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Elliptic $U(2)$ Quantum Group and Elliptic Hypergeometric Series

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Abstract: We investigate an elliptic quantum group introduced by Felder and Varchenko, which is constructed from the R-matrix of the Andrews–Baxter–Forrester model, containing both spectral and dynamical parameter. We explicitly compute the matrix elements of certain corepresentations and obtain orthogonality relations for these elements. Using dynamical representations these orthogonality relations give discrete biorthogonality relations for terminating very-well-poised balanced elliptic hypergeometric series, previously obtained by Frenkel and Turaev and by Spiridonov and Zhedanov in different contexts.

1. Introduction

Elliptic functions appear in various solvable models in statistical mechanics and other areas of physics. A famous example is Baxter's 8-vertex model [2], whose R-matrix, containing the Boltzmann weights, is an elliptic solution of the Yang–Baxter equation. A related face model was introduced by Andrews, Baxter and Forrester [1]. In this case the R-matrix satisfies a modified, "dynamical", version of the Yang–Baxter equation, generalizing Wigner's hexagon identity for the classical $6j$-symbols of quantum mechanics.

In the early 1980's, the algebraic study of the Yang–Baxter equation lead to the introduction of quantum groups. The most well understood quantum groups are those constructed from the simplest, constant, solutions. Quantum groups connected to more complicated solutions, and in particular to elliptic solutions, have been more difficult to construct and study. One reason for this is that elliptic quantum groups are not Hopf algebras. Various approaches have been tried for finding a substitute; cf. [5, 8, 10, 12, 20]. In the dynamical case, a decisive step was taken by Felder and Varchenko [9], who

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introduced the algebra that we will study here. This example motivated Etingof and Varchenko [7] to introduce $\mathfrak{h}$-Hopf algebroids, a generalization of Hopf algebras adapted to studying dynamical R-matrices; cf. [15, 19] for further additions to this framework.

An important mathematical application of quantum groups is their relation to basic hypergeometric series (or q-series), a class of special functions going back to work of Cauchy and Heine in the 1840's. The input from quantum group theory has been important for the rapid development of this field during the last 20 years. To our knowledge, nobody has so far associated special functions to elliptic quantum groups in an analogous way. There is, however, a natural candidate for the special functions that should appear, namely, the elliptic or modular hypergeometric series of Frenkel and Turaev [11]. This type of sums may be used to express the elliptic $6j$-symbols of Date et al. [4], which are solutions to the Yang–Baxter equation that greatly generalize the Andrews–Baxter–Forrester solution. For more information on elliptic hypergeometric series we refer to [13, 18, 17, 21, 23, 22, 24, 25, 27].

Our main aim is to give an explicit link from elliptic quantum groups to elliptic hypergeometric series. Namely, we show that ${}_{10}\omega_9$ sums, or elliptic $6j$-symbols, appear as matrix elements for an elliptic quantum group which we denote by $\mathcal{F}_R(U(2))$, which is the algebra of Felder and Varchenko with some extra structure. To achieve this, we first construct finite-dimensional corepresentations of $\mathcal{F}_R(U(2))$, analogous to the standard representations of $SU(2)$ on spaces of homogeneous polynomials in two variables. A main result, Theorem 3.4, is an explicit expression for the matrix elements of these corepresentations. We can then calculate the action of the matrix elements in representations found by Felder and Varchenko, and show that it is given in terms of elliptic hypergeometric series.

The matrix elements satisfy orthogonality relations in the non-commutative algebra $\mathcal{F}_R(U(2))$. Evaluating these in a representation leads to bi-orthogonality relations for ${}_{10}\omega_9$ series. These relations were found already by Frenkel and Turaev [11]; cf. also [25].

Our new derivation of the bi-orthogonality relations shows that they can be viewed as analogues of the orthogonality relations for Krawtchouk polynomials, see [26] for the Lie group $SU(2)$. For the quantum $SU(2)$ group the same approach leads to quantum q-Krawtchouk polynomials, see [16]. For the dynamical quantum $SU(2)$ group, i.e. corresponding to a trigonometric dynamical R-matrix, we get the orthogonality relations for q-Racah polynomials, see [15, §4]. So the above cases can be considered as limiting cases of the bi-orthogonality relations for elliptic $6j$-symbols.

The paper is organized as follows. In Sect. 2 we recall the definition of an $\mathfrak{h}$-Hopf algebroid and the generalized FRST-construction from [7]. Then we describe the elliptic quantum group $\mathcal{F}_R(U(2))$, which is obtained from the R-matrix of the Andrews–Baxter–Forrester model. In Sect. 3 we define finite-dimensional corepresentations of $\mathcal{F}_R(U(2))$ and compute their matrix elements explicitly. In Sect. 4 we consider representations of $\mathcal{F}_R(U(2))$, from which we obtain commutative versions of the orthogonality relations for matrix elements of the corepresentations. It turns out that these are in fact bi-orthogonality relations for terminating very-well-poised balanced elliptic hypergeometric ${}_{10}\omega_9$-series (or elliptic $6j$-symbols).

Notation. We denote by $\theta(z)$ the normalized Jacobi theta function

$$\theta(z) = \prod_{j=0}^{\infty} \left(1 - zp^j\right)\left(1 - p^{j+1}/z\right), \qquad |p| < 1,$$

where p is a fixed parameter that is suppressed from the notation. It satisfies

$$\theta(pz) = \theta(z^{-1}) = -z^{-1}\theta(z),$$

and the addition formula

$$\theta(xy, x/y, zw, z/w) = \theta(xw, x/w, zy, z/y) + (z/y)\theta(xz, x/z, yw, y/w), \quad (1.1)$$

where we use the notation

$$\theta(a_1, \ldots, a_n) = \theta(a_1) \cdots \theta(a_n).$$

We define elliptic Pochhammer symbols by

$$(a)_n = \prod_{i=0}^{n-1} \theta(aq^{2i}),$$

with q another fixed parameter. We will frequently write

$$(a_1, a_2, \ldots, a_k)_n = (a_1)_n \cdots (a_k)_n.$$

Elliptic binomial coefficients are defined by

$$\begin{bmatrix} k \\ l \end{bmatrix} = \prod_{i=1}^{l} \frac{\theta(q^{2(k-l+i)})}{\theta(q^{2i})}.$$

Finally, the balanced very-well-poised elliptic hypergeometric series is defined by [11]

$${}_{r+1}\omega_r(a_1; a_4, a_5, \ldots, a_{r+1}) = \sum_{k=0}^{\infty} \frac{\theta(a_1 q^{4k})}{\theta(a_1)} \frac{(a_1, a_4, \ldots, a_{r+1})_k q^{2k}}{(q^2, a_1 q^2/a_4, \ldots, a_1 q^2/a_{r+1})_k}, \quad (1.2)$$

where $(a_4 \cdots a_{r+1})^2 = a_1^{r-3} q^{2(r-5)}$. In this paper all series terminate, i.e. one of the a_i is of the form q^{-2n} with n a nonnegative integer, so there are no convergence problems. Let us emphasize that in this paper all elliptic factorials and elliptic hypergeometric series are in base q^2, p.

