(\mu_1\mathbf{k}) + O(\mathbf{k}^3)$, $c_1(\mathbf{k}) = O(\mathbf{k}^2)$ and $d_1(\mathbf{k}) = O(\mu_1\mathbf{k}^2)$.

A2.2. Symmetry properties. In this section we identify some symmetries of model (2.7) and we prove that the quadratic and quartic terms in $\mathcal{V}^{(1)}$ have the structure described in (2.22), (2.23) and (2.24).

The formal action appearing in (2.7) (see also (2.2) and (2.9) for an explicit form) is invariant under the following transformations:

1) *Parity*: $H_{\mathbf{x}}^{(j)} \to \overline{H}_{-\mathbf{x}}^{(j)}$, $\overline{H}_{\mathbf{x}}^{(j)} \to -H_{-\mathbf{x}}^{(j)}$ (the same for V and $\overline{V}$). In terms of the variables $\hat\psi_{\omega,\mathbf{k}}^{\alpha}$, this transformation is equivalent to $\hat\psi_{\omega,\mathbf{k}}^{\alpha} \to i\omega\hat\psi_{\omega,-\mathbf{k}}^{\alpha}$ (the same for χ) and we shall call it *parity*.
2) *Complex conjugation*: $\hat\psi_{\omega,\mathbf{k}}^{\alpha} \to \hat\psi_{-\omega,\mathbf{k}}^{-\alpha}$ (the same for χ) and $c \to c^*$, where c is a generic constant appearing in the formal action and c^* is its complex conjugate. Note that (2.10) is left invariant by this transformation that we shall call *complex conjugation*.
3) *Hole-particle*: $H_{\mathbf{x}}^{(j)} \to (-1)^{j+1}H_{\mathbf{x}}^{(j)}$ (the same for $\overline{H}$, V, $\overline{V}$). This transformation is equivalent to $\hat\psi_{\omega,\mathbf{k}}^{\alpha} \to \hat\psi_{\omega,-\mathbf{k}}^{-\alpha}$ (the same for χ) and we shall call it *hole-particle*.
4) *Rotation*: $H_{x,x_0}^{(j)} \to i\overline{V}_{-x_0,-x}^{(j)}$, $\overline{H}_{x,x_0}^{(j)} \to iV_{-x_0,-x}^{(j)}$, $V_{x,x_0}^{(j)} \to i\overline{H}_{-x_0,-x}^{(j)}$, $\overline{V}_{x,x_0}^{(j)} \to iH_{-x_0,-x}^{(j)}$. This transformation is equivalent to
$$\hat\psi_{\omega,(k,k_0)}^{\alpha} \to -\omega e^{-i\omega\pi/4}\hat\psi_{-\omega,(-k_0,-k)}^{\alpha} \quad , \quad \hat\chi_{\omega,(k,k_0)}^{\alpha} \to \omega e^{-i\omega\pi/4}\hat\chi_{-\omega,(-k_0,-k)}^{\alpha}, \tag{A2.17}$$
and we shall call it *rotation*.
5) *Reflection*: $H_{x,x_0}^{(j)} \to i\overline{H}_{-x,x_0}^{(j)}$, $\overline{H}_{x,x_0}^{(j)} \to iH_{-x,x_0}^{(j)}$, $V_{x,x_0}^{(j)} \to -iV_{-x,x_0}^{(j)}$, $\overline{V}_{x,x_0}^{(j)} \to i\overline{V}_{-x,x_0}^{(j)}$. This transformation is equivalent to $\hat\psi_{\omega,(k,k_0)}^{\alpha} \to i\hat\psi_{-\omega,(-k,k_0)}^{\alpha}$ (the same for χ) and we shall call it *reflection*.
6) *The* $(1) \longleftrightarrow (2)$ *symmetry:* $H_{\mathbf{x}}^{(1)} \longleftrightarrow H_{\mathbf{x}}^{(2)}$, $\overline{H}_{\mathbf{x}}^{(1)} \longleftrightarrow \overline{H}_{\mathbf{x}}^{(2)}$, $V_{\mathbf{x}}^{(1)} \longleftrightarrow V_{\mathbf{x}}^{(2)}$, $\overline{V}_{\mathbf{x}}^{(1)} \longleftrightarrow \overline{V}_{\mathbf{x}}^{(2)}$, $u \to -u$. This transformation is equivalent to $\hat\psi_{\omega,\mathbf{k}}^{\alpha} \to -i\alpha\hat\psi_{\omega,-\mathbf{k}}^{-\alpha}$ (the same for χ) together with $u \to -u$ and we shall call it $(1) \longleftrightarrow (2)$ *symmetry*.

It is easy to verify that the quadratic forms $P(d\chi)$, $P(d\psi)$ and $P_{Z_1,\sigma_1,\mu_1,C_1}(d\psi)$ are separately invariant under the symmetries above. Then the effective action $\mathcal{V}^{(1)}(\psi)$ is still invariant under the same symmetries. Using the invariance of $\mathcal{V}^{(1)}$ under transformations (1)–(6), we now prove that the structure of its quadratic and quartic terms is the one described in Theorem 2.1, see in particular (2.22), (2.23) and (2.24).

Quartic term. The term $\sum_{\mathbf{k}_i} W(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\hat\psi_{1,\mathbf{k}_1}^{+}\hat\psi_{-1,\mathbf{k}_2}^{+}\hat\psi_{-1,\mathbf{k}_3}^{-}\hat\psi_{1,\mathbf{k}_4}^{-}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)$ under complex conjugation becomes equal to $\sum_{\mathbf{k}_i} W^*(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\hat\psi_{-1,\mathbf{k}_1}^{-}\hat\psi_{1,\mathbf{k}_2}^{-}\hat\psi_{1,\mathbf{k}_3}^{+}\hat\psi_{-1,\mathbf{k}_4}^{+}\delta(\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_1-\mathbf{k}_2)$, so that $W(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = W^*(\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_1,\mathbf{k}_2)$. Then, defining $L_1 = W(\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++})$, where $\bar{\mathbf{k}}_{++} = (\pi/M,\pi/M)$, and $l_1 = \mathcal{P}_0 L_1 \stackrel{def}{=} L_1\big|_{\sigma_1=\mu_1=0}$, we see that L_1 and l_1 are real. From the explicit computation of the lower order term we find $l_1 = \tilde\lambda/Z_1^2 + O(\lambda^2)$.

Quadratic terms. We distinguish 4 cases (items (a)–(d) below).
a) Let $\alpha_1 = -\alpha_2 = +$ and $\omega_1 = -\omega_2 = \omega$ and consider the expression $\sum_{\omega,\mathbf{k}} W_\omega(\mathbf{k};\mu_1)$ $\hat\psi^+_{\omega,\mathbf{k}}\hat\psi^-_{-\omega,\mathbf{k}}$. Under parity it becomes
$\sum_{\omega,\mathbf{k}} W_\omega(\mathbf{k};\mu_1)(i\omega)\hat\psi^+_{\omega,-\mathbf{k}}(-i\omega)\hat\psi^-_{-\omega,-\mathbf{k}} = \sum_{\omega,\mathbf{k}} W_\omega(-\mathbf{k};\mu_1)\hat\psi^+_{\omega,\mathbf{k}}\hat\psi^-_{-\omega,\mathbf{k}}$,
so that $W_\omega(\mathbf{k};\mu_1)$ is even in $\mathbf{k}$.
Under complex conjugation it becomes
$\sum_{\omega,\mathbf{k}} W_\omega(\mathbf{k};\mu_1)^*\hat\psi^-_{-\omega,\mathbf{k}}\hat\psi^+_{\omega,\mathbf{k}} = -\sum_{\omega,\mathbf{k}} W_\omega(\mathbf{k};\mu_1)^*\hat\psi^+_{\omega,\mathbf{k}}\hat\psi^-_{-\omega,\mathbf{k}}$,
so that $W_\omega(\mathbf{k};\mu_1)$ is purely imaginary.
Under hole-particle it becomes
$\sum_{\omega,\mathbf{k}} W_\omega(\mathbf{k};\mu_1)\hat\psi^-_{\omega,-\mathbf{k}}\hat\psi^+_{-\omega,-\mathbf{k}} = -\sum_{\omega,\mathbf{k}} W_{-\omega}(\mathbf{k};\mu_1)\hat\psi^+_{\omega,\mathbf{k}}\hat\psi^-_{-\omega,\mathbf{k}}$,
so that $W_\omega(\mathbf{k};\mu_1)$ is odd in ω.
Under (1) $\longleftrightarrow$ (2) it becomes
$\sum_{\omega,\mathbf{k}} W_\omega(\mathbf{k};-\mu_1)(-i)\hat\psi^-_{-\omega,-\mathbf{k}}(i)\hat\psi^+_{\omega,-\mathbf{k}} = \sum_{\omega,\mathbf{k}} W_\omega(\mathbf{k};-\mu_1)\hat\psi^+_{\omega,\mathbf{k}}\hat\psi^-_{-\omega,\mathbf{k}}$,
so that $W_\omega(\mathbf{k};\mu_1)$ is even in μ_1. Let us define $S_1 = i\omega/2\sum_{\eta,\eta'=\pm1} W_\omega(\bar{\mathbf{k}}_{\eta\eta'})$, where $\bar{\mathbf{k}}_{\eta\eta'} = (\eta\pi/M, \eta'\pi/M)$, and $\gamma n_1 = \mathcal{P}_0 S_1$, $s_1 = \mathcal{P}_1 S_1 = \sigma_1\partial_{\sigma_1}S_1\big|_{\sigma_1=\mu_1=0} + \mu_1\partial_{\mu_1}S_1\big|_{\sigma_1=\mu_1=0}$. From the previous discussion we see that S_1, s_1 and n_1 are real and s_1 is independent of μ_1. From the computation of the lower order terms we find $s_1 = O(\lambda\sigma_1)$ and $\gamma n_1 = \nu/Z_1 + c_1^\nu\lambda + O(\lambda^2)$, for some constant c_1^ν independent of λ. Note that since $W_\omega(\mathbf{k};\mu_1)$ is even in $\mathbf{k}$ (so that in particular no linear terms in $\mathbf{k}$ appear) in real space no terms of the form $\psi^+_{\omega,\mathbf{x}}\partial\psi^-_{-\omega,\mathbf{x}}$ can appear.
b) Let $\alpha_1 = \alpha_2 = \alpha$ and $\omega_1 = -\omega_2 = \omega$ and consider the expression $\sum_{\omega,\alpha,\mathbf{k}} W^\alpha_\omega(\mathbf{k};\mu_1)$ $\hat\psi^\alpha_{\omega,\mathbf{k}}\hat\psi^\alpha_{-\omega,-\mathbf{k}}$. We proceed as in item (a) and, by using parity, we see that $W^\alpha_\omega(\mathbf{k};\mu_1)$ is even in $\mathbf{k}$ and odd in ω.
By using complex conjugation, we see that $W^\alpha_\omega(\mathbf{k};\mu_1) = -W^{-\alpha}_\omega(\mathbf{k};\mu_1)^*$.
By using hole-particle, we see that $W^\alpha_\omega(\mathbf{k};\mu_1)$ is even in α and $W^\alpha_\omega(\mathbf{k};\mu_1) = -W^{-\alpha}_\omega(\mathbf{k};\mu_1)^*$ implies that $W^\alpha_\omega(\mathbf{k};\mu_1)$ is purely imaginary.
By using (1) $\longleftrightarrow$ (2) we see that $W^\alpha_\omega(\mathbf{k};\mu_1)$ is odd in μ_1.

