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Ramanujan's "Lost Notebook" and the Virasoro Algebra

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Abstract: By using the theory of vertex operator algebras, we gave a new proof of the famous Ramanujan's modulus 5 modular equation from his "Lost Notebook" (p. 139 in [R]). Furthermore, we obtained an infinite list of q-identities for all odd moduli; thus, we generalized the result of Ramanujan.

1. Introduction

According to Hardy (cf. [A1], p.177): "..If I had to select one formula from all Ramanujan's work, I would agree with Major MacMahon in selecting...

$$\sum_{n\geq 0} p(5n+4)q^n = \frac{5(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{1.1}$$

where $p(n)$ is the number of partitions of n", and

$$(a;q)_\infty = (1-a)(1-aq)\cdots .$$

Closely related to formula (1.1) is a pair of q–identities recorded by Ramanujan in his "Lost Notebook" (cf. p.139–140, [R]):

$$1 - 5\sum_{n\geq 1}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n} = \frac{(q;q)_\infty^5}{(q^5;q^5)_\infty} \tag{1.2}$$

and

$$\sum_{n\geq 1}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2} = \frac{q(q^5;q^5)_\infty^5}{(q;q)_\infty}, \tag{1.3}$$

where $\left(\frac{n}{5}\right)$ is the Legendre symbol. As a matter of fact, we can talk about a single identity (cf. [C2]) because (1.3) (and subsequently (1.1)) can be obtained easily from (1.2) by

⋆ *Current address:* Department of Mathematics, University at Albany (SUNY), 1400 Washington Road, Albany, NY 12222, USA. E-mail: amilas@math.albany.edu applying a classical result of Hecke (cf. p. 119 in [My]). By now, there are several proofs of (1.2) and (1.3) in the literature. The first proof was given by Bailey ([B1, B2]) by using the ${}_6\Psi_6$–summation. For recent proofs see, for instance, [Ra, C1] and references therein. For an extensive account on Ramanujan's modular identities see [Br] (see also [A2, BrO]).

Compared to these conventional approaches (e.g., hypergeometric q–series, modular forms), our approach to Ramanujan's modular identities is based on completely different ideas. Let us elaborate on recent developments and results that brought us to Ramanujan's "Lost Notebook" [R] and in particular to (1.1).

It is well known that infinite-dimensional Lie theoretical methods can be used to conjecture, interpret and ultimately prove series of combinatorial and q–series identities related to partitions. This direction was initiated by Lepowsky and Wilson in [LW] and it is based on explicit constructions of integrable highest weight representations for affine Lie algebras. In addition, various dilogarithm techniques, crystal bases and path representation techniques that originate in conformal field theory and statistical physics, led to interesting combinatorial and q–series identities. Besides affine Lie algebras there is another important algebraic structure closely related to affine Lie algebras: the Virasoro algebra (cf. [FFu1, FFu2, KR, KW]). Even though the Virasoro algebra and its representation theory are well-understood (including character formulas and their modular properties), the "smallest" representations of the Virasoro algebra (i.e, the minimal models [BPZ]) have no known explicit constructions.

Our motivation for studying Ramanujan's "Lost Notebook" identities stems from the following fact: A large part of Ramanujan's work concerns modular equations closely related to some of his q–identities and continued fractions. Up to now there has not been any work done in the direction of understanding these identities from a conformal field theoretical point of view. This is surprising because the modular invariance of characters holds for a large class of vertex operator algebras (VOA) [Z]. The Virasoro algebra is already included in the definition of VOA, so it appears very natural to seek for modular identities in connection with irreducible Virasoro algebra modules (e.g., minimal models).

Let us briefly outline the content of the paper. In the first part we consider the simplest (yet quite involved) minimal models with exactly two irreducible modules and with $c = \frac{-22}{5}$ (i.e., the Lee-Yang model). This model is related to Rogers-Ramanujan identities. The main idea is to show that the irreducible characters satisfy a second order linear differential equation with coefficients being certain Eisenstein series. In order to achieve this we use the theory of vertex operator algebras (especially Zhu's work [Z]). When we combine the character formulas for the Virasoro minimal models obtained in [FFu1, FFu3] with some standard ODE techniques to obtain (1.2) (cf. Theorem 6.2).

In the second part we provide a generalization of the formula (1.2). As in the $c = \frac{-22}{5}$ case, the key idea is to consider a series of ODEs satisfied by irreducible characters. It is a highly nontrivial problem to compute these ODEs explicitly. Luckily, for our present purposes, it was enough to compute only the first two leading coefficients. As a consequence, we obtain the following family of q-identities (the $k = 2$ case corresponds to Ramanujan's modular equation (1.2)):

Fix $k \in \mathbb{N}$, $k \geq 2$. For every $i = 1, ..., k$, let

$$A_i(q) = \frac{6i^2 - 6i + 1 + 6k^2 - 12ki + 5k}{12(2k+1)} + \sum_{n \geq 0, n \neq \pm i, 0 \bmod (2k+1)} \frac{nq^n}{1-q^n}.$$

Then

$$\begin{vmatrix} 1 & 1 & .. & 1 \\ A_1(q) & A_2(q) & .. & A_k(q) \\ \bar{P}_2(A_1(q)) & \bar{P}_2(A_2(q)) & .. & \bar{P}_2(A_k(q)) \\ . & . & .. & . \\ \bar{P}_{k-1}(A_1(q)) & \bar{P}_{k-1}(A_2(q)) & .. & \bar{P}_{k-1}(A_k(q)) \end{vmatrix}$$

$$= \frac{\prod_{i=1}^{k-1}(2i)!}{(-4k-2)^{\frac{k(k-1)}{2}}} \left(\frac{(q;q)_\infty^{2k+1}}{(q^{2k+1};q^{2k+1})_\infty} \right)^{k-1}, \tag{1.4}$$

where $\bar{P}_j$'s are certain (shifted) Faà di Bruno operators (polynomials) defined in (8.10). We should mention that our identities do not have obvious modular properties (at least not the determinant side), so it is an open question to express (1.4) in a more explicit way (for related work see [M2]).

It appears that our modulus 7 identity ($k = 3$, see Corollary 2) is related to another pair of modular identities recorded by Ramanujan (cf. p.145 in [R]). We shall treat a possible connection in our future publications.

2. The Virasoro Algebra and Minimal Models

The Virasoro algebra Vir (cf. [FFu1, KR], etc.) is defined as the unique non-trivial central extension of the Lie algebra of polynomial vector fields on $\mathbb{C}^*$. It is generated by L_n, $n \in \mathbb{Z}$ and C, with bracket relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C, \tag{2.1}$$

where $\delta_{m+n,0}$ is the Kronecker symbol and C is the central element. Let us fix a triangular decomposition

$$\mathrm{Vir} = \mathrm{Vir}_+ \oplus \mathrm{Vir}_0 \oplus \mathrm{Vir}_-,$$

where Vir_+ is spanned by L_i, $i > 1$, Vir_- is spanned by L_i, $i < 0$ and Vir_0 is spanned by C and L_0. Let M be a Vir–module. We shall denote the action of L_n on M by $L(n)$. Let $U(\mathrm{Vir})$ denote the enveloping algebra of Vir. Thus

$$U(\mathrm{Vir}) = U(\mathrm{Vir}_-) \otimes U(\mathrm{Vir}_0) \otimes U(\mathrm{Vir}_+).$$

The enveloping algebra $U(\mathrm{Vir}_-)$ is equipped with the natural filtration

$$\mathbb{C} = U(\mathrm{Vir}_-)_0 \subset U(\mathrm{Vir}_-)_1 \subset \cdots .$$

It follows from PBW theorem that every element of $U(\mathrm{Vir}_-)_k$ is spanned by elements of the form

$$L(-i_1)\cdots L(-i_r), \quad 0 \leq r \leq k, \; i_j > 0, \; j = 1, ..., r.$$

We shall denote by $\mathrm{Vir}_{\leq -n}$ (resp. $\mathrm{Vir}_{\geq n}$) a Lie subalgebra spanned by L_i, $i \leq -n$ (resp. $i \geq n$).

