

# Critical Percolation in Annuli and $SLE_6$

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Received: 4 June 2003 / Accepted: 29 August 2003  
Published online: 20 January 2004 – © Springer-Verlag 2004

**Abstract:** Building on the identification of the scaling limit of the critical percolation exploration process as a Schramm-Loewner Evolution, we derive a PDE characterization for the crossing probability of an annulus.

## 1. Introduction

Percolation is arguably the simplest example of a planar “critical” model, i.e. a random planar graph (generally speaking a subgraph of a regular lattice, eventually with spins on sites) with a small mesh such that the probabilities of geometrically meaningful macroscopic events have non-degenerate limits when the mesh goes to zero. Recall that percolation consists in removing each edge (or each vertex) in a lattice with a given probability  $p$ ; bond percolation on the square lattice and site percolation on the triangular lattice are critical for  $p = 1/2$ .

The probabilities of macroscopic events are generally believed to be conformally invariant: the limiting probability when the mesh of the lattice goes to zero should not change if one applies a conformal equivalence to the corresponding geometric configuration. An important related example is the conformal invariance of hitting probabilities (harmonic measure) for planar Brownian motion, which is the scaling limit of simple random walks. Using techniques from Conformal Field Theory (CFT) and Coulomb gas models, physicists have proposed several intriguing formulas for the limiting probabilities of critical percolation. Unfortunately, it does not seem easy to make their arguments rigorous. We now review some of these predictions.

- **Simply connected domains**

Mark four points on the boundary of a simply connected domain to get a conformal rectangle; up to conformal equivalence, there is only one degree of freedom, namely the aspect ratio of the rectangle. The probability that two opposite sides of the rectangle are connected by a percolation cluster is given by Cardy’s formula ([Ca92]).

Watts' formula ([Wa96]) describes the probability of a double crossing, i.e. the two pairs of opposite sides are all connected by one cluster. Cardy has also studied the occurrence of  $n$  disjoint clusters connecting two opposite sides ([Ca01]).

- Multiply connected domains

A  $n$ -connected domain is a plane connected domain the complement in  $\mathbb{C}$  of which has  $n$  connected components (its fundamental group is the free group on  $(n - 1)$  generators). A natural question concerns the probability that two given connected components of the boundary are connected by a cluster inside the domain. The simplest case is the existence of a crossing from the outer to the inner boundary of an annulus. More generally, a law for the number of crossings of alternate colors has been proposed by Cardy ([Ca02]).

- Compact Riemann surfaces

Here there is no boundary; tori are the simplest example. Since the conformal structure is of crucial importance, it may be better to think of (complex) elliptic curves. Pinson ([Pin94]) has studied the image of the (first) homology group of clusters of a given color in the homology group of the torus (i.e. a random subgroup of  $\mathbb{Z} \times \mathbb{Z}$ ).

Since a continuous random graph does not really make sense, one has first to clarify what these limiting probabilities do correspond to. A natural way to understand the “continuous scaling limit of critical percolation” is to focus on the interfaces between clusters, which are random curves. The limiting probabilistic objects (random curves) are the Schramm-Loewner Evolution (SLE), introduced by Schramm in [Sch00] (see [RohSch01, Wer02] for some background on SLE). It describes the only possible conformally invariant scaling limits of these interfaces.

For critical site percolation on the triangular lattice, Smirnov ([Smi01]) has proved that the cluster interface indeed converges to the so-called chordal SLE<sub>6</sub> process. His proof in fact relies on establishing directly Cardy's formula for single crossings in conformal rectangles.

Hence it may be interesting to derive some percolation probabilities in the SLE<sub>6</sub> framework. Cardy's formula itself is easily derived for the SLE<sub>6</sub>, as pointed out by Schramm (see for instance [Wer02]). The general idea is that if one wants to compute probabilities of macroscopic events in terms of the SLE<sub>6</sub> process, one uses the fact that the conditional probabilities (using the filtration associated to the SLE<sub>6</sub>) of this event are a martingale. This leads (usually) to a partial differential equation (in terms of the “conformal coordinates” of the problem), that characterize fully these probabilities. This method has been used in [Sch01] to derive a new formula for critical percolation, and also in [LSW01a, LSW01b, LSW02a, LSW02e] to derive the value of the corresponding critical exponents (that describe the asymptotic decay of the probabilities of some events).

