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A New Cohomology Theory of Orbifold*

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Abstract: Based on the orbifold string theory model in physics, we construct a new cohomology ring for any almost complex orbifold. The key theorem is the associativity of this new ring. Some examples are computed.

Contents

1. Introduction

An orbifold is a topological space locally modeled on the quotient of a smooth manifold by a finite group. Therefore, orbifolds belong to one of the simplest kinds of singular spaces. Orbifolds appear naturally in many branches of mathematics. For example,

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symplectic reduction often gives rise to orbifolds. An algebraic 3-fold with terminal singularities can be deformed into a symplectic orbifold. Orbifold also appears naturally in string theory, where many known Calabi-Yau 3-folds are the so-called crepant resolutions of a Calabi-Yau orbifold. The physicists even attempted to formulate string theories on Calabi-Yau orbifolds which are expected to be "equivalent" to the string theories on its crepant resolutions [DHVW]. As a consequence of this orbifold string theory consideration, one has the following prediction that *"orbifold quantum cohomology" is "isomorphic" to the ordinary quantum cohomology of its crepant resolutions.* At this moment, even the physical idea around this subject is still vague and incomplete, particularly for the possible isomorphism. However, it seems that there are interesting new mathematical structures that are behind such orbifold string theories.

This article is the first paper of a program to understand these new mathematical treasures behind orbifold string theory. We introduce *orbifold cohomology groups* of an almost complex orbifold, and *orbifold Dolbeault cohomology groups* of a complex orbifold. The main result of this paper is the construction of orbifold cup products on orbifold cohomology groups and orbifold Dolbeault cohomology groups, which make the corresponding total orbifold cohomology into a ring with unit. We will call the resulting rings *orbifold cohomology ring* or *orbifold Dolbeault cohomology ring*. (See Theorems 4.1.5 and 4.1.7 for details.) In the case when the almost complex orbifold is closed and symplectic, the orbifold cohomology ring corresponds to the "classical part" of the orbifold quantum cohomology ring constructed in [CR]. Originally, this article is a small part of the much longer paper [CR] regarding the theory of orbifold quantum cohomology. However, we feel that the classical part (i.e. the orbifold cohomology) of the orbifold quantum cohomology is interesting in its own right, and technically, it is also much simpler to construct. Therefore, we decided to put it in a separate paper.

A brief history is in order. In the case of Gorenstein global quotients, orbifold Euler characteristic-Hodge numbers have been extensively studied in the literature (see [RO, BD, Re] for a more complete reference). However, we would like to point out that (i) our orbifold cohomology is well-defined for any almost complex orbifold which may or may not be Gorenstein. Furthermore, it has an interesting feature that an orbifold cohomology class of a non-Gorenstein orbifold could have a rational degree (see examples in Sect. 5); (ii) Even in the case of Gorensterin orbifolds, the orbifold cohomology ring contains much more information than just orbifold Betti-Hodge numbers. In the case of global quotients, some constructions of this paper are already known to physicists. A notable exception is the orbifold cup product. On the other hand, many interesting orbifolds are not global quotients in general. For examples, most of Calabi-Yau hypersurfaces of weighted projective spaces are not global quotients. In this article, we systematically developed the theory (including the construction of orbifold cup products) for general orbifolds. Our construction of orbifold cup products is motivated by the construction of orbifold quantum cohomology.

2. Recollections on Orbifold

In this section, we review basic definitions in the theory of orbifold. A systematic treatment of various aspects of differential geometry on orbifolds is contained in our forthcoming paper [CR]. The notion of orbifold was first introduced by Satake in [S], where a different name, *V-manifold*, was used. Our current definition is taken from [K1].

Let $\mathcal{M}$ be the category of connected smooth manifolds and open embeddings. Then, we define a category $\mathcal{M}_S$ (the category of manifolds with finite symmetries) as follows:

The objects of $\mathcal{M}_S$ are the class of pairs (M, G), where M is a connected smooth manifold of dimension n (uniformizing system) and G is a finite group acting on M. Here we assume throughout that *the fixed-point set of each element of the group is either the whole space or of codimension at least two*. In particular, the action of G does not have to be effective. This is the case, for example, when the action is orientation-preserving. This requirement has a consequence that the non-fixed-point set is locally connected if it is not empty. We will call the subgroup of G, which consists of elements fixing the whole space V, the *kernel* of the action. Let (M, G) and (M', G') be two objects. Then, a morphism $\{\phi\} : (M, G) \to (M', G')$ is a family of open embeddings $\phi : M \to M'$ (injections) satisfying

(i) For each $\phi \in \{\phi\}$, there is a group homomorphism $\lambda_\phi : G \to G'$ that makes ϕ to be λ_ϕ-equivariant. Furthermore, λ_ϕ induces an isomorphism from $ker(G)$ to $ker(G')$.
(ii) If $g\phi(M) \cap \phi(M) \neq \emptyset$ for some $g \in G'$, then g is in the image of λ_ϕ.
(iii) G' acts on the set $\{\phi\}$ simply transitively. ($(g\phi)(x) = g\phi(x)$, for $x \in M$ and $g \in G'$.)

The morphism $\{\phi\}$ induces a unique open embedding $i_\phi : M/G \to M'/G'$ of orbit spaces. We denote by $\mathcal{J}$ the category of connected topological spaces and open embeddings. Then we have a functor $\mathcal{L} : \mathcal{M}_S \to \mathcal{J}$ defined by $\mathcal{L}(M, G) = M/G$ and $\mathcal{L}\{\phi\} = i_\phi$.

Definition 2.1. *Let X be a paracompact Hausdorff space and let $\mathcal{U}$ be a covering of X consisting of connected open subsets. We assume $\mathcal{U}$ satisfies the conditions:*

(*) For any $x \in U \cap U'$, $U, U' \in \mathcal{U}$, there is $U" \in \mathcal{U}$ such that $x \in U" \subset U \cap U'$.

Let $\mathcal{J}(\mathcal{U})$ be the subcategory of $\mathcal{J}$ consisting of all the elements of $\mathcal{U}$ and the inclusions. Then, an **Orbifold Structure** *$\mathcal{V}$ is a functor $\mathcal{V} : \mathcal{J}(\mathcal{U}) \to \mathcal{M}_S$ such that $\mathcal{L} \circ \mathcal{V} = I_{\mathcal{J}(\mathcal{U})}$ (the identity functor).*

If $\mathcal{U}'$ is a refinement of $\mathcal{U}$ satisfying (*), then there is an orbifold structure $\mathcal{V}' : \mathcal{J}(\mathcal{U}') \to \mathcal{M}_S$ such that $\mathcal{V} \cup \mathcal{V}' : \mathcal{J}(\mathcal{U} \cup \mathcal{U}') \to \mathcal{M}_S$ is an orbifold structure. We consider $\mathcal{V}$, $\mathcal{V}'$ to be equivalent. Such an equivalent class is called *an orbifold structure over X*. So we may choose $\mathcal{U}$ arbitrarily fine.

Let $p \in X$. By choosing a small neighborhood $U_p \in \mathcal{U}$, we may assume that its uniformizing system $\mathcal{V}(U_p) = (V_p, G_p)$ has the property that V_p is a n-ball centered at origin o and $\pi_p^{-1}(p) = o$, where $\pi_p : V_p \to U_p = V_p/G_p$ is the projection map. In particular, the origin o is fixed by G_p. We called G_p *the local group at p*. If G_p acts effectively for every p, we call X a *reduced orbifold*.

Now we consider a class of continuous maps between two orbifolds which respect the orbifold structures in a certain sense. Let U be uniformized by (V, G, π) and U' by (V', G', π'), and $f : U \to U'$ be a continuous map. A C^l *lifting,* $0 \leq l \leq \infty$, of f is a C^l map $\tilde{f} : V \to V'$ such that $\pi' \circ \tilde{f} = f \circ \pi$, and for any $g \in G$, there is $g' \in G'$ so that $g' \cdot \tilde{f}(x) = \tilde{f}(g \cdot x)$ for any $x \in V$. Two liftings $\tilde{f}_i : (V_i, G_i, \pi_i) \to (V_i', G_i', \pi_i')$, $i = 1, 2$, are *isomorphic* if there exist isomorphisms $(\phi, \tau) : (V_1, G_1, \pi_1) \to (V_2, G_2, \pi_2)$ and $(\phi', \tau') : (V_1', G_1', \pi_1') \to (V_2', G_2', \pi_2')$ such that $\phi' \circ \tilde{f}_1 = \tilde{f}_2 \circ \phi$.

Let $p \in U$ be any point. Then for any uniformized neighborhood U_p of p and uniformized neighborhood $U_{f(p)}$ of $f(p)$ such that $f(U_p) \subset U_{f(p)}$, a lifting $\tilde{f}$ of f will induce a lifting $\tilde{f}_p$ for $f|_{U_p} : U_p \to U_{f(p)}$ as follows: For any injection $(\phi, \tau) : (V_p, G_p, \pi_p) \to (V, G, \pi)$, consider the map $\tilde{f} \circ \phi : V_p \to V'$. Observe that the inclusion $\pi' \circ \tilde{f} \circ \phi(V_p) \subset U_{f(p)}$ implies that $\tilde{f} \circ \phi(V_p)$ lies in $(\pi')^{-1}(U_{f(p)})$. Therefore there is an injection $(\phi', \tau') : (V_{f(p)}, G_{f(p)}, \pi_{f(p)}) \to (V', G', \pi')$ such that $\tilde{f} \circ \phi(V_p) \subset \phi'(V_{f(p)})$. We define $\tilde{f}_p = (\phi')^{-1} \circ \tilde{f} \circ \phi$. In this way we obtain a lifting $\tilde{f}_p : (V_p, G_p, \pi_p) \to (V_{f(p)}, G_{f(p)}, \pi_{f(p)})$ for $f|_{U_p} : U_p \to U_{f(p)}$. We can verify that different choices give isomorphic liftings. We define the *germ* of liftings as follows: two liftings are *equivalent at p* if they induce isomorphic liftings on a smaller neighborhood of p.

Let $f : X \to X'$ be a continuous map between orbifolds X and X'. A *lifting* of f consists of the following data: for any point $p \in X$, there exist charts (V_p, G_p, π_p) at p and $(V_{f(p)}, G_{f(p)}, \pi_{f(p)})$ at $f(p)$ and a lifting $\tilde{f}_p$ of $f|_{\pi_p(V_p)} : \pi_p(V_p) \to \pi_{f(p)}(V_{f(p)})$ such that for any $q \in \pi_p(V_p)$, $\tilde{f}_p$ and $\tilde{f}_q$ induce the same germ of liftings of f at q. We can define the *germ* of liftings in the sense that two liftings of f, $\{\tilde{f}_{p,i} : (V_{p,i}, G_{p,i}, \pi_{p,i}) \to (V_{f(p),i}, G_{f(p),i}, \pi_{f(p),i}) : p \in X\}$, $i = 1, 2$, are *equivalent* if for each $p \in X$, $\tilde{f}_{p,i}$, $i = 1, 2$, induce the same germ of liftings of f at p.

Definition 2.2. *A C^l map ($0 \leq l \leq \infty$) between orbifolds X and X' is a germ of C^l liftings of a continuous map between X and X'.* □

We denote by $\tilde{f}$ a C^l map which is a germ of liftings of a continuous map f. Our definition of C^l maps corresponds to the notion of *V-maps* in [S].

Next we shall define orbifold bundles. We regard a smooth fibre bundle as a structure over a smooth manifold. It is pull-back by an open embedding. We denote $\mathcal{E}$ the category of smooth fibre bundles and bundle maps over open embeddings. Then, we have the category $\mathcal{E}_S$ of smooth fibre bundles with finite symmetries. The object of $\mathcal{E}_S$ is a smooth fibre bundle $E \to M$ with an action of a finite group G as the local transformation group for both base and total space. We have a forgetful functor $\mathcal{F} : \mathcal{E}_S \to \mathcal{M}_S$ defined by $(E \to M, G) \to (M, G)$.

Definition 2.3. *Let $(X, \mathcal{V})$ be an orbifold with orbifold structure $\mathcal{V}$. An orbifold-bundle $\mathcal{B}$ over $(X, \mathcal{V})$ is a functor $\mathcal{B} : \mathcal{J}(\mathcal{U}) \to \mathcal{E}_S$ such that $\mathcal{F} \circ \mathcal{B} = \mathcal{V}$. We call $\mathcal{B}$ an orbifold vector bundle if $E \to M$ is a vector bundle and G acts linearly on the fiber.*

For each $U \in \mathcal{U}$, we denote $\mathcal{B}(U) = (\tilde{E} \to \tilde{U}, G_U)$. If $U \subset U', U, U' \in \mathcal{U}$, then $\mathcal{B}(U \subset U')$ is a family $\{\Phi\}$ of bundle maps $\Phi : \tilde{E}_U \to \tilde{E}_{U'}$. The family $\{\Phi\}$ induces a unique open embedding $i_\phi : \tilde{E}_U/G_U \to \tilde{E}_{U'}/G_{U'}$ of orbit spaces. By these embeddings we can glue together all $\tilde{E}_U/G_U$'s ($U \in \mathcal{U}$) to form a topological space $E = E(\mathcal{B})$. $E = E(\mathcal{B})$ is called *the total space of* $\mathcal{B}$. The projection $\tilde{p}_U : \tilde{E}_U \to \tilde{U}$ induces a map $p : E \to X$ called *the projection of* $\mathcal{B}$. In general $p : E \to X$ is not a fibre bundle.

A C^l map $\tilde{s}$ from X to an orbifold bundle $pr : E \to X$ is called a C^l *section* if locally $\tilde{s}$ is given by $\tilde{s}_p : V_p \to V_p \times \mathbf{R}^k$, where $\tilde{s}_p$ is G_p-equivariant and $\tilde{pr} \circ \tilde{s}_p = Id$ on V_p. We observe that

1. For each point p, $s(p)$ lies in E^p, the linear subspace of fixed points of G_p.
2. The space of all C^l sections of E, denoted by $C^l(E)$, has a structure of vector space over $\mathbf{R}$ (or $\mathbf{C}$) as well as a $C^l(X)$-module structure.
3. The C^l sections $\tilde{s}$ are in $1 : 1$ correspondence with the underlying continuous maps s.

Orbifold bundles are more conveniently described by transition maps, e.g. as in [S]. More precisely, an orbifold bundle over an orbifold X can be constructed from the following data: A compatible cover $\mathcal{U}$ of X such that for any injection $i : (V', G', \pi') \to$

(V, G, π), there is a smooth map $g_i : V' \to Aut(\mathbf{R}^k)$ giving an open embedding $V' \times \mathbf{R}^k \to V \times \mathbf{R}^k$ by $(x, v) \to (i(x), g_i(x)v)$, and for any composition of injections $j \circ i$, we have

$$g_{j \circ i}(x) = g_j(i(x)) \circ g_i(x). \tag{2.1}$$

Two collections of maps $g^{(1)}$ and $g^{(2)}$ define isomorphic orbifold bundles if there are maps $\delta_V : V \to Aut(\mathbf{R}^k)$ such that for any injection $i : (V', G', \pi') \to (V, G, \pi)$, we have

$$g_i^{(2)}(x) = \delta_V(i(x)) \circ g_i^{(1)}(x) \circ (\delta_{V'}(x))^{-1}, \forall x \in V'. \tag{2.2}$$

Since Eq. (2.1) behaves naturally under constructions of vector spaces such as the tensor product, exterior product, etc., we can define the corresponding constructions for orbifold bundles.

Example 2.4. For an orbifold X, the tangent bundle TX can be constructed because the differential of any injection satisfies Eq. (2.1). Likewise, we define the cotangent bundle T^*X, the bundles of exterior power or tensor product. The C^∞ sections of these bundles give us vector fields, differential forms or tensor fields on X. We remark that if ω is a differential form on X' and $\tilde{f} : X \to X'$ is a C^∞ map, then there is a pull-back form $\tilde{f}^*\omega$ on X.