Let $c, h \in \mathbb{C}$. Let $\mathbb{C}v_{c,h}$ denote a one-dimensional $U(\mathrm{Vir}_{\geq 0})$–module such that

$$L(n)v_{c,h} = 0, \quad n > 0,$$

$$C \cdot v_{c,h} = cv_{c,h}, \quad L(0)v_{c,h} = hv_{c,h}.$$

Consider the Verma module [KR]

$$M(c,h) = U(\mathrm{Vir}) \otimes_{U(\mathrm{Vir}_+)} \mathbb{C}v_{c,h}. \tag{2.2}$$

We shall say that $M(c,h)$ has *central charge* c and *weight* h. In particular $M(c,h)$ has the maximal submodule $M^{(1)}(c,h)$ and the corresponding irreducible quotient

$$L(c,h) = M(c,h)/M^{(1)}(c,h).$$

In the 1980s Feigin and Fuchs provided a detailed embedding structure for Verma modules for all values of c and h [FFu1, FFu2].

There is an infinite, distinguished, family of irreducible highest weight modules $L(c_{p.q}, h^{m,n}_{p,q})$ (*minimal models*) parameterized by the central charge

$$c_{p,q} = 1 - \frac{6(p-q)^2}{pq},$$

where $p, q \in \mathbb{N}$, $p, q \geq 2$, $(p,q) = 1$, and with weights

$$h^{m,n}_{p,q} = \frac{(np-mq)^2 - (p-q)^2}{4pq},$$

where $1 \leq m < p$, $1 \leq n < q$. Notice that for certain pairs (m,n) and (m',n'),

$$h^{m,n}_{p,q} = h^{m',n'}_{p,q}.$$

More precisely, there are exactly

$$\frac{(p-1)(q-1)}{2}$$

different values of $h_{m,n}$ for $1 \leq m < p$, $1 \leq n < q$. Because of

$$M(c,h) \cong U(\mathrm{Vir}_-)$$

and the fact that $M^{(1)}(c,h)$ is graded it is clear that $L(c_{p,q}, h_{m,n})$ is naturally $\mathbb{Q}$-graded with respect to the action of $L(0)$. Moreover, the graded subspaces are finite–dimensional. Hence to every highest weight module M we can associate its graded dimension, q–trace, or simply its *character*

$$\mathrm{tr}|_M q^{L(0)}, \tag{2.3}$$

where (unless otherwise stated) q is just a formal variable [1]. In the case of minimal models we shall write

$$\mathrm{ch}_{c_{p,q}, h^{m,n}_{p,q}}(q) = \mathrm{tr}|_{L(c_{p,q}, h^{m,n}_{p,q})} q^{L(0)}.$$

It is not hard to see that

$$\mathrm{tr}|_{M(c,h)} q^{L(0)} = \frac{q^h}{(q;q)_\infty}. \tag{2.4}$$

However, computing $\mathrm{tr}|_{L(c,h)} q^{L(0)}$ is a much more difficult problem. The only known proof uses a complete BGG-type resolution for irreducible highest weight modules due to Feigin and Fuchs [FFu1]. By using their result it is a straightforward task to obtain explicit formulas for

[1] Formal variable q has nothing to do with the integer q used for parameterization of the central charge.

$$\mathrm{ch}_{c_{p.q},h_{p,q}^{m,n}}(q), \tag{2.5}$$

where p, q, m and n are as above. We should mention that some partials result have been known prior to their result.

For present purposes the expression (2.5) has to be modified (actually, this modification turns out to be essential). Let

$$\bar{\mathrm{ch}}_{c_{p.q},h_{p,q}^{m,n}}(q) = \mathrm{tr}|_{L(c_{p.q},h_{p,q}^{m,n})}q^{L(0)-\frac{c_{p,q}}{24}}. \tag{2.6}$$

From now on we will consider a one-parameter family of central charges

$$c_{2,2k+1} = 1 - \frac{6(2k-1)^2}{(4k+2)}, \quad k \geq 2$$

and the corresponding weights

$$h_{2,2k+1}^{1,i} = \frac{(2(k-i)+1))^2 - (2k-1)^2}{8(2k+1)}, \quad i = 1, ..., k.$$

The $c = c_{2,3}$ case is not interesting because it gives the trivial module.

An important observation (cf. [RC]) is that characters of minimal models with the central charge $c_{2,2k+1}$, $k \geq 1$, can be expressed as infinite products (see [FFr, RC, KW]). We will explore this fact in the later sections.

3. The $c = c_{2,5}$ Case

The simplest non-trivial minimal models occur for $c = c_{2,5} = -\frac{22}{5}$. In this case there are exactly two irreducible modules:

$$L\left(\frac{-22}{5}, 0\right) \text{ and } L\left(\frac{-22}{5}, \frac{-1}{5}\right).$$

The corresponding characters (written as infinite products) are essentially product sides appearing in Rogers-Ramanujan identities [FFr, RC]. More precisely (cf. [KW, FFr, RC]):

$$\mathrm{ch}_{-22/5,0}(q) = \prod_{n\geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} \tag{3.1}$$

and

$$\mathrm{ch}_{-22/5,-1/5}(q) = q^{-1/5} \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}. \tag{3.2}$$

We will show that these character formulas can be used to obtain Ramanujan's modular equation mentioned in the introduction.

4. Vertex Operator Algebras and Modular Invariance

In this section we shall recall some of results from the theory of vertex operator algebras. For the definition of vertex operator algebras, modules for vertex operator algebras and irreducible modules see [FHL] or [FLM]. It is well known (cf. [FZ, W]) that the so–called *vacuum* module

$$V(c, 0) = M(c, 0)/\langle L(-1)v_{c,0}\rangle$$

can be equipped with a vertex operator algebra structure such that

$$\mathbf{1} = v_{c,0}$$

and

$$\omega = L(-2)\mathbf{1}.$$

By quotienting $V(c, 0)$ by the maximal ideal we obtain a simple vertex operator algebra $L(c, 0)$. However, $L(c, 0)$ is not very interesting for all values of c. In the $c = c_{2,2k+1}$ case (and more generally $c = c_{p,q}$), the representation theory of $L(c_{2,2k+1}, 0)$ becomes surprisingly simple (cf. [W], see also [FZ]).

Theorem 4.1. *For every $k \geq 1$, the vertex operator algebra $L(c_{2,2k+1}, 0)$ is rational (in the sense of [DLM2]). Moreover, a complete list of (inequivalent) irreducible $L(c_{2,2k+1}, 0)$-modules is given by*

$$L(c_{2,2k+1}, h^{1,i}_{2,2k+1}), \quad i = 1, ..., k.$$

In particular, the only irreducible $L\left(\frac{-22}{5}, 0\right)$–modules are (up to isomorphism) $L\left(\frac{-22}{5}, 0\right)$ and $L\left(\frac{-22}{5}, \frac{-1}{5}\right)$.

