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Determinantal Processes with Number Variance Saturation

Kurt Johansson★

Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden.
E-mail: kurtj@math.kth.se

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Dedicated to Freeman J. Dyson on his 80th birthday

Abstract: Consider Dyson's Hermitian Brownian motion model after a finite time S, where the process is started at N equidistant points on the real line. These N points after time S form a determinantal process and has a limit as $N \to \infty$. This limting determinantal process has the interesting feature that it shows number variance saturation. The variance of the number of particles in an interval converges to a limiting value as the length of the interval goes to infinity. Number variance saturation is also seen for example in the zeros of the Riemann ζ-function, [21, 2]. The process can also be constructed using non-intersecting paths and we consider several variants of this construction. One construction leads to a model which shows a transition from a non-universal behaviour with number variance saturation to a universal sine-kernel behaviour as we go up the line.

1. Introduction

The Bohigas-Gianonni-Schmidt conjecture, [5], says that the spectrum $\{E_j\}_{j\geq 1}$, $E_j \to \infty$ as $j \to \infty$, of a quantum system whose classical dynamics is fully chaotic, has random matrix statistics in the large energy limit, $E_j \to \infty$. For finite E_j the spectrum has non-universal features depending on the particular system, but as we go higher up in the spectrum the statistical properties become more and more like those from the universal point processes obtained from random matrix theory. The zeros of Riemann's ζ-function, $\{E_j\}_{j\geq 1}$, $0 < E_1 < E_2 < \dots$ (assuming the Riemann hypothesis) show a similar behaviour and has been popular as a model system in quantum chaos, [3], since there are many analogies. The number of zeros $\leq E$, denoted by $\mathcal{N}(E)$, is approximately $\frac{E}{2\pi} \log \frac{E}{2\pi} - \frac{E}{2\pi}$. If we unfold the zeros by letting $x_j = \mathcal{N}(E_j)$, so that the mean spacing becomes 1, it is conjectured by Montgomery, [20], and tested numerically by Odlyzko,

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[21], that the statistics of the x_j's as $j \to \infty$, is like the statistics of a determinantal point process with correlation functions

$$\det \left(\frac{\sin \pi(x_i - x_j)}{\pi(x_i - x_j)} \right)_{i,j=1}^{m}, \tag{1.1}$$

$m \geq 1$. Weak forms of this have been proved in [13, 22].

This determinantal point process is obtained as a scaling limit of GUE or the classical compact groups, e.g. $U(n)$, and also as the universal scaling limit of many other hermitian random matrix ensembles. If we count the number of particles (eigenvalues) in an interval of length L in a process on $\mathbb{R}$ with correlation functions (1.1) we get a random variable whose variance is called the *number variance*, and was first introduced in random matrix theory by Dyson and Mehta, [10]. For the sine kernel point process the number variance goes like $\frac{1}{\pi^2} \log L$ as $L \to \infty$ (an exact formula for finite L is given in (2.33) below). A feature in the quantum chaos model and for the Riemann zeros is *number variance saturation*, [2, 23]. If we consider the Riemann zeros and intervals of length L at height E, where $L \ll E$, and compute the variance by considering many disjoint intervals of length L, the dependence on L is such that for small $L < \log \frac{E}{2\pi}$ it behaves like $\frac{1}{\pi^2} \log L$ but as L grows it saturates, actually oscillates around an average value which is approximately $\frac{1}{\pi^2} \log(\log \frac{E}{2\pi})$, see the work of Berry, [2], for interesting precise predictions. Hence the sine kernel determinantal point process is only a good model in a restricted range which becomes longer as we go up the line. The question that we address in this paper is whether it is possible to construct a determinantal process which shows number variance saturation? Can we construct a determinantal process on $[0, \infty)$ which shows a transition from a non-universal regime to a universal regime described by (1.1) as we go further and further away from the origin? It is not possible to get number variance saturation with a translation invariant kernel, like in (1.1), since the sine kernel is the kernel with the slowest growth of the number variance among all translation invariant kernels which define a determinantal point process, [24].

In this paper we will construct models having these properties by taking suitable scaling limits of determinantal process defined using non-intersecting Brownian motions. These models will not be translation invariant. We can restore translation invariance by averaging, but then we will no longer have a determinantal point process. One of the kernels obtained is given approximately by

$$\frac{\sin \pi(x-y)}{\pi(x-y)} + \frac{d \cos \pi(x+y) + (y-x) \sin \pi(y+x)}{\pi(d^2 + (y-x)^2)}, \tag{1.2}$$

where $d > 0$ is a parameter, see (2.23) below. (The exact kernel has corrections of order $\exp(-Cd)$.) This model will have a number variance with saturation level $\sim \frac{1}{\pi^2} \log(2\pi d)$.

There are other connections between L-functions and random matrix theory. Katz and Sarnak, [16], study low-lying zeros of families of L-functions and connect their statistical behaviour with that obtained for the eigenvalues close to special points of random matrices from the compact classical groups with respect to Haar measure. This leads to a classification of the L-functions into different symmetry classes. Three different laws for the distribution of the lowest zero are obtained. We will see below that these three laws can also be obtained from the non-intersecting Brownian motions by choosing different boundary conditions, see also [11]. Another recent development is the study of characteristic polynomials of matrices from the classical groups which have been used to model L-functions, and led to interesting conjectures for their moments, see [17]. See also [8] for a discussion of linear statistics of zeros.

In this paper we will have nothing to say about quantum chaos or L-functions. The above discussion only serves as a background and a motivation for discussing the probabilistic models that we will introduce. For a discussion of bounded variance in another context see [1].

2. Models and Results

2.1. The model. A point process on $B \subseteq \mathbb{R}$ with correlation functions $\rho_n(x_1, \dots, x_n)$, $n \geq 1$, [24], has determinantal correlation functions if there is a function $K : B \times B \to \mathbb{R}$, the *correlation kernel*, such that

$$\rho_n(x_1, \dots, x_n) = \det(K(x_i, x_j))_{i,j=1}^n, \tag{2.1}$$

$n \geq 1$. The interpretation of ρ_n is that $\rho_n(x_1, \dots, x_n)dx_1 \dots dx_n$ is the probability of finding particles in infinitesimal intervals $dx_1, \dots, dx_n$ around $x_1, \dots x_n$. In particular $\rho_1(x) = K(x, x)$ is the local density at x.

Below we will construct kernels K by taking appropriate limits of other kernels and it is natural to ask if there is a determinantal point process whose correlation kernel is K. This can be answered using the following theorem.

Theorem 2.1. *Let $\rho_{k,N}(x_1, \dots, x_k)$, $k \geq 1$, be the correlation functions of a determinantal point process on $I \subseteq \mathbb{R}$ with continuous correlation kernel $K_N(x, y)$, $N \geq 1$. Assume that $K_N(x, y) \to K(x, y)$ uniformly on compact subsets of I^2. Then there is a point process on I with correlation functions*

$$\rho_k(x_1, \dots, x_k) = \det(K(x_i.x_j))_{i,j=1}^k, \tag{2.2}$$

$k \geq 1$.

The theorem will be proved at the end of Sect. 3.

Let $\phi_i(t)$, $\psi_i(t)$, $i \geq 1$ be functions in $L^2(X, \mu)$. Then, [6, 25],

$$u_N(x)d^N\mu(x) = \frac{1}{Z}\det(\phi_i(x_j))_{i,j=1}^N \det(\psi_i(x_j))_{i,j=1}^N d^N\mu(x) \tag{2.3}$$

defines a measure on X^N with determinantal correlation functions. If we have $u_N(x) \geq 0$ for all $x \in X^N$ and

$$Z = \int_{X^N} \det(\phi_i(x_j))_{i,j=1}^N \det(\psi_i(x_j))_{i,j=1}^N d^N\mu(x) > 0, \tag{2.4}$$

we get a probability measure. We can think of a symmetric probability measure on X^N as a point process on X with exactly N particles. In this paper we will have $X = \mathbb{R}$ or $[0, \infty)$ and μ will be Lebesgue measure. The correlation kernel is given by

$$K_N(x, y) = \sum_{i,j=1}^N \psi_i(x)(A^{-1})_{ij}\phi_j(y), \tag{2.5}$$

where $A = (\int_X \phi_i(x)\psi_j(x)d\mu(x))_{i,j=1}^N$. Note that $\det A = Z > 0$.

One natural way to obtain probability measures of the form (2.3) is from non-intersecting paths using the Karlin-McGregor theorem, [15]. Consider N one-dimensional Brownian motions started at $y_1 < \dots < y_N$ at time 0 and conditioned to stop at $z_1 < \dots < z_N$ at time $S + T$ and not to intersect (coincide) in the whole time interval $[0, S + T]$. The induced measure on the positions $x_1, \dots, x_N$ at time S is then

$$p_{N,S,T}(x) = \frac{1}{Z}\det(p_S(y_i,x_j))_{i,j=1}^N \det(p_T(x_i,z_j))_{i,j=1}^N, \tag{2.6}$$

where

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$$

is the transition kernel for one-dimensional Brownian motion. From the results discussed above it follows that (2.6) defines a point process on $\mathbb{R}$ with determinantal correlation functions. The correlation kernel is given by

$$K_{N,S,T}(u,v) = \sum_{k,j=1}^{N} p_S(y_k,u)(A^{-1})_{kj} p_T(v,z_j), \tag{2.7}$$

where

$$A = \det(p_{S+T}(y_i,z_j))_{i,j=1}^N. \tag{2.8}$$

In the limit $T \to \infty$ this model converges to Dyson's Brownian motion model, [9], with $\beta = 2$, and was considered in [14], see also [12].

We can also consider the same type of measure but on $[0,\infty)$ and with appropriate boundary conditions at the origin. We will consider reflecting or absorbing boundary conditions, where we have the transition kernels, [7],

$$p_t^{\mathrm{re}}(x,y) = p_t(x,y) + p_t(x,-y) \tag{2.9}$$

and

$$p_t^{\mathrm{ab}}(x,y) = p_t(x,y) - p_t(x,-y) \tag{2.10}$$

respectively. We simply replace p_S, p_T in (2.6) with p_S^{re}, p_T^{re} or p_S^{ab}, p_T^{ab}. In these cases we have initial points $0 < y_1 < \cdots < y_N$ and final points $0 < z_1 < \cdots < z_N$. We will be interested in these models as $N \to \infty$ with fixed S, T or with S fixed and $T \to \infty$.

2.2. Correlation kernels. We want to obtain useful expressions for the correlation kernel (2.7) with equidistant final positions, compare [14], Prop. 2.3.

Theorem 2.2. *Let $\Gamma_L : \mathbb{R} \ni t \to L+it$, $L \in \mathbb{R}$, and let γ be a simple closed curve that surrounds $y_0 < \cdots < y_{2n}$; L is so large that Γ_L and γ do not intersect. Set $z_j = a(j-n)$ for some $a > 0$, $0 \le j \le 2n$. Consider the model (2.6) with $(y_i)_{i=0}^{2n}$ as initial conditions, and $(z_i)_{i=0}^{2n}$ as final points, and with no boundary. Then*

$$K_{2n+1,S,T}(u,v) = \frac{ae^{-(u^2+v^2)/2T}}{(2\pi i)^2 S(S+T)} \int_{\Gamma_L} dw \int_{\gamma} dz \frac{1}{e^{a(w-z)/(T+S)}-1} \tag{2.11}$$
$$e^{-\frac{T}{2S(T+S)}[(w-\frac{T+S}{T}v)^2+(z-\frac{T+S}{T}u)^2]+\frac{an}{T+S}(z-w)} \prod_{j=0}^{2n} \frac{e^{aw/(T+S)}-e^{ay_j/(T+S)}}{e^{az/(T+S)}-e^{ay_j/(T+S)}}.$$

The theorem will be proved in Sect. 3.

The $T \to \infty$ limit of this formula appears in [14], compare Theorem 2.3 below. There are analogues of the formula (2.11) for the absorbing and reflecting cases. We have not been able to write down a useful formula for the case of general final positions. The expression in (2.11) is more useful computationally than (2.5) but still rather complicated. We will obtain simpler formulas in certain special cases. First we will give a double contour integral formula for the case $T = \infty$ in the absorbing and reflecting cases. We will also consider the $N \to \infty$ formula in the absorbing case. Then we will specialize to the case when the initial points are also equidistant and $T = \infty$ or $T = S$. In these last two cases we can obtain very nice formulas that are not in terms of contour integrals. The next theorem gives the analogue of Proposition 2.3 in [14] in the absorbing and reflecting cases.

Theorem 2.3. *Let Γ_L be as in Theorem 2.2 and assume that γ surrounds $y_1, \dots, y_N$ and does not intersect Γ_L. Set $z_j = j - 1$ and assume $0 < y_1 < \cdots < y_N$. Then, uniformly for (u, v) in a compact set in $[0, \infty)^2$,*

$$\lim_{T\to\infty} K^{ab}_{N,S,T}(u, v) = K^{ab}_{N,S}(u, v) \doteq \frac{1}{(2\pi i)^2 S} \int_{\Gamma_L} dw \int_{\gamma} dz e^{(w-v)^2/2S} (e^{-(z-u)^2/2S} - e^{-(z+u)^2/2S}) \frac{2w}{w^2 - z^2} \prod_{j=1}^{N} \frac{w^2 - y_j^2}{z^2 - y_j^2} \tag{2.12}$$

and

$$\lim_{T\to\infty} K^{re}_{N,S,T}(u, v) = K^{re}_{N,S}(u, v) \doteq \frac{1}{(2\pi i)^2 S} \int_{\Gamma_L} dw \int_{\gamma} dz e^{(w-v)^2/2S} (e^{-(z-u)^2/2S} + e^{-(z+u)^2/2S}) \frac{2z}{w^2 - z^2} \prod_{j=1}^{N} \frac{w^2 - y_j^2}{z^2 - y_j^2}. \tag{2.13}$$

The theorem will be proved in Sect. 3.

We will also write down a contour integral formula for the $N \to \infty$ limit of $K^{\mathrm{ab}}_{N,S}(u, v)$ under an assumption on the y_j's. (We could write a similar formula in the reflecting case, but we will omit it.)

Theorem 2.4. *Let Γ_L be as above, L arbitrary, and γ_M the two lines $\mathbb{R} \ni t \to \mp t \pm iM$ with $M > 0$. Let $0 < y_1 < y_2 < \dots$ and assume that $\sum_{j=1}^{\infty} 1/y_j^2 < \infty$. Define*

$$F(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{y_j^2}\right), \tag{2.14}$$

which converges uniformly on all compact subsets of $\mathbb{C}$. Set

$$K^*_{S,1}(u, v) = \frac{1}{(2\pi i)^2 S} \int_{\Gamma_L} dw \int_{\gamma_M} dz e^{(w-v)^2/2S - (z-u)^2/2S} \frac{1}{z - w} \frac{wF(w)}{zF(z)}, \tag{2.15}$$

$$K^*_{S,2}(u, v) = \frac{1}{2\pi i} \int_{L-Mi}^{L+Mi} e^{(w-v)^2/2S - (w-u)^2/2S} dw, \tag{2.16}$$

and $K^*_S(u, v) = K^*_{S,1}(u, v) + K^*_{S,2}(u, v)$. Then uniformly on compact subsets of $[0, \infty)^2$,

$$\lim_{N\to\infty} K^{ab}_{N,S}(u, v) = K^{ab}_S(u, v) \doteq K^*_S(u, v) - K^*_S(-u, v). \tag{2.17}$$

The theorem will be proved in Section 3.

We now come to the case when the initial points y_j are equidistant. The next theorem is what makes it possible to compute the number variance in this case. To compute the number variance using the double contour integrals seems difficult.