In the present paper we derive analytic characterizations of some percolation probabilities in annuli using Smirnov's result on convergence of the percolation exploration process to SLE<sub>6</sub>. The computations are somewhat tedious, but the underlying probabilistic ideas are extremely simple, and may be of use for more general problems (in particular for  $n$ -connected domains,  $n \geq 3$ ).

Let  $U$  be an  $n$ -connected domain and  $\partial$  be one connected component of its boundary (a Jordan closed curve, say). Pick two points  $x$  and  $y$  on  $\partial$  and set the following boundary conditions on  $\partial$ : the arc  $(x, y)$  is colored in blue and the arc  $(y, x)$  in yellow (in clockwise order). A *chordal event* is a percolation event depending only on  $(U, x, y)$ . For instance, consider the event that there exists a blue crossing between  $(x, y)$  and  $\partial_1$  and a yellow crossing between  $(y, x)$  and  $\partial_2$ , where  $\partial_i$  are connected components of

$\partial U$ . Then “grow” a small percolation (resp. SLE<sub>6</sub>) hull at  $x$ . Let  $K_t$  be this hull ( $t$  is a measure of its size) and  $x_t$  be its “tip” (see e.g. [Smi01]); then  $(U \setminus K_t, x_t, y)$  is the perturbed domain. For a well chosen chordal event, the event holds in  $(U, x, y)$  if and only if it holds in  $(U \setminus K_t, x_t, y)$ .

Now consider the set of the  $(g + 1)$ -connected domain with two points marked on a connected component of the boundary modulo conformal equivalence (a Teichmüller space); it is a manifold of dimension  $(3g - 1)$  if  $g \geq 2$ . Then  $t \mapsto (U \setminus K_t, x_t, y)$  (modulo conformal equivalence) should be, up to time change, a diffusion in the Teichmüller space; as percolation is local, it does not “feel” the boundary before actually touching it. The probability of the chordal event defines a function on this space and this function is harmonic for the diffusion. If one is able to compute a SDE for the diffusion and, crucially, to work out boundary conditions for the harmonic function, this yields an analytic characterization of the chordal event probability (as a function on the Teichmüller space).

In this paper, we essentially carry out this program for annuli. In the first section, we recall some facts of complex analysis on the annulus (mainly the solution to the Dirichlet problem). Next, we derive the SDE for the diffusion in the Teichmüller space (which is isomorphic to  $\mathbb{R}_- \times (0, 2\pi)$ ). In the third section, we characterize the law of the number of nested clusters of alternate colors wrapped around the inner disk of an annulus, and make the connection with Cardy’s results (see [Ca02]).

Defining SLE on Riemann surfaces has also been recently (and independently) the subject of [Z03, FK03], in different settings.

## 2. Annuli

Let  $q < 1$ ; define the annulus

$$A_q = \{z : q < |z| < 1\}.$$

It is classical that for  $q \neq q'$ , the annuli  $A_q$  and  $A_{q'}$  are not conformally equivalent, and that the conformal automorphisms of  $A_q$  are the maps  $z \mapsto uz$  and  $z \mapsto quz^{-1}$ ,  $u \in \mathbb{U}$ . Moreover, any doubly connected domain (i.e. a connected domain the complement in  $\mathbb{C}$  of which has two connected components) is conformally equivalent to an annulus. Thus, one may identify the Teichmüller space of doubly connected domains with the set  $\{A_q\}_{0 < q < 1}$ .

Let us briefly recall Villat’s solution of the Dirichlet problem in an annulus [Vil12]. Let  $\phi, \psi$  be two continuous, real-valued  $2\pi$ -periodic functions. The Dirichlet problem consists in finding a real harmonic function  $f$  defined in  $A_q$  with boundary values given by  $\phi$  and  $\psi$ . The classical Dirichlet problem in the disk may be solved using the Poisson kernel. In the case of annuli, one may also exhibit a kernel, which involves elliptic functions (see e.g. [Cha84]). Let  $\omega_1, \omega_2$  be two numbers such that  $\omega_1$  is real positive,  $\omega_2$  is imaginary, and

$$q = \exp\left(-\frac{\pi \omega_2}{i \omega_1}\right).$$

We shall consider elliptic functions with basic periods  $(2\omega_1, 2\omega_2)$ . Recall that the Weierstrass zeta-function is given by:

