

Polyvector Super-Poincaré Algebras

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Abstract: A class of \mathbb{Z}_2 -graded Lie algebra and Lie superalgebra extensions of the pseudo-orthogonal algebra of a spacetime of arbitrary dimension and signature is investigated. They have the form $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$, with $\mathfrak{g}_0 = \mathfrak{so}(V) + W_0$ and $\mathfrak{g}_1 = W_1$, where the algebra of generalized translations $W = W_0 + W_1$ is the maximal solvable ideal of \mathfrak{g} , W_0 is generated by W_1 and commutes with W . Choosing W_1 to be a spinorial $\mathfrak{so}(V)$ -module (a sum of an arbitrary number of spinors and semispinors), we prove that W_0 consists of polyvectors, i.e. all the irreducible $\mathfrak{so}(V)$ -submodules of W_0 are submodules of $\wedge V$. We provide a classification of such Lie (super)algebras for all dimensions and signatures. The problem reduces to the classification of $\mathfrak{so}(V)$ -invariant $\wedge^k V$ -valued bilinear forms on the spinor module S .

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1. Introduction

A *superextension* of a Lie algebra \mathfrak{h} is a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$, such that $\mathfrak{h} \subset \mathfrak{g}_0$. If the Lie algebra $\mathfrak{g}_0 \supset \mathfrak{h}$ and a \mathfrak{g}_0 -module \mathfrak{g}_1 are given, then a superextension is determined by the Lie superbracket in the odd part, which is a \mathfrak{g}_0 -equivariant linear map

$$\vee^2 \mathfrak{g}_1 \rightarrow \mathfrak{g}_0, \tag{1.1}$$

satisfying the Jacobi identity for $X, Y, Z \in \mathfrak{g}_1$, where \vee denotes the symmetric tensor product. Similarly, a \mathbb{Z}_2 -graded extension (or simply *Lie extension*) of \mathfrak{h} is a \mathbb{Z}_2 -graded Lie algebra \mathfrak{g} , i.e. a Lie algebra with a \mathbb{Z}_2 -grading $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ compatible with the Lie bracket: $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$, such that $\mathfrak{g}_0 \supset \mathfrak{h}$. As above, a \mathbb{Z}_2 -graded extension is determined by the Lie bracket in \mathfrak{g}_1 , which defines a \mathfrak{g}_0 -equivariant linear map,

$$\wedge^2 \mathfrak{g}_1 \rightarrow \mathfrak{g}_0, \tag{1.2}$$

satisfying the Jacobi identity. For instance, consider a super vector space $V_0 + V_1$ endowed with a scalar superproduct $g = g_0 + g_1$, i.e. g_0 is a (possibly indefinite) scalar product on V_0 and g_1 is a nondegenerate skewsymmetric bilinear form on V_1 . The Lie algebra $\mathfrak{h} = \mathfrak{g}_0 = \mathfrak{so}(V_0) \oplus \mathfrak{sp}(V_1)$ of infinitesimal even automorphisms of $(V_0 + V_1, g)$ has a natural extension with $\mathfrak{g}_1 = V_0 \otimes V_1$, where the Lie superbracket is given by:

$$[v_0 \otimes v_1, v'_0 \otimes v'_1] := g_1(v_1, v'_1)v_0 \wedge v'_0 + g_0(v_0, v'_0)v_1 \vee v'_1.$$

This is the orthosymplectic Lie superalgebra $\mathfrak{osp}(V_0|V_1)$. One can also define an analogous Lie superalgebra $\mathfrak{spo}(V_0|V_1)$, starting from a symplectic super vector space $(V_0 + V_1, \omega = \omega_0 + \omega_1)$, such that $\mathfrak{spo}(V_0|V_1) = \mathfrak{osp}(V_1|V_0)$.

Similarly, for a \mathbb{Z}_2 -graded vector space $V_0 + V_1$ endowed with a scalar product $g = g_0 + g_1$ (respectively, a symplectic form $\omega = \omega_0 + \omega_1$) we have a natural \mathbb{Z}_2 -graded extension $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 = \mathfrak{so}(V_0 + V_1)$ (respectively, $\mathfrak{g} = \mathfrak{sp}(V_0 + V_1)$) of the Lie algebra $\mathfrak{h} = \mathfrak{g}_0 = \mathfrak{so}(V_0) \oplus \mathfrak{so}(V_1)$ (respectively, of $\mathfrak{h} = \mathfrak{sp}(V_0) \oplus \mathfrak{sp}(V_1)$).

For a pseudo-Euclidean space-time $V = \mathbb{R}^{p,q}$ (with p positive and q negative directions), Nahm [N] classified superextensions \mathfrak{g} of the pseudo-orthogonal Lie algebra $\mathfrak{so}(V)$ under the assumptions that $q \leq 2$, \mathfrak{g} is simple, \mathfrak{g}_0 is a direct sum of ideals, $\mathfrak{g}_0 = \mathfrak{so}(V) \oplus \mathfrak{k}$, where \mathfrak{k} is reductive and \mathfrak{g}_1 is a spinorial module (i.e. its irreducible summands are spinors or semi-spinors). These algebras for $q = 2$ are usually considered as superconformal algebras for Minkowski spacetimes, by virtue of the identification $\text{conf}(p - 1, 1) = \mathfrak{so}(p, 2)$.

In this paper, we shall consider both super and Lie extensions (which we call ϵ -extensions) of the pseudo-orthogonal Lie algebra $\mathfrak{so}(V)$, with $\epsilon = +1$ corresponding to superextensions and $\epsilon = -1$ to Lie extensions. Here $V = \mathbb{R}^{p,q}$ or $V = \mathbb{C}^n$ is a vector space endowed with a scalar product. In the case $\mathfrak{g}_0 = \mathfrak{so}(V) + V$ (Poincaré Lie algebra), ϵ -extensions $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ such that \mathfrak{g}_1 is a spinorial module and $[\mathfrak{g}_1, \mathfrak{g}_1] \subset V$ were classified in [AC]. In the case $\epsilon = -1$ such extensions clearly do not respect the conventional field theoretical spin–statistics relationship. However, in order to classify super-Poincaré algebras ($\epsilon = +1$) with an arbitrary number of irreducible spinorial submodules in \mathfrak{g}_1 we need to classify Lie extensions as well as superextensions with irreducible \mathfrak{g}_1 .

We study \mathbb{Z}_2 -graded Lie algebras and Lie superalgebras, $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{so}(V) + W_0$, $\mathfrak{g}_1 = W_1$, such that $\mathfrak{so}(V)$ is a maximal semisimple Lie subalgebra of \mathfrak{g} and

$W = W_0 + W_1$ is its maximal solvable ideal. If W_0 contains $[W_1, W_1]$ and commutes with W , we call \mathfrak{g} an ϵ -extension of translational type. If moreover, $W_0 = [W_1, W_1]$, we call \mathfrak{g} an ϵ -transalgebra. Our main result is the classification of ϵ -extended polyvector Poincaré algebras, i.e. ϵ -extensions of translational type in the case when $W_1 = S$, the spinor $\mathfrak{so}(V)$ -module, or, more generally, an arbitrary spinorial module. Here V is an arbitrary pseudo-Euclidean vector space $\mathbb{R}^{p,q}$. We prove that, under these assumptions, any irreducible $\mathfrak{so}(V)$ -submodule of W_0 is of the form $\wedge^k V$ or $\wedge_{\pm}^m V$, where $m = (p + q)/2$ and $\wedge_{\pm}^m V$ are the eigenspaces of the Hodge star operator on $\wedge^m V$.

If $\mathfrak{g} = \mathfrak{so}(V) + W_0 + S$ is an ϵ -transalgebra, then the (super) Lie bracket defines an $\mathfrak{so}(V)$ -equivariant surjective map $\Gamma_{W_0} : S \otimes S \rightarrow W_0$. If K is the kernel of this map, then there exists a complementary submodule \tilde{W}_0 such that $S \otimes S = \tilde{W}_0 + K$ and we can identify \tilde{W}_0 with W_0 . We note that we can choose $\tilde{W}_0 \subset S \wedge S$ in the Lie algebra case and $\tilde{W}_0 \subset S \vee S$ in the Lie superalgebra case. Conversely, if we have a decomposition $S \otimes S = W_0 + K$ into a sum of two $\mathfrak{so}(V)$ -submodules and moreover $W_0 \subset S \wedge S$ or $W_0 \subset S \vee S$, then the projection Γ_{W_0} onto W_0 with the kernel K defines an $\mathfrak{so}(V)$ -equivariant bracket

$$\begin{aligned}
 [,] : S \otimes S &\rightarrow W_0 \\
 [s, t] &= \Gamma_{W_0}(s \otimes t)
 \end{aligned}
 \tag{1.3}$$

which is skewsymmetric or symmetric, respectively. More generally, if A is an endomorphism of W_0 that commutes with $\mathfrak{so}(V)$, then the twisted projection $A \circ \Gamma_{W_0}$ is another $\mathfrak{so}(V)$ -equivariant bracket and any bracket can be obtained in this way. Together with the action of $\mathfrak{so}(V)$ on W_0 and S , this defines the structure of an ϵ -transalgebra $\mathfrak{g} = \mathfrak{so}(V) + W_0 + S$, since the Jacobi identity for $X, Y, Z \in \mathfrak{g}_1$ follows from $[\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]] = 0$. The classification problem then reduces essentially to the decomposition of $S \wedge S, S \vee S$ into irreducible $\mathfrak{so}(V)$ -submodules and the description of the projection Γ_{W_0} . In this paper, we resolve both these matters. In all cases the irreducible $\mathfrak{so}(V)$ -submodules occurring in the tensor product $S \otimes S$ are k -forms $\wedge^k V$, with the exception of the case of even dimensions $n = p+q = 2m$ with signature $s = p-q$ divisible by 4. In the latter case the m -form module splits into irreducible selfdual and anti-selfdual submodules $\wedge_{\pm}^m V$. The multiplicities of any irreducible $\mathfrak{so}(V)$ -submodules of $S \otimes S$ take values 1,2,4 or 8. For example if $V = \mathbb{C}^n$, $n = 2m + 1$ or if $V = \mathbb{R}^{m,m+1}$, we have (cf. [OV])

$$\begin{aligned}
 S \otimes S &= \sum_{k=0}^m \wedge^k V, \\
 S \vee S &= \sum_{k=0}^{[m/4]} \wedge^{m-4k} V + \sum_{k=0}^{[(m-3)/4]} \wedge^{m-3-4k} V, \\
 S \wedge S &= \sum_{k=0}^{[(m-2)/4]} \wedge^{m-2-4k} V + \sum_{k=0}^{[(m-1)/4]} \wedge^{m-1-4k} V.
 \end{aligned}$$

The vector space of $\epsilon = -1$ -extensions of translational type of the form $\mathfrak{g} = \mathfrak{so}(V) + \wedge^k V + S$ is identified with the vector space $\text{Bil}_{-}^k(S)^{\mathfrak{so}(V)} := \text{Hom}_{\mathfrak{so}(V)}(S \wedge S, \wedge^k V)$ of $\wedge^k V$ -valued invariant skewsymmetric bilinear forms on S . Similarly, the vector space of $\epsilon = +1$ -extensions of translational type of the form $\mathfrak{g} = \mathfrak{so}(V) + \wedge^k V + S$ is identified with the vector space $\text{Bil}_{+}^k(S)^{\mathfrak{so}(V)} := \text{Hom}_{\mathfrak{so}(V)}(S \vee S, \wedge^k V)$.

The main problem is the description of these spaces of invariant $\wedge^k V$ -valued bilinear forms. For $k = 0, 1$ this problem was solved in [AC], where three invariants, σ, τ and ι , were defined for bilinear forms on the spinor module.

Following [AC], a nondegenerate $\mathfrak{so}(V)$ -invariant (scalar) bilinear form β on the spinor module S is called *admissible* if it has the following properties:

- 1) β is either symmetric or skewsymmetric, $\beta(s, t) = \sigma(\beta)\beta(t, s)$, $s, t \in S$, $\sigma(\beta) = \pm 1$. We define $\sigma(\beta)$ to be the *symmetry* of β .
- 2) Clifford multiplication by $v \in V$,

$$\gamma(v) : S \rightarrow S, \quad s \mapsto \gamma(v)s = v \cdot s,$$

is either β -symmetric or β -skewsymmetric, i.e.

$$\beta(vs, t) = \tau(\beta)\beta(s, vt), \quad s, t \in S,$$

with $\tau(\beta) = +1$ or -1 , respectively. We define $\tau(\beta)$ to be the *type* of β .

- 3) If the spinor module is reducible, $S = S^+ + S^-$, then the semispinor modules S^+ and S^- are either mutually orthogonal or isotropic. We define the *isotropy* of β to be $\iota(\beta) = +1$ if $\beta(S_+, S_-) = 0$ or $\iota(\beta) = -1$ if $\beta(S_\pm, S_\pm) = 0$.

In [AC], a basis β_i of the space $\text{Bil}(S)^{\mathfrak{so}(V)} := \text{Bil}^0(S)^{\mathfrak{so}(V)}$ of scalar-valued invariant forms was constructed explicitly, which consists of admissible forms. These are tabulated in the appendix (Table A.3). The dimension $N(s) = \dim \text{Bil}(S)^{\mathfrak{so}(V)}$ depends only on the signature $s = p - q$ of V (see Table A.1 in the Appendix). We associate with a bilinear form β on S the $\wedge^k V$ -valued bilinear form $\Gamma_\beta^k : S \otimes S \rightarrow \wedge^k V$, defined by the following fundamental formula:

$$\langle \Gamma_\beta^k(s \otimes t), v_1 \wedge \cdots \wedge v_k \rangle = \sum_{\pi \in \mathfrak{S}_k} \text{sgn}(\pi) \beta(\gamma(v_{\pi(1)}) \cdots \gamma(v_{\pi(k)})s, t) \quad s, t \in S, v_i \in V,$$

which extends the formula given in [AC] from $k = 1$ to arbitrary k . For $k = 0$ we have that $\Gamma_\beta^0 = \beta$.

We shall prove that the map $\beta \mapsto \Gamma_\beta^k$ is $\mathfrak{so}(V)$ -equivariant and induces an isomorphism

$$\Gamma^k : \text{Bil}(S)^{\mathfrak{so}(V)} \xrightarrow{\sim} \text{Bil}^k(S)^{\mathfrak{so}(V)}$$

onto the vector space of $\wedge^k V$ -valued invariant bilinear forms on S . This was proven for $k = 1$ in [AC].

The definitions of the invariants σ, τ, ι make sense for $\wedge^k V$ -valued bilinear forms as well. If $\sigma(\Gamma_\beta^k) = -1$, the form Γ_β^k is skewsymmetric and hence defines a Lie algebra structure on $\mathfrak{g} = \mathfrak{so}(V) + \wedge^k V + S$. If $\sigma(\Gamma_\beta^k) = +1$, it defines a Lie superalgebra structure on $\mathfrak{g} = \mathfrak{so}(V) + \wedge^k V + S$. We shall prove, for admissible β , that

$$\sigma(\Gamma_\beta^k) = \sigma(\beta)\tau(\beta)^k(-1)^{k(k-1)/2}. \tag{1.4}$$

In the cases when semi-spinors exist, we shall prove that

$$\iota(\Gamma_\beta^k) = \iota(\beta)(-1)^k. \tag{1.5}$$

For $k > 0$ the bilinear forms Γ_β^k associated with an admissible bilinear form β have neither value of the type τ . Clearly, the formulae for the invariants show that $\sigma(\Gamma_\beta^k)$ and $\iota(\Gamma_\beta^k)$ depend only on k modulo 4. We tabulate these invariants for $\Gamma_{\beta_i}^k$ for $k = 0, 1, 2, 3$ in the Appendix.

Let the number $N_k^\epsilon(s, n)$ denote the dimension of the vector space $\text{Bil}_\epsilon^k(S)^{\mathfrak{so}(V)}$ of ϵ -extended k -polyvector Poincaré algebra structures on $\mathfrak{g} = \mathfrak{so}(V) + \wedge^k V + S$. We shall see that the sum

$$N_k(s, n) = N_k^+(s, n) + N_k^-(s, n) = N(s) = \dim \text{Bil}(S)^{\mathfrak{so}(V)}$$

depends only on the signature s . We shall also verify the following remarkable shift formula:

$$N_k^\pm(s, n + 2k) = N_0^\pm(s, n), \tag{1.6}$$

which reduces the calculation of these numbers to the case of zero forms. The function $N^\pm(s, n) := N_0^\pm(s, n)$ has the following symmetries:

a) Periodicity modulo 8 in s and n :

$$N^\pm(s + 8a, n + 8b) = N^\pm(s, n), \quad a, b \in \mathbb{Z}.$$

Using this, we can extend the functions $N^\pm(s, n)$ to all integer values of s and n .

b) Symmetry with respect to reflection in signature 3:

$$N^\pm(-s + 6, n) = N^\pm(s, n).$$

c) The mirror symmetries:

$$N^\pm(s, n + 4) = N^\mp(s, n), \tag{1.7}$$

$$N^\pm(s, -n + 4) = N^\mp(s, n). \tag{1.8}$$

Due to the shift formula (1.6), all these identities yield corresponding identities for $N_k^\pm(s, n)$ for any k . For example the mirror identity (1.8) gives the mirror symmetry for $k=1$ (reflection with respect to zero dimension),

$$N_1^\pm(s, -n) = N_1^\mp(s, n),$$

which was discovered in [AC].

In Appendix B, we summarise our results in language more familiar to the physics community.