Theorem 2.5. *Let $y_j = \Delta + a(j-n)$, $1 \le j \le 2n-1$, $0 \le \Delta < a$, $a > 0$. Set*

$$d = \frac{2\pi S}{a^2}. \tag{2.18}$$

Then, uniformly for (u, v) in a compact subset of $\mathbb{R}^2$,

$$\lim_{N\to\infty} K_{N,S}(u, v) = K_S(u-\Delta, v-\Delta), \tag{2.19}$$

where $K_S(u, v) = a^{-1}L_S(a^{-1}u, a^{-1}v)$, and

$$L_S(x, y) = \frac{1}{\pi} Re \sum_{n\in\mathbb{Z}} e^{-\pi dn(n-1)} \frac{e^{\pi i(y+(2n-1)x)}}{nd + i(y-x)}. \tag{2.20}$$

Furthermore, if $y_j = aj$, $j \ge 1$, then uniformly on compact subsets of $[0, \infty)^2$,

$$\lim_{N\to\infty} K^{ab}_{N,S}(u, v) = K^{ab}_S(u, v) \doteq K_S(u, v) - K_S(-u, v), \tag{2.21}$$

and

$$\lim_{N\to\infty} K^{re}_{N,S}(u, v) = K^{re}_S(u, v) \doteq K_S(u, v) + K_S(-u, v). \tag{2.22}$$

The theorem will be proved in Sect. 3.

The leading contribution to (2.20) comes from the terms $n = 0$ and $n = 1$. The other terms are exponentially small in d. The leading part is

$$L_{S,\mathrm{appr}}(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)} + \frac{d\cos\pi(x+y) + (y-x)\sin\pi(y+x)}{\pi(d^2 + (y-x)^2)}, \tag{2.23}$$

so we have the ordinary sine kernel plus a non-translation invariant term. In particular for the density we have

$$K_{S,\mathrm{appr}}(u, u) = \frac{1}{a}\left(1 + \frac{\cos(2\pi u/a)}{\pi d}\right), \tag{2.24}$$

so we have an oscillating density reflecting the initial configuration.

It is also possible to give a more explicit formula in the case $T = S$. We have the following theorem.

Theorem 2.6. *Let $\theta_i(x;\omega)$, $i = 1, 2, 3, 4$, be the Jacobi theta functions, see (3.43), let $y_j = \Delta + a(j-n)$, $1 \le j \le 2n-1$, $0 \le \Delta < a$, $a > 0$, and let d be as in (2.18). Then, uniformly on compact subsets of $\mathbb{R}$,*

$$\lim_{N\to\infty} K_{N,S,S}(u, v) = K_{S,S}(u-\Delta, v-\Delta), \tag{2.25}$$

where $K_{S,S}(u, v) = a^{-1}L_{S,S}(a^{-1}u, a^{-1}v)$ and

$$\begin{aligned} L_{S,S}(u, v) =& \frac{1}{\theta_2(0; id)\sqrt{\theta_3(0; id)\theta_4(0; id)}} \\ &\times \left(\frac{\theta_3(u+v; 2id)\theta_1(u-v; 2id)}{\sinh(\pi(u-v)/2d)} + \frac{\theta_2(u+v; 2id)\theta_4(u-v; 2id)}{\cosh(\pi(u-v)/2d)}\right). \end{aligned} \tag{2.26}$$

If we neglect contributions which are exponentially small in d the leading part of (2.26) is

$$L_{S,S,\mathrm{appr}}(u,v)=\frac{\sin\pi(u-v)}{2d\sinh(\pi(u-v)/2d)}+\frac{\cos\pi(u+v)}{2d\cosh(\pi(u-v)/2d)}. \tag{2.27}$$

Note that as $d\to\infty$ both L_S and $L_{S,S}$ converge to the sine kernel. We see from (2.23) and (2.27) that $L_{S,S}$ decays much faster than L_S at long distances.

We can also consider an averaged model by averaging over Δ in Theorems 2.5 and 2.6. The averaged model has correlation functions

$$\frac{1}{a}\int_0^a \det(K(x_i-\Delta,x_j-\Delta))_{i,j=1}^m d\Delta, \tag{2.28}$$

where K is the appropriate kernel K_S or $K_{S,S}$. This averaging will restore translation invariance. The density in the averaged process will be a constant equal to $1/a$. In particular we have

$$\frac{1}{a}\int_0^a K_S(x-\Delta,y-\Delta)K_S(y-\Delta,x-\Delta)d\Delta=\frac{\sin^2\frac{\pi(x-y)}{a}}{\pi^2(x-y)^2}+\frac{d^2-\left(\frac{x-y}{a}\right)^2}{2\pi^2a^2\left(d^2+\left(\frac{x-y}{a}\right)^2\right)^2}. \tag{2.29}$$

plus terms exponentially small in d. If we instead consider $K_{S,S}$ we get

$$\frac{\sin^2\frac{\pi(x-y)}{a}}{4a^2d^2\sinh^2\frac{\pi(x-y)}{2ad}}+\frac{1}{8a^2d^2\cosh^2\frac{\pi(x-y)}{2ad}}. \tag{2.30}$$

2.3. The number variance. Let $I\subseteq\mathbb{R}$ be an interval and denote by $\#I$ the number of particles contained in I. We are interested in the variance, $Var_K(\#I)$, of this random variable in the determinantal point process with kernel K. The kernel $K=K_{N,S,T}$ that we have considered above is a reproducing kernel, i.e.

$$\int_{\mathbb{R}} K(x,y)K(y,z)dy=K(x,z). \tag{2.31}$$

This is immediately clear from (2.7) and (2.8) and the same also holds in the absorbing and reflecting cases with integration over $[0,\infty)$ instead, and the reproducing property is inherited by the limiting kernels obtained above. Using (2.31) and the determinantal form of the correlation functions it follows that

$$Var_K(\#I)=\int_I dx\int_{I^c} dyK(x,y)K(y,x). \tag{2.32}$$

The sine kernel $\frac{\sin\pi(x-y)/a}{\pi(x-y)}$ with density $1/a$ has the number variance, $I=[0,L]$,

$$V_{\mathrm{sinekernel}}(L)=\frac{1}{\pi^2}\left[\log\frac{2\pi L}{a}+\gamma+1+\frac{2\pi L}{a}\left(\frac{\pi}{2}-\mathrm{Si}\left(\frac{2\pi L}{a}\right)\right)-\cos\frac{2\pi L}{a}-\mathrm{Ci}\left(\frac{2\pi L}{a}\right)\right], \tag{2.33}$$

where γ is Eulers constant. The Sine and Cosine integrals, Si and Ci, are defined in (4.16) and (4.17). In the averaged models we get (we denote the averaging over Δ by $< >$),

$$< \mathrm{Var}_K(\#I) >= \int_I dx \int_{I^c} dy \frac{1}{a} \int_0^a K(x-\Delta, y-\Delta)K(y-\Delta, x-\Delta)d\Delta, \quad (2.34)$$

where K is K_S or $K_{S,S}$. The formulas above for the correlation functions and the formulas for the number variance are used to prove the next theorems. We will only consider the contributions from the leading parts of the kernels, (2.23) and (2.27). Also, we will not use the reflecting and absorbing kernels. If the intervals are high up, $I = [R, R+L]$ with R large, then the contribution from $K_S(-u, v)$ in (2.21) and (2.22) will be small (like $O(1/R)$).

Theorem 2.7. *Consider the kernel K_S of Theorem 2.5. Define θ and ϕ by $R/a-[R/a] = \theta/\pi$ and $L/a - [L/a] = \phi/\pi$. The contribution to $Var_{K_S}(\#[R, R+L])$ coming from the leading part of the kernel (2.23) is, $A = \pi L/a$,*

$$\begin{aligned}
&\frac{1}{\pi^2}\left(1+\frac{\cos 2(\theta+\phi)+\cos 2\theta}{\pi d}\right)\left(\log\frac{2\pi A d}{\sqrt{A^2+\pi^2 d^2}}+\gamma-Ci(2A)\right)\\
&+\frac{1}{\pi^2}\left(1+2A\left(\frac{\pi}{2}-Si(2A)\right)-\cos 2A\right)\\
&+\frac{1}{2\pi^3 d}\{(\cos 2(\theta+\phi)+\cos 2\theta)(h_3(2i\pi d)-h_3(2A+2i\pi d))\\
&+(\sin 2(\theta+\phi)-\sin 2\theta)(h_1(2A+2i\pi d)+\pi-2Si(2A))\}\\
&+\frac{\sin 2\phi}{8\pi^3 d}\{h_2(2A+2i\pi d)(\sin 4(\theta+\phi)-\sin 4\theta)\\
&-h_4(2A+2i\pi d)(\cos 4(\theta+\phi)+\cos 4\theta)\}, \qquad (2.35)
\end{aligned}$$

where $h_1(z) = 2Re\, f(z)$, $h_2(z) = 2Im\, f(z)$, $h_3(z) = 2Re\, g(z)$, $h_4(z) = 2Im\, g(z)$ and

$$f(z) = \int_0^\infty \frac{\sin t}{t+z}dt \quad , \quad g(z) = \int_0^\infty \frac{\cos t}{t+z}dt, \qquad (2.36)$$

for $Re\, z > 0$ or $Re\, z = 0$, $Im\, z \neq 0$. These functions have the asymptotics $f(z) = 1/z - 2/z^3 + O(1/z^4)$, $g(z) = 1/z^2 + O(1/z^4)$ as $z \to \infty$.

The theorem will be proved in Sect. 4.

If we average the expression for the variance over θ or equivalently consider the averaged model, we get that the contribution to the number variance from the leading part is

$$\frac{1}{\pi^2}\left[\log\frac{2\pi L d/a}{\sqrt{L^2/a^2+d^2}}+\gamma+1-\mathrm{Ci}\left(\frac{2\pi L}{a}\right)+\frac{2\pi L}{a}\left(\frac{\pi}{2}-\mathrm{Si}\left(\frac{2\pi L}{a}\right)\right)-\cos\frac{2\pi L}{a}\right]. \qquad (2.37)$$

Apart from the logarithmic term we have exactly the same formula as for the sine kernel, (2.33). It is not difficult to obtain (2.37) using the averaged correlation functions, (2.29). The proof is then analogous to that in the sine kernel case, see Sect. 4. The proof of (2.34) is a lengthy but rather straightforward computation.

Note that when L is small compared to d the leading term in (2.37) is $\frac{1}{\pi^2}\log(2\pi L/a)$, which is what we have for the sine kernel. When $L \to \infty$ the expression (2.37) converges to

$$\frac{1}{\pi^2}(\log(2\pi d) + \gamma + 1) \tag{2.38}$$

so the number variance saturates. Note that the saturation level does not depend directly on the mean spacing. If we rescale the model, see below, we get the same saturation level.

We can also compute the number variance for the (S,S)-model where we have the kernel $K_{S,S}$. In this case we will only consider the averaged model.

Theorem 2.8. *Denote by $V_d(L)$ the leading part of the averaged number variance of an interval of length L for the model with kernel $K_{S,S}$, i.e. (2.27), where d is given by (2.18). Then*

$$\begin{aligned} V_d(L) = & \frac{2}{\pi^2}\int_0^{\pi L/a}\left(\frac{u/2\pi d}{\sinh(u/2\pi d)}\right)^2 \frac{\sin^2 u}{u}du \\ & + \frac{2L}{\pi a}\int_{\pi L/a}^{\infty}\left(\frac{u/2\pi d}{\sinh(u/2\pi d)}\right)^2 \frac{\sin^2 u}{u^2}du \\ & + \frac{1}{\pi^2}\int_0^{\infty}\frac{\min(u, L/2ad)}{\cosh^2 u}du. \end{aligned} \tag{2.39}$$

Also,

$$\lim_{d\to\infty}\frac{1}{\frac{1}{\pi^2}\log d}\lim_{L\to\infty} V_d(L) = 1. \tag{2.40}$$

If we compare (2.39) with the integrals which lead to (2.33), we see that there is a truncation effect which depends on d and which is responsible for the saturation. The limit (2.40) shows that the saturation level is similar to (2.38) for large d.

As mentioned in the introduction the unitary group $U(n)$ has been used in [17] to model the ζ-function at height T, where $n = \log(T/2\pi)$. This n is obtained by equating the mean spacing in $U(n)$, $2\pi/n$, with the mean spacing of the zeros at height T, which is $2\pi(\log(T/2\pi))^{-1}$. Note that the eigenvalues of a random matrix from $U(n)$ with respect to the Haar measure also show a kind of number variance saturation. The variance for the number of eigenvalues in an interval on $\mathbb{T}$ of length a, $0 \le a \le 2\pi$ is given by

$$\mathrm{Var}_{U(n)}(a) = \frac{na}{2\pi} - \frac{na^2}{4\pi^2} - \frac{2}{\pi^2}\sum_{k=1}^{n-1}\frac{n-k}{k^2}\sin^2\frac{ka}{2}. \tag{2.41}$$

This increases as a function of a for $0 \le a \le \pi$ and then decreases symmetrically. We have a maximum variance when $a = \pi$, i.e when we have a half circle. This maximum variance is

$$\frac{1}{\pi^2}(\log(2n) + \gamma + 1) + O\left(\frac{1}{n}\right), \tag{2.42}$$

which is analogous to (2.38) if we set $\pi d = n$. Note that the number variance for the Riemann zeros at height T saturates at the mean level $\frac{1}{\pi^2}\log(\log(T/2\pi)) + \text{const}$, [2], so equating the saturation levels (disregarding constant terms) leads to $n = \log(T/2\pi)$ again. This may be a more natural argument in a sense since it is not changed under rescaling.

2.4. Approximation. Consider the situation in Theorem 2.4, the absorbing (S, ∞)-model, where the y_j's are not equally spaced but are given by $y_j = F^{-1}(j)$ for some increasing function F. If F is nice and does not vary too quickly, the y_j's will be almost equally spaced for long stretches of j, and hence we expect that the kernel (2.17) should be well approximated by (2.21) in a region where the average spacing is a. We will not attempt to make this clear in the greatest possible generality. Our goal is an approximation theorem valid for a certain class of functions F. Denote the correlation kernel $K_S^{\text{ab}}(u, v)$ with initial points $\underline{y} = (y_j)_{j=1}^{\infty}$ by $K_S^{\text{ab}}(\underline{y}, u, v)$ to indicate the dependence on $\underline{y}$. We will prove the following approximation theorem:

Theorem 2.9. *Assume that $F : [0, \infty) \to [0, \infty)$ is a C^2-function that satisfies*

(i) $F(x) \le Cx^{1+\delta}$, for all $x \ge 0$, for some $\delta \in (0, 1)$ and some constant C,
(ii) $F'(0) = 0$ and $F'(x) > 0$ for $x > 0$,
(iii) F'' is decreasing.

Fix α (large) and $\epsilon > 0$ so that $1 + \delta + \epsilon < 2$. Also, fix $S \in [F'(F(\alpha), 1]$. Define $\underline{y} = (y_j)_{j=1}^{\infty}$ by $y_j = F^{-1}(j)$, $j \ge 1$ and $y_{-j} = -y_j$ for $j \ge 0$. For $m \ge 1$ set

$$\xi_m = \sum_{j=1}^{\infty} \frac{2y_m - y_{m-j} - y_{m+j}}{(y_m - y_{m-j})(y_{m+j} - y_m)}, \tag{2.43}$$

$\lambda_m^{-1} = F'(y_m)$ and $\eta_m = F''(y_m)$. Also let $\zeta_m = y_m - \xi_m S$. There is an $m = m(\alpha) \ge 1$ such that $|\zeta_m - \alpha| \le \lambda_m$. We have roughly $m(\alpha) \sim F(\alpha)$. Set $\lambda(\alpha) = \lambda_{m(\alpha)} \sim (F'(F^{-1}(\alpha))^{-1}$, $\eta(\alpha) = \eta_{m(\alpha)} \sim F''(F^{-1}(\alpha))$ and

$$T_0(\alpha) = \min\left(\frac{1}{4\sqrt{\eta(\alpha)}}, m(\alpha)^{(1+\delta)^2/2(1-\delta)}\right). \tag{2.44}$$

Define $\underline{\tilde{y}} = \{\tilde{y}_j\}_{j=1}^{\infty}$, by $\tilde{y}_j = \lambda(\alpha)j$, $j \ge 1$. Assume that $0 < T \le T_0(\alpha)$. There are constants c_0, C, which depend on F and ϵ, but not α, such that

$$\begin{aligned} &\left| e^{-\xi(\alpha)(u-v)} K_S^{ab}(\underline{y}; u, v) - K_S^{ab}(\underline{\tilde{y}}; u - \zeta(\alpha), v - \zeta(\alpha)) \right| \\ &\quad \le C\big[\lambda(\alpha)S^{-3/2}e^{-R^2/8S} + (T^2 + R^2)m(\alpha)^{-(1-\delta)^2/(1+\delta)}S^{-1}\big], \end{aligned} \tag{2.45}$$

for all $u, v \in [\alpha - T, \alpha + T]$ provided R^2 lies in the interval

$$\left[c_0^2 \max\left(\left(\frac{S}{\lambda(\alpha)}\right)^{2/(1-\epsilon)}, T^{1+\delta+\epsilon}S, \frac{T^{1+\epsilon}S}{\lambda(\alpha)}\right), m(\alpha)^{(1-\delta)^2/(1+\delta)}\right]. \tag{2.46}$$

Note that $K_S^{\text{ab}}(\underline{\tilde{y}}; x, y)$ is given by (2.21) with $a = \lambda(\alpha)$, so provided the right hand side of (2.45) is small we have an approximation with a kernel having equally spaced initial points. The factor $\exp(-\xi(\alpha)(u - v))$ in front of $K_S^{\text{ab}}(\underline{y}; u, v)$ in (2.45) does not affect the correlation functions corresponding to this kernel since it cancels in the determinant. Let us consider two examples of Theorem 2.8.