2. Elliptic $U(2)$ Quantum Group

In this section we recall the definition of $\mathfrak{h}$-Hopf algebroids (also known as dynamical quantum groups) and the FRST-construction. We start with the definition of the quantum dynamical Yang-Baxter equation with spectral parameter and give in (2.2) the R-matrix to which we apply the FRST-construction. The generators and relations for the resulting $\mathfrak{h}$-Hopf algebroid have been studied by Felder and Varchenko [9].

Let $\mathfrak{h}$ be a finite dimensional complex vector space, viewed as a commutative Lie algebra and $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$ a diagonalizable $\mathfrak{h}$-module. The quantum dynamical Yang-Baxter equation with spectral parameter (QDYBE) is given by

$$\begin{aligned} &R^{12}(\lambda - h^{(3)}, z_{12}) R^{13}(\lambda, z_{13}) R^{23}(\lambda - h^{(1)}, z_{23}) \\ &\quad = R^{23}(\lambda, z_{23}) R^{13}(\lambda - h^{(2)}, z_{13}) R^{12}(\lambda, z_{12}). \end{aligned} \quad (2.1)$$

Here $R : \mathfrak{h}^* \times \mathbb{C} \to \mathrm{End}(V \otimes V)$ is a meromorphic function, h indicates the action of $\mathfrak{h}$, the upper indices are leg-numbering notation for the tensor product and $z_{ij} = z_i/z_j$. For instance, $R^{12}(\lambda - h^{(3)}, z)$ denotes the operator $R^{12}(\lambda - h^{(3)}, z)(u \otimes v \otimes w) = R(\lambda - \mu, z)(u \otimes v) \otimes w$ for $w \in V_\mu$. An R-matrix is by definition a solution of the QDYBE (2.1) which is $\mathfrak{h}$-invariant.

In the example we study, $\mathfrak{h}$ is one-dimensional. We identify $\mathfrak{h} = \mathfrak{h}^* = \mathbb{C}$ and take V to be the two-dimensional $\mathfrak{h}$-module $V = \mathbb{C}e_1 \oplus \mathbb{C}e_{-1}$. In the basis $e_1 \otimes e_1, e_1 \otimes e_{-1}, e_{-1} \otimes e_1, e_{-1} \otimes e_{-1}$ the R-matrix is given by

$$R(\lambda, z) = R(\lambda, z, p, q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(\lambda, z) & b(\lambda, z) & 0 \\ 0 & c(\lambda, z) & d(\lambda, z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.2}$$

where

$$\begin{aligned} a(\lambda, z) &= \frac{\theta(z)\theta(q^{2(\lambda+2)})}{\theta(q^2 z)\theta(q^{2(\lambda+1)})}, & b(\lambda, z) &= \frac{\theta(q^2)\theta(q^{-2(\lambda+1)}z)}{\theta(q^2 z)\theta(q^{-2(\lambda+1)})}, \\ c(\lambda, z) &= \frac{\theta(q^2)\theta(q^{2(\lambda+1)}z)}{\theta(q^2 z)\theta(q^{2(\lambda+1)})}, & d(\lambda, z) &= \frac{\theta(z)\theta(q^{-2\lambda})}{\theta(q^2 z)\theta(q^{-2(\lambda+1)})}. \end{aligned} \tag{2.3}$$

The R-matrix defined by (2.2) satisfies the QDYBE (2.1), see [3, 1, 9, 12].

2.1. $\mathfrak{h}$-Hopf algebroids. In this subsection we recall the notion of $\mathfrak{h}$-bialgebroids and $\mathfrak{h}$-Hopf algebroids (or dynamical quantum groups) originally introduced by Etingof and Varchenko [7], see also [6]. For the definition of the antipode in an $\mathfrak{h}$-Hopf algebroid we follow [15]. We discuss the FRST-construction that associates an $\mathfrak{h}$-bialgebroid to an $\mathfrak{h}$-invariant matrix, see [7, 6].

Let $\mathfrak{h}$ be a finite dimensional complex vector space. Denote the field of meromorphic functions on the dual of $\mathfrak{h}$ by $M_{\mathfrak{h}^*}$.

Definition 2.1. *An $\mathfrak{h}$-algebra is a complex associative algebra A with 1, which is bigraded over $\mathfrak{h}^*$, $A = \oplus_{\alpha,\beta \in \mathfrak{h}^*} A_{\alpha\beta}$, and equipped with two algebra embeddings μ_l, $\mu_r : M_{\mathfrak{h}^*} \to A_{00}$ (the left and right moment map) such that*

$$\mu_l(f)a = a\mu_l(T_\alpha f), \qquad \mu_r(f)a = a\mu_r(T_\beta f), \text{ for all } a \in A_{\alpha\beta},\ f \in M_{\mathfrak{h}^*},$$

where T_α denotes the automorphism $(T_\alpha f)(\lambda) = f(\lambda + \alpha)$.

A morphism of $\mathfrak{h}$-algebras is an algebra homomorphism preserving the moment maps.

Let A and B be two $\mathfrak{h}$-algebras. The matrix tensor product $A \tilde{\otimes} B$ is the $\mathfrak{h}^*$-bigraded vector space with $(A \tilde{\otimes} B)_{\alpha\beta} = \bigoplus_{\gamma \in \mathfrak{h}^*} (A_{\alpha\gamma} \otimes_{M_{\mathfrak{h}^*}} B_{\gamma\beta})$, where $\otimes_{M_{\mathfrak{h}^*}}$ denotes the usual tensor product modulo the relations

$$\mu_r^A(f)a \otimes b = a \otimes \mu_l^B(f)b, \text{ for all } a \in A, b \in B, f \in M_{\mathfrak{h}^*}. \tag{2.4}$$

The multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$ for $a, c \in A$ and $b, d \in B$ and the moment maps $\mu_l(f) = \mu_l^A(f) \otimes 1$ and $\mu_r(f) = 1 \otimes \mu_r^B(f)$ make $A \tilde{\otimes} B$ into an $\mathfrak{h}$-algebra.