$$\zeta(z) = \zeta(z; 2\omega_1, 2\omega_2) = \frac{1}{z} + \sum_{\omega \neq 0} \left( \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right),$$

where the sum is on the vertices of the lattice  $2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$ . Then, for any  $z$ ,  $\zeta(z + 2\omega_1) = \zeta(z) + 2\eta_1$ ,  $\zeta(z + 2\omega_2) = \zeta(z) + 2\omega_2$ , with  $\eta_1 = \zeta(\omega_1)$ ,  $\eta_2 = \zeta(\omega_2)$ . Let  $\omega_3 = \omega_1 + \omega_2$  and  $\eta_3 = \eta_1 + \eta_2$ . Define also:  $\zeta_3(z) = \eta_3 - \zeta(\omega_3 - u)$ .

Note that  $z \mapsto \log|z|$  defines a real harmonic function on an annulus which is not the real part of an holomorphic function on this annulus. So assume that:

$$\int_0^{2\pi} \phi(\theta)d\theta = \int_0^{2\pi} \psi(\theta)d\theta.$$

Then one may define

$$\begin{aligned} \Omega(z) &= \frac{i\omega_1}{\pi^2} \int_0^{2\pi} \phi(\theta)\zeta\left(\frac{\omega_1}{i\pi}\log(z) - \frac{\omega_1}{\pi^2}\theta\right) d\theta \\ &\quad - \frac{i\omega_1}{\pi^2} \int_0^{2\pi} \psi(\theta)\zeta_3\left(\frac{\omega_1}{i\pi}\log(z) - \frac{\omega_1}{\pi}\theta\right) d\theta. \end{aligned}$$

Along a clockwise loop in  $A_q$ , the first integral is increased by  $i\omega_1/\pi^2 \int_0^{2\pi} \phi(\theta)\eta_1 d\theta$  and the second integral is increased by the same amount; so there is no problem with the logarithm's determination.

This function  $\Omega$  is holomorphic on  $A_q$ ; moreover, for all  $\theta$ ,

$$\Re\Omega(\exp(i\theta)) = \phi(\theta) \text{ and } \Re\Omega(q \exp(i\theta)) = \psi(\theta).$$

We apply this result with the conditions  $\phi(\theta)d\theta = 2\pi\delta_{\theta_0}$ , and  $\psi(\theta) = 1$  for all  $\theta$ , thus getting a holomorphic function  $\Omega$  which is well defined up to the addition of an imaginary constant. Let  $x = \exp(i\theta_0)$  and  $y \in \mathbb{U}$ ,  $y \neq x$ . The holomorphic vector field:

$$\begin{aligned} V_{x,y}(z) &= \frac{2i\omega_1}{\pi} z \left( \zeta\left(\frac{\omega_1}{i\pi}\log(z/x)\right) - \zeta\left(\frac{\omega_1}{i\pi}\log(y/x)\right) \right) \\ &\quad - \frac{2i\omega_1}{\pi} z \left( \int_0^{2\pi} \zeta_3\left(\frac{\omega_1}{i\pi}\log(ze^{-i\theta})\right) - \zeta_3\left(\frac{\omega_1}{i\pi}\log(ye^{-i\theta})\right) \frac{d\theta}{2\pi} \right) \end{aligned}$$

is equal to  $z\Omega(z)$ , so it is well defined by the following properties:

- $V_{x,y}$  is holomorphic on  $A_q$ ,
- $V_{x,y}$  may be extended continuously to the boundary, except at  $x$ ,
- $\Re(V_{x,y}(z)/z)$  is constant on  $q\mathbb{U}$ ,
- $\Re(V_{x,y}(z)/z) = 0$  on  $\mathbb{U} \setminus \{x\}$ ,
- $V_{x,y}(y) = 0$ , and
- $V_{x,y}$  has a simple pole at  $x$  with residue  $-2x^2$ .

### 3. SLE<sub>6</sub> in an Annulus

A chordal SLE <sub>$\kappa$</sub>  going from  $x \in \mathbb{U}$  to 1 in the unit disk  $D$  may be defined (up to linear time change) by the equations:

$$dg_t(w) = L_{W_t}(g_t(w))dt,$$

where  $W$  is  $\sqrt{\kappa}$  times a standard real Brownian motion starting from  $i(x + 1)/(1 - x)$  and the holomorphic vector field  $L_y$  is defined on  $D$  by:

$$L_y(w) = -\frac{i(1-w)^2}{i(1+w)/(1-w) - y}.$$

The map  $g_t$  is a conformal equivalence between  $D \setminus K_t$  and  $D$  that fixes 1, where  $K_t$  is the SLE hull at time  $t$ . The time parameter corresponds to half of the capacity of the image of  $K_t$  under the homography  $w \mapsto i(w+1)/(1-w)$  seen from infinity in the half-plane.