Recently, there have been many discussions (e.g. [AI, CAIP, DFLV, DN, FV, Sc, Sh, V, VV]) of generalizations of spacetime supersymmetry algebras which go beyond Nahm's classification. Of particular interest has been the M-theory algebra, which extends the $d=11$ super Poincaré algebra by two-form and five-form brane charges. In the important paper [DFLV], the authors study superconformal Lie algebras and polyvector super-Poincaré algebras $\mathfrak{g} = \mathfrak{so}(V) + \wedge^k V + W_1$, where $W_1 = S$ or $W_1 = S_\pm$. They propose an approach for the classification of such Lie superalgebras \mathfrak{g} which consists essentially of the following two steps: first describe the space $\text{Hom}_{\mathfrak{so}(V^{\mathbb{C}})}(\mathbb{S} \vee \mathbb{S}, \wedge^k V^{\mathbb{C}})$, if the complex spinor module \mathbb{S} is irreducible, and the spaces $\text{Hom}_{\mathfrak{so}(V^{\mathbb{C}})}(\mathbb{S}_\pm \vee \mathbb{S}_\pm, \wedge^k V^{\mathbb{C}})$

and $\text{Hom}_{\mathfrak{so}(V^{\mathbb{C}})}(\mathbb{S}_+ \otimes \mathbb{S}_-, \wedge^k V^{\mathbb{C}})$ if the complex spinor module $\mathbb{S} = \mathbb{S}_+ + \mathbb{S}_-$ is reducible, then describe $\mathfrak{so}(V)$ -invariant reality conditions. They determine the dimension of the above vector spaces, which is always zero or one and discuss the second problem. In the present paper we start from the real spinor module S and, in particular, describe explicitly the real vector space $H = \text{Hom}_{\mathfrak{so}(V)}(W_1 \vee W_1, \wedge^k V)$ for an arbitrary spinorial module W_1 . We shall see that even if W_1 is an irreducible spinor module S , the dimension of H can be 0, 1, 2 or 3. Polyvector super-Poincaré algebras were also considered in [CAIP] for Lorentzian signature $(1, q)$ in the dual language of left-invariant one-forms on the supergroup of supertranslations.

2. ϵ -Extensions of $\mathfrak{so}(V)$

Let V be a real or complex vector space endowed with a scalar product and W_1 an $\mathfrak{so}(V)$ -module.

Definition 1. A **superextension ($\epsilon = +1$ -extension) of $\mathfrak{so}(V)$ of type W_1** is a Lie superalgebra \mathfrak{g} satisfying the conditions

- i) $\mathfrak{so}(V) \subset \mathfrak{g}_0$ as a subalgebra,
- ii) $\mathfrak{g}_1 = W_1$, a \mathfrak{g}_0 -module.

A **Lie extension ($\epsilon = -1$ -extension) of $\mathfrak{so}(V)$ of type W_1** is a \mathbb{Z}_2 -graded Lie algebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$, also satisfying i) and ii). Further, an ϵ -extension is called **minimal** if it does not contain a proper subalgebra which is also an ϵ -extension of type W_1 ; more precisely, if $\mathfrak{g}' = \mathfrak{g}'_0 + \mathfrak{g}_1 \subset \mathfrak{g}$, $\mathfrak{so}(V) \subset \mathfrak{g}'_0$, then $\mathfrak{g}' = \mathfrak{g}$.

The Lie superalgebras classified by Nahm are examples of superextensions of $\mathfrak{so}(\mathbb{R}^{p,q})$ of spinor type $W_1 = S$.

Let $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$, be an ϵ -extension of $\mathfrak{so}(V)$, $\mathfrak{g}_0 = \mathfrak{so}(V) + W_0$, with W_0 an $\mathfrak{so}(V)$ -submodule that is complementary to $\mathfrak{so}(V)$ in \mathfrak{g}_0 and $\mathfrak{g}_1 = W_1$. There are two extremal classes of such algebras:

- E1: \mathfrak{g} is semi-simple, i.e. does not contain any proper solvable ideal,
- E2: \mathfrak{g} is of *semi-direct type*, i.e. \mathfrak{g} is maximally non-semi-simple, in the sense that $\mathfrak{so}(V)$ is its largest semi-simple super Lie subalgebra, $\mathfrak{g} = \mathfrak{so}(V) + W_0 + W_1$ and $W = W_0 + W_1$ is a solvable ideal.

3. Extensions of Translational Type and ϵ -Transalgebras

Definition 2. Let $\mathfrak{g} = \mathfrak{so}(V) + W_0 + W_1$ be an ϵ -extension of $\mathfrak{so}(V)$. If $[W_0, W] = 0$ and $[W_1, W_1] \subset W_0$ then the extension $\mathfrak{g} = \mathfrak{so}(V) + W_0 + W_1$ is called an **ϵ -extension of $\mathfrak{so}(V)$ of translational type** and the (nilpotent) ideal $W = W_0 + W_1$ is called the **algebra of generalized translations**. If it is minimal, in the sense of Definition 1, then it is called an **ϵ -transalgebra**.

We note that such an extension is automatically of semi-direct type, provided that $\dim V \geq 3$, which we assume in this section. We also assume for definiteness that V is a real vector space. The minimality condition is equivalent to $[W_1, W_1] = W_0$ and means that even translations are generated by odd translations. The construction of ϵ -extensions of $\mathfrak{so}(V)$ of translational type with given $\mathfrak{so}(V)$ -modules W_0 and W_1 reduces to the construction of $\mathfrak{so}(V)$ -equivariant linear maps $\vee^2 W_1 \rightarrow W_0$ and $\wedge^2 W_1 \rightarrow W_0$. The Jacobi

identity for the Lie bracket associated to such a map follows from the $\mathfrak{so}(V)$ -equivariance. Now we show that the description of ϵ -extensions of $\mathfrak{so}(V)$ of translational type reduces to that of minimal ones (i.e.transalgebras). Let $\mathfrak{g} = \mathfrak{so}(V) + W_0 + W_1$ be an ϵ -extension of $\mathfrak{so}(V)$ of translational type. Then $\mathfrak{g}' := \mathfrak{so}(V) + [W_1, W_1] + W_1$ is an ϵ -transalgebra and $\mathfrak{g} = \mathfrak{g}' + \mathfrak{a}$ is the semi-direct sum of the subalgebra \mathfrak{g}' and an (even) Abelian ideal \mathfrak{a} , where $\mathfrak{a} \subset W_0$ is an $\mathfrak{so}(V)$ -submodule complementary to $[W_1, W_1] \subset W_0$. Conversely, if $\mathfrak{g}' = \mathfrak{so}(V) + W'_0 + W_1$ is an ϵ -transalgebra and \mathfrak{a} is an $\mathfrak{so}(V)$ -module then the semi-direct sum $\mathfrak{g} := \mathfrak{g}' + \mathfrak{a}$ is an ϵ -extension of $\mathfrak{so}(V)$ of translational type, where $W_0 := W'_0 + \mathfrak{a}$.

Proposition 1. *Let W_1 be an $\mathfrak{so}(V)$ -module. Then there exists a unique (up to isomorphism) ϵ -transalgebra of maximal dimension with $\mathfrak{g}_1 = W_1$:*

$$\mathfrak{g}^\epsilon = \mathfrak{g}^\epsilon(W_1) = \mathfrak{g}_0^\epsilon + \mathfrak{g}_1^\epsilon = (\mathfrak{so}(V) + W_0^\epsilon) + W_1,$$

where $W_0^+ = \sphericalangle^2 W_1$ and $W_0^- = \wedge^2 W_1$. The Lie (super)bracket $[\cdot, \cdot] : W_1 \otimes W_1 \rightarrow W_0^\epsilon$ is the projection onto the corresponding summand of $W_1 \otimes W_1 = \sphericalangle^2 W_1 \oplus \wedge^2 W_1$. Moreover, any ϵ -transalgebra with $\mathfrak{g}_1 = W_1$ is isomorphic to a contraction of $\mathfrak{g}^\epsilon(W_1)$.

Proof. It is clear that \mathfrak{g}^ϵ is a maximal ϵ -transalgebra. Let $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ be a maximal ϵ -transalgebra with $\mathfrak{g}_1 = W_1$ and $\mathfrak{g}_0 = \mathfrak{so}(V) + W_0$. The Lie (super)bracket $[\cdot, \cdot] : W_1 \otimes W_1 \rightarrow W_0$ defines an $\mathfrak{so}(V)$ -equivariant isomorphism from $\sphericalangle^2 W_1$ or $\wedge^2 W_1$ onto W_0 . This isomorphism extends to an isomorphism $\mathfrak{g}^\epsilon \rightarrow \mathfrak{g}$, which is the identity on $\mathfrak{so}(V) + W_1$. Similarly for any ϵ -transalgebra with $\mathfrak{g}_1 = W_1$ the (super) Lie bracket $[\cdot, \cdot] : W_1 \otimes W_1 \rightarrow W_0$ defines an $\mathfrak{so}(V)$ -equivariant epimorphism φ from $\sphericalangle^2 W_1$ or $\wedge^2 W_1$ onto W_0 . The kernel K is an $\mathfrak{so}(V)$ -submodule of $\sphericalangle^2 W_1$ or $\wedge^2 W_1$, respectively. Since $\mathfrak{so}(V)$ is semi-simple, there exists a complementary submodule \widetilde{W}_0 isomorphic to W_0 . We can identify the $\mathfrak{so}(V)$ -module \widetilde{W}_0 with W_0 by means of the isomorphism $\varphi|_{\widetilde{W}_0}$. Then the Lie bracket corresponds to the projection $\pi_{\widetilde{W}_0}^\pm : \sphericalangle^2 W_1 = K + \widetilde{W}_0 \rightarrow \widetilde{W}_0$ or $\pi_{\widetilde{W}_0}^- : \wedge^2 W_1 = K + \widetilde{W}_0 \rightarrow \widetilde{W}_0$. This defines an ϵ -transalgebra $\mathfrak{g}^\epsilon(W_1, \widetilde{W}_0) = \mathfrak{so}(V) + \widetilde{W}_0 + W_1$, whose bracket is the above projection $\pi_{\widetilde{W}_0}^\epsilon$. The isomorphism $\varphi|_{\widetilde{W}_0} : \widetilde{W}_0 \rightarrow W_0$ of $\mathfrak{so}(V)$ -modules extends trivially to an isomorphism $\mathfrak{g}^\epsilon(W_1, \widetilde{W}_0) \rightarrow \mathfrak{g}$. This shows that any ϵ -transalgebra is isomorphic to an ϵ -transalgebra of the form $\mathfrak{g}^\epsilon(W_1, \widetilde{W}_0)$, where $\widetilde{W}_0 \subset W_1 \otimes W_1$ is an $\mathfrak{so}(V)$ -submodule contained in $\sphericalangle^2 W_1$ or $\wedge^2 W_1$, respectively. Consider now the one-parameter family of Lie brackets $[\cdot, \cdot]_t := t(\text{Id} - \pi_{\widetilde{W}_0}^\epsilon) + \pi_{\widetilde{W}_0}^\epsilon$. This defines a family of ϵ -transalgebras $(\mathfrak{g}^\epsilon(W_1), [\cdot, \cdot]_t)$. For $t \neq 0$ they are isomorphic to the original ($t = 1$) ϵ -transalgebra. In the limit $t \rightarrow 0$ we obtain the ϵ -transalgebra $\mathfrak{g}^\epsilon(W_1, \widetilde{W}_0)$ as a contraction of $\mathfrak{g}^\epsilon(W_1)$. \square

The following proposition describes the structure of extensions of semi-direct type under the additional assumption that the $\mathfrak{so}(V)$ -module W_1 is irreducible. We denote by $\rho : \mathfrak{g}_0 \rightarrow \mathfrak{gl}(W_1)$ the adjoint representation of $\mathfrak{g}_0 = \mathfrak{so}(V) + W_0$ on W_1 and by K the kernel of $\rho|_{W_0}$, which is clearly an ideal of \mathfrak{g} . Thus, ρ is the action of ad_{W_0} on W_1 and K are all the generators of W_0 that commute with W_1 .

Proposition 2. *Let $\mathfrak{g} = \mathfrak{so}(V) + W_0 + W_1$ be an ϵ -extension of semi-direct type. Assume that the $\mathfrak{so}(V)$ -module W_1 is irreducible of dimension at least 3 if it does not admit an $\mathfrak{so}(V)$ -invariant complex structure and $\dim_{\mathbb{C}} W_1 \geq 3$ if it does and that $\dim V \geq 3$. Consider the decomposition of \mathfrak{g} into a direct sum of $\mathfrak{so}(V)$ -submodules, $\mathfrak{g} = \mathfrak{so}(V) + A + K + W_1$, where A is an $\mathfrak{so}(V)$ -invariant complement to K in $W_0 = A + K$. Then the*

dimension $\dim A = 0, 1, 2$ and the irreducible linear Lie algebra $\rho(\mathfrak{g}_0) = \rho(\mathfrak{so}(V)) + Z$, where the centre $Z \cong W_0/K$ is either $0, \mathbb{R}\cdot\text{Id}$ or $\mathbb{C}\cdot\text{Id}$. Moreover,

$$[A, A] \subset K \quad , \quad [\mathfrak{so}(V), A] = 0 \quad , \quad [W_1, W_1] \subset K.$$

Proof. By assumption, the linear Lie algebra $\rho(\mathfrak{g}_0) \subset \mathfrak{gl}(W_1)$ is irreducible and hence reductive. Since any solvable ideal of a reductive Lie algebra belongs to the centre, we conclude that the solvable ideal $\rho(W_0) \subset \rho(\mathfrak{g}_0)$ is in fact Abelian and consists of operators commuting with $\mathfrak{so}(V)$. Now Schur’s Lemma implies that $\rho(W_0) = 0, \mathbb{R}\cdot\text{Id}$, or $\mathbb{C}\cdot\text{Id}$. The inclusion $[A, A] \subset K$ follows from the fact that $\rho(A)$ is in the centre of \mathfrak{g}_0 . Since the restriction of ρ to $\mathfrak{so}(V) + A$ is faithful and $[\rho(\mathfrak{so}(V)), \rho(A)] = 0$, we conclude that $[\mathfrak{so}(V), A] = 0$. From the assumptions it follows that there exist three vectors $x, y, z \in W_1$, which are linearly independent over the reals if W_1 has no invariant complex structure and over the complex numbers if W_1 has an invariant complex structure J . For any three linearly independent vectors (over \mathbb{R} or \mathbb{C}) $x, y, z \in W_1$, the Jacobi identity gives

$$\begin{aligned} 0 &= [[x, y], z] + [[y, z], x] + [[z, x], y] \\ &= \rho([x, y])z + \rho([y, z])x + \rho([z, x])y. \end{aligned}$$

Since $[W_1, W_1] \subset W_0$ and $\rho(W_0) = \rho(A)$ consists of scalar operators (over \mathbb{R} or \mathbb{C}), we have that $\rho([x, y]) = 0$, i.e. $[W_1, W_1] \subset K = \ker \rho$. \square

Note that \mathfrak{g} is a transalgebra if and only if $A=0$.

The following corollary gives sufficient conditions for extensions of semi-direct type to be transalgebras.

Corollary 1. *Under the assumptions of the previous proposition, assume moreover that $\mathfrak{g} = \mathfrak{so}(V) + W_0 + W_1$ is a minimal extension of type W_1 . Then \mathfrak{g} is a transalgebra.*

Proof. Minimality implies $W_0 = [W_1, W_1]$ and, by the above proposition, $[W_1, W_1]$ commutes with W_1 . Now the Jacobi identity for $x, y \in W_1$ and $z \in W_0$ yields $[W_0, W_0] = 0$. \square

Instead of minimality we may assume the irreducibility of the $\mathfrak{so}(V)$ -module W_0 .

Proposition 3. *Let $\mathfrak{g} = \mathfrak{so}(V) + W_0 + W_1$ be an ϵ -extension of semi-direct type, with $\dim V \geq 3$. Assume that W_0 and W_1 are irreducible $\mathfrak{so}(V)$ -modules. Then either \mathfrak{g} is of translational type, i.e. $[W_0, W] = 0$, or $W_0 \cong \mathbb{R}$ (considered as a real Lie algebra) is the centre of $\mathfrak{g}_0 = \mathfrak{so}(V) + W_0$ and $\text{ad } w_0$ acts on W_1 by scalars.*

Proof. Let W_0, W_1 be irreducible $\mathfrak{so}(V)$ modules. Since the algebra $W = W_0 + W_1$ is solvable, $[W_0, W_0]$ is a proper $\mathfrak{so}(V)$ submodule of W_0 , hence $[W_0, W_0] = 0$. The kernel K of the adjoint representation $\rho : W_0 \rightarrow \mathfrak{gl}(W_1)$ is an $\mathfrak{so}(V)$ -submodule of W_0 . Hence $K = W_0$ or 0 . In the first case, \mathfrak{g} is of translational type. In the second case, the representation ρ is faithful and $\rho(W_0)$ commutes with $\rho(\mathfrak{so}(V))$, hence $[\mathfrak{so}(V), W_0] = 0$. On the other hand the $\mathfrak{so}(V)$ -module W_0 is irreducible, so $W_0 \cong \mathbb{R}$. \square

4. Extended Polyvector Poincaré Algebras and $\wedge^k V$ -Valued Invariant Bilinear Forms on the Spinor Module S

In this and the next two sections, we devote ourselves to the classification of ϵ -transalgebras $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ with $\mathfrak{g}_1 = W_1 = S$, the spinor $\mathfrak{so}(V)$ -module. We take V to be the pseudo-Euclidean space $\mathbb{R}^{p,q}$ of dimension $n = p+q$ and signature $s = p-q$. In other words, we consider (super) Lie algebras $\mathfrak{g} = (\mathfrak{so}(V) + W_0) + S$ with

$$[W_0, W_0 + S] = 0 \quad , \quad W_0 = [S, S].$$

The (super) Lie bracket defines an $\mathfrak{so}(V)$ -equivariant surjective map $\Gamma_{W_0} : S \otimes S \rightarrow W_0$. If K is the kernel of this map, then $S \otimes S = \tilde{W}_0 + K$, where \tilde{W}_0 is an $\mathfrak{so}(V)$ -submodule equivalent to W_0 such that $W_0 \subset S \wedge S$ in the Lie algebra case and $W_0 \subset S \vee S$ in the superalgebra case. Conversely, if we have a decomposition $S \otimes S = W_0 + K$ into a sum of two $\mathfrak{so}(V)$ -submodules and moreover $W_0 \subset S \wedge S$ or $W_0 \subset S \vee S$, then the projection Γ_{W_0} onto W_0 with the kernel K defines an $\mathfrak{so}(V)$ -equivariant bracket

$$\begin{aligned} [,] : S \otimes S &\rightarrow W_0 \\ [s, t] &= \Gamma_{W_0}(s \otimes t) \end{aligned} \tag{4.1}$$

which is skewsymmetric or symmetric, respectively. More generally, if A is an endomorphism of W_0 which commutes with $\mathfrak{so}(V)$, then the twisted projection $A \circ \Gamma_{W_0}$ is another $\mathfrak{so}(V)$ -equivariant bracket and any bracket can be obtained in this way. Together with the action of $\mathfrak{so}(V)$ on W_0 and S , this defines the structure of an ϵ -transalgebra $\mathfrak{g} = \mathfrak{so}(V) + W_0 + S$. We therefore have a 1–1 correspondence between ϵ -transalgebras of the form $\mathfrak{g} = \mathfrak{so}(V) + W_0 + S$, where W_0 is a submodule of $S \vee S$ (for $\epsilon = 1$) or $S \wedge S$ (for $\epsilon = -1$), and equivariant surjective maps $\Gamma_{W_0} : S \otimes S \rightarrow W_0$, whose kernel contains $S \vee S$ if $\epsilon = -1$ and $S \wedge S$ if $\epsilon = 1$. The problem thus reduces to the description of the decomposition of $S \wedge S$ and $S \vee S$ into irreducible $\mathfrak{so}(V)$ -submodules and the determination of the twisted projections $A \circ \Gamma_{W_0}$. We consider these projections as equivariant W_0 -valued symmetric or skewsymmetric bilinear forms on S . In the next section we show that the irreducible submodules of $S \otimes S$ are of the form $\wedge^k V$ or $\wedge_{\pm}^m V$ ((anti)selfdual m -forms) if $n = 2m$ and s is divisible by 4. We denote by

$$\text{Bil}^k(S) = \text{Hom}(S \otimes S, \wedge^k V),$$

the vector space of $\wedge^k V$ -valued bilinear forms on S . It can be decomposed, $\text{Bil}^k(S) = \text{Bil}_+^k(S) \oplus \text{Bil}_-^k(S)$, into the sum of the vector spaces of symmetric (+) and skewsymmetric (–) bilinear forms.