Example. Let $D_{\mathfrak{h}}$ be the algebra of difference operators in $M_{\mathfrak{h}^*}$, consisting of the operators $\sum_i f_i T_{\beta_i}$, with $f_i \in M_{\mathfrak{h}^*}$ and $\beta_i \in \mathfrak{h}^*$. This is an $\mathfrak{h}$-algebra with the bigrading defined by $fT_{-\beta} \in (D_{\mathfrak{h}})_{\beta\beta}$ and both moment maps equal to the natural embedding.

For any $\mathfrak{h}$-algebra A, there are canonical isomorphisms $A \cong A\tilde{\otimes} D_{\mathfrak{h}} \cong D_{\mathfrak{h}}\tilde{\otimes} A$, defined by

$$x \cong x \otimes T_{-\beta} \cong T_{-\alpha} \otimes x, \text{ for all } x \in A_{\alpha\beta}. \tag{2.5}$$

The algebra $D_{\mathfrak{h}}$ plays the role of the unit object in the category of $\mathfrak{h}$-algebras.

Definition 2.2. *An $\mathfrak{h}$-bialgebroid is an $\mathfrak{h}$-algebra A equipped with two $\mathfrak{h}$-algebra homomorphisms $\Delta : A \to A\tilde{\otimes} A$ (the comultiplication) and $\varepsilon : A \to D_{\mathfrak{h}}$ (the counit) such that $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ and $(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta$ (under the identifications* (2.5)*).*

Definition 2.3. *An $\mathfrak{h}$-Hopf algebroid is an $\mathfrak{h}$-bialgebroid A equipped with a $\mathbb{C}$-linear map $S : A \to A$, the antipode, such that*

$$S(\mu_r(f)a) = S(a)\mu_l(f), \ S(a\mu_l(f)) = \mu_r(f)S(a), \text{ for all } a \in A, f \in M_{\mathfrak{h}^*},$$
$$m \circ (id \otimes S) \circ \Delta(a) = \mu_l(\varepsilon(a)1), \text{ for all } a \in A,$$
$$m \circ (S \otimes id) \circ \Delta(a) = \mu_r(T_\alpha(\varepsilon(a)1)), \text{ for all } a \in A_{\alpha\beta},$$

where $m : A\tilde{\otimes} A \to A$ denotes the multiplication and $\varepsilon(a)1$ is the result of applying the difference operator $\varepsilon(a)$ to the constant function $1 \in M_{\mathfrak{h}^*}$.

If there exists an antipode on an $\mathfrak{h}$-bialgebroid, it is unique. Furthermore, the antipode is anti-multiplicative, anti-comultiplicative, unital, counital and interchanges the moment maps μ_l and μ_r, see [15, Prop. 2.2].

The FRST-construction provides many examples of $\mathfrak{h}$-bialgebroids, see [7, 6, 9, 15]. We recall the construction.

Let $\mathfrak{h}$ and $M_{\mathfrak{h}^*}$ be as before, $V = \bigoplus_{\alpha\in\mathfrak{h}^*} V_\alpha$ be a finite-dimensional diagonalizable $\mathfrak{h}$-module and $R : \mathfrak{h}^* \times \mathbb{C} \to \mathrm{End}_{\mathfrak{h}}(V \otimes V)$ a meromorphic function that commutes with the $\mathfrak{h}$-action on $V \otimes V$. Let $\{e_x\}_{x\in X}$ be a homogeneous basis of V, where X is an index set. Write $R^{ab}_{xy}(\lambda, z)$ for the matrix elements of R,

$$R(\lambda, z)(e_a \otimes e_b) = \sum_{x,y\in X} R^{ab}_{xy}(\lambda, z)e_x \otimes e_y,$$

and define $\omega : X \to \mathfrak{h}^*$ by $e_x \in V_{\omega(x)}$. Let A_R be the unital complex associative algebra generated by the elements $\{L_{xy}(z)\}_{x,y\in X}$, with $z \in \mathbb{C}$, together with two copies of $M_{\mathfrak{h}^*}$, embedded as subalgebras. The elements of these two copies will be denoted by $f(\lambda)$ and $f(\mu)$, respectively. The defining relations of A_R are $f(\lambda)g(\mu) = g(\mu)f(\lambda)$,

$$f(\lambda)L_{xy}(z) = L_{xy}(z)f(\lambda + \omega(x)), \quad f(\mu)L_{xy}(z) = L_{xy}(z)f(\mu + \omega(y)), \tag{2.6}$$

for all $f, g \in M_{\mathfrak{h}^*}$, together with the RLL-relations

$$\sum_{x,y\in X} R^{xy}_{ac}(\lambda, z_1/z_2)L_{xb}(z_1)L_{yd}(z_2) = \sum_{x,y\in X} R^{bd}_{xy}(\mu, z_1/z_2)L_{cy}(z_2)L_{ax}(z_1), \tag{2.7}$$

for all $z_1, z_2 \in \mathbb{C}$ and $a, b, c, d \in X$. The bigrading on A_R is defined by $L_{xy}(z) \in A_{\omega(x),\omega(y)}$ and $f(\lambda), f(\mu) \in A_{0,0}$. The moment maps defined by $\mu_l(f) = f(\lambda)$,

$\mu_r(f) = f(\mu)$ make A_R into a $\mathfrak{h}$-algebra. The $\mathfrak{h}$-invariance of R ensures that the bigrading is compatible with the RLL-relations (2.7). Finally the co-unit and co-multiplication defined by

$$\varepsilon(L_{ab}(z)) = \delta_{ab} T_{-\omega(a)}, \quad \varepsilon(f(\lambda)) = \varepsilon(f(\mu)) = f, \tag{2.8}$$

$$\begin{aligned} \Delta(L_{ab}(z)) &= \sum_{x\in X} L_{ax}(z) \otimes L_{xb}(z), \\ \Delta(f(\lambda)) &= f(\lambda) \otimes 1, \Delta(f(\mu)) = 1 \otimes f(\mu), \end{aligned} \tag{2.9}$$

equip A_R with the structure of an $\mathfrak{h}$-bialgebroid, see [7].

2.2. Elliptic $U(2)$ quantum group. We now give the results of the generalized FRST-construction when applied to the R-matrix (2.2), see [9]. Let $0 < q < 1$, $0 < p < 1$. We assume that p, q are generic, it suffices to take p and q algebraically independent over $\mathbb{Q}$. Let $X = \{-1, 1\}$ and define $\omega : \{-1, 1\} \to \mathbb{C}$ by $\omega(x) = x$. We denote the corresponding $\mathfrak{h}$-bialgebroid by $\mathcal{F}_R(M(2))$: it is an elliptic analogue of the algebra of polynomials on the space of complex 2×2-matrices. In the rest of this paper the four L-generators will be denoted by $\alpha(z) = L_{1,1}(z)$, $\beta(z) = L_{1,-1}(z)$, $\gamma(z) = L_{-1,1}(z)$ and $\delta(z) = L_{-1,-1}(z)$.