Example 2.10. Let $F(x) = x^{1+\delta}$ with $0 < \delta < 1$ and fix $\epsilon > 0$ small. Then $\lambda(\alpha)^{-1} \sim (1+\delta)\alpha^{\delta/(1+\delta)}$, $\eta(\alpha) \sim \delta(1+\delta)\alpha^{-(1-\delta)/(1+\delta)}$ and $m(\alpha) \sim \alpha^{1+\delta}$ as $\alpha \to \infty$. Choose $S = 1$. If $0 < T \le \alpha^{(1-\delta)^2/2}$, then

$$\left| e^{-\xi(\alpha)(u-v)} K_1^{\text{ab}}(\underline{y}; u, v) - K_1^{\text{ab}}(\underline{\tilde{y}}; u - \zeta(\alpha), v - \zeta(\alpha)) \right| \le C \frac{T^2}{\alpha^{(1-\delta)^2 - 2\delta/[(1-\epsilon)(1+\delta)]}}, \tag{2.47}$$

for all $u, v \in [\alpha - T, \alpha + T]$. We see that this is only interesting if $\delta < 2 - \sqrt{3}$, otherwise the right hand side of (2.47) does not go to zero as $\alpha \to \infty$ unless we also let $T \to 0$. It also follows that

$$\lim_{\alpha\to\infty} e^{-\xi(\alpha)(x-y)/\lambda(\alpha)} K_1(\underline{y}; \alpha + \frac{x}{\lambda(\alpha)}, \alpha + \frac{y}{\lambda(\alpha)}) = \frac{\sin \pi(x-y)}{\pi(x-y)} \tag{2.48}$$

uniformly for x, y in a compact set, so as we go up the line we see the sine kernel process. Note that when x and y belong to a compact set we can take $T = C\lambda(\alpha)$ and the right-hand side of (2.47) goes to zero as $\alpha \to \infty$ for $\delta < 1$ provided we choose ϵ small enough. Thus we can extend (2.48) to all $0 < \delta < 1$ We see a transition from a non-universal regime for small α to the universal sine kernel regime for large α.

Example 2.11. Let $F(x) = \frac{x}{2\pi} \log \frac{x}{2\pi} - \frac{x}{2\pi}$. This F does not satisfy all the conditions in Theorem 2.9 but we can modify it for small x so that it does without changing the y_j's. Then $\lambda(\alpha)\frac{1}{2\pi} \log \frac{\alpha}{2\pi} \to 1$, $2\pi\alpha\eta(\alpha) \to 1$ and $m(\alpha)/F(\alpha) \to 1$ as $\alpha \to \infty$. Fix $\epsilon_0 > 0$ small and let $\delta = \epsilon$, where $(1-\epsilon)^2/(1+\epsilon) - 2\epsilon = 1 - \epsilon_0$. Pick $R = c_0 T^{1/2+\epsilon}\alpha^{\epsilon}$. The theorem then shows that if $1 \le T \le \alpha^{1/2-\epsilon}$, then ($S = 1$)

$$\left| e^{-\xi(u-v)} K_1(\underline{y}; u, v) - K_1(\underline{\tilde{y}}; u, v) \right| \le CT^2\alpha^{-1+\epsilon} \tag{2.49}$$

for $u, v \in [\alpha - T, \alpha + T]$. We see that the parameter d for the approximate kernel is

$$d(\alpha) = \frac{2\pi}{\lambda(\alpha)^2} \sim \frac{8\pi^3}{(\log(\alpha/2\pi))^2}.$$

The saturation level is the (disregarding constants)

$$\sim \frac{1}{\pi^2} \log d(\alpha) \sim \frac{2}{\pi^2} \log\left(\log \frac{\alpha}{2\pi} \right). \tag{2.50}$$

This differs by a factor 2 from what we would like to have if we want to model the ζ-function. Note that the mean spacing at height α is the same as for the ζ-function. However the saturation level does not change in the equidistant case if we rescale the point process linearly (the distance a between the y_j's is not changed, it is the constructed point process that is rescaled). Hence the mean spacing and the saturation level are independent. We can make a better model of the ζ-function by picking a suitable F, see below, and then make a non-linear rescaling. Below we will construct a model for the unfolded zeros $s_j = \mathcal{N}(E_j)$.

Consider a determinantal point process on $[0, \infty)$ with correlation kernel K and let $G(x)$ be a strictly increasing C^1-function. Then

$$\tilde{K}(x, y) = K(G(x), G(y))\sqrt{G'(x)G'(y)} \tag{2.51}$$

defines a new, rescaled point process. The density for the new process is $\rho(G(x))G'(x)$ if $\rho(x)$ is the density for the original process.

Let $y_j = F^{-1}(j)$, $j \geq 1$, where F satisfies the conditions of Theorem 2.9. The density at x is then $F'(x)$ and if we rescale with G we get $F'(G(x))G'(x) = \frac{d}{dx}F(G(x))$. Hence to get a constant density we should pick $G(x) = F^{-1}(x)$. Choose

$$F(x) = 1 + \int_{2\pi e}^{x} \left(\log \frac{t}{2\pi}\right)^{1/2} dt \tag{2.52}$$

for $x \geq 2\pi e$ and define it for $0 \leq x \leq 2\pi e$ so that the conditions in Theorem 2.9 are satisfied. Consider the model corresponding to this F and rescale it using $G = F^{-1}$ as in (2.51). We will say "approximately" below without being too precise. The estimates involved can be made precise with a little effort. By Theorem 2.9 at height T, the kernel will be approximately ($S = 1$)

$$K_1(\underline{\tilde{y}}; F^{-1}(x) - F^{-1}(T), F^{-1}(y) - F^{-1}(T))\sqrt{(F^{-1})'(x)(F^{-1})'(y)} \tag{2.53}$$

for x, y in a neighbourhood of T, where $\tilde{y}_j = \lambda(F^{-1}(T))j$, $j \geq 1$, $\lambda(\alpha)^{-1} \sim F'(\alpha)$. (We have approximated $\zeta(\alpha)$ with α.) Now, for x close to T we have $F^{-1}(x) - F^{-1}(T) \approx (x-T)(F^{-1})'(T)$. Set $b = (F^{-1})'(T) \approx \lambda(T)$. By (2.53) and (2.23) at height T (large) the kernel will approximately equal

$$\begin{aligned} L_{S,\mathrm{appr}}(x-T, y-T) = &\frac{\sin \pi(x-y)}{\pi(x-y)} \\ &+ \frac{d \cos \pi(x+y-2T) + (y-x)\sin \pi(x+y-2T)}{\pi(d^2 + (y-x)^2)}, \end{aligned} \tag{2.54}$$

where $d = 2\pi/\lambda(T)^2$. We have $\lambda(T) \sim (\log \frac{T}{2\pi})^{-1/2}$. As $T \to \infty$, $d \to \infty$ and we see that the kernel in (2.54) converges to the sine kernel. The point process we have constructed thus has the correct universal asymptotics as we go up the line, and it is non-universal for small T, since it depends on the particular F we have chosen. The saturation level for the kernel in (2.54) is

$$\frac{1}{\pi^2}\log(2\pi d) \sim \frac{1}{\pi^2}\log\left(\log\frac{T}{2\pi}\right), \tag{2.55}$$

when T is large, which is exactly what we would like to have at height T. We have thus constructed a determinantal point process in $[0, \infty)$ which has, in some aspects, similar behaviour to the unfolded zeros of the ζ-function.

2.5. Correlation kernels close to the origin. Consider the kernels K_S on $\mathbb{R}$, and K_S^{ab} and K_S^{re} on $[0, \infty)$ in Theorem 2.5. If we are interested in, say, the distribution of the first particle to the right of the origin we have to compute the probability of having no particle in $[0, \xi]$. Let X be the position of the first particle to the right of the origin. Then, if the correlation kernel is K,

$$\begin{aligned} \mathbb{P}[X \leq \xi] &= 1 - \mathbb{P}[\text{no particle in } [0, \xi]] \\ &= 1 - \det(I - K)_{L^2([0,\xi])}, \end{aligned} \tag{2.56}$$

where the second equality is a standard result for determinantal point processes. If we vary a and S in such a way that $S/a^2 \to \infty$, it follows from Theorem 2.5 that

$$\lim_{S/a^2 \to \infty} aK_S(au, av) = \frac{\sin \pi(u-v)}{\pi(u-v)}, \tag{2.57}$$

$$\lim_{S/a^2 \to \infty} aK_S^{\mathrm{ab}}(au, av) = \frac{\sin \pi(u-v)}{\pi(u-v)} - \frac{\sin \pi(u+v)}{\pi(u+v)} \tag{2.58}$$

and

$$\lim_{S/a^2 \to \infty} aK_S^{\mathrm{re}}(au, av) = \frac{\sin \pi(u-v)}{\pi(u-v)} + \frac{\sin \pi(u+v)}{\pi(u+v)}. \tag{2.59}$$

These kernels can also be obtained from the classical compact groups and have been used by Katz and Sarnak to model the lowest zeros in families of L-functions, see [16]. The above results show that they can also be obtained in a natural way from non-intersecting paths with appropriate boundary conditions.

The kernels in the right hand side of (2.58) and (2.59) are directly related to special instances of the Bessel kernel,

$$B_\nu(x, y) = \frac{x^{1/2} J_{\nu+1}(x^{1/2}) J_\nu(y^{1/2}) - J_\nu(x^{1/2}) y^{1/2} J_{\nu+1}(y^{1/2})}{2(x-y)}, \tag{2.60}$$

where $J_\nu(x)$ is the ordinary Bessel function. In fact a simple computation shows that if we define the rescaled Bessel kernel $\tilde{B}_\nu$ by

$$\tilde{B}_\nu(x, y) = \sqrt{2\pi^2 x 2\pi^2 y}\, B_\nu(\pi^2 x^2, \pi^2 y^2), \tag{2.61}$$

then

$$\tilde{B}_{\pm 1/2}(x, y) = \frac{\sin \pi(u-v)}{\pi(u-v)} \mp \frac{\sin \pi(u+v)}{\pi(u+v)}. \tag{2.62}$$

When ν is an integer the kernel B_ν appears in the scaling limit around the smallest eigenvalue in LUE, the Laguerre Unitary Ensemble. If $x_1, \dots, x_N$ are the eigenvalues of M^*M, where M is a $(\nu + N) \times N$ complex matrix with independent standard complex Gaussian elements, $N(0, 1/2) + iN(0, 1/2)$, then $x_1, \dots, x_N$ is a finite determinantal point process with correlation kernel K_N^ν and

$$\lim_{N \to \infty} \frac{1}{4N} K_N^\nu\left(\frac{x}{4N}, \frac{y}{4N}\right) = B_\nu(x, y). \tag{2.63}$$

This interpretation does not work for $\nu = \pm 1/2$.

3. Computation of the Correlation Functions

In this section we will use the formula (2.7) to compute the correlation functions. If A is a matrix and b a column vector we will denote by $(A|b)_k$ the matrix where column k in A is replaced by b. Let

$$p = (p_T(v, z_0) \dots p_T(v, z_N))^T,$$

where $z_j = a(j-n)$, $0 \le j \le N = 2n$. By (2.7) and Kramers rule we have

$$K_{2n,S,T}(\underline{y}; u, v) = \sum_{k=0}^{2n} p_S(y_k, u) \frac{\det(A|p)_k}{\det A}, \tag{3.1}$$

where $A = (p_{S+T}(y_i, z_k))_{j,k=0}^{2n}$. If C_M is the contour $t \to t + iM$, $M \in \mathbb{R}$, $t \in \mathbb{R}$, we have

$$p_T(v, z_i) = \sqrt{\frac{T+S}{2\pi T}} e^{-v^2/2T} \int_{C_M} e^{-\tau^2/2} p_{T+S}(\tilde{y}_k(\tau), z_i) e^{\tilde{y}_k(\tau)^2/2(T+S)} d\tau, \tag{3.2}$$

where $\tilde{y}_k = \tilde{y}_k(\tau) = v(T+S)/T + i\tau\sqrt{S(T+S)/T}$; set also $\tilde{y}_i = y_i$ if $i \neq k$. Then,

$$\frac{\det(A|p)_k}{\det A} = \sqrt{\frac{T+S}{2\pi T}} e^{-v^2/2T} \int_{C_M} e^{-\tau^2/2 + \tilde{y}_k(\tau)^2/2(T+S)} \frac{\det(p_{T+S}(\tilde{y}_j, z_i))_{i,j=0}^{2n}}{\det(p_{T+S}(y_j, z_i))_{i,j=0}^{2n}} d\tau. \tag{3.3}$$

Since $z_j = a(j-n)$ the determinants in the quotient in the right hand side of (3.3) can be computed using Vandermonde's determinant and we find

$$e^{(y_k^2 - \tilde{y}_k^2)/2(T+S) + an(y_k - \tilde{y}_k)/2(T+S)} \prod_{j \neq k} \frac{e^{a\tilde{y}_k/(T+S)} - e^{ay_j/(T+S)}}{e^{ay_k/(T+S)} - e^{ay_j/(T+S)}}.$$

Inserting this into (3.2) and making the change of variables $w = \tilde{y}_k(\tau)$ we obtain

$$\begin{aligned} \frac{\det(A|p)_k}{\det A} &= \frac{1}{i\sqrt{2\pi S}} e^{-v^2/2T + y_k^2/2(T+S) + ny_k a/(T+S)} \\ &\times \int_{\Gamma_L} dw e^{T(w-(T+S)v/T)^2/2S(T+S) - nwa/(T+S)} \prod_{j \neq k} \frac{e^{aw/(T+S)} - e^{ay_j/(T+S)}}{e^{ay_k/(T+S)} - e^{ay_j/(T+S)}}. \end{aligned} \tag{3.4}$$

We can now use the expression in (3.4) and insert it into (3.1) to get

$$\begin{aligned} K_{2n,S,T}(\underline{y}; u, v) &= \frac{1}{2\pi S} e^{-v^2/2T + u^2/2T} \sum_{k=0}^{2n} e^{-T(y_k - (T+S)u/T)^2/2S(T+S)} \\ &\times \int_{\Gamma_L} dw e^{T(w-(T+S)v/T)^2/2S(T+S)} e^{na(y_k - w)/(T+S)} \prod_{j \neq k} \frac{e^{aw/(T+S)} - e^{ay_j/(T+S)}}{e^{ay_k/(T+S)} - e^{ay_j/(T+S)}}. \end{aligned} \tag{3.5}$$

This is the basic formula from which the others will be derived. It is now straightforward to prove Theorem 2.2.

Proof of Theorem 2.2. If we apply the residue theorem in (2.11) we get the expression (3.5). Since Γ_L and γ do not intersect the $w = z$ singularity does not contribute. □

We turn next to Theorem 2.3.