Let  $q < 1$ . For small enough  $t$ ,  $D \setminus g_t(qD)$  is a doubly connected set; hence there exists a unique  $q_t \geq q$  and a unique conformal equivalence  $h_t$  between  $D \setminus g_t(qD)$  and  $A_{q_t}$  such that  $h_t(1) = 1$ . Let  $\xi_t = (W_t - i)/(W_t + i)$ ,  $\lambda_t = h_t(\xi_t)$ , and  $f_t = h_t \circ g_t$ .

Now  $w \mapsto (df_t)(f_t^{-1}(w))$  defines a holomorphic vector field on  $A_{q_t}$ . It is easily seen that this vector field is proportional to  $V_{\lambda_t,1}$  (we omit the  $q_t$  parameter). Thus, there exists a function  $a_t$  such that:

$$d(f_t(w)) = V_{\lambda_t,1}(f_t(w))da_t.$$

Moreover,  $d \log(q_t) = da_t$ , so one may pick  $a(t) = \log(q(t))$ . Using the chain rule, one gets:

$$d(f_t(w)) = (dh_t)(g_t(w)) + h'_t(g_t(w))dg_t(w).$$

It follows that:

$$dh_t(w) = V_{\lambda_t,1}(h_t(w))da_t - L_{W_t}(w)h'_t(w)dt.$$

As the poles in this expression should cancel out, necessarily:

$$da_t = h'_t(\xi_t)^2 \frac{(1 - \xi_t)^4}{4\lambda_t^2} dt,$$

and this is indeed real. Now let  $g_t(w) = \xi_t + u$ , with small  $u$ . Then,

$$L_{W_t}(\xi_t + u) = -\frac{(1 - \xi_t)^4}{2u} + \frac{3}{2}(1 - \xi_t)^3 + O(u),$$

$$\begin{aligned} & h'_t(\xi_t + u)L_{W_t}(\xi_t + u) \\ &= -h'_t(\xi_t) \frac{(1 - \xi_t)^4}{2u} - h''_t(\xi_t) \frac{(1 - \xi_t)^4}{2} + \frac{3}{2}h'_t(\xi_t)(1 - \xi_t)^3 + O(u) \end{aligned}$$

and

$$\zeta \left( \frac{\omega_1}{i\pi} \log \left( \frac{h_t(\xi_t + u)}{\lambda_t} \right) \right) = \frac{i\pi\lambda_t}{\omega_1 h'_t(\xi_t)} \left( \frac{1}{u} - \frac{\lambda_t}{2h'_t(\xi_t)} \left( \frac{h''_t(\xi_t)}{\lambda_t} - \frac{h'_t(\xi_t)^2}{\lambda_t^2} \right) \right) + O(u).$$

Finally

$$\begin{aligned} & V_{\lambda_t,1}(h_t(\xi_t + u)) \\ &= \lambda_t \left[ -\frac{2\lambda_t}{h'_t(\xi_t)} \left( \frac{1}{u} - \frac{\lambda_t}{2h'_t(\xi_t)} \left( \frac{h''_t(\xi_t)}{\lambda_t} - \frac{h'_t(\xi_t)^2}{\lambda_t^2} \right) \right) + \frac{2i\omega_1}{\pi} \zeta \left( \frac{\omega_1}{i\pi} \log(\lambda_t) \right) \right. \\ & \quad \left. + \frac{2i\omega_1}{\pi} \int_0^{2\pi} \left( \zeta_3 \left( \frac{\omega_1}{i\pi} \log(e^{-i\theta}) \right) - \zeta_3 \left( \frac{\omega_1}{i\pi} \log(\lambda_t e^{-i\theta}) \right) \right) \frac{d\theta}{2\pi} \right] - 2\lambda_t + O(u). \end{aligned}$$

Notice that:

$$\int_0^{2\pi} \left( \zeta_3 \left( \frac{\omega_1}{i\pi} \log(e^{-i\theta}) \right) - \zeta_3 \left( \frac{\omega_1}{i\pi} \log(\lambda_t e^{-i\theta}) \right) \right) \frac{d\theta}{2\pi} = -\frac{\log(\lambda_t)}{2i\pi} 2\eta_1.$$