For $W_0 = \wedge^k V$, the space of ϵ -transalgebras ($\epsilon = \pm$) is identified with the space $\text{Bil}_{\pm}^k(S)^{\mathfrak{so}(V)}$ of $\mathfrak{so}(V)$ -invariant symmetric ($\epsilon = +$) or skewsymmetric ($\epsilon = -$) bilinear forms. Hence:

The classification of ϵ -transalgebras $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ with $\mathfrak{g}_1 = S$ reduces to the description of the spaces $\text{Bil}_{\pm}^k(S)^{\mathfrak{so}(V)}$ of $\wedge^k V$ -valued invariant bilinear forms on the spinor module S .

The following formula associates a $\wedge^k V$ -valued bilinear form $\Gamma_{\beta}^k \in \text{Bil}^k(S)$ to every (scalar) bilinear form $\beta \in \text{Bil}(S)$:

$$\langle \Gamma_{\beta}^k(s \otimes t), v_1 \wedge \cdots \wedge v_k \rangle = \sum_{\pi \in \mathfrak{S}_k} \text{sgn}(\pi) \beta(\gamma(v_{\pi(1)}) \cdots \gamma(v_{\pi(k)})s, t) \quad s, t \in S, v_i \in V,$$

where the sum is over permutations π of $\{1, \dots, k\}$. Our classification is based on the following theorem.

Theorem 1. *For any pseudo-Euclidean vector space $V \cong \mathbb{R}^{p,q}$, the map*

$$\begin{aligned} \Gamma^k &: \text{Bil}(S) \rightarrow \text{Bil}^k(S) \\ \beta &\mapsto \Gamma_\beta^k \end{aligned}$$

is a $\text{Spin}(V)$ -equivariant monomorphism and it induces an isomorphism

$$\Gamma^k : \text{Bil}(S)^{\mathfrak{so}(V)} \xrightarrow{\sim} \text{Bil}^k(S)^{\mathfrak{so}(V)}$$

of vector spaces.

Proof. It is known that Clifford multiplication $\gamma : V \rightarrow \text{End } S$ is $\text{Spin}(V)$ -equivariant, i.e.

$$\gamma(gv) = g\gamma(v)g^{-1}, \quad g \in \text{Spin}(V), \quad v \in V.$$

Using this property we now check that the map Γ^k is also $\text{Spin}(V)$ -equivariant:

$$\Gamma_{g \cdot \beta}^k = g \cdot \Gamma_\beta^k,$$

where $(g \cdot \beta)(s, t) = \beta(g^{-1}s, g^{-1}t)$ and $(g \cdot \Gamma_\beta^k)(s, t) = g\Gamma_\beta^k(g^{-1}s, g^{-1}t)$. We calculate

$$\begin{aligned} \langle \Gamma_{g \cdot \beta}^k(s, t), v_1 \wedge \dots \wedge v_k \rangle &= \sum_{\pi \in \mathfrak{S}_k} \text{sgn}(\pi) \beta \left(g^{-1} \gamma(v_{\pi(1)}) \dots \gamma(v_{\pi(k)}) s, g^{-1} t \right) \\ &= \sum_{\pi \in \mathfrak{S}_k} \text{sgn}(\pi) \beta \left(\gamma(g^{-1} v_{\pi(1)}) \dots \gamma(g^{-1} v_{\pi(k)}) g^{-1} s, g^{-1} t \right) \\ &= \langle \Gamma_\beta^k(g^{-1} s, g^{-1} t), g^{-1} v_1 \wedge \dots \wedge g^{-1} v_k \rangle \\ &= \langle g \Gamma_\beta^k(g^{-1} s, g^{-1} t), v_1 \wedge \dots \wedge v_k \rangle \\ &= \langle (g \cdot \Gamma_\beta^k)(s, t), v_1 \wedge \dots \wedge v_k \rangle. \end{aligned} \tag{4.2}$$

Next, we prove that Γ is injective. For $\beta \in \text{Bil}(S)$ the bilinear form Γ_β^k is zero if and only if

$$\beta \left(\sum_{\pi} \text{sgn}(\pi) \gamma(v_{\pi(1)}) \dots \gamma(v_{\pi(k)}) S, S \right) = 0$$

or

$$\sum_{\pi} \text{sgn}(\pi) \gamma(v_{\pi(1)}) \dots \gamma(v_{\pi(k)}) S \subset \ker(\beta)$$

for any vectors v_1, \dots, v_k . If the vectors v_1, \dots, v_k are orthogonal, then the endomorphisms $\gamma(v_1), \dots, \gamma(v_k)$ anticommute and the endomorphism $\sum_{\pi} \text{sgn}(\pi) \gamma(v_{\pi(1)}) \dots \gamma(v_{\pi(k)}) = k! \gamma(v_1) \dots \gamma(v_k)$ is invertible. This implies that $\ker(\beta) = S$ and so $\beta = 0$.

To complete the proof of the theorem, we need to check that

$$\dim \text{Bil}^k(S)^{\mathfrak{so}(V)} = \dim \text{Bil}(S)^{\mathfrak{so}(V)} =: N(p - q).$$

In fact, $\dim \text{Bil}^k(S)^{\mathfrak{so}(V)} = \mu(k) \dim \mathcal{C}(\wedge^k V)$, where $\mu(k)$ is the multiplicity of $\wedge^k V$ in $S \otimes S$ and $\mathcal{C}(M) = \text{End}_{\mathfrak{so}(V)}(M)$ denotes the Schur algebra of an $\mathfrak{so}(V)$ -module

M . If the signature $s = p - q$ is divisible by 4 and $k = m = n/2$, then $\wedge^m V = \wedge^m_+ V \oplus \wedge^m_- V$ is the sum of two inequivalent irreducible $\mathfrak{so}(V)$ -modules of real type and hence $\mathcal{C}(\wedge^m V) \cong \mathbb{R} \oplus \mathbb{R}$. If the signature s is even but not divisible by 4 and $k = m = n/2$, then $\wedge^m V$ is an irreducible $\mathfrak{so}(V)$ -module of complex type, with the complex structure defined by the Hodge star operator and hence $\mathcal{C}(\wedge^m V) \cong \mathbb{C}$. In both cases

$$\dim \text{Bil}^m(S)^{\mathfrak{so}(V)} = \mu(m) \dim \mathcal{C}(\wedge^m V) = 2\mu(m) = N(s),$$

where the last equation follows from Table A.1 in the appendix. In all other cases, $\wedge^k V$ is an irreducible module of real type and $\mathcal{C}(\wedge^k V) = \mathbb{R}$. Therefore, using Table A.1, we obtain

$$\dim \text{Bil}^k(S)^{\mathfrak{so}(V)} = \mu(k) \dim \mathcal{C}(\wedge^k V) = \mu(k) = N(s).$$

□

In the Introduction we defined the three $\mathbb{Z}/2\mathbb{Z}$ -valued invariants for $\wedge^k V$ -valued bilinear forms on the spinor module: symmetry, type and isotropy. We say that a non-zero $\wedge^k V$ -valued bilinear form $\Gamma \in \text{Bil}^k(S)$, $k > 0$, is *admissible* if it is either symmetric or skewsymmetric and, in the cases when semispinor modules exist, if the two semispinor modules are either isotropic or mutually orthogonal with respect to Γ . Recall that in the case of scalar-valued bilinear forms ($k = 0$), admissibility requires, in addition, that the bilinear form has a specific type τ . The invariants of admissible $\wedge^k V$ -valued bilinear forms in terms of the invariants of the scalar-valued admissible bilinear forms are given by:

Proposition 4. *Let $\beta \in \text{Bil}(S)$ be an admissible scalar bilinear form and Γ_β^k the associated $\wedge^k V$ -valued bilinear form. Then Γ_β^k is admissible and its invariants, the symmetry $\sigma(\Gamma_\beta^k)$ and the isotropy $\iota(\Gamma_\beta^k)$, can be calculated as follows:*

$$\sigma(\Gamma_\beta^k) = \sigma(\beta)\tau(\beta)^k(-1)^{k(k-1)/2}, \tag{4.3}$$

$$\iota(\Gamma_\beta^k) = \iota(\beta)(-1)^k. \tag{4.4}$$

For $k > 0$ the bilinear forms $\Gamma_\beta^k \neq 0$ have neither type.

Proof. Let $s, t \in S$ and $e_1, \dots, e_k \in V$ be orthogonal vectors. We put $\gamma_i := \gamma_{e_i}$ and compute

$$\begin{aligned} \langle \Gamma_\beta^k(s \otimes t), e_1 \wedge \dots \wedge e_k \rangle &= k! \beta(\gamma_1 \dots \gamma_k s, t) \\ &= k! \tau(\beta)^k \beta(s, \gamma_k \dots \gamma_1 t) \\ &= k! \tau(\beta)^k (-1)^{k(k-1)/2} \beta(s, \gamma_1 \dots \gamma_k t) \\ &= k! \tau(\beta)^k (-1)^{k(k-1)/2} \sigma(\beta) \beta(\gamma_1 \dots \gamma_k t, s) \\ &= k! \tau(\beta)^k (-1)^{k(k-1)/2} \sigma(\beta) \langle \Gamma_\beta^k(t \otimes s), e_1 \wedge \dots \wedge e_k \rangle. \end{aligned}$$

This proves Eq. (4.3). Equation (4.4) follows from the fact that Clifford multiplication γ_v maps S_+ to S_- and vice versa. □

5. Decomposition of the Tensor Square of the Spinor Module of Spin(V) into Irreducible Components: Complex Case

In this section we consider the spinor module S of a complex Euclidean vector space $V = \mathbb{C}^n$ and we derive the decompositions of $S \otimes S$, $S \vee S$ and $S \wedge S$ into inequivalent irreducible Spin(V)-submodules. These decompositions also yield the corresponding decompositions for the cases when S is a spinor module of a real vector space $V = \mathbb{R}^{m,m}$ if $n = 2m$ and $V = \mathbb{R}^{m,m+1}$ if $n = 2m + 1$. We shall use the well known facts summarised in the following lemma, see e.g. [OV].

Lemma 1. *Let V be an n -dimensional complex Euclidean vector space or a real pseudo-Euclidean vector space of signature (p, q) , $p + q = n$, $p - q = s$. If $n = 2m + 1$, then the decomposition of $\wedge V$ into irreducible pairwise inequivalent $\mathfrak{so}(V)$ -submodules is given by*

$$\wedge V = \sum_{k=0}^n \wedge^k V = \sum_{k=0}^m \wedge^k V + \sum_{k=0}^m * \wedge^k V = 2 \sum_{k=0}^m \wedge^k V. \tag{5.1}$$

If $n = 2m$ then we have the following decomposition into irreducible pairwise inequivalent $\mathfrak{so}(V)$ -submodules:

$$\wedge V = \begin{cases} 2 \sum_{k=0}^{m-1} \wedge^k V + \wedge^m V & \text{if } s/2 \text{ is odd} \\ 2 \sum_{k=0}^{m-1} \wedge^k V + \wedge^m_+ V + \wedge^m_- V & \text{if } s/2 \text{ is even or if } V \text{ is complex.} \end{cases} \tag{5.2}$$

Here $\wedge^\pm_m V$ are selfdual and anti-selfdual m -forms, the ± 1 -eigenspaces of the Hodge $*$ -operator, which acts isometrically on $\wedge^m V$, with $*^2 = (-1)^{m+q} = (-1)^{s/2} = +1$ if $s/2$ is even .

In particular, the $\mathfrak{so}(V)$ -module $\wedge^k V$ is irreducible, unless $n = 2m$, $s/2$ is even and $k = m$, in which case $\wedge^m V = \wedge^m_+ V + \wedge^m_- V$, where $\wedge^m_+ V$ and $\wedge^m_- V$ are irreducible inequivalent modules.

Theorem 2. (i) *The Spin(V)-module $S \otimes S$ contains all modules $\wedge^k V$ which are irreducible.*
 (ii) *If V is a complex vector space of dimension $n = 2m + 1$ or if V is real of signature $(m, m + 1)$ then*

$$\begin{aligned} S \otimes S &= \sum_{k=0}^m \wedge^k V, \\ S \vee S &= \sum_{i=0}^{[m/4]} \wedge^{m-4i} V + \sum_{i=0}^{[(m-3)/4]} \wedge^{m-3-4i} V, \\ S \wedge S &= \sum_{i=0}^{[(m-2)/4]} \wedge^{m-2-4i} V + \sum_{i=0}^{[(m-1)/4]} \wedge^{m-1-4i} V. \end{aligned}$$

Proof. (i) Theorem 1 associates a $\text{Spin}(V)$ -equivariant linear map

$$(\Gamma_\beta^k)^* : \wedge^k V \cong \wedge^k V^* \rightarrow S^* \otimes S^* \cong S \otimes S$$

with any invariant bilinear form β on S . In particular, if $\wedge^k V$ is irreducible and $\beta \neq 0$ then $(\Gamma_\beta^k)^*$ embeds $\wedge^k V$ into $S \otimes S$ as a submodule. It was proven in [AC] that a non-zero invariant bilinear form β on S always exists. This shows that $S \otimes S \supset \sum_{k=0}^m \wedge^k V$.

(ii) If $n = 2m + 1$ then the right-hand side has dimension $\frac{1}{2}2^n = 4^m$ and under the assumptions on V we have that $\dim S = 2^m$. Hence $\dim S \otimes S = 4^m$, so the inclusion is an equality. The decompositions of $S \vee S$ and $S \wedge S$ can either be read off the tables in [OV] or they follow from Proposition 4 using the invariants of the admissible scalar-valued form, which in this case is unique up to scale [AC] (see the tables in the appendix). \square

Now, we consider the case when V is complex of dimension $n = 2m$ or real of signature (m, m) . In this case, $\wedge^m V = \wedge_+^m V \oplus \wedge_-^m V$ and the spinor module splits as a sum $S = S_+ + S_-$ of inequivalent irreducible semi-spinor modules S_\pm of dimension 2^{m-1} .

Theorem 3. *Let V be complex of dimension $n = 2m$ or real of signature (m, m) . Then the decompositions of the $\text{Spin}(V)$ -modules $S_+ \otimes S_-$ and $S_\pm \otimes S_\pm$ into inequivalent irreducible submodules are given by:*

$$S_+ \otimes S_- = \sum_{i=0}^{[(m-1)/2]} \wedge^{m-1-2i} V, \tag{5.3}$$

$$S_\pm \otimes S_\pm = \wedge_\pm^m V + \sum_{i=0}^{[(m-2)/2]} \wedge^{m-2-2i} V, \tag{5.4}$$

$$S \otimes S = S_+ \otimes S_+ + 2S_+ \otimes S_- + S_- \otimes S_- = \wedge V. \tag{5.5}$$

Further, for any admissible bilinear form β on S , the equivariant maps $\Gamma_\beta^k|_{S_\pm \otimes S_\pm}$ and $\Gamma_\beta^k|_{S_+ \otimes S_-}$ have the following images:

$$\Gamma_\beta^m(S_\pm \otimes S_\pm) = \wedge_\pm^m V, \tag{5.6}$$

$$\Gamma_\beta^{m\pm(2i+2)}(S_\pm \otimes S_\pm) = \wedge^{m\pm(2i+2)} V, \quad 0 \leq i \leq [\frac{m-2}{2}], \tag{5.7}$$

$$\Gamma_\beta^{m\pm(2i+1)}(S_+ \otimes S_-) = \wedge^{m\pm(2i+1)} V, \quad 0 \leq i \leq [\frac{m-1}{2}], \tag{5.8}$$

$$\Gamma_\beta^{m\pm(2i+1)}(S_\pm \otimes S_\pm) = 0, \quad 0 \leq i \leq [\frac{m-1}{2}], \tag{5.9}$$

$$\Gamma_\beta^{m\pm 2i}(S_+ \otimes S_-) = 0, \quad 0 \leq i \leq [\frac{m}{2}]. \tag{5.10}$$

Proof. To prove the theorem we use the following model for the spinor module of an even dimensional complex Euclidean space V or of a pseudo-Euclidean space V with split signature (m, m) : $V = U \oplus U^*$, where U is an m -dimensional vector space and the scalar product is defined by the natural pairing between U and the dual space U^* . Then the spinor module is given by $S = \wedge U = \wedge^{\text{ev}} U + \wedge^{\text{odd}} U = S_+ + S_-$, where the

semi-spinor modules S_{\pm} consist of even and odd forms. The Clifford multiplication is given by exterior and interior multiplication:

$$\begin{aligned} u \cdot s &:= u \wedge s & \text{for } u \in U, s \in S, \\ u^* \cdot s &:= \iota_u^* s & \text{for } u^* \in U^*, s \in S. \end{aligned}$$

There exist exactly two independent admissible bilinear forms f and $f_E = f(E \cdot, \cdot)$ on the spinor module, where $E|_{S_{\pm}} = \pm \text{Id}$, and the form f is given by

$$\begin{aligned} f(\wedge^i U, \wedge^j U) &= 0, & \text{if } i + j \neq m, \\ f(s, t) \text{ vol}_U &= (-1)^{i(i+1)/2} s \wedge t, & s \in \wedge^i U, \quad t \in \wedge^{m-i} U, \end{aligned} \tag{5.11}$$

where $\text{vol}_U \in \wedge^m U$ is a fixed volume form of U^* . We note that the symmetry, type and isotropy of the admissible basis (f, f_E) of $\text{Bil}(S)^{\mathfrak{so}(V)}$ are given by

$$\begin{aligned} \sigma(f) &= (-1)^{m(m+1)/2}, & \sigma(f_E) &= (-1)^{m(m-1)/2}, \\ \tau(f) &= -1, & \tau(f_E) &= +1, & \iota(f) &= \iota(f_E) = (-1)^m. \end{aligned}$$

From this and Proposition 4 it follows that

$$\sigma(\Gamma_f^k) = (-1)^{(m(m+1)+k(k+1))/2}, \quad \sigma(\Gamma_{f_E}^k) = (-1)^{(m(m-1)+k(k-1))/2}, \tag{5.12}$$

$$\iota(\Gamma_f^k) = \iota(\Gamma_{f_E}^k) = (-1)^{m+k}. \tag{5.13}$$

The formulae (5.7)-(5.10) and the fact that $\Gamma_{\beta}^m(S_{\pm} \otimes S_{\pm}) \neq 0$ follow from the formulae for the isotropy of Γ_f^k and $\Gamma_{f_E}^k$.