Proof of Theorem 2.3. Consider first the absorbing case. We use the formulas (3.1) and (3.3) but with p_t^{ab} instead of p_t. The evaluation of the two determinants can now be done using the following Vandermonde type identity

$$\begin{aligned}&\det(e^{(2i-1)x_j}-e^{-(2i-1)x_j})_{i,j=1}^{N}\\&\quad=\prod_{i=1}^{N}(e^{x_i}-e^{-x_i})\prod_{1\le i<j\le N}((e^{x_j}-e^{-x_j})^2-(e^{x_i}-e^{-x_i})^2).\end{aligned}\tag{3.6}$$

Using this identity we find

$$\begin{aligned}\frac{\det(A|p^{\mathrm{ab}})_k}{\det A}&=\sqrt{\frac{T+S}{2\pi T}}e^{\frac{y_k^2}{2(T+S)}-\frac{v^2}{2T}}\int_{C_M}e^{-\frac{\tau^2}{2}}\frac{e^{\frac{v}{T}+i\sqrt{\frac{S}{T(T+S)}}}-e^{-\frac{v}{T}-i\sqrt{\frac{S}{T(T+S)}}}}{e^{\frac{y_k}{T+S}}-e^{-\frac{y_k}{T+S}}}\\&\quad\times\prod_{j\neq k}\frac{\left(e^{\frac{v}{T}+i\sqrt{\frac{S}{T(T+S)}}}-e^{-\frac{v}{T}-i\sqrt{\frac{S}{T(T+S)}}}\right)^2-\left(e^{\frac{y_j}{T+S}}-e^{-\frac{y_j}{T+S}}\right)^2}{\left(e^{\frac{y_k}{T+S}}-e^{-\frac{y_k}{T+S}}\right)^2-\left(e^{\frac{y_j}{T+S}}-e^{-\frac{y_j}{T+S}}\right)^2}.\end{aligned}\tag{3.7}$$

In this expression we can expand the exponentials and take the $T\to\infty$ limit. We find that

$$\lim_{T\to\infty}\frac{\det(A|p^{\mathrm{ab}})_k}{\det A}=\frac{1}{i\sqrt{2\pi S}}\int_{\Gamma_L}e^{(w-v)^2/2S}\frac{w}{y_k}\prod_{j\neq k}\frac{w^2-y_j^2}{y_k^2-y_j^2}dw\tag{3.8}$$

uniformly for (u,v) in a compact set. Here we have changed the integration variable by putting $w=v+i\tau\sqrt{S}$. Hence

$$\begin{aligned}\lim_{T\to\infty}K_{N,S,T}^{\mathrm{ab}}(u,v)&=\frac{1}{2\pi iS}\sum_{k=1}^{N}\left(e^{-(y_k-u)^2/2S}-e^{-(y_k+u)^2/2S}\right)\\&\quad\times\int_{\Gamma_L}e^{(w-v)^2/2S}\frac{w}{y_k}\prod_{j\neq k}\frac{w^2-y_j^2}{y_k^2-y_j^2}dw.\end{aligned}\tag{3.9}$$

That this expression equals the expression in (2.12) follows from the residue theorem.

The proof of (2.13) is completely analogous. Instead of (3.6) we use the identity

$$\det\left(e^{(j-1)x_i}-e^{-(j-1)x_i}\right)_{i,j=1}^{N}=\prod_{1\le i<j\le N}((e^{x_j}+e^{-x_j})-(e^{x_i}-e^{-x_i})).\tag{3.10}$$

□

Proof (of Theorem 2.5). Set

$$F_N(z)=\prod_{j=1}^{N}\left(1-\frac{z^2}{y_j^2}\right),\tag{3.11}$$

so that, by (2.12),

$$K^{\mathrm{ab}}_{N,S}(u,v)=\frac{1}{(2\pi i)^2 S}\int_{\Gamma_0}dw\int_{\gamma^+_{M,r}}dze^{\frac{(w-v)^2}{2S}}\left(e^{\frac{(z-u)^2}{2S}}-e^{\frac{(z+u)^2}{2S}}\right)\frac{2w}{w^2-z^2}\frac{F_N(w)}{F_N(z)}. \tag{3.12}$$

Here $\gamma^+_{M,r}$ is the curve given by $t\to -t\pm im$, $t\geq r$ and $s\to r\mp is$, $-M\leq s\leq M$, where $0<r<y_1$. The fact that $F_N(z)\to F(z)$ uniformly on compact subsets of $\mathbb{C}$, together with estimates like (5.10) and (5.11) below, which can be used to restrict the $z-$ and $w-$ integrations, shows that

$$\lim_{N\to\infty}K^{\mathrm{ab}}_{N,S}(u,v)=\frac{1}{(2\pi i)^2 S}\int_{\Gamma_0}dw\int_{\gamma^+_{M,r}}dze^{\frac{(w-v)^2}{2S}}$$
$$\times\left(e^{\frac{(z-u)^2}{2S}}-e^{\frac{(z+u)^2}{2S}}\right)\frac{2w}{w^2-z^2}\frac{F(w)}{F(z)}\doteq K^{\mathrm{ab}}_S(u,v), \tag{3.13}$$

uniformly for (u,v) in a compact set. Let $\gamma^-_{M,r}$ be the contour which is the image of $\gamma^+_{M,r}$ under $z\to -z$. Using

$$\frac{2w}{w^2-z^2}=\frac{w}{z}\left(\frac{1}{w-z}-\frac{1}{w+z}\right),$$

we see that $K^{\mathrm{ab}}_S(u,v)=K^*_S(u,v)-K^*_S(-u,v)$, where

$$K^*_S(u,v)=\frac{1}{(2\pi i)^2 S}\int_{\Gamma_0}dw\int_{\gamma_{M,r}}dze^{\frac{(w-v)^2}{2S}-\frac{(z-u)^2}{2S}}\frac{1}{z-w}\frac{wF(w)}{zF(z)},$$

$\gamma_{M,r}=\gamma^+_{M,r}+\gamma^-_{M,r}$. If we let $r\to 0+$ we pick up a contribution from the pole $z=w$. This leads to $K^*_S=K^*_{S,1}+K^*_{S,2}$ as stated in the theorem. □

Next we consider Theorem 2.5.

Proof of Theorem 2.5. It suffices to consider the case $\Delta=0$, otherwise we replace (u,v) with $(u-\Delta,v-\Delta)$. From the proof of Proposition 2.3 in [14], the $T\to\infty$ limit of (3.5) is

$$K_{N,S}(u,v)=\lim_{T\to\infty}K_{N,S,T}(u,v) \tag{3.14}$$
$$=\frac{1}{2\pi i S}\sum_{k=-n}^{n}e^{-\frac{(ak-u)^2}{2S}}\int_{\Gamma_L}dwe^{\frac{(w-v)^2}{2S}}\prod_{j=-n,j\neq k}^{n}\frac{w-aj}{ak-aj}.$$

Now,

$$\prod_{j=-n,j\neq k}^{n}\frac{w-aj}{ak-aj}=\frac{(-1)^k w}{w-ak}\prod_{j=1}^{n}\left(1-\frac{w^2}{(aj)^2}\right)\prod_{j=0}^{k-1}\frac{n-j}{n+1+j}. \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\lim_{N\to\infty} K_{N,S}(u,v) = K_S(u,v)$$
$$= \frac{1}{2\pi i S}\sum_{k\in\mathbb{Z}} e^{-\frac{(ak-u)^2}{2S}} \int_{\Gamma_L} dw e^{\frac{(w-v)^2}{2S}} \prod_{j=1}^{\infty}\left(1-\frac{w^2}{(aj)^2}\right) \tag{3.16}$$

uniformly for (u, v) in a compact set. To prove the convergence we need some estimates so that we can cut off the k-summation and the w-integration. We omit the details. Now,

$$w\prod_{j=1}^{\infty}\left(1-\frac{w^2}{(aj)^2}\right) = \frac{a}{\pi}\sin\frac{\pi}{a}w. \tag{3.17}$$

Hence,

$$K_S(u,v) = \frac{a}{2\pi^2 i S}\sum_{k\in\mathbb{Z}}\int_{\Gamma_L} e^{\frac{(w-v)^2-(ak-u)^2}{2S}} \frac{(-1)^k}{w-ak}\sin\frac{\pi w}{a}dw, \tag{3.18}$$

where L is arbitrary. Replace w by $w + v - u + ak$ in the integral in (3.18) and use Cauchy's theorem to get

$$\begin{aligned} K_S(u,v) &= \frac{a}{2\pi^2 i S}\sum_{k=0}^{\infty}\int_{\Gamma_{-1}} e^{\frac{w^2-2uw}{2S}} e^{\frac{kaw}{S}} \frac{\sin\frac{\pi}{a}(w+v-u)}{w+v-u}dw \\ &\quad + \frac{a}{2\pi^2 i S}\sum_{k=-\infty}^{-1}\int_{\Gamma_{1}} e^{\frac{w^2-2uw}{2S}} e^{\frac{kaw}{S}} \frac{\sin\frac{\pi}{a}(w+v-u)}{w+v-u}dw \\ &= \frac{a}{2\pi^2 i S}\int_{\Gamma} \frac{e^{(w^2-2uwD)/2S}}{e^{aw/S}-1}\frac{\sin\frac{\pi}{a}(w+v-u)}{w+v-u}dw, \end{aligned} \tag{3.19}$$

where $\Gamma = \Gamma_{-1} + \Gamma_1$. The function $e^{aw/S} - 1$ has simple zeros at $w = 2\pi Sni/a, n \in \mathbb{Z}$ and hence by the residue theorem applied to the last integral in (3.19),

$$K_S(u,v) = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} e^{-2\pi^2 n^2 S/a^2 - 2\pi uni/a}\frac{\sin\frac{\pi}{a}\left(\frac{2\pi nS}{a}i+v-u\right)}{\frac{2\pi nS}{a}i+v-u}. \tag{3.20}$$

If we set $L(x, y) = aK(ax, ay)$ and define d as in (2.18) we find

$$\begin{aligned} L(x,y) &= \frac{1}{\pi}\sum_{n\in\mathbb{Z}} e^{-\pi dn^2-2\pi nix}\frac{\sin\pi(ndi+y-x)}{ndi+y-x} \\ &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\frac{e^{-\pi dn^2-\pi nd}}{-nd+i(y-x)}e^{\pi i(y-(2n+1)x)} \\ &\quad -\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\frac{e^{-\pi dn^2+\pi nd}}{-nd+i(y-x)}e^{\pi i(y+(2n-1)x)}. \end{aligned} \tag{3.21}$$

If we change n to $-n$ in the first sum we get (2.20).

In the absorbing case we have by (3.9), (3.11) and the identity

$$\frac{w}{y_k}\prod_{j\neq k}\frac{w^2-y_j^2}{y_k^2-y_j^2}=\left(\frac{1}{w-y_k}-\frac{1}{w+y_k}\right)\frac{F_N(w)}{F_N'(y_k)},$$

that

$$\lim_{N\to\infty}K_{N,S}^{\text{ab}}(u,v)=\frac{a}{2\pi^2 iS}\sum_{k=1}^{\infty}\int_{\Gamma_L}\left(e^{-\frac{(ak-u)^2}{2S}}-e^{-\frac{(ak+u)^2}{2S}}\right)$$
$$\times\left(\frac{1}{w-y_k}-\frac{1}{w+y_k}\right)(-1)^k\sin\frac{\pi}{a}w\,dw. \qquad (3.22)$$

If we now note that

$$\sum_{k=1}^{\infty}(-1)^k\left(e^{-\frac{(ak-u)^2}{2S}}-e^{-\frac{(ak+u)^2}{2S}}\right)\left(\frac{1}{w-y_k}-\frac{1}{w+y_k}\right)$$
$$=\sum_{k\in\mathbb{Z}}e^{-\frac{(ak-u)^2}{2S}}\frac{(-1)^k}{w-ak}-\sum_{k\in\mathbb{Z}}e^{-\frac{(ak+u)^2}{2S}}\frac{(-1)^k}{w-ak} \qquad (3.23)$$

then (3.18) and (3.22) give (2.21). The proof of (2.22) is completely analogous. □

The proof of Theorem 2.6 is a similar but somewhat more complicated computation where we have to use θ-function identities.

Proof of Theorem 2.6. Our starting point is the formula (3.5) with $T=S$. We can assume that $\Delta=0$ so that $y_j=a(j-n)$, $0\leq j\leq 2n$. Set

$$F_n(z;q)=\prod_{j=0}^{n}(1-q^jz)\prod_{j=1}^{n}(1-q^j/z), \qquad (3.24)$$

where

$$q=e^{-a^2/2S}=e^{-\pi/d}. \qquad (3.25)$$

Then (3.5) and a short computation gives

$$K_{2n+1,S,S}(u,v)=\frac{ae^{-(v^2-u^2)/2S}}{2(2\pi i)^2S^2}\int_{\Gamma_L}dw\int_{\gamma}dz\frac{1}{e^{a(w-z)/2S}-1}$$
$$\times e^{[(w-2v)^2-(z-2u)^2]/2S}\frac{F_n(e^{aw/2S};q)}{F_n(e^{az/2S};q)}, \qquad (3.26)$$

where γ surrounds $y_j=a(j-n)$, $0\leq j\leq 2n$ and does not interesect Γ_L. Let γ_M be $t\to\mp t\pm iM$. Replacing γ by γ_M in (3.26) we pick up a contribution from the pole $z=w$, when $|\mathrm{Im}\,w|\leq M$. Hence

$$K_{2n+1,S,S}(u,v)=\frac{ae^{-(v^2-u^2)/2S}}{2(2\pi i)^2S^2}\int_{\Gamma_L}dw\int_{\gamma_M}dz\frac{1}{e^{a(w-z)/2S}-1}$$
$$\times e^{[(w-2v)^2-(z-2u)^2]/4S}\frac{F_n(e^{aw/2S};q)}{F_n(e^{az/2S};q)}$$
$$-\frac{e^{(v^2-u^2)/2S}}{2\pi i}\int_{L-iM}^{L+iM}e^{[(w-2v)^2-(z-2u)^2]/4S}dw. \qquad (3.27)$$

In this expression we can control the $n \to \infty$ limit (using some estimates of F_n, compare (5.10), (5.11)). Set

$$F(z;q) = \prod_{j=0}^{\infty}(1-q^j z)\prod_{j=1}^{\infty}(1-q^j/z). \tag{3.28}$$

It follows from (3.27) that

$$\begin{aligned} K_{S,S}(u,v) = \lim_{N\to\infty} K_{2n+1,S,S}(u,v) &= \frac{ae^{-(v^2-u^2)/2S}}{2(2\pi i)^2 S^2}\int_{\Gamma_L} dw \int_{\gamma_M} dz \frac{1}{e^{a(w-z)/2S}-1} \\ &\quad \times e^{[(w-2v)^2-(z-2u)^2]/4S}\frac{F(e^{aw/2S};q)}{F(e^{az/2S};q)} \\ &\quad -\frac{e^{(v^2-u^2)/2S}}{2\pi i}\int_{L-iM}^{L+iM} e^{[(w-2v)^2-(z-2u)^2]/4S} dw. \end{aligned} \tag{3.29}$$

We now compute the z-integral in (3.29) using the residue theorem. Apart from the pole $z = w$ if $|\mathrm{Im}\, w| \le M$ we have simple poles at $z = ak$, $k \in \mathbb{Z}$. We obtain

$$\begin{aligned} K_{S,S}(u,v) &= \frac{e^{-(v^2-u^2)/2S}}{2\pi i S}\sum_{k\in\mathbb{Z}}\int_{\Gamma_L} dw e^{[(w-2v)^2-(z-2u)^2]/4S} \\ &\quad \times \frac{1}{q^{-w/a+k}}\frac{F(q^{-w/a})}{q^{-k}F'(q^{-k})}. \end{aligned} \tag{3.30}$$

In the integral in (3.30) we make the change of variables $w \to a(w+k)$ and use Cauchy's theorem. A computation shows that

$$F(q^{-k}q^{-w}) = \frac{(-1)^k}{q^{k(k+1)/2}} q^{-kw} F(q^{-w}), \tag{3.31}$$

$$q^{-k}F'(q^{-k}) = \frac{(-1)^{k-1}}{q^{k(k+1)/2}}\prod_{n=1}^{\infty}(1-q^n)^2, \tag{3.32}$$

which gives

$$\frac{1}{q^{-w}-1}\frac{F(q^{-k}q^{-w})}{q^{-k}F'(q^{-k})} = \prod_{n=1}^{\infty}(1-q^n)^{-2}\frac{q^{-kw}}{1-q^{-w}}F(q^{-w}).$$

If we set

$$G(z;q) = \prod_{j=1}^{\infty}(1-q^j z)(1-q^j/z), \tag{3.33}$$

we obtain

$$\begin{aligned} K_{S,S}(u,v) &= \frac{ae^{-(v^2-u^2)/2S}}{2\pi i S}\sum_{k\in\mathbb{Z}}\int_{\Gamma_L} dw e^{\frac{1}{4S}[(aw+ak-2v)^2-(ak-2u)^2]} \\ &\quad \times e^{ka^2 w/2S}\frac{G(q^{-w};q)}{G(1;q)}. \end{aligned} \tag{3.34}$$