Remark 2.4. Since $a(\lambda, q^2) = c(\lambda, q^2)$ and $b(\lambda, q^2) = d(\lambda, q^2)$ we see that the R-matrix (2.2) is singular for $z = q^2$. Using (1.1) we compute

$$\det \begin{pmatrix} a(\lambda, z) & b(\lambda, z) \\ c(\lambda, z) & d(\lambda, z) \end{pmatrix} = q^2 \frac{\theta(zq^{-2})}{\theta(zq^2)}.$$

We find that $z = q^2$, up to powers of p, is the only zero of the determinant of R. For $(a, b, c, d) = (1, 1, 1, -1)$ and $(1, -1, 1, 1)$ in (2.7) we obtain

$$\begin{aligned} \alpha(z_1)\beta(z_2) &= a(\mu, z_1/z_2)\beta(z_2)\alpha(z_1) + c(\mu, z_1/z_2)\alpha(z_2)\beta(z_1), \\ \beta(z_1)\alpha(z_2) &= b(\mu, z_1/z_2)\beta(z_2)\alpha(z_1) + d(\mu, z_1/z_2)\alpha(z_2)\beta(z_1). \end{aligned} \tag{2.10}$$

In the case $z_1/z_2 = q^2$ the right-hand side of the relations in (2.10) are multiples of each other, so this also holds for the left-hand sides giving $b(\mu, q^2)\alpha(q^2 z)\beta(z) = a(\mu, q^2)\beta(q^2 z)\alpha(z)$. Simplifying this identity and doing the same for (2.7) for the cases $(a, b, c, d) = (\pm 1, 1, \mp 1, 1)$, $(\pm 1, -1, \mp 1, -1)$ and $(-1, \pm 1, -1, \mp 1)$ we find

$$\theta(q^{-2\mu})\, \alpha(q^2 z)\, \beta(z) = q^2 \theta(q^{-2(\mu+2)})\, \beta(q^2 z)\, \alpha(z), \tag{2.11a}$$

$$\gamma(z)\, \alpha(q^2 z) = \alpha(z)\, \gamma(q^2 z), \tag{2.11b}$$

$$\delta(z)\, \beta(q^2 z) = \beta(z)\, \delta(q^2 z), \tag{2.11c}$$

$$\theta(q^{-2\mu})\, \gamma(q^2 z)\, \delta(z) = q^2 \theta(q^{-2(\mu+2)})\, \delta(q^2 z)\, \gamma(z). \tag{2.11d}$$

From the relations (2.7) for $(a, b, c, d) = (\pm 1, \pm 1, \mp 1, \mp 1)$, $(\pm 1, \mp 1, \mp 1, \pm 1)$ we obtain three independent relations

$$\begin{aligned} a(\lambda, q^2)\alpha(q^2 z)\delta(z) + b(\lambda, q^2)\gamma(q^2 z)\beta(z) &= a(\mu, q^2)[\gamma(z)\beta(q^2 z) + \delta(z)\alpha(q^2 z)], \\ a(\lambda, q^2)\beta(q^2 z)\gamma(z) + b(\lambda, q^2)\delta(q^2 z)\alpha(z) &= b(\mu, q^2)[\gamma(z)\beta(q^2 z) + \delta(z)\alpha(q^2 z)], \\ \alpha(z)\delta(q^2 z) + \beta(z)\gamma(q^2 z) &= \gamma(z)\beta(q^2 z) + \delta(z)\alpha(q^2 z). \end{aligned} \tag{2.12}$$

From (2.3) we see that $a(\lambda, z)$, $b(\lambda, z)$, $c(\lambda, z)$ and $d(\lambda, z)$ have a simple pole for $z = q^{-2}$. The residual relations of (2.7) are the relations obtained by multiplying by $(z_1/z_2) - q^{-2}$ and taking the limit $z_1/z_2 \to q^{-2}$, see [9]. By convention, we interpret (2.7) so that these are also supposed to hold. The residual relations of (2.7) for $(a, b, c, d) = (1, \pm 1, 1, \mp 1)$, respectively $(\pm 1, 1, \mp 1, 1)$, $(\pm 1, -1, \mp 1, -1)$ and $(-1, \pm 1, -1, \mp 1)$, are linearly dependent and simplify to (2.11). The residual relations of (2.7) for $(a, b, c, d) = (\pm 1, \pm 1, \mp 1, \mp 1)$, $(\pm 1, \mp 1, \mp 1, \pm 1)$ reduce to three independent relations of which two can be derived from (2.12). The independent relation can be written as

$$b(\mu, q^2)\gamma(q^2 z)\beta(z) - a(\mu, q^2)\delta(q^2 z)\alpha(z) = a(\lambda, q^2)[\gamma(z)\beta(q^2 z) - \alpha(z)\delta(q^2 z)]. \tag{2.13}$$

Note that Δ, ε preserve the commutation relations (2.11)–(2.13).

Lemma 2.5. *The element*

$$\begin{aligned}
\det(z) &= \frac{F(\mu)}{F(\lambda)}\left[\alpha(z)\delta(q^2 z) - \gamma(z)\beta(q^2 z)\right] \\
&= \frac{F(\mu)}{F(\lambda)}\left[\delta(z)\alpha(q^2 z) - \beta(z)\gamma(q^2 z)\right] \\
&= \frac{q^\mu}{q^\lambda}\left[\frac{\theta(q^{-2(\mu+2)})}{\theta(q^{-2(\lambda+2)})}\delta(q^2 z)\alpha(z) - \frac{\theta(q^{-2\mu})}{q^2\theta(q^{-2(\lambda+2)})}\gamma(q^2 z)\beta(z)\right] \\
&= \frac{q^\mu}{q^\lambda}\left[\frac{\theta(q^{-2\mu})}{\theta(q^{-2\lambda})}\alpha(q^2 z)\delta(z) - \frac{q^2\theta(q^{-2(\mu+2)})}{\theta(q^{-2\lambda})}\beta(q^2 z)\gamma(z)\right],
\end{aligned}$$

where $F(\mu) = q^\mu\theta(q^{-2(\mu+1)})$, *is a central element of* $\mathcal{F}_R(M(2))$. *Moreover,* $\Delta(\det(z)) = \det(z) \otimes \det(z)$ *and* $\varepsilon(\det(z)) = 1$.