Integration over orbifolds is defined as follows. Let U be a connected n-dimensional orbifold, which is uniformized by (V, G, π), with the kernel of the action of G on V denoted by K. For any compact supported differential n-form ω on U, which is, by definition, a G-equivariant compact supported n-form $\tilde{\omega}$ on V, the integration of ω on U is defined by

$$\int_U^{orb} \omega := \frac{1}{|G|} \int_V \tilde{\omega}, \tag{2.3}$$

where $|G|$ is the order of the group G. In general, let X be an orbifold. Fix a C^∞ partition of unity $\{\rho_i\}$ subordinated to $\{U_i\}$, where each U_i, is a uniformized open set in X. Then the integration over X is defined by

$$\int_X^{orb} \omega := \sum_i \int_{U_i}^{orb} \rho_i \ \omega, \tag{2.4}$$

which is independent of the choice of the partition of unity $\{\rho_i\}$. We remark that it is important throughout this paper that we adopt the integration over orbifolds as in (2.3) and (2.4), where we divide the integral over the uniformizer V by the group order $|G|$ instead of $|G|/|K|$ (K is the kernel of the action). As a result, the fundamental class of an orbifold is rational in general. The integration $\int^{orb}$ coincides with the usual measure-theoretic integration if and only if the orbifold is reduced.

The de Rham cohomology groups of an orbifold are defined similarly through differential forms, which are naturally isomorphic to the singular cohomology groups with real coefficients. For an oriented, closed orbifold, the singular cohomology groups are naturally isomorphic to the intersection homology groups, both with rational coefficients, for which the Poincaré duality is valid [GM].

Characteristic classes (*Euler class* for oriented orbifold bundles, *Chern classes* for complex orbifold bundles, and *Pontrjagin classes* for real orbifold bundles) are well-defined for orbifold bundles. One way to define them is through Chern-Weil theory, so that the characteristic classes take values in the deRham cohomology groups. Another way to define them is through the transgressions in the Serre spectral sequences with rational coefficients of the associated Stiefel orbifold bundles, so that these characteristic classes are defined over the rationals [K1].

3. Orbifold Cohomology Groups

In this section, we introduce the main object of study, the *orbifold cohomology groups* of an almost complex orbifold.

3.1. Twisted sectors. Let X be an orbifold. For any point $p \in X$, let (V_p, G_p, π_p) be a local chart at p. Consider the set of pairs:

$$\widetilde{X} = \{(p, (g)_{G_p}) | p \in X, g \in G_p\}, \tag{3.1.1}$$

where $(g)_{G_p}$ is the conjugacy class of g in G_p. If there is no confusion, we will omit the subscript G_p to simplify the notation. There is a surjective map $\pi : \widetilde{X} \to X$ defined by $(p, (g)) \mapsto p$.

Lemma 3.1.1. (Kawasaki,[K1]). *The set $\widetilde{X}$ is naturally an orbifold (not necessarily connected) with an orbifold structure given by*

$$\{\pi_{p,g} : (V_p^g, C(g)) \to V_p^g / C(g) : p \in X, g \in G_p.\},$$

where V_p^g is the fixed-point set of g in V_p, $C(g)$ is the centralizer of g in G_p. Moreover, if X is closed, so is $\widetilde{X}$. Under this orbifold structure, the map $\pi : \widetilde{X} \to X$ is a C^∞ map.

Proof. First we identify a point $(q, (h))$ in $\widetilde{X}$ as a point in $\bigsqcup_{\{(g), g \in G_p\}} V_p^g / C(g)$ if $q \in U_p$ for some $p \in X$. Pick a representative $y \in V_p$ such that $\pi_p(y) = q$. Then this gives rise to a monomorphism $\lambda_y : G_q \to G_p$. Pick a representative $h \in G_q$ for (h) in G_q, we let $g = \lambda_y(h)$. Then $y \in V_p^g$. So we have a map $\Phi : (q, h) \to (y, g) \in (V_p^g, G_p)$. If we change h by a $h' = a^{-1}ha \in G_q$ for $a \in G_q$, then g is changed to $\lambda_y(a^{-1}ha) = \lambda_y(a)^{-1} g \lambda_y(a)$. So we have $\Phi : (q, a^{-1}ha) \to (y, \lambda_y(a)^{-1} g \lambda_y(a)) \in (V_p^{\lambda_y(a)^{-1} g \lambda_y(a)}, G_p)$. (Note that λ_y is determined up to conjugacy by an element in G_q.) If we take a different representative $y' \in V_p$ such that $\pi_p(y') = q$, and suppose $y' = b \cdot y$ for some $b \in G_p$, then we have a different identification $\lambda_{y'} : G_q \to G_p$ of G_q as a subgroup of G_p, where $\lambda_{y'} = b \cdot \lambda_y \cdot b^{-1}$. In this case, we have $\Phi : (q, h) \to (y', bgb^{-1}) \in (V_p^{bgb^{-1}}, G_p)$. If $g = bgb^{-1}$, then $b \in C(g)$. In any event, Φ induces a map ϕ sending $(q, (h))$ to a point in $\bigsqcup_{\{(g), g \in G_p\}} V_p^g / C(g)$. It is one to one because if $\phi(q_1, (h_1)) = \phi(q_2, (h_2))$, then we may assume that $\Phi(q_1, h_1) = \Phi(q_2, h_2)$ after applying conjugations. But this means that $(q_1, h_1) = (q_2, h_2)$. It is easily seen that this map ϕ is also onto. Hence we have shown that $\widetilde{X}$ is covered by $\bigsqcup_{\{p \in X\}} \bigsqcup_{\{(g), g \in G_p\}} V_p^g / C(g)$.

We define a topology on $\widetilde{X}$ so that each $V_p^g/C(g)$ is an open subset for any (p, g), where $p \in X$ and $g \in G_p$. We also uniformize $V_p^g/C(g)$ by $(V_p^g, C(g))$. It remains to show that these charts fit together to form an orbifold structure on $\widetilde{X}$. Let $x \in V_p^g/C(g)$ and take a representative $\tilde{x}$ in V_p^g. Let H_x be the isotropy subgroup of $\tilde{x}$ in $C(g)$. Then $(V_p^g, C(g))$ induces a germ of the uniformizing system at x as (B_x, H_x), where B_x is a small ball in V_p^g centered at $\tilde{x}$. Let $\pi_p(\tilde{x}) = q$. We need to write (B_x, H_x) as $(V_q^h, C(h))$ for some $h \in G_q$. We let $\lambda_x : G_q \to G_p$ be an induced monomorphism which resulted from choosing $\tilde{x}$ as the representative of q in V_p. We define $h = \lambda_x^{-1}(g)$ (g is in $\lambda_x(G_q)$ since $\tilde{x} \in V_p^g$ and $\pi_p(\tilde{x}) = q$.) Then we can identify B_x as V_q^h. We also see that $H_x = \lambda_x(C(h))$. Therefore (B_x, H_x) is identified as $(V_q^h, C(h))$.

The map $\pi : \widetilde{X} \to X$ is obviously continuous with the given topology of $\widetilde{X}$, and actually is a C^∞ map with the given orbifold structure on $\widetilde{X}$ with the local liftings given by embeddings $V_p^g \hookrightarrow V_p$.

We finish the proof by showing that $\widetilde{X}$ is Hausdorff and second countable with the given topology. Let $(p, (g))$ and $(q, (h))$ be two distinct points in $\widetilde{X}$. When $p \neq q$, there are U_p, U_q such that $U_p \cap U_q = \emptyset$ since X is Hausdorff. It is easily seen that in this case $(p, (g))$ and $(q, (h))$ are separated by disjoint neighborhoods $\pi^{-1}(U_p)$ and $\pi^{-1}(U_q)$, where $\pi : \widetilde{X} \to X$. When $p = q$, we must then have $(g) \neq (h)$. In this case, $(p, (g))$ and $(q, (h))$ lie in different open subsets $V_p^g/C(g)$ and $V_q^h/C(h)$ respectively. Hence $\widetilde{X}$ is Hausdorff. The second countability of $\widetilde{X}$ follows from the second countability of X and the fact that $\pi^{-1}(U_p)$ is a finite union of open subsets of $\widetilde{X}$ for each $p \in X$ and a uniformized neighborhood U_p of p. □

Next, we would like to describe the connected components of $\widetilde{X}$. Recall that every point p has a local chart (V_p, G_p, π_p) which gives a local uniformized neighborhood $U_p = \pi_p(V_p)$. If $q \in U_p$, up to conjugation, there is an injective homomorphism $G_q \to G_p$. For $g \in G_q$, the conjugacy class $(g)_{G_p}$ is well-defined. We define an equivalence relation $(g)_{G_q} \sim (g)_{G_p}$. Let T be the set of equivalence classes. To abuse the notation, we often use (g) to denote the equivalence class which $(g)_{G_q}$ belongs to. It is clear that $\widetilde{X}$ is decomposed as a disjoint union of connected components

$$\widetilde{X} = \bigsqcup_{(g) \in T} X_{(g)}, \tag{3.1.2}$$

where

$$X_{(g)} = \{(p, (g')_{G_p}) | g' \in G_p, (g')_{G_p} \in (g)\}. \tag{3.1.3}$$

Definition 3.1.2. *$X_{(g)}$ for $g \neq 1$ is called a* **twisted sector**. *Furthermore, we call $X_{(1)} = X$ the* **nontwisted sector**.

Example 3.1.3. Consider the case that the orbifold $X = Y/G$ is a global quotient. We will show that $\widetilde{X}$ can be identified with $\bigsqcup_{\{(g), g \in G\}} Y^g/C(g)$, where Y^g is the fixed-point set of element $g \in G$.

Let $\pi : \widetilde{X} \to X$ be the surjective map defined by $(p, (g)) \mapsto p$. Then for any $p \in X$, the preimage $\pi^{-1}(p)$ in $\widetilde{X}$ has a neighborhood described by $W_p = \bigsqcup_{\{(g), g \in G_p\}} V_p^g/C(g)$, which is uniformized by $\widehat{W}_p = \bigsqcup_{\{(g), g \in G_p\}} V_p^g$. For each $p \in X$, pick a $y \in Y$ that represents p, and an injection $(\phi_p, \lambda_p) : (V_p, G_p) \to (Y, G)$ whose image is centered at y. This induces an open embedding $\tilde{f}_p : \widehat{W}_p \to \bigsqcup_{\{(\lambda_p(g)), \lambda_p(g) \in G\}} Y^{\lambda_p(g)} \subset$

$\bigsqcup_{\{(g),g\in G\}} Y^g$, which induces a homeomorphism f_p from W_p into $\bigsqcup_{\{(g),g\in G\}} Y^g/C(g)$ that is independent of the choice of y and (ϕ_p, λ_p). These maps $\{f_p; p \in X\}$ fit together to define a map $f : \widetilde{X} \to \bigsqcup_{\{(g),g\in G\}} Y^g/C(g)$ which we can verify to be a homeomorphism. □

Remark 3.1.4. There is a natural C^∞ map $I : \widetilde{X} \to \widetilde{X}$ defined by

$$I((p, (g)_{G_p})) = (p, (g^{-1})_{G_p}). \tag{3.1.4}$$

The map I is an involution (i.e., $I^2 = Id$) which induces an involution on the set T of equivalence classes of relations $(g)_{G_q} \sim (g)_{G_p}$. We denoted by (g^{-1}) the image of (g) under this induced map.

3.2. Degree shifting and orbifold cohomology group. For the rest of the paper, we will assume that X is an almost complex orbifold with an almost complex structure J. Recall that an almost complex structure J on X is a smooth section of the orbifold bundle $End(TX)$ such that $J^2 = -Id$. Observe that $\widetilde{X}$ naturally inherits an almost complex structure from the one on X, and the map $\pi : \widetilde{X} \to X$ defined by $(p, (g)_{G_p}) \to p$ is naturally pseudo-holomorphic, i.e., its differential commutes with the almost complex structures on $\widetilde{X}$ and X.

An important feature of orbifold cohomology groups is degree shifting, which we shall explain now. Let p be any point of X. The almost complex structure on X gives rise to a representation $\rho_p : G_p \to GL(n, \mathbf{C})$ (here $n = \dim_{\mathbf{C}} X$). For any $g \in G_p$, we write $\rho_p(g)$ as a diagonal matrix

$$diag(e^{2\pi i m_{1,g}/m_g}, \cdots, e^{2\pi i m_{n,g}/m_g}),$$

where m_g is the order of $\rho_p(g)$, and $0 \le m_{i,g} < m_g$. This matrix depends only on the conjugacy class $(g)_{G_p}$ of g in G_p. We define a function $\iota : \widetilde{X} \to \mathbf{Q}$ by

$$\iota(p, (g)_{G_p}) = \sum_{i=1}^{n} \frac{m_{i,g}}{m_g}.$$

It is straightforward to show the following

Lemma 3.2.1. *The function $\iota : X_{(g)} \to \mathbf{Q}$ is constant. Its constant value, which will be denoted by $\iota_{(g)}$, satisfies the following conditions:*

- *$\iota_{(g)}$ is integral if and only if $\rho_p(g) \in SL(n, \mathbf{C})$.*
-

$$\iota_{(g)} + \iota_{(g^{-1})} = rank(\rho_p(g) - I), \tag{3.2.1}$$

which is the "complex codimension" $\dim_{\mathbf{C}} X - \dim_{\mathbf{C}} X_{(g)} = n - \dim_{\mathbf{C}} X_{(g)}$ of $X_{(g)}$ in X. As a consequence, $\iota_{(g)} + \dim_{\mathbf{C}} X_{(g)} < n$ when $\rho_p(g) \neq I$.

Definition 3.2.2. *$\iota_{(g)}$ is called a* **degree shifting number**.

In the definition of orbifold cohomology groups, we will shift up the degree of cohomology classes of $X_{(g)}$ by $2\iota_{(g)}$. The reason for such a degree shifting will become clear after we discuss the dimension of moduli space of ghost maps (see formula (4.2.14)).

An orbifold X is called a *SL-orbifold* if $\rho_p(g) \in SL(n, \mathbf{C})$ for all $p \in X$ and $g \in G_p$, and called a *SP-orbifold* if $\rho_p(g) \in SP(n, \mathbf{C})$. In particular, a Calabi-Yau orbifold is a SL-orbifold, and a holomorphic symplectic orbifold or hyperkahler orbifold is a SP-orbifold. By Lemma 3.2.1, $\iota_{(g)}$ is integral if and only if X is a SL-orbifold.

We observe that although the almost complex structure J is involved in the definition of degree shifting numbers $\iota_{(g)}$, they do not depend on J because locally the parameter space of almost complex structures, which is the coset $SO(2n, \mathbf{R})/U(n, \mathbf{C})$, is connected.

Definition 3.2.3. *We define the orbifold cohomology groups $H^d_{orb}(X)$ of X by*

$$H^d_{orb}(X) = \oplus_{(g)\in T} H^{d-2\iota_{(g)}}(X_{(g)})) \tag{3.2.2}$$

and orbifold Betti numbers $b^d_{orb} = \sum_{(g)} \dim H^{d-2\iota_{(g)}}(X_{(g)})$.

Here each $H^*(X_{(g)})$ is the singular cohomology of $X_{(g)}$ with real coefficients, which is isomorphic to the corresponding de Rham cohomology group. As a consequence, the cohomology classes can be represented by closed differential forms on $X_{(g)}$. Note that, in general, orbifold cohomology groups are rationally graded.

Suppose X is a complex orbifold with an integrable complex structure J. Then each twisted sector $X_{(g)}$ is also a complex orbifold with the induced complex structure. We consider the Čech cohomology groups on X and each $X_{(g)}$ with coefficients in the sheaves of holomorphic forms (in the orbifold sense). These Čech cohomology groups are identified with the Dolbeault cohomology groups of (p, q)-forms (in the orbifold sense). When X is closed, the harmonic theory [Ba] can be applied to show that these groups are finite dimensional, and there is a Kodaira-Serre duality between them. When X is a closed Kahler orbifold (so is each $X_{(g)}$), these groups are then related to the singular cohomology groups of X and $X_{(g)}$ as in the smooth case, and the Hodge decomposition theorem holds for these cohomology groups.

Definition 3.2.4. *Let X be a complex orbifold. We define, for $0 \leq p, q \leq \dim_{\mathbf{C}} X$, orbifold Dolbeault cohomology groups*

$$H^{p,q}_{orb}(X) = \oplus_{(g)} H^{p-\iota_{(g)}, q-\iota_{(g)}}(X_{(g)}). \tag{3.2.3}$$

We define orbifold Hodge numbers by $h^{p,q}_{orb}(X) = \dim H^{p,q}_{orb}(X)$.