Make the change of variables $w \to w + (v-u)/a$ and perform the k-summation to get

$$K_{S,S}(u,v) = -\frac{ae^{-(v^2+u^2)/2S}}{2\pi i S}\int_\Gamma \frac{e^{(aw-u-v)^2/S}}{e^{-a^2w/S}-1}\frac{G(q^{-w+(u-v)/a};q)}{G(1;q)}dw, \tag{3.35}$$

where $\Gamma = \Gamma_1 - \Gamma_{-1}$ as before. The integrand in (3.35) has simple poles at $w = dni$, $n \in \mathbb{Z}$, and the residue theorem gives

$$K_{S,S}(u,v) = -\frac{e^{-(v^2+u^2)/2S}}{a}\sum_{n\in\mathbb{Z}} e^{(adni-u-v)^2/S}\frac{G(q^{-dni+(u-v)/a};q)}{G(1;q)}. \tag{3.36}$$

A computation leads to

$$\begin{aligned} L_{S,S}(u,v) &= aK_{S,S}(au,av) \\ &= e^{-\pi(u-v)^2/2d}\sum_{n\in\mathbb{Z}} e^{-\pi dn^2/2-\pi ni(u+v)}\frac{G((-1)^n e^{-\pi(u-v)/d};q)}{G(1;q)}. \end{aligned} \tag{3.37}$$

If we write $q = e^{\pi i\omega}$, $\omega = i/d$, $p = e^{\pi ix}$, $C_0 = \prod_{n=1}^\infty(1-q^{2n})$, it follows from the product representations of the θ-functions that

$$\begin{aligned} G(p^2;q) &= \frac{1}{2C_0^2q^{1/4}}\frac{\theta_1(x;\omega)\theta_4(x;\omega)}{\sin\pi x}, \\ G(-p^2;q) &= \frac{1}{2C_0^2q^{1/4}}\frac{\theta_2(x;\omega)\theta_3(x;\omega)}{\cos\pi x}. \end{aligned} \tag{3.38}$$

Write $\omega' = -2/\omega = 2id$. If we insert the formulas (3.38) into (3.37) we obtain

$$\begin{aligned} L_S(u,v) &= \frac{e^{-\frac{\pi(u-v)^2}{2d}}}{2C_0^2q^{1/4}G(1;q)}\sum_{n\in\mathbb{Z}}\Bigg\{e^{i\pi\omega'n^2-2\pi ni(u+v)}\frac{\theta_1(\frac{\omega}{2}(u-v);\omega)\theta_4(\frac{\omega}{2}(u-v);\omega)}{\sin\frac{\pi\omega}{2}(u-v)} \\ &\quad + e^{i\pi\omega'(n-\frac12)^2-\pi(2n-1)i(u+v)}\frac{\theta_2(\frac{\omega}{2}(u-v);\omega)\theta_3(\frac{\omega}{2}(u-v);\omega)}{\cos\frac{\pi\omega}{2}(u-v)}\Bigg\} \\ &= \frac{e^{-\frac{\pi(u-v)^2}{2d}}}{2C_0^2q^{1/4}G(1;q)}\Bigg\{\frac{\theta_3(u+v;\omega')\theta_1(\frac{\omega}{2}(u-v);\omega)\theta_4(\frac{\omega}{2}(u-v);\omega)}{\sin\frac{\pi\omega}{2}(u-v)} \\ &\quad + \frac{\theta_2(u+v;\omega')\theta_2(\frac{\omega}{2}(u-v);\omega)\theta_3(\frac{\omega}{2}(u-v);\omega)}{\cos\frac{\pi\omega}{2}(u-v)}\Bigg\}, \end{aligned} \tag{3.39}$$

where we have used the series expansions of the θ-functions, see (3.43) below.

We can simplify (3.39) somewhat by using some θ-function identities. The first is Jacobi's transformations:

$$\theta_k(\omega x;\omega) = \frac{\theta_k(x;-1/\omega)}{\alpha_k(-i\omega)^{1/2}e^{\pi i\omega x^2}},$$

$k = 1,2,3,4$, where $\alpha_1 = -i$ and $\alpha_2 = \alpha_3 = \alpha_4 = 1$, and the formulas [18], p. 17,

$$\begin{aligned} \theta_1(x;\omega)\theta_2(x;\omega) &= \theta_1(2x;2\omega)\theta_4(0;2\omega), \\ \theta_3(x;\omega)\theta_4(x;\omega) &= \theta_4(2x;2\omega)\theta_4(0;2\omega). \end{aligned} \tag{3.40}$$

This leads to

$$L_S(u,v)=\frac{i\theta_4(0;-2/\omega)}{2\omega C_0^2 q^{1/4}G(1;q)}\left\{\frac{\theta_3(u+v;\omega')\theta_1(u-v;\omega')}{-i\sin\frac{\pi\omega}{2}(u-v)}+\frac{\theta_2(u+v;\omega')\theta_4(u-v;\omega')}{-i\cos\frac{\pi\omega}{2}(u-v)}\right\}. \tag{3.41}$$

Now, by the product formulas for the values of the θ-functions at the origin we have

$$G(1;q)=\prod_{n=1}^{\infty}(1-q^n)^2=\frac{\theta_4(0;\omega)\theta_1'(0,\omega)}{2\pi C_0^2 q^{1/4}}$$

and consequently

$$\begin{aligned}2C_0^2 q^{1/4}G(1;q)&=\frac{1}{\pi}\theta_4(0;\omega)\theta_1'(0;\omega)\\&=-\frac{1}{\omega^2}\theta_4(0;-\frac{1}{\omega})\theta_3(0;-\frac{1}{\omega})\theta_2(0;-\frac{1}{\omega})^2\end{aligned} \tag{3.42}$$

by Jacobi's transformation and the formula

$$\theta_1'(0;\omega)=\pi\theta_2(0;\omega)\theta_3(0;\omega)\theta_4(0;\omega).$$

By Landen's transformation

$$\theta_4(0;-2/\omega)=\sqrt{\theta_3(0;-1/\omega)\theta_4(0;-1/\omega)},$$

and hence

$$\frac{i\theta_4(0;-2/\omega)}{2\omega C_0^2 q^{1/4}G(1;q)}=-\frac{i\omega}{\theta_2(0;-1/\omega)^2\sqrt{\theta_3(0;-1/\omega)\theta_4(0;-1/\omega)}}.$$

If we insert this into (3.27) we obtain (2.26) and the theorem is proved. That the leading behaviour of the kernel is given by (2.27) follows from the series expansions of the θ-functions:

$$\begin{aligned}\theta_1(x;\tau)&=i\sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i\tau(n-1/2)^2+\pi i(2n-1)x},\\\theta_2(x;\tau)&=\sum_{n\in\mathbb{Z}}e^{\pi i\tau(n-1/2)^2+\pi i(2n-1)x},\\\theta_3(x;\tau)&=\sum_{n\in\mathbb{Z}}e^{\pi i\tau n^2+2\pi inx},\\\theta_4(x;\tau)&=\sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i\tau n^2+2\pi inx}.\end{aligned} \tag{3.43}$$

□

We should also prove Theorem 2.1.

Proof. We will use the following criterion of Lenard, [19, 24]: A family of locally integrable functions $\rho_k: I^k\to\mathbb{R}$, $k=1,2,\dots$ are the correlation functions of some point process if and only if the following two conditions are satisfied

a) (Symmetry) For any $\sigma \in S_k$,

$$\rho_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \rho_k(x_1, \dots, x_k).$$

b) (Positivity) For any finite set of measurable bounded functions $\phi_k : I^k \to \mathbb{R}$, $k = 0, \dots, N$, with compact support, such that

$$\phi_0 + \sum_{k=1}^{N} \sum_{i_1 \neq \dots \neq i_k} \phi_k(x_{i_1}, \dots, x_{i_k}) \geq 0 \tag{3.44}$$

for all $(x_1, \dots, x_N) \in I^N$ it holds that

$$\phi_0 + \sum_{k=1}^{N} \int_{I^k} \phi_k(x_1, \dots, x_k)\rho_k(x_1, \dots, x_k)dx_1 \dots dx_k \geq 0. \tag{3.45}$$

The uniform convergence of K_N to K on compact sets implies that K is continuous and hence ρ_k is locally integrable. It is also symmetric. We know that (3.45) holds with $\rho_{k,N}$ instead of ρ_k since $\rho_{k,N}$ are the correlation functions of a point process. Since all ϕ_k have compact support and are bounded we can use the uniform convergence of K_N to K and take $N \to \infty$ to get (3.45). This completes the proof. $\square$

4. Computation of the Number Variance

We will first show how (2.37) can be obtained from (2.29). By (2.34) we want to compute

$$\int_0^L dx \int_L^{\infty} dy f(x-y) + \int_0^L dx \int_{-\infty}^{0} dy f(x-y), \tag{4.1}$$

where

$$f(x) = \frac{\sin^2 \frac{\pi x}{a}}{\pi^2 x^2} + \frac{d^2 - (x/a)^2}{2\pi^2 a^2 (d^2 + (x/a)^2)^2}. \tag{4.2}$$

Since $f(x)$ is even we see that (4.1) equals

$$\begin{aligned} 2\int_{-L}^{0} dx \int_0^{\infty} dy f(x-y) &= 2\int_0^L \int_0^{\infty} dy f(x+y) \\ &= 2\int_0^L \left(\int_x^{\infty} f(y)dy \right) dx \\ &= 2\int_0^L x f(x) dx + 2L \int_L^{\infty} f(x)dx. \end{aligned} \tag{4.3}$$

If we take $f(x) = (\sin^2 \pi x)/\pi^2 x^2$ in (4.3) we get (2.33) using (4.16) and (4.17) below. Inserting (4.2) into (4.3) and computing the integrals we obtain (2.37). A similar computation using (2.29) leads to (2.39). To prove (2.40) we have to show that

$$\lim_{d \to \infty} \frac{2}{\log d} \int_0^{\infty} \frac{u^2}{\sinh^2 u} \frac{\sin^2 2\pi u d}{u} du = 1. \tag{4.4}$$

Using $\sinh^2 u \geq u^2$ we get

$$\begin{aligned}\int_0^\infty \frac{2u^2}{\sinh^2 u}\frac{\sin^2 2\pi ud}{u}du &\leq \int_0^1 \frac{1-\cos 2\pi ud}{u}du + \int_1^\infty \frac{du}{\sinh^2 u}\\ &= \int_0^{2\pi d} \frac{1-\cos x}{x}dx + \int_1^\infty \frac{du}{\sinh^2 u}\\ &= \log(2\pi d) + \gamma - \mathrm{Ci}\,(2\pi d) + \int_1^\infty \frac{du}{\sinh^2 u},\end{aligned} \tag{4.5}$$

and hence

$$\limsup_{d\to\infty} \frac{2}{\log d}\int_0^\infty \frac{u^2}{\sinh^2 u}\frac{\sin^2 2\pi ud}{u}du \leq 1.$$

Given $\epsilon > 0$ we can choose $\delta > 0$ so that $|x^{-1}\sinh x - 1| \leq \epsilon$ if $|x| \leq \delta$. Thus

$$\begin{aligned}\int_0^\infty \frac{2u^2}{\sinh^2 u}\frac{\sin^2 2\pi ud}{u}du &\geq \frac{2}{(1+\epsilon)^2}\int_0^\delta \frac{\sin^2 2\pi ud}{u}du\\ &= \frac{1}{(1+\epsilon)^2}(\log(2\pi d\delta) + \gamma - \mathrm{Ci}\,(2\pi d\delta)),\end{aligned} \tag{4.6}$$

which gives

$$\liminf_{d\to\infty} \frac{2}{\log d}\int_0^\infty \frac{u^2}{\sinh^2 u}\frac{\sin^2 2\pi ud}{u}du \geq 1,$$

and we have proved (4.4). This completes the proof of Theorem 2.8.

Proof of Theorem 2.7. We have

$$\begin{aligned}\mathrm{Var}_{K_S}(\#[R, R+L]) = &\int_0^{R+L} dx \int_{R+L}^\infty dy K(x,y)K(y,x)\\ &+ \int_R^{R+L} dx \int_{-\infty}^R dy K(x,y)K(y,x).\end{aligned}$$

Set $L^*(x,y) = \frac{1}{\pi}L(\frac{x}{\pi}, \frac{y}{\pi})$, and

$$v(A,\theta) = \int_0^A dx \int_0^\infty dy L^*(-x+\theta, y+\theta)L^*(y+\theta, -x+\theta).$$

A computation shows that

$$\mathrm{Var}_{K_S}(\#[R, R+L]) = v\left(\frac{\pi L}{a}, \theta + \phi\right) + v\left(\frac{\pi L}{a}, -\theta\right). \tag{4.7}$$

Hence we have to compute $v(A,\theta)$. If we set

$$G(n,m,k) = \int_0^A dx \int_0^\infty dy \frac{e^{-2i((n+k)x+(m+k)y)}}{(\pi dn + i(y+x))(\pi dm + i(y+x))}, \tag{4.8}$$

then

$$v(A,\theta)=\frac{1}{4\pi^2}\sum_{m,n\in\mathbb{Z}}e^{-dn(n-1)}e^{2i(n-m)\theta}$$
$$\times\left[e^{-dm(m+1)}G(n,m,0)+e^{-dm(m-1)}G(n,m,-1)\right]+\text{c.c.} \quad (4.9)$$

If we neglect the terms that are exponentially small in d we get

$$\begin{aligned}4\pi^2 v(A,\theta)=&-G(0,0,0)+G(0,0,-1)+G(1,1,-1)\\&+e^{2i\theta}[G(1,0,-1)-G(0,-1,0)-G(1,0,0)]+e^{-2i\theta}G(0,1,-1)\\&-e^{4i\theta}G(1,-1,0)+c.c.\end{aligned} \quad (4.10)$$

(Note that $G(0,0,0)$ and $G(0,0,-1)$ are not individually convergent but have to be considered together.)