Hence  $w \mapsto dh_t(w)$  is smooth at  $\xi_t$  and:

$$\begin{aligned} (dh_t)(\xi_t) &= \frac{h'_t(\xi_t)^2(1-\xi_t)^4}{4\lambda_t} \left[ \frac{2i\omega_1}{\pi} \zeta \left( \frac{\omega_1}{i\pi} \log(\lambda_t) \right) - \frac{2\omega_1 \log(\lambda_t)\eta_1}{\pi^2} \right] dt \\ &\quad + \left[ \frac{3}{4}h''_t(\xi_t)(1-\xi_t)^4 - \frac{3h'_t(\xi_t)^2(1-\xi_t)^4}{4\lambda_t} - \frac{3}{2}h'_t(\xi_t)(1-\xi_t)^3 \right] dt. \end{aligned}$$

Recall that  $\xi_t = (W_t - i)/(W_t + i)$ ,  $dW_t = \sqrt{\kappa}dB_t$ , where  $B$  is a standard real Brownian motion. Hence:

$$d\xi_t = \frac{(1-\xi_t)^2}{2i} \sqrt{\kappa}dB_t + \frac{(1-\xi_t)^3}{4} \kappa dt.$$

Since  $\lambda_t = h_t(\xi_t)$ , an appropriate version of Itô’s formula yields:

$$\begin{aligned} d\lambda_t &= (dh_t)(\xi_t) + h'_t(\xi_t)d\xi_t + \frac{1}{2}h''_t(\xi_t)d\langle \xi_t \rangle \\ &= \frac{h'_t(\xi_t)(1-\xi_t)^2}{2i} \sqrt{\kappa}dB_t \\ &\quad + \frac{h'_t(\xi_t)^2(1-\xi_t)^4}{4\lambda_t} \left[ \frac{2i\omega_1}{\pi} \zeta \left( \frac{\omega_1}{i\pi} \log(\lambda_t) \right) - \frac{2\omega_1 \log(\lambda_t)\eta_1}{\pi^2} \right] dt \\ &\quad + \left( h''_t(\xi_t) \frac{(1-\xi_t)^4}{2} - h'_t(\xi_t)(1-\xi_t)^3 \right) \left( \frac{3}{2} - \frac{\kappa}{4} \right) dt - \frac{3h'_t(\xi_t)^2(1-\xi_t)^4}{4\lambda_t} dt. \end{aligned}$$

Let  $\kappa = 6$  and  $\exp(iv) = \lambda$ ,  $v \in [0, 2\pi]$ . We perform a time change using the increasing function  $t \mapsto a(t)$ :

$$dv_a = -\sqrt{\kappa}d\tilde{B}_a + \frac{2\omega_1}{\pi} \left( \zeta \left( \omega_1 \frac{v_a}{\pi} \right) - \frac{v_a}{\pi} \zeta(\omega_1) \right) da,$$

where  $(\tilde{B}_a)_{a \geq 0}$  is a standard real Brownian motion. Note that the zeta-functions, as well as the half-period  $\omega_1$  depend implicitly on  $a$ . Using Jacobi’s theta-function, one may rewrite this SDE as (see e.g. [Cha84], Theorem V.2):

$$dv_a = -\sqrt{\kappa}d\tilde{B}_a + \frac{1}{\pi} \frac{\theta'}{\theta} \left( \frac{v_a}{2\pi}, -\frac{ia}{\pi} \right) da,$$

where  $\theta'$  designates the derivative of the bivariate function  $\theta(v, z)$  with respect to the first variable. Note that this SDE is invariant under  $v \leftrightarrow 2\pi - v$ , which is obvious from the definition of  $v$ .

The principle of this computation is closely related to the approach of locality/restriction in [LSW02d]. The fact that one gets a (time-inhomogeneous) diffusion for  $v$  is a feature of locality for  $\kappa = 6$ . Note also that we could have begun with any configuration conformally equivalent to  $(A_q, \lambda, 1)$ , getting the same dynamics for  $v$ , hence the same percolation probabilities.