Remark 3.2.5. We can define compact supported orbifold cohomology groups $H^*_{orb,c}(X)$, $H^{*,*}_{orb,c}(X)$ in the obvious fashion.

3.3. Poincaré duality. Recall that there is a natural C^∞ map $I : X_{(g)} \to X_{(g^{-1})}$ defined by $(p, (g)) \mapsto (p, (g^{-1}))$, which is an automorphism of $\widetilde{X}$ as an orbifold and $I^2 = Id$ (Remark 3.1.4).

Proposition 3.3.1 (Poincaré duality). *For any* $0 \leq d \leq 2n$, *the pairing*

$$< \ >_{orb}: H^d_{orb}(X) \times H^{2n-d}_{orb,c}(X) \to \mathbf{R}$$

defined by the direct sum of

$$< \ >^{(g)}_{orb}: H^{d-2\iota_{(g)}}(X_{(g)}) \times H_c^{2n-d-2\iota_{(g^{-1})}}(X_{(g^{-1})}) \to \mathbf{R},$$

where

$$< \alpha, \beta >^{(g)}_{orb} = \int^{orb}_{X_{(g)}} \alpha \wedge I^*(\beta) \qquad (3.3.4)$$

for $\alpha \in H^{d-2\iota_{(g)}}(X_{(g)})$, $\beta \in H_c^{2n-d-2\iota_{(g^{-1})}}(X_{(g^{-1})})$ *is nondegenerate. Here the integral in the right hand side of* (3.3.4) *is defined using* (2.4).

Note that $< \ >_{orb}$ equals the ordinary Poincaré pairing when restricted to the non-twisted sectors $H^*(X)$.

Proof. By (3.2.1), we have

$$2n - d - 2\iota_{(g^{-1})} = \dim X_{(g)} - d - 2\iota_{(g)}.$$

Furthermore, $I|_{X_{(g)}} : X_{(g)} \to X_{(g^{-1})}$ is a homeomorphism. Under this homeomorphism, $< \ >^{(g)}_{orb}$ is isomorphic to the ordinary Poincaré pairing on $X_{(g)}$. Hence $< \ >_{orb}$ is nondegenerate. □

For the case of orbifold Dolbeault cohomology, the following proposition is straightforward.

Proposition 3.3.2. *Let X be an n-dimensional complex orbifold. There is a Kodaira-Serre duality pairing*

$$< \ >_{orb}: H^{p,q}_{orb}(X) \times H^{n-p,n-q}_{orb,c}(X) \to \mathbf{C}$$

similarly defined as in the previous proposition. When X is closed and Kahler, the following is true:

- $H^r_{orb}(X) \otimes \mathbf{C} = \oplus_{r=p+q} H^{p,q}_{orb}(X)$,
- $H^{p,q}_{orb}(X) = \overline{H^{q,p}_{orb}(X)}$,

and the two pairings (Poincaré and Kodaira-Serre) coincide.

4. Orbifold Cup Product and Orbifold Cohomology Ring

4.1. Orbifold cup product. In this section, we give an explicit definition of the orbifold cup product. Its interpretation in terms of Gromov-Witten invariants and the proof of associativity of the product will be given in subsequent sections.

Let X be an orbifold, and (V_p, G_p, π_p) be a uniformizing system at point $p \in X$. We define the *k-multi-sector* of X, which is denoted by $\widetilde{X}_k$, to be the set of all pairs $(p, (\mathbf{g}))$, where $p \in X$, $\mathbf{g} = (g_1, \cdots, g_k)$ with each $g_i \in G_p$, and $(\mathbf{g})$ stands for the conjugacy class of $\mathbf{g} = (g_1, \cdots, g_k)$. Here two k-tuple $(g_1^{(i)}, \cdots, g_k^{(i)})$, $i = 1, 2$, are conjugate if there is a $g \in G_p$ such that $g_j^{(2)} = g g_j^{(1)} g^{-1}$ for all $j = 1, \cdots, k$.

Lemma 4.1.1. *The k-multi-sector $\widetilde{X}_k$ is naturally an orbifold, with the orbifold structure given by*

$$\{\pi_{p,\mathbf{g}} : (V_p^{\mathbf{g}}, C(\mathbf{g})) \to V_p^{\mathbf{g}}/C(\mathbf{g})\}, \tag{4.1.1}$$

where $V_p^{\mathbf{g}} = V_p^{g_1} \cap V_p^{g_2} \cap \cdots \cap V_p^{g_k}$, $C(\mathbf{g}) = C(g_1) \cap C(g_2) \cap \cdots \cap C(g_k)$. Here $\mathbf{g} = (g_1, \cdots, g_k)$, V_p^g stands for the fixed-point set of $g \in G_p$ in V_p, and $C(g)$ for the centralizer of g in G_p. For each $i = 1, \cdots, k$, there is a C^∞ map $e_i : \widetilde{X}_k \to \widetilde{X}$ defined by sending $(p, (\mathbf{g}))$ to $(p, (g_i))$, where $\mathbf{g} = (g_1, \cdots, g_k)$. When X is almost complex, $\widetilde{X}_k$ inherits an almost complex structure from X, and when X is closed, $\widetilde{X}_k$ is a finite disjoint union of closed orbifolds.

Proof. The proof is parallel to the proof of Lemma 3.1.1 where $\widetilde{X}$ is shown to be an orbifold.

First we identify a point $(q, (\mathbf{h}))$ in $\widetilde{X}_k$ as a point in $\bigsqcup_{\{(p,(\mathbf{g}))\in\widetilde{X}_k\}} V_p^{\mathbf{g}}/C(\mathbf{g})$ if $q \in U_p$. Pick a representative $y \in V_p$ such that $\pi_p(y) = q$. Then this gives rise to a monomorphism $\lambda_y : G_q \to G_p$. Pick a representative $\mathbf{h} = (h_1, \cdots, h_k) \in G_q \times \cdots \times G_q$ for $(\mathbf{h})$, we let $\mathbf{g} = \lambda_y(\mathbf{h})$. Then $y \in V_p^{\mathbf{g}}$. So we have a map $\theta : (q, \mathbf{h}) \to (y, \mathbf{g})$. If we change $\mathbf{h}$ by $\mathbf{h}' = a^{-1}\mathbf{h}a$ for some $a \in G_q$, then $\mathbf{g}$ is changed to $\lambda_y(a^{-1}\mathbf{h}a) = \lambda_y(a)^{-1}\mathbf{g}\lambda_y(a)$. So we have $\theta : (q, a^{-1}\mathbf{h}a) \to (y, \lambda_y(a)^{-1}\mathbf{g}\lambda_y(a))$, where y is regarded as a point in $V_p^{\lambda_y(a)^{-1}\mathbf{g}\lambda_y(a)}$. (Note that λ_y is determined up to conjugacy by an element in G_q.) If we take a different representative $y' \in V_p$ such that $\pi_p(y') = q$, and suppose $y' = b \cdot y$ for some $b \in G_p$. Then we have a different identification $\lambda_{y'} : G_q \to G_p$ of G_q as a subgroup of G_p, where $\lambda_{y'} = b \cdot \lambda_y \cdot b^{-1}$. In this case, we have $\theta : (q, \mathbf{h}) \to (y', b\mathbf{g}b^{-1})$, where $y' \in V_p^{b\mathbf{g}b^{-1}}$. If $\mathbf{g} = b\mathbf{g}b^{-1}$, then $b \in C(\mathbf{g})$. Therefore we have shown that θ induces a map sending $(q, (\mathbf{h}))$ to a point in $\bigsqcup_{\{(p,(\mathbf{g}))\in\widetilde{X}_k\}} V_p^{\mathbf{g}}/C(\mathbf{g})$, which can be similarly shown to be one to one and onto. Hence we have shown that $\widetilde{X}_k$ is covered by $\bigsqcup_{\{(p,(\mathbf{g}))\in\widetilde{X}_k\}} V_p^{\mathbf{g}}/C(\mathbf{g})$.

We define a topology on $\widetilde{X}_k$ so that each $V_p^{\mathbf{g}}/C(\mathbf{g})$ is an open subset for any $(p, \mathbf{g})$. We also uniformize $V_p^{\mathbf{g}}/C(\mathbf{g})$ by $(V_p^{\mathbf{g}}, C(\mathbf{g}))$. It remains to show that these charts fit together to form an orbifold structure on $\widetilde{X}_k$. Let $x \in V_p^{\mathbf{g}}/C(\mathbf{g})$ and take a representative $\tilde{x}$ in $V_p^{\mathbf{g}}$. Let H_x be the isotropy subgroup of $\tilde{x}$ in $C(\mathbf{g})$. Then $(V_p^{\mathbf{g}}, C(\mathbf{g}))$ induces a germ of uniformizing system at x as (B_x, H_x), where B_x is a small ball in $V_p^{\mathbf{g}}$ centered at $\tilde{x}$. Let $\pi_p(\tilde{x}) = q$. We need to write (B_x, H_x) as $(V_q^{\mathbf{h}}, C(\mathbf{h}))$ for some $\mathbf{h} \in G_q \times \cdots \times G_q$. We let $\lambda_x : G_q \to G_p$ be an induced monomorphism resulting from choosing $\tilde{x}$ as the representative of q in V_p. We define $\mathbf{h} = \lambda_x^{-1}(\mathbf{g})$ (each g_i is in $\lambda_x(G_q)$ since $\tilde{x} \in V_p^{\mathbf{g}}$ and $\pi_p(\tilde{x}) = q$). Then we can identify B_x as $V_q^{\mathbf{h}}$. We also see that $H_x = \lambda_x(C(\mathbf{h}))$. Therefore (B_x, H_x) is identified as $(V_q^{\mathbf{h}}, C(\mathbf{h}))$. Hence we proved that $\widetilde{X}_k$ is naturally an orbifold with the orbifold structure described above ($\widetilde{X}_k$ is Hausdorff and second countable with the given topology for similar reasons). The rest of the lemma is obvious. □

We can also describe the components of $\widetilde{X}_k$ in the same fashion. Using the conjugacy class of monomorphisms $\pi_{pq} : G_q \to G_p$ in the patching condition, we can define an equivalence relation $(\mathbf{g})_{G_q} \sim (\pi_{pq}(\mathbf{g}))_{G_p}$ similarly. Let T_k be the set of equivalence classes. We will write a general element of T_k as $(\mathbf{g})$. Then $\widetilde{X}_k$ is decomposed as a disjoint union of connected orbifolds

$$\widetilde{X}_k = \bigsqcup_{(\mathbf{g})\in T_k} X_{(\mathbf{g})}, \tag{4.1.2}$$

where

$$X_{(\mathbf{g})} = \{(p, (\mathbf{g}')_{G_p}) | (\mathbf{g}')_{G_p} \in (\mathbf{g})\}. \tag{4.1.3}$$

There is a map $o : T_k \to T$ induced by the map $o : (g_1, g_2, \cdots, g_k) \mapsto g_1 g_2 \cdots g_k$. We set $T_k^o = o^{-1}((1))$. Then $T_k^o \subset T_k$ is the subset of equivalence classes $(\mathbf{g})$ such that $\mathbf{g} = (g_1, \cdots, g_k)$ satisfies the condition $g_1 \cdots g_k = 1$. Finally, we set

$$\widetilde{X}_k^o := \bigsqcup_{(\mathbf{g})\in T_k^o} X_{(\mathbf{g})}. \tag{4.1.4}$$

In order to define the orbifold cup product, we need a digression on a few classical results about *reduced* 2-dimensional orbifolds (cf. [Th, Sc]). Every closed orbifold of dimension 2 is complex, whose underlying topological space is a closed Riemann surface. More concretely, a closed, reduced 2-dimensional orbifold consists of the following data: a closed Riemann surface Σ with complex structure j, a finite subset of distinct points $\mathbf{z} = (z_1, \cdots, z_k)$ on Σ, each with a multiplicity $m_i \geq 2$ (let $\mathbf{m} = (m_1, \cdots, m_k)$), such that the orbifold structure at z_i is given by the ramified covering $z \to z^{m_i}$. We will also call a closed, reduced 2-dimensional orbifold a *complex orbicurve* when the underlying complex analytic structure is emphasized.

A C^∞ map $\tilde{\pi}$ between two reduced connected 2-dimensional orbifolds is called an *orbifold covering* if the local liftings of $\tilde{\pi}$ are either a diffeomorphism or a ramified covering. It is shown that the universal orbifold covering exists, and its group of deck transformations is defined to be the *orbifold fundamental group* of the orbifold. In fact, given a reduced 2-orbifold Σ, with orbifold fundamental group denoted by $\pi_1^{orb}(\Sigma)$, for any subgroup Γ of $\pi_1^{orb}(\Sigma)$, there is a reduced 2-orbifold $\widetilde{\Sigma}$ and an orbifold covering $\tilde{\pi} : \widetilde{\Sigma} \to \Sigma$ such that $\tilde{\pi}$ induces an injective homomorphism $\pi_1^{orb}(\widetilde{\Sigma}) \to \pi_1^{orb}(\Sigma)$ with image $\Gamma \subset \pi_1^{orb}(\Sigma)$. The orbifold fundamental group of a reduced, closed 2-orbifold $(\Sigma, \mathbf{z}, \mathbf{m})$ has a presentation

$$\pi_1^{orb}(\Sigma) = \{x_i, y_i, \lambda_j, i = 1, \cdots, g, j = 1, \cdots, k | \prod_i x_i y_i x_i^{-1} y_i^{-1} \prod_j \lambda_j = 1, \lambda_j^{m_j} = 1\},$$

where g is the genus of Σ, $\mathbf{z} = (z_1, \cdots, z_k)$ and $\mathbf{m} = (m_1, \cdots, m_k)$.

The remaining ingredient is to construct an "obstruction bundle" $E_{(\mathbf{g})}$, over each component $X_{(\mathbf{g})}$, where $(\mathbf{g}) \in T_3^o$. For this purpose, we consider the Riemann sphere S^2 with three distinct marked points $\mathbf{z} = (0, 1, \infty)$. Suppose $(\mathbf{g})$ is represented by $\mathbf{g} = (g_1, g_2, g_3)$ and the order of g_i is m_i for $i = 1, 2, 3$. We give a reduced orbifold structure on S^2 by assigning $\mathbf{m} = (m_1, m_2, m_3)$ as the multiplicity of $\mathbf{z}$. The orbifold fundamental group $\pi_1^{orb}(S^2)$ has the following presentation:

$$\pi_1^{orb}(S^2) = \{\lambda_1, \lambda_2, \lambda_3 | \lambda_i^{m_i} = 1, \lambda_1 \lambda_2 \lambda_3 = 1\},$$

where each generator λ_i is geometrically represented by a loop around the marked point z_i (here recall that $(z_1, z_2, z_3) = (0, 1, \infty)$).

Now for each point $(p, (\mathbf{g})_{G_p}) \in X_{(\mathbf{g})}$, fix a representation $\mathbf{g}$ of $(\mathbf{g})_{G_p}$, where $\mathbf{g} = (g_1, g_2, g_3)$, we define a homomorphism $\rho_{p,\mathbf{g}} : \pi_1^{orb}(S^2) \to G_p$ by sending λ_i to g_i, which is possible since $g_1g_2g_3 = 1$. Let $G \subset G_p$ be the image of $\rho_{p,\mathbf{g}}$. There is a reduced 2-orbifold Σ and an orbifold covering $\tilde{\pi} : \Sigma \to S^2$, which induces the following short exact sequence:

$$1 \to \pi_1(\Sigma) \to \pi_1^{orb}(S^2) \to G \to 1.$$

The group G acts on Σ as the group of deck transformations, whose finiteness implies that Σ is closed. Moreover, Σ actually has a trivial orbifold structure (i.e. Σ is a Riemann surface) since each map $\lambda_i \mapsto g_i$ is injective, and we can assume G acts on Σ holomorphically. In the end, we obtained a uniformizing system $(\Sigma, G, \tilde{\pi})$ of $(S^2, \mathbf{z}, \mathbf{m})$, which depends on $(p, \mathbf{g})$, but is locally constant.