Proof. The equality of the four expressions follows from (2.12), (2.13). The remainder of the lemma follows from [9, Theorem 13]. □

To $\mathcal{F}_R(M(2))$ we adjoin the central element $\det^{-1}(z)$ subject to the relation $\det(z)\det^{-1}(z) = 1$. The comultiplication and counit extend by

$$\Delta(\det^{-1}(z)) = \det^{-1}(z) \otimes \det^{-1}(z), \qquad \varepsilon(\det^{-1}(z)) = 1.$$

It is easily checked that the resulting algebra, denoted by $\mathcal{F}_R(GL(2, \mathbb{C}))$, is an $\mathfrak{h}$-bialgebroid. Note that $\det(z)$ and $\det^{-1}(z)$ have $(0, 0)$-bigrading. In the dynamical representations we consider later, $\det(z)$ does not act as id, see Remark 4.3. Therefore we do not put $\det(z) = 1$.

Lemma 2.6. *The* $\mathfrak{h}$*-bialgebroid* $\mathcal{F}_R(GL(2, \mathbb{C}))$ *is an* $\mathfrak{h}$*-Hopf algebroid with the antipode* S *defined by* $S(\det^{-1}(z)) = \det(z)$,

$$\begin{aligned}
S(\alpha(z)) &= \frac{F(\mu)}{F(\lambda)}\det{}^{-1}(q^{-2}z)\delta(q^{-2}z), \quad S(\beta(z)) = -\frac{F(\mu)}{F(\lambda)}\det{}^{-1}(q^{-2}z)\beta(q^{-2}z), \\
S(\gamma(z)) &= -\frac{F(\mu)}{F(\lambda)}\det{}^{-1}(q^{-2}z)\gamma(q^{-2}z), \quad S(\delta(z)) = \frac{F(\mu)}{F(\lambda)}\det{}^{-1}(q^{-2}z)\alpha(q^{-2}z), \\
&S(f(\lambda)) = f(\mu), \quad S(f(\mu)) = f(\lambda),
\end{aligned} \tag{2.14}$$

on the generators and extended as an algebra antihomomorphism.

Proof. By Proposition 2.2 of [15] we only have to check that on the generators we have

$$\begin{pmatrix} S(\alpha(z)) & S(\beta(z)) \\ S(\gamma(z)) & S(\delta(z)) \end{pmatrix} \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix} \begin{pmatrix} S(\alpha(z)) & S(\beta(z)) \\ S(\gamma(z)) & S(\delta(z)) \end{pmatrix}, \tag{2.15}$$

and that the antipode preserves the defining relations of the algebra. The proof is straightforward, using the RLL-relations and Lemma 2.5. Note that we need the residual relation (2.13) for the second equality in (2.15). □

Next we give a $*$-structure to the obtained $\mathfrak{h}$-Hopf algebroid. Therefore we recall the definition of a $*$-structure on $\mathfrak{h}$-bialgebroids, see [15]. Assuming $^{-} : \mathfrak{h} \to \mathfrak{h}$ is a conjugation, we put $\overline{f}(\lambda) = \overline{f(\overline{\lambda})}$, $f \in M_{\mathfrak{h}^*}$. A $*$-operator on an $\mathfrak{h}$-bialgebroid A is a $\mathbb{C}$-antilinear, antimultiplicative involution on A satisfying $\mu_l(\overline{f}) = \mu_l(f)^*$, $\mu_r(\overline{f}) = \mu_r(f)^*$ and $(* \otimes *) \circ \Delta = \Delta \circ *$, $\varepsilon \circ * = *^{D_\mathfrak{h}} \circ \varepsilon$, where $*^{D_\mathfrak{h}}$ is defined by $(f T_\alpha)^* = T_{-\overline{\alpha}} \overline{f}$. We use complex conjugation on $\mathfrak{h} \cong \mathbb{C}$.

Lemma 2.7. *The $\mathfrak{h}$-Hopf algebroid $\mathcal{F}_R(GL(2, \mathbb{C}))$ has a $*$-structure defined on the generators by* $\det^{-1}(z)^* = \det^{-1}(q^{-2}/\overline{z})$,

$$\alpha(z)^* = \delta(1/\overline{z}), \quad \beta(z)^* = -\gamma(1/\overline{z}), \quad \gamma(z)^* = -\beta(1/\overline{z}), \quad \delta(z)^* = \alpha(1/\overline{z}). \tag{2.16}$$

We call this $\mathfrak{h}$-Hopf algebroid the elliptic $U(2)$ quantum group and denote it by $\mathcal{F}_R(U(2))$.

Proof. We can easily check that this definition preserves the defining relations of the algebra, that it is an involution and that we have $(* \otimes *) \circ \Delta = \Delta \circ *$ and $\varepsilon \circ * = *^{D_\mathfrak{h}} \circ \varepsilon$.
□

Remark 2.8. Note that $S(\det(z)) = \det^{-1}(z)$ and $\det(z)^* = \det(q^{-2}/\overline{z})$.

3. Corepresentations of the Elliptic $U(2)$ Quantum Group

Before discussing a special corepresentation of the elliptic $U(2)$ quantum group, we recall the general definition of a corepresentation of an $\mathfrak{h}$-bialgebroid on an $\mathfrak{h}$-space, see [15].

Definition 3.1. *An $\mathfrak{h}$-space is a vector space over $M_{\mathfrak{h}^*}$ which is also a diagonalizable $\mathfrak{h}$-module, $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$, with $M_{\mathfrak{h}^*} V_\alpha \subseteq V_\alpha$ for all $\alpha \in \mathfrak{h}^*$. A morphism of $\mathfrak{h}$-spaces is an $\mathfrak{h}$-invariant (i.e. grade preserving) $M_{\mathfrak{h}^*}$-linear map.*

We next define the tensor product of an $\mathfrak{h}$-bialgebroid A and an $\mathfrak{h}$-space V. Put $A \tilde{\otimes} V = \bigoplus_{\alpha,\beta \in \mathfrak{h}^*} (A_{\alpha\beta} \otimes_{M_{\mathfrak{h}^*}} V_\beta)$, where $\otimes_{M_{\mathfrak{h}^*}}$ denotes the usual tensor product modulo the relations $\mu_r^A(f) a \otimes v = a \otimes f v$. The grading $A_{\alpha\beta} \otimes_{M_{\mathfrak{h}^*}} V_\beta \subseteq (A \tilde{\otimes} V)_\alpha$ and the extension of scalars $f(a \otimes v) = \mu_l^A(f) a \otimes v$, $a \in A$, $v \in V$, $f \in M_{\mathfrak{h}^*}$, make $A \tilde{\otimes} V$ into an $\mathfrak{h}$-space.