If we define $M_1 = -i\omega/2\sum_{\eta,\eta'} W^\alpha_\omega(\bar{\mathbf{k}}_{\eta\eta'};\mu_1)$ and $m_1 = \mathcal{P}_1 M_1$, from the previous properties it follows that M_1 and m_1 are real, m_1 is independent of σ_1 and, from the computation of its lower order, $m_1 = O(\lambda\mu_1)$. Note that since $W^\alpha_\omega(\mathbf{k};\mu_1)$ is even in $\mathbf{k}$ (so that in particular no linear terms in $\mathbf{k}$ appear) in real space no terms of the form $\psi^\alpha_{\omega,\mathbf{x}}\partial\psi^\alpha_{-\omega,\mathbf{x}}$ can appear.
c) Let $\alpha_1 = -\alpha_2 = +$, $\omega_1 = \omega_2 = \omega$ and consider the expression $\sum_{\omega,\mathbf{k}} W_\omega(\mathbf{k};\mu_1)\hat\psi^+_{\omega,\mathbf{k}}$ $\hat\psi^-_{\omega,\mathbf{k}}$. By using parity we see that $W_\omega(\mathbf{k};\mu_1)$ is odd in $\mathbf{k}$.
By using reflection we see that $W_\omega(k,k_0;\mu_1) = W_{-\omega}(k,-k_0;\mu_1)$.
By using complex conjugation we see that $W_\omega(k,k_0;\mu_1) = W^*_\omega(-k,k_0;\mu_1)$.
By using rotation we find $W_\omega(k,k_0;\mu_1) = -i\omega W_\omega(k_0,-k;\mu_1)$.
By using (1) $\longleftrightarrow$ (2) we see that $W_\omega(\mathbf{k};-\mu_1)$ is even in μ_1.

If we define
$$G_1(\mathbf{k}) = \frac{1}{4}\sum_{\eta,\eta'} W_\omega(\bar{\mathbf{k}}_{\eta\eta'};\mu_1)(\eta\frac{\sin k}{\sin\pi/M} + \eta'\frac{\sin k_0}{\sin\pi/M})$$
$$= a_\omega\sin k + b_\omega\sin k_0\ , \tag{A2.18}$$
it can be easily verified that the previous properties imply that
$$a_\omega = a_{-\omega} = -a^*_\omega = i\omega b_\omega \stackrel{def}{=} ia \quad , \quad b_\omega = -b_{-\omega} = b^*_\omega = -i\omega a_\omega \stackrel{def}{=} \omega b = -i\omega ia \tag{A2.19}$$

with $a = b$ real and independent of ω. As a consequence, $G_1(\mathbf{k}) = G_1(i \sin k + \omega \sin k_0)$ for some real constant G_1. If $z_1 \stackrel{def}{=} \mathcal{P}_0 G_1$ and we compute the lowest order contribution to z_1, we find $z_1 = O(\lambda^2)$.
d) Let $\alpha_1 = \alpha_2 = \alpha, \omega_1 = \omega_2 = \omega$ and consider the expression $\sum_{\alpha,\omega,\mathbf{k}} W^\alpha_\omega(\mathbf{k}; \mu_1)\hat\psi^\alpha_{\omega,\mathbf{k}}$ $\hat\psi^\alpha_{\omega,-\mathbf{k}}$. Repeating the proof in item (c) we see that $W^\alpha_\omega(\mathbf{k}; \mu_1)$ is odd in $\mathbf{k}$ and in μ_1 and, if we define $F_1(\mathbf{k}) = \frac{1}{4}\sum_{\eta,\eta'} W^\alpha_\omega(\bar{\mathbf{k}}_{\eta\eta'}; \mu_1)(\eta \frac{\sin k}{\sin \pi/M} + \eta' \frac{\sin k_0}{\sin\pi/M})$, we can rewrite $F_1(\mathbf{k}) = F_1(i \sin k + \omega \sin k_0)$. Since $W^\alpha_\omega(\mathbf{k}; \mu_1)$ is odd in μ_1, we find $F_1 = O(\lambda\mu_1)$.

Note that with the definition of $\mathcal{L}$ introduced in §3.2, the result of the previous discussion is the following:

$$\mathcal{L}\mathcal{V}^{(1)}(\psi) = (s_1 + \gamma n_1)F^{(\le 1)}_\sigma + m_1 F^{(\le 1)}_\mu + l_1 F^{(\le 1)}_\lambda + z_1 F^{(\le 1)}_\zeta\ , \tag{A2.20}$$

where s_1, n_1, m_1, l_1 and z_1 are real constants and: s_1 is linear in σ_1 and independent of μ_1; m_1 is linear in μ_1 and independent of σ_1; n_1, l_1, z_1 are independent of σ_1, μ_1; moreover $F^{(\le 1)}_\sigma$, $F^{(\le 1)}_\mu$, $F^{(\le 1)}_\lambda$, $F^{(\le 1)}_\zeta$ are defined by (3.8) with $h = 1$.

Proof of Lemma 3.1. The symmetries (1)–(6) discussed above are preserved by the iterative integration procedure. In fact it is easy to verify that $\mathcal{L}\mathcal{V}^{(h)}$, $\mathcal{R}\mathcal{V}^{(h)}$ and $P_{Z_{h-1},\sigma_{h-1},\mu_{h-1},\tilde f_h}(d\psi^{(h)})$ are, step by step, separately invariant under the transformations (1)–(6). Then Lemma 3.1 can be proven exactly in the same way (A2.20) was proven above. □

Proof of Lemma 3.2. It is sufficient to note that the symmetry properties discussed above imply that $\mathcal{L}_1 W_{2,\underline{\alpha},\underline{\omega}} = 0$ if $\omega_1 + \omega_2 = 0$; $\mathcal{L}_0 W_{2,\underline{\alpha},\underline{\omega}} = 0$ if $\omega_1 + \omega_2 \neq 0$; $\mathcal{P}_0 W_{2,\underline{\alpha},\underline{\omega}} = 0$ if $\alpha_1 + \alpha_2 \neq 0$; and use the definitions of $\mathcal{R}_i, \mathcal{S}_i, i = 1, 2$. □

Appendix A3. Proof of Lemma 3.3

The propagators $g^{(h)}_{\underline{a},\underline{a}'}(\mathbf{x})$ can be written in terms of the propagators $g^{(j,h)}_{\omega,\omega'}(\mathbf{x})$, $j = 1, 2$, see (3.16) and the following lines; $g^{(j,h)}_{\omega,\omega'}(\mathbf{x})$ are given by

$$\begin{aligned} g^{(j,h)}_{\omega,\omega}&(\mathbf{x}-\mathbf{y}) \\ &= \frac{2}{M^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\tilde f_h(\mathbf{k})\frac{-i\sin k + \omega \sin k_0 + a^{-(j)}_{h-1}(\mathbf{k})}{\sin^2 k + \sin^2 k_0 + \big(\overline{m}^{(j)}_{h-1}(\mathbf{k})\big)^2 + \delta B^{(j)}_{h-1}(\mathbf{k})}, \\ g^{(j,h)}_{\omega,-\omega}&(\mathbf{x}-\mathbf{y}) \\ &= \frac{2}{M^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\tilde f_h(\mathbf{k})\frac{-i\omega\overline{m}^{(j)}_{h-1}(\mathbf{k})}{\sin^2 k + \sin^2 k_0 + \big(\overline{m}^{(j)}_{h-1}(\mathbf{k})\big)^2 + \delta B^{(j)}_{h-1}(\mathbf{k})}, \end{aligned} \tag{A3.1}$$

where

$$\begin{aligned} a^{\omega(j)}_{h-1}(\mathbf{k}) &\stackrel{def}{=} -a^\omega_{h-1}(\mathbf{k}) + (-1)^j b^\omega_{h-1}(\mathbf{k}) \quad , \quad c^{(j)}_{h-1}(\mathbf{k}) \stackrel{def}{=} c_{h-1}(\mathbf{k}) + (-1)^j d_{h-1}(\mathbf{k}), \\ m^{(j)}_{h-1}(\mathbf{k}) &\stackrel{def}{=} \sigma_{h-1}(\mathbf{k}) + (-1)^j \mu_{h-1}(\mathbf{k}) \quad , \quad \overline{m}^{(j)}_{h-1}(\mathbf{k}) \stackrel{def}{=} m^{(j)}_{h-1}(\mathbf{k}) + c^{(j)}(\mathbf{k}), \\ \delta B^{(j)}_{h-1}(\mathbf{k}) &\stackrel{def}{=} \sum_\omega \big[a^{\omega(j)}_{h-1}(\mathbf{k})(i\sin k - \omega \sin k_0) + a^{\omega(j)}_{h-1}(\mathbf{k})a^{-\omega(j)}_{h-1}(\mathbf{k})/2\big]\,. \end{aligned} \tag{A3.2}$$

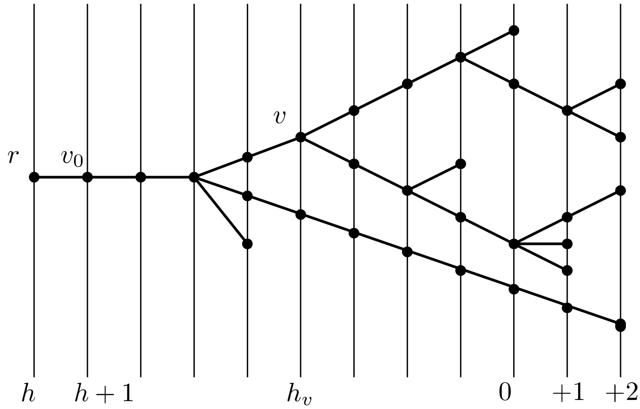

Fig. 4. A tree with its scale labels

In order to bound the propagators defined above, we need estimates on $\sigma_h(\mathbf{k})$, $\mu_h(\mathbf{k})$ and on the "corrections" $a^\omega_{h-1}(\mathbf{k})$, $b^\omega_{h-1}(\mathbf{k})$, $c_{h-1}(\mathbf{k})$, $d_{h-1}(\mathbf{k})$. As regarding $\sigma_h(\mathbf{k})$ and $\mu_h(\mathbf{k})$, in [BM] is proved (see Proof of Lemma 2.6) that, on the support of $f_h(\mathbf{k})$, for some c, $c^{-1}|\sigma_h| \le |\sigma_{h-1}(\mathbf{k})| \le c|\sigma_h|$ and $c^{-1}|\mu_h| \le |\mu_{h-1}(\mathbf{k})| \le c|\mu_h|$. Note also that, if $h \ge \bar{h}$, using the first two of (3.18), we have $\frac{|\sigma_h|+|\mu_h|}{\gamma^h} \le 2C_1$. As regarding the corrections, using their iterative definition (3.11), the asymptotic estimates near $\mathbf{k} = \mathbf{0}$ of the corrections on scale $h = 1$ (see lines after (2.19)) and the hypothesis (3.18), we easily find that, on the support of $f_h(\mathbf{k})$:

$$
\begin{aligned}
a^\omega_{h-1}(\mathbf{k}) &= O(\sigma_h \gamma^{(1-2c|\lambda|)h}) + O(\gamma^{(3-c|\lambda|^2)h}) \quad , \\
b^\omega_h(\mathbf{k}) &= O(\mu_h \gamma^{(1-2c|\lambda|)h}) + O(\gamma^{(3-c|\lambda|^2)h}) \, , \\
c_h(\mathbf{k}) &= O(\gamma^{(2-c|\lambda|^2)h}) \quad , \quad d_h(\mathbf{k}) = O(\mu_h \gamma^{(2-2c|\lambda|)h}) \, .
\end{aligned} \tag{A3.3}
$$

The bounds on the propagators follow from the remark that, as a consequence of the estimates discussed above, the denominators in (A3.1) are $O(\gamma^{2h})$ on the support of f_h.