The "obstruction bundle" $E_{(\mathbf{g})}$ over $X_{(\mathbf{g})}$ is constructed as follows. On the local chart $(V_p^{\mathbf{g}}, C(\mathbf{g}))$ of $X_{(g)}$, $E_{(\mathbf{g})}$ is given by $(H^1(\Sigma) \otimes TV_p)^G \times V_p^{\mathbf{g}} \to V_p^{\mathbf{g}}$, where $(H^1(\Sigma) \otimes TV_p)^G$ is the invariant subspace of G. We define an action of $C(\mathbf{g})$ on $H^1(\Sigma) \otimes TV_p$, which is trivial on the first factor and the usual one on TV_p, then it is clear that $C(\mathbf{g})$ commutes with G, hence $(H^1(\Sigma) \otimes TV_p)^G$ is invariant under $C(\mathbf{g})$. In summary, we have obtained an action of $C(\mathbf{g})$ on $(H^1(\Sigma)\otimes TV_p)^G \times V_p^{\mathbf{g}} \to V_p^{\mathbf{g}}$, extending the usual one on $V_p^{\mathbf{g}}$, and it is easily seen that these trivializations fit together to define the bundle $E_{(\mathbf{g})}$ over $X_{(\mathbf{g})}$. If we set $e : X_{(\mathbf{g})} \to X$ to be the C^∞ map $(p, (\mathbf{g})_{G_p}) \mapsto p$, one may think of $E_{(\mathbf{g})}$ as $(H^1(\Sigma) \otimes e^*TX)^G$.

Since we do not assume that X is compact, $X_{(\mathbf{g})}$ could be a non-compact orbifold in general. The Euler class of $E_{(\mathbf{g})}$ depends on a choice of a connection on $E_{(\mathbf{g})}$. Let $e_A(E_{(\mathbf{g})})$ be the Euler form computed from a connection A by Chern-Weil theory.

Definition 4.1.2. *For $\alpha, \beta \in H^*_{orb}(X)$, and $\gamma \in H^*_{orb,c}(X)$, we define a 3-point function*

$$< \alpha, \beta, \gamma >_{orb} = \sum_{(\mathbf{g})\in T_3^0} \int^{orb}_{X_{(\mathbf{g})}} e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge e_A(E_{(\mathbf{g})}), \tag{4.1.5}$$

where each $e_i : X_{(\mathbf{g})} \to \widetilde{X}$ is the C^∞ map defined by $(p, (\mathbf{g})_{G_p}) \mapsto (p, (g_i)_{G_p})$ for $\mathbf{g} = (g_1, g_2, g_3)$. Integration over orbifolds is defined by Eq. (2.4).

Note that since γ is compact supported, each integral is finite, and the summation is over a finite subset of T_3^o. Moreover, if we choose a different connection A', $e_A(E_{(\mathbf{g})}), e_{A'}(E_{(\mathbf{g})})$ differ by an exact form. Hence the 3-point function is independent of the choice of the connection A.

Definition 4.1.3. *We define the orbifold cup product on $H^*_{orb}(X)$ by the relation*

$$< \alpha \cup_{orb} \beta, \gamma >_{orb} \; = \; < \alpha, \beta, \gamma >_{orb} . \tag{4.1.6}$$

Next we shall give a decomposition of the orbifold cup product $\alpha \cup_{orb} \beta$ according to the decomposition $H^*_{orb}(X) = \oplus_{(g)\in T} H^{*-2\iota_{(g)}}(X_{(g)})$, when α, β are homogeneous, i.e. $\alpha \in H^*(X_{(g_1)})$ and $\beta \in H^*(X_{(g_2)})$ for some $(g_1), (g_2) \in T$. We need to introduce some notation first. Given $(g_1), (g_2) \in T$, let $T((g_1), (g_2))$ be the subset of T_2 which consists of $(\mathbf{h})$, where $\mathbf{h} = (h_1, h_2)$ satisfies $(h_1) = (g_1)$ and $(h_2) = (g_2)$. Recall that there is a map $o : T_k \to T$ defined by sending $(g_1, g_2, \cdots, g_k)$ to $g_1g_2 \cdots g_k$. We define a map $\delta : \mathbf{g} \mapsto (\mathbf{g}, o(\mathbf{g})^{-1})$, which clearly induces a one to one correspondence between T_k and T_{k+1}^o. We also denote by δ the resulting isomorphism $\widetilde{X}_k \cong \widetilde{X}_{k+1}^o$. Finally, we set $\delta_i = e_i \circ \delta$.

Decomposition Lemma 4.1.4. *For any* $\alpha \in H^*(X_{(g_1)})$, $\beta \in H^*(X_{(g_2)})$,

$$\alpha \cup_{orb} \beta = \sum_{(\mathbf{h}) \in T((g_1),(g_2))} (\alpha \cup_{orb} \beta)_{(\mathbf{h})}, \tag{4.1.7}$$

where $(\alpha \cup_{orb} \beta)_{(\mathbf{h})} \in H^*(X_{o((\mathbf{h}))})$ *is defined by the relation*

$$< (\alpha \cup_{orb} \beta)_{o((\mathbf{h}))}, \gamma >_{orb} = \int_{X_{(\mathbf{h})}}^{orb} \delta_1^* \alpha \wedge \delta_2^* \beta \wedge \delta_3^* \gamma \wedge e_A(\delta^* E_{\delta(\mathbf{h})}), \tag{4.1.8}$$

for $\gamma \in H_c^*(X_{(o(\mathbf{h})^{-1})})$.

In the subsequent sections, we shall describe the 3-point function and orbifold cup product in terms of Gromov-Witten invariants. In fact, we will prove the following

Theorem 4.1.5. *Let X be an almost complex orbifold with almost complex structure J and* $\dim_{\mathbf{C}} X = n$. *The orbifold cup product preserves the orbifold grading, i.e.,*

$$\cup_{orb} : H^p_{orb}(X) \times H^q_{orb}(X) \to H^{p+q}_{orb}(X)$$

for any $0 \le p, q \le 2n$ *such that* $p + q \le 2n$, *and has the following properties:*

1. *The total orbifold cohomology group* $H^*_{orb}(X) = \oplus_{0 \le d \le 2n} H^d_{orb}(X)$ *is a ring with unit* $e^0_X \in H^0(X)$ *under* $\cup_{orb}$, *where* e^0_X *is the Poincaré dual to the fundamental class* $[X]$. *In particular,* $\cup_{orb}$ *is associative.*
2. *When X is closed, for each* $H^d_{orb}(X) \times H^{2n-d}_{orb}(X) \to H^{2n}_{orb}(X)$, *we have*

$$\int_X^{orb} \alpha \cup_{orb} \beta = < \alpha, \beta >_{orb} . \tag{4.1.9}$$

3. *The cup product* $\cup_{orb}$ *is invariant under deformation of J.*
4. *When X is of integral degree shifting numbers, the total orbifold cohomology group* $H^*_{orb}(X)$ *is integrally graded, and we have supercommutativity*

$$\alpha_1 \cup_{orb} \alpha_2 = (-1)^{\deg \alpha_1 \cdot \deg \alpha_2} \alpha_2 \cup_{orb} \alpha_1.$$

5. *Restricted to the nontwisted sectors, i.e., the ordinary cohomologies* $H^*(X)$, *the cup product* $\cup_{orb}$ *equals the ordinary cup product on X.*

When X is a complex orbifold, the definition of orbifold cup product $\cup_{orb}$ on the total orbifold Dolbeault cohomology group of X is completely parallel. We observe that in this case all the objects we have been dealing with are holomorphic, i.e., $\widetilde{X}_k$ is a complex orbifold, the "obstruction bundles" $E_{(\mathbf{g})} \to X_{(\mathbf{g})}$ are holomorphic orbifold bundles, and the evaluation maps e_i are holomorphic.

Definition 4.1.6. *For any* $\alpha_1 \in H^{p,q}_{orb}(X)$, $\alpha_2 \in H^{p',q'}_{orb}(X)$, *we define a 3-point function and orbifold cup product in the same fashion as in Definitions 4.1.2, 4.1.3.* □

Note that since the top Chern class of a holomorphic orbifold bundle can be represented by a closed (r, r)-form, where r is the (complex) rank of the bundle, it follows that the orbifold cup product preserves the orbifold bi-grading, i.e., $\cup_{orb} : H^{p,q}_{orb}(X) \times H^{p',q'}_{orb}(X) \to H^{p+p',q+q'}_{orb}(X)$.

The following theorem can be similarly proved.

Theorem 4.1.7. *Let X be a n-dimensional complex orbifold with complex structure J. The orbifold cup product*

$$\cup_{orb} : H^{p,q}_{orb}(X) \times H^{p',q'}_{orb}(X) \to H^{p+p',q+q'}_{orb}(X)$$

has the following properties:

1. *The total orbifold Dolbeault cohomology group is a ring with unit $e^0_X \in H^{0,0}_{orb}(X)$ under $\cup_{orb}$, where e^0_X is the class represented by the equaling-one constant function on X.*
2. *When X is closed, for each $H^{p,q}_{orb}(X) \times H^{n-p,n-q}_{orb}(X) \to H^{n,n}_{orb}(X)$, the integral $\int_X \alpha \cup_{orb} \beta$ equals the Kodaira-Serre pairing $< \alpha, \beta >_{orb}$.*
3. *The cup product $\cup_{orb}$ is invariant under deformation of J.*
4. *When X is of integral degree shifting numbers, the total orbifold Dolbeault cohomology group of X is integrally graded, and we have supercommutativity*

$$\alpha_1 \cup_{orb} \alpha_2 = (-1)^{\deg\alpha_1 \cdot \deg\alpha_2} \alpha_2 \cup_{orb} \alpha_1.$$

5. *Restricted to the nontwisted sectors, i.e., the ordinary Dolbeault cohomologies $H^{*,*}(X)$, the cup product $\cup_{orb}$ coincides with the ordinary wedge product on X.*
6. *When X is closed Kahler or projective, the cup product $\cup_{orb}$ coincides with the orbifold cup product on the total orbifold cohomology group $H^*_{orb}(X)$ under the relation*

$$H^r_{orb}(X) \otimes \mathbf{C} = \oplus_{p+q=r} H^{p,q}_{orb}(X),$$

and hence is associative.

4.2. Moduli space of ghost maps. We first give a classification of rank-n complex orbifold bundles over a closed, *reduced*, 2-dimensional orbifold.

Let $(\Sigma, \mathbf{z}, \mathbf{m})$ be a closed, reduced, 2-dimensional orbifold, where $\mathbf{z} = (z_1, \cdots, z_k)$ and $\mathbf{m} = (m_1, \cdots, m_k)$. Let E be a complex orbifold bundle of rank n over $(\Sigma, \mathbf{z}, \mathbf{m})$. Then at each singular point z_i, $i = 1, \cdots, k$, E determines a representation $\rho_i : \mathbf{Z}_{m_i} \to Aut(\mathbf{C}^n)$ so that over a disc neighborhood of z_i, E is uniformized by $(D \times \mathbf{C}^n, \mathbf{Z}_{m_i}, \pi)$, where the action of $\mathbf{Z}_{m_i}$ on $D \times \mathbf{C}^n$ is given by

$$e^{2\pi i/m_i} \cdot (z, w) = (e^{2\pi i/m_i} z, \rho_i(e^{2\pi i/m_i}) w) \tag{4.2.1}$$

for any $w \in \mathbf{C}^n$. Each representation ρ_i is uniquely determined by a n-tuple of integers $(m_{i,1}, \cdots, m_{i,n})$ with $0 \le m_{i,j} < m_i$, as it is given by the matrix

$$\rho_i(e^{2\pi i/m_i}) = diag(e^{2\pi i m_{i,1}/m_i}, \cdots, e^{2\pi i m_{i,n}/m_i}). \tag{4.2.2}$$

Over the punctured disc $D_i \setminus \{0\}$ at z_i, E inherits a specific trivialization from $(D \times \mathbf{C}^n, \mathbf{Z}_{m_i}, \pi)$ as follows: We define a $\mathbf{Z}_{m_i}$-equivariant map $\Psi_i : D \setminus \{0\} \times \mathbf{C}^n \to D \setminus \{0\} \times \mathbf{C}^n$ by

$$(z, w_1, \cdots, w_n) \to (z^{m_i}, z^{-m_{i,1}} w_1, \cdots, z^{-m_{i,n}} w_n), \tag{4.2.3}$$

where Z_{m_i} acts trivially on the second $D \setminus \{0\} \times \mathbf{C}^n$. Hence Ψ_i induces a trivialization $\psi_i : E_{D_i \setminus \{0\}} \to D_i \setminus \{0\} \times \mathbf{C}^n$. We can extend the smooth complex vector bundle $E_{\Sigma \setminus \mathbf{z}}$ over $\Sigma \setminus \mathbf{z}$ to a smooth complex vector bundle over Σ by using these trivializations ψ_i. We call the resulting complex vector bundle the *de-singularization* of E, and denote it by $|E|$.

Proposition 4.2.1. *The space of isomorphism classes of complex orbifold bundles of rank n over a closed, reduced, 2-dimensional orbifold $(\Sigma, \mathbf{z}, \mathbf{m})$, where $\mathbf{z} = (z_1, \cdots, z_k)$ and $\mathbf{m} = (m_1, \cdots, m_k)$, is in 1:1 correspondence with the set of $(c, (m_{1,1}, \cdots, m_{1,n}), \cdots, (m_{k,1}, \cdots, m_{k,n}))$ for $c \in \mathbf{Q}$, $m_{i,j} \in \mathbf{Z}$, where c and $m_{i,j}$ are confined by the following condition:*

$$0 \leq m_{i,j} < m_i \quad \textit{and} \quad c \equiv \sum_{i=1}^{k}\sum_{j=1}^{n} \frac{m_{i,j}}{m_i} \quad (\text{mod } \mathbf{Z}). \tag{4.2.4}$$

In fact, c is the first Chern number of the orbifold bundle and $c - (\sum_{i=1}^{k}\sum_{j=1}^{n}\frac{m_{i,j}}{m_i})$ is the first Chern number of its de-singularization.

Proof. We only need to show the relation:

$$c_1(E)([\Sigma]) = c_1(|E|)([\Sigma]) + \sum_{i=1}^{k}\sum_{j=1}^{n} \frac{m_{i,j}}{m_i}. \tag{4.2.5}$$

We take a connection ∇_0 on $|E|$ which equals d on a disc neighborhood D_i of each $z_i \in \mathbf{z}$ so that $c_1(|E|)([\Sigma]) = \int_\Sigma c_1(\nabla_0)$. We use ∇_0' to denote the pull-back connection $br_i^*\nabla_0$ on $D \setminus \{0\} \times \mathbf{C}^n$ via $br_i : D \to D_i$ by $z \to z^{m_i}$. On the other hand, on each uniformizing system $(D \times \mathbf{C}^n, \mathbf{Z}_{m_i}, \pi)$, we take the trivial connection $\nabla_i = d$ which is obvious $\mathbf{Z}_{m_i}$-equivariant. Furthermore, we take a $\mathbf{Z}_{m_i}$-equivariant cut-off function β_i on D which equals one in a neighborhood of the boundary ∂D. We are going to paste these connections together to get a connection ∇ on E. We define ∇ on each uniformizing system $(D \times \mathbf{C}^n, \mathbf{Z}_{m_i}, \pi)$ by

$$\nabla_v u = (1 - \beta_i)(\nabla_i)_v u + \beta_i \bar{\psi}_i^{-1}(\nabla_0)_{\bar{\psi}_i v}\bar{\psi}_i u, \tag{4.2.6}$$

where $\bar{\psi}_i : D \setminus \{0\} \times \mathbf{C}^n \to D \setminus \{0\} \times \mathbf{C}^n$ is given by

$$(z, w_1, \cdots, w_n) \to (z, z^{-m_{i,1}}w_1, \cdots, z^{-m_{i,n}}w_n). \tag{4.2.7}$$

One easily verifies that $F(\nabla) = F(\nabla_0)$ on $\Sigma \setminus (\cup_i D_i)$ and

$$F(\nabla) = -diag(d(\beta_i m_{i,1}dz/z), \cdots, d(\beta_i m_{i,n}dz/z))$$

on each uniformizing system $(D, \mathbf{Z}_{m_i}, \pi)$. So

$$\begin{aligned} c_1(E)([\Sigma]) &= \int_\Sigma^{orb} c_1(\nabla) \\ &= \int_{\Sigma\setminus(\cup_i D_i)} c_1(\nabla_0) + \sum_{i=1}^{k}\frac{1}{m_i}\int_D c_1(\nabla) \\ &= c_1(|E|)([\Sigma]) + \sum_{i=1}^{k}\sum_{j=1}^{n}\frac{m_{i,j}}{m_i}. \end{aligned}$$

Here the integraton over Σ, $\int_\Sigma^{orb}$, should be understood as in (2.4). $\square$

We will need the following index formula.