The previous result is a reformulation in the language of vertex operator algebras of a result due to Feigin and Fuchs [FFu3].

5. A Change of "Coordinate" for Vertex Operator Algebras

Let V be an arbitrary vertex operator algebra and suppose that $u \in V$ is a homogeneous element (i.e., an eigenvector for $L(0)$). Let

$$Y[u, y] = e^{y\deg(u)} Y(u, e^y - 1),$$

where

$$L(0)u = (\deg(u))u,$$

y is a formal variable and $(e^y - 1)^{-n-1}$, $n \in \mathbb{Z}$, is expanded inside $\mathbb{C}((y))$, truncated Laurent series in y. Extend $Y[u, y]$ to all $u \in V$ by the linearity. Let

$$Y[u, y] = \sum_{n\in\mathbb{Z}} u[n] y^{-n-1}, \;\; u[n] \in \mathrm{End}(V).$$

The following theorem was proven in [H] (see also [Z] and [L]).

Theorem 5.1. *Let $(V, Y(\cdot, y), \omega, \mathbf{1})$ be a vertex operator algebra and*

$$\tilde{\omega} = L[-2]\mathbf{1} = (L(-2) - \frac{c}{24})\mathbf{1} \in V.$$

The quadruple $(V, Y[\cdot, y], \tilde{\omega}, \mathbf{1})$ has a vertex operator algebra structure isomorphic to the vertex operator algebra $(V, Y(\cdot, y), \omega, \mathbf{1})$. In particular, if we let

$$Y[L[-2]\mathbf{1}, x] = \sum_{n\in\mathbb{Z}} L[n] x^{-n-2}$$

then

$$[L[m], L[n]] = (m - n)L[m + n] + \frac{m^3 - m}{12}\delta_{m+n,0}c.$$

The following lemma is from [Z] (see also Chapter 7 in [H]).

Lemma 1. *For every $n \geq 0$ there are sequences*

$$\{c_n^{(i)}\},\ c_n^{(i)} \in \mathbb{Q},\ i \geq (n+1)$$

and

$$\{d_n^{(i)}\},\ d_n^{(i)} \in \mathbb{Q},\ i \geq (n+1),$$

such that

$$L(n) = L[n] + \sum_{i \geq (n+1)} c_n^{(i)} L[i] \tag{5.1}$$

and

$$L[n] = L(n) + \sum_{i \geq (n+1)} d_n^{(i)} L(i). \tag{5.2}$$

Now, we specialize $V = V(c, 0)$. The following lemma is essentially from [Z].

Lemma 2. *Let $v \in V(c, 0)$ be a singular vector, i.e., a homogeneous vector annihilated by $L(i)$, $i > 0$. Suppose that*

$$v = \sum a_I L(-i_1)L(-i_2) \cdots L(-i_k)\mathbf{1},$$

where $a_I \in \mathbb{C}$, and the summation goes over all indices $i_1 \geq i_2 \geq \cdots \geq i_k$, such that $i_1 + \cdots + i_k = \deg(v)$. Then

$$v = \sum a_I L[-i_1]L[-i_2] \cdots L[-i_k]\mathbf{1}.$$

Informally speaking, singular vectors in the vertex operator algebra $V(c, 0)$ are invariant with respect to the change of coordinate $x \mapsto e^x - 1$.

Proof. The proof is a consequence of Huang's theorem [H] concerning an arbitrary change of coordinate induced by a conformal transformation vanishing at 0 (see [H] for details, cf. also [Z]). Let Ψ_{e^x-1} be the isomorphism of V induced by the change of variable $x \longrightarrow e^x - 1$. Let us compute $\Psi_{e^x-1}(v)$,

$$v = v_{-1}\mathbf{1} \mapsto e^{\sum_{i \geq 1} r_i L_i} v = \Psi_{e^x-1} v = v[-1]\mathbf{1}$$

for some $r_i \in \mathbb{C}$. Every singular vector v satisfies

$$L(i)v = 0, \quad i \geq 1.$$

Hence

$$v \mapsto v, \ (\text{under } \ \Psi_{e^x-1}).$$

Because of the previous corollary (in fact Theorem 5.1) v is also a singular vector with respect to $L[i]$ generators. On the other hand, by the definition of isomorphism for VOA,

$$\Psi_{e^x-1} : V \longrightarrow V, \quad \Psi_{e^x-1}(Y(u, x)v) = Y[\Psi_{e^x-1}(u), x]\Psi_{e^x-1}(v).$$

Therefore

$$\Psi_{e^x-1}(L(-i_1) \cdots L(-i_k)\mathbf{1}) = L[-i_1] \cdots L[-i_k]\mathbf{1},$$

for any choice of $i_1, ..., i_k$. $\square$

Remark 1. It is important to mention that the previous construction has been known by physicists since the early eighties (after the seminal work [BPZ]). Invariance of singular vectors (or *primary fields*) with respect to conformal transformations is one of the most important features in conformal field theory.

Because of $L[-2]\mathbf{1} = (L(-2) - \frac{c}{24})\mathbf{1}$, it is convenient to introduce

$$\bar{L}(0) = L(0) - \frac{c}{24}.$$

This transformation corresponds to cylindrical change of coordinates. Notice also that $L[0] \neq L(0) - \frac{c}{24}$. The following theorem is essentially due to Zhu [Z] (for further generalizations and modifications see [DLM1]).

Theorem 5.2. *Let V be a rational vertex operator algebra which satisfies the C_2–condition* [2]*. Let $M_1, \ldots, M_k$ be a list of all (inequivalent) irreducible V–modules. Then the vector space spanned by*

$$\mathrm{tr}|_{M_1} q^{\bar{L}(0)}, \ldots, \mathrm{tr}|_{M_k} q^{\bar{L}(0)},$$

is modular invariant with respect to $\Gamma(1)$, where γ acts on the modulus τ ($q = e^{2\pi i\tau}$) in the standard way

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1).$$

Now, we let $V = L\left(\frac{-22}{5}, 0\right)$, where

$$\bar{L}(0) = L(0) + \frac{11}{60}.$$

The previous theorem implies the following result (even though we will not use it in the rest of the paper).

Corollary 1. *The vector space spanned by*

$$\bar{\mathrm{ch}}_{-22/5,0}(q) = q^{11/60} \prod_{n \geq 0} \frac{1}{(1 - q^{5n+2})(1 - q^{5n+3})} \tag{5.3}$$

and

$$\bar{\mathrm{ch}}_{-22/5,-1/5}(q) = q^{-1/60} \prod_{n \geq 0} \frac{1}{(1 - q^{5n+1})(1 - q^{5n+4})} \tag{5.4}$$

is modular invariant.

Let $\tilde{G}_{2k}(q)$, $k \geq 1$, denote (normalized) Eisenstein series given by their q–expansions

$$\tilde{G}_{2k}(q) = \frac{-B_{2k}}{(2k)!} + \frac{2}{(2k-1)!} \sum_{n \geq 0} \frac{n^{2k-1} q^n}{1 - q^n},$$

[2] C_2 condition: The vector space spanned by $u_{-2}v$, $u, v \in V$, has a finite codimension [DLM1].

where B_{2k}, $k \geq 1$, are Bernoulli numbers. In particular,

$$\tilde{G}_2(q) = \frac{-1}{12} + 2\sum_{n\geq 0} \frac{nq^n}{1-q^n}$$

and

$$\tilde{G}_4(q) = \frac{1}{720} + \frac{1}{3}\sum_{n\geq 0} \frac{n^3q^n}{1-q^n}.$$

Our normalization is convenient because all the coefficients in the q–expansion of $\tilde{G}_{2k}(q)$ are rational numbers (notice that Zhu [Z] used a different normalization, cf. [M1]).