Appendix A4. Analyticity of the Effective Potentials

It is possible to write $\mathcal{V}^{(h)}$ (3.3) in terms of *Gallavotti-Nicolo' trees.* See Fig. 4.

We need some definitions and notations.

1) Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \ge 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. Then the number of unlabeled trees with n end-points is bounded by 4^n.
2) We associate a label $h \le 0$ with the root and we denote $\mathcal{T}_{h,n}$ the corresponding set of labeled trees with n endpoints. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in $[h, 2]$, and we represent any tree $\tau \in \mathcal{T}_{h,n}$ so that, if v is an endpoint or a non-trivial vertex, it is contained in a vertical line with

index $h_v > h$, to be called the *scale* of v, while the root is on the line with index h. There is the constraint that, if v is an endpoint, $h_v > h + 1$; if there is only one end-point its scale must be equal to $h + 2$, for $h \le 0$.

Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint; its scale is $h + 1$.
3) With each endpoint v of scale $h_v = +2$ we associate one of the contributions to $\mathcal{V}^{(1)}$ given by (2.21); with each endpoint v of scale $h_v \le 1$ one of the terms in $\mathcal{L}\mathcal{V}^{(h_v-1)}$ defined in (3.7). Moreover, we impose the constraint that, if v is an endpoint and $h_v \le 1$, $h_v = h_{v'} + 1$, if v' is the non-trivial vertex immediately preceding v.
4) We introduce a *field label* f to distinguish the field variables appearing in the terms associated with the endpoints as in item 3); the set of field labels associated with the endpoint v will be called I_v. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\mathbf{x}(f)$, $\sigma(f)$ and $\omega(f)$ will denote the space-time point, the σ index and the ω index, respectively, of the field variable with label f.
5) We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \dots, v_{s_v}$ are the s_v vertices immediately following it, then $P_v \subset \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. We shall denote Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The subsets $P_{v_i} \backslash Q_{v_i}$, whose union will be made, by definition, of the *internal fields* of v, have to be non empty, if $s_v > 1$, that is if v is a non trivial vertex. Given $\tau \in \mathcal{T}_{j,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with the previous constraints; let us call $\mathbf{P}$ one of these choices. Given $\mathbf{P}$, we consider the family $\mathcal{G}_\mathbf{P}$ of all connected Feynman graphs, such that, for any $v \in \tau$, the internal fields of v are paired by propagators of scale h_v, so that the following condition is satisfied: for any $v \in \tau$, the subgraph built by the propagators associated with all vertices $v' \ge v$ is connected. The sets P_v have, in this picture, the role of the external legs of the subgraph associated with v. The graphs belonging to $\mathcal{G}_\mathbf{P}$ will be called *compatible with* $\mathbf{P}$ and we shall denote $\mathcal{P}_\tau$ the family of all choices of $\mathbf{P}$ such that $\mathcal{G}_\mathbf{P}$ is not empty.
6) We associate with any vertex v an index $\rho_v \in \{s, p\}$ and correspondingly an operator $\mathcal{R}_{\rho_v}$, where $\mathcal{R}_s$ or $\mathcal{R}_p$ are defined as

$$\mathcal{R}_s \stackrel{def}{=} \begin{cases} \mathcal{S}_2 & \text{if } n = 1 \text{ and } \omega_1 + \omega_2 = 0, \\ \mathcal{R}_1\mathcal{S}_1 & \text{if } n = 1 \text{ and } \omega_1 + \omega_2 \ne 0, \\ \mathcal{S}_1 & \text{if } n = 2, \\ 1 & \text{if } n > 2; \end{cases} \tag{A4.1}$$

and

$$\mathcal{R}_p \stackrel{def}{=} \begin{cases} \mathcal{R}_2(\mathcal{P}_0 + \mathcal{P}_1) & \text{if } n = 1 \text{ and } \omega_1 + \omega_2 = 0, \\ \mathcal{R}_2\mathcal{P}_0 & \text{if } n = 1, \omega_1 + \omega_2 \ne 0 \text{ and } \alpha_1 + \alpha_2 = 0, \\ 0 & \text{if } n = 1, \omega_1 + \omega_2 \ne 0 \text{ and } \alpha_1 + \alpha_2 \ne 0, \\ \mathcal{R}_1\mathcal{P}_0 & \text{if } n = 2, \\ 0 & \text{if } n > 2. \end{cases} \tag{A4.2}$$

Note that $\mathcal{R}_s + \mathcal{R}_p = \mathcal{R}$, see Lemma 3.2.

The effective potential can be written in the following way:

$$\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) + M^2 \tilde{E}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau, \sqrt{Z_h}\psi^{(\le h)}), \tag{A4.3}$$

where, if v_0 is the first vertex of τ and $\tau_1, \ldots, \tau_s$ are the subtrees of τ with root v_0, $\mathcal{V}^{(h)}(\tau, \sqrt{Z_h}\psi^{(\le h)})$ is defined inductively by the relation

$$\mathcal{V}^{(h)}(\tau, \sqrt{Z_h}\psi^{(\le h)}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}^T_{h+1}[\bar{V}^{(h+1)}(\tau_1, \sqrt{Z_h}\psi^{(\le h+1)}); \ldots ; \bar{V}^{(h+1)}(\tau_s, \sqrt{Z_h}\psi^{(\le h+1)})] , \tag{A4.4}$$

and $\bar{V}^{(h+1)}(\tau_i, \sqrt{Z_h}\psi^{(\le h+1)})$:

a) is equal to $\mathcal{R}_{\rho_{v_i}} \widehat{\mathcal{V}}^{(h+1)}(\tau_i, \sqrt{Z_h}\psi^{(\le h+1)})$ if the subtree τ_i with first vertex v_i is not trivial (see (3.12) for the definition of $\widehat{\mathcal{V}}^{(h)}$);

b) if τ_i is trivial and $h \le -1$, it is equal to one of the terms in $\mathcal{L}\widehat{\mathcal{V}}^{(h+1)}$, see (3.12), or, if $h = 0$, to one of the terms contributing to $\widehat{\mathcal{V}}^{(1)}(\sqrt{Z_1}\psi^{\le 1})$.

A4.1. The explicit expression for the kernels of $\mathcal{V}^{(h)}$ can be found from (A4.3) and (A4.4) by writing the truncated expectations of monomials of ψ fields using the analogue of (A2.2): if $\widetilde{\psi}(P_{v_i}) = \prod_{f \in P_{v_i}} \psi^{\alpha(f)(h_v)}_{\mathbf{x}(f),\omega(f)}$, the following identity holds:

$$\mathcal{E}^T_{h_v}(\widetilde{\psi}(P_{v_1}),\ldots,\widetilde{\psi}(P_{v_s}))=\Big(\frac{1}{Z_{h_v-1}}\Big)^n \sum_{T_v} \alpha_{T_v} \prod_{\ell \in T_v} g^{(h_v)}(f^1_\ell, f^2_\ell) \int dP_{T_v}(\mathbf{t}) \mathrm{Pf}\, G^{T_v}(\mathbf{t}), \tag{A4.5}$$

where $g^{(h)}(f, f') = g_{\underline{a}(f),\underline{a}(f')}(\mathbf{x}(f) - \mathbf{x}(f'))$ and the other symbols in a.1 have the same meaning as those in A2.2.

Using iteratively A4.5 we can express the kernels of $\mathcal{V}^{(h)}$ as sums of products of propagators of the fields (the ones associated to the anchored trees T_v) and Pfaffians of matrices G^{T_v}.

A4.2. If the $\mathcal{R}$ operator were not applied to the vertices $v \in \tau$ then the result of the iteration would lead to the following relation:

$$\mathcal{V}^*_h(\tau, \sqrt{Z_h}\psi^{(\le h)}) = \sqrt{Z_h}^{|P_{v_0}|} \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sum_{T \in \mathbf{T}} \int d\mathbf{x}_{v_0} W^*_{\tau,\mathbf{P},\mathbf{T}}(\mathbf{x}_{v_0}) \Big\{ \prod_{f \in P_{v_0}} \psi^{\alpha(f)(\le h)}_{\mathbf{x}(f),\omega(f)} \Big\} , \tag{A4.6}$$

where $\mathbf{x}_{v_0}$ is the set of integration variables asociated to τ and $T = \bigcup_v T_v$; $W^*_{\tau,\mathbf{P},\mathbf{T}}$ is given by

$$W^*_{\tau,\mathbf{P},\mathbf{T}}(\mathbf{x}_{v_0}) = \Big[\prod_{v\,\text{not e.p.}} \Big(\frac{Z_{h_v}}{Z_{h_v-1}}\Big)^{\frac{|P_v|}{2}} \Big]\Big[\prod_{i=1}^n K^{h_i}_{v^*_i}(\mathbf{x}_{v^*_i})\Big]\Big\{ \prod_{v\,\text{not e.p.}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \cdot \mathrm{Pf}\, G^{h_v,T_v}(\mathbf{t}_v)\Big[\prod_{l \in T_v} g^{(h_v)}(f^1_l, f^2_l)\Big]\Big\} , \tag{A4.7}$$

where: $e.p.$ is an abbreviation of "end points"; $v_1^*, \dots, v_n^*$ are the endpoints of τ, $h_i = h_{v_i^*}$ and $K_v^{h_v}(\mathbf{x}_v)$ are the corresponding kernels (equal to $\lambda_{h_v-1}\delta(\mathbf{x}_v)$ or $\nu_{h_v-1}\delta(\mathbf{x}_v)$ if v is an endpoint of type λ or ν on scale $h_v \le 1$; or equal to one of the kernels of $\mathcal{V}^{(1)}$ if $h_v = 2$).

We can bound (A4.7) using (3.20) and the Gram–Hadamard inequality, see Appendix A2, we would find:

$$\int d\mathbf{x}_{v_0} |W^*_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0})| \le C^n M^2 |\lambda|^n \gamma^{-h(-2+|P_{v_0}|/2)}$$
$$\times \prod_{v \text{ not e.p.}} \left\{ \frac{1}{s_v!} \left(\frac{Z_{h_v}}{Z_{h_v-1}} \right)^{\frac{|P_v|}{2}} \gamma^{-[-2+\frac{|P_v|}{2}]} \right\} . \tag{A4.8}$$

We call $D_v = -2 + \frac{|P_v|}{2}$ the *dimension* of v, depending on the number of the external fields of v. If $D_v < 0$ for any v one can sum over $\tau, \mathbf{P}, T$ obtaining convergence for λ small enough; however $D_v \le 0$ when there are two or four external lines. We will take now into account the effect of the $\mathcal{R}$ operator and we will see how the bound (A4.21) is improved.