We will now outline how (4.10) can be computed without giving all the details. Set

$$H_1(A;n,m)=\int_0^A\frac{1-e^{-2inx}}{\pi dm+ix}dx, \quad (4.11)$$

$A>0$, $n,m\in\mathbb{Z}$, $d>0$ and

$$H_2(A;n,m)=\int_A^\infty\frac{e^{-2inx}}{\pi dm+ix}dx, \quad (4.12)$$

$A\geq 0$, $d>0$, $n\neq 0$, $n,m\in\mathbb{Z}$. Some computation now gives

$$\begin{aligned}G(n,m,k)=&\frac{1-e^{-2i(n-m)A}}{2\pi id(n-m)^2}[H_2(A;m+k,m)-H_2(A;m+k,n)]\\&+\frac{1}{2\pi id(n-m)^2}[H_1(A;m+k,n)-H_1(A;m+k,m)\\&+H_1(A;n+k,m)-H_1(A;n+k,n)]\end{aligned} \quad (4.13)$$

if $n\neq m$,

$$\begin{aligned}G(n,n,k)=&1-e^{-2i(n+k)A}-\frac{1}{2}\log(A^2+\pi^2d^2)\\&+i\arctan\frac{\pi dn}{A}+\log(\pi d)-i\frac{\pi}{2}\operatorname{sgn}(n)\\&-2(A-\pi idn)(n+k)H_2(A;n+k,n)\\&-2\pi idn(n+k)H_2(0;n+k,n)+iH_1(A;n+k,n)\end{aligned} \quad (4.14)$$

if $n\neq 0$ and

$$G(0,0,k)-G(0,0,0)=1-e^{-2ikA}-2kAH_2(A;k,0)+H_1(A;k,0). \quad (4.15)$$

The next step is to express the functions H_1 and H_2 in terms of $f(z)$ and $g(z)$ defined by (2.36), and in terms of the sine and cosine integrals:

$$\mathrm{Si}\,(A)=\int_0^A\frac{\sin y}{y}dy \quad ; \quad \frac{\pi}{2}-\mathrm{Si}\,(A)=\int_A^\infty\frac{\sin y}{y}dy, \quad (4.16)$$

$$\int_0^A\frac{1-\cos y}{y}dy=\gamma+\log A-\mathrm{Ci}\,(A) \quad ; \quad \mathrm{Ci}\,(A)=-\int_A^\infty\frac{\cos y}{y}dy. \quad (4.17)$$

After some computation we obtain

$$\begin{aligned}H_1(A;n,m) &= -i\log\sqrt{A^2+\pi^2d^2m^2}+i\log|\pi dm|+\arg(A-\pi idm)\\&\quad+\arg(\pi idm)+iG(-2\pi idm|n|)-i(\cos 2nA)g(2A|n|-2\pi idm|n|)\\&\quad+i(\sin 2|n|A)f(2A|n|-2\pi idm|n|)+f(-2\pi idm|n|)\mathrm{sgn}\,(n)\\&\quad-(\cos 2nA)f(2A|n|-2\pi idm|n|)\mathrm{sgn}\,(n)-(\sin 2nA)g(2A|n|-2\pi idm|n|),\end{aligned} \tag{4.18}$$

if $n\neq 0, m\neq 0$,

$$H_1(A;n,0) = -i(\gamma+\log(2A|n|)-\mathrm{Ci}\,(2|n|A)-\mathrm{sgn}\,(n)\mathrm{Si}\,(2A|n|), \tag{4.19}$$

$$\begin{aligned}H_2(A;n,m) &= -i(\cos 2nA)g(2A|n|-2\pi idm|n|)\\&\quad+i(\sin 2|n|A)f(2A|n|-2\pi idm|n|)\\&\quad-(\cos 2nA)gf(2A|n|-2\pi idm|n|)\mathrm{sgn}\,(n)\\&\quad-(\sin 2nA)g(2A|n|-2\pi idm|n|),\end{aligned} \tag{4.20}$$

if $m\neq 0, n\neq 0$,

$$H_2(A;n,0) = i\mathrm{Ci}\,(2A|n|)-\mathrm{sgn}\,(n)\left(\frac{\pi}{2}-\mathrm{Si}\,(2A|n|)\right) \tag{4.21}$$

if $n\neq 0$, and finally

$$H_2(A;0,0)-H_2(A;0,1) = i\log A - i\log\sqrt{A^2+\pi^2d^2}+\arg(A-id). \tag{4.22}$$

If we use these formulas in (4.10), (4.13), (4.14) and (4.15) we end up with (2.35). The asymptotics for $f(z)$ and $g(z)$ are easy to obtain using integration by parts. □

5. Proof of the Approximation Theorem

5.1. Main part of proof. We will use the formulas (2.15)–(2.17) for $K_S^{\mathrm{ab}}(u,v)$. Given a sequence $\{c_j\}_{j\geq 1}$ of complex numbers $\neq 0$, we define the counting function,

$$n_c(t) = \#\{j\geq 1\,;\ |c_j|\leq t\}.$$

If $n_c(t)\leq Ct^{1+\delta}$ for some $\delta, 0\leq\delta<1$, we can define the convergent canonical product

$$P_c(z)=\prod_{j=1}^{\infty}\left(1-\frac{z}{c_j}\right)e^{z/c_j}. \tag{5.1}$$

It follows from Lemma 5.1 below that ξ_m, as defined by (2.43) is finite. Define

$$c_j=c_{j,m}=y_{j+m}-y_m \quad ; \quad b_j=b_{j,m}=y_m-y_{m-j}. \tag{5.2}$$

We will first show that

$$\frac{(w+y_m)F(w+y_m)}{(z+y_m)F(z+y_m)} = e^{\xi_m(w-z)}\frac{wP_c(w)P_b(-w)}{zP_c(z)P_b(-z)}, \tag{5.3}$$

provided z is not a zero of P_c or $-z$ a zero of P_b. Note that P_b and P_c are well defined by assumption (i) in the theorem. The left hand side of (5.3) is

$$\begin{aligned}
&\lim_{N\to\infty} \frac{w+y_m}{z+y_m}\prod_{j=1}^{N}\frac{y_j^2-(w+y_m)^2}{y_j^2-(z+y_m)^2}\\
&=\lim_{N\to\infty} \frac{w+y_m}{z+y_m}\prod_{j=1}^{N}\frac{(y_j-(y_m+w))(y_j+y_m+w)}{(y_j-(y_m+z))(y_j+y_m+z)}\\
&=\lim_{N\to\infty} \frac{w+y_m}{z+y_m}\prod_{j=-N}^{N}\frac{y_j-y_m-w}{y_j-y_m-z}\\
&=\lim_{N\to\infty} \frac{w}{z}\prod_{j=-N}^{m-1}\frac{y_j-y_m-w}{y_j-y_m-z}\prod_{j=m+1}^{N}\frac{y_j-y_m-w}{y_j-y_m-z}\\
&=\lim_{N\to\infty} \frac{w}{z}\prod_{j=1}^{N+m}\frac{y_m-y_{m-j}+w}{y_m-y_{m-j}+z}\prod_{j=1}^{N-m}\frac{y_{j+m}-y_m-w}{y_{j+m}-y_m-z}\\
&=\lim_{N\to\infty} \frac{w}{z}\prod_{j=1}^{N+m}\frac{1+w/b_j}{1+z/b_j}\prod_{j=1}^{N-m}\frac{1-w/c_j}{1-z/c_j}\\
&=\lim_{N\to\infty} \frac{w}{z}\prod_{j=1}^{N+m}\frac{e^{-w(\frac{1}{c_j}-\frac{1}{b_j})}}{e^{-z(\frac{1}{c_j}-\frac{1}{b_j})}}\prod_{j=N-m+1}^{N+m}e^{\frac{1}{c_j}(z-w)}\frac{P_b(-w)P_c(w)}{P_b(-z)P_c(z)},
\end{aligned} \tag{5.4}$$

which gives the right hand side of (5.3) since, $\sum_{j=N-m+1}^{N+m}\frac{1}{c_j}\to 0$ as $N\to\infty$ for a fixed m.

Write $u_m = u-\zeta_m$, $v_m = v-\zeta_m$ (we write just m instead of $m(\alpha)$). Then

$$|u_m|\le T+\lambda_m \quad , \quad |v_m|\le T+\lambda_m. \tag{5.5}$$

Note that

$$\frac{(w+y_m-v)^2}{2S}-\frac{(z+y_m-u)^2}{2S}=\frac{(w-v_m)^2}{2S}-\frac{(z-u_m)^2}{2S}+\xi_m(w-z+u-v). \tag{5.6}$$

If we make the change of variables $z\to z+y_m+u_m$, $w\to w+y_m+v_m$ in (2.15) we obtain, using (5.3),

$$\begin{aligned}
K_{S,1}^*(u,v)&=\frac{e^{\xi_m(u-v)}}{(2\pi i)^2S}\int_{\Gamma_\sigma}dw\int_{\gamma_M}e^{w^2/2S-z^2/2S}\\
&\times\frac{1}{z-w+u-v}\,\frac{(w+v_m)P_c(w+v_m)P_b(-w-v_m)}{(z+u_m)P_c(z+u_m)P_b(-z-u_m)},
\end{aligned} \tag{5.7}$$

where we have taken $L=\sigma+y_m+v_m$. The number σ will be specified later and satisfies $|\sigma|\le\lambda_m$. We will choose $M=\pi S/\lambda_m$ for reasons that will be clear below. Note that $M\ge 1$ if α is large enough by our assumption on the allowed values of S.

Below we will need the following estimates of the canonical products. Fix an $\epsilon > 0$. There are constants $c_1, c_2 > 0$ such that

$$|P_b(w)|, |P_c(w)| \le c_1 e^{c_2(\lambda_m^{-1}|w|+|w|^{1+\delta})}, \tag{5.8}$$

for all $w \in \mathbb{C}$ and

$$|P_b(z)|, |P_c(z)| \ge c_1^{-1} e^{-c_2(\lambda_m^{-1}|z|^{1+\epsilon}+|z|^{1+\delta+\epsilon})}, \tag{5.9}$$

if $|\mathrm{Im}\, z| \ge 1$. These estimates are proved using the estimate

$$n_b(t) \le n_c(t) \le \lambda^{-1} t + C t^{1+\delta},$$

see (5.52) below, and the following inequalities in [4], pp. 19–22. If $x = \{x_k\}_{k=1}^{\infty}$ satisfies $n_x(t) \le C t^{1+\delta}$, $0 \le \delta < 1$, $t \ge 0$ and $r = |z|$, then

$$\log |P_x(z)| \le 8 \left\{ r \int_0^r \frac{n_x(t)}{t^2} dt + r^2 \int_r^{\infty} \frac{n_x(t)}{t^3} dt \right\} \tag{5.10}$$

for all $z \in \mathbb{C}$, and

$$\log |P_x(z)| \ge -n_x(2r)\log(2r) + \int_0^{2r} \frac{n_x(t)}{t} dt - 8r^2 \int_{2r}^{\infty} \frac{n_x(t)}{t^3} dt - 2r \int_0^{2r} \frac{n_x(t)}{t^2} dt, \tag{5.11}$$

provided $|z - x_j| \ge 1$ for all $j \ge 1$.

Next we will prove an estimate which allows us to restrict the domain of integration in (5.7). Fix $R > 0$. Introduce the following contours:

$$\begin{aligned} \Gamma_{\sigma,R} &: [-R, R] \ni t \to \sigma + it, \\ \Gamma_{\sigma,R}^c &: \mathbb{R} \setminus [-R, R] \ni t \to \sigma + it, \\ \gamma_{M,R} &: [-R, R] \ni t \to \mp t \pm iM, \\ \gamma_{M,R}^c &: \mathbb{R} \setminus [-R, R] \ni t \to \mp t \pm iM. \end{aligned} \tag{5.12}$$

Let (γ, γ') denote either $(\Gamma_\sigma, \gamma_{M,R}^c)$ or $(\Gamma_{\sigma,R}^c, \gamma_M)$. We will show that there is a constant c_0 such that if

$$R \ge c_0 \max \left\{ \left(\frac{S}{\lambda_m}\right)^{1/(1-\epsilon)}, (T^{1+\delta+\epsilon} S)^{1/2}, \left(\frac{T^{1+\epsilon} S}{\lambda_m}\right)^{1/2} \right\} \tag{5.13}$$

then

$$\begin{aligned} &\frac{1}{S} \int_\gamma |dw| \int_{\gamma'} |dz| \left| e^{(w^2 - z^2)/2S} \right| \frac{1}{|z - w + u - v|} \frac{|w + v_m|}{|z + u_m|} \\ &\times \left| \frac{P_c(w + v_m) P_b(-w - v_m)}{P_c(z + u_m) P_b(-z - u_m)} \right| \le \frac{C\lambda_m}{S^{3/2}} e^{-R^2/8S}. \end{aligned} \tag{5.14}$$

Write $g(t) = \lambda_m^{-1} t + t^{1+\delta}$ and $h(t) = \lambda_m^{-1} t^{1+\epsilon} + t^{1+\delta+\epsilon}$, and $I_R = [-R, R]$. It follows from (5.8) and (5.9) that the integral in (5.14) is

$$\begin{aligned} &\le \frac{C}{MS} \int_{\mathbb{R}, I_R} ds \int_{I_R^c, \mathbb{R}} dt e^{(-s^2 + \sigma^2 - t^2 + M^2)/2S} \frac{1}{\sqrt{(\mp t - \sigma + u - v)^2 + (\mp s + M)^2}} \\ &\times e^{cg(\sqrt{(\sigma + v_m)^2 + s^2}) + ch(\sqrt{(\mp t + u_m)^2 + M^2})}, \end{aligned} \tag{5.15}$$

for some constant $c > 0$. Here we have used $|z + u_m| \geq M$ and $t \leq \exp(g(t))$. We can now use $g(t+s) \leq 2(g(t) + g(s))$ and similarly for h to see that the integral in (5.15) is

$$\begin{aligned}
&\leq \frac{C}{MS} \exp\left(\frac{\sigma^2 + M^2}{2S} + C(g(|\sigma|) + g(|v_m|) + h(|u_m|) + h(M))\right) \\
&\quad \times \int_{\mathbb{R}, I_R^c} ds \int_{I_R^c, \mathbb{R}} dt \frac{\exp\left(-\frac{s^2+t^2}{2S} + C(g(|s|) + h(|t|))\right)}{\sqrt{(t - \sigma + u - v)^2 + (s - M)^2}} \\
&\leq \frac{C}{MS} \exp\left(\frac{\sigma^2 + M^2}{2S} - \frac{R^2}{4S} + C(g(|\sigma|) + g(|v_m|) + h(|u_m|) + h(M))\right) \\
&\quad \times \int_{\mathbb{R}} ds \int_{\mathbb{R}} dt \frac{\exp\left(-\frac{s^2+t^2}{4S} + C(g(|s|) + h(|t|))\right)}{\sqrt{(t - \sigma + u - v)^2 + (s - M)^2}} \\
&\leq \frac{C}{MS^{1/2}} \exp\left(\frac{\sigma^2 + M^2}{2S} - \frac{R^2}{4S} + C(g(|\sigma|) + g(|v_m|) + h(|u_m|) + h(M)) + \Delta\right),
\end{aligned} \tag{5.16}$$

where

$$\Delta = \max_{(s,t)\in\mathbb{R}^2} \left(-\frac{s^2 + t^2}{4} + C(g(|s|\sqrt{S}) + h(|t|\sqrt{S}))\right) \leq CS^{\frac{1+\epsilon}{1-\epsilon}} \lambda_m^{-\frac{2}{1-\epsilon}},$$

since $1 \geq S \geq \lambda_m$ (essentially). We see that we need

$$\frac{R^2}{8S} \geq \frac{\sigma^2 + M^2}{2S} + C(g(|\sigma|) + g(|v_m|) + h(|u_m|) + h(M)) + CS^{\frac{1+\epsilon}{1-\epsilon}} \lambda_m^{-\frac{2}{1-\epsilon}},$$

which holds if R satisfies (5.13). Here we have used $|\sigma| \leq \lambda_m$, (5.5) and $M = \pi S/\lambda_m$. This proves (5.14).

We will also need the following estimate. There is a constant C such that

$$\frac{1}{S} \int_{\Gamma_\sigma} |dw| \int_{\gamma_M} |dz| \left| e^{\frac{w^2 - z^2}{2S}} \right| \frac{1}{|z - w + u - v|} \left| \frac{\sin \frac{\pi(w + v_m)}{\lambda_m}}{\sin \frac{\pi(z + u_m)}{\lambda_m}} \right| \leq \frac{C}{S}. \tag{5.17}$$

The left-hand side of (5.17) is

$$\frac{1}{S} \int_{\mathbb{R}} d\tau \int_{\mathbb{R}} dt \frac{e^{(-\tau^2 - t^2 + \sigma^2 + M^2)/2S}}{|\mp t + u - v + i(\sigma + \tau \mp M)|} \left| \frac{\sin \frac{\pi}{\lambda_m}(v_m + \sigma + i\tau)}{\sin \frac{\pi}{\lambda_m}(u_m \pm t \mp M)} \right|. \tag{5.18}$$

Now,

$$\left| \sin \frac{\pi}{\lambda_m}(v_m + \sigma + i\tau) \right|^2 \leq 2 \cosh \frac{2\pi\tau}{\lambda_m} \leq 4e^{\frac{2\pi\tau}{\lambda_m}}, \tag{5.19}$$

and hence

$$\left| \sin \frac{\pi}{\lambda_m}(v_m + \sigma + i\tau) \right| \leq 2e^{\frac{\pi\tau}{\lambda_m}}.$$

Also, by our choice of M,

$$\left|\sin \frac{\pi}{\lambda_m}(u_m \pm t \mp M)\right| \geq \sinh \frac{\pi M}{\lambda_m} = \sinh \frac{\pi^2 S}{\lambda_m^2}. \tag{5.20}$$

Hence the integral in (5.18) is

$$\leq \frac{C}{S} \frac{e^{\frac{\pi^2 S}{\lambda_m^2}}}{\sinh \frac{\pi^2 S}{\lambda_m^2}} \int_{\mathbb{R}^2} \frac{e^{-\frac{1}{2S}(\tau - \frac{\pi S}{\lambda_m})^2 - \frac{t^2}{2S}}}{\sqrt{(t-(u-v))^2 + (\tau \pm \sigma \pm \frac{\pi S}{\lambda_m})^2}} d\tau dt. \tag{5.21}$$

The contribution to the integral from $(t-(u-v))^2 + (\tau \pm \sigma \pm \pi S/\lambda_m)^2 \leq 1$ is $\leq C$ and from the contribution from the complementary region is $\leq CS$. This proves (5.17).