### 4. Crossing of an Annulus

The previous computations may be used to study various critical percolation probabilities in the annulus. For instance, consider the following crossing probability:  $F(v, a)$  is the probability that there exists a blue crossing between the arc  $(1, \exp(i\nu)) \subset \mathbb{U}$  and the inner circle  $q\mathbb{U}$  in the annulus  $A_q$  and a yellow crossing between  $(\exp(i\nu), 1)$  and  $q\mathbb{U}$ , with  $a = \log(q)$ . It follows from the previous section that:

$$3F_{v,\nu} + \frac{1}{\pi} \frac{\theta'}{\theta} \left( \frac{\nu_a}{2\pi}, -\frac{ia}{\pi} \right) F_\nu + F_a = 0 \tag{4.1}$$

and  $F$  satisfies the boundary conditions:  $F(0, a) = F(2\pi, a) = 0$ ,  $F(\nu, 0) = 1$  for all  $a < 0$ ,  $\nu \in (0, 2\pi)$ . This last condition is a consequence of the Russo-Seymour-Welsh theory.

In the rest of this section, we discuss events related to nested circuits of alternate colors around the inner disk of an annulus for critical percolation.

Consider  $(\nu_a, a)_a$  as a two dimensional diffusion in the half-strip  $S = \{z : \Re z < 0, 0 < \Im z < 2\pi\}$ . This half-strip may be identified with the Teichmüller space of doubly connected plane domains with two marked boundary points (on the same connected component of the boundary) modulo conformal equivalence. The diffusion is stopped when it hits  $\mathbb{R}_- \cup [0, 2i\pi]$  and is instantaneously reflected on  $2i\pi + \mathbb{R}_-$ . For any  $z \in S$ , this defines a harmonic measure seen from  $z$  on  $\mathbb{R}_- \cup [0, 2i\pi]$ . The restriction of this probability measure to  $\mathbb{R}_-$  is a finite measure (not a probability measure), which we shall note  $\tilde{K}(z, \cdot)$ . For  $y \in \mathbb{R}_-$ , note  $K(y, \cdot) = \tilde{K}(y + 2i\pi, \cdot)$ . Thus we have defined a defective Markov kernel on  $\mathbb{R}_-$  (identified with doubly connected domains modulo conformal equivalence). From a probabilistic point of view, this is equivalent to a (discrete time) Markov chain on  $\mathbb{R}_- \cup \{\partial\}$ , where  $\partial$  is a cemetery state. The number of steps taken by this Markov chain (starting from  $y < 0$ ) before reaching  $\partial$  corresponds to the number of downcrossings and upcrossings of  $\nu$  (starting from  $2\pi$  at time  $y$ ). For a given  $y$ ,  $(\nu, a) \mapsto \tilde{K}(a + i\nu, y)$  is a solution of the PDE (4.1) with boundary conditions:  $\tilde{K}(i\nu, y) = 0$  for all  $\nu \in (0, 2\pi)$ , and, on  $\mathbb{R}_-$ ,  $\tilde{K}(\cdot, y) = \delta_y$ , the Dirac mass at  $y$ . On  $2i\pi + \mathbb{R}_-$ , there is a Neumann boundary condition:  $\tilde{K}_\nu(2i\pi + x, y)$  for all  $x < 0$ , just as in [LSW02e], Lemma 2.3.

We now interpret downcrossings and upcrossings of  $\nu$  in terms of the exploration process. See also [LSW02e] for a discussion of percolation events in annuli in relation with the exploration process (and nice figures !). For the sake of simplicity, consider critical site percolation on the triangular lattice: each vertex of the triangular lattice (or each hexagon of a honeycomb lattice) is colored in blue or yellow with probability  $1/2$ . Consider a portion of this lattice (with small mesh) that approximates the annulus  $A_q$ . Two points are marked on the outer boundary, say  $x$  and  $1$ ; the arc  $(1, x)$  is colored in blue and the arc  $(x, 1)$  is colored in yellow. The exploration process from  $x$  to  $1$  (which is well defined as long as it does not hit  $q\mathbb{U}$ ) is the path with only blue hexagons on its left-hand and yellow ones on its right-hand. The exploration process completes a clockwise loop if there is a blue circuit of hexagons around the inner disk  $qD$  and a counterclockwise loop if there is a yellow circuit. It hits the inner disk before completing a circuit if there is a blue crossing from  $(1, x)$  to  $q\mathbb{U}$  and a yellow one from  $(x, 1)$  to  $q\mathbb{U}$ . In the following, a circuit will always be a circuit around  $qD$ , and a crossing will always be a crossing between  $\mathbb{U}$  and  $q\mathbb{U}$  in  $A_q$ .