A4.3. The effect of application of $\mathcal{P}_j$ and $\mathcal{S}_j$ is to replace a kernel $W^{(h)}_{2n,\underline{\sigma},\underline{j},\underline{\alpha},\underline{\omega}}$ with $\mathcal{P}_j W^{(h)}_{2n,\underline{\sigma},\underline{j},\underline{\alpha},\underline{\omega}}$ and $\mathcal{S}_j W^{(h)}_{2n,\underline{\sigma},\underline{j},\underline{\alpha},\underline{\omega}}$. If inductively, starting from the end–points, we write the kernels $W^{(h)}_{2n,\underline{\sigma},\underline{j},\underline{\alpha},\underline{\omega}}$ in a form similar to (A4.7), we easily realize that, eventually, $\mathcal{P}_j$ or $\mathcal{S}_j$ will act on some propagator of an anchored tree or on some Pfaffian Pf G^{T_v}, for some v. It is easy to realize that $\mathcal{P}_j$ and $\mathcal{S}_j$, when applied to Pfaffians, do not break the Pfaffian structure. In fact the effect of $\mathcal{P}_j$ on the Pfaffian of an antisymmetric matrix G with elements $G_{f,f'}$, $f, f' \in J$, $|J| = 2k$, is the following (the proof is trivial):

$$\mathcal{P}_0 \text{Pf}\, G = \text{Pf}\, G^0 \quad , \quad \mathcal{P}_1 \text{Pf}\, G = \frac{1}{2} \sum_{f_1, f_2 \in J} \mathcal{P}_1 G_{f_1,f_2} (-1)^\pi \text{Pf}\, G_1^0 \, , \tag{A4.9}$$

where G^0 is the matrix with elements $\mathcal{P}_0 G_{f,f'}$, $f, f' \in J$; G_1^0 is the matrix with elements $\mathcal{P}_0 G_{f,f'}$, $f, f' \in J_1 \stackrel{def}{=} J \setminus \{f_1 \cup f_2\}$ and $(-1)^\pi$ is the sign of the permutation leading from the ordering J of the labels f in the l.h.s. to the ordering f_1, f_2, J_1 in the r.h.s. The effect of $\mathcal{S}_j$ is the following, see Appendix A7 for a proof:

$$\mathcal{S}_1 \text{Pf}\, G = \frac{1}{2 \cdot k!} \sum_{f_1, f_2 \in J} \mathcal{S}_1 G_{f_1,f_2} \sum^{*}_{J_1 \cup J_2 = J \setminus \cup_i f_i} (-1)^\pi k_1! k_2! \,\text{Pf}\, G_1^0 \,\text{Pf}\, G_2 \, , \tag{A4.10}$$

where the $*$ on the sum means that $J_1 \cap J_2 = \emptyset$; $|J_i| = 2k_i$, $i = 1, 2$; $(-1)^\pi$ is the sign of the permutation leading from the ordering J of the field labels on the l.h.s. to the ordering f_1, f_2, J_1, J_2 on the r.h.s.; G_1^0 is the matrix with elements $\mathcal{P}_0 G_{f,f'}$, $f, f' \in J_1$; G_2 is the matrix with elements $G_{f,f'}$, $f, f' \in J_2$. The effect of $\mathcal{S}_2$ on Pf G^T is given by a formula similar to (A4.10). Note that the number of terms in the sums appearing in (A4.9), (A4.10) (and in the analogous equation for $\mathcal{S}_2$Pf G^T), is bounded by c^k for some constant c.

A4.4. It is possible to show that the $\mathcal{R}_j$ operators produce derivatives applied to the propagators of the anchored trees and on the elements of G^{T_v}; and a product of "zeros" of the form $d_j^b(\mathbf{x}(f_\ell^1) - \mathbf{x}(f_\ell^2))$, $j = 0, 1$, $b = 0, 1, 2$, associated to the lines $\ell \in T_v$. This is a well known result, and a very detailed discussion can be found in §3 of [BM]. By such analysis, and using (A4.9),(A4.10), we get the following expression for $\mathcal{R}\mathcal{V}^{(h)}(\tau, \sqrt{Z_h}\psi^{(\le h)})$:

$$\mathcal{R}\mathcal{V}^{(h)}(\tau, \sqrt{Z_h}\psi^{(\le h)}) = \sqrt{Z_h}^{|P_{v_0}|} \sum_{\mathbf{P}\in\mathcal{P}_\tau} \sum_{T\in\mathbf{T}} \sum_{\beta\in B_T} \int d\mathbf{x}_{v_0} W_{\tau,\mathbf{P},\mathbf{T},\beta}(\mathbf{x}_{v_0}) \Big\{ \prod_{f\in P_{v_0}} \hat\partial^{q_\beta(f)}_{j_\beta(f)} \psi^{\alpha(f)(\le h)}_{\mathbf{x}_\beta(f),\omega(f)} \Big\}, \tag{A4.11}$$

where B_T is a set of indices which allows to distinguish the different terms produced by the non trivial $\mathcal{R}$ operations; $\mathbf{x}_\beta(f)$ is a coordinate obtained by interpolating two points in $\mathbf{x}_{v_0}$, in a suitable way depending on β; $q_\beta(f)$ is a nonnegative integer ≤ 2; $j_\beta(f) = 0, 1$ and $\hat\partial^q_j$ is a suitable differential operator, dimensionally equivalent to ∂^q_j (see [BM] for a precise definition); $W_{\tau,\mathbf{P},\mathbf{T},\beta}$ is given by:

$$W_{\tau,\mathbf{P},\mathbf{T},\beta}(\mathbf{x}_{v_0}) = \Big[\prod_{v \text{ not e.p.}} \Big(\frac{Z_{h_v}}{Z_{h_v-1}}\Big)^{\frac{|P_v|}{2}}\Big]\Big[\prod_{i=1}^n d^{b_\beta(v_i^*)}_{j_\beta(v_i^*)}(\mathbf{x}^i_\beta, \mathbf{y}^i_\beta)\mathcal{P}^{C_\beta(v_i^*)}_{I_\beta(v_i^*)} \mathcal{S}^{c_\beta(v_i^*)}_{i_\beta(v_i^*)} K^{h_i}_{v_i^*}(\mathbf{x}_{v_i^*})\Big] \cdot\Big\{\prod_{v \text{ not e.p.}} \frac{1}{s_v!}\int dP_{T_v}(\mathbf{t}_v)\mathcal{P}^{C_\beta(v)}_{I_\beta(v)}\mathcal{S}^{c_\beta(v)}_{i_\beta(v)} \mathrm{Pf}\, G^{h_v,T_v}_\beta(\mathbf{t}_v)\cdot \cdot\Big[\prod_{l\in T_v} \hat\partial^{q_\beta(f_l^1)}_{j_\beta(f_l^1)}\hat\partial^{q_\beta(f_l^2)}_{j_\beta(f_l^2)}[d^{b_\beta(l)}_{j_\beta(l)}(\mathbf{x}_l, \mathbf{y}_l)\mathcal{P}^{C_\beta(l)}_{I_\beta(l)}\mathcal{S}^{c_\beta(l)}_{i_\beta(l)} g^{(h_v)}(f_l^1, f_l^2)]\Big]\Big\}, \tag{A4.12}$$

where $v_1^*, \dots, v_n^*$ are the endpoints of τ; $b_\beta(v)$, $b_\beta(l)$, $q_\beta(f_l^1)$ and $q_\beta(f_l^2)$ are nonnegative integers ≤ 2; $j_\beta(v)$, $j_\beta(f_l^1)$, $j_\beta(f_l^2)$ and $j_\beta(l)$ can be 0 or 1; $i_\beta(v)$ and $i_\beta(l)$ can be 1 or 2; $I_\beta(v)$ and $I_\beta(l)$ can be 0 or 1; $C_\beta(v)$, $c_\beta(v)$, $C_\beta(l)$ and $c_\beta(l)$ can be 0, 1 and $\max\{C_\beta(v) + c_\beta(v), C_\beta(l) + c_\beta(l\}) \le 1$; $G^{h_v,T_v}_\beta(\mathbf{t}_v)$ is obtained from $G^{h_v,T_v}(\mathbf{t}_v)$ by substituting the element $t_{i(f),i(f')}g^{(h_v)}(f, f')$ with $t_{i(f),i(f')}\hat\partial^{q_\beta(f)}_{j_\beta(f)}\hat\partial^{q_\beta(f')}_{j_\beta(f')}g^{(h_v)}(f, f')$.

It would be very difficult to give a precise description of the various contributions of the sum over B_T, but fortunately we only need to know some very general properties, which easily follow from the construction in §3.

1) There is a constant C such that, $\forall T \in \mathbf{T}_\tau$, $|B_T| \le C^n$; for any $\beta \in B_T$, the following inequality is satisfied:

$$\Big[\prod_{f\in\cup_v P_v} \gamma^{h(f)q_\beta(f)}\Big]\Big[\prod_{l\in T}\gamma^{-h(l)b_\beta(l)}\Big] \le \prod_{v \text{ not e.p.}} \gamma^{-z(P_v)}, \tag{A4.13}$$

where $h(f) = h_{v_0} - 1$ if $f \in P_{v_0}$, otherwise it is the scale of the vertex where the field with label f is contracted; $h(l) = h_v$, if $l \in T_v$ and

$$z(P_v) = \begin{cases} 1 & if |P_v| = 4 \; and \; \rho_v = p, \\ 2 & if |P_v| = 2 \; and \; \rho_v = p, \\ 1 & if |P_v| = 2, \rho_v = s \; and \; \sum_{f\in P_v}\omega(f) \ne 0, \\ 0 & \text{otherwise.}\end{cases} \tag{A4.14}$$

2) If we define

$$\prod_{v\in\tau}\Big[\Big(\frac{|\sigma_{h_v}|+|\mu_{h_v}|}{\gamma^{h_v}}\Big)^{c_\beta(v)i_\beta(v)}\prod_{\ell\in T_v}\Big(\frac{|\sigma_{h_v}|+|\mu_{h_v}|}{\gamma^{h_v}}\Big)^{c_\beta(\ell)i_\beta(\ell)}\Big]$$
$$\stackrel{def}{=}\prod_{v\in V_\beta}\Big(\frac{|\sigma_{h_v}|+|\mu_{h_v}|}{\gamma^{h_v}}\Big)^{i(v,\beta)}, \tag{A4.15}$$

the indices $i(v,\beta)$ satisfy, for any B_T, the following property:

$$\sum_{w\geq v} i(v,\beta)\geq z'(P_v)\ , \tag{A4.16}$$

where

$$z'(P_v)=\begin{cases}1 & \text{if } |P_v|=4 \text{ and } \rho_v=s\ ,\\ 2 & \text{if } |P_v|=2 \text{ and } \rho_v=s and \sum_{f\in P_v}\omega(f)=0\ ,\\ 1 & \text{if } |P_v|=2,\ \rho_v=s \text{ and } \sum_{f\in P_v}\omega(f)\neq 0\ ,\\ 0 & \text{otherwise.}\end{cases} \tag{A4.17}$$