Let V be a vertex operator algebra and M a V–module. Also, let $u \in V$ be a homogeneous element. Define

$$o(u) = u_{\mathrm{wt}-1} \in \mathrm{End}(M).$$

For instance

$$o(\omega) = L(0).$$

Extend this definition for every $u \in V$ by linearity. Also,

$$o(\tilde{\omega}) = L(0) - \frac{c}{24}.$$

The following result was proven in [Z] (also it is a consequence of a more general result obtained in [M1]; see also [DMN]).

Theorem 5.3. *For every $u, v \in V$,*

$$\mathrm{tr}|_M o(u[0]v)q^{\bar{L}(0)} = 0 \tag{5.5}$$

and

$$\begin{aligned}&\mathrm{tr}|_M o(u[-1]v)q^{\bar{L}(0)}\\ &\quad = \mathrm{tr}|_M o(u)o(v)q^{\bar{L}(0)} + \sum_{k\geq 1} \tilde{G}_{2k}(q)\mathrm{tr}|_M X(u[2k-1]v, x)q^{\bar{L}(0)}.\end{aligned} \tag{5.6}$$

6. Differential Equation

In this section we obtain a second order linear differential equation with a fundamental system of solutions given by $\bar{\mathrm{ch}}_{-22/5,0}(q)$ and $\bar{\mathrm{ch}}_{-22/5,-1/5}(q)$. We should mention that Kaneko and Zagier (cf. [KZ]) considered related second order differential equations from a different point of view.

Theorem 6.1. *Let $\tilde{G}_2(q)$ and $\tilde{G}_4(q)$ be as above and $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$. Then $\bar{ch}_{-22/5,0}(q)$ and $\bar{ch}_{-22/5,-1/5}(q)$ form a fundamental system of solutions of*

$$\left(q\frac{d}{dq}\right)^2 F(q) + 2\tilde{G}_2(q)\left(q\frac{d}{dq}\right)F(q) - \frac{11}{5}\tilde{G}_4(q)F(q) = 0. \tag{6.1}$$

Proof. It is enough to show that both $\bar{ch}_{-22/5,0}(q)$ and $\bar{ch}_{-22/5,-1/5}(q)$ satisfy the equation (6.1). Firstly, from the structure of Verma modules for the Virasoro algebra [FFu1] it follows that

$$v = (L^2(-2) - \frac{3}{5}L(-4))\mathbf{1}$$

is a singular vector inside $V(-22/5, 0)$. This vector generates the maximal submodule of the vertex operator algebra $V(-22/5, 0)$, i.e.,

$$L\Big(\frac{-22}{5},0\Big)=V\Big(\frac{-22}{5},0\Big)/\langle v\rangle,$$

where $\langle S\rangle$ denotes the Vir–submodule generated by the set S. By using Lemma 2, it follows that

$$v=\Big(L^2[-2]-\frac{3}{5}L[-4]\Big)\mathbf{1}.$$

Let M be an arbitrary $L(-22/5,0)$–module. Then

$$Y_M\Big(\Big(L^2[-2]-\frac{3}{5}L[-4]\Big)\mathbf{1},x\Big)=0 \tag{6.2}$$

inside

$$\mathrm{End}(M)[[x,x^{-1}]].$$

Hence

$$\mathrm{tr}|_M o\Big(\Big(L^2[-2]-\frac{3}{5}L[-4]\Big)\mathbf{1}\Big)q^{L(0)}=0. \tag{6.3}$$

In particular, we may set $M=L\left(\frac{-22}{5},0\right)$ or $M=L\left(\frac{-22}{5},\frac{-1}{5}\right)$. Now, we apply the formula (5.5) and get

$$\mathrm{tr}|_M o(\tilde{\omega}[0]v)q^{\bar{L}(0)}=\mathrm{tr}|_M o(L[-1]v)q^{L(0)}=0,$$

for every $v\in V$. We shall pick $v=L[-3]\mathbf{1}$, which implies

$$\mathrm{tr}|_M o(L[-1]L[-3]\mathbf{1})q^{L(0)}=2\mathrm{tr}|_M o(L[-4]\mathbf{1})q^{L(0)}=0.$$

The previous formula and (6.3) imply

$$\mathrm{tr}|_M o(L^2[-2]\mathbf{1},x)q^{L(0)}=0. \tag{6.4}$$

From

$$\mathrm{tr}|_M o(L[-2]L[-2]\mathbf{1})q^{\bar{L}(0)}=\mathrm{tr}|_M o(\tilde{\omega}[-1]L[-2]\mathbf{1})q^{\bar{L}(0)}$$

and

$$o(L[-2]\mathbf{1})=L(0)+\frac{11}{60},$$

we get

$$\begin{aligned}
&\mathrm{tr}|_M o(L[-2]L[-2]\mathbf{1})q^{\bar{L}(0)}\\
&=\mathrm{tr}|_M o(L[-2]\mathbf{1})o(L[-2]\mathbf{1})q^{\bar{L}(0)}+2\tilde{G}_2(q)\mathrm{tr}|_M o(L[-2]\mathbf{1})q^{\bar{L}(0)}\\
&-\frac{11}{5}\tilde{G}_4(q)\mathrm{tr}|_M q^{\bar{L}(0)}\\
&=\mathrm{tr}|_M\Big(L(0)-\frac{c_{2,5}}{24}\Big)^2 q^{\bar{L}(0)}+2\tilde{G}_2(q)\mathrm{tr}|_M\Big(L(0)-\frac{c_{2,5}}{24}\Big)q^{\bar{L}(0)}\\
&-\frac{11}{5}\tilde{G}_4(q)\mathrm{tr}|_M q^{\bar{L}(0)}\\
&=\Big(q\frac{d}{dq}\Big)^2\mathrm{tr}|_M q^{\bar{L}(0)}+2\tilde{G}_2(q)\Big(q\frac{d}{dq}\Big)\mathrm{tr}|_M q^{\bar{L}(0)}-\frac{11}{5}\tilde{G}_4(q)\mathrm{tr}|_M q^{\bar{L}(0)}.
\end{aligned} \tag{6.5}$$

□

Remark 2. The property (6.4) is closely related to a "difference–two condition at the distance one" property (cf. [A1]). More precisely, Feigin and Frenkel [FFr] used (6.2) (and the sewing rules obtained in [BFM]) to give a conformal field theoretical proof of Rogers–Ramanujan identities and their generalizations (due to Gordon and Andrews).

Let

$$\left(\frac{n}{5}\right)$$

be the Legendre symbol. In his Lost Notebook (p. 139, [R]), S. Ramanujan recorded

Theorem 6.2.

$$1 - 5\sum_{n\geq 1}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n} = \frac{(q;q)_\infty^5}{(q^5,q^5)_\infty}. \tag{6.6}$$

We will need a simple lemma concerning infinite products and their logarithmic derivatives.