Proposition 4.2.2. *Let E be a holomorphic orbifold bundle of rank n over a complex orbicurve $(\Sigma, \mathbf{z}, \mathbf{m})$ of genus g. Then $\mathcal{O}(E) = \mathcal{O}(|E|)$, where $\mathcal{O}(E)$, $\mathcal{O}(|E|)$ are sheaves of holomorphic sections of E, $|E|$. Hence,*

$$\chi(\mathcal{O}(E)) = \chi(\mathcal{O}(|E|)) = c_1(|E|)([\Sigma]) + n(1-g). \tag{4.2.9}$$

If E corresponds to $(c, (m_{1,1}, \cdots, m_{1,n}), \cdots, (m_{k,1}, \cdots, m_{k,n}))$ (cf. Proposition 4.2.1), then we have

$$\chi(\mathcal{O}(E)) = n(1-g) + c_1(E)([\Sigma]) - \sum_{i=1}^{k}\sum_{j=1}^{n}\frac{m_{i,j}}{m_i}.$$

Proof. By construction, we have $\mathcal{O}(E) = \mathcal{O}(|E|)$. Hence

$$\chi(\mathcal{O}(E)) = \chi(\mathcal{O}(|E|)) = c_1(|E|)([\Sigma]) + n(1-g). \tag{4.2.10}$$

By Proposition 4.2.1, we have

$$\chi(\mathcal{O}(E)) = n(1-g) + c_1(E)([\Sigma]) - \sum_{i=1}^{k}\sum_{j=1}^{n}\frac{m_{i,j}}{m_i},$$

if E corresponds to $(c, (m_{1,1}, \cdots, m_{1,n}), \cdots, (m_{k,1}, \cdots, m_{k,n}))$. □

Now we come to the main issue of this section. Suppose $f : X \to X'$ is a C^∞ map between manifolds and E is a smooth vector bundle over X', then there is a smooth pull-back vector bundle f^*E over X and a bundle morphism $\bar{f} : f^*E \to E$ which covers the map f. However, if instead, we have a C^∞ map $\tilde{f}$ between orbifolds X and X', and an orbifold bundle E over orbifold X', the question whether there is a pull-back orbifold bundle E^* over X' and an orbifold bundle morphism $\bar{f} : E^* \to E$ covering the map $\tilde{f}$ is a quite complicated issue: (1) What is the precise meaning of pull-back orbifold bundle E^*, (2) E^* might not exist, or even if it exists, it might not be unique. Understanding this question is the first step in our establishment of an *orbifold Gromov-Witten theory* in [CR].

In the present case, given a constant map $f : \Sigma \to X$ from a marked Riemann surface Σ with marked-point set $\mathbf{z}$ into an almost complex orbifold X, we need to settle the existence and classification problem of pull-back orbifold bundles via f, with some reduced orbifold structure on Σ, whose set of orbifold points is contained in the given marked-point set $\mathbf{z}$.

Let $(S^2, \mathbf{z})$ be a genus-zero Riemann surface with k-marked points $\mathbf{z} = (z_1, \cdots, z_k)$, $p \in X$ any point in an almost complex orbifold X with $\dim_{\mathbf{C}} X = n$, and (V_p, G_p, π_p) a local chart at p. Then for any k-tuple $\mathbf{g} = (g_1, \cdots, g_k)$ where $g_i \in G_p$, $i = 1, \cdots, k$, there is an orbifold structure on S^2 so that it becomes a complex orbicurve $(S^2, \mathbf{z}, \mathbf{m})$, where $\mathbf{m} = (|g_1|, \cdots, |g_k|)$ (here $|g|$ stands for the order of g). If further assuming that $o(\mathbf{g}) = g_1 g_2 \cdots g_k = 1_{G_p}$, one can construct a rank-n holomorphic orbifold bundle $E_{p,\mathbf{g}}$ over $(S^2, \mathbf{z}, \mathbf{m})$, together with an orbifold bundle morphism $\Phi_{p,\mathbf{g}} : E_{p,\mathbf{g}} \to TX$ covering the constant map from S^2 to $p \in X$, as we shall see next.

Denote $\mathbf{1}_{G_p} = (1_{G_p}, \cdots, 1_{G_p})$. The case $\mathbf{g} = \mathbf{1}_{G_p}$ is trivial; we simply take the rank-n trivial holomorphic bundle over S^2. Hence in what follows, we assume that $\mathbf{g} \neq \mathbf{1}_{G_p}$. We recall that the orbifold fundamental group of $(S^2, \mathbf{z}, \mathbf{m})$ is given by

$$\pi_1^{orb}(S^2) = \{\lambda_1, \lambda_2, \cdots, \lambda_k | \lambda_i^{|g_i|} = 1, \lambda_1\lambda_2\cdots\lambda_k = 1\},$$

where each generator λ_i is geometrically represented by a loop around the marked point z_i. We define a homomorphism $\rho : \pi_1^{orb}(S^2) \to G_p$ by sending each λ_i to $g_i \in G_p$ (note that we assumed that $g_1g_2\cdots g_k = 1_{G_p}$). There is a closed Riemann surface Σ and a finite group G acting on Σ holomorphically, such that (Σ, G) uniformizes $(S^2, \mathbf{z}, \mathbf{m})$ and $\pi_1(\Sigma) = \ker\ \rho$ with $G = Im\rho \subset G_p$. We identify $(TV_p)_p$ with $\mathbf{C}^n$ and denote the rank-n trivial holomorphic vector bundle on Σ by $\underline{\mathbf{C}^n}$. The representation $G \to Aut((TV_p)_p)$ defines a holomorphic action on the holomorphic vector bundle $\underline{\mathbf{C}}^n$. We take $E_{p,\mathbf{g}}$ to be the corresponding holomorphic orbifold bundle uniformized by $(\underline{\mathbf{C}}^n, G, \tilde{\pi})$, where $\tilde{\pi} : \underline{\mathbf{C}}^n \to \underline{\mathbf{C}}^n/G$ is the quotient map. There is a natural orbifold bundle morphism $\Phi_{p,\mathbf{g}} : E_{p,\mathbf{g}} \to TX$ sending Σ to the point p.

By the nature of construction, if $\mathbf{g} = (g_1, \cdots, g_k)$ and $\mathbf{g}' = (g_1', \cdots, g_k')$ are conjugate, i.e., there is an element $g \in G_p$ such that $g_i' = g^{-1}g_ig$, then there is an isomorphism $\psi : E_{p,\mathbf{g}} \to E_{p,\mathbf{g}'}$ such that $\Phi_{p,\mathbf{g}} = \Phi_{p,\mathbf{g}'} \circ \psi$.

If there is an isomorphism $\psi : E_{p,\mathbf{g}} \to E_{p,\mathbf{g}'}$ such that $\Phi_{p,\mathbf{g}} = \Phi_{p,\mathbf{g}'} \circ \psi$, then there is a lifting $\tilde{\psi} : \tilde{E}_{p,\mathbf{g}} \to \tilde{E}_{p,\mathbf{g}'}$ of ψ and an automorphism $\phi : TV_p \to TV_p$, such that $\phi \circ \tilde{\Phi}_{p,\mathbf{g}} = \tilde{\Phi}_{p,\mathbf{g}'} \circ \tilde{\psi}$. If ϕ is given by the action of an element $g \in G_p$, then we have $gg_ig^{-1} = g_i'$ for all $i = 1, \cdots, k$.

Lemma 4.2.3. *Let E be a rank-n holomorphic orbifold bundle over $(S^2, \mathbf{z}, \mathbf{m})$ (for some $\mathbf{m}$). Suppose that there is an orbifold bundle morphism $\Phi : E \to TX$ covering a constant map from S^2 into X. Then there is a $(p, \mathbf{g})$ such that $(E, \Phi) = (E_{p,\mathbf{g}}, \Phi_{p,\mathbf{g}})$.*

Proof. Let E be a rank-n holomorphic orbifold bundle over $(S^2, \mathbf{z}, \mathbf{m})$ with a morphism $\Phi : E \to TX$ covering the constant map to a point p in X. We will find a $\mathbf{g}$ so that $(E, \Phi) = (E_{p,\mathbf{g}}, \Phi_{p,\mathbf{g}})$. For this purpose, we again consider the uniformizing system (Σ, G, π) of $(S^2, \mathbf{z}, \mathbf{m})$, where Σ is a closed Riemann surface with a holomorphic action by a finite group G. Then there is a holomorphic vector bundle $\tilde{E}$ over Σ with a compatible action of G so that $(\tilde{E}, G)$ uniformizes the holomorphic orbifold bundle E. Moreover, there is a vector bundle morphism $\tilde{\Phi} : \tilde{E} \to TV_p$, which is a lifting of Φ so that for any $a \in G$, there is a $\tilde{\lambda}(a)$ in G_p such that $\tilde{\Phi} \circ a = \tilde{\lambda}(a) \circ \Phi$. In fact, $a \to \tilde{\lambda}(a)$ defines a homomorphism $\tilde{\lambda} : G \to G_p$. Since $\tilde{\Phi}$ covers a constant map from Σ into V_p, the holomorphic vector bundle $\tilde{E}$ is in fact a trivial bundle. Recall that G is the quotient group of $\pi_1^{orb}(S^2)$ by the normal subgroup $\pi_1(\Sigma)$. Let λ be the induced homomorphism $\pi_1^{orb}(S^2) \to G_p$, and let $g_i = \lambda(\gamma_i)$. Then we have $g_1g_2\cdots g_k = 1_{G_p}$. We simply define $\mathbf{g} = (g_1, g_2, \cdots, g_k)$. It is easily seen that $(E, \Phi) = (E_{p,\mathbf{g}}, \Phi_{p,\mathbf{g}})$. □

Definition 4.2.4. *Given a genus-zero Riemann surface with k-marked points $(\Sigma, \mathbf{z})$, where $\mathbf{z} = (z_1, \cdots, z_k)$, we call each equivalence class $[E_{p,\mathbf{g}}, \Phi_{p,\mathbf{g}}]$ of pair $(E_{p,\mathbf{g}}, \Phi_{p,\mathbf{g}})$ a* **ghost map** *from $(\Sigma, \mathbf{z})$ into X. A ghost map $[E, \Phi]$ from $(\Sigma, \mathbf{z})$ is said to be equivalent to a ghost map $[E', \Phi']$ from $(\Sigma', \mathbf{z}')$ ($\mathbf{z}' = (z_1', \cdots, z_k')$) if there is a holomorphic orbifold bundle morphism $\tilde{\psi} : E \to E'$ covering a biholomorphism $\psi : \Sigma \to \Sigma'$ such* *that $\psi(z_i) = z'_i$ and $\Phi = \Phi' \circ \tilde{\psi}$. An equivalence class of ghost maps is called a* **ghost curve** *(with k-marked points). We denote by $\mathcal{M}_k$ the moduli space of ghost curves with k-marked points.* □

As a consequence, we obtain

Proposition 4.2.5. *Let X be an almost complex orbifold. For any $k \geq 0$, the moduli space of ghost curves with k-marked points $\mathcal{M}_k$ is naturally an almost complex orbifold. When $k \geq 4$, $\mathcal{M}_k$ can be identified with $\mathcal{M}_{0,k} \times \widetilde{X}^o_k$, where $\mathcal{M}_{0,k}$ is the moduli space of genus-zero curve with k-marked points. It has a natural partial compactification $\overline{\mathcal{M}}_k$, which is an almost complex orbifold and can be identified with $\overline{\mathcal{M}}_{0,k} \times \widetilde{X}^o_k$, where $\overline{\mathcal{M}}_{0,k}$ is the Deligne-Mumford compactification of $\mathcal{M}_{0,k}$.*

Remark 4.2.6. (i) The natural partial compactification $\overline{\mathcal{M}}_k$ of $\mathcal{M}_k$ ($k \geq 4$) can be interpreted geometrically as adding *nodal* ghost curves into $\mathcal{M}_k$.
(ii) The space $\widetilde{X}^o_2$ is naturally identified with the graph of the map $I : \widetilde{\Sigma X} \to \widetilde{\Sigma X}$ in $\widetilde{\Sigma X} \times \widetilde{\Sigma X}$, where I is defined by $(p, (g)) \to (p, (g^{-1}))$.

Next, we construct a complex orbifold bundle E_k, a kind of obstruction bundle in nature, over the moduli space $\mathcal{M}_k$ of ghost curves with k-marked points. The rank of E_k may vary over different connected components of $\mathcal{M}_k$. When $k = 3$, the restriction of E_3 to each component gives a geometric construction of the obstruction bundle $E_{(\mathbf{g})}$ in the last section under identification $\mathcal{M}_3 = \widetilde{X}^o_3$.

Let us consider the space $\mathcal{C}_k$ of all triples $((\Sigma, \mathbf{z}), E_{p,\mathbf{g}}, \Phi_{p,\mathbf{g}})$, where $(\Sigma, \mathbf{z})$ is a genus-zero curve with k-marked points $\mathbf{z} = (z_1, \cdots, z_k)$, $E_{p,\mathbf{g}}$ is a rank-n holomorphic orbifold bundle over Σ, and $\Phi_{p,\mathbf{g}} : E_{p,\mathbf{g}} \to TX$ a morphism covering the constant map sending Σ to the point p in X. To each point $x \in \mathcal{C}_k$ we assign a complex vector space V_x, which is the cokernel of the operator

$$\bar{\partial} : \Omega^{0,0}(E_{p,\mathbf{g}}) \to \Omega^{0,1}(E_{p,\mathbf{g}}). \tag{4.2.11}$$

We introduce an equivalence relation $\sim$ amongst pairs (x, v) where $x \in \mathcal{C}_k$ and $v \in V_x$ as follows: Let $x = ((\Sigma, \mathbf{z}), E_{p,\mathbf{g}}, \Phi_{p,\mathbf{g}})$ and $x' = ((\Sigma', \mathbf{z}'), E_{p',\mathbf{g}'}, \Phi_{p',\mathbf{g}'})$, then $(x, v) \sim (x', v')$ if there is a morphism $\tilde{\psi} : E_{p,\mathbf{g}} \to E_{p',\mathbf{g}'}$ such that $\Phi_{p,\mathbf{g}} = \Phi_{p',\mathbf{g}'} \circ \tilde{\psi}$ and $\tilde{\psi}$ covers a biholomorphism $\psi : \Sigma \to \Sigma'$ satisfying $\psi(\mathbf{z}) = \mathbf{z}'$ (as ordered sets), and $v' = \psi_*(v)$, where $\psi_* : V_x \to V_{x'}$ is induced by $\tilde{\psi}$. We define E_k to be the quotient space of all (x, v) under this equivalence relation. There is obviously a surjective map $pr : E_k \to \mathcal{M}_k$ induced by the projection $(x, v) \to x$.

Lemma 4.2.7. *The space E_k can be given a topology such that $pr : E_k \to \mathcal{M}_k$ is a complex orbifold bundle over $\mathcal{M}_k$.*

Proof. First we show that the dimension of V_x is a local constant function of the equivalence class $[x]$ in $\mathcal{M}_k$. Recall a neighborhood of $[x]$ in $\mathcal{M}_k$ is given by $\mathcal{O} \times V^{\mathbf{g}}_p / C(\mathbf{g})$, where $\mathcal{O}$ is a neighborhood of the genus-zero curve with k-marked points $(\Sigma, \mathbf{z})$ in the moduli space $\mathcal{M}_{0,k}$. In fact, we will show that the kernel of (4.2.11) is identified with $(TV^{\mathbf{g}}_p)_p$, whose dimension is a local constant. Then it follows that $\dim V_x$ is locally constant as the dimension of cokernel of (4.2.11), since by Proposition 4.2.2, the index of (4.2.11) is locally constant.