It follows from (5.14) that

$$\begin{aligned} e^{-\xi_m(u-v)} K_{S,1}^*(\underline{y}; u, v) = \frac{1}{(2\pi i)^2 S} \int_{\Gamma_{\sigma,R}} dw \int_{\gamma_{M,R}} dz e^{\frac{w^2-z^2}{2S}} \frac{1}{z-w+u-v} \\ \times \frac{(w+v_m)P_c(w+v_m)P_b(-w-v_m)}{(z+u_m)P_c(z+u_m)P_b(-z-u_m)} + \mathcal{R}_1, \end{aligned} \tag{5.22}$$

where

$$|\mathcal{R}_1| \leq \frac{C\lambda_m}{S^{3/2}} e^{-R^2/8S}, \tag{5.23}$$

provided R satisfies (5.13). Let $a_k = \lambda_m k$, $k \geq 1$. Then

$$z P_a(z) P_a(-z) = \sin \frac{\pi z}{\lambda_m}, \tag{5.24}$$

and we write

$$\begin{aligned} &\frac{(w+v_m)P_c(w+v_m)P_b(-w-v_m)}{(z+u_m)P_c(z+u_m)P_b(-z-u_m)} = \frac{\sin \frac{\pi}{\lambda_m}(w+v_m)}{\sin \frac{\pi}{\lambda_m}(z+u_m)} \\ &+ \left(\frac{P_c(w+v_m)}{P_a(w+v_m)} \frac{P_b(-w-v_m)}{P_a(-w-v_m)} \frac{P_a(z+u_m)}{P_c(z+u_m)} \frac{P_a(-z-u_m)}{P_b(-z-u_m)} - 1 \right) \frac{\sin \frac{\pi}{\lambda_m}(w+v_m)}{\sin \frac{\pi}{\lambda_m}(z+u_m)}. \end{aligned} \tag{5.25}$$

The argument above with $F(t) = \lambda_m^{-1} t$ gives

$$K_{S,1}^*(\underline{\tilde{y}}; u, v) = \int_{\Gamma_{\sigma,R}} dw \frac{1}{(2\pi)^2 S} \int_{\gamma_{M,R}} dz e^{\frac{w^2-z^2}{2S}} \frac{1}{z-w+u-v} \frac{\sin \frac{\pi}{\lambda_m}(w+v_m)}{\sin \frac{\pi}{\lambda_m}(z+u_m)} + \mathcal{R}_2, \tag{5.26}$$

where

$$|\mathcal{R}_2| \leq \frac{C\lambda_m}{S^{3/2}} e^{-R^2/8S}, \tag{5.27}$$

provided R satisfies (5.13) (with an appropriate constant c_0 that does not depend on λ_m). Here $K_{S,1}^*(\underline{\tilde{y}}; u, v)$ is given by (2.15) with $L = \sigma + v_m$.

Set

$$\mathcal{R}_3 = \frac{1}{(2\pi i)^2 S}\int_{\Gamma_{\sigma,R}} dw \int_{\gamma_{M,R}} dz e^{\frac{w^2-z^2}{2S}} \frac{1}{z-w+u-v} \frac{\sin\frac{\pi}{\lambda_m}(w+v_m)}{\sin\frac{\pi}{\lambda_m}(z+u_m)}$$
$$\times\left(\frac{P_c(w+v_m)}{P_a(w+v_m)}\frac{P_b(-w-v_m)}{P_a(-w-v_m)}\frac{P_a(z+u_m)}{P_c(z+u_m)}\frac{P_a(-z-u_m)}{P_b(-z-u_m)}-1\right). \quad (5.28)$$

Then, by (5.22), (5.25) and (5.26),

$$e^{-\xi_m(u-v)}K^*_{S,1}(\underline{y};u,v) = K^*_{S,1}(\underline{\tilde{y}};u,v)+\mathcal{R}_1-\mathcal{R}_2+\mathcal{R}_3. \quad (5.29)$$

We need an estimate of $\mathcal{R}_3$. For this we need estimates of

$$\left|\frac{P_c(w+v_m)}{P_a(w+v_m)}-1\right| \quad \text{and} \quad \left|\frac{P_a(z+u_m)}{P_c(z+u_m)}-1\right| \quad (5.30)$$

and the same expression with b instead of c and a change of sign. We have the identity

$$\frac{P_c(w+v_m)}{P_a(w+v_m)} = \exp\left[\int_0^\infty \frac{(w+v_m)^2}{w+v_m-t}\frac{n_c(t)-n_a(t)}{t^2}dt\right]. \quad (5.31)$$

Hence

$$\left|\frac{P_c(w+v_m)}{P_a(w+v_m)}-1\right| \le \exp\left[\int_0^\infty \left|\frac{(w+v_m)^2}{w+v_m-t}\right|\frac{|n_c(t)-n_a(t)|}{t^2}dt\right]-1. \quad (5.32)$$

Here we can use Lemma 5.2 below with

$$g(t) = \frac{(\sigma+v_m)^2+s^2}{\sqrt{(\sigma+v_m-t)^2+s^2}}, \quad (5.33)$$

where $w=\sigma+is$. Let $k_m=[v_m/\lambda_m-1/2]$ and choose $\sigma=\sigma_m=\lambda_m(k_m+1/2)-v_m$. Then $|\sigma_m|\le\lambda_m$ and we have, by (5.5), $|\sigma_m+v_m|\le T+2\lambda_m$. By assumption $T\le T_0\le 1/4\sqrt{\eta_m}$, (2.44), and it follows that $k_m\le K=[(2\lambda_m\sqrt{\eta_m})^{-1}]$ if m (i.e. α) is large enough. Using the notation of Lemma 5.2 we see that if $|t-\lambda_m k|\le\alpha_k$, then

$$|\sigma_m+v_m-t| \ge \frac{\lambda_m}{8}(|k-k_m|+1), \quad (5.34)$$

$1\le k\le K$. Here we have used that $\alpha_k\le\lambda_m/4$ if $1\le k\le K$. If $t\ge K\lambda_m$, then

$$|\sigma_m+v_m-t| \ge \frac{t}{4} \quad (5.35)$$

if m is large enough. It follows from (5.34) and (5.35) that

$$g(t) \le 8\frac{(T+2\lambda_m)^2+s^2}{\lambda_m(|k-k_m|+1)} \quad (5.36)$$

if $|t-\lambda_m k|<a_k$ for $1\le k\le K$, and

$$g(t) \le 4\frac{(T+2\lambda_m)^2+s^2}{t} \quad (5.37)$$

if $t \geq K\lambda_m$. We can now use Lemma 5.2 to conclude that

$$\int_0^\infty g(t)\frac{|n_c(t)-n_a(t)|}{t^2}dt \leq \left[(T+2\lambda_m)^2+R^2\right]\left(\eta_m \log(\frac{1}{\lambda_m\sqrt{\eta_m}})+(\lambda_m\eta_m)^{1-\delta}\right)$$
$$\leq C\left[(T+2\lambda_m)^2+R^2\right]m^{-\frac{(1-\delta)^2}{1+\delta}}, \tag{5.38}$$

where we have used $|s| \leq R$, $\lambda_m^{-1} \leq Cm^\delta$, $\eta_m \leq Cm^{-\frac{1-\delta}{1+\delta}}$ and $\lambda_m \leq C$. Hence, by (5.32),

$$\left|\frac{P_c(w+v_m)}{P_a(w+v_m)}-1\right| \leq \exp\left(Cm^{-\frac{(1-\delta)^2}{1+\delta}}\left[(T+2\lambda_m)^2+R^2\right]\right)-1. \tag{5.39}$$

A very similar computation using Lemma 5.4 instead gives

$$\left|\frac{P_b(-w-v_m)}{P_a(-w-v_m)}-1\right| \leq \exp\left(Cm^{-\frac{(1-\delta)^2}{1+\delta}}\left[(T+2\lambda_m)^2+R^2\right]\right)-1. \tag{5.40}$$

We also have the estimate

$$\left|\frac{P_a(z+u_m)}{P_c(z+u_m)}-1\right| \leq \exp\left[\int_0^\infty g(t)\frac{|n_c(t)-n_a(t)|}{t^2}dt\right]-1, \tag{5.41}$$

where now

$$g(t)=\left|\frac{(z+u_m)^2}{z+u_m-t}\right| \leq \frac{(-s+u_m)^2+M^2}{\sqrt{(-s+u_m-t)^2+M^2}}, \tag{5.42}$$

if z belongs to the upper part of $\gamma_{M,R}$ (the other case is completely analogous). We have $|s| \leq R$ and $2(R+T+\lambda_m) < 1/2\sqrt{\eta_m}$ if m is sufficiently large by our assumptions on R and the fact that $T \leq T_0$. If $t \geq 1/2\sqrt{\eta_m}$ we get

$$g(t) \leq C\frac{R^2+(T+\lambda_m)^2}{t}. \tag{5.43}$$

If $0 \leq t \leq 1/2\sqrt{\eta_m}$, it could happen that $-s+u_m$ is close to t. Here we use a similar estimate as above,

$$g(t) \leq C\frac{R^2+(T+\lambda_m)^2}{\sqrt{\lambda_m^2|k-k_*|^2+M^2}} \tag{5.44}$$

with an appropriate k_* (depending on $-s+u_m$). Using the estimates (5.43) and (5.44) in (5.41) and Lemma 5.2 we again get

$$\left|\frac{P_a(z+u_m)}{P_c(z+u_m)}-1\right| \leq \exp\left(Cm^{-\frac{(1-\delta)^2}{1+\delta}}[(T+2\lambda_m)^2+R^2]\right)-1, \tag{5.45}$$

and similarly, using Lemma 5.4 instead,

$$\left|\frac{P_a(-z-u_m)}{P_b(-z-u_m)}-1\right| \leq \exp\left(Cm^{-\frac{(1-\delta)^2}{1+\delta}}[(T+2\lambda_m)^2+R^2]\right)-1. \tag{5.46}$$

By our assumptions on R and T we see that the expression in the exponent in (5.46) is bounded by a constant. Hence by (5.17), (5.28), (5.39), (5.40), (5.45) and (5.46) we get

$$|\mathcal{R}_3| \leq Cm^{-\frac{(1-\delta)^2}{1+\delta}}(T^2+R^2). \tag{5.47}$$

From (5.29), (5.23), (5.27) and (5.47) it follows that

$$\left|e^{-\xi_m(u-v)}K^*_{S,1}(\underline{y};u,v) - K^*_{S,1}(\underline{\tilde{y}};u_m,v_m)\right| \leq \frac{C}{S}m^{-\frac{(1-\delta)^2}{1+\delta}}(T^2+R^2) + \frac{C\lambda_m}{S^{3/2}}e^{-R^2/8S}. \tag{5.48}$$

Here $K^*_{S,1}(\underline{y};u,v)$ is given by (2.15) with $L=\sigma+y_m+v_m$ and $K^*_{S,1}(\underline{\tilde{y}};u_m,v_m)$ is given by (2.15) with $L=\sigma+v_m$. Thus,

$$\begin{aligned} e^{-\xi_m(u-v)}K^*_{S,2}(\underline{y};u,v) &= \frac{1}{2\pi i}e^{-\xi_m(u-v)}\int_{\sigma+y_m+v_m-Mi}^{\sigma+y_m+v_m+Mi} e^{\frac{1}{2S}((w-v)^2-(w-u)^2)}dw \\ &= \frac{1}{2\pi i}\int_{\sigma+v_m-Mi}^{\sigma+v_m+Mi} e^{\frac{1}{2S}((w-v)^2-(w-u)^2)}dw = K^*_{S,2}(\underline{\tilde{y}};u_m,v_m). \end{aligned} \tag{5.49}$$

Hence (5.48) also gives

$$\left|e^{-\xi_m(u-v)}K^*_{S}(\underline{y};u,v) - K^*_{S}(\underline{\tilde{y}};u_m,v_m)\right| \leq \frac{C}{S}m^{-\frac{(1-\delta)^2}{1+\delta}}(T^2+R^2) + \frac{C\lambda_m}{S^{3/2}}e^{-R^2/8S}. \tag{5.50}$$

Note that

$$\begin{aligned} K^*_{S,1}(\underline{y};-u,v) &= \frac{1}{(2\pi i)^2 S}\int_{\Gamma_L}dw\int_{\gamma_M}dz e^{\frac{1}{2S}((w-v)^2-(z+u)^2)}\frac{1}{z-w}\frac{wF(z)}{zF(z)} \\ &= \frac{1}{(2\pi i)^2 S}\int_{\Gamma_L}dw\int_{\gamma_M}dz e^{\frac{1}{2S}((w-v)^2-(z-u)^2)}\frac{1}{z+w}\frac{wF(z)}{zF(z)} \end{aligned}$$

since $F(-z)=F(z)$. We can carry out the same type of computation as above to see that (5.50) also holds for $K^{\mathrm{ab}}_S(\underline{y},u,v)$. Now $K^{\mathrm{ab}}_S(\underline{\tilde{y}},u_m,v_m)$ is approximated by $K_S(u_m,v_M)$, given by $K_S(u,v)=a^{-1}L_S(a^{-1}u,a^{-1}v)$ and L_S as in (2.20), with error $\leq \frac{C}{S\alpha}$, which is smaller than the error term we have in the theorem. This completes the proof of the approximation theorem.

5.2. Some lemmas. In the proof above we need some facts about certain numbers defined in the theorem.

Lemma 5.1. *The number ξ_m defined by (2.43) is finite. Also if we set $\zeta_m = y_m - \xi_m S$, there is, for sufficiently large α, an $m=m(\alpha)$ such that*

$$|\zeta_{m(\alpha)} - \alpha| \leq \lambda_{m(\alpha)}. \tag{5.51}$$

Proof. In the proof we will need some rather simple facts which we will prove later. They are immediate consequences of our assumptions on F.

(a) $F'(t+s) \le F'(t) + F'(s)$, for all $t, s \ge 0$.
(b) $F^{-1}(t+s) \le F^{-1}(t) + F^{-1}(s)$, for all $t, s \ge 0$.
(c) $tF'(t) \le 4F(t)$, for all $t \ge 0$.
(d) $tF''(t) \le F'(t)$, for all $t \ge 0$.
(e) $|F(F^{-1}(m)+t) - m - \lambda_m^{-1}| \le \eta_m t^2$ for all $m \ge 1, t \ge 0$.
(f) $|F(F^{-1}(m)+t) - m| \le \lambda_m^{-1} t + Ct^{1+\delta}$ for all $m \ge 1, t \ge 0$.
(g) $\lambda_m - \lambda_{m+j} \le \eta_m \lambda_M^3 j,\ j \ge 1$.
(h) $\lambda_{m+1} \le y_{m+1} - y_m \le \lambda_m$.

Let $c_{j,m}$ and $b_{j,m}$ be defined by (5.2). Then $n_c(t) = [F(F^{-1}(m)+t) - m]$, where $[\cdot]$ denotes the integer part, and hence by (f),

$$n_c(t) \le \lambda^{-1} t + Ct^{1+\delta} \tag{5.52}$$

for $t \ge 0$. We have

$$\xi_m = \sum_{j=1}^{\infty} \left(\frac{1}{c_{j,m}} - \frac{1}{b_{j,m}} \right), \tag{5.53}$$

and since F is convex, $b_{j,m} \ge c_{j,m}$. Thus $0 \le b_{j,m} - c_{j,m} = 2y_m + y_{j,m} - y_{j+m} \le 2y_m$, and we see from (5.53) that $\xi_m \ge 0$ and

$$\xi_m = \sum_{j=1}^{\infty} \frac{2y_m}{(y_{j+m} - y_m)(y_m + y_{j-m})}. \tag{5.54}$$

Since $F(x) \le Cx^{1+\delta}$ we have

$$y_j = F^{-1}(j) \ge Cj^{1/(1+\delta)}, \tag{5.55}$$

and it follows that the series in (5.54) is convergent.