A4.5. We can bound any $|\mathcal{P}_{I_\beta(v)}^{C_\beta(v)}\mathcal{S}_{i_\beta(v)}^{c_\beta(v)}\mathrm{Pf}\,G_\beta^{h_v,T_v}|$ in (A4.12), with $C_\beta(v)+c_\beta(v)=0,1$, by using (A4.9), (A4.10) and Gram inequality, as illustrated in Appendix A2 for the case of the integration of the χ fields. Using that the elements of G are all propagators on scale h_v, dimensionally bounded as in Lemma 3.3, we find:

$$|\mathcal{P}_{I_\beta(v)}^{C_\beta(v)}\mathcal{S}_{i_\beta(v)}^{c_\beta(v)}\mathrm{Pf}\,G_\beta^{h_v,T_v}|\leq C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)}$$
$$\cdot\gamma^{\frac{h_v}{2}\left(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)\right)}\Big[\prod_{f\in J_v}\gamma^{h_v q_\beta(f)}\Big]\Big(\frac{|\sigma_{h_v}|+|\mu_{h_v}|}{\gamma^{h_v}}\Big)^{c_\beta(v)i_\beta(v)+C_\beta(v)I_\beta(v)}, \tag{A4.18}$$

where $J_v=\cup_{i=1}^{s_v}P_{v_i}\setminus Q_{v_i}$. We will bound the factors $\Big(\frac{|\sigma_{h_v}|+|\mu_{h_v}|}{\gamma^{h_v}}\Big)^{C_\beta(v)I_\beta(v)}$ using (3.19) by a constant.

If we call

$$J_{\tau,\mathbf{P},T,\beta}=\int d\mathbf{x}_{v_0}\Big|\Big[\prod_{i=1}^n d_{j_\beta(v_i^*)}^{b_\beta(v_i^*)}(\mathbf{x}_\beta^i,\mathbf{y}_\beta^i)\mathcal{P}_{I_\beta(v_i^*)}^{C_\beta(v_i^*)}\mathcal{S}_{i_\beta(v_i^*)}^{c_\beta(v_i^*)}K_{v_i^*}^{h_i}(\mathbf{x}_{v_i^*})\Big]$$
$$\cdot\Big\{\prod_{v\text{ not e.p.}}\frac{1}{s_v!}\Big[\prod_{l\in T_v}\hat\partial_{j_\beta(f_l^1)}^{q_\beta(f_l^1)}\hat\partial_{j_\beta(f_l^2)}^{q_\beta(f_l^2)}[d_{j_\beta(l)}^{b_\beta(l)}(\mathbf{x}_l,\mathbf{y}_l)\mathcal{P}_{I_\beta(l)}^{C_\beta(l)}\mathcal{S}_{i_\beta(l)}^{c_\beta(l)}g^{(h_v)}(f_l^1,f_l^2)]\Big]\Big\}\Big|, \tag{A4.19}$$

we have, under the hypothesis (3.24),

$$J_{\tau,\mathbf{P},T,\alpha} \leq C^n M^2 |\lambda|^n \Big[\prod_{i=1}^n \Big(\frac{|\sigma_{h_i^*}| + |\mu_{h_i^*}|}{\gamma^{h_i^*}}\Big)^{c_\beta(v_i^*) i_\beta(v_i^*)}\Big] \cdot$$
$$\cdot \Big\{ \prod_{v \text{ not e.p.}} \frac{1}{s_v!} C^{2(s_v-1)} \gamma^{h_v n_\nu(v)} \gamma^{-h_v \sum_{l\in T_v} b_\beta(l)} \gamma^{-h_v \sum_{i=1}^n b_\beta(v_i^*)} \gamma^{-h_v(s_v-1)} \cdot$$
$$\cdot \gamma^{h_v \sum_{l\in T_v}[q_\beta(f_l^1)+q_\beta(f_l^2)]}\Big\}\Big[\prod_{\ell\in T}\Big(\frac{|\sigma_{h_v}| + |\mu_{h_v}|}{\gamma^{h_v}}\Big)^{c_\beta(\ell) i_\beta(\ell)}\Big], \tag{A4.20}$$

where $n_\nu(v)$ is the number of vertices of type ν with scale $h_v + 1$.

Now, substituting (A4.18), (A4.20) into (A4.12), using (A4.13), we find that:

$$\int d\mathbf{x}_{v_0} |W_{\tau,\mathbf{P},T,\beta}(\mathbf{x}_{v_0})| \leq C^n M^2 |\lambda|^n \gamma^{-hD_k(|P_{v_0}|)} \prod_{v\in V_\beta} \Big(\frac{|\sigma_{h_v}| + |\mu_{h_v}|}{\gamma^{h_v}}\Big)^{i(v,\beta)}$$
$$\cdot \prod_{v \text{ not e.p.}} \left\{ \frac{1}{s_v!} C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|} \Big(\frac{Z_{h_v}}{Z_{h_v-1}}\Big)^{\frac{|P_v|}{2}} \gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)]} \right\}, \tag{A4.21}$$

where, if $k = \sum_{f\in P_{v_0}} q_\beta(f)$, $D_k(p) = -2 + p + k$ and we have used (A4.15). Note that given $v \in \tau$ and $\tau \in \mathcal{T}_{h,n}$ and using (3.19) together with the first two of (3.18),

$$\frac{|\sigma_{h_v}|}{\gamma^{h_v}} = \frac{|\sigma_h|}{\gamma^h}\frac{|\sigma_{h_v}|}{|\sigma_h|}\gamma^{h-h_v} \leq \frac{|\sigma_h|}{\gamma^h}\gamma^{(h-h_v)(1-c|\lambda|)} \leq C_1 \gamma^{(h-h_{\bar v})(1-c|\lambda|)},$$
$$\frac{|\mu_{h_v}|}{\gamma^{h_v}} = \frac{|\mu_h|}{\gamma^h}\frac{|\mu_{h_v}|}{|\mu_h|}\gamma^{h-h_v} \leq \frac{|\mu_h|}{\gamma^h}\gamma^{(h-h_v)(1-c|\lambda|)} \leq C_1 \gamma^{(h-h_v)(1-c|\lambda|)}. \tag{A4.22}$$

Moreover the indices $i(v,\beta)$ satisfy, for any B_T, (A4.17) so that, using (A4.22) and (A4.16), we find

$$\prod_{v\in V_\beta}\Big(\frac{|\sigma_{h_v}| + |\mu_{h_v}|}{\gamma^{h_v}}\Big)^{i(v,\beta)} \leq C_1^n \prod_{v \text{ not e.p.}} \gamma^{-z'(P_v)}. \tag{A4.23}$$

Substituting (A4.22) into (A4.21) and using (A4.16), we find:

$$\int d\mathbf{x}_{v_0} |W_{\tau,\mathbf{P},T,\beta}(\mathbf{x}_{v_0})| \leq C^n M^2 |\lambda|^n \gamma^{-hD_k(|P_{v_0}|)}$$
$$\cdot \prod_{v \text{ not e.p.}} \left\{ \frac{1}{s_v!} C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|} \Big(\frac{Z_{h_v}}{Z_{h_v-1}}\Big)^{\frac{|P_v|}{2}} \gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)+(1-c|\lambda|)z'(P_v)]} \right\}, \tag{A4.24}$$

where

$$D_v \stackrel{def}{=} -2 + \frac{|P_v|}{2} + z(P_v) + (1-c|\lambda|)z'(P_v) \geq \frac{|P_v|}{6}. \tag{A4.25}$$

Then (3.25) in Theorem 3.1 follows from the previous bounds and the remark that

$$\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{\mathbf{P}\in\mathcal{P}_\tau}\sum_{T\in\mathbf{T}}\sum_{\beta\in B_T}\prod_v \frac{1}{s_v!}\gamma^{-\frac{|P_v|}{6}} \le c^n , \tag{A4.26}$$

for some constant c, see [BM] or [GM] for further details.

The bound on $\tilde{E}_h$, t_h, (3.26) and (3.27) follow from a similar analysis. The remarks following (3.26) and (3.27) follow from noticing that in the expansion for $\mathcal{L}\mathcal{V}^{(h)}$ only propagators of type $\mathcal{P}_0 g^{(h_v)}_{\underline{a},\underline{a}'}$ or $\mathcal{P}_1 g^{(h_v)}_{\underline{a},\underline{a}'}$ appear (in order to bound these propagators we do not need (3.19), see the last statement in Lemma 3.3). Furthermore, by construction l_h, n_h and z_h are independent of σ_k, μ_k, so that, in order to prove (3.27) we do not even need the first two inequalities in (3.18). □

A4.6. The sum over all the trees with root scale h and with at least a v with $h_v = k$ is $O(|\lambda|\gamma^{\frac{1}{2}(h-k)})$; this follows from the fact that the bound (A4.26) holds, for some $c = O(1)$, even if $\gamma^{-|P_v|/6}$ is replaced by $\gamma^{-\kappa|P_v|}$, for any constant $\kappa > 0$ independent of λ; and that D_v, instead of using (A4.25), can also be bounded as $D_v \ge 1/2+|P_v|/12$. This property is called *short memory property.*

Appendix A5. Proof of Theorem 4.1 and Lemma 4.2

We consider the space $\mathfrak{M}_\vartheta$ of sequences $\underline{\nu} = \{\nu_h\}_{h\le 1}$ such that $|\nu_h| \le c|\lambda|\gamma^{(\vartheta/2)h}$; we shall think $\mathfrak{M}_\vartheta$ as a Banach space with norm $||\cdot||_\vartheta$, where $||\underline{\nu}||_\vartheta \overset{def}{=} \sup_{k\le 1}|\nu_k|\gamma^{-(\vartheta/2)k}$. We will proceed as follows: we first show that, for any sequence $\underline{\nu} \in \mathfrak{M}_\vartheta$, the flow equation for ν_h, the hypothesis (3.17), (3.18) and the property $|\lambda_h(\underline{\nu})| \le c|\lambda|$ are verified, uniformly in $\underline{\nu}$. Then we fix $\underline{\nu} \in \mathcal{M}_\vartheta$ via an exponentially convergent iterative procedure, in such a way that the flow equation for ν_h is satisfied.

A5.1. Proof of Theorem 4.1. Given $\underline{\nu} \in \mathfrak{M}_\vartheta$, let us suppose inductively that (3.17), (3.18) and that, for $k > \bar{h}+1$,

$$|\lambda_{k-1}(\underline{\nu}) - \lambda_k(\underline{\nu})| \le c_0|\lambda|^2\gamma^{(\vartheta/2)k} , \tag{A5.1}$$

for some $c_0 > 0$. Note that (A5.1) is certainly true for $h = 1$ (in that case the r.h.s. of (A5.1) is just the bound on β^1_λ). Note also that (A5.1) implies that $|\lambda_k| \le c|\lambda|$, for any $k > \bar{h}$.

Using (3.26), the second of (3.27) and (4.1) we find that (3.17), (3.18) are true with $\bar{h}$ replaced by $\bar{h}-1$.