Lemma 3. *Let*

$$\mathcal{A}(q) = q^r \prod_{n\geq 0}\frac{1}{(1-q^n)^{a_n}},$$

where $a_n \in \mathbb{Z}$ and $r \in \mathbb{C}$. Then

$$\left(q\frac{d}{dq}\right)\mathcal{A}(q) = \left(r + \sum_{n\geq 0} a_n \frac{nq^n}{1-q^n}\right)\mathcal{A}(q)$$

or (equivalently)

$$\left(\frac{d}{d\tau}\right)\mathcal{A}(\tau) = 2\pi i\left(r + \sum_{n\geq 0} a_n \frac{nq^n}{1-q^n}\right)\mathcal{A}(\tau).$$

Proof of the theorem. The proof will follow from the following elementary (but fundamental) result due to Abel [Hi]. Let $\Omega \subset \mathbb{C}$ be a domain and $P_1(z)$ and $P_2(z)$ be two holomorphic functions inside D. Suppose that y_1 and y_2 form a fundamental system of solutions of the differential equation

$$y'' + P_1(z)y' + P_2(z)y = 0.$$

Then we have the formula

$$W(z) = W(z_0)e^{-\int_{z_0}^{z} P_1(t)dt}, \tag{6.7}$$

where

$$W(z) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

is the Wronskian of the system, $z_0 \in \Omega$, and the integration goes along any rectifiable path in Ω. We will apply Abel's formula (6.7) to (6.1), i.e.,

$$\left(q\frac{d}{dq}\right)^2 F(q) + 2\tilde{G}_2(q)\left(q\frac{d}{dq}\right)F(q) - \frac{11}{5}\tilde{G}_4(q)F(q) = 0. \tag{6.8}$$

Firstly, notice that our differential equation (6.8) can be written in terms of τ, rather than q, where we can take Ω to be the upper-half plane and

$$' = \frac{1}{2\pi i}\frac{d}{d\tau} = \left(q\frac{d}{dq}\right).$$

Now, for the fundamental system of solutions of (6.8) we pick (cf. Theorem 6.1)

$$y_1(\tau) = \bar{\mathrm{ch}}_{-22/5,0}(\tau) \text{ and } y_2(\tau) = \bar{\mathrm{ch}}_{-22/5,-1/5}(\tau).$$

It is easy to compute the Wronskian by using the infinite product expressions for $\bar{\mathrm{ch}}_{-22/5,0}(\tau)$ and $\bar{\mathrm{ch}}_{-22/5,-1/5}(\tau)$ (see formulas (5.3) and (5.4)). We have

$$\begin{aligned} &y_1(\tau)y_2'(\tau) - y_1'(\tau)y_2(\tau) \\ &\quad = \left\{\frac{-1}{5} + \frac{11}{60} + \sum_{n\geq 0}\left(\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}} + \frac{(5n+4)q^{5n+4}}{1-q^{5n+4}}\right) - \frac{11}{60}\right. \\ &\quad \left. - \sum_{n\geq 0}\left(\frac{(5n+2)q^{5n+2}}{1-q^{5n+2}} + \frac{(5n+3)q^{5n+3}}{1-q^{5n+3}}\right)\right\} y_1(\tau)y_2(\tau). \end{aligned} \tag{6.9}$$

By combining (6.9) and (6.8) together with the formula (6.7) we get

$$\begin{aligned} &\left\{\frac{-1}{5} + \sum_{n\geq 0}\left(\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}} + \frac{(5n+4)q^{5n+4}}{1-q^{5n+4}}\right.\right. \\ &\left.\left. - \frac{(5n+2)q^{5n+2}}{1-q^{5n+2}} - \frac{(5n+3)q^{5n+3}}{1-q^{5n+3}}\right)\right\}\frac{\eta(5\tau)}{\eta(\tau)} \\ &= W(\tau_0)e^{-2\int_{\tau_0}^{\tau}\tilde{G}_2(\tau)d(2\pi i\tau)}, \end{aligned} \tag{6.10}$$

where we used the fact

$$H_1(\tau)H_2(\tau) = \frac{q^{1/6}(q^5;q^5)_\infty}{(q;q)_\infty} = \frac{\eta(5\tau)}{\eta(\tau)}.$$

By Lemma 3

$$\left(\frac{1}{2\pi i}\frac{d}{d\tau}\right)\eta^4(\tau) = \left(\frac{4}{24} - 4\sum_{n\geq 1}\frac{nq^n}{1-q^n}\right)\eta^4(\tau) = -2\tilde{G}_2(\tau)\eta^4(\tau) \tag{6.11}$$

or

$$\eta^4(\tau) = e^{-2\int_{\tau_0}^{\tau}\tilde{G}_2(t)d(2\pi i\tau)}.$$

The previous formula implies

$$W(\tau_0)e^{-2\int_{\tau_0}^{\tau}\tilde{G}_2(\tau)d(2\pi i\tau)} = W(\tau_0)e^{\int_{\tau_0}^{\tau}\left(\frac{1}{6} - 4\sum_{n\geq 0}\frac{nq^n}{1-q^n}\right)d(2\pi i\tau)} = C\eta^4(\tau), \tag{6.12}$$

where C is some constant which does not depend on τ. Now, (6.10) and (6.12) imply that

$$\sum_{n\geq 0}\Bigg(\frac{-1}{5}+\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}}+\frac{(5n+4)q^{5n+4}}{1-q^{5n+4}} -\frac{(5n+2)q^{5n+2}}{1-q^{5n+2}}-\frac{(5n+3)q^{5n+3}}{1-q^{5n+3}}\Bigg)\frac{\eta(5\tau)}{\eta(\tau)} = C\eta^4(\tau). \quad (6.13)$$

Therefore

$$\frac{-1}{5}+\sum_{n\geq 0}\left(\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}}+\frac{(5n+4)q^{5n+4}}{1-q^{5n+4}}-\frac{(5n+2)q^{5n+2}}{1-q^{5n+2}}-\frac{(5n+3)q^{5n+3}}{1-q^{5n+3}}\right) = \frac{-1}{5}+\sum_{n\geq 0}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n} = C\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}. \quad (6.14)$$

By comparing the first terms on both sides we get

$$C=\frac{-1}{5}.$$

If we multiply (6.14) by -5 we get

$$1-5\sum_{n\geq 0}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n} = \frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}. \quad (6.15)$$

□

7. A Recursion Formula

In this section, as a byproduct of Theorem 6.2, we obtain two recursion formulas for coefficients in the q–expansions of the Rogers-Ramanujan's q–series [3].