To prove (5.6) we first show that for any admissible form β , the image $\Gamma_{\beta}^m(S_+ \otimes S_+)$ contains $\wedge_+^m V$ and the image $\Gamma_{\beta}^m(S_- \otimes S_-)$ does not contain $\wedge_+^m V$. For this we need to show that for any $a \in \wedge_+^m V$ there exist spinors $s, t \in S_+ = \wedge^{\text{ev}} U$ such that the scalar product $\langle \Gamma_{\beta}^m(s \otimes t), a \rangle \neq 0$, and that there exists an element $a \in \wedge_+^m V$ such that $\langle \Gamma_{\beta}^m(s \otimes t), a \rangle = 0$ for any $s, t \in S_- = \wedge^{\text{odd}} U$. Since $\wedge_+ V$ is an irreducible $\mathfrak{so}(V)$ -module, it follows that if a single element a of $\wedge_+^m V$ is contained in the $\mathfrak{so}(V)$ -module $\Gamma_{\beta}(S_+ \otimes S_-)$, then all of $\wedge_+^m V$ is contained in it. Therefore, it will suffice to prove the first statement for just one choice of a .

We shall use the following lemma.

Lemma 2. *Let $V = U \oplus U^*$ as above. Then $\wedge^m U \subset \wedge_+^m V$.*

Proof. Let (u_1, \dots, u_m) be a basis of U and (u_1^*, \dots, u_m^*) the dual basis of U^* . Then, up to a sign factor, the volume form is given by $\text{vol} = u_1 \wedge \dots \wedge u_m \wedge u_1^* \wedge \dots \wedge u_m^*$. Now, using the definition of the Hodge star operator, $\langle * \alpha, \beta \rangle \text{vol} = \alpha \wedge \beta$, we may immediately check that $\ast(u_1 \wedge \dots \wedge u_m) = u_1 \wedge \dots \wedge u_m$. \square

Let us consider $a = \text{vol}_U$. By the lemma, $a \in \wedge_+^m V$. Then for $s = t = 1 \in S_+$ we have

$$\langle \Gamma_{\beta}^m(s \otimes t), a \rangle = \beta(a \wedge s, t) = \beta(a, 1) = \pm 1 \neq 0.$$

Similarly, for any $s, t \in S_-$,

$$\langle \Gamma_{\beta}^m(s \otimes t), a \rangle = \beta(a \wedge s, t) = 0,$$

since $\deg(a \wedge s) > m = \dim U$ and hence $a \wedge s = 0$. This proves both the above statements and hence $\Gamma_\beta^m(S_+ \otimes S_+) = \wedge_+^m V$. Since the image $\Gamma_\beta^m(S_- \otimes S_-)$ is non-zero and does not contain $\wedge_+^m V$, we also have $\Gamma_\beta^m(S_- \otimes S_-) = \wedge_-^m V$. This proves (5.6).

We now prove (5.3). By (5.8), we have the inclusion $\sum_{i=0}^{[(m-1)/2]} \wedge^{m-1-2i} V \subset S_+ \otimes S_-$. To prove equality we compare dimensions. Using the identity $\binom{2m}{m-1-2i} = \binom{2m-1}{m-1-2i} + \binom{2m-1}{m-2-2i}$, we calculate:

$$\begin{aligned} \dim \left(\sum_{i=0}^{[(m-1)/2]} \wedge^{m-1-2i} V \right) &= \sum_{i=0}^{[(m-1)/2]} \binom{2m}{m-1-2i} = \sum_{i=0}^{m-1} \binom{2m-1}{i} \quad (5.14) \\ &= \frac{1}{2} \sum_{i=0}^{2m-1} \binom{2m-1}{i} = 2^{2m-2} = \dim(S_+ \otimes S_-) \quad (5.15) \end{aligned}$$

since $\dim S_\pm = 2^{m-1}$. This proves (5.3).

Similarly, by (5.6) and (5.7), we have $S_\pm \otimes S_\pm \supset \wedge_\pm^m V + \sum_{i=0}^{[(m-2)/2]} \wedge^{m-2i-2} V$. To prove (5.4) we compare dimensions:

$$\begin{aligned} \dim \left(\wedge_\pm^m V + \sum_{i=0}^{[(m-2)/2]} \wedge^{m-2i-2} V \right) &= \sum_{i=0}^{[m/2]} \binom{2m}{m-2i} - \frac{1}{2} \binom{2m}{m} \\ &= \sum_{i=0}^{[m/2]} \left(\binom{2m-1}{m-2i} + \binom{2m-1}{m-2i-1} \right) \\ &\quad \times -\frac{1}{2} \binom{2m}{m} \\ &= \sum_{i=0}^m \binom{2m-1}{m-i} - \frac{1}{2} \binom{2m}{m} \\ &= \frac{1}{2} \sum_{i=0}^{2m-1} \binom{2m-1}{i} = 2^{2m-2} = 2^{m-1} \cdot 2^{m-1} \\ &= \dim(S_\pm \otimes S_\pm). \end{aligned}$$

This proves (5.4) and (5.5). \square

Corollary 2. (i) Let V be either complex of even dimension or real of signature (m, m) and β an admissible bilinear form on the spinor module $S = S_+ + S_-$. Then for all k the image of Γ_β^k restricted to $S_+ \otimes S_+$, $S_- \otimes S_-$ and $S_+ \otimes S_-$ is an irreducible $\text{Spin}(V)$ -module and the $\text{Spin}(V)$ -module $S \otimes S$ is isomorphic to $\wedge V$.

(ii) Let V be either complex of odd dimension or real of signature $(m, m + 1)$ and β an admissible bilinear form on the spinor module S . Then for all k the image $\Gamma_\beta^k(S \otimes S)$ is irreducible and the $\text{Spin}(V)$ -module $2S \otimes S$ is isomorphic to $\wedge V$.

Corollary 3. *Let V be complex of dimension $n = 2m$ or real of signature (m, m) . Then we have*

$$S_{\pm} \vee S_{\pm} = \wedge_{\pm}^m V + \sum_{i=0}^{[(m-4)/4]} \wedge^{m-4-4i} V, \tag{5.16}$$

$$S_{\pm} \wedge S_{\pm} = \sum_{i=0}^{[(m-2)/4]} \wedge^{m-2-4i} V. \tag{5.17}$$

Proof. These decompositions follow from (5.4) and (5.12). \square

6. Decomposition of the Tensor Square of the Spinor Module of $\text{Spin}(V)$ into Irreducible Components: Real Case

In this section we describe the decompositions of $S \otimes S, S \vee S$ and $S \wedge S$ into inequivalent irreducible $\text{Spin}(V)$ -submodules, where S is the spinor module of a pseudo-Euclidean vector space $V = \mathbb{R}^{p,q}$ of arbitrary signature $s = p - q$ and dimension $n = p + q$. We obtain these decompositions in two steps: First, we describe the complexification $S^{\mathbb{C}}$ of the spinor module S . Second, using the decomposition of the tensor square $\mathbb{S} \otimes \mathbb{S}$ of the complex spinor module \mathbb{S} , we decompose $S^{\mathbb{C}} \otimes S^{\mathbb{C}}$ into complex irreducible $\text{Spin}(V^{\mathbb{C}})$ -submodules and then we take real forms. We recall that the complex spinor module \mathbb{S} associated to the complex Euclidean space $\mathbb{V} = V^{\mathbb{C}} = V \otimes \mathbb{C} = \mathbb{C}^n$ is the restriction to $\text{Spin}(\mathbb{V})$ of an irreducible representation of the complex Clifford algebra $\mathcal{Cl}(\mathbb{V})$.

Depending on the signature $s \equiv p - q \pmod{8}$, the complexification $S^{\mathbb{C}}$ of the spinor module S is given by either

- i) $S^{\mathbb{C}} = \mathbb{S}$, where we denote by \mathbb{S} the spinor module of the complex Euclidean space $\mathbb{V} = V^{\mathbb{C}} = V \otimes \mathbb{C} = \mathbb{C}^n$, or
- ii) $S^{\mathbb{C}} = \mathbb{S} + \overline{\mathbb{S}}$, where $\overline{\mathbb{S}}$ is the complex conjugated module of \mathbb{S} .

In the latter case S admits a $\text{Spin}(V)$ -invariant complex structure J and \mathbb{S} is identified with the complex space (S, J) and $\overline{\mathbb{S}}$ with $(S, -J)$. In the next lemma we specify the signatures for which the cases i) or ii) occur. For this we use Table 1, in which we have collected important information about the real and complex Clifford algebras and spinor modules.

Now, we define the notion of $\text{Type}_{\mathcal{Cl}^0(V)}(\mathbb{S}, \mathbb{S}_{\pm})$ used in Table 1. If s is odd, then the complex spinor module \mathbb{S} is irreducible (as a complex module of the real even Clifford algebra $\mathcal{Cl}^0(V)$). In this case we define $\text{Type}_{\mathcal{Cl}^0(V)}(\mathbb{S}) := \mathbb{K} \in \{\mathbb{R}, \mathbb{H}\}$ if the $\mathcal{Cl}^0(V)$ -module \mathbb{S} is of real or quaternionic type, i.e. it admits a real or quaternionic structure commuting with $\mathcal{Cl}^0(V)$. For even s the complex spinor module $\mathbb{S} = \mathbb{S}_+ + \mathbb{S}_-$ and \mathbb{S}_{\pm} are irreducible complex $\mathcal{Cl}^0(V)$ -modules. We put $\text{Type}_{\mathcal{Cl}^0(V)}(\mathbb{S}, \mathbb{S}_{\pm}) = (l\mathbb{K}, \mathbb{K}')$, where \mathbb{K} and \mathbb{K}' are the types of \mathbb{S} and \mathbb{S}_{\pm} , respectively, further $l = 1$ if \mathbb{S} is irreducible and $l = 2$ if \mathbb{S}_+ and \mathbb{S}_- are not equivalent as complex $\mathcal{Cl}^0(V)$ -modules. Note that if the semispinor modules are of complex type ($s = 2, 6$), then they are complex-conjugates of each other: $\overline{\mathbb{S}_{\pm}} \cong \mathbb{S}_{\mp}$. If \mathbb{S}_{\pm} are of real ($s = 0$) or quaternionic ($s = 4$) type, then they are selfconjugate: $\overline{\mathbb{S}_{\pm}} \cong \mathbb{S}_{\pm}$.

We now explain how Table 1 has been obtained. The first two columns have been extracted from [LM] and imply the third column. Passing to the complexification of the

Table 1. Clifford Modules $Cl_{p,q}$, their even parts $Cl_{p,q}^0$, the Schur algebra $\mathcal{C} = \text{End}_{Cl^0(V)}(S)$, the complex Spinor Module \mathbb{S} , the complex Semispinor Modules \mathbb{S}_\pm , the Type of these $Cl^0(V)$ -modules and physics terminology: M stands for Majorana, W for Weyl and Symp for symplectic (i.e. quaternionic) spinors; $s = p - q \pmod 8$, $n = p + q$, $N = 2^{\lfloor n/2 \rfloor}$. Note that p is the number of negative eigenvalues of the product of two gamma matrices, and q the number of positive eigenvalues, see Appendix B.1

s	$Cl_{p,q}$	$Cl_{p,q}^0$	\mathcal{C}	\mathbb{S}	\mathbb{S}_\pm	Type $_{Cl^0(V)}(\mathbb{S}, \mathbb{S}_\pm)$	Name
0	$\mathbb{R}(N)$	$2\mathbb{R}(N/2)$	$2\mathbb{R}$	$S \otimes \mathbb{C}$	$S_\pm \otimes \mathbb{C}$	$(2\mathbb{R}, \mathbb{R})$	M-W
1	$\mathbb{C}(N)$	$\mathbb{R}(N)$	$\mathbb{R}(2)$	$S = S_\pm \otimes \mathbb{C}$		\mathbb{R}	M
2	$\mathbb{H}(N/2)$	$\mathbb{C}(N/2)$	$\mathbb{C}(2)$	$S = S_\pm \otimes \mathbb{C}$	S_\pm	$(2\mathbb{C}, \mathbb{C})$	M, W
3	$2\mathbb{H}(N/2)$	$\mathbb{H}(N/2)$	\mathbb{H}	S		\mathbb{H}	Symp
4	$\mathbb{H}(N/2)$	$2\mathbb{H}(N/4)$	$2\mathbb{H}$	S	S_\pm	$(2\mathbb{H}, \mathbb{H})$	Symp-W
5	$\mathbb{C}(N)$	$\mathbb{H}(N/2)$	\mathbb{H}	S		\mathbb{H}	Symp
6	$\mathbb{R}(N)$	$\mathbb{C}(N/2)$	\mathbb{C}	$S \otimes \mathbb{C}$	S	(\mathbb{R}, \mathbb{C})	M, W
7	$2\mathbb{R}(N)$	$\mathbb{R}(N)$	\mathbb{R}	$S \otimes \mathbb{C}$		\mathbb{R}	M

Clifford algebras we have: $Cl(V) \otimes \mathbb{C} = Cl(V \otimes \mathbb{C})$ and $Cl^0(V) \otimes \mathbb{C} = Cl^0(V \otimes \mathbb{C})$. From this we can describe the complex spinor module \mathbb{S} and semispinor modules \mathbb{S}_\pm and determine the relation between S, S_\pm and $\mathbb{S}, \mathbb{S}_\pm$. This gives the fourth, fifth and sixth columns of the table. Using this table we prove the following lemma, which describes the complex $Spin(\mathbb{V})$ -module $S^\mathbb{C}$:

Lemma 3.

$$S^\mathbb{C} = \begin{cases} \mathbb{S} + \bar{\mathbb{S}} & \text{if } s = p - q \equiv 1, 2, 3, 4, 5 \pmod 8 \\ \mathbb{S} & \text{if } s \equiv 6, 7, 8 \pmod 8. \end{cases}$$

Proof. According to Table 1, if $s \equiv 6, 7, 8 \pmod 8$ we have $\mathbb{S} = S \otimes \mathbb{C}$. In all other cases there exists a $Spin(V)$ -invariant complex structure J and the complex space (S, J) is identified with \mathbb{S} . Then $S^\mathbb{C} = \mathbb{S} + \bar{\mathbb{S}}$. \square

Remark 1. We note that a $Spin(V)$ -invariant real or quaternionic structure φ on \mathbb{S} (i.e. an antilinear map with $\varphi^2 = +1$ or -1 , respectively) defines an isomorphism $\varphi : \mathbb{S} \rightarrow \bar{\mathbb{S}}$. From Table 1 it follows that if $s \equiv 1, 2 \pmod 8$, then there exists a $Spin(V)$ -invariant real structure and if $s \equiv 3, 4, 5 \pmod 8$, then there exists a $Spin(V)$ -invariant quaternionic structure on \mathbb{S} .

Now, using the results of the previous section for the complex case, we decompose $S^\mathbb{C} \otimes S^\mathbb{C}$ into complex irreducible $Spin(\mathbb{V})$ -submodules. If $S^\mathbb{C} \otimes S^\mathbb{C} = \sum \mathbb{W}_i$ and all submodules \mathbb{W}_i are of real type (i.e. complexifications of irreducible real $Spin(V)$ -submodules W_i), then $S \otimes S = \sum W_i$ is the desired decomposition. In odd dimensions all modules $\mathbb{W}_i = \wedge^i \mathbb{V}$ are of real type. This is also the case in even dimensions $n = 2m$, with one exception: the modules $\wedge_\pm^m \mathbb{V} \subset S^\mathbb{C} \otimes S^\mathbb{C}$ are not of real type if $*^2 = -1$, i.e. if $s/2$ is odd. Then, $\wedge^m \mathbb{V} = \wedge_+^m \mathbb{V} + \wedge_-^m \mathbb{V}$ is the complexification of the irreducible $Spin(V)$ -module $\wedge^m V$, which has the $Spin(V)$ -invariant complex structure $*$.

The decompositions of $S \vee S$ and $S \wedge S$ can be obtained using the same method. Using this approach, we describe in detail all these decompositions for any pseudo-Euclidean vector space $V = \mathbb{R}^{p,q}$ in the next three subsections.

6.1. *Odd dimensional case: $\dim V = 2m + 1$.* We now describe the decomposition of $S \otimes S = S \vee S + S \wedge S$ for all signatures $s = 1, 3, 5, 7 \pmod{8}$ in the odd dimensional case.