We now consider the equation $\lambda_{h-1} = \lambda_h + \beta^h_\lambda(\lambda_h, \nu_h; \dots ; \lambda_1, \nu_1)$, $h > \bar{h}$. The function β^h_λ can be expressed as a convergent sum over tree diagrams, as described in Appendix A4; note that it depends on $(\lambda_h, \nu_h; \dots ; \lambda_1, \nu_1)$ directly through the end–points of the trees and indirectly through the factors Z_h/Z_{h-1}.

We can write $\mathcal{P}_0 g^{(h)}_{(+,\omega),(-,\omega)}(\mathbf{x}-\mathbf{y}) = g^{(h)}_{L,\omega}(\mathbf{x}-\mathbf{y}) + r^{(h)}_\omega(\mathbf{x}-\mathbf{y})$, where

$$g^{(h)}_{L,\omega}(\mathbf{x}-\mathbf{y}) \overset{def}{=} \frac{4}{M^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\tilde{f}_h(\mathbf{k})\frac{1}{ik+\omega k_0} \tag{A5.2}$$

and $r_\omega^{(h)}$ is the rest, satisfying the same bound as $g^{(h)}_{(+,\omega),(-,\omega)}$, times a factor γ^h. This decomposition induces the following decomposition for β^h_λ:

$$\beta^h_\lambda(\lambda_h,\nu_h;\dots;\lambda_1,\nu_1)=\beta^h_{\lambda,L}(\lambda_h,\dots,\lambda_h)+\sum_{k=h+1}^{1}D^{h,k}_\lambda+r^h_\lambda(\lambda_h,\dots,\lambda_1)+\sum_{k\ge h}\nu_k\tilde\beta^{h,k}_\lambda(\lambda_k,\nu_k;\dots;\lambda_1,\nu_1)\ ,\tag{A5.3}$$

with

$$|\beta^h_{\lambda,L}|\le c|\lambda|^2\gamma^{\vartheta h}\ ,\qquad |D^{h,k}_\lambda|\le c|\lambda|\gamma^{\vartheta(h-k)}|\lambda_k-\lambda_h|\ ,$$
$$|r^h_\lambda|\le c|\lambda|^2\gamma^{(\vartheta/2)h}\ ,\qquad |\tilde\beta^{h,k}_\lambda|\le c|\lambda|\gamma^{\vartheta(h-k)}\ .\tag{A5.4}$$

The first two terms in (A5.3) $\beta^h_{\lambda,L}$ collect the contributions obtained by posing $r_\omega^{(k)}=0$, $k\ge h$ and substituting the discrete δ function defined after (3.8) with $M^2\delta_{\mathbf{k},\mathbf{0}}$. The first of (A5.4) is called the *vanishing of the Luttinger model Beta function* property, see [BGPS, GS, BM1] (or [BeM1] for a simplified proof), and it is a crucial property of interacting fermionic systems in $d=1$.

Using the decomposition (A5.3) and the bounds (A5.4) we prove the following bounds for $\lambda_{\bar h}(\underline{\nu})$, $\underline{\nu}\in\mathfrak{M}_\vartheta$:

$$|\lambda_{\bar h}(\underline{\nu})-\lambda_1(\underline{\nu})|\le c_0|\lambda|^2\quad ,\quad |\lambda_{\bar h}(\underline{\nu})-\lambda_{\bar h+1}(\underline{\nu})|\le c_0|\lambda|^2\gamma^{(\vartheta/2)\bar h}\ ,\tag{A5.5}$$

for some $c_0>0$. Moreover, given $\underline{\nu},\underline{\nu}'\in\mathfrak{M}_\vartheta$, we show that:

$$|\lambda_{\bar h}(\underline{\nu})-\lambda_{\bar h}(\underline{\nu}')|\le c|\lambda|\,||\underline{\nu}-\underline{\nu}'||_0\ ,\tag{A5.6}$$

where $||\underline{\nu}-\underline{\nu}'||_0\stackrel{def}{=}\sup_{h\le 1}|\nu_h-\nu'_h|$.

Proof of (A5.5). We decompose $\lambda_{\bar h}-\lambda_{\bar h+1}=\beta^{\bar h+1}_\lambda$ as in (A5.3). Using the bounds (A5.4) and the inductive hypothesis (A5.1), we find:

$$|\lambda_{\bar h}(\underline{\nu})-\lambda_{\bar h+1}(\underline{\nu})|\le c|\lambda|^2\gamma^{\vartheta(\bar h+1)}+\sum_{k\ge\bar h+2}c|\lambda|\gamma^{\vartheta(\bar h+1-k)}\sum_{k'=\bar h+2}^{k}c_0|\lambda|^2\gamma^{(\vartheta/2)k'}+c|\lambda|^2\gamma^{(\vartheta/2)(\bar h+1)}+\sum_{k\ge\bar h+1}c^2|\lambda|^2\gamma^{(\vartheta/2)k}\gamma^{(\vartheta(\bar h+1-k))}\ ,\tag{A5.7}$$

which, for c_0 big enough, immediately implies the second of (A5.5) with $h\to h-1$; from this bound and the hypothesis (A5.1) follows the first of (A5.5). $\square$

Proof of (A5.6). If we take two sequences $\underline{\nu},\underline{\nu}'\in\mathfrak{M}_\vartheta$, we easily find that the beta function for $\lambda_{\bar h}(\underline{\nu})-\lambda_{\bar h}(\underline{\nu}')$ can be represented by a tree expansion similar to the one for β^h_λ, with the property that the trees giving a non vanishing contribution have necessarily one end–point on scale $k\ge h$ associated to a coupling constant $\lambda_k(\underline{\nu})-\lambda_k(\underline{\nu}')$ or $\nu_k-\nu'_k$. Then we find:

$$\lambda_{\bar h}(\underline{\nu})-\lambda_{\bar h}(\underline{\nu}')=\lambda_1(\underline{\nu})-\lambda_1(\underline{\nu}')+\sum_{\bar h+1\le k\le 1}[\beta^k_\lambda(\lambda_k(\underline{\nu}),\nu_k;\dots;\lambda_1,\nu_1)-\beta^k_\lambda(\lambda_k(\underline{\nu}'),\nu'_k;\dots;\lambda_1,\nu'_1)].\tag{A5.8}$$

Note that $|\lambda_1(\underline{\nu}) - \lambda_1(\underline{\nu}')| \leq c_0|\lambda||\nu_1 - \nu_1'|$, because $\lambda_1 = \lambda/Z_1^2 + O(\lambda^2/Z_1^4)$ and $Z_1 = \sqrt{2} - 1 + \nu/2$. If we inductively suppose that, for any $k > \bar{h}$, $|\lambda_k(\underline{\nu}) - \lambda_k(\underline{\nu}')| \leq 2c_0|\lambda|||\underline{\nu} - \underline{\nu}'||_0$, we find, by using the decomposition (A5.3):

$$|\lambda_{\bar{h}}(\underline{\nu}) - \lambda_{\bar{h}}(\underline{\nu}')| \leq c_0|\lambda||\nu_1 - \nu_1'| + c|\lambda| \times \sum_{k\geq\bar{h}+1} \gamma^{(\vartheta/2)k} \sum_{k'\geq k} \gamma^{\vartheta(k-k')}\Big[2c_0|\lambda|\,||\underline{\nu} - \underline{\nu}'||_0 + |\nu_k - \nu_k'|\Big]\,. \quad \text{(A5.9)}$$

Choosing c_0 big enough, (A5.6) follows. □

We are now left with fixing the sequence $\underline{\nu}$ in such a way that the flow equation for ν is satisfied. Since we want to fix $\underline{\nu}$ in such a way that $\nu_{-\infty} = 0$, we must have:

$$\nu_1 = -\sum_{k=-\infty}^{1} \gamma^{k-2}\beta_\nu^k(\lambda_k, \nu_k; \ldots; \lambda_1, \nu_1)\,. \quad \text{(A5.10)}$$

If we manage to fix ν_1 as in (A5.10), we also get:

$$\nu_h = -\sum_{k\leq h}\gamma^{k-h-1}\beta_\nu^k(\lambda_k, \nu_k; \ldots; \lambda_1, \nu_1)\,. \quad \text{(A5.11)}$$

We look for a fixed point of the operator $\mathbf{T} : \mathfrak{M}_\vartheta \to \mathfrak{M}_\vartheta$ defined as:

$$(\mathbf{T}\underline{\nu})_h = -\sum_{k\leq h}\gamma^{k-h-1}\beta_\nu^k(\lambda_k(\underline{\nu}), \nu_k; \ldots; \lambda_1, \nu_1)\,. \quad \text{(A5.12)}$$

where $\lambda_k(\underline{\nu})$ is the solution of the first line of (4.2), obtained as a function of the *parameter* $\underline{\nu}$, as described above.

If we find a fixed point $\underline{\nu}^*$ of (A5.12), the first two lines in (4.2) will be simultaneously solved by $\underline{\lambda}(\underline{\nu}^*)$ and $\underline{\nu}^*$ respectively, and the solution will have the desired smallness properties for λ_h and ν_h.

First note that, if $|\lambda|$ is sufficiently small, then $\mathbf{T}$ leaves $\mathfrak{M}_\vartheta$ invariant: in fact, as a consequence of parity cancellations, the ν–component of the Beta function satisfies:

$$\beta_\nu^h(\lambda_h, \nu_h; \ldots; \lambda_1, \nu_1) = \beta_{\nu,1}^h(\lambda_h; \ldots; \lambda_1) + \sum_k \nu_k\tilde{\beta}_\nu^{h,k}(\lambda_h, \nu_h; \ldots; \lambda_1, \nu_1), \quad \text{(A5.13)}$$

where, if c_1, c_2 are suitable constants

$$|\beta_{\nu,1}^h| \leq c_1|\lambda|\gamma^{\vartheta h} \qquad |\tilde{\beta}_\nu^{h,k}| \leq c_2|\lambda|\gamma^{\vartheta(h-k)}\,. \quad \text{(A5.14)}$$

By using (A5.13) and choosing $c = 2c_1$ we obtain

$$|(\mathbf{T}\nu)_h| \leq \sum_{k\leq h} 2c_1|\lambda|\gamma^{(\vartheta/2)k}\gamma^{k-h} \leq c|\lambda|\gamma^{(\vartheta/2)h}\,. \quad \text{(A5.15)}$$