Let $b(n)$ denote the number of partitions of n in parts of the form $5i+1$ and $5i+4$, $i\geq 0$, and let $a(n)$ denote the number of partitions of n in parts of the form $5i+2$ and $5i+3$, $i\geq 0$. If we recall Rogers-Ramanujan identities [A1], $b(n)$ is the number of partitions of n satisfying the "difference two condition at the distance one" and $a(n)$ is the number of partitions of n satisfying the "difference two condition at the distance one" with the smallest part >1. Let

$$\sigma_k(n)=\sum_{d|n} d^k.$$

Clearly (cf. [A1])

$$L_1(q)=\sum_{n\geq 0}a(n)q^n=\prod_{n\geq 0}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}=\sum_{n\geq 0}\frac{q^{n^2+n}}{(q)_n}$$

[3] These formulas are inefficient for computation though.

and

$$L_2(q) = \sum_{n\geq 0} b(n)q^n = \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} = \sum_{n\geq 0} \frac{q^{n^2}}{(q)_n}.$$

Theorem 7.1. *For every $n \geq 1$,*

$$a(n) = \frac{11\sum_{k\geq 0}^{n-1} (\sigma_3(n-k) - \sigma_1(n-k))\, a(k) - 60\sum_{k\geq 0}^{n-1} ka(k)\sigma_1(n-k)}{15n^2+3n}, \tag{7.1}$$

$$b(n) = \frac{\sum_{k\geq 0}^{n-1} (11\sigma_3(n-k) + \sigma_1(n-k))\, b(k) - 60\sum_{k\geq 0}^{n-1} kb(k)\sigma_1(n-k)}{15n^2-3n}. \tag{7.2}$$

Proof. From the differential equation (6.1) and

$$\bar{\text{ch}}|_{-22/5,0}(q) = q^{11/60}L_1(q), \quad \bar{\text{ch}}|_{-22/5,-1/5}(q) = q^{-1/60}L_2(q)$$

we obtain a pair of second order differential equations satisfied by $L_1(q)$ and $L_2(q)$. These differential equations are explicitly given by:

$$\left(q\frac{d}{dq}\right)^2 F(q) + \left(\frac{1}{5} + 4\sum_{n\geq 1}\sigma_1(n)q^n\right)\left(q\frac{d}{dq}\right)F(q) + \frac{11}{15}\left(\sum_{n\geq 1}\left(\sigma_1(n)q^n - \sigma_3(n)q^n\right)\right)F(q) = 0, \tag{7.3}$$

with a solution being $L_1(q)$, and

$$\left(q\frac{d}{dq}\right)^2 F(q) + \left(\frac{-1}{5} + 4\sum_{n\geq 1}\sigma_1(n)q^n\right)\left(q\frac{d}{dq}\right)F(q) - \frac{1}{15}\left(\sum_{n\geq 1}\sigma_1(n)q^n + 11\sum_{n\geq 1}\sigma_3(n)q^n\right)F(q) = 0, \tag{7.4}$$

with a solution being $L_2(q)$. From these differential equations and initial conditions

$$L_1(0) = 1, \;\; L_1'(0) = 0 \;\text{ and }\; L_2(0) = 1, \;\; L_2'(0) = 1,$$

we can compute $a(n)$'s and $b(n)$'s by taking Coeff_{q^n} in (7.3) and (7.4), respectively. This gives formulas (7.1) and (7.2). □

8. The General Case

In this part we generalize Ramanujan's modulus 5 identity for all odd moduli.

Our starting point are certain infinite products that appear in Gordon–Andrews' identities [A1] (a generalization of Rogers-Ramanujan identities for odd moduli). These infinite products are given by

$$\prod_{n \neq \pm i, 0 \bmod (2k+1)} \frac{1}{(1-q^n)},$$

where $i = 1, ..., k$. It is known (cf. [FFu1, FFu2, FFr, KW]) that these expressions are closely related to graded dimensions of minimal models with the central charge $c_{2,2k+1}$, $k \geq 2$. More precisely,

$$\bar{\mathrm{ch}}_{c_{2,2k+1}, h^{1,i}_{2,2k+1}}(q) = q^{h^{1,i}_{2,2k+1} - \frac{c_{2,2k+1}}{24}} \prod_{n \neq \pm i, 0 \bmod (2k+1)} \frac{1}{(1-q^n)},$$

where $i = 1, ..., k$. Let us recall (cf. Theorem 4.1) that there are exactly k (non-equivalent) irreducible modules for the vertex operator algebra $L(c_{2,2k+1}, 0)$. Let us *multiply* the modified characters:

$$\prod_{i=1}^{k} \bar{\mathrm{ch}}_{c_{2,2k+1}, h^{1,i}_{2,2k+1}}(q) = q^{\sum_{i=1}^{k}(h^{1,1}_{2,2k+1} - c_{2,2k+1}/24)} \left(\frac{(q^{2k+1}; q^{2k+1})_\infty}{(q;q)_\infty} \right)^{k-1}. \quad (8.1)$$

Miraculously,

Lemma 4.

$$\sum_{i=1}^{k} \left(h^{1,i}_{2,2k+1} - \frac{c_{2,2k+1}}{24} \right) = \frac{k(k-1)}{12} = \frac{2k(k-1)}{24}. \quad (8.2)$$

The previous lemma implies

$$q^{\sum_{i=1}^{k}(h^{1,i}_{2,2k+1} - c_{2,2k+1}/24)} \left(\frac{(q^{2k+1}; q^{2k+1})_\infty}{(q;q)_\infty} \right)^{k-1} = \left(\frac{\eta((2k+1)\tau)}{\eta(\tau)} \right)^{k-1}. \quad (8.3)$$

This expression indicates that the product of all modified characters exhibits nice modular properties for every $c_{2,2k+1}$ value. The next lemma can be obtained by straightforward computation (via the Vandermonde determinant formula). It will be useful for computation of the constant factor for our higher moduli identities.

Lemma 5. *Denote by*

$$\bar{h}^{1,i}_{2,2k+1} = h^{1,i}_{2,2k+1} - \frac{c_{2,2k+1}}{24} = \frac{6i^2 - 6i + 1 + 6k^2 - 12ki + 5k}{12(2k+1)},$$

where $i = 1, ..., k$. *Then*

$$\begin{vmatrix} 1 & 1 & \cdot\cdot & 1 \\ \bar{h}^{1,1}_{2,2k+1} & \bar{h}^{1,2}_{2,2k+1} & \cdot\cdot & \bar{h}^{1,k}_{2,2k+1} \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot\cdot & \cdot \\ (\bar{h}^{1,1}_{2,2k+1})^{k-1} & (\bar{h}^{1,2}_{2,2k+1})^{k-1} & \cdot\cdot & (\bar{h}^{1,k}_{2,2k+1})^{k-1} \end{vmatrix} = \frac{\prod_{i=1}^{k-1}(2i)!}{(-4k-2)^{\frac{k(k-1)}{2}}}.$$

In order to obtain differential equations with a fundamental system of solutions being $\bar{\mathrm{ch}}_{c_{2,2k+1}, h^{1,i}_{2,2k+1}}(q)$, $i = 1, ..., k$, we need precise information regarding singular vectors in the vertex operator algebra $V(c_{2,2k+1}, 0)$.

The following lemma will be crucial for our considerations. Feigin and Frenkel exploited this fact in [FFr] to obtain an upper bound for the characters expressed as sum sides in Gordon-Andrews identities [A1].

Lemma 6. *For every $k \geq 2$, the module $V(c_{2k+1}, 0)$ contains a singular vector of degree $2k$ of the form*

$$v_{sing,2k+1} = \left(L^k[-2] + \cdots\right) \cdot \mathbf{1},$$

where the dots denote the lower order terms with respect to the filtration of $U(\mathrm{Vir}_{\leq -2})$.

Proof. It follows directly from Lemma 2 and the description of singular vectors in [FFu1] (see [FFr] for application in our situation). □

The previous lemma implies that

$$Y_M(v_{sing,2k+1}, x) = Y_M\left(\left(L^k[-2] + \cdots\right) \cdot \mathbf{1}, x\right) = 0, \tag{8.4}$$

for every $L(c_{2,2k+1}, 0)$–module M.