Theorem 4. *Let $V = \mathbb{R}^{p,q}$ be a pseudo-Euclidean vector space of dimension $n = p + q = 2m+1$. Then the decompositions of the $\text{Spin}(V)$ -modules $S \otimes S$, $S \vee S$ and $S \wedge S$ into inequivalent irreducible submodules is given by the following:
If the signature $s = p - q \equiv 1, 3, 5 \pmod{8}$, we have*

$$S \otimes S = 2(\wedge V) = 4 \sum_{i=0}^m \wedge^i V, \tag{6.1}$$

$$S \vee S = 3 \sum_{i=0}^{[m/4]} \wedge^{m-4i} V + 3 \sum_{i=0}^{[(m-3)/4]} \wedge^{m-3-4i} V + \sum_{i=0}^{[(m-2)/4]} \wedge^{m-2-4i} V + \sum_{i=0}^{[(m-1)/4]} \wedge^{m-1-4i} V, \tag{6.2}$$

$$S \wedge S = 3 \sum_{i=0}^{[(m-2)/4]} \wedge^{m-2-4i} V + 3 \sum_{i=0}^{[(m-1)/4]} \wedge^{m-1-4i} V + \sum_{i=0}^{[m/4]} \wedge^{m-4i} V + \sum_{i=0}^{[(m-3)/4]} \wedge^{m-3-4i} V. \tag{6.3}$$

If the signature $s \equiv 7 \pmod{8}$, we have

$$S \otimes S = \sum_{i=0}^m \wedge^i V, \tag{6.4}$$

$$S \vee S = \sum_{i=0}^{[m/4]} \wedge^{m-4i} V + \sum_{i=0}^{[(m-3)/4]} \wedge^{m-3-4i} V, \tag{6.5}$$

$$S \wedge S = \sum_{i=0}^{[(m-2)/4]} \wedge^{m-2-4i} V + \sum_{i=0}^{[(m-1)/4]} \wedge^{m-1-4i} V. \tag{6.6}$$

Moreover, S is an irreducible $\text{Spin}(V)$ -module for $s = 3, 5, 7$ and for $s = 1$, $S = S_+ + S_-$ is the sum of two equivalent semi-spinor modules.

Proof. The signature $s \equiv 7 \pmod{8}$ corresponds to $\mathbb{R}^{m,m+1}$, which was already discussed in Theorem 2.

For $s \equiv 1, 3, 5 \pmod{8}$, the spinor module S has an invariant complex structure J and (S, J) is identified with the complex spinor module \mathbb{S} . We denote by $\bar{\mathbb{S}} = (S, -J)$ the module conjugate to \mathbb{S} . According to Lemma 3 and Remark 1, \mathbb{S} and $\bar{\mathbb{S}}$ are equivalent as complex modules of the real spin group $\text{Spin}(V)$ and $\mathbb{S}^{\mathbb{C}} = \mathbb{S} + \bar{\mathbb{S}} = 2\mathbb{S}$. Hence,

$$\begin{aligned} (S \otimes_{\mathbb{R}} S)^{\mathbb{C}} &= S^{\mathbb{C}} \otimes_{\mathbb{C}} S^{\mathbb{C}} = 4\mathbb{S} \otimes_{\mathbb{C}} \mathbb{S}, \\ (\vee^2 S)^{\mathbb{C}} &= \vee^2(\mathbb{S} + \bar{\mathbb{S}}) = \vee^2(2\mathbb{S}) = 3 \vee^2 \mathbb{S} + \wedge^2 \mathbb{S}, \\ (\wedge^2 S)^{\mathbb{C}} &= 3 \wedge^2 \mathbb{S} + \vee^2 \mathbb{S}. \end{aligned}$$

This implies the theorem. For example,

$$(S \otimes_{\mathbb{R}} S)^{\mathbb{C}} = 4\mathbb{S} \otimes_{\mathbb{C}} \mathbb{S} = 4 \sum_{i=0}^m \wedge^i \mathbb{V} = 4 \sum_{i=0}^m (\wedge^i V)^{\mathbb{C}},$$

by virtue of Theorem 2 and the real part gives (6.1). \square

6.2. *Even dimensional case: $\dim V = 2m$.* In this subsection, we describe the decomposition of $S \otimes S = S \vee S + S \wedge S$ for all signatures $s = 0, 2, 4, 6 \pmod{8}$ in the even dimensional case.

Theorem 5. *Let $V = \mathbb{R}^{p,q}$ be a pseudo-Euclidean vector space of dimension $n = p + q = 2m$. Then the decompositions of the $\text{Spin}(V)$ -module $S \otimes S$ into inequivalent irreducible submodules is given by the following:*

If the signature $s = p - q \equiv 2, 4 \pmod{8}$, we have

$$S \otimes S = 4(\wedge V) = \begin{cases} 8 \sum_{i=0}^{m-1} \wedge^i V + 4 \wedge^m V & \text{if } s \equiv 2 \pmod{8} \\ 8 \sum_{i=0}^{m-1} \wedge^i V + 4 \wedge_+^m V + 4 \wedge_-^m V & \text{if } s \equiv 4 \pmod{8}. \end{cases} \tag{6.7}$$

If the signature $s \equiv 0, 6 \pmod{8}$, we have

$$S \otimes S = \wedge V = \begin{cases} 2 \sum_{i=0}^{m-1} \wedge^i V + \wedge_+^m V + \wedge_-^m V & \text{if } s \equiv 0 \pmod{8} \\ 2 \sum_{i=0}^{m-1} \wedge^i V + \wedge^m V & \text{if } s \equiv 6 \pmod{8}. \end{cases} \tag{6.8}$$

Proof. Similarly to the odd dimensional case, we have

$$(S \otimes_{\mathbb{R}} S)^{\mathbb{C}} = \mathbb{S} \otimes_{\mathbb{C}} \mathbb{S} = \wedge \mathbb{V} = 2 \sum_{i=0}^m \wedge^i \mathbb{V} + \wedge_+^m \mathbb{V} + \wedge_-^m \mathbb{V}. \tag{6.9}$$

Now we note that $(\wedge^m V)^{\mathbb{C}} = \wedge^m \mathbb{V}$ and $*^2|_{\wedge^m V} = (-1)^{s/2}$. If $s/2$ is even, then $\wedge^m V = \wedge_+^m V + \wedge_-^m V$, where $\wedge_{\pm}^m V$ are irreducible submodules, which are ± 1 -eigenspaces of $*$. Then, $(\wedge_{\pm}^m V)^{\mathbb{C}} = \wedge_{\pm}^m \mathbb{V}$. In this case, the real part of (6.9) gives the first part of (6.8). If $s/2$ is odd, then $*$ is a complex structure on $\wedge^m V$, which is irreducible since it has complex structure and its complexification $\wedge^m \mathbb{V}$ has only two irreducible components $\wedge_{\pm}^m \mathbb{V}$. In this case the real part of (6.9) gives the second part of (6.8). \square

The decompositions of $S \vee S$ and $S \wedge S$ for the cases when semispinor modules exist, in particular for $s = 0, 2, 4 \pmod{8}$, will be given in the next subsection. Therefore it is now sufficient to determine these decompositions for $s = 6 \pmod{8}$.

Corollary 4. *Let S be the spinor module of a pseudo-Euclidean vector space V of signature $s \equiv 6 \pmod{8}$ and dimension $n = 2m$. Then we have*

$$S \vee S = \wedge^m V + 2 \sum_{i=0}^{[(m-4)/4]} \wedge^{m-4-4i} V + \sum_{i=0}^{[(m-1)/2]} \wedge^{m-1-2i} V, \quad (6.10)$$

$$S \wedge S = 2 \sum_{i=0}^{[(m-2)/4]} \wedge^{m-2-4i} V + \sum_{i=0}^{[(m-1)/2]} \wedge^{m-1-2i} V. \quad (6.11)$$

Proof. This follows by complexification of Eq. (6.8), using Lemma 3, Eq. (5.3) and Corollary 3. \square

6.3. Decomposition of tensor square of semi-spinors. According to Table 1, semi-spinor modules S_{\pm} exist if the signature $s \equiv 0, 1, 2, 4 \pmod{8}$. More precisely, we list below whether S_{\pm} are equivalent $\text{Spin}(V)$ -modules and we give $S_{\pm}^{\mathbb{C}}$.

s	S_{\pm}	$S_{\pm}^{\mathbb{C}}$
0	inequivalent	$S_{\pm} \otimes \mathbb{C} = \mathbb{S}_{\pm}$
1	equivalent	$S_{\pm} \otimes \mathbb{C} = \mathbb{S}$
2	equivalent	$S_{\pm} \otimes \mathbb{C} = \mathbb{S}$
4	inequivalent	$S_{\pm} + \bar{S}_{\pm} = 2\mathbb{S}_{\pm}$

For $s \equiv 1, 2, 4 \pmod{8}$ we have $\mathbb{S} = S$, whereas for $s \equiv 0 \pmod{8}$ we have $\mathbb{S} = S^{\mathbb{C}}$. We also note that for $s \equiv 2 \pmod{8}$, although the $\text{Spin}(V)$ -modules S_{\pm} are equivalent, the $\text{Spin}(V)$ -modules \mathbb{S}_{+} and $\mathbb{S}_{-} = \bar{\mathbb{S}}_{+}$ are not equivalent. For $s = 4$, $S_{\pm} = \bar{S}_{\pm}$ admits a $\text{Spin}(V)$ -invariant quaternionic structure.

Using the above description for $S_{\pm}^{\mathbb{C}}$ and the decompositions of the tensor squares of complex spinor and semi-spinor modules, we obtain the following:

Theorem 6. *Let $S = S_{+} + S_{-}$ be the spinor module of a pseudo-Euclidean vector space V of signature $s \equiv 0, 1, 2, 4 \pmod{8}$ and dimension $n = 2m$ or $n = 2m + 1$. Then we have the following decomposition of $\text{Spin}(V)$ -modules $S_{\pm} \otimes S_{\pm}$ and $S_{+} \otimes S_{-}$ into inequivalent irreducible submodules:*

For $s \equiv 0 \pmod{8}$:

$$S_{+} \otimes S_{-} = \sum_{i=0}^{[(m-1)/2]} \wedge^{m-1-2i} V,$$

$$S_{\pm} \otimes S_{\pm} = \wedge_{\pm}^m V + \sum_{i=0}^{[(m-2)/2]} \wedge^{m-2-2i} V,$$

$$S_{\pm} \vee S_{\pm} = \wedge_{\pm}^m V + \sum_{i=0}^{[(m-4)/4]} \wedge^{m-4-4i} V,$$

$$S_{\pm} \wedge S_{\pm} = \sum_{i=0}^{[(m-2)/4]} \wedge^{m-2-4i} V.$$

For $s \equiv 1 \pmod{8}$:

$$\begin{aligned} S_{\pm} \otimes S_{\pm} &= S_+ \otimes S_- = \sum_{i=0}^m \wedge^i V, \\ S_{\pm} \vee S_{\pm} &= \sum_{i=0}^{[m/4]} \wedge^{m-4i} V + \sum_{i=0}^{[(m-3)/4]} \wedge^{m-3-4i} V, \\ S_{\pm} \wedge S_{\pm} &= \sum_{i=0}^{[(m-2)/4]} \wedge^{m-2-4i} V + \sum_{i=0}^{[(m-1)/4]} \wedge^{m-1-4i} V. \end{aligned}$$

For $s \equiv 2 \pmod{8}$:

$$\begin{aligned} S_{\pm} \otimes S_{\pm} &= S_+ \otimes S_- = \wedge^m V + 2 \sum_{i=0}^{m-1} \wedge^i V, \\ S_{\pm} \vee S_{\pm} &= \wedge^m V + 2 \sum_{i=0}^{[(m-4)/4]} \wedge^{m-4-4i} V + \sum_{i=0}^{[(m-1)/2]} \wedge^{m-1-2i} V, \\ S_{\pm} \wedge S_{\pm} &= 2 \sum_{i=0}^{[(m-2)/4]} \wedge^{m-2-4i} V + \sum_{i=0}^{[(m-1)/2]} \wedge^{m-1-2i} V. \end{aligned}$$

For $s \equiv 4 \pmod{8}$:

$$\begin{aligned} S_+ \otimes S_- &= 4 \sum_{i=0}^{[(m-1)/2]} \wedge^{m-1-2i} V, \\ S_{\pm} \otimes S_{\pm} &= 4 \wedge_{\pm}^m V + 4 \sum_{i=0}^{[(m-2)/2]} \wedge^{m-2-2i} V, \\ S_{\pm} \vee S_{\pm} &= 3 \wedge_{\pm}^m V + 3 \sum_{i=0}^{[(m-4)/4]} \wedge^{m-4-4i} V + \sum_{i=0}^{[(m-2)/4]} \wedge^{m-2-4i} V, \\ S_{\pm} \wedge S_{\pm} &= 3 \sum_{i=0}^{[(m-2)/4]} \wedge^{m-2-4i} V + \wedge_{\pm}^m V + \sum_{i=0}^{[(m-4)/4]} \wedge^{m-4-4i} V. \end{aligned}$$

Proof. The case $s = 0 \pmod{8}$ follows from the complex case (see Theorem 3 and Corollary 3). For $s = 1, 2 \pmod{8}$ the modules S_+ and S_- are isomorphic. Hence $S = S_+ + S_- = 2S_+$ and $S \otimes S = 4S_+ \otimes S_+$. Since $S \otimes S = 4 \sum_{i=0}^m \wedge^i V$ we have

$$S_+ \otimes S_+ = S_- \otimes S_- = S_+ \otimes S_- = \sum_{i=0}^m \wedge^i V.$$

The splitting into symmetric and skew parts of these tensor products follows that in the complex cases, see Theorem 2 and Corollary 3. For $s = 4 \pmod{8}$ the semi-spinor modules S_{\pm} are not equivalent but $S_{\pm}^{\mathbb{C}} = \mathbb{S} = \mathbb{S}_+ + \mathbb{S}_-$. The result follows from the decomposition in the complex case (Theorem 3 and Corollary 3) on taking the real parts.

□

7. *N*-Extended Polyvector Poincaré Algebras

In the previous sections we have classified ϵ -transalgebras of the form $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$, $\mathfrak{g}_0 = \mathfrak{so}(V) + W_0$, with $\mathfrak{g}_1 = W_1 = S$ the spinor module. As in [AC] we can easily extend this classification to the case where W_1 is a general spinorial module, i.e. $W_1 = NS = S \otimes \mathbb{R}^N$, if S is irreducible, or $W_1 = N_+S_+ \oplus N_-S_- = S_+ \otimes \mathbb{R}_+^N \oplus S_- \otimes \mathbb{R}_-^N$ if semi-spinors exist. An ϵ -extension of translational type of the above form is called an ***N*-extended polyvector Poincaré algebra** if $W_1 = NS$ and an ***(N₊, N₋)-extended polyvector Poincaré algebra*** if $W_1 = N_+S_+ \oplus N_-S_-$. Consider first the case $W_1 = NS$. As before, the classification reduces to the decomposition of $\vee^2 W_1$ and $\wedge^2 W_1$ into irreducible submodules. These decompositions follow from the decompositions of $\vee^2 S$ and $\wedge^2 S$ obtained in the previous sections, together with the decompositions

$$\begin{aligned} \vee^2 W_1 &= \vee^2 S \otimes \vee^2 \mathbb{R}^N \oplus \wedge^2 S \otimes \wedge^2 \mathbb{R}^N, \\ \wedge^2 W_1 &= \wedge^2 S \otimes \vee^2 \mathbb{R}^N \oplus \vee^2 S \otimes \wedge^2 \mathbb{R}^N. \end{aligned}$$

In particular, this implies that the multiplicities $\mu_+(k, N)$ and $\mu_-(k, N)$ of the module $\wedge^k V$ in $\vee^2 W_1$ and $\wedge^2 W_1$, respectively, are given by

$$\begin{aligned} \mu_+(k, N) &= \mu_+(k) \frac{N(N+1)}{2} + \mu_-(k) \frac{N(N-1)}{2}, \\ \mu_-(k, N) &= \mu_-(k) \frac{N(N+1)}{2} + \mu_+(k) \frac{N(N-1)}{2}, \end{aligned}$$

where $\mu_+(k)$ and $\mu_-(k)$ are the multiplicities of $\wedge^k V$ in $\vee^2 S$ and $\wedge^2 S$, respectively. The vector space of *N*-extended polyvector Poincaré ϵ -algebra structures with $W_0 = \wedge^k V$ is identified with the space $\text{Bil}_\epsilon^k(W_1)^{\mathfrak{so}(V)}$ of invariant $\wedge^k V$ -valued bilinear forms on W_1 . Its dimension is given by

$$\dim \text{Bil}_\epsilon^k(W_1)^{\mathfrak{so}(V)} = \mu_\epsilon(k, N) \dim \mathcal{C}(\wedge^k V),$$

where the Schur algebra $\mathcal{C}(\wedge^k V) = \mathbb{R}, \mathbb{R} \oplus \mathbb{R}$ or \mathbb{C} , see the proof of Theorem 1. Any element $\Gamma_\epsilon \in \text{Bil}_\epsilon^k(W_1)^{\mathfrak{so}(V)}$ can be represented as

$$\begin{aligned} \Gamma_+ &= \sum_i \Gamma_{\beta_+^i}^k \otimes b_+^i + \sum_j \Gamma_{\beta_-^j}^k \otimes b_-^j, \\ \Gamma_- &= \sum_i \Gamma_{\beta_-^i}^k \otimes b_+^i + \sum_j \Gamma_{\beta_+^j}^k \otimes b_-^j, \end{aligned}$$

where $\beta_\pm^i \in \text{Bil}_\pm^k(S)^{\mathfrak{so}(V)}$ and b_+^i and b_-^i are, respectively, symmetric and skewsymmetric bilinear forms on \mathbb{R}^N . We note also that there exists a unique minimal (i.e. $W_0 = [W_1, W_1]$) *N*-extended polyvector Poincaré ϵ -algebra with $W_0 = \mu_\epsilon(k, N) \wedge^k V$. The Lie (super)bracket is given, up to a twist by an invertible element of the Schur algebra of W_0 , by the projection onto the corresponding maximal isotypical submodule of $\vee^2 W_1$ or $\wedge^2 W_1$, respectively.