To prove the other statement in the lemma, (5.51), we want to estimate $|\xi_m - \xi_{m+1}|$. From (5.53) we have

$$\xi_m - \xi_{m+1} = \sum_{j=1}^{\infty} \left[\frac{\Delta y_{m+j} - \Delta y_m}{c_{j,m} c_{j,m+1}} + \frac{\Delta y_{m-j} - \Delta y_m}{b_{j,m} b_{j,m+1}} \right], \tag{5.56}$$

where we have used the notation $\Delta y_k = y_{k+1} - y_k$. If we take $t = \Delta y_m$ in (e) and use (h) we get

$$|\lambda_m - \Delta y_m| \le \lambda_m^3 \eta_m, \tag{5.57}$$

and together with (g) this gives

$$|\Delta y_{m+j} - \Delta y_m| \le \lambda_m^3 \eta_m (j+2), \tag{5.58}$$

for $j, m \ge 1$. Since λ_m is decreasing in m, (h) gives

$$c_{j,m} = y_{m+j} - y_m \ge j\lambda_{m+j}. \tag{5.59}$$

Hence, if $1 \le j \le m$,

$$c_{j,m} c_{j,m+1} \ge j^2 \lambda_{2m}^2 \ge \frac{1}{4} j^2 \lambda_m^2. \tag{5.60}$$

Here we have used

$$\frac{\lambda_m}{\lambda_{2m}} = \frac{F'(F^{-1}(2m))}{F'(F^{-1}(m))} \le \frac{F'(F^{-1}(m) + F^{-1}(m))}{F'(F^{-1}(m))} \le 2 \tag{5.61}$$

by (a) and (b). Combining (5.58) and (5.60) we get

$$\left| \sum_{j=1}^{m-1} \frac{\Delta y_{m+j} - \Delta y_m}{c_{j,m} c_{j,m+1}} \right| \le 4\lambda_m \eta_m \sum_{j=1}^{m} \frac{j+2}{j^2} \le 36 \lambda_m \eta_m \log m \tag{5.62}$$

if $m \ge 2$. From (c), (d) and (5.55) we get

$$\eta_m = F''(y_m) \le \frac{4F(y_m)}{y_m^2} \le C m^{-\frac{1-\delta}{1+\delta}}, \tag{5.63}$$

and thus (5.62) gives

$$\left| \sum_{j=1}^{m-1} \frac{\Delta y_{m+j} - \Delta y_m}{c_{j,m} c_{j,m+1}} \right| \le C (\log m) m^{-\frac{1-\delta}{1+\delta}} \lambda_m. \tag{5.64}$$

By (h) and the fact that λ_m is decreasing we get $|\Delta y_{m+j} - \Delta y_m| \le 2\lambda_m$. Hence, by (5.52),

$$\begin{aligned} \left| \sum_{j=m}^{\infty} \frac{\Delta y_{m+j} - \Delta y_m}{c_{j,m} c_{j,m+1}} \right| &\le 2\lambda_m \sum_{j=m}^{\infty} \frac{1}{c_{j,m}^2} = 2\lambda_m \int_{c_{m,m}-}^{\infty} \frac{dn_c(t)}{t^2} \\ &\le 4\lambda_m \int_{c_{m,m}}^{\infty} \frac{\lambda_m^{-1} + Ct^{1+\delta}}{t^3} dt \le C\lambda_m \left(\frac{1}{\lambda_m c_{m,m}} + \frac{1}{c_{m,m}^{1-\delta}} \right). \end{aligned} \tag{5.65}$$

By (h) and (5.61), $c_{m,m} = y_{2m} - y_m \ge \lambda_{2m} \ge \frac{1}{2} m \lambda_m$. It follows that the right hand side of (5.65) is $\le C\lambda_m ((m\lambda_m^2)^{-1} + (m\lambda_m)^{\delta-1})$. Now, by (c),

$$\frac{F(t)}{F'(t)^2} = \frac{1}{F(t)} \left(\frac{F(t)}{F'(t)} \right)^2 \ge \frac{t^2}{16F(t)} \ge Ct^{1-\delta},$$

and consequently by (5.55),

$$m\lambda_m^2 = \frac{F(F^{-1}(m))}{F'(F^{-1}(m))^2} \ge C y_m^{1-\delta} \ge C m^{\frac{1-\delta}{1+\delta}}. \tag{5.66}$$

We find

$$\left| \sum_{j=m}^{\infty} \frac{\Delta y_{m+j} - \Delta y_m}{c_{j,m} c_{j,m+1}} \right| \le C m^{-\frac{1-\delta}{1+\delta}} \lambda_m. \tag{5.67}$$

Combining (5.64) and (5.67) we find

$$\left| \sum_{j=1}^{\infty} \frac{\Delta y_{m+j} - \Delta y_m}{c_{j,m} c_{j,m+1}} \right| \le C m^{-\frac{1-\delta}{1+\delta}} \lambda_m. \tag{5.68}$$

The same argument that led to (5.64) gives

$$\left|\sum_{j=1}^{m/2} \frac{\Delta y_{m-j} - \Delta y_m}{b_{j,m} b_{j,m+1}}\right| \le C m^{-\frac{1-\delta}{1+\delta}} \lambda_m. \tag{5.69}$$

Using $b_{j,m} b_{j,m+1} \ge c_{j,m} c_{j,m+1}$ and $\Delta y_{m-j} = \Delta y_{j-m-1}$ for $j > m$, we see by similar arguments as above that

$$\left|\sum_{j=3m/2}^{\infty} \frac{\Delta y_{m-j} - \Delta y_m}{b_{j,m} b_{j,m+1}}\right| + \sum_{j=m/2}^{\infty} \frac{\Delta y_m}{b_{j,m} b_{j,m+1}} \le C m^{-\frac{1-\delta}{1+\delta}} \lambda_m. \tag{5.70}$$

It remains to consider

$$\sum_{j=m/2}^{3m/2} \frac{|\Delta y_{m-j}|}{b_{j,m} b_{j,m+1}} \le 2 \sum_{j=1}^{m/2} \frac{\Delta y_j}{b_{m-j,m} b_{m-j,m+1}}. \tag{5.71}$$

Now, by (h), $b_{m-j,m} = y_m - y_j \ge (m-j)\lambda_m$ and hence, by (5.66), (c) and straightforward estimates,

$$\sum_{j=1}^{m/2} \frac{\Delta y_j}{b_{m-j,m} b_{m-j,m+1}} \le C \frac{y_m}{m\lambda_m} m^{-\frac{1-\delta}{1+\delta}} \lambda_m \le C m^{-\frac{1-\delta}{1+\delta}} \lambda_m. \tag{5.72}$$

Combining this with (5.69), (5.70) and (5.71) we obtain

$$\left|\sum_{j=1}^{\infty} \frac{\Delta y_{m-j} - \Delta y_m}{b_{j,m} b_{j,m+1}}\right| \le C(\log m) m^{-\frac{1-\delta}{1+\delta}} \lambda_m. \tag{5.73}$$

From (5.56), (5.68) and (5.73) we now get the desired estimate

$$|\xi_m - \xi_{m+1}| \le C(\log m + 1) m^{-\frac{1-\delta}{1+\delta}} \lambda_m \tag{5.74}$$

for $m \ge 1$. Using (5.74) and (h) we get, for $1 \le r \le m-1$,

$$\begin{aligned} |\xi_m| &\le \xi_1 + \sum_{j=1}^{m-1} |\xi_{k+1} - \xi_k| \le \xi_1 + C \sum_{j=1}^{m-1} (\log m + 1) m^{-\frac{1-\delta}{1+\delta}} \Delta y_{k-1} \\ &\le \xi_1 + C y_r + C(\log r + 1) r^{-\frac{1-\delta}{1+\delta}} (y_{m-1} - y_{r-1}). \end{aligned} \tag{5.75}$$

Consequently $\xi_m / y_m \to 0$ as $m \to \infty$ and since $S \le 1$, we see that $\zeta_m = y_m - S\xi_m \to \infty$ as $m \to \infty$. Since $\Delta y_m / \lambda_m \to 1$ as $m \to \infty$ it follows from (5.74) that

$$\left|\frac{\zeta_{m+1} - \zeta_m}{\lambda_m}\right| = \left|\frac{\Delta y_m}{\lambda_m} - \frac{S(\xi_{m+1} - \xi_m)}{\lambda_m}\right| \to 1$$

as $m \to \infty$. Hence $|\zeta_{m+1} - \zeta_m| \le 3\lambda_m/2$ if m is sufficiently large. If we take α sufficiently large the closest ξ_m is thus within distance $3\lambda_m/4$ or $3\lambda_{m+1}/4$, which is $\le \lambda_m$ and we take this m as our $m(\alpha)$. $\square$

Our next lemma is

Lemma 5.2. *Let $a_j = \lambda_m j$, $j \geq 1$ and $c_j = c_{j,m}$, $j \geq 1$ as above. If g is a bounded measurable function on* $[0, \infty)$ *we have*

$$\left|\int_0^\infty g(t)\frac{|n_c(t)-n_a(t)|}{t^2}dt\right| \leq \sum_{k=1}^{K}\int_{k\lambda_m-\alpha_k}^{k\lambda_m+\alpha_k}\frac{|g(t)|}{t^2}dt$$
$$+5\eta_m\int_{1/2\sqrt{\eta_m}}^{1/\lambda_m\eta_m}|g(t)|dt+\frac{2}{\lambda_m}\int_{1/\lambda_m\eta_m}^{\infty}\frac{|g(t)|}{t}dt+C\int_{1/\lambda_m\eta_m}^{\infty}\frac{|g(t)|}{t^{1-\delta}}dt, \quad (5.76)$$

where $\alpha_k = \lambda_m^3\eta_m(k+1)^2$, $K = [(2\lambda_m\sqrt{\eta_m})^{-1}]$.

Proof. We have $n_c(t) = [F(t+F^{-1}(m))-m]$ and $n_a(t) = [\lambda_m^{-1}t]$. The proof of (5.76) is based on the following claim which we will prove below.

Claim 5.3. Assume that $0 \leq t \leq \lambda_m K, 1 \leq k \leq K$. Then $n_c(t) = n_a(t)$ if $|t-\lambda_m k| \geq \alpha_k$ and $|n_c(t)-n_a(t)| \leq 1$ if $|t-\lambda_m k| \leq \alpha_k$.

Using the claim we have that the left hand side of (5.76) is

$$\leq \sum_{k=1}^{K}\int_{k\lambda_m-\alpha_k}^{k\lambda_m+\alpha_k}\frac{|g(t)|}{t^2}dt+\int_{\lambda_m K}^{\infty}|g(t)|\frac{|n_c(t)-n_a(t)|}{t^2}dt. \quad (5.77)$$

It follows from (e) that $|n_c(t)-n_a(t)| \leq \eta_m t^2+1$ and from (f) we get

$$|n_c(t)-n_a(t)| \leq n_c(t)+n_a(t) \leq \frac{2}{\lambda_m}t+Ct^{1+\delta}.$$

Using these estimates in the second integral in (5.77) we obtain (5.76). □

Proof of Claim 5.3. If $0 \leq t \leq \lambda_m K$, then $t \in [\lambda_m k, \lambda_m(k+1)]$ for some $k, 0 \leq k < K$. If furthermore $t \in [\lambda_m k+\alpha_k, \lambda_m(k+1)-\alpha_{k+1}]$, then $[\lambda^{-1}t] = k$, since $\lambda_m^{-1}\alpha_k = \lambda_m^2\eta_m(k+1)^2 \leq \lambda_m^2\eta_m K^2 \leq 1/4$, by our choice of K. Also, $\lambda_m^{-1}t-[\lambda_m^{-1}t] \geq \lambda_m^{-1}\alpha_k > \eta_m t^2$ since $t < \lambda_m(k+1)$, and $\lambda_m^{-1}t-([\lambda_m^{-1}t]+1) \leq -\lambda_m^{-1}\alpha_{k+1} < -\eta_m t^2$ since $t < \lambda_m(k+1)$. Combined with (e) this gives $0 < F(t+F^{-1}(m))-m-[\lambda_m^{-1}t] < 1$, i.e. $n_c(t) = n_a(t)$. On the other hand, if $0 \leq t \leq \lambda_m K$ and $|t-\lambda_m k| < \alpha_k$, then $n_c(t)$ and $n_a(t)$ can differ by at most 1. □

We have a similar lemma for n_b instead.

Lemma 5.4. *Using the same notation as in Lemma 5.2 we have*

$$\left|\int_0^\infty g(t)\frac{|n_b(t)-n_a(t)|}{t^2}dt\right| \leq \sum_{k=1}^{K}\int_{k\lambda_m-\alpha_k}^{k\lambda_m+\alpha_k}\frac{|g(t)|}{t^2}dt$$
$$+\int_{1/4\sqrt{\eta_{m/2}}}^{m\lambda_m/2}|g(t)|(\frac{1}{t^2}+2\eta_{m/2})dt+\frac{2}{\lambda_m}\int_{m\lambda_m/2}^{\infty}\frac{|g(t)|}{t}dt+C\int_{m\lambda_m/2}^{\infty}\frac{|g(t)|}{t^{1-\delta}}dt, \quad (5.78)$$

where the second integral in the right hand side is present only if $m\lambda_m/2 > 1/4\sqrt{\eta_{m/2}}$.

Proof. An integration by parts and the fact that F'' is decreasing gives

$$|m - F(F^{-1}(m) - t) - \lambda_m^{-1} t| \leq 2\eta_{m/2} t^2$$

if $0 \leq t \leq y_m - y_{m/2}$. Note that by (h), $y_m - y_{m/2} \geq m\lambda_m/2$. Now, $n_b(t)$ equals $[m - F(F^{-1}(m) - t)]$ if $0 \leq t \leq y_m$ and $m + n_y(t - y_m)$ if $t > y_m$. It is clear that $n_b(t) \leq \lambda_m^{-1} t$ for $0 \leq t \leq y_m$. If $t > y_m$, then using $n_y(t) \leq F(t)$ we find

$$n_b(t) \leq F(F^{-1}(m)) + F(t - F^{-1}(m)) \leq F(t) \leq Ct^{1+\delta}.$$

The proof of (5.78) now proceeds in the same way as the proof of Lemma 5.2. □

It remains to prove the statements in the beginning of this subsection. (a) and (b) are immediate consequences of our assumptions on F. To prove (c) write

$$F(t) = \int_0^t F'(s)ds \geq \int_{t/2}^t F'(s)ds \geq \frac{t}{2} F'\Big(\frac{t}{2}\Big),$$

since F' is increasing. By (a), $F'(t) \leq 2F'(t/2)$, and (c) follows. The statement (d) follows from the fact that F'' is decreasing. To prove (e) write

$$\begin{aligned} |F(F^{-1}(m) + t) - m - \lambda_m^{-1} t| &= \left| \int_{F^{-1}(m)}^{F^{-1}(m)+t} (F^{-1}(m) + t - s) F''(s) ds \right| \\ &\leq F''(F^{-1}(m)) t^2 = \eta_m t^2, \end{aligned} \tag{5.79}$$

since F'' is decreasing. For (f) write

$$\begin{aligned} F(F^{-1}(m) + t) - m &= \int_{F^{-1}(m)}^{F^{-1}(m)+t} F'(s) ds \leq t F'(F^{-1}(m) + t) \\ &\leq t F'((F^{-1}(m)) + t F'(t) \leq \lambda_m^{-1} t + C t^{1+\delta}, \end{aligned} \tag{5.80}$$

by (a), (c) and the fact that F' is increasing. To prove (g) write

$$\begin{aligned} \lambda_m - \lambda_{m+j} &= \frac{F'(F^{-1}(m+j)) - F'(F^{-1}(m))}{F'(F^{-1}(m)) F'(F^{-1}(m+j))} \leq \frac{1}{F'(y_m)^2} \int_{y_m}^{y_{m+j}} F''(s) ds \\ &\leq \eta_m \lambda_m^2 (y_{m+j} - y_m) = \eta_m \lambda_m^2 \int_m^{m+j} \frac{dt}{F'(F^{-1}(t))} \leq \eta_m \lambda_m^3 j. \end{aligned} \tag{5.81}$$

Finally, to prove (h) we write

$$y_{m+1} - y_m = \int_m^{m+1} \frac{dt}{F'(F^{-1}(t))}.$$

Since $F' \circ F^{-1}$ is increasing the right hand side is $\leq \lambda_m$ and $\geq \lambda_{m+1}$.

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