Furthermore, using (A5.13) and (A5.6), we find that $\mathbf{T}$ is a contraction on $\mathfrak{M}_\vartheta$:

$$
\begin{aligned}
|(\mathbf{T}\nu)_h - (\mathbf{T}\underline{\nu}')_h| &\le \sum_{k\le h} \gamma^{k-h-1} |\beta_\nu^k(\lambda_k(\underline{\nu}), \nu_k; \ldots ; \lambda_1, \nu_1) - \beta_\nu^k(\lambda_k(\underline{\nu}'), \nu_k'; \ldots ; \lambda_1, \nu_1')| \\
&\le c \sum_{k\le h} \gamma^{k-h-1} \left[\gamma^{\vartheta k} \sum_{k'=k}^{1} |\lambda_{k'}(\underline{\nu}) - \lambda_{k'}(\underline{\nu}')| + \sum_{k'=k}^{1} \gamma^{\vartheta(k-k')} |\lambda| |\nu_{k'} - \nu_{k'}'| \right] \\
&\le c' \sum_{k\le h} \gamma^{k-h-1} \Big[|k| \gamma^{\vartheta k} |\lambda| \, ||\underline{\nu} - \underline{\nu}'||_0 + \sum_{k'=k}^{1} \gamma^{\vartheta(k-k')} |\lambda| \gamma^{(\vartheta/2)k'} ||\underline{\nu} - \underline{\nu}'||_\vartheta \\
&\le c'' |\lambda| \gamma^{(\vartheta/2)h} ||\underline{\nu} - \underline{\nu}'||_\vartheta,
\end{aligned}
\tag{A5.16}
$$

hence $||(\mathbf{T}\nu) - (\mathbf{T}\underline{\nu}')||_\vartheta \le c''|\lambda| ||\underline{\nu} - \underline{\nu}'||_\vartheta$. Then, a unique fixed point $\underline{\nu}^*$ for $\mathbf{T}$ exists on $\mathfrak{M}_\vartheta$. Proof of Theorem 4.1 is concluded by noticing that $\mathbf{T}$ is analytic (in fact β_ν^h and $\underline{\lambda}$ are analytic in $\underline{\nu}$ in the domain $\mathfrak{M}_\vartheta$). □

A5.2.Proof of Lemma 4.2. From now on we shall think of λ_h and ν_h fixed, with ν_1 conveniently chosen as above ($\nu_1 = \nu_1^*(\lambda)$). Then we have $|\lambda_h| \le c|\lambda|$ and $|\nu_h| \le c|\lambda|\gamma^{(\vartheta/2)h}$, for some $c, \vartheta > 0$. Having fixed ν_1 as a convenient function of λ, we can also think of λ_h and ν_h as functions of λ.

The flow of Z_h. The flow of Z_h is given by the first of (4.1) with z_h independent of σ_k, μ_k, $k \ge h$. By Theorem 3.1 we have that $|z_h| \le c|\lambda|^2$, uniformly in h. Again, as for λ_h and ν_h, we can formally study this equation up to $h = -\infty$. We define $\gamma^{-\eta_z} \stackrel{def}{=} \lim_{h\to-\infty} 1 + z_h$, so that

$$
\log_\gamma Z_h = \sum_{k\ge h+1} \log_\gamma(1 + z_k) = \eta_z(h-1) + \sum_{k\ge h+1} r_\zeta^k \quad , \quad r_\zeta^k \stackrel{def}{=} \log_\gamma \left(1 + \frac{z_k - z_{-\infty}}{1 + z_{-\infty}}\right). \tag{A5.17}
$$

Using the fact that $z_{k-1} - z_k$ is necessarily proportional to $\lambda_{k-1} - \lambda_k$ or to $\nu_{k-1} - \nu_k$ and that $\lambda_{k-1} - \lambda_k$ is bounded as in (A5.1), we easily find: $|r_\zeta^k| \le c \sum_{k'\le k} |z_{k'-1} - z_{k'}| \le c'|\lambda|^2 \gamma^{(\vartheta/2)k}$. So, if $F_\zeta^h \stackrel{def}{=} \sum_{k\ge h+1} r_\zeta^k$ and $F_\zeta^1 = 0$, then $F_\zeta^h = O(\lambda)$ and $Z_h = \gamma^{\eta_z(h-1)+F_\zeta^h}$. Clearly, by definition, η_z and F_ζ^h only depend on $\lambda_k, \nu_k, k \le 1$, so they are independent of t and u.

The flow of μ_h. The flow of μ_h is given by the last of (4.1). One can easily show inductively that $\mu_k(\mathbf{k})/\mu_h$, $k \ge h$, is independent of μ_1, so that one can think that μ_{h-1}/μ_h is just a function of λ_h, ν_h. Then, again we can study the flow equation for μ_h up to $h \to -\infty$. We define $\gamma^{-\eta_\mu} \stackrel{def}{=} \lim_{h\to-\infty} 1 + (m_h/\mu_h - z_h)/(1 + z_h)$, so that, proceeding as for Z_h, we see that

$$
\mu_h = \mu_1 \gamma^{\eta_\mu(h-1)+F_\mu^h} , \tag{A5.18}
$$

for a suitable $F_\mu^h = O(\lambda)$. Of course η_μ and F_μ^h are independent of t and u.

The flow of σ_h. The flow of σ_h can be studied as the one of μ_h. If we define $\gamma^{-\eta_\sigma} \stackrel{def}{=} \lim_{h\to-\infty} 1 + (s_h/\sigma_h - z_h)/(1 + z_h)$, we find that

$$\sigma_h = \sigma_1 \gamma^{\eta_\sigma(h-1)+F_\sigma^h} , \tag{A5.19}$$

for a suitable $F_\sigma^h = O(\lambda)$. Again, η_σ and F_σ^h are independent of t, u.

We are left with proving that $\eta_\sigma - \eta_\mu \neq 0$. It is sufficient to note that, by direct computation of the lowest order terms, for some $\vartheta > 0$, (4.1) can be written as:

$$\begin{aligned} z_h &= b_1\lambda_h^2 + O(|\lambda|^2\gamma^{\vartheta h}) + O(|\lambda|^3) \quad , \quad b_1 > 0, \\ s_h/\sigma_h &= -b_2\lambda_h + O(|\lambda|\gamma^{\vartheta h}) + O(|\lambda|^2) \quad , \quad b_2 > 0, \\ m_h/\mu_h &= b_2\lambda_h + O(|\lambda|\gamma^{\vartheta h}) + O(|\lambda|^2) \quad , \quad b_2 > 0 , \end{aligned} \tag{A5.20}$$

where b_1, b_2 are constants independent of λ and h. Using (A5.20) and the definitions of η_μ and η_σ we find: $\eta_\sigma - \eta_\mu = (2b_2/\log\gamma)\lambda + O(\lambda^2)$. □

Appendix A6. Proof of Lemma 5.3

Proceeding as in §4 and Appendix A5, we first solve the equations for Z_h and $\widehat{m}_h^{(2)}$ parametrically in $\underline{\pi} = \{\pi_h\}_{h\le h_1^*}$. If $|\pi_h| \le c|\lambda|\gamma^{(\vartheta/2)(h-h_1^*)}$, the first two assumptions of (5.14) easily follow. Now we will construct a sequence $\underline{\pi}$ such that $|\pi_h| \le c|\lambda|\gamma^{(\vartheta/2)(h-h_1^*)}$ and satisfying the flow equation $\pi_{h-1} = \gamma^h\pi_h + \beta_\pi^h(\pi_h, \ldots, \pi_{h_1^*})$.

A6.1. Tree expansion for β_π^h. β_π^h can be expressed as a sum over tree diagrams, similar to those used in Appendix A4. The main difference is that they have vertices on scales k between h and $+2$. The vertices on scales $h_v \ge h_1^* + 1$ are associated to the truncated expectations (A4.4); the vertices on scale $h_v = h_1^*$ are associated to truncated expectations w.r.t. the propagators $g_{\omega_1,\omega_2}^{(1,h_1^*)}$; the vertices on scale $h_v < h_1^*$ are associated to truncated expectations w.r.t. the propagators $g_{\omega_1,\omega_2}^{(2,h_v+1)}$. Moreover the end–points on scale $\ge h_1^* + 1$ are associated to the couplings λ_h or ν_h, as in Appendix A4; the end–points on scales $h \le h_1^*$ are necessarily associated to the couplings π_h.

A6.2. Bounds on β_π^h. The non-vanishing trees contributing to β_π^h must have at least one vertex on scale $\ge h_1^*$: in fact the diagrams depending only on the vertices of type π are vanishing (they are chains, so they are vanishing, because of the compact support property of the propagator). This means that, by the short memory property, see the Remark at the end of Appendix A4: $|\beta_\pi^h| \le c|\lambda|\gamma^{\vartheta(h-h_1^*)}$.

A6.3. Fixing the counterterm. We now proceed as in Appendix A5 but the analysis here is easier, because no λ end–points can appear and the bound $|\beta_\pi^h| \le c|\lambda|\gamma^{\vartheta(h-h_1^*)}$ holds. As in Appendix A5, we can formally consider the flow equation up to $h = -\infty$, even if h_2^* is a finite integer. This is because the beta function is independent of $\widehat{m}_k^{(2)}$, $k \le h_1^*$ and admits bounds uniform in h. If we want to fix the counterterm $\pi_{h_1^*}$ in such a way that $\pi_{-\infty} = 0$, we must have, for any $h \le h_1^*$:

$$\pi_h = -\sum_{k\le h} \gamma^{k-h-1}\beta^k_\pi(\pi_k,\dots,\pi_{h_1^*})\,. \tag{A6.1}$$

Let $\tilde{\mathfrak{M}}$ be the space of sequences $\underline{\pi}=\{\pi_{-\infty},\dots,\pi_{h_1^*}\}$ such that $|\pi_h|\le c|\lambda|\gamma^{-(\vartheta/2)(h-h_1^*)}$. We look for a fixed point of the operator $\tilde{\mathbf{T}}:\tilde{\mathfrak{M}}\to\tilde{\mathfrak{M}}$ defined as:

$$(\tilde{\mathbf{T}}\underline{\pi})_h = -\sum_{k\le h} \gamma^{k-h-1}\beta^k_\pi(\pi_k;\dots;\pi_{h_1^*})\,. \tag{A6.2}$$

Using that β^k_π is independent from $\hat m^{(2)}_k$ and the bound on the beta function, choosing λ small enough and proceeding as in the proof of Theorem 4.1, we find that $\tilde{\mathbf{T}}$ is a contraction on $\tilde{\mathfrak{M}}$, so that we find a unique fixed point, and the first of (5.16) follows.

A6.4. The flows of Z_h and $\widehat{m}^{(2)}_h$. Once $\pi_{h_1^*}$ is fixed via the iterative procedure of §A6.3, we can study in more detail the flows of Z_h and $\widehat{m}^{(2)}_h$ given by (5.10). Note that z_h and s_h can be again expressed as a sum over tree diagrams and, as discussed for β^h_π, see §A6.2, any non-vanishing diagram must have at least one vertex on scale $\ge h_1^*$. Then, by the short memory property, see §A4.6, we have $z_h = O(\lambda^2\gamma^{\vartheta(h-h_1^*)})$ and $s_h = O(\lambda\widehat{m}^{(2)}_h\gamma^{\vartheta(h-h_1^*)})$ and, repeating the proof of Lemma 4.1, we find the second and third of (5.16).