Definition 3.2. *A (left) corepresentation of an* $\mathfrak{h}$*-bialgebroid* A *on an* $\mathfrak{h}$*-space* V *is an* $\mathfrak{h}$*-space morphism* $\rho : V \to A\tilde{\otimes}V$ *such that*

$$(\Delta \otimes id) \circ \rho = (id \otimes \rho) \circ \rho, \qquad (\varepsilon \otimes id) \circ \rho = id. \tag{3.1}$$

The first equality is in the sense of the natural isomorphism $(A\tilde{\otimes}A)\tilde{\otimes}V \cong A\tilde{\otimes}(A\tilde{\otimes}V)$ *and in the second identity we use the identification* $V \simeq D_{\mathfrak{h}}\tilde{\otimes}V$ *defined by* $fT_{-\alpha} \otimes v \cong fv$*, for all* $f \in M_{\mathfrak{h}^*}$*,* $v \in V_\alpha$*.*

Choose a homogeneous basis $\{v_k\}_k$ of V over $M_{\mathfrak{h}^*}$, $v_k \in V_{\omega(k)}$, and introduce the corresponding matrix elements of a corepresentation ρ by $\rho(v_k) = \sum_j t_{kj} \otimes v_j$. For these matrix elements we have from (3.1),

$$\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}, \qquad \varepsilon(t_{ij}) = \delta_{ij} T_{-\omega(i)},$$

and Definition 2.3 implies

$$\delta_{kl} = \sum_j S(t_{kj})t_{jl} = \sum_j t_{kj} S(t_{jl}). \tag{3.2}$$

Our next objective is to construct explicit corepresentations of $\mathcal{F}_R(U(2))$. Define, with the convention that the empty product is 1,

$$\begin{aligned} v_k = v_k(z) = \gamma(z)\gamma(q^2z)\cdots\gamma(q^{2(N-k-1)}z) \\ \times \alpha(q^{2(N-k)}z)\cdots\alpha(q^{2(N-1)}z), \qquad k \in \{0, 1, \dots, N\}, \end{aligned} \tag{3.3}$$

and put $V_{2k-N} = \mu_l(M_{\mathfrak{h}^*})v_k$, $V = V^N = \bigoplus_{k=0}^N V_{2k-N}$. Then V is an $\mathfrak{h}$-space. Note that the grading on V is compatible with the grading of $\mathcal{F}_R(M(2))$. We show that $\Delta : V^N \to \mathcal{F}_R(U(2))\tilde{\otimes}V^N$, making V^N a corepresentation of $\mathcal{F}_R(U(2))$, see Theorem 3.4. We start with the following preparatory lemma.

Lemma 3.3. *In the* $\mathfrak{h}$*-Hopf algebroid* $\mathcal{F}_R(U(2))$ *we have*

$$\begin{aligned} &\alpha(q^{2k}z)\beta(q^{2(l-1)}z)\cdots\beta(z) \\ &= \frac{\theta(q^{2(k-l+1)}, q^{2(\mu+l+1)})}{\theta(q^{2(k+1)}, q^{2(\mu+1)})}\beta(z)\cdots\beta(q^{2(l-1)}z)\alpha(q^{2k}z) \\ &\quad + \frac{\theta(q^2, q^{2(\mu+k+1)})}{\theta(q^{2(k+1)}, q^{2(\mu+1)})}\sum_{i=0}^{l-1}\beta(z)\cdots\alpha(q^{2i}z)\cdots\beta(q^{2(l-1)}z)\beta(q^{2k}z), \end{aligned}$$

for all $k \geq l \geq 1$*.*

Proof. For $l = 1$ this is (2.10). In order to provide for the induction step we interchange the order of the β's since $\beta(z)\beta(w) = \beta(w)\beta(z)$ for all $z, w \in \mathbb{C}$, use the case $l = 1$ and then the induction hypothesis with $z \mapsto q^2z$, $k \mapsto k - 1$ finishes the proof using (2.6) and (2.3). □

Theorem 3.4. *In the* $\mathfrak{h}$*-Hopf algebroid* $\mathcal{F}_R(U(2))$*, with* $v_k(z)$ *defined by* (3.3)*, we have*

$$\Delta(v_k(z)) = \sum_{j=0}^{N} t_{kj}^N(\mu, z) \otimes v_j(z), \tag{3.4}$$

where the matrix-elements $t_{kj}^N(\mu, z)$ *are given by*

$$\begin{aligned} t_{kj}^N(\mu, z) = & \sum_{l=\max(0,k+j-N)}^{\min(k,j)} \begin{bmatrix} k \\ l \end{bmatrix} \begin{bmatrix} N-k \\ j-l \end{bmatrix} \frac{(q^{2(\mu+N-k-2j+l+2)})_l}{(q^{2(\mu+N-2j+2)})_l} \frac{(q^{2(\mu+l-j+2)})_{j-l}}{(q^{2(\mu+N-2j-k+2l+2)})_{j-l}} \\ & \times \gamma(q^{2(N-k-1)}z) \cdots \gamma(q^{2(N-j-k+l)}z) \delta(q^{2(N-j-k+l-1)}z) \cdots \delta(z) \\ & \times \alpha(q^{2(N-1)}z) \cdots \alpha(q^{2(N-l)}z) \beta(q^{2(N-l-1)}z) \cdots \beta(q^{2(N-k)}z). \end{aligned}$$

Proof. We first deal with the cases $k = N$ and $k = 0$, and get the general result from the homomorphism property of the comultiplication Δ.

Claim. For all $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \Delta(\alpha(z) \cdots \alpha(q^{2(k-1)}z)) = & \sum_{l=0}^{k} C_{kl}(\mu) \alpha(q^{2(k-1)}z) \cdots \alpha(q^{2(k-l)}z) \beta(q^{2(k-l-1)}z) \cdots \beta(z) \\ & \otimes \gamma(z) \cdots \gamma(q^{2(k-l-1)}z) \alpha(q^{2(k-l)}z) \cdots \alpha(q^{2(k-1)}z), \end{aligned} \tag{3.5}$$

where the coefficients $C_{kl}(\mu) \in M_{\mathfrak{h}^*}$ are given by

$$C_{kl}(\mu) = \begin{bmatrix} k \\ l \end{bmatrix} \frac{(q^{2(\mu-l+2)})_l}{(q^{2(\mu+k-2l+2)})_l}.$$