For the identification of the kernel of (4.2.11), recall that the holomorphic orbifold bundle $E_{p,\mathbf{g}}$ over the genus-zero curve Σ is uniformized by the trivial holomorphic vector bundle $\underline{\mathbf{C}}^n$ over a Riemann surface $\tilde{\Sigma}$ with a holomorphic action of a finite group G. Hence the kernel of (4.2.11) is identified with the G-invariant holomorphic sections of the trivial bundle $\underline{\mathbf{C}}^n$, which are constant sections invariant under G. Through morphism $\Phi_{p,\mathbf{g}} : E_{p,\mathbf{g}} \to TX$, the kernel of $\bar{\partial}$ is then identified with $(TV_p^{\mathbf{g}})_p$.

Recall that the moduli space $\mathcal{M}_{0,k}$ is a smooth complex manifold. Let $\mathcal{O}$ be a neighborhood of $(\Sigma_0, \mathbf{z}_0)$ in $\mathcal{M}_{0,k}$. Then a neighborhood of $[x_0] = [(\Sigma_0, \mathbf{z}_0), E_{p,\mathbf{g}}, \Phi_{p,\mathbf{g}}]$ in $\mathcal{M}_k$ is uniformized by $(\mathcal{O} \times V_p^{\mathbf{g}}, C(\mathbf{g}))$ (cf. Lemma 4.1.1). More precisely, to any $((\Sigma, \mathbf{z}), y) \in \mathcal{O} \times V_p^{\mathbf{g}}$, we associate a rank-n holomorphic orbifold bundle over $(\Sigma, \mathbf{z})$ as follows: Let $q = \pi_p(y) \in U_p$, then the pair $(y, \mathbf{g})$ canonically determines a $\mathbf{h}_y \in G_q \times \cdots \times G_q$, and there is a canonically constructed holomorphic orbifold bundle $E_{q,\mathbf{h}_y}$ over $(\Sigma, \mathbf{z})$ with morphism $\Phi_{q,\mathbf{h}_y} : E_{q,\mathbf{h}_y} \to TX$ covering the constant map to q. Hence we have a family of holomorphic orbibundles over genus-zero curve with k-marked points, which are parametrized by $\mathcal{O} \times V_p^{\mathbf{g}}$. Moreover, it depends on the parameter in $\mathcal{O}$ holomorphically and the action of $C(\mathbf{g})$ on $V_p^{\mathbf{g}}$ coincides with the equivalence relation between the pairs of holomorphic orbifold bundle and morphism $(E_{q,\mathbf{h}_y}, \Phi_{q,\mathbf{h}_y})$. Now we put a Kahler metric on each genus-zero curve in $\mathcal{O}$ which is compatible to the complex structure and depends smoothly on the parameter in $\mathcal{O}$, and we also put a hermitian metric on X. Then we have a family of first order elliptic operators depending smoothly on the parameters in $\mathcal{O} \times V_p^{\mathbf{g}}$:

$$\bar{\partial}^* : \Omega^{0,1}(E_{q,\mathbf{h}_y}) \to \Omega^{0,0}(E_{q,\mathbf{h}_y})$$

and whose kernel gives rise to a complex vector bundle E_{x_0} over $\mathcal{O} \times V_p^{\mathbf{g}}$. The finite group $C(\mathbf{g})$ naturally acts on the complex vector bundle which coincides with the equivalence relation amongst the pairs (x, v), where $x \in \mathcal{C}_k$ and $v \in V_x$. Hence $(E_{x_0}, C(\mathbf{g}))$ is a uniformizing system for $pr^{-1}(\mathcal{O} \times V_p^{\mathbf{g}}/C(\mathbf{g}))$, which fits together to give an orbifold bundle structure for $pr : E_k \to \mathcal{M}_k$. $\square$

Remark 4.2.8. Recall that each holomorphic orbifold bundle $E_{p,\mathbf{g}}$ over $(S^2, \mathbf{z}, \mathbf{m})$ can be uniformized by a trivial holomorphic vector bundle $\underline{\mathbf{C}}^n$ over a Riemann surface Σ with a holomorphic group action by G. Hence each element ξ in the kernel of

$$\bar{\partial}^* : \Omega^{0,1}(E_{p,\mathbf{g}}) \to \Omega^{0,0}(E_{p,\mathbf{g}})$$

can be identified with a G-invariant harmonic $(0, 1)$-form on Σ with value in $(TV_p)_p$ (here we identify each fiber of $\underline{\mathbf{C}}^n$ with $(TV_p)_p$ through the morphism $\Phi_{p,\mathbf{g}}$), i.e., $\xi = w \otimes \alpha$ where $w \in (TV_p)_p$, α is a harmonic $(0, 1)$-form on $\tilde{\Sigma}$, and ξ is G-invariant. Therefore, when $k = 3$, it agrees with $E_{(\mathbf{g})}$. We observe that with respect to the taken hermitian metric on X, $w \in (TV_p)_p$ must lie in the orthogonal complement of $(TV_p^{\mathbf{g}})_p$ in $(TV_p)_p$. This is because: For any $u \in (TV_p^{\mathbf{g}})_p$ and a harmonic $(0, 1)$-form β on Σ, if $u \otimes \beta$ is G-invariant, then β is G-invariant too, which means that β descends to a harmonic $(0, 1)$-form on S^2, and β must be identically zero. $\square$

Recall the cup product is defined by equation

$$< \alpha_1 \cup_{orb} \alpha_2, \gamma >_{orb} = \left(\int_{\mathcal{M}_3}^{orb} e_1^*(\alpha_1) \cup e_2^*(\alpha_2) \cup e_3^*(\gamma) \cup e(E_3) \right),$$

where $e(E_3)$ is the Euler form of the complex orbifold bundle E_3 over $\mathcal{M}_3$ and $\gamma \in H^*_{orb,c}(X)$.

We take a basis $\{e_j\}$, $\{e^o_k\}$ of the total orbifold cohomology group $H^*_{orb}(X)$, $H^*_{orb,c}(X)$ such that each e_j, e^o_k is of homogeneous degree. Let $< e_j, e^o_k >_{orb} = a_{jk}$ be the Poincare pairing matrix and (a^{jk}) be the inverse. It is easy to check that the Poincaré dual of the graph of I in $\widetilde{\Sigma}^2$ can be written as $\sum_{j,k} a^{jk} e_j \otimes e^o_k$. Then,

$$\alpha_1 \cup_{orb} \alpha_2 = \sum_{j,k} e_j a^{kj} (\int^{orb}_{\mathcal{M}_3} e^*_1(\alpha_1) \cup e^*_2(\alpha_2) \cup e^*_3(e^o_k) \cup e(E_3)). \tag{4.2.12}$$

Proof of Theorem 4.1.5. We postpone the proof of associativity of $\cup_{orb}$ to the next subsection.

We first show that if $\alpha_1 \in H^p_{orb}(X)$ and $\alpha_2 \in H^q_{orb}(X)$, then $\alpha_1 \cup_{orb} \alpha_2$ is in $H^{p+q}_{orb}(X)$. For the integral in (4.2.12) to be nonzero,

$$\deg(e^*_1(\alpha_1)) + \deg(e^*_2(\alpha_2)) + \deg(e^*_3(e^o_k)) + \deg(e(E_3)) = 2 \dim_{\mathbf{C}} \mathcal{M}_3. \tag{4.2.13}$$

Here deg stands for the degree of a cohomology class without degree shifting. The degree of the Euler class $e(E_3)$ is equal to the dimension of the cokernel of (4.2.11), which by the index formula (cf. Proposition 4.2.2) equals $2 \dim_{\mathbf{C}} \mathcal{M}^{(i)}_3 - (2n - 2\sum^3_{j=1} \iota(p, g_j))$ on a connected component $\mathcal{M}^{(i)}_3$ containing the point $(p, (\mathbf{g}))$, where $\mathbf{g} = (g_1, g_2, g_3)$. Hence (4.2.13) becomes

$$\deg(\alpha_1) + deg(\alpha_2) + deg(e^o_k) + 2\sum^3_{j=1} \iota(p, g_j) = 2n, \tag{4.2.14}$$

from which it is easily seen that $\alpha_1 \cup_{orb} \alpha_2$ is in $H^{p+q}_{orb}(X)$.

Next we show that e^0_X is a unit with respect to $\cup_{orb}$, i.e., $\alpha \cup_{orb} e^0_X = e^0_X \cup_{orb} \alpha = \alpha$. First observe that there are connected components of $\mathcal{M}_3$ consisting of points $(p, (\mathbf{g}))$ for which $\mathbf{g} = (g_1, g_2, g_3)$ satisfies the condition that one of the g_i is 1_{G_p}. Over these components the Euler class $e(E_3) = 1$ in the 0^{th} cohomology group since (4.2.11) has zero cokernel. Let $\alpha \in H^*(X_{(g)})$. Then $e^*_1(\alpha) \cup e^*_2(e^0_X) \cup e^*_3(e^o_k)$ is non-zero only on the connected component of $\mathcal{M}_3$ which is the image of the embedding $X_{(g)} \to \mathcal{M}_3$ given by $(p, (g)_{G_p}) \to (p, ((g, 1_{G_p}, g^{-1})))$ and e^o_k must be in $H^*_c(X_{(g^{-1})})$. Moreover, we have

$$\begin{aligned}
\alpha \cup_{orb} e^0_X &:= \sum_{j,k} (\int^{orb}_{\mathcal{M}_3} e^*_1(\alpha) \cup e^*_2(e^0_X) \cup e^*_3(e^o_k) \cup e(E_3)) a^{kj} e_j \\
&= \sum_{j,k} (\int^{orb}_{X_{(g)}} \alpha \cup I^*(e^o_k)) a^{kj} e_j \\
&= \alpha.
\end{aligned}$$

Similarly, we can prove that $e^0_X \cup_{orb} \alpha = \alpha$.

Now we consider the case $\cup_{orb} : H^d_{orb}(X) \times H^{2n-d}_{orb}(X) \to H^{2n}_{orb}(X) = H^{2n}(X)$. Let $\alpha \in H^d_{orb}(X)$ and $\beta \in H^{2n-d}_{orb}(X)$, then $e^*_1(\alpha) \cup e^*_2(\beta) \cup e^*_3(e^0_X)$ is non-zero only on those connected components of $\mathcal{M}_3$ which are images under embedding $\widetilde{X} \to \mathcal{M}_3$ given by

$(p,(g)) \to (p,((g,g^{-1},1_{G_p})))$, and if α is in $H^*(X_{(g)})$, β must be in $H^*(X_{(g^{-1})})$. Moreover, let e_X^{2n} be the generator in $H^{2n}(X)$ such that $e_X^{2n}\cdot[X]=1$, then we have

$$\begin{aligned}
\alpha \cup_{orb} \beta &:= \sum_{j,k}\Big(\int^{orb}_{\mathcal{M}_3} e_1^*(\alpha)\cup e_2^*(\beta)\cup e_3^*(e_k^o)\cup e(E_3)\Big)a^{kj}e_j\\
&= \Big(\int^{orb}_{\mathcal{M}_3} e_1^*(\alpha)\cup e_2^*(\beta)\cup e_3^*(e_X^0)\cup e(E_3)\Big)\cdot e_X^{2n}\\
&= \Big(\int^{orb}_{\widetilde{X}} \alpha\cup I^*(\beta)\Big)\cdot e_X^{2n}\\
&= <\alpha,\beta>_{orb}\, e_X^{2n}
\end{aligned}$$

from which we see that $\int_X \alpha\cup_{orb}\beta = <\alpha,\beta>_{orb}$.

The rest of the assertions are obvious. $\square$

4.3. Proof of associativity. In this subsection, we give a proof of associativity of the orbifold cup products $\cup_{orb}$ defined in the last subsection. We will only present the proof for the orbifold cohomology groups $H^*_{orb}(X)$. The proof for orbifold Dolbeault cohomology is the same. We leave it to readers.

Recall the moduli space of ghost curves with k-marked points $\mathcal{M}_k$ for $k\geq 4$ can be identified with $\mathcal{M}_{0,k}\times \widetilde{X}_k^o$ which admits a natural partial compactification $\overline{\mathcal{M}}_{0,k}\times\widetilde{X}_k^o$ by adding nodal ghost curves. We will first give a detailed analysis on this for the case when $k=4$.

Let Δ be the graph of map $I:\widetilde{\Sigma X}\to\widetilde{\Sigma X}$ in $\widetilde{\Sigma X}\times\widetilde{\Sigma X}$ given by $I:(p,(g))\to(p,(g^{-1}))$. To obtain the orbifold structure, one can view Δ as the orbifold fiber product of identify map and I, which has an induced orbifold structure since both the identify and I are so-called "good maps" (see [CR]). Consider map $\Lambda:\tilde{X}_3^o\times\tilde{X}_3^o\to\widetilde{\Sigma X}\times\widetilde{\Sigma X}$ given by $((p,(\mathbf{g})),(q,(\mathbf{h})))\to((p,(g_3)),(q,(h_1)))$. We wish to consider the preimage of Δ.

Remark. Suppose that we have two maps

$$f:X\to Z,\, g:Y\to Z.$$

In general, the ordinary fiber product $X\times_Z Y$ may not have a natural orbifold structure. The correct formulation is to use the "good map" introduced in [CR]. If f,g are good maps, there is a canonical orbifold fiber product (still denoted by $X\times_Z Y$) obtained by taking the fiber product on the uniformizing system. It has an induced orbifold structure and there are good map projections to both X, Y to make the appropriate diagram commute. However, as a set, such an orbifold fiber product is not the usual fiber product. Throughout this paper, we will use $X\times_Z Y$ to denote orbifold fiber product only.

It is clear that the pre-image of Δ can be viewed as the fiber product of

$$e_3,\, I\circ e_1:\tilde{X}_3^0\to\widetilde{X}.$$

Then, we define the pre-image $\Lambda^{-1}(\Delta)$ as the orbifold fiber product of $e_3,\,,I\circ e_1$. It is easy to check that $\Lambda^{-1}(\Delta)=\tilde{X}_4^o$. Next, we describe explicitly the compactification $\overline{\mathcal{M}}_4$ of $\mathcal{M}_4$.

Recall the moduli space of genus-zero curves with 4-marked points $\mathcal{M}_{0,4}$ can be identified with $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ by fixing the first three marked points to be $\{0, 1, \infty\}$. The Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,4}$ is then identified with $\mathbf{P}^1$, where each point of $\{0, 1, \infty\}$ corresponds to a nodal curve obtained as the last marked point is running into this point. It is easy to see that part of the compactification $\overline{\mathcal{M}}_4$ by adding a copy of $\widetilde{X}^o_4$ at $\infty \in \overline{\mathcal{M}}_{0,4} = \mathbf{P}^1$, where intuitively we associate $(g_1g_2)^{-1}$, g_1g_2 at the nodal point. In the same way, the compactification at 0 is by adding a copy of $\widetilde{X}^o_4$ where we associate $(g_1g_4)^{-1}$, g_1g_4 at the nodal point, and at 1 by associating $(g_1g_3)^{-1}$, g_1g_3 at the nodal point.

Next, we define an orbifold bundle to measure the failure of transversality of Λ to Δ.