Similarly, in the case when the spinor module $S = S_+ \oplus S_-$ is reducible, we can reduce the description of all *(N₊, N₋)-extended polyvector Poincaré ϵ -algebras $\mathfrak{g} = \mathfrak{so}(V) + \wedge^k V + W_1$ such that $W_1 = N_+S_+ + N_-S_-$ to the chiral cases $(N_+, N_-) = (1, 0)$ or $(0, 1)$ and to the isotropic case: $(N_+, N_-) = (1, 1)$ and $[S_+, S_+] = [S_-, S_-] = 0$.*

Table A.1. The numbers $(N^+(s, n), N^-(s, n))$ of independent symmetric and skewsymmetric bilinear forms on S

$s \setminus n$	0	1	2	3	4	5	6	7	$N(s)$	$\mu(k) = \mu(0)$	$\mu(m)$
0	2,0		1,1		0,2		1,1		2	2	1*
1		3,1		1,3		1,3		3,1	4	4	
2	6,2		4,4		2,6		4,4		8	8	4
3		3,1		1,3		1,3		3,1	4	4	
4	6,2		4,4		2,6		4,4		8	8	4*
5		3,1		1,3		1,3		3,1	4	4	
6	2,0		1,1		0,2		1,1		2	2	1
7		1,0		0,1		0,1		1,0	1	1	

Let β be an admissible bilinear form on $S = S_+ \oplus S_-$ and $\Gamma_\beta^k \in \text{Bil}_\epsilon^k(S)^{\mathfrak{so}(V)}$ the corresponding admissible $\wedge^k V$ -valued bilinear form. Its restriction to S_+ (or S_-) defines a $(1, 0)$ -extended (respectively, $(0, 1)$ -extended) k -polyvector Poincaré ϵ -algebra if and only if $\iota(\Gamma_\beta^k) = +1$. If $\iota(\Gamma_\beta^k) = -1$ then we obtain an isotropic $(1, 1)$ -extended k -polyvector Poincaré ϵ -algebra, i.e. $[S_+, S_+] = [S_-, S_-] = 0$. The values of the invariants $\sigma(\Gamma_\beta^k) = \epsilon$ and $\iota(\Gamma_\beta^k)$ can be read off Tables A.3–A.6 in Appendix A.

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A. Admissible $\wedge^k V$ -Valued Bilinear Forms on S

In Table A.1 we give the numbers $(N^+(s, n), N^-(s, n))$ of independent symmetric (+) and skewsymmetric (−) invariant bilinear forms on S , i.e. $N^\pm(s, n) = \dim \text{Bil}_\pm(S)^{\mathfrak{so}(V)}$. They were computed in [AC] and are periodic with period 8 in the signature $s = p - q$ and in the dimension $n = p + q$ of $V = \mathbb{R}^{p,q}$. The entry $N(s) = N^+(s, n) + N^-(s, n)$ is the total number of invariant bilinear forms and $\mu(k)$ is the multiplicity of the irreducible submodule $\wedge^k V$ in $S \otimes S$. For $k \neq n/2$, it does not depend on k , i.e. $\mu(k) = \mu(0)$. In the case $n = 2m$ the multiplicities $\mu(m)$ with a * indicate that the module $\wedge^m V$ is reducible as $\wedge^m_+ V + \wedge^m_- V$.

From Table A.1, we note the following symmetries:

a) Modulo 8-symmetry:

$$N^\pm(s + 8a, n + 8b) = N^\pm(s, n) \quad , \quad a, b \in \mathbb{Z}.$$

b) Reflection with respect to the horizontal line $s = 3$,

$$N^\pm(-s + 6, n) = N^\pm(s, n)$$

and reflection with respect to the vertical line $n = 0$,

$$N^\pm(s, n) = N^\pm(s, -n).$$

c) The reflection with respect to the vertical line $n = 2$ is a mirror symmetry, i.e.it interchanges N^+ and N^- :

$$N^\pm(s, n) = N^\mp(s, -n + 4).$$

It is the same as the mirror symmetry with respect to $n = 6$. We note that the composition of reflections in $n = 0$ and $n = 2$ gives the translation $n \mapsto n + 4$:

$$N^\pm(s, n) = N^\mp(s, n + 4).$$

More generally, we consider the dimension

$$\tilde{N}_k^\pm(p, q) := N_k^\pm(s, n) := \dim \text{Bil}_\pm^k(S)^{\text{so}(V)}$$

of the vector space of invariant $\wedge^k V$ -valued bilinear forms on S . By Theorem 1, the sum $N_k(s, n) = N_k^+(s, n) + N_k^-(s, n) = N(s, n)$ does not depend on k . From Table A.1, we see that it depends only on the signature s . As a corollary of Proposition 4, the numbers $\tilde{N}_k^\pm(p, q)$ are periodic modulo 4 in k :

$$\tilde{N}_k^\pm(p, q) = \tilde{N}_{k+4}^\pm(p, q).$$

Moreover, we have the following periodicities in (p, q) :

$$\tilde{N}_k^\pm(p, q) = \tilde{N}_k^\pm(p + 8, q) = \tilde{N}_k^\pm(p, q + 8) = \tilde{N}_k^\pm(p + 4, q + 4).$$

In fact, it was proven in [AC] that, for any given symmetry σ_0 , type τ_0 and isotropy ι_0 (if defined), the number of bilinear forms β with $\sigma(\beta) = \sigma_0$, $\tau(\beta) = \tau_0$ and $\iota(\beta) = \iota_0$ in a basis (β_i) of $\text{Bil}(S)^{\text{so}(V)}$ consisting of admissible forms is (8, 0)-, (0, 8)- and (4, 4)-periodic in (p, q) . By Proposition 4, this implies that for any given symmetry σ'_0 and isotropy ι'_0 (if defined), the number of $\wedge^k V$ -valued bilinear forms $\Gamma_{\beta_i}^k$ with $\sigma(\Gamma_{\beta_i}^k) = \sigma'_0$, and $\iota(\Gamma_{\beta_i}^k) = \iota'_0$ is (8, 0)-, (0, 8)- and (4, 4)-periodic in (p, q) .

Finally, we have the following shift formula:

$$N_k^\pm(s, n + 2k) = N_0^\pm(s, n) := N^\pm(s, n),$$

which we can write also as

$$\tilde{N}_k^\pm(p + k, q + k) = \tilde{N}_0^\pm(p, q).$$

This shift formula follows from the tables below.

In Table A.3 we describe a basis of $\text{Bil}(S)^{\text{so}(V)}$, which consists of admissible forms and indicate the values of the three invariants σ , τ and ι . In the three following tables we give the invariants σ and ι for the corresponding bases of $\text{Bil}^k(S)^{\text{so}(V)}$, $k = 1, 2, 3$ modulo 4, denoted for simplicity by the same symbols. Due to the above periodicity properties, we can calculate, from these tables, the values of the invariants for the corresponding bases of $\text{Bil}^k(S)^{\text{so}(V)}$ for all $k \in \mathbb{N}$ and $V = \mathbb{R}^{p,q}$.

For any $V = \mathbb{R}^{p,q}$ an explicit basis of $\text{Bil}^k(S)^{\text{so}(V)}$ consisting of admissible forms was constructed for $k = 0$ and $k = 1$, in terms of appropriate models of the spinor module in [AC]. It was proven there that any admissible V -valued bilinear form on S is of the form Γ_β^1 , where β is a linear combination of admissible scalar-valued forms. By Theorem 1 and Proposition 4, this result extends to $k > 1$, namely any admissible $\wedge^k V$ -valued

bilinear form is of the form Γ_β^k , where β is a linear combination of admissible scalar-valued forms. This provides a basis of the vector space $\text{Bil}^k(S)^{\text{so}(V)} \cong \text{Bil}(S)^{\text{so}(V)}$ consisting of admissible forms. The dimension of this space is equal to the dimension of the Schur algebra $\mathcal{C}(S)$, which depends only on the signature $s = p - q$ modulo 8, see [AC].

In the tables below we use the notation of [AC]. We use the fact that any pseudo-Euclidean vector space can be written as $V = V_1 \oplus V_2$, where $V_1 = \mathbb{R}^{r,r}$ and $V_2 = \mathbb{R}^{0,k}$ or $V_2 = \mathbb{R}^{k,0}$. Then [LM]

$$\mathcal{C}l(V) \cong \mathcal{C}l(V_1) \widehat{\otimes} \mathcal{C}l(V_2),$$

where $\widehat{\otimes}$ denotes the $\mathbb{Z}/2\mathbb{Z}$ -graded tensor product. Let S_1 be the spinor module of $\text{Spin}(V_1)$ and S_2 the spinor module of $\text{Spin}(V_2)$. Then we always have that $S_1 = S_1^+ + S_1^-$ is a $\mathbb{Z}/2\mathbb{Z}$ -graded module of the $\mathbb{Z}/2\mathbb{Z}$ -graded algebra $\mathcal{C}l(V_1)$. The spinor module S of $\text{Spin}(V)$ can be described in terms of S_1 and S_2 as follows, see Proposition 2.3 of [AC]. Consider first the case when $S_2 = S_2^+ + S_2^-$ is reducible. In this case S_2 is a $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathcal{C}l(V_2)$ -module and the spinor module $S = S_1 \widehat{\otimes} S_2$ of $\text{Spin}(V)$ is obtained as the $\mathbb{Z}/2\mathbb{Z}$ -graded tensor product of the modules S_1 and S_2 . It is again $\mathbb{Z}/2\mathbb{Z}$ -graded with even part $S_+ = S_1^+ \otimes S_2^+ + S_1^- \otimes S_2^-$ and odd part $S_- = S_1^+ \otimes S_2^- + S_1^- \otimes S_2^+$. If S_2 is an irreducible $\text{Spin}(V)$ -module, then the spinor module of $\text{Spin}(V)$ is given by

$$S = S_1 \widehat{\otimes} S_2 = S_1^+ \otimes S_2 + S_1^- \otimes S_2$$

with the action

$$(a \widehat{\otimes} b) \cdot (s_1^\pm \otimes s_2) = (-1)^{\text{deg}(s_1^\pm)\text{deg}(b)} a s_1^\pm \otimes b s_2,$$

where $a \in \mathcal{C}l(V_1)$, $b \in \mathcal{C}l(V_1)$, $s_1^\pm \in S_1^\pm$, $s_2 \in S_2$, $\text{deg}(s_1^+) = 0$ and $\text{deg}(s_1^-) = 1$. In this case S is an irreducible $\text{Spin}(V)$ -module.

As discussed in Sect. 5, for the case of split signature, $V = \mathbb{R}^{r,r}$, there exist two independent admissible bilinear forms f and $f_E = f(E \cdot, \cdot)$ on the spinor module, where $E|_{S_\pm} = \pm \text{Id}$. Their invariants are given in Table A.3. If V is positive or negative definite, then there exists a unique (up to scale) $\text{Pin}(V)$ -invariant scalar product g on the spinor module S and any admissible, hence invariant, bilinear form on S is of the form $g_A = g(A \cdot, \cdot)$, where A is an admissible element of the Schur algebra $\mathcal{C}(S)$. The admissibility of A means that A is either symmetric or skewsymmetric with respect to g , that it either commutes or anticommutes with the Clifford multiplication γ_v and that it preserves or interchanges S_+ and S_- if they exist [AC].

Returning to the general case $V = \mathbb{R}^{p,q} = V_1 \oplus V_2$, $S = S_1 \widehat{\otimes} S_2$, as above, the admissible bilinear forms on S can be described as follows. Let (g_{A_i}) be a basis of $\text{Bil}(S_2)^{\text{so}(V_2)}$ consisting of admissible elements. Inspection of Table A.3 shows that for any g_{A_i} there exists a unique element $\phi_i \in \{f, f_E\}$ which satisfies the condition $\tau(\phi_i) = \iota(g_{A_i})\tau(g_{A_i})$. By virtue of Proposition 3.4 of [AC], the tensor products $\phi_i \otimes g_{A_i}$ provide a basis of $\text{Bil}(S)^{\text{so}(V)}$ consisting of admissible elements. The corresponding basis $\Gamma_{\phi_i \otimes g_{A_i}}^k$ of $\text{Bil}^k(S)^{\text{so}(V)}$ and its invariants are tabulated below. For simplicity the symbols Γ^k and \otimes are omitted.

We use the following bases for the Schur algebra $\mathcal{C}(S)$ (see Table A.2). If the signature $s = 0, 1, 2$ or 4 , then $S = S_+ \oplus S_-$, and we put $E = \text{diag}(\text{Id}_{S_+}, \text{Id}_{S_-})$. In the cases $s = 1$ and $s = 6$ we denote by I the standard complex structure in $\mathcal{C}(S) = \mathbb{R}(2)$ and $\mathcal{C}(S) = \mathbb{C}$, respectively. In fact, in the case $s = 1$, $S = \mathbb{C}^N = \mathbb{R}^N \oplus \mathbb{R}^N$ has a $\mathcal{C}l(V)$ -invariant

Table A.2. Standard bases for the Schur algebras $\mathcal{C}(S)$

s	$\mathcal{C}(S)$	basis of $\mathcal{C}(S)$
0	$2\mathbb{R}$	Id, E
1	$\mathbb{R}(2)$	Id, E, I, EI
2	$\mathbb{C}(2)$	Id, I, J, K, E, EI, EJ, EK
3	\mathbb{H}	Id, I, J, K
4	$2\mathbb{H}$	Id, I, J, K, E, EI, EJ, EK
5	\mathbb{H}	Id, I, J, K
6	\mathbb{C}	Id, I
7	\mathbb{R}	Id

complex structure. In the cases $s = 2, 3, 4$ and 5 , we denote by $I, J, K = IJ \in \mathcal{C}(S)$ the canonical $\mathcal{C}l^0(V)$ -invariant hypercomplex structure of S (3 anticommuting complex structures). In fact, in the case $s = 4$, $S = \mathbb{H}^{N/2} = \mathbb{H}^{N/4} \oplus \mathbb{H}^{N/4}$ has a $\mathcal{C}l(V)$ -invariant hypercomplex structure.

B. Reformulation for Physicists

In this appendix, we reformulate our results in a language that may be more familiar to physicists. It is useful first to review some properties of Clifford algebras, in particular those that concern the real Clifford algebras.

B.1. Complex and real Clifford algebras. We use here the terminology of Clifford algebras, spinors and gamma matrices as used in physics. Results for the real case are dependent on the signature. We remark that in the main text Clifford algebras have been taken with a minus sign:

$$\gamma^a \gamma^b + \gamma^b \gamma^a = -2\eta^{ab}. \tag{B.1}$$

The signature s has been introduced as $p - q$ modulo 8, where p and q are the number of $+1$, respectively -1 eigenvalues of the metric η^{ab} . Thus,

$$\begin{aligned} p &= \text{number of negative eigenvalues of } (\gamma^a \gamma^b + \gamma^b \gamma^a), \\ q &= \text{number of positive eigenvalues of } (\gamma^a \gamma^b + \gamma^b \gamma^a), \\ s &= p - q \pmod{8}, \quad n = p + q. \end{aligned} \tag{B.2}$$

This is important in order to interpret Table 1.

The bilinear form β corresponds to the charge conjugation matrix \mathcal{C} , or for spinors s and t , we have $\beta(s, t) = s^T \mathcal{C}t$. If $v = e^a$, a basis vector, then the operation $\gamma(v)$ is the gamma matrix $\gamma(v = e^a) = \gamma^a$. The invariant $\sigma(\mathcal{C})$ indicates the symmetry of \mathcal{C} , while $\tau(\mathcal{C})$ indicates the symmetry of $\mathcal{C}\gamma^a$:

$$\mathcal{C}^T = \sigma(\mathcal{C}) \mathcal{C}, \quad (\mathcal{C}\gamma^a)^T = \tau(\mathcal{C}) \mathcal{C}\gamma^a. \tag{B.3}$$

Table A.3. Admissible bilinear forms β on S and their invariants (σ, τ, ι)

$p \backslash q$		0		1		2		3
0	f	$+-+$	g	$++$	fg	$--$	f_{Eg}	$-+$
	f_E	$+++$			f_{EgI}	$++$	f_{EgI}	$++$
1	g	$+-+$	f	$---$	g	$--$	f_{EgJ}	$--$
	g_E	$+++$	f_E	$++-$			f_{EgK}	$--$
	g_I	$---$					f_{Eg}	$-+$
	g_{IE}	$++-$					f_{gI}	$--$
2	g	$+-+$	f_{Eg}	$++-$	f	$---+$	g	$-+$
	g_I	$---+$	f_{EgI}	$---+$	f_E	$---+$		
	g_J	$---$	f_{gE}	$---$				
	g_K	$---$	f_{gIE}	$---+$				
	g_E	$+++$						
	g_{IE}	$-++$						
	g_{JE}	$++-$						
3	g	$+-$	f_{Eg}	$++-$	fg	$---+$	f	$+-$
	g_I	$--$	f_{EgI}	$-+-$	f_{gI}	$+-$	f_E	$-+-$
	g_J	$--$	f_{EgJ}	$-++$	f_{EgE}	$++$		
	g_K	$--$	f_{EgK}	$-++$	f_{EgIE}	$-+-$		
			f_{gE}	$---$				
			f_{gIE}	$+-$				
			f_{gJE}	$-+$				
4	g	$+-+$	f_{Eg}	$++$	fg	$---+$	f_{Eg}	$-+-$
	g_I	$---+$	f_{EgI}	$-+$	f_{gI}	$+-+$	f_{EgI}	$+++$
	g_J	$---+$	f_{EgJ}	$-+$	f_{gJ}	$+-$	f_{gE}	$+-$
	g_K	$---+$	f_{EgK}	$-+$	f_{gK}	$+-$	f_{gIE}	$+-+$
	g_E	$+++$			f_{EgE}	$-++$		
	g_{IE}	$-++$			f_{EgIE}	$+++$		
	g_{JE}	$-++$			f_{EgJE}	$-+-$		
	g_{KE}	$-++$			f_{EgKE}	$-+-$		
5	g	$+-$	f_{Eg}	$++-$	fg	$--$	f_{Eg}	$-+-$
	g_I	$--$	f_{EgI}	$-+-$	f_{gI}	$+-$	f_{EgI}	$+-$
	g_J	$-+$	f_{EgJ}	$-+-$	f_{gJ}	$+-$	f_{EgJ}	$+++$
	g_K	$-+$	f_{EgK}	$-+-$	f_{gK}	$+-$	f_{EgK}	$+++$
			f_{gE}	$---$			f_{gE}	$+-$
			f_{gIE}	$+-$			f_{gIE}	$---$
			f_{gJE}	$+-$			f_{gJE}	$+-+$
6	g	$+-$	f_{Eg}	$++$	fg	$---+$	f_{Eg}	$-+$
	g_I	$-+$	f_{EgI}	$-+$	f_{gI}	$+-+$	f_{EgI}	$++$
			f_{gJ}	$+-$	f_{gJ}	$+-+$	f_{EgJ}	$++$
			f_{gI}	$+-$	f_{gK}	$+-+$	f_{EgK}	$++$
					f_{EgE}	$-++$		
					f_{EgIE}	$+++$		
					f_{EgJE}	$+++$		
7	g	$+-$	f_{Eg}	$++$	fg	$--$	f_{Eg}	$-+-$
			f_{gI}	$+-$	f_{gI}	$+-$	f_{EgI}	$+-$
					f_{gE}	$++$	f_{EgJ}	$+-$
					f_{EgI}	$++$	f_{EgK}	$+-$
							f_{gE}	$+-$
							f_{gIE}	$---$
							f_{gJE}	$---$
						f_{gKE}	$---$	