Note that $C_{k,0} = C_{k,k} = 1$. We prove the claim by induction on k. For $k = 1$ this is just (2.9) on $\alpha(z)$. Assume that the claim is true for k. Then we obtain from (2.9) and repeated application of (2.11b),

$$\begin{aligned} \Delta(\alpha(z) \cdots \alpha(q^{2k}z)) = & \ \Delta(\alpha(z) \cdots \alpha(q^{2(k-1)}z)) \Delta(\alpha(q^{2k})z) \\ = & \sum_{l=0}^{k} C_{k,l}(\mu) \alpha(q^{2(k-1)}z) \cdots \alpha(q^{2(k-l)}z) \\ & \times \beta(q^{2(k-l-1)}z) \cdots \beta(z) \beta(q^{2k}z) \\ & \otimes \gamma(z) \cdots \gamma(q^{2(k-l)}z) \alpha(q^{2(k-l+1)}z) \cdots \alpha(q^{2k}z) \\ & + \sum_{l=1}^{k+1} C_{k,l-1}(\mu) \alpha(q^{2(k-1)}z) \cdots \alpha(q^{2(k-l+1)}z) \\ & \times \beta(q^{2(k-l)}z) \cdots \beta(z) \alpha(q^{2k}z) \\ & \otimes \gamma(z) \cdots \gamma(q^{2(k-l)}z) \alpha(q^{2(k-l+1)}z) \cdots \alpha(q^{2k}z). \end{aligned}$$

For $l = 0$ and $l = k + 1$, we have $C_{k+1,0}(\mu) = C_{k,0}(\mu) = 1$ and $C_{k+1,k+1}(\mu) = C_{k,k}(\mu) = 1$ respectively. So it remains to prove that for $1 \leq l \leq k$ we have

$$\begin{aligned} &C_{k+1,l}(\mu)\alpha(q^{2k}z)\cdots\alpha(q^{2(k-l+1)}z)\beta(q^{2(k-l)}z)\cdots\beta(z) \\ &\quad= C_{k,l}(\mu)\alpha(q^{2(k-1)}z)\cdots\alpha(q^{2(k-l)}z)\beta(q^{2(k-l-1)}z)\cdots\beta(z)\beta(q^{2k}z) \\ &\qquad+ C_{k,l-1}(\mu)\alpha(q^{2(k-1)}z)\cdots\alpha(q^{2(k-l+1)}z) \\ &\qquad\times\beta(q^{2(k-l)}z)\cdots\beta(z)\alpha(q^{2k}z). \end{aligned} \tag{3.6}$$

Using

$$C_{k,l-1}(\mu) = C_{k,l}(\mu)\frac{\theta(q^{2l}, q^{2(\mu+k-2l+3)}, q^{2(\mu+k-2l+2)})}{\theta(q^{2(k-l+1)}, q^{2(\mu+k-l+2)}, q^{2(\mu-l+2)})},$$

and (2.6) we obtain that the right-hand side of (3.6) equals

$$\begin{aligned} &C_{kl}(\mu)\alpha(q^{2(k-1)}z)\cdots\alpha(q^{2(k-l+1)}z) \\ &\quad\times\Big[\alpha(q^{2(k-l)}z)\beta(q^{2(k-l-1)}z)\cdots\beta(z)\beta(q^{2k}z) \\ &\quad+\frac{\theta(q^{2l}, q^{2(\mu+k-l+2)}, q^{2(\mu+k-l+1)})}{\theta(q^{2(k-l+1)}, q^{2(\mu+k+1)}, q^{2(\mu+1)})}\beta(q^{2(k-l)}z)\cdots\beta(z)\alpha(q^{2k}z)\Big]. \end{aligned}$$

By Lemma 3.3, with (k, l) replaced by $(k - l, k - l)$ the term in square brackets equals

$$\begin{aligned} &\frac{\theta(q^{2(k+1)}, q^{2(\mu+k-l+1)})}{\theta(q^{2(k-l+1)}, q^{2(\mu+k+1)})}\left[\frac{\theta(q^{2}, q^{2(\mu+k+1)})}{\theta(q^{2(k+1)}, q^{2(\mu+1)})}\sum_{n=0}^{k-l}\beta(z)\cdots\alpha(q^{2n}z)\cdots\beta(q^{2(k-l)}z)\right. \\ &\qquad\left.\times\beta(q^{2k}z)+\frac{\theta(q^{2l}, q^{2(\mu+k-l+2)})}{\theta(q^{2(k+1)}, q^{2(\mu+1)})}\beta(z)\cdots\beta(q^{2(k-l)z})\alpha(q^{2k}z)\right] \\ &\quad=\frac{\theta(q^{2(k+1)}, q^{2(\mu+k-l+1)})}{\theta(q^{2(k-l+1)}, q^{2(\mu+k+1)})}\alpha(q^{2k}z)\beta(q^{2(k-l)}z)\cdots\beta(z), \end{aligned}$$

where we use Lemma 3.3 with (k, l) replaced by $(k, k - l + 1)$ in the last step. Using $\alpha(z)\alpha(w) = \alpha(w)\alpha(z)$ for all $w, z \in \mathbb{C}$ and (2.6) we see that the right-hand side of (3.6) equals the left-hand side using

$$C_{k+1,l}(\mu) = C_{k,l}(\mu)\frac{\theta(q^{2(k+1)}, q^{2(\mu+k-2l+2)})}{\theta(q^{2(k-l+1)}, q^{2(\mu+k-l+2)})}.$$

This proves the claim.

Since the (α,β)- and the (γ,δ)-commutation relations are similar we analogously have

$$\begin{aligned} \Delta(\gamma(z)\cdots\gamma(q^{2(k-1)}z)) = &\sum_{l=0}^{k} C_{kl}(\mu)\gamma(q^{2(k-1)}z)\cdots\gamma(q^{2(k-l)}z)\delta(q^{2(k-l-1)}z)\cdots\delta(z) \\ &\otimes\gamma(z)\cdots\gamma(q^{2(k-l-1)}z)\alpha(q^{2(k-l)}z)\cdots\alpha(q^{2(k-1)}z). \end{aligned} \tag{3.7}$$

Using (3.5), (3.7) and that the comultiplication Δ is a morphism we find

$$\begin{aligned}
&\Delta(\gamma(z)\cdots\gamma(q^{2(N-k-1)}z)\alpha(q^{2(N-k)}z)\cdots\alpha(q^{2(N-1)}z))\\
&\quad=\sum_{l=0}^{k}\sum_{m=0}^{N-k} C_{N-k,m}(\mu)C_{k,l}(\mu-2m+N-k)\gamma(q^{2(N-k-1)}z)\cdots\gamma(q^{2(N-k-m)}z)\\
&\qquad\times\delta(q^{2(N-k-m-1)}z)\cdots\delta(z)\alpha(q^{2(N-1)}z)\cdots\alpha(q^{2(N-l)}z)\\
&\qquad\times\beta(q^{2(N-l-1)}z)\cdots\beta(q^{2(N-k)}z)\\
&\qquad\otimes\gamma(z)\cdots\gamma(q^{2(N-m-l-1)}z)\alpha(q^{2(N-m-l)}z)\cdots\alpha(q^{2(N-1)}z),
\end{aligned}$$

where we use (2.6), (2.11b). Substituting $m=j-l$ proves the theorem. □

In the next proposition we prove that this corepresentation is unitary in a certain sense. Note that this property is an extension of unitarizability of a corepresentation introduced in [15].