Definition 4.3.1. *We define a complex orbifold bundle ν over $\Lambda^{-1}(\Delta)_{(g_1,g_2,g_3,g_4)}$ as follows: over each uniformizing system $(V^{\mathbf{g}}_p, C(\mathbf{g}))$ of $\Lambda^{-1}(\Delta_{(\mathbf{g})})$, where $\mathbf{g} = (g_1, g_2, g_3, g_4)$, we regard $V^{\mathbf{g}}_p$ as the intersection of $V^{g_1}_p \cap V^{g_2}_p$ with $V^{g_3}_p \cap V^{g_4}_p$ in V^g_p, where $g = (g_1g_2)^{-1}$. We define ν to be the complex orbifold bundle over $\Lambda^{-1}(\Delta)$ whose fiber is the orthogonal complement of $V^{g_1}_p \cap V^{g_2}_p + V^{g_3}_p \cap V^{g_4}_p$ in V^g_p.*

The associativity is based on the following

Lemma 4.3.2. *The complex orbifold bundle $pr : E_4 \to \mathcal{M}_4$ can be extended over the compactification $\overline{\mathcal{M}}_4$, denoted by $\bar{pr} : \overline{E}_4 \to \overline{\mathcal{M}}_4$, such that $\overline{E}_4|_{\{*\}\times\widetilde{X}^o_4} = (E_3 \oplus E_3)|_{\Lambda^{-1}(\Delta)} \oplus \nu$ under the above identification, where $\{*\}$ represents a point in $\{0, 1, \infty\} \subset \overline{\mathcal{M}}_{0,4}$.*

Proof of Theorem 4.1.5. We fix an identification of infinite cylinder $\mathbf{R} \times S^1$ with $\mathbf{C}^* \setminus \{0\}$ via the biholomorphism defined by $t + is \to e^{-(t+is)}$, where $t \in \mathbf{R}$ and $s \in S^1 = \mathbf{R}/2\pi\mathbf{Z}$. Through this identification, we regard a punctured Riemann surface as a Riemann surface with cylindrical ends. A neighborhood of a point $* \in \{0, 1, \infty\} \subset \overline{\mathcal{M}}_{0,4}$, as a family of isomorphism classes of genus-zero curves with 4-marked points, can be described by a family of curves $(\Sigma_{r,\theta}, \mathbf{z})$ obtained by gluing of two genus-zero curves with a cylindrical end and two marked points on each, parametrized by (r, θ), where $0 \le r \le r_0$ and $\theta \in S^1$, as we glue the two curves by self-biholomorphisms of $(-\ln r, -3\ln r) \times S^1$ defined by $(t, s) \to (-4\ln r - t, -(s + \theta))$ ($r = 0$ represents the nodal curve $*$). Likewise, thinking of points in $\mathcal{M}_4$ as equivalence classes of triples $((\Sigma, \mathbf{z}), E_{p,\mathbf{g}}, \Phi_{p,\mathbf{g}})$, where $(\Sigma, \mathbf{z})$ is a genus-zero curve of 4-marked points $\mathbf{z}$, a neighborhood of $\{*\} \times (X \sqcup \widetilde{X}^o_4)$ in $\overline{\mathcal{M}}_4$ is described by a family of holomorphic orbifold bundles on $(\Sigma_{r,\theta}, \mathbf{z})$ with morphisms obtained by gluing two holomorphic orbifold bundles on genus-zero curves with two marked points and one cylindrical end on each. We denote them by $(E_{r,\theta}, \Phi_{r,\theta})$.

The key is to construct a family of isomorphisms of complex orbifold bundle

$$\Psi_{r,\theta} : E_3 \oplus E_3 \oplus \nu|_{\Lambda^{-1}(\Delta)} \to E_4$$

for $(r, \theta) \in (0, r_0) \times S^1$. Recall the fiber of E_3 and E_4 is given by kernels of the $\bar{\partial}^*$ operators. In fact, $\Psi_{r,\theta}$ are given by gluing maps of kernels of $\bar{\partial}^*$ operators.

More precisely, suppose $((\Sigma_{r,\theta}, z), E_{r,\theta}, \Phi_{r,\theta})$ are obtained by gluing $((\Sigma_1, \mathbf{z}_1), E_{p,\mathbf{g}}, \Phi_{p,\mathbf{g}})$ and $((\Sigma_2, \mathbf{z}_2), E_{p,\mathbf{h}}, \Phi_{p,\mathbf{h}})$, where $\mathbf{g} = (g_1, g_2, g)$ and $\mathbf{h} = (g^{-1}, h_2, h_3)$. Let $m = |g|$. Then $E_{r,\theta}|_{(-\ln r,-3\ln r)\times S^1}$ is uniformized by $(-\frac{\ln r}{m}, -\frac{3\ln r}{m}) \times S^1 \times TV_p$ with an obvious action by $\mathbf{Z}_m = \langle g\rangle$.

Let $\xi_1 \in \Omega^{0,1}(E_{p,\mathbf{g}})$, $\xi_2 \in \Omega^{0,1}(E_{p,\mathbf{h}})$ such that $\bar{\partial}^*\xi_i = 0$ for $i = 1, 2$. On the cylindrical end, if we fix the local coframe $d(t + is)$, then each ξ_i is a TV_p-valued, exponentially decaying holomorphic function on the cylindrical end. We fix a cut-off function $\rho(t)$ such that $\rho(t) \equiv 1$ for $t \leq 0$ and $\rho(t) \equiv 0$ for $t \geq 1$. We define the gluing of ξ_1 and ξ_2, which is a section of $\Omega^{0,1}(E_{r,\theta})$ and denoted by $\xi_1\#\xi_2$, by

$$\xi_1\#\xi_2 = \rho(-2\ln r + t)\xi_1 + (1 - \rho(-2\ln r + t))\xi_2$$

on the cylindrical end. Let $\Psi_{r,\theta}(\xi_1, \xi_2)$ be the L^2-projection of $\xi_1\#\xi_2$ onto $\ker \bar{\partial}^*$, then the difference $\eta = \xi_1\#\xi_2 - \Psi_{r,\theta}(\xi_1, \xi_2)$ satisfies the estimate $||\bar{\partial}^*\eta||_{L^2} \leq Cr^\delta(||\xi_1|| + ||\xi_2||)$ for some $\delta = \delta(\xi_1, \xi_2) > 0$. Hence $||\eta||_{L^2} \leq C|\ln r|r^\delta(||\xi_1|| + ||\xi_2||)$ (cf. [Ch]), from which it follows that for small enough r, $\Psi_{r,\theta}$ is an injective linear map.

Now given any $\xi \in V_p^g$ which is orthogonal to both $V_p^{g_1} \cap V_p^{g_2}$ and $V_p^{g_3} \cap V_p^{g_4}$, we define $\Psi_{r,\theta}(\xi)$ as follows: fixing a cut-off function, we construct a section u_ξ over the cylindrical neck $(-\ln r, -3\ln r) \times S^1$ with support in $(-\ln r + 1, -3\ln r - 1) \times S^1$ and equals ξ on $(-\ln r + 2, -3\ln r - 2) \times S^1$. We write $\bar{\partial}^* u_\xi = v_{\xi,1} + v_{\xi,2}$, where $v_{\xi,1}$ is supported in $(-\ln r + 1, -\ln r + 2) \times S^1$ and $v_{\xi,2}$ in $(-3\ln r - 2, -3\ln r - 1) \times S^1$. Since ξ is orthogonal to both $V_p^{g_1} \cap V_p^{g_2}$ and $V_p^{g_3} \cap V_p^{g_4}$, we can arrange so that $v_{\xi,1}$ is L^2-orthogonal to $V_p^{g_1} \cap V_p^{g_2} \cap V_p^g$ and $v_{\xi,2}$ is L^2-orthogonal to $V_p^{g^{-1}} \cap V_p^{g_3} \cap V_p^{g_4}$, which are the kernels of the $\bar{\partial}$ operators on Σ_1 and Σ_2 acting on sections of $E_{p,\mathbf{g}}$ and $E_{p,\mathbf{h}}$ respectively. Hence there exist $\alpha_1 \in \Omega^{0,1}(E_{p,\mathbf{g}})$ and $\alpha_2 \in \Omega^{0,1}(E_{p,\mathbf{h}})$ such that $\bar{\partial}^*\alpha_i = v_{\xi,i}$ and α_i are L^2-orthogonal to the kernels of the $\bar{\partial}^*$ operators respectively. We define $\Psi_{r,\theta}(\xi)$ to be the L^2-orthogonal projection of $u_\xi - \alpha_1\#\alpha_2$ onto $\ker \bar{\partial}^*$, then $\Psi_{r,\theta}(\xi)$ is linear on ξ. On the other hand, observe that $||\bar{\partial}^*(u_\xi - \alpha_1\#\alpha_2)||_{L^2} \leq Cr^\delta||\xi||$ for some $\delta > 0$, if we let η be the difference of $\Psi_{r,\theta}(\xi)$ and $u_\xi - \alpha_1\#\alpha_2$, then $||\eta||_{L^2} \leq C|\ln r|r^\delta||\xi||$ (cf. [Ch]), from which we see that for sufficiently small $r > 0$, $\Psi_{r,\theta}(\xi) \neq 0$ if $\xi \neq 0$.

Hence we construct a family of injective morphisms

$$\Psi_{r,\theta} : E_3 \oplus E_3 \oplus \nu|_{\Lambda^{-1}(\Delta)} \rightarrow E_4$$

for $(r, \theta) \in (0, r_0) \times S^1$. We will show next that each $\Psi_{r,\theta}$ is actually an isomorphism.

We denote by $\bar{\partial}_i$ the $\bar{\partial}$ operator on Σ_i, and $\bar{\partial}_{r,\theta}$ the $\bar{\partial}$ operator on $\Sigma_{r,\theta}$. Then the index formula tells us that (cf. Proposition 4.2.2)

$$index\ \bar{\partial}_1 = n - \sum_{j=1}^{3} \iota(p, g_j),$$

$$index\ \bar{\partial}_2 = n - \sum_{j=1}^{3} \iota(p, h_j),$$

$$index\ \bar{\partial}_{r,\theta} = n - (\iota(p, g_1) + \iota(p, g_2) + \iota(p, h_2) + \iota(p, h_3)),$$

from which we see that $index\ \bar{\partial}_1 + index\ \bar{\partial}_2 = index\ \bar{\partial}_{r,\theta} + \dim_{\mathbf{C}} V_p^g$. Since $\dim\ker\bar{\partial}_1 + \dim\ker\bar{\partial}_2 = \dim\ker\bar{\partial}_{r,\theta} + \dim_{\mathbf{C}} V_p^g - rank\ \nu$, we have

$$\dim coker\bar{\partial}_1 + \dim coker\bar{\partial}_2 + rank\ \nu = \dim coker\bar{\partial}_{r,\theta}.$$

Hence $\Psi_{r,\theta}$ is an isomorphism for each (r, θ). $\square$

Before we prove the associativity, let's review some of the basic construction of the smooth manifold and its orbifold analogue. Recall that if $Z \subset X$ is a submanifold, then the Poincare dual of Z can be constructed by the Thom form of the normal bundle N_Z via the natural identification between the normal bundle and tubular neighborhood of Z. Here, the Thom form Θ_Z is a close form such that its restriction on each fiber is a compact supported form of top degree with volume one. In orbifold category, the same is true provided that we interpret "suborbifold" correctly. Here, a suborbifold is a good map $f : Z \to X$ such that locally, f can be lifted to a G-invariant embedding to the "general" uniformizing system $\tilde{f} : (U_Z, G, \pi_Z) \to (U_X, G, \pi_X)$. Here, "general" means that U_Z, U_X could be disconnected. For example, the orbifold fiber product $\Lambda^{-1}(\Delta)$ is a suborbifold of $\tilde{X}_3^o \times \tilde{X}_3^0$. It is clear that the Poincaré dual of Z can be represented by the Thom class of a normal bundle Z.

Proposition 4.3.3. *Choose a basis* $\{e_j\}, \{e_k^o\}$ *of the total orbifold cohomology group* $H^*_{orb}(X)$, $H^*_{orb,c}(X)$ *such that each* e_j, e_k^o *is of homogeneous degree. Let* $< e_j, e_k^o >_{orb} = a_{jk}$ *be the Poincaré pairing matrix and* (a^{jk}) *be the inverse. Then,*

$$\int^{orb}_{(\tilde{X}_4^o)_{(\mathbf{g})}} e_1^*(\alpha_1) \cup e_2^*(\alpha_2) \cup e_3^*(\alpha_3) \cup e_4^*(e_l^o) \cup e(E_4)$$

$$= \sum_{j,k} \left(\int^{orb}_{\tilde{X}_3^o} e_1^*(\alpha_1) \cup e_2^*(\alpha_2) \cup e_3^*(e_k^o) \cup e(E_3)\right) \cdot \left(\int^{orb}_{\tilde{X}_3^o} e_1^*(e_j) \cup e_2^*(\alpha_3) \cup e_3^*(e_l^o) \cup e(E_3)\right) \cdot a^{kj}.$$

Proof. Key observation is $\Lambda^* N_\Delta = N_{\Lambda^{-1}(\Delta)} \oplus \nu$. Hence, $\Lambda^*\Theta_\Delta = \Theta_{\Lambda^{-1}(\Delta)} \cup \Theta_\nu$.

$$\begin{aligned}
&\textstyle\int^{orb}_{\tilde{X}_4^o} e_1^*(\alpha_1) \cup e_2^*(\alpha_2) \cup e_3^*(\alpha_3) \cup e_4^*(e_l^o) \cup e(E_4) \\
&= \textstyle\int^{orb}_{\Lambda^{-1}(\Delta)} e_1^*(\alpha_1) \cup e_2^*(\alpha_2) \cup e_3^*(\alpha_3) \cup e_4^*(e_l^o) \cup e(E_3) \cup e(E_3) \cup e(\nu) \\
&= \textstyle\int^{orb}_{\tilde{X}_3^o \times \tilde{X}_3^o} e_1^*(\alpha_1) \cup e_2^*(\alpha_2) \cup e_3^*(\alpha_3) \cup e_4^*(e_l^o) \cup e(E_3) \cup e(E_3) \cup \Theta_\nu \cup \Theta_{\Lambda^{-1}(\Delta)} \\
&= \textstyle\int^{orb}_{\tilde{X}_3^o \times \tilde{X}_3^o} e_1^*(\alpha_1) \cup e_2^*(\alpha_2) \cup e_3^*(\alpha_3) \cup e_4^*(e_l^o) \cup e(E_3) \cup e(E_3) \cup \Lambda^*\Theta_\Delta \\
&= \textstyle\sum_{j,k} (\int^{orb}_{\tilde{X}_3^o} e_1^*(\alpha_1) \cup e_2^*(\alpha_2) \cup e_3^*(e_k^o) \cup e(E_3))) \\
&\quad \cdot (\textstyle\int^{orb}_{\tilde{X}_3^o} e_1^*(e_j) \cup e_2^*(\alpha_3) \cup e_3^*(e_l^o) \cup e(E_3)) \cdot a^{kj}.
\end{aligned}$$

Now we are ready to prove

Proposition 4.3.4. *The cup product* $\cup_{orb}$ *is associative, i.e., for any* α_i, $i = 1, 2, 3$, *we have*

$$(\alpha_1 \cup_{orb} \alpha_2) \cup_{orb} \alpha_3 = \alpha_1 \cup_{orb} (\alpha_2 \cup_{orb} \alpha_3).$$

Proof. By definition of the cup product $\cup_{orb}$, we have $(\alpha_1 \cup_{orb} \alpha_2) \cup_{orb} \alpha_3$ equals

$$\sum_{j,k,l,s} \left(\int^{orb}_{\mathcal{M}_3} e_1^*(\alpha_1) \cup e_2^*(\alpha_2) \cup e_3^*(e_k^o) \cup e(E_3)\right)$$

$$\cdot \left(\int^{orb}_{\mathcal{M}_3} e_1^*(e_j) \cup e_2^*(\alpha_3) \cup e_3^*(e_l^o) \cup e(E_3)\right) \cdot a^{kj} a^{ls} e_s$$

and $\alpha_1 \cup_{orb} (\alpha_2 \cup_{orb} \alpha_3)$ equals

$$\sum_{j,k,l,s} \left(\int^{orb}_{\mathcal{M}_3} e_1^*(\alpha_1) \cup e_2^*(e_j) \cup e_3^*(e^o_l) \cup e(E_3)\right)$$
$$\cdot\left(\int^{orb}_{\mathcal{M}_3} e_1^*(\alpha_2) \cup e_2^*(\alpha_3) \cup e_3^*(e^o_k) \cup e(E_3)\right) \cdot a^{kj} a^{ls} e_s.$$

By Proposition 4.3.3,

$$\sum_{j,k} \left(\int^{orb}_{\mathcal{M}_3} e_1^*(\alpha_1) \cup e_2^*(\alpha_2) \cup e_3^*(e^o_k) \cup e(E_3)\right) \cdot \left(\int^{orb}_{\mathcal{M}_3} e_1^*(e_j) \cup e_2^*(\alpha_3) \cup e_3^*(e^o_l) \cup e(E_3)\right) \cdot a^{kj}$$

equals

$$\int^{orb}_{\{\infty\}\times \widetilde{X}^o_4} e_1^*(\alpha_1) \cup e_2^*(\alpha_2) \cup e_3^*(\alpha_3) \cup e_4^*(e^o_l) \cup e(E_4),$$

and

$$\sum_{j,k} \left(\int^{orb}_{\mathcal{M}_3} e_1^*(\alpha_1) \cup e_2^*(e_j) \cup e_3^*(e^o_l) \cup e(E_3)\right) \cdot \left(\int^{orb}_{\mathcal{M}_3} e_1^*(\alpha_2) \cup e_2^*(\alpha_3) \cup e_3^*(e^o_k) \cup e(E_3)\right) \cdot a^{kj}$$

equals

$$\int^{orb}_{\{0\}\times \widetilde{X}^o_4} e_1^*(\alpha_1) \cup e_2^*(\alpha_2) \cup e_3^*(\alpha_3) \cup e_4^*(e^o_l) \cup e(E_4).$$

Hence $(\alpha_1 \cup_{orb} \alpha_2) \cup_{orb} \alpha_3 = \alpha_1 \cup_{orb} (\alpha_2 \cup_{orb} \alpha_3)$. $\square$

5. Examples

In general, it is easy to compute orbifold cohomology once we know the action of the local group.