Table A.4. $\wedge^k V$ -valued admissible bilinear forms on S and their invariants (σ, ι) for $k \equiv 1(4)$

$p \backslash q$		0		1		2		3
0	f	--	g	+	fg	+	f_{Eg}	-
	f_E	+-			f_{EgI}	+	f_{EgI}	+
1	g	--	f	++	g	+	f_{Eg}	-
	g_E	+-	f_E	++			f_{gI}	+
	g_I	++						
	g_{IE}	++						
2	g	--	f_{Eg}	++	f	+-	g	-
	g_I	+-	f_{EgI}	--	f_E	--		
	g_J	++	f_{gE}	++				
	g_K	++	f_{gIE}	+-				
	g_E	+-						
	g_{IE}	--						
	g_{JE}	++						
3	g	-	f_{Eg}	++	fg	+-	f	--
	g_I	+	f_{EgI}	+-	f_{gI}	--	f_E	--
	g_J	+	f_{EgJ}	--	f_{EgE}	--		
	g_K	+	f_{EgK}	--	f_{EgIE}	--		
			f_{gE}	++				
			f_{gIE}	+-				
			f_{gJE}	+-				
4	g	--	f_{Eg}	+	fg	+-	f_{Eg}	--
	g_I	+-	f_{EgI}	-	f_{gI}	--	f_{EgI}	+-
	g_J	+-	f_{EgJ}	-	f_{gJ}	+-	f_{gE}	--
	g_K	+-	f_{EgK}	-	f_{gK}	+-	f_{gIE}	--
	g_E	+-			f_{EgE}	--		
	g_{IE}	--			f_{EgIE}	+-		
	g_{JE}	--			f_{EgJE}	+-		
	g_{KE}	--			f_{EgKE}	+-		
	5	g	-	f_{Eg}	++	fg	+	f_{Eg}
g_I		+	f_{EgI}	+-	f_{gI}	-	f_{EgI}	++
g_J		-	f_{EgJ}	+-	f_{gJ}	-	f_{EgJ}	+-
g_K		-	f_{EgK}	+-	f_{gK}	-	f_{EgK}	+-
			f_{gE}	++			f_{gE}	--
			f_{gIE}	+-			f_{gIE}	++
			f_{gJE}	+-			f_{gJE}	--
6	g	-	f_{Eg}	+	fg	+-	f_{Eg}	-
	g_I	-	f_{EgI}	-	f_{gI}	--	f_{EgI}	+
			f_g	-	f_{gJ}	--	f_{EgJ}	+
			f_{gI}	-	f_{gK}	--	f_{EgK}	+
					f_{EgE}	--		
					f_{EgIE}	+-		
					f_{EgJE}	+-		
7	g	-	f_{Eg}	+	fg	+	f_{Eg}	--
			f_{gI}	-	f_{gI}	-	f_{EgI}	++
					f_{Eg}	+	f_{EgJ}	++
					f_{EgI}	+	f_{EgK}	++
							f_{gE}	+-
							f_{gIE}	++
							f_{gJE}	++
						f_{gKE}	++	

Table A.5. $\wedge^k V$ -valued admissible bilinear forms on S and their invariants (σ, ι) for $k \equiv 2(4)$

$p \backslash q$		0		1		2		3
0	f	-+	g	-	fg	+	$f_{E\mathcal{E}}$	+
	f_E	-+			$f_{E\mathcal{E}I}$	-	$f_{E\mathcal{E}I}$	-
							$f_{E\mathcal{E}J}$	+
1	g	-+	f	+ -	g	+	$f_{E\mathcal{E}}$	+
	g_E	-+	f_E	--			$f_{\mathcal{E}I}$	+
	g_I	+-						
	g_{IE}	--						
2	g	-+	$f_{E\mathcal{E}}$	--	f	++	g	+
	g_I	++	$f_{E\mathcal{E}I}$	++	f_E	++		
	g_J	+ -	$f_{\mathcal{E}E}$	+ -				
	g_K	+ -	$f_{\mathcal{E}IE}$	++				
	g_E	-+						
	g_{IE}	++						
	g_{JE}	--						
3	g	-	$f_{E\mathcal{E}}$	--	fg	++	f	--
	g_I	+	$f_{E\mathcal{E}I}$	++	$f_{\mathcal{E}I}$	--	f_E	+ -
	g_J	+	$f_{E\mathcal{E}J}$	++	$f_{E\mathcal{E}E}$	++		
	g_K	+	$f_{E\mathcal{E}K}$	++	$f_{E\mathcal{E}IE}$	+ -		
			$f_{\mathcal{E}E}$	+ -				
			$f_{\mathcal{E}IE}$	--				
			$f_{\mathcal{E}JE}$	++				
			$f_{\mathcal{E}KE}$	++				
4	g	-+	$f_{E\mathcal{E}}$	-	fg	++	$f_{E\mathcal{E}}$	+ -
	g_I	++	$f_{E\mathcal{E}I}$	+	$f_{\mathcal{E}I}$	-+	$f_{E\mathcal{E}I}$	-+
	g_J	++	$f_{E\mathcal{E}J}$	+	$f_{\mathcal{E}J}$	--	$f_{\mathcal{E}E}$	--
	g_K	++	$f_{E\mathcal{E}K}$	+	$f_{\mathcal{E}K}$	--	$f_{\mathcal{E}IE}$	-+
	g_E	-+			$f_{E\mathcal{E}E}$	++		
	g_{IE}	++			$f_{E\mathcal{E}IE}$	-+		
	g_{JE}	++			$f_{E\mathcal{E}JE}$	+ -		
	g_{KE}	++			$f_{E\mathcal{E}KE}$	+ -		
			$f_{E\mathcal{E}}$	--	$f_{\mathcal{E}}$	+	$f_{E\mathcal{E}}$	+ -
5	g_I	+	$f_{E\mathcal{E}I}$	+ -	$f_{\mathcal{E}I}$	-	$f_{E\mathcal{E}I}$	--
	g_J	+	$f_{E\mathcal{E}J}$	+ -	$f_{\mathcal{E}J}$	-	$f_{E\mathcal{E}J}$	-+
	g_K	+	$f_{E\mathcal{E}K}$	+ -	$f_{\mathcal{E}K}$	-	$f_{E\mathcal{E}K}$	-+
			$f_{\mathcal{E}E}$	+ -			$f_{\mathcal{E}E}$	--
			$f_{\mathcal{E}IE}$	--			$f_{\mathcal{E}IE}$	+ -
			$f_{\mathcal{E}JE}$	--			$f_{\mathcal{E}JE}$	-+
			$f_{\mathcal{E}KE}$	--			$f_{\mathcal{E}KE}$	-+
			$f_{E\mathcal{E}}$	-	fg	++	$f_{E\mathcal{E}}$	+
6	g_I	+	$f_{E\mathcal{E}I}$	+	$f_{\mathcal{E}I}$	-+	$f_{E\mathcal{E}I}$	-
			$f_{\mathcal{E}}$	-	$f_{\mathcal{E}J}$	-+	$f_{E\mathcal{E}J}$	-
			$f_{\mathcal{E}I}$	-	$f_{\mathcal{E}K}$	-+	$f_{E\mathcal{E}K}$	-
					$f_{E\mathcal{E}E}$	++		
					$f_{E\mathcal{E}IE}$	-+		
					$f_{E\mathcal{E}JE}$	-+		
					$f_{E\mathcal{E}KE}$	-+		
			$f_{E\mathcal{E}}$	-	fg	+	$f_{E\mathcal{E}}$	+ -
			$f_{\mathcal{E}I}$	-	$f_{\mathcal{E}I}$	-	$f_{E\mathcal{E}I}$	--
7	g	-	$f_{E\mathcal{E}}$	-	fg	+	$f_{E\mathcal{E}}$	+ -
			$f_{\mathcal{E}I}$	-	$f_{\mathcal{E}I}$	-	$f_{E\mathcal{E}I}$	--
					$f_{\mathcal{E}}$	-	$f_{E\mathcal{E}J}$	--
					$f_{E\mathcal{E}I}$	-	$f_{E\mathcal{E}K}$	--
							$f_{\mathcal{E}E}$	--
							$f_{\mathcal{E}IE}$	+ -
							$f_{\mathcal{E}JE}$	+ -
							$f_{\mathcal{E}KE}$	+ -

Table A.6. $\wedge^k V$ -valued admissible bilinear forms on S and their invariants (σ, ι) for $k \equiv 3(4)$

$p \backslash q$		0		1		2		3	
0	f	+-	g	-	fg	-	f_{Eg}	+	
	f_E	--			f_{EgI}	-	f_{EgI}	-	
1	g	+-	f	-+	g	-	f_{Eg}	+	
	g_E	--	f_E	-+			f_{gI}	-	
	g_I	+-					f_{EgJ}	-	
	g_{IE}	+-					f_{EgK}	-	
2	g	+-	f_{Eg}	-+	f	--	g	+	
	g_I	--	f_{EgI}	+-	f_E	+-			
	g_J	+-	f_{gE}	+-					
	g_K	+-	f_{gIE}	--					
	g_E	--							
	g_{IE}	+-							
	g_{JE}	+-							
	g_{KE}	+-							
3	g	+	f_{Eg}	-+	fg	--	f	++	
	g_I	-	f_{EgI}	++	f_{gI}	++	f_E	++	
	g_J	-	f_{EgJ}	+-	f_{EgE}	+-			
	g_K	-	f_{EgK}	+-	f_{EgIE}	++			
			f_{gE}	-+					
			f_{gIE}	++					
			f_{gJE}	--					
			f_{gKE}	--					
4	g	+-	f_{Eg}	-	fg	--	f_{Eg}	++	
	g_I	--	f_{EgI}	+	f_{gI}	+-	f_{EgI}	--	
	g_J	--	f_{EgJ}	+	f_{gJ}	++	f_{gE}	++	
	g_K	--	f_{EgK}	+	f_{gK}	++	f_{gIE}	+-	
	g_E	--			f_{EgE}	+-			
	g_{IE}	+-			f_{EgIE}	--			
	g_{JE}	+-			f_{EgJE}	++			
	g_{KE}	+-			f_{EgKE}	++			
	5	g	+	f_{Eg}	-+	fg	-	f_{Eg}	++
		g_I	-	f_{EgI}	++	f_{gI}	+	f_{EgI}	+-
g_J		+	f_{EgJ}	++	f_{gJ}	+	f_{EgJ}	--	
g_K		+	f_{EgK}	++	f_{gK}	+	f_{EgK}	--	
			f_{gE}	-+			f_{gE}	++	
			f_{gIE}	++			f_{gIE}	-+	
			f_{gJE}	++			f_{gJE}	+-	
			f_{gKE}	++			f_{gKE}	+-	
6	g	+	f_{Eg}	-	fg	--	f_{Eg}	+	
	g_I	+	f_{EgI}	+	f_{gI}	+-	f_{EgI}	-	
			fg	+	f_{gJ}	+-	f_{EgJ}	-	
			f_{gI}	+	f_{gK}	+-	f_{EgK}	-	
					f_{EgE}	+-			
					f_{EgIE}	--			
					f_{EgJE}	--			
					f_{EgKE}	--			
7	g	+	f_{Eg}	-	fg	-	f_{Eg}	++	
			f_{gI}	+	f_{gI}	+	f_{EgI}	-+	
					f_{Eg}	-	f_{EgJ}	-+	
					f_{EgI}	-	f_{EgK}	-+	
							f_{gE}	++	
							f_{gIE}	-+	
							f_{gJE}	-+	
							f_{gKE}	-+	

When we can define chiral spinors, called semi-spinors here, the invariant ι indicates whether the charge conjugation matrix maps between spinors of equal chirality $\iota(\beta) = 1$ or different chirality $\iota(\beta) = -1$.

For **complex gamma matrices**, there are many references, and one can compare e.g. with [VP]. The invariants σ and τ and ι are related to the two numbers ϵ and η of [VP] as

$$\sigma(\mathcal{C}) = -\epsilon, \quad \tau(\mathcal{C}) = -\eta. \tag{B.4}$$

The main results depend on the dimension n . For n **odd**, there is one charge conjugation matrix (i.e.1 bilinear form \mathcal{C}) and

$$\begin{aligned} n = 1 \pmod 8 : & \quad \sigma(\mathcal{C}) = 1, \quad \tau(\mathcal{C}) = 1, \\ n = 3 \pmod 8 : & \quad \sigma(\mathcal{C}) = -1, \quad \tau(\mathcal{C}) = -1, \\ n = 5 \pmod 8 : & \quad \sigma(\mathcal{C}) = -1, \quad \tau(\mathcal{C}) = 1, \\ n = 7 \pmod 8 : & \quad \sigma(\mathcal{C}) = 1, \quad \tau(\mathcal{C}) = -1. \end{aligned} \tag{B.5}$$

For **even** n we can define a charge conjugation matrix for either sign of τ . We define

$$\gamma_* \equiv (-i)^{n/2+p} \gamma_1 \dots \gamma_n, \quad \gamma_* \gamma_* = 1. \tag{B.6}$$

Now, if \mathcal{C} is a good charge conjugation matrix, then $\mathcal{C}' = \mathcal{C} \gamma_*$ is a charge conjugation matrix as well, with $\tau(\mathcal{C}') = -\tau(\mathcal{C})$. The value of σ is

$$\begin{aligned} n = 0 \pmod 8 : & \quad \sigma(\mathcal{C}) = 1, \quad n = 2 \pmod 8 : \sigma(\mathcal{C}) = \tau(\mathcal{C}), \\ n = 4 \pmod 8 : & \quad \sigma(\mathcal{C}) = -1, \quad n = 6 \pmod 8 : \sigma(\mathcal{C}) = -\tau(\mathcal{C}). \end{aligned} \tag{B.7}$$

Using $(1 \pm \gamma_*)/2$, we can define chiral spinors in this case, and we find

$$\iota(\mathcal{C}) = \overleftarrow{n} \equiv (-1)^{n(n-1)/2}. \tag{B.8}$$

Here, \overleftarrow{n} is the sign change on reversing n indices of an antisymmetric tensor.

In this paper we more often make use of **real Clifford algebras**. Explicit results on real Clifford algebras can be found in [O]. Here we give some key results. Only in the cases $s = 0, 6, 7$, can the matrices of the complex Clifford algebra (of dimension $2^{\lfloor n/2 \rfloor}$, where the Gauss bracket $\lfloor x \rfloor$ denotes the integer part of x) be chosen to be real. This is called the *normal type*.

In the other cases, we can get real matrices of dimension twice that of the complex Clifford algebra. Many representations contain only pure real or pure imaginary gamma matrices. A simple way to obtain real matrices of double dimension is to use the matrices

$$\Gamma^a = \gamma^a \otimes \mathbb{1}_2 \quad \text{if } \gamma^a \text{ is real,} \quad \Gamma^a = \gamma^a \otimes \sigma_2 \quad \text{if } \gamma^a \text{ is imaginary.} \tag{B.9}$$

In the cases $s = 1, 5$, called the *almost complex type*, there is one matrix that commutes with all gamma matrices. It is

$$J \equiv \Gamma^1 \dots \Gamma^n, \quad J^2 = -\mathbb{1}. \tag{B.10}$$

Note that in the complex case the product of all the γ^a 's is proportional to the identity for odd dimensions, but this is not so for these larger and real Γ^a 's. (For $s = 3, 7$ the

product of all real gamma matrices is also $\pm \mathbb{1}$). In this case there is always a charge conjugation matrix C with¹

$$\sigma(C) = \sigma(\mathcal{C}), \quad \tau(C) = -1, \quad (\text{B.11})$$

where \mathcal{C} indicates the charge conjugation matrix for the complex case. There is also a matrix D that satisfies the properties

$$D\Gamma^a + \Gamma^a D = 0, \quad D^T = C D C^{-1}, \quad \begin{aligned} D^2 &= \mathbb{1} & \text{if } s = 1, \\ D^2 &= -\mathbb{1} & \text{if } s = 5. \end{aligned} \quad (\text{B.12})$$

In the remaining cases, $s = 2, 3, 4$, called the *quaternionic type*, there are 3 matrices, which commute with all the Γ^a 's, denoted E_i for $i = 1, 2, 3$. They satisfy

$$[E_i, \Gamma^a] = 0, \quad E_i E_j = -\delta_{ij} \mathbb{1} + \varepsilon_{ijk} E_k, \quad E_i^T = -C E_i C^{-1}, \quad (\text{B.13})$$

where a charge conjugation matrix C is used that satisfies

$$\sigma(C) = -\sigma(\mathcal{C}), \quad \tau(C) = \tau(\mathcal{C}). \quad (\text{B.14})$$

With these properties, we can obtain the following consequences for bilinear forms:

$s = 0, 6$. We have the normal type. The two charge conjugation matrices of the complex Clifford algebra can be used (possibly multiplied by i to make them real, but an overall factor is not important), having opposite values of τ . For σ one can use (B.7). For $s = 0$ there is no imaginary factor in (B.6), and thus γ_* is a real matrix that can be used to define real chiral spinors (*Majorana-Weyl spinors*). The value of ι is then as in the complex case, see (B.8). For $s = 6$ there is no projection possible in this real case. The fact that the Clifford algebra is real reflects that the irreducible spinors are *Majorana spinors*.

$s = 7$. The real Clifford representation is also of the normal type. With the odd dimension there is only one charge conjugation matrix, and no chiral projection. The values of σ and τ are as in (B.5). Again, the reality reflects the property of *Majorana spinors*.

$s = 1, 5$. The real Clifford representation is of the 'almost complex type'. We have 4 choices for the charge conjugation matrix: $C, C J, C D$ and $C D J$. We can derive from the given properties that

$$\begin{aligned} \sigma(\mathcal{C}) = \sigma(C) &= -\overleftarrow{n} \sigma(C J) = \sigma(C D) = \overleftarrow{n} \sigma(C D J), \\ \tau(C) = \tau(C J) &= -1, \quad \tau(C D) = \tau(C D J) = 1. \end{aligned} \quad (\text{B.15})$$

If $s = 1$ then (B.12) says that $\frac{1}{2}(\mathbb{1} \pm D)$ are good projection operators, and can be used to define semispinors. These (real) semispinors have the same dimension as the original complex ones and are the *Majorana spinors*. It is straightforward to check that

$$\iota(C) = \iota(D) = 1, \quad \iota(C J) = \iota(C D J) = -1. \quad (\text{B.16})$$

¹ In this explanation of properties of real Clifford algebras, we will always denote by C a specific choice of charge conjugation matrix for the real Clifford algebra, and by \mathcal{C} the one for the complex gamma matrices. In general we use \mathcal{C} for any choice of charge conjugation matrix.