In the continuous setting, starting from  $\exp(i\nu_a) = x$ , if  $\nu$  reaches  $0$ , then the boundary “seen from the inner disk” is completely yellow; this means that the exploration

process has completed a counterclockwise loop (including the yellow arc  $(x, 1)$ ). Hence, starting from  $\nu_a = 2i\pi$ , the process  $\nu$  reaches 0 before time 0 if and only if there is a yellow circuit inside the annulus (the outer circle is blue). Then  $\nu$  reflects instantaneously on  $\mathbb{R}_-$ , which means that the exploration process proceeds towards  $qD$  in a conformal annulus the outer boundary of which (i.e. the circuit) is yellow. Thus a downcrossing for  $nu$  means that there exists a yellow circuit (here  $\mathbb{U}$  is blue); a downcrossing followed by an upcrossing means that there exists a yellow circuit and a blue circuit nested in the yellow one; and so on.

Now set free boundary conditions. Let  $N$  denote the event that there is no circuit (blue or yellow); equivalently, there is a blue crossing and a yellow one. For  $n \geq 1$ , let  $B_n$  be the event that there is exactly  $n$  disjoint clusters of alternate colors wrapped around  $qD$  and the outermost is blue;  $Y_n$  is the corresponding events with colors changed (see figure).

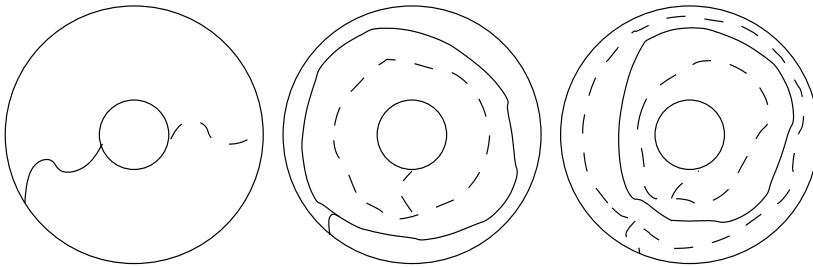


Fig. 1. The events  $N, B_2, Y_3$

Obviously  $\mathbb{P}(B_i) = \mathbb{P}(Y_i)$  and:

$$\mathbb{P}(N) + 2 \sum_{n=1}^{\infty} \mathbb{P}(B_i) = 1.$$

It follows from the previous discussion that the probability  $c_n$  that  $\nu$  starting from  $y + 2i\pi$  makes at least  $n$  alternate downcrossings and upcrossings is:

$$c_n = (K^{\star n})(y, \mathbf{1}),$$

where  $\star$  designates the convolution of Markov kernels and  $\mathbf{1}$  is the constant function. Note that we have analytically characterized  $K$ . Here one should be cautious because of the boundary conditions. The number  $c_n$  is the probability to see at least  $n$  circuits of alternate colors, not counting the outermost one if it is blue. This outermost circuit is either blue or yellow, so:

$$\begin{aligned} c_n &= \sum_{k=n}^{\infty} \mathbb{P}(Y_k) + \sum_{k=n+1}^{\infty} \mathbb{P}(B_k) \\ &= \mathbb{P}(B_n) + 2 \sum_{k=n+1}^{\infty} \mathbb{P}(B_k). \end{aligned}$$



Determining  $\mathbb{P}(N), \mathbb{P}(B_i)$  from the  $c_i$  is then a trivial problem of (infinite dimensional) linear algebra. In the Fréchet space of rapidly decaying series, consider the vectors  $v = (\mathbb{P}(N), \mathbb{P}(B_1), \dots, \mathbb{P}(B_n) \dots)$  and  $w = (1, c_1, \dots, c_n, \dots)$ , and the bounded operator:

$$M = \text{Id} + 2J + 2J^2 + \dots + 2J^n + \dots,$$

where  $J$  is the left shift:  $J(u_0 \dots u_n \dots) = (u_1 \dots u_{n+1} \dots)$ . Then  $Mv = w$ . As (a little formally)  $M = 2(\text{Id} - J)^{-1} - \text{Id}$ , one gets  $M^{-1} = \text{Id} - 2(\text{Id} + J)^{-1}$ . Hence:

$$\mathbb{P}(B_i) = c_i + 2 \sum_{n=1}^{\infty} (-1)^n c_{n+i}$$

with the conventions  $B_0 = N, c_0 = 1$ . Note that  $(c_n)$  decays exponentially: once a circuit has been completed, the “new annulus” has a greater modulus than the initial one, and the greater the modulus, the lesser the probability that there exist circuits. Hence  $c_n(q) < c_1(q)^n$ , and  $c_1(q) < 1$  from the RSW theory (one may also argue using the BK inequality rather than the previous “renewal” argument). Similarly,  $(\mathbb{P}(B_n))$  decays exponentially, which justifies the previous formal argument.