Example 5.1-Kummer surface. Consider the Kummer surface $X = T^4/\tau$, where τ is the involution $x \to -x$. τ has 16 fixed points, which give 16 twisted sectors. It is easily seen that $\iota_{(\tau)} = 1$. Hence, we should shift the cohomology classes of a twisted sector by 2 to obtain 16 degree two classes in orbifold cohomology. The cohomology classes of the nontwisted sector come from invariant cohomology classes of T^4. It is easy to compute that $H^0(X, \mathbf{R})$, $H^4(X, \mathbf{R})$ has dimension one and $H^2(X, \mathbf{R})$ has dimension 6. Hence, we obtain

$$b_0^{orb} = b_4^{orb} = 1, b_1^{orb} = b_3^{orb} = 0, b_2^{orb} = 22.$$

Note that the orbifold cohomology group of T^4/τ is isomorphic to the ordinary cohomology of the $K3$-surface, which is the crepant resolution of T^4/τ. However, it is easy to compute that Poincaré pairing of $H^*_{orb}(T^4/\tau, \mathbf{R})$ is different from Poincaré pairing of the $K3$-surface. We leave it to readers.

Example 5.2-Borcea-Voisin threefold. An important class of Calabi-Yau 3-folds due to Borcea-Voisin is constructed as follows: Let E be an elliptic curve with an involution τ and S be a $K3$-surface with an involution σ acting by (-1) on $H^{2,0}(S)$. Then, $\tau \times \sigma$

is an involution of $E \times S$, and $X = E \times S / < \tau \times \sigma >$ is a Calabi-Yau orbifold. The crepant resolution $\widetilde{X}$ of X is a smooth Calabi-Yau 3-fold. This class of Calabi-Yau 3-folds occupies an important place in mirror symmetry. Now, we want to compute the orbifold Dolbeault cohomology of X to compare with Borcea-Voisin's calculation of Dolbeault cohomology of $\widetilde{X}$.

Let's give a brief description of X. Our reference is [Bo]. τ has 4 fixed points. (S, σ) is classified by Nikulin. Up to deformation, it is decided by three integers (r, a, δ) with the following geometric meaning. Let L^{σ} be the fixed part of $K3$-lattice. Then,

$$r = rank(L^{\sigma}), (L^{\sigma})^{*}/L^{\sigma} = (\mathbf{Z}/2\mathbf{Z})^{a}. \tag{5.1}$$

$\delta = 0$ if the fixed locus S_{σ} of σ represents a class divisible by 2. Otherwise $\delta = 1$. There is a detail table for possible values of (r, a, δ) [Bo].

The cases we are interested in are $(r, a, \delta) \neq (10, 10, 0)$, where $S_{\sigma} \neq \emptyset$. When $(r, a, \delta) \neq (10, 8, 0)$,

$$S_{\sigma} = C_g \cup E_1 \cdots, \cup E_k \tag{5.2}$$

is a disjoint union of a curve C_g of genus

$$g = \frac{1}{2}(22 - r - a)$$

and k rational curves E_i, with

$$k = \frac{1}{2}(r - a).$$

For $(r, a, \delta) = (10, 8, 0)$,

$$S_{\sigma} = C_1 \cup \tilde{C}_1,$$

the disjoint union of two elliptic curves.

Now, let's compute its orbifold Dolbeault cohomology. We assume that $(r, a, \delta) \neq (10, 8, 0)$. The case that $(r, a, \delta) = (10, 8, 0)$ can be computed easily as well. We leave it as an exercise for the readers.

An elementary computation yields

$$h^{1,0}(X) = h^{2,0}(X) = 0, h^{3,0}(X) = 1, h^{1,1}(X) = r + 1, h^{2,1}(X) = 1 + (20 - r). \tag{5.3}$$

Note that twisted sectors consist of 4 copies of S_{σ},

$$h^{0,0}(S_{\sigma}) = k + 1, h^{1,0}(S_{\sigma}) = g. \tag{5.4}$$

It is easy to compute that the degree shifting number for twisted sectors is 1. Therefore, we obtain

$$h^{1,0}_{orb} = h^{2,0}_{orb} = 0, h^{3,0}_{orb} = 1, h^{1,1}_{orb} = 1 + r + 4(k + 1), h^{2,1}_{orb} = 1 + (20 - r) + 4g. \tag{5.5}$$

Compared with the calculation for $\widetilde{X}$, we get precise agreement.

Next, we compute the triple product on $H^{1,1}_{orb}$. $H^{1,1}_{orb}$ consists of contributions from the nontwisted sector with dimension $1 + r$ and twisted sectors with dimension $4(k + 1)$. The only nontrivial one is the classes from the twisted sector. Recall that we need to consider the moduli space of 3-point ghost maps with weight g_1, g_2, g_3 at three marked points satisfying the condition $g_1g_2g_3 = 1$. In our case, the only possibility is $g_1 = g_2 = g_3 = \tau \times \sigma$. But $(\tau \times \sigma)^3 = \tau \times \sigma \neq 1$. Therefore, for any class α from twisted sectors, $\alpha^3 = 0$. On the other hand, we know the triple product of the exceptional divisor of $\widetilde{X}$ is never zero. Hence, X, $\widetilde{X}$ have a different cohomology ring. Borcea-Voisin examples show that the relation between the orbifold cohomology and the cohomology of its crepant resolution is rather subtle. See further comments in the next section.

Example 5.3-Weighted projective space. The examples we compute so far are global quotient. Weighted projective spaces are the easiest examples of non-global quotient orbifolds. Let's consider the weighted projective space $CP(d_1, d_2)$, where $(d_1, d_2) = 1$. Thurston's famous tear drop is $CP(1, d)$. $CP(d_1, d_2)$ can be defined as the quotient of S^3 by S^1, where S^1 acts on the unit sphere of $\mathbf{C}^2$ by

$$e^{i\theta}(z_1, z_2) = (e^{id_1\theta}z_1, e^{id_2\theta}z_2). \tag{5.6}$$

$CP(d_1, d_2)$ has two singular points $x = [1, 0]$, $y = [0, 1]$. x, y gives rise to d_2-1, d_1-1 many twisted sectors indexed by the elements of the isotropy subgroup. The degree shifting numbers are $\frac{i}{d_2}, \frac{j}{d_1}$ for $1 \leq i \leq d_2 - 1, 1 \leq j \leq d_1 - 1$. Hence, the orbifold cohomology are

$$h^0_{orb} = h^2_{orb} = h^{\frac{2i}{d_2}}_{orb} = h^{\frac{2j}{d_1}}_{orb} = 1. \tag{5.7}$$

Note that orbifold cohomology classes from twisted sectors have rational degree. Let $\alpha \in H^{\frac{2}{d_1}}_{orb}$, $\beta \in H^{\frac{2}{d_2}}_{orb}$ be the generators corresponding to $1 \in H^0(pt, \mathbf{C})$. An easy computation yields that orbifold cohomology is generated by $\{1, \alpha^j, \beta^i\}$ with relation

$$\alpha^{d_1} = \beta^{d_2}, \alpha^{d_1+1} = \beta^{d_2+1} = 0. \tag{5.8}$$

The Poincaré pairing is for $1 \leq i_1, i_2, i < d_2 - 1, 1 \leq j_1, j_2, j < d_1 - 1$,

$$< \beta^i, \alpha^j >_{orb} = 0, < \beta^{i_1}, \beta^{i_2} >_{orb} = \delta_{i_1, d_2 - i_2}, < \alpha^{j_1}, \alpha^{j_2} >_{orb} = \delta_{j_1, d_1 - j_2}.$$

The last two examples are local examples in nature. But they exhibit a strong relation with group theory.

Example 5.4. The easiest example is probably a point with a trivial group action of G. In this case, a sector $X_{(g)}$ is a point with the trivial group action of $C(g)$. Hence, orbifold cohomology is generated by conjugacy classes of elements of G. All the degree shifting numbers are zero. Only Poincaré pairing and cup products are interesting. Poincaré pairing is obvious. Let's consider the cup product. First we observe that $X_{(g_1, g_2, (g_1g_2)^{-1})}$ is a point with the trivial group action of $C(g_1) \cap C(g_2)$. We choose a basis $\{x_{(g)}\}$ of the orbifold cohomology group where $x_{(g)}$ is given by the constant function 1 on $X_{(g)}$. Then the inverse of the intersection matrix $(< x_{(g_1)}, x_{(g_2)} >_{orb})$ has $a^{x_{(g)}x_{(g^{-1})}} = |C(g)|$.

Now by Lemma 4.1.4 and Eq. (4.2.12), we have

$$x_{(g_1)} \cup x_{(g_2)} = \sum_{(h_1, h_2), h_1 \in (g_1), h_2 \in (g_2)} \frac{|C(h_1h_2)|}{|C(h_1) \cap C(h_2)|} x_{(h_1h_2)},$$

where (h_1, h_2) is the conjugacy class of the pair h_1, h_2.

On the other hand, recall that the center $Z(\mathbf{C}[G])$ of the group algebra $\mathbf{C}[G]$ is generated by $\sum_{h\in(g)} h$. We can define a map from the orbifold cohomology group onto $Z(\mathbf{C}[G])$ by

$$\Psi : x_{(g)} \mapsto \sum_{h\in(g)} h. \tag{5.9}$$

The map Ψ is a ring homomorphism, which can be seen as follows:

$$(\sum_{h\in(g_1)} h)(\sum_{k\in(g_2)} k) = \sum_{h\in(g_1),k\in(g_2)} hk = \sum_{(h_1,h_2),h_1\in(g_1),h_2\in(g_2)} \frac{A}{B}(\sum_{h\in(h_1h_2)} h), \tag{5.10}$$

where $A = \frac{|G|}{|C(h_1)\cap C(h_2)|}$ is the number of elements in the orbit of (h_1, h_2) of the action of G given by $g\cdot(h_1,h_2) = (gh_1g^{-1}, gh_2g^{-1})$, and $B = \frac{|G|}{|C(h_1h_2)|}$ is the number of elements in the orbit of h_1h_2 of the action of G given by $g\cdot h = ghg^{-1}$. Therefore, the orbifold cup product is the same as the product of $Z(\mathbf{C}[G])$, and the orbifold cohomology ring can be identified with the center $Z(\mathbf{C}[G])$ of the group algebra $\mathbf{C}[G]$ via (5.9).

Example 5.5. Suppose that $G \subset SL(n, \mathbf{C})$ is a finite subgroup. Then, $\mathbf{C}^n/G$ is an orbifold. $H^{p,q}(X_{(g)}, \mathbf{C}) = 0$ for $p > 0$ or $q > 0$ and $H^{0,0}(X_{(g)}, \mathbf{C}) = \mathbf{C}$. Therefore, $H^{p,q}_{orb} = 0$ for $p \neq q$ and $H^{p,p}_{orb}$ is a vector space generated by the conjugacy class of g with $\iota_{(g)} = p$. Therefore, we have a natural decomposition

$$H^*_{orb}(\mathbf{C}^n/G, \mathbf{C}) = Z[\mathbf{C}[G]) = \sum_p H_p, \tag{5.11}$$

where H_p is generated by conjugacy classes of g with $\iota_{(g)} = p$. The ring structure is also easy to describe. Let $x_{(g)}$ be the generator corresponding to the zero cohomology class of twisted sector $X_{(g)}$. We would like to get a formula for $x_{(g_1)} \cup x_{(g_2)}$. As we showed before, the multiplication of conjugacy classes can be described in terms of the center of group algebra $Z(\mathbf{C}[G])$. But we have further restrictions in this case. Let's first describe the moduli space $X_{(h_1,h_2,(h_1h_2)^{-1})}$ and its corresponding GW-invariants. It is clear

$$X_{(h_1,h_2,(h_1h_2)^{-1})} = X_{h_1} \cap X_{h_2}/C(h_1, h_2).$$

To have nonzero invariant, we require that

$$\iota_{(h_1h_2)} = \iota_{(h_1)} + \iota_{(h_2)}. \tag{5.12}$$

Then, we need to compute

$$\int^{orb}_{X_{h_1}\cap X_{h_2}/C(h_1,h_2)} e_3^*(vol_c(X_{h_1h_2})) \wedge e(E), \tag{5.13}$$

where $vol_c(X_{h_1h_2})$ is the compact supported $C(h_1h_2)$-invariant top form with volume one on $X_{h_1h_2}$. It is also viewed as a form on $X_{h_1} \cap X_{h_2}/C(h_1) \cap C(h_2)$. However,

$$X_{h_1} \cap X_{h_2} \subset X_{h_1h_2}$$

is a submanifold. Therefore, (5.13) is zero unless

$$X_{h_1} \cap X_{h_2} = X_{h_1 h_2}. \tag{5.14}$$

In this case, we call (h_1, h_2) transverse. In this case, it is clear that the obstruction bundle is trivial. Let

$$I_{g_1,g_2} = \{(h_1, h_2); h_i \in (g_i), \iota_{(h_1)} + \iota_{(h_2)} = \iota_{(h_1 h_2)}, (h_1, h_2) - transverse\}. \tag{5.15}$$

Then, using decomposition Lemma 4.1.4,

$$x_{(g_1)} \cup x_{(g_2)} = \sum_{(h_1,h_2)\in I_{g_1,g_2}} d_{(h_1,h_2)} x_{(h_1 h_2)}. \tag{5.16}$$

A similar computation as the previous example yields $d_{(h_1,h_2)} = \frac{|C(h_1 h_2)|}{|C(h_1)\cap C(h_2)|}$.

6. Some General Remarks

Physics indicated that orbifold quantum cohomology should be "equivalent" to ordinary quantum cohomology of crepant resolution. As the Borcea-Voisin example indicated, they are not equal. It is a highly nontrivial problem to find the precise mathematical relations between orbifold quantum cohomology with the quantum cohomology of a crepant resolution. We leave it to a future research. At the classical level, there is an indication that equivariant K-theory is better suited for this purpose. For GW-invariant, the orbifold GW-invariant defined in [CR] seems to be equivalent to the relative GW-invariant of pairs studied by Li-Ruan [LR]. We hope that we will have a better understanding of this relation in the near future.

There are many interesting problems in this orbifold cohomology theory. As we mentioned at the beginning, many Calabi-Yau 3-folds are constructed as crepant resolutions of Calabi-Yau orbifolds. The orbifold string theory suggests that there might be a mirror symmetry phenomenon for Calabi-Yau orbifolds. Another interesting question is the relation between quantum cohomology and birational geometry [R, LR]. In fact, this was our original motivation. Namely, we want to investigate the change of quantum cohomology under birational transformations. Birational transformation corresponds to wall crossing phenomenon for symplectic quotients. Here, the natural category is symplectic orbifolds instead of smooth manifolds. From our work, it is clear that we should replace quantum cohomology by orbifold quantum cohomology. Then, it is a challenging problem to calculate the change of orbifold quantum cohomology under birational transformation. The first step is to investigate the change of orbifold cohomology under birational transformation. This should be an interesting problem in its own right.

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