Proposition 3.5. *The matrix elements $t_{kj}^N(\mu,z)$ of the corepresentation in Theorem 3.4 satisfy*

$$\Gamma_k(\mu)S(t_{kj}^N(\mu,z))^* = \Gamma_j(\lambda)t_{jk}^N(\mu,q^{-2(N-2)}/\overline{z})\prod_{i=0}^{N-1}\det{}^{-1}(q^{-2i}/\overline{z}),$$

with

$$\Gamma_k(\mu)=\begin{bmatrix}N\\k\end{bmatrix}\frac{(q^{2(\mu-k+2)})_k}{(q^{2(\mu+N-2k+2)})_k}\prod_{i=0}^{N-k-1}\frac{q^{-(\mu+N-2k-i)}}{\theta(q^{-2(\mu+N-2k-i+1)})}\prod_{i=0}^{k-1}\frac{q^{-(\mu-k+i)}}{\theta(q^{-2(\mu-k+i+1)})}.$$

Proof. To simplify the formulas in the proof we denote $D=\prod_{i=0}^{N-1}\det^{-1}(q^{-2i}/\overline{z})$, $G_{Nk}(\mu)=\begin{bmatrix}N\\k\end{bmatrix}\frac{(q^{2(\mu-k+2)})_k}{(q^{2(\mu-N-2k+2)})_k}$ and $F_k(\mu)=\prod_{i=0}^{k-1}F(\mu+i)$, where F is defined in Lemma 2.5.

From Theorem 3.4 we see that the matrix elements $t_{kj}^N(\mu,z)$ for k or j equal to 0 or N consist of a single term. Using Lemmas 2.6, 2.7 and the commutation relations of the elliptic quantum group proves the proposition in case $j=N$,

$$\begin{aligned}
[S(t_{kN}^N(\mu,z))]^* &= D\,\frac{F_{N-k}(\mu-k+1)}{F_{N-k}(\lambda-N)}\frac{F_k(\lambda-k)}{F_k(\mu-k)}\\
&\quad\times\alpha(q^2/\overline{z})\cdots\alpha(q^{-2(k-2)}/\overline{z})\beta(q^{-2(k-1)}/\overline{z})\cdots\beta(q^{-2(N-2)}/\overline{z})\\
&= D\,\frac{F_{N-k}(\mu-k+1)}{F_{N-k}(\lambda-N)}\frac{F_k(\lambda-k)}{F_k(\mu-k)}G_{Nk}(\mu)^{-1}t_{Nk}^N(\mu,q^{-2(N-2)}/\overline{z}).
\end{aligned}$$

From $\Delta(t_{kN}^N(\mu,z))=\sum_{j=0}^N t_{kj}^N(\mu,z)\otimes t_{jN}^N(\mu,z)$ and $\sigma\circ((*\circ S)\otimes(*\circ S))\circ\Delta=\Delta\circ(*\circ S)$ we obtain

$$\sum_{j=0}^{N} S(t_{jN}^N(\mu,z))^*\otimes S(t_{kj}^N(\mu,z))^* = \Delta(S(t_{kN}^N(\mu,z))^*).$$

This relation gives

$$\sum_{j=0}^{N} F_{N-j}(\mu-j+1)F_j(\mu-j)G_{Nj}(\mu)^{-1}t_{Nj}^N(\mu,q^{-2(N-2)}/\bar{z})\otimes S(t_{kj}^N(\mu,z))^*$$
$$=[1\otimes D\,F_{N-k}(\mu-k+1)F_k(\mu-k)G_{Nk}(\mu)^{-1}]$$
$$\times\sum_{j=0}^{N} t_{Nj}^N(\mu,q^{-2(N-2)}/\bar{z})\otimes\, t_{jk}^N(\mu,q^{-2(N-2)}/\bar{z}).$$

Since $\{t_{Nj}^N(\mu,z)\}_{j=0}^N$ are linearly independent (this follows easily from Proposition 4.2 and Lemma 4.4), the identity holds termwise. So (2.4) proves the proposition. □

4. Discrete Bi-Orthogonality for Elliptic Hypergeometric Series

Using Proposition 3.5 we can reformulate the orthogonality relations (3.2) for the matrix elements as

$$\delta_{kl}=\sum_{j=0}^{N}(t_{jl}^N(\mu,z))^*\frac{\Gamma_j(\lambda)}{\Gamma_k(\mu)}t_{jk}(\mu,q^{-2(N-2)}/\bar{z})\prod_{i=0}^{N-1}\det{}^{-1}(q^{-2i}/\bar{z}) \tag{4.1a}$$
$$=\sum_{j=0}^{N}\frac{\Gamma_l(\lambda)}{\Gamma_j(\mu)}t_{lj}^N(\mu,q^{-2(N-2)}/\bar{z})(t_{kj}^N(\mu,z))^*\prod_{i=0}^{N-1}\det{}^{-1}(q^{-2i}/\bar{z}). \tag{4.1b}$$

To obtain commutative versions of (4.1), we need to represent the algebra $\mathcal{F}_R(U(2))$ explicitly. For this we need the notion of a dynamical representation of an $\mathfrak{h}$-algebra, see [6, 7, 9, 15].

Let $V=\bigoplus_{\alpha\in\mathfrak{h}^*}V_\alpha$ be an $\mathfrak{h}$-space and let $(D_{\mathfrak{h},V})_{\alpha\beta}$ be the space of $\mathbb{C}$-linear operators U on V such that $U(gv)=T_{-\beta}(g)U(v)$ and $U(V_\gamma)\subseteq V_{\gamma+\beta-\alpha}$ for all $g\in M_{\mathfrak{h}^*}$, $v\in V_\beta$, $\gamma\in\mathfrak{h}^*$. Then the space $D_{\mathfrak{h},V}=\bigoplus_{\alpha,\beta\in\mathfrak{h}^*}(D_{\mathfrak{h},V})_{\alpha,\beta}$ is an $\mathfrak{h}$-algebra with the moment maps $\mu_l,\mu_r: M_{\mathfrak{h}^*}\to(D_{\mathfrak{h},V})_{00}$ given by $\mu_l(f)(v)=T_{-\alpha}(f)(v)$ and $\mu_r(f)(v)=fv$ for all $v\in V_\alpha$.

Definition 4.1. *A dynamical representation of an $\mathfrak{h}$-algebra A on an $