Lemma 7. *The condition (8.4) yields a degree k homogeneous linear differential equation*

$$\left(q\frac{d}{dq}\right)^k F(q) + k(k-1)\,\tilde{G}_2(q)\left(q\frac{d}{dq}\right)^{k-1} F(q) + \cdots + P_k(\tau)F(q) = 0, \tag{8.5}$$

with a fundamental system of solutions being

$$\bar{\mathrm{ch}}_{c_{2,2k+1}, h^{1,i}_{2,2k+1}}(q), \quad i = 1, ..., k.$$

Proof. The idea is similar as in the case of $c = \frac{-22}{5}$. However, this time we are unable to obtain explicit formulas for differential equations; rather we obtained first two leading derivatives, which is enough for our purposes. The existence of a homogeneous differential equation of degree k satisfied by

$$\mathrm{tr}|_M q^{\bar{L}(0)},$$

M being a $L(c_{2,2k+1}, 0)$–module, was already proven in [Z]. Therefore we only need to analyze the $(k-1)^{\mathrm{st}}$ coefficient in

$$\left(q\frac{d}{dq}\right)^k \mathrm{tr}|_M q^{\bar{L}(0)} + A_1(q)\left(q\frac{d}{dq}\right)^{k-1} \mathrm{tr}|_M q^{\bar{L}(0)} + \cdots = 0. \tag{8.6}$$

An important observation is that $A_1(q)$ is independent of the lower order terms in (8.4). This can be easily seen by using the formula (5.6) or [Z] (cf. [DMN, M1]). We will be using Theorem 5.6 repeatedly to compute

$$\mathrm{tr}|_M o(L^k[-2]\mathbf{1}) q^{\bar{L}(0)}.$$

Firstly,

$$\mathrm{tr}|_M o(L^k[-2]\mathbf{1}) q^{\bar{L}(0)} = \left(q\frac{d}{dq}\right)^k \mathrm{tr}|_M q^{\bar{L}(0)} + \cdots, \tag{8.7}$$

where dots involve lower order derivatives of $\mathrm{tr}|_M q^{\bar{L}(0)}$, multiplied (possibly) with certain Eisenstein series. We will prove by the induction on k that

$$A_1(q) = k(k-1)\tilde{G}_2(q), \; k \geq 2.$$

For $k = 2$ this is true (cf. Theorem 6.2). We compute

$$\begin{aligned}\mathrm{tr}|_M o(L^k[-2]\mathbf{1})q^{\bar{L}(0)} &= \mathrm{tr}|_M o(L[-2]\mathbf{1})o(L^{k-1}[-2]\mathbf{1})q^{\bar{L}(0)} \\ &\quad +2(k-1)\tilde{G}_2(q)\mathrm{tr}|_M o(L^{k-1}[-2]\mathbf{1})q^{\bar{L}(0)} + \cdots, \end{aligned} \tag{8.8}$$

where the dots denote terms that do not contribute to $A_1(q)$. Because of (8.7), the second term contributes to $A_1(q)$ with

$$2(k-1)\tilde{G}_2(q).$$

We shall work out the second term in (8.8),

$$\begin{aligned}&\mathrm{tr}|_M o(L[-2]\mathbf{1})o(L^{k-1}[-2]\mathbf{1})q^{\bar{L}(0)} = \mathrm{tr}|_M o(L^{k-1}[-2]\mathbf{1})o(L[-2]\mathbf{1})q^{\bar{L}(0)} \\ &= \left(q\frac{d}{dq}\right)\left\{\mathrm{tr}|_M o(L^{k-1}[-2]\mathbf{1})q^{\bar{L}(0)}\right\}. \end{aligned} \tag{8.9}$$

Now, if we use the induction hypothesis

$$\begin{aligned}&\mathrm{tr}|_M o(L^{k-1}[-2]\mathbf{1})q^{\bar{L}(0)} \\ &= \left(q\frac{d}{dq}\right)^{k-1}\mathrm{tr}|_M q^{\bar{L}(0)} + (k-1)(k-2)\tilde{G}_2(q)\left(q\frac{d}{dq}\right)^{k-2}\mathrm{tr}|_M q^{\bar{L}(0)} + \cdots,\end{aligned}$$

where the dots denote terms with derivatives of $\mathrm{tr}|_M q^{\bar{L}(0)}$ being less then or equal to $(k-3)$. By combining the previous equation and (8.9) we obtain

$$A_1(q) = (2(k-1) + (k-1)(k-2))\tilde{G}_2(q) = k(k-1)\tilde{G}_2(q).$$

This proves the lemma. □

For every $n \in \mathbb{N}$, define a nonlinear differential operator $P_n(\cdot)$ in the following way:

$$(f(g(q)))^{[n]} = P_n(g(q))f^{[n]}(g(q)),$$

where $(h(q))^{[n]} := \left(q\frac{d}{dq}\right)^n h(q)$, for any functions $f(q)$ and $g(q)$. For instance,

$$P_1(g(q)) = \left(q\frac{d}{dq}\right)g(q)$$

and

$$P_2(g(q)) = \left(\left(q\frac{d}{dq}\right)g(q)\right)^2 + \left(q\frac{d}{dq}\right)^2 g(q).$$

By using the Faà di Bruno formula we get

$$P_n(\cdot) = \sum_{i_1,\ldots,i_n}\frac{n!}{i_1!\cdots i_n!}\left(\frac{1}{1!}\left(q\frac{d}{dq}\right)^1(\cdot)\right)^{i_1}\cdots\left(\frac{1}{n!}\left(q\frac{d}{dq}\right)^n(\cdot)\right)^{i_n},$$

where the summation goes over all the n–tuples $i_1, \ldots, i_n \geq 0$ such that

$$n = i_1 + 2i_2 + \cdots + ni_n.$$

We will need certain *shifted* Faà di Bruno operators which we define as

$$\bar{P}_n(\cdot\,) = \sum_{i_1,\dots,i_n} \frac{n!}{i_1!\cdots i_n!} \left(\frac{1}{1!}(\cdot\,)^1\right)^{i_1} \cdots \left(\frac{1}{n!}\left(q\frac{d}{dq}\right)^{n-1}(\cdot\,)\right)^{i_n}, \tag{8.10}$$

where, $n \geq 1$, and again the summation goes over all the n–tuples $i_1, \dots, i_n \geq 0$, such that $n = i_1 + 2i_2 + \cdots + ni_n$. For instance

$$\bar{P}_1(f(q)) = f(q)$$

and

$$\bar{P}_2(f(q)) = (f(q))^2 + \left(q\frac{d}{dq}\right) f(q).$$

Lemma 8. *Fix $k \geq 2$. For every $i = 1, \dots, k$, let*

$$y_i(\tau) = \bar{\mathrm{ch}}_{c_{2,2k+1}, h^{1,i}_{2,2k+1}}(q),$$

and let

$$W(y_1, \dots, y_n) = \begin{vmatrix} y_1(\tau) & y_2(\tau) & .. & y_k(\tau) \\ y_1'(\tau) & y_2'(\tau) & .. & y_k'(\tau) \\ y_1''(\tau) & y_2''(\tau) & .. & y_k''(\tau) \\ . & . & .. & . \\ y_1^{(k-1)}(\tau) & y_2^{(k-1)}(\tau) & .. & y_k^{(k-1)}(\tau) \end{vmatrix} \tag{8.11}$$