A6.5 The Lipshitz property (5.17). Clearly, $\pi^*_{h_1^*}(\lambda,\sigma_1,\mu_1)-\pi^*_{h_1^*}(\lambda,\sigma_1',\mu_1')$ can be expressed via a tree expansion similar to the one discussed above; in the trees with non-vanishing value, there is either a difference of propagators at scale $h\ge h_1^*$ with couplings σ_h,μ_h and σ_h',μ_h', giving in the dimensional bounds an extra factor $O(|\sigma_h-\sigma_h'|\gamma^{-h})$ or $O(|\mu_h-\mu_h'|\gamma^{-h})$; or a difference of propagators at scale $h\le h_1^*$ (computed by definition at $\widehat{m}^{(2)}_h=0$) with the "corrections" a^ω_h, c_h associated to σ_1,μ_1 or σ_1',μ_1', giving in the dimensional bounds an extra factor $O(|\sigma_1-\sigma_1'|)$ or $O(|\mu_1-\mu_1'|)$. Then,

$$\left|\pi_{h_1^*}(\lambda,\sigma_1,\mu_1)-\pi_{h_1^*}(\lambda,\sigma_1',\mu_1')\right| \le c|\lambda|\sum_{k\le h_1^*}\gamma^{k-h_1^*-1} \cdot\Big[\sum_{h\ge h_1^*}\Big(\frac{|\sigma_h-\sigma_h'|}{\gamma^h}+\frac{|\mu_h-\mu_h'|}{\gamma^h}\Big)+\sum_{k\le h\le h_1^*}\big(|\sigma_1-\sigma_1'|+|\mu_1-\mu_1'|\big)\Big]\,, \tag{A6.3}$$

from which, using (A5.18) and (A5.19), we easily get (5.17).

Appendix A7. Proof of (A4.10)

We have, by definition $\mathrm{Pf}\,G = (2^k k!)^{-1}\sum_{\mathbf{p}}(-1)^{\mathbf{p}}G_{p(1)p(2)}\cdots G_{p(2k-1)p(2k)}$, where $\mathbf{p}=(p(1),\dots\dots,p(|J|))$ is a permutation of the indices $f\in J$ (we suppose $|J|=2k$) and $(-1)^{\mathbf{p}}$ its sign.

If we apply $\mathcal{S}_1 = 1 - \mathcal{P}_0$ to Pf G and we call $G^0_{f,f'} \stackrel{def}{=} \mathcal{P}_0 G_{f,f'}$, we find that $\mathcal{S}_1$Pf G is equal to

$$\frac{1}{2^k k!}\sum_{\mathbf{p}}(-1)^{\mathbf{p}}\Big[G_{p(1)p(2)}\cdots G_{p(2k-1)p(2k)} - G^0_{p(1)p(2)}\cdots G^0_{p(2k-1)p(2k)}\Big]$$
$$= \frac{1}{2^k k!}\sum_{\mathbf{p}}(-1)^{\mathbf{p}}\sum_{j=1}^{k}\cdot\Big(G^0_{p(1)p(2)}\cdots G^0_{p(2j-3)p(2j-2)}\Big)$$
$$\times \mathcal{S}_1 G_{p(2j-1)p(2j)}\Big(G_{p(2j+1)p(2j+2)}\cdots G_{p(2k-1)p(2k)}\Big)\,, \tag{A7.1}$$

where in the last sum the meaningless factors must be put equal to 1. We rewrite the two sums over $\mathbf{p}$ and j in the following way:

$$\sum_{\mathbf{p}}\sum_{j=1}^{k} = \sum_{j=1}^{k}\sum_{\substack{f_1,f_2\in J\\ f_1\neq f_2}}\sum_{J_1,J_2}^{*}\sum_{\mathbf{p}}^{**}\,, \tag{A7.2}$$

where the $*$ on the second sum means that the sets J_1 and J_2 are s.t. (f_1, f_2, J_1, J_2) is a partition of J; the $**$ on the second sum means that $p(1),\ldots,p(2j-2)$ belong to J_1, $(p(2j-1), p(2j)) = (f_1, f_2)$ and $p(2j+1),\ldots,p(2k)$ belong to J_2. Using (A7.2) we can rewrite (A7.1) as

$$\mathcal{S}_1\text{Pf}\, G = \frac{1}{2^k k!}\sum_{j=1}^{k}\sum_{\substack{f_1,f_2\in J\\ f_1\neq f_2}}(-1)^{\pi}\mathcal{S}_1 G_{f_1,f_2}\sum_{J_1,J_2}^{*}$$
$$\cdot\sum_{\mathbf{p}_1,\mathbf{p}_2}(-1)^{\mathbf{p}_1+\mathbf{p}_2}\Big(G^0_{p_1(1)p_1(2)}\cdots G^0_{p_1(2k_1-1)p(2k_1)}\Big)$$
$$\times\Big(G_{p_2(1)p_2(2)}\cdots G_{p_2(2k_2-1)p(2k_2)}\Big)\,, \tag{A7.3}$$

where $(-1)^{\pi}$ is the sign of the permutation leading from the ordering J to the ordering (f_1, f_2, J_1, J_2); $\mathbf{p}_i$, $i = 1, 2$ is a permutation of the labels in J_i (we suppose $|J_i| = 2k_i$) and $(-1)^{\mathbf{p}_i}$ is its sign. It is clear that (A7.3) is equivalent to (A4.10).

Acknowledgements. AG thanks Prof. J. L. Lebowitz for his invitation at Rutgers University, where part of this work was done; and acknowledges the NSF Grant DMR 01–279–26, which partially supported his work. VM thanks Prof. T. Spencer for his nice invitation to the Institute for Advanced Studies, in Princeton, where part of this work was done. We both thank Prof. G. Gallavotti for many important remarks and suggestions.

References

[AT] Ashkin, J., Teller, E.: Statistics of Two-Dimensional Lattices with Four Components. Phys. Rev. **64**, 178–184 (1943)

[B] Baxter, R.J.: Eight-Vertex Model in Lattice Statistics. Phys. Rev. Lett. **26**, 832–833 (1971)

[Ba] Baxter, R.: Exactly solved models in statistical mechanics. London-New York: Academic Press, 1982

[BG1] Benfatto, G., Gallavotti, G.: Perturbation Theory of the Fermi Surface in Quantum Liquid. A General Quasiparticle Formalism and One-Dimensional Systems. J. Stat. Phys. **59**, 541–664 (1990)

[BG] Benfatto, G., Gallavotti, G.: Renormalization group. Physics notes **1**, Princeton, NJ: Princeton University Press 1995

[BGPS] Benfatto, G., Gallavotti, G., Procacci, A., Scoppola, B.: Beta function and Schwinger functions for a Many Fermions System in One Dimension. Commun. Math. Phys. **160**, 93–171 (1994)

[BM] Benfatto, G., Mastropietro, V.: Renormalization group, hidden symmetries and approximate Ward identities in the XYZ model. Rev. Math. Phys. **13** no 11, 1323–143 (2001); Commun. Math. Phys. **231**, 97–134 (2002)

[BM1] Bonetto, F., Mastropietro, V.: Beta function and Anomaly of the Fermi Surface for a d=1 System of interacting Fermions in a Periodic Potential. Commun. Math. Phys. **172**, 57–93 (1995)

[Bad] Badehdah, M., et al.: Physica B, **291**, 394 (2000)

[Bez] Bezerra, C.G., Mariz, A.M.: The anisotropic Ashkin-Teller model: a renormalization Group study. Physica A **292**, 429–436 (2001)

[Bar] Bartelt, N.C., Einstein, T.L., et al.: Phys. Rev. B **40**, 10759 (1989)

[Be] Bekhechi, S., et al.: Physica A **264**, 503 (1999)

[BeM1] Benfatto, G., Mastropietro, V.: Ward identities and Dyson equations in interacting Fermi systems. To appear in J. Stat. Phys.

[DR] Domany, E., Riedel, E.K.: Phys. Rev. Lett. **40**, 562 (1978)

[GM] Gentile, G., Mastropietro, V.: Renormalization group for one-dimensional fermions. A review on mathematical results. Phys. Rep. **352**(4–6), 273–243 (2001)

[GS] Gentile, G., Scoppola, B.: Renormalization Group and the ultraviolet problem in the Luttinger model. Commun. Math. Phys. **154**, 153–179 (1993)

[K] Kadanoff, L.P.: Connections between the Critical Behavior of the Planar Model and That of the Eight-Vertex Model. Phys. Rev. Lett. **39**, 903–905 (1977)

[KW] Kadanoff, L.P., Wegner, F.J.: Phys. Rev. B **4**, 3989–3993 (1971)

[Ka] Kasteleyn, P.W.: Dimer Statistics and phase transitions. J. Math.Phys. **4**, 287 (1963)

[F] Fan, C.: On critical properties of the Ashkin-Teller model. Phys. Lett. **39A**, 136–138 (1972)

[H] Hurst, C.: New approach to the Ising problem. J.Math. Phys. **7**(2), 305–310 (1966)

[ID] Itzykson, C., Drouffe, J.: Statistical field theory: 1, Cambridge: Cambridge Univ. Press, 1989

[Le] Lesniewski, A.: Effective action for the Yukawa 2 quantum field Theory. Commun. Math. Phys. **108**, 437–467 (1987)

[Li] Lieb, H.: Exact solution of the problem of entropy of two-dimensional ice. Phys. Rev. Lett. **18**, 692–694 (1967)

[LP] Luther, A., Peschel, I.: Calculations of critical exponents in two dimension from quantum field theory in one dimension. Phys. Rev. B **12**, 3908–3917 (1975)

[M1] Mastropietro, V.: Ising models with four spin interaction at criticality. Commun. Math. Phys **244**, 595–642 (2004)

[ML] Mattis, D., Lieb, E.: Exact solution of a many fermion system and its associated boson field. J. Math. Phys. **6**, 304–312 (1965)

[MW] McCoy, B., Wu, T.: *The two-dimensional Ising model*. Cambridge, MA: Harvard Univ. Press, 1973

[MPW] Montroll, E., Potts, R., Ward, J.: Correlation and spontaneous magnetization of the two dimensional Ising model. J. Math. Phys. **4**, 308 (1963)

[N] den Nijs, M.P.M.: Derivation of extended scaling relations between critical exponents in two dimensional models from the one dimensional Luttinger model. Phys. Rev. B **23**(11), 6111–6125 (1981)

[O] Onsager, L.: Critical statistics. A two dimensional model with an order-disorder transition. Phys. Rev. **56**, 117–149 (1944)

[PB] Pruisken, A.M.M., Brown, A.C.: Universality for the critical lines of the eight vertex, Ashkin-Teller and Gaussian models. Phys. Rev. B, **23**(3), 1459–1468 (1981)

[PS] Pinson, H., Spencer, T.: Universality in 2D critical Ising model. To appear in Commun. Math. Phys.

[S] Samuel, S.: The use of anticommuting variable integrals in statistical mechanics. J. Math. Phys. **21** 2806 (1980)

[Su] Sutherland, S.B.: Two-Dimensional Hydrogen Bonded Crystals. J. Math. Phys. **11**, 3183–3186 (1970)

[Spe] Spencer, T.: A mathematical approach to universality in two dimensions. Physica A **279**, 250–259 (2000)

[SML] Schultz, T., Mattis, D., Lieb, E.: Two-dimensional Ising model as a soluble problem of many Fermions. Rev. Mod. Phys. **36**, 856 (1964)

[W] Wegner, F.J.: Duality relation between the Ashkin-Teller and the eight-vertex model. J. Phys. C **5**, L131–L132 (1972)

[WL] Wu, F.Y., Lin, K.Y.: Two phase transitions in the Ashkin-Teller model. J. Phys. C **5**, L181–L184 (1974)
[Wu] Wu, F.W.: The Ising model with four spin interaction. Phys. Rev. B **4**, 2312–2314 (1971)

Communicated by J.Z. Imbrie