Let  $M$  be the number of downcrossings and upcrossings completed by the diffusion  $\nu$  starting from  $2i\pi + \log q$ , and  $\epsilon$  be the random sign:

$$\epsilon = 1 + 2 \sum_{i=1}^{\infty} \mathbf{1}_{\{M \geq i\}}.$$

The sum is finite a.s. From the preceding discussion,  $\mathbb{P}(N) = \mathbb{E}(\epsilon)$  as functions of  $a = \log q$ . Note that  $\epsilon = 1$  if the last visited boundary half-line is the top one, and  $\epsilon = -1$  in the other case. Since  $\nu$  and  $2i\pi - \nu$  have the same law, it follows that:

$$\mathbb{E}_{i\pi + \log q'}(\epsilon) = 0$$

for all  $q' \in (0, 1)$ . Moreover, if  $\nu$  does not reach the half-line  $i\pi + \mathbb{R}_-$ , then  $\epsilon = 1$  (since paths are continuous). Hence, by the Markov property:

$$\mathbb{P}(N)(q) = \mathbb{E}_{2i\pi + \log q}(\epsilon) = \mathbb{P}_{2i\pi + \log q}(\nu \text{ does not reach } i\pi + \mathbb{R}_-).$$

So we can characterize  $\mathbb{P}(N)$  as the solution of a first exit problem: consider the diffusion  $\nu$  starting from  $z, \Re z < 0, 0 < \Im z < \pi$  (using the symmetry  $\nu \leftrightarrow 2i\pi - \nu$ ), stopped at time  $\tau$  when it exits the half-strip  $\{z : \Re z < 0, 0 \leq \Im z < \pi\}$ , the bottom part of the boundary being instantaneously reflecting; either  $\Im \nu_\tau = \pi$  or  $\Re \nu_\tau = 0$  a.s. Then:

$$\mathbb{P}(N)(q) = \mathbb{P}_{\log q}(\Re \nu_\tau = 0).$$

Let  $H(z) = \mathbb{P}_z(\Re \nu_\tau = 0)$ ,  $z = a + i\nu$ , defined on the half-strip  $\{z : \Re z < 0, 0 < \Im z < \pi\}$ . It is easily seen that in the interior of the domain,

$$3H_{\nu, \nu} + \frac{1}{\pi} \frac{\theta'}{\theta} \left( \frac{\nu_a}{2\pi}, -\frac{ia}{\pi} \right) H_\nu + H_a = 0,$$

and  $H$  satisfies the boundary conditions:  $H$  equals 1 on  $i(0, \pi)$ , 0 on  $i\pi + \mathbb{R}_-$ , and the normal derivative of  $H$  vanishes on  $\mathbb{R}_-$ . This mixed Dirichlet-Neumann problem characterizes completely the function  $q \mapsto \mathbb{P}(N)(q)$ . This should be compared with the following formula, derived by Cardy ([Ca02]) using Coulomb gas techniques, in link with Conformal Field Theory:

*Conjecture 1 (Cardy).* Let  $\tau = -ia/2\pi$ . With the previous notations:

$$\mathbb{P}(N) = \sqrt{3} \frac{\eta(\tau)\eta(6\tau)^2}{\eta(3\tau)\eta(2\tau)^2},$$

where  $\eta$  designates Dedekind's eta function.

So far, we have not (yet?) been able to derive this from our characterization of  $\mathbb{P}(N)$ . In a subsequent paper ([Dub03]), we intend to study percolation problems in multiply-connected domains, as well as  $\text{SLE}_{8/3}$  in such domains.

*Acknowledgements.* I wish to thank Wendelin Werner for his help and advice, and Vincent Beffara for very enlightening discussions.

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Communicated by A. Kupiainen