be the Wronskian associated to $\{y_1, \dots, y_k\}$. Here

$$' = \frac{1}{2\pi i}\frac{d}{d\tau} = \left(q\frac{d}{dq}\right).$$

Then

$$W(y_1, \dots, y_n) = \left(\prod_{i=1}^k y_i(\tau)\right) \begin{vmatrix} 1 & 1 & .. & 1 \\ A_1(q) & A_2(q) & .. & A_k(q) \\ \bar{P}_2(A_1(q)) & \bar{P}_2(A_2(q)) & .. & \bar{P}_2(A_k(q)) \\ . & . & .. & . \\ \bar{P}_{k-1}(A_1(q)) & \bar{P}_{k-1}(A_2(q)) & .. & \bar{P}_{k-1}(A_k(q)) \end{vmatrix},$$

where

$$A_i(q) = \frac{y_i'(\tau)}{y_i(\tau)},\ i = 1, \dots, k.$$

Proof. We know that every $y_i(\tau)$ admits an infinite product form, which implies, because of Lemma 3, that

$$y_i(\tau) = e^{\int_{\tau_0}^{\tau} A_i(\tau) d(2\pi i \tau)}.$$

The Faà di Bruno formula now gives

$$y_i^{(j)}(\tau) = P_j\left(\int_{\tau_0}^{\tau} A_i(\tau) d\tau\right) e^{\int_{\tau_0}^{\tau} A_j(\tau) d(2\pi i \tau)} = \bar{P}_j(A_i(\tau)) y_i(\tau)$$

for every $j = 1, \dots, k$. Finally, observe that in (8.11) we can factor y_i from the i^{th} column. This explains the multiplicative factor $\prod_{i=1}^k y_i(\tau)$ and proves the lemma. □

Here is our main result:

Theorem 8.1. *Fix $k \geq 2$. For every $i = 1, ...k$, let*

$$A_i(q) = \bar{h}_{1,i} + \sum_{n\geq 0, n\neq \pm i, 0 \bmod (2k+1)} \frac{nq^n}{1-q^n}.$$

Then

$$\begin{vmatrix} 1 & 1 & .. & 1 \\ A_1(q) & A_2(q) & .. & A_k(q) \\ \bar{P}_2(A_1(q)) & \bar{P}_2(A_2(q)) & .. & \bar{P}_2(A_k(q)) \\ . & . & .. & . \\ \bar{P}_{k-1}(A_1(q)) & \bar{P}_{k-1}(A_2(q)) & .. & \bar{P}_{k-1}(A_k(q)) \end{vmatrix}$$
$$= \frac{\prod_{i=1}^{k-1}(2i)!}{(-4k-2)^{\frac{k(k-1)}{2}}} \left(\frac{(q;q)_\infty^{2k+1}}{(q^{2k+1};q^{2k+1})_\infty} \right)^{k-1}. \tag{8.12}$$

Proof. As in the $k = 2$ case we apply Abel's theorem for the k^{th} order linear differential equation (8.5) in Lemma 7. The same lemma implies that for a fundamental system of solutions we can take

$$H_i(\tau) = \bar{ch}_{c_{2,2k+1}, h^{1,i}_{2,2k+1}}(\tau), \quad i = 1, ..., k.$$

With this choice

$$W(H_1(\tau), ..., H_k(\tau)) = Ce^{-\int_{\tau_0}^{\tau} k(k-1)\tilde{G}_2(\tau) d(2\pi i \tau)},$$

where C is a constant which does not depend on τ. The last expression is by Lemma 3 equal to

$$C\eta(\tau)^{2k(k-1)}.$$

After we apply Lemma 8 we get

$$\left(\prod_{i=1}^{k} H_i(\tau) \right) \begin{vmatrix} 1 & 1 & .. & 1 \\ A_1(q) & A_2(q) & .. & A_k(q) \\ \bar{P}_2(A_1(q)) & \bar{P}_2(A_2(q)) & .. & \bar{P}_2(A_k(q)) \\ . & . & .. & . \\ \bar{P}_{k-1}(A_1(q)) & \bar{P}_{k-1}(A_2(q)) & .. & \bar{P}_{k-1}(A_k(q)) \end{vmatrix}$$
$$= C\eta(\tau)^{2k(k-1)}. \tag{8.13}$$

Formulas (8.1)–(8.3) imply

$$\left(\frac{\eta((2k+1)\tau)}{\eta(\tau)} \right)^{k-1} \begin{vmatrix} 1 & 1 & .. & 1 \\ A_1(q) & A_2(q) & .. & A_k(q) \\ \bar{P}_2(A_1(q)) & \bar{P}_2(A_2(q)) & .. & \bar{P}_2(A_k(q)) \\ . & . & .. & . \\ \bar{P}_{k-1}(A_1(q)) & \bar{P}_{k-1}(A_2(q)) & .. & \bar{P}_{k-1}(A_k(q)) \end{vmatrix}$$
$$= C\eta(\tau)^{2k(k-1)}.$$

Hence

$$\begin{vmatrix} 1 & 1 & .. & 1 \\ A_1(q) & A_2(q) & .. & A_k(q) \\ \bar{P}_2(A_1(q)) & \bar{P}_2(A_2(q)) & .. & \bar{P}_2(A_k(q)) \\ . & . & .. & . \\ \bar{P}_{k-1}(A_1(q)) & \bar{P}_{k-1}(A_2(q)) & .. & \bar{P}_{k-1}(A_k(q)) \end{vmatrix} = C\left(\frac{\eta(\tau)^{2k+1}}{\eta((2k+1)\tau)}\right)^{k-1}.$$

To figure the constant C we use Lemma 5. The proof follows. □

9. Example: A Modulus 7 Identity

Here, we derive a q–identity in the $c_{2,7} = -\frac{68}{7}$ case. There are three (inequivalent) minimal models:

$$L\left(\frac{-68}{7}, 0\right), \quad \left(\frac{-68}{7}, \frac{-2}{7}\right) \text{ and } \left(\frac{-68}{7}, \frac{-3}{7}\right).$$

If we apply Theorem 8.1 we get

Corollary 2. *Let*

$$' = q\frac{d}{dq}$$

and

$$A_1(q) = \frac{17}{42} + \sum_{n\geq 0, n=2,3,4,5 \bmod 7} \frac{nq^n}{1-q^n},$$

$$A_2(q) = \frac{5}{42} + \sum_{n\geq 0, n=1,3,4,6 \bmod 7} \frac{nq^n}{1-q^n},$$

$$A_3(q) = \frac{-1}{42} + \sum_{n\geq 0, n=1,2,5,6 \bmod 7} \frac{nq^n}{1-q^n}.$$

Then

$$\begin{vmatrix} 1 & 1 & 1 \\ A_1(q) & A_2(q) & A_3(q) \\ A_1'(q) & A_2'(q) & A_3'(q) \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ A_1(q) & A_2(q) & A_3(q) \\ A_1^2(q) & A_2^2(q) & A_3^2(q) \end{vmatrix} = -\frac{6}{7^3}\left(\frac{(q;q)_\infty^7}{(q^7;q^7)_\infty}\right)^2. \quad (9.1)$$

10. Future Work

(a) After we finished the first draft of the paper, S. Milne pointed out to us that his recent work [Mi] might be related to our determinantal identities. It would be nice to understand this more precisely but perhaps in the framework of vertex operator superalgebras (e.g. for $N = 1$ and $N = 2$ superconformal models). Zhu's work [Z] (cf. [DLM1]) indicates that the C_2–condition implies existence of certain differential equation so, hopefully, one can obtain many interesting modular identities [M2].
(b) (Added in the final version) The methods of this paper can be extended to all $c_{p,q}$–series [M2]. Our main result in [M2] is an extension of certain Dyson-Macdonald's identities.

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