If $s = 5$ no such projection is possible. The size of the spinor representation is doubled by the procedure (B.9), and this reflects the fact that we have *symplectic-Majorana* spinors.

$s = 2, 4$. The real Clifford representation is of the quaternionic type. With dimension even, we start from the two charge conjugation matrices of the complex case. For each of them, we can construct 3 extra ones by multiplying with the imaginary units E_i , bringing the total to 8 invariant bilinear forms. From (B.13) and (B.14) it follows that

$$\begin{aligned} \sigma(\mathcal{C}) &= -\sigma(C) = \sigma(CE_i), \\ \tau(\mathcal{C}) &= \tau(C) = \tau(CE_i). \end{aligned} \tag{B.17}$$

The definition of chiral spinors as in the complex case is only possible if γ_* in (B.6) is real. Thus if $\frac{1}{2}n + p$ is even, i.e. $s = 4$, the product of all the Γ^a 's is a good chiral projection operator. The projected spinors are the components of *symplectic Majorana-Weyl spinors*. If γ_* is imaginary, i.e. $s = 2$, the product of all the Γ^a 's squares to $-\mathbb{1}$. Using one of the complex structures, say E_1 , then gives chiral projections of the form $\frac{1}{2}(\mathbb{1} \pm i\Gamma_* E_1)$. In this case

$$\iota(C) = \iota(CE_1) = -\iota(\mathcal{C}), \quad \iota(CE_2) = \iota(CE_3) = \iota(\mathcal{C}). \tag{B.18}$$

The projected spinors are the components of *Majorana spinors*.

$s = 3$. Here also, the real Clifford algebra is of the quaternionic type, but since the dimension is odd, there is only one charge conjugation matrix in the complex Clifford algebra. For this, (B.17) applies. There is no chiral projection, and the components correspond to *symplectic Majorana spinors*.

The results can be seen in Table A.3 (though the names for the different bilinear forms are unrelated to what has been explained here). Table A.1 gives the number of solutions for charge conjugation matrices that have $\beta = 1, \beta = -1$.

The map Γ_{β}^k in the main text corresponds to the mapping from two spinors s and t to the form with components $s^T \mathcal{C} \Gamma_{a_1 \dots a_k} t$ (where \mathcal{C} denotes now any choice as explained in footnote 1), and the number $\sigma(\Gamma_{\mathcal{C}}^k)$ gives the symmetry of this bispinor (for commuting spinors) under interchange of s and t , while $\iota(\Gamma_{\mathcal{C}}^k)$ tells whether s and t have the same chirality. They are related to $\sigma(\mathcal{C}), \tau(\mathcal{C})$ and $\iota(\mathcal{C})$ by (1.4) and (1.5):

$$\sigma(\Gamma_{\mathcal{C}}^k) = \sigma(\mathcal{C}) \tau^k(\mathcal{C}) \overleftarrow{k}, \quad \iota(\Gamma_{\mathcal{C}}^k) = (-)^k \iota(\mathcal{C}). \tag{B.19}$$

For real Clifford algebras they are given explicitly in Tables A.4– A.6.

B.2. Summary of the results for the algebras. This paper treats algebras that consist of an even sector $\mathfrak{g}_0 = \mathfrak{so}(p, q) + W_0$, and an odd sector $\mathfrak{g}_1 = W_1$ consisting of a representation of $\mathfrak{so}(p, q)$. The group $\mathfrak{so}(p, q)$ is denoted as $\mathfrak{so}(V)$, and V denotes its vector representation. We consider either the usual case where the odd generators are fermionic ($\epsilon = 1$, and we have a superalgebra), or they can be bosonic ($\epsilon = -1$, and we have a ‘ \mathbb{Z}_2 -graded Lie algebra’). We will use the word ‘commutator’ in all cases, though this is obviously an anticommutator for $[W_1, W_1]$ in the superalgebra case. These algebras are called ϵ -**extensions of $\mathfrak{so}(V)$** . We use the following terminologies for special cases:

Poincaré superalgebras or Lie algebras: W_0 are the translations in n dimensions ($n = p + q$), which are denoted by V [and thus $\mathfrak{g}_0 = \mathfrak{so}(V) + V$] and W_1 is a spinorial representation.

Algebra of translational type: all generators in $[W_1, W_1]$ belong to W_0 :

$$[W_1, W_1] \subset W_0, \quad [W_1, W_0] = 0, \quad [W_0, W_0] = 0. \tag{B.20}$$

This part $W_0 + W_1$ is called the ‘algebra of generalized translations’.

Transalgebra: algebra of translational type where all the generators of W_0 appear in $[\mathfrak{g}_1, \mathfrak{g}_1]$, i.e.

$$[W_1, W_1] = W_0. \tag{B.21}$$

ϵ -extended polyvector Poincaré algebras: Algebra of translational type where W_1 is a (possibly reducible) spinorial representation (includes chiral and extended supersymmetry).

There are 2 extreme cases: one in which the full \mathfrak{g} is semisimple, which is the case of the Nahm superalgebras, and the **algebras of semi-direct type**, where $\mathfrak{so}(p, q)$ is its largest semisimple subalgebra.

Apart from degenerate cases where $n \leq 2$, any transalgebra is of semi-direct type.

Transalgebras are *minimal* cases of algebras of translational type in the sense that there are no proper subalgebras, see Definition 1. In fact, any algebra of translational type can be written as $\mathfrak{g} = \mathfrak{g}' + \mathfrak{a}$, where $\mathfrak{a} \subset W_0$ is an $\mathfrak{so}(p, q)$ representation, which is irrelevant in the sense that all its generators commute with all of W_1 and W_0 and do not appear in $[W_1, W_1]$. The algebra \mathfrak{g}' is a transalgebra.

For any choice of W_1 there is a unique transalgebra where W_0 has all the $\mathfrak{so}(p, q)$ representations that appear in the (anti)symmetric product of W_1 with itself. The (anti)commutators of W_1 are then

$$[W_1, W_1] = \sum_{\text{all } r} W_0^{(r)}. \tag{B.22}$$

Here r labels all representations that appear in the symmetric product for $\epsilon = 1$, i.e. superalgebras, and in the antisymmetric product for $\epsilon = -1$, i.e. for Lie algebras.

Any other transalgebra can be obtained by removing an arbitrary number of terms in (B.22). We can consider these to be contractions of this basic transalgebra, where the representations to be removed are multiplied by some parameter t and the limit $t \rightarrow 0$ is taken.

Any ϵ -extension of semi-direct type with W_1 irreducible and of dimension at least 3, is of the following form:

$$\begin{aligned} W_0 &= A + K, & [K, W_1] &= 0, & [A, W_1] &= \rho W_1, \\ [\mathfrak{so}(V), A] &= 0, & [A, A] &\subset K, & [W_1, W_1] &\subset K, \end{aligned} \tag{B.23}$$

where $\rho = 0$ or $\mathbb{R}\cdot\text{Id}$ or $\mathbb{C}\cdot\text{Id}$. This is thus a transalgebra iff $A = 0$. Also, if the algebra is minimal, then $[W_1, W_1] = W_0$ and it is a transalgebra.

Table B.1. The values of k in (B.24) for the case of complex spinors. n is the dimension of the vector space. i can be 0, 1, . . . limited by the fact that obviously $k \geq 0$. For the even-dimensional case, we split the (anti)commutator between spinors of different and equal chirality. For equal chirality, the $k = m$ generator is either selfdual or antiselfdual

	Superalgebra: $\sigma = +1$	Lie algebra: $\sigma = -1$
$n = 2m + 1$	$k = m - 4i$ $k = m - 3 - 4i$	$k = m - 1 - 4i$ $k = m - 2 - 4i$
$n = 2m$		
$[S_{\pm}, S_{\pm}]$	$k = m - 4 - 4i$ $k = m$ (anti)selfdual	$k = m - 2 - 4i$
$[S_+, S_-]$	$k = m - 1 - 2i$	$k = m - 1 - 2i$

Also, if W_0 and W_1 are irreducible $\mathfrak{so}(p, q)$ representations, then either the algebra is of translational type, i.e. $[W_0, W] = 0$, or W_0 is an abelian generator and $[W_0, W_1] = a W_1$, where a is a number.

We now restrict ourselves to transalgebras where $W_1 = S$, the irreducible spinor representation of $\mathfrak{so}(p, q)$. Then the representations that appear in the right-hand side of (B.22) are either k -forms or, in the case that $s = p - q$ is divisible by 4, also (anti-)selfdual $(n/2)$ -forms. Thus, the unique maximal transalgebra has (anti)commutators

$$[S_{\alpha}, S_{\beta}] = \sum_k (\mathbb{C}\Gamma^{a_1 \dots a_k})_{\alpha\beta} W_0^k{}_{a_1 \dots a_k}, \tag{B.24}$$

where α, β denote spinor indices. The classification of transalgebras with $W_1 = S$ reduces to the description of all the charge conjugation matrices \mathcal{C} and the specification of the range of the summation over k . The relevant issue is the symmetry for a particular k , i.e. the $\sigma(\Gamma_{\mathcal{C}}^k)$ of the previous subsection. When there are chiral spinors involved, the chirality should be respected, which is related to $\iota(\Gamma_{\mathcal{C}}^k)$.

First, in Sect. 5, the complex case is discussed. That means that there are no reality conditions on bosonic or fermionic generators. When the dimension is odd, the result is given in Theorem 2. There is only one charge conjugation matrix, and the result can be understood from (B.19) and (B.5). For even dimensions the result is given in Theorem 3. This depends mainly on (B.7) and (B.8). Here the spinors can be split into chiral spinors, and we can separately consider the commutators between spinor generators of the same and of opposite chirality. The result for allowed values of k in (B.24) can be found also in Table B.1.

As an example we may check that in 11 dimensions we can indeed have P, Z^{ab} and $Z^{a_1 \dots a_5}$ generators in W_0 , as is the case of the M-algebra, and the classification implies that we can consistently put any one of these to zero.

For the case of real generators, it is important to note that (anti)selfdual tensors in even dimensions are only consistent for $s/2$ even. We now discuss the algebras according to the 8 different values of s . The results are shown in Table B.2.

$s = 0$ (Majorana-Weyl spinors). There are chiral spinors and we can split the commutators. The k values that appear in Tables A.3–A.6 with $\iota = 1$ can appear in commutators of equal chirality. The value of σ indicates whether they appear in superalgebras ($\sigma = 1$) or in Lie algebras ($\sigma = -1$). Those with $\iota = -1$ appear in the same way in commutators of

Table B.2. The values of k in (B.24) for the case of real spinors. n is the dimension of the vector space. i can be 0, 1, . . . limited by the fact that obviously $k \geq 0$ (and $k \leq n$ for $s = 2, 6$). In cases where there are real Weyl spinors, we split the (anti)commutator between spinors of different and equal chirality, and the $k = m$ generator is either selfdual or antiselfdual. When there are symplectic spinors, the right-hand side of (B.24) contains for some k 's triplets of the automorphism group $\mathfrak{su}(2)$, and singlets for other k 's. The types of real spinors, Majorana (M), symplectic-Majorana (SM), or symplectic Majorana-Weyl (SMW) are indicated

	Superalgebra : $\sigma = +1$	Lie algebra : $\sigma = -1$	
$n = 2m + 1$			
$s = 1, 7(\text{M})$	$k = m - 4i$	$k = m - 1 - 4i$	
	$k = m - 3 - 4i$	$k = m - 2 - 4i$	
$s = 3, 5(\text{SM})$	$k = m - 4i$	triplet $k = m - 4i$	singlet
	$k = m - 3 - 4i$	triplet $k = m - 3 - 4i$	singlet
	$k = m - 1 - 4i$	singlet $k = m - 1 - 4i$	triplet
	$k = m - 2 - 4i$	singlet $k = m - 2 - 4i$	triplet
$n = 2m$			
$s = 0(\text{MW})$			
$[S_{\pm}, S_{\pm}]$	$k = m - 4 - 4i$	$k = m - 2 - 4i$	
	$k = m$ (anti)selfdual		
$[S_+, S_-]$	$k = m - 1 - 2i$	$k = m - 1 - 2i$	
$s = 2, 6(\text{M})$	$k = m - 4i, m + 4 + 4i$	$k = m - 1 - 4i, m + 3 + 4i$	
	$k = m - 3 - 4i, m + 1 + 4i$	$k = m - 2 - 4i, m + 2 + 4i$	
$s = 4(\text{SMW})$			
$[S_{\pm}, S_{\pm}]$	$k = m - 4 - 4i$	triplet $k = m - 2 - 4i$	triplet
	$k = m$ (anti)selfdual	triplet $k = m$ (anti)selfdual	singlet
	$k = m - 2 - 4i$	singlet $k = m - 4 - 4i$	singlet
$[S_+, S_-]$	$k = m - 1 - 2i$	2×2 $k = m - 1 - 2i$	2×2

different chirality. The (anti)selfdual tensors appear in the commutators between spinors of the same chirality.

$s = 1$ (Majorana spinors). The two projections to semispinors mentioned above (B.16), lead to equivalent spinors. We thus consider only the commutator between these irreducible spinors (including the others is contained in the ‘extended algebras’ discussed below). In Tables A.3– A.6 we thus consider the $\iota = 1$ cases. We can check that $\iota = -1$ always allows both $\sigma = 1$ and $\sigma = -1$ as this concerns commutators between unrelated but equivalent spinors.

$s = 2$ (Majorana spinors). The two projections to semispinors lead to equivalent spinors. We thus consider only the commutator between these irreducible spinors. Note that in the table we indicate here also forms with $k > m$. These are dual to $k < m$ forms, and this duality has been used in the formulation of the $s = 2$ part of Theorem 6. The formulation here shows the gamma matrices completely, e.g. the appearance of $\Gamma_{abc} = \varepsilon_{abcd}\gamma^d$ in 4 dimensions.

$s = 3, s = 5$ (Symplectic-Majorana spinors). The symplectic spinors are in a doublet of $\mathfrak{su}(2)$. According to the value of σ for a particular k we find either a triplet or a singlet of generators in the superalgebra or in the Lie algebra.

$s = 4$ (Symplectic Majorana-Weyl spinors). In the commutators between generators of equal chirality (which are again doublets of $\mathfrak{su}(2)$), we find either triplets (symmetric) or singlet (antisymmetric) generators. For commutators between generators of different

chirality no symmetry or antisymmetry can be defined, and the generators allowed by the chirality ($t = -1$) appear in the superalgebra as well as in the Lie algebra.

$s = 6$ (Majorana spinors). This case is straightforward from the tables and the spinors are just real and not projected. Remark that the result is then the same as for the projected ones of $s = 2$. The same remark about showing tensors with $k > m$ holds here too. These are dualized in the formulation in Corollary 4.

$s = 7$ (Majorana spinors). Here also, the tables straightforwardly lead to the same result as for the projected spinors of $s = 1$.

We remark that the result is the same for s and for $-s$, which shows that the conventional choices discussed at the beginning of Sect. B.1. do not influence the final algebras.

Finally, in Sect. 7, results are obtained for **N -extended polyvector Poincaré algebras**. This means that W_1 consists of N copies of the irreducible spinor S . In cases where there are two inequivalent copies (complex even dimensional, or real with $s = 0$ or $s = 4$) we have $(N_+, N_-)(N_+, N_-)$ -**extended polyvector Poincaré algebras**.

The results are straightforward from the above tables and this shows why it has been useful to include the Lie algebra case. The generators in W_1 are in an N -representation of the automorphism algebra that acts on the copies of S .

For the complex odd-dimensional case and real $s = 1, 2, 6, 7$ (Majorana): We just have to split the $N \times N$ representations into the symmetric and antisymmetric ones.

$$\begin{aligned}
 \text{for superalgebras:} & \quad \frac{N(N+1)}{2} \text{ copies of the } \sigma = 1 \text{ generators} \\
 & \quad + \frac{N(N-1)}{2} \text{ copies of the } \sigma = -1 \text{ generators,} \\
 \text{for Lie algebras:} & \quad \frac{N(N-1)}{2} \text{ copies of the } \sigma = 1 \text{ generators} \\
 & \quad + \frac{N(N+1)}{2} \text{ copies of the } \sigma = -1 \text{ generators.} \quad (\text{B.25})
 \end{aligned}$$

For the complex even-dimensional case and real $s = 0$ (Weyl): We have (N_+, N_-) algebras. We use the above rule separately for the commutators between the N_+ chiral generators and between the N_- antichiral ones. Furthermore there are N_+N_- copies of the generators that appear in $[S_+, S_-]$ in Tables B.1 and B.2. As an example, the $(2, 1)$ superalgebra in 8-dimensional $(4,4)$ space contains: three selfdual 4-forms, and one anti-selfdual 4-form, four 0-forms (three in $[2S_+, 2S_+]$ and one in $[S_-, S_-]$, one 2-form (in $[2S_+, 2S_+]$) and two 3-forms and 1-forms in $[2S_+, S_-]$.

For the symplectic real case $s = 3, 5$: The automorphism algebra is already $\mathfrak{sp}(2) = \mathfrak{su}(2)$ for the simple algebras discussed above. For the extended algebras it is $\mathfrak{sp}(N)$ where N is even. The simple case is thus similar to (B.25) with $N = 2$, and the ‘triplet’ and ‘singlet’ indications in Table B.2 reflect this. Therefore for higher N (always even) we replace in Table B.2 the ‘triplet’ by $N(N + 1)/2$ and the ‘singlet’ by $N(N - 1)/2$.

For the symplectic Majorana-Weyl case $s = 4$: We merely need to combine the remarks above for the symplectic case and the Weyl case. Extended algebras are of the form (N_+, N_-) where both numbers are even. The ‘triplet’ indication in Table B.2 is replaced by $N_+(N_+ + 1)/2$ and $N_-(N_- + 1)/2$ and ‘singlet’ is replaced by $N_+(N_+ - 1)/2$ and $N_-(N_- - 1)/2$. The mixed commutators are multiplied by N_+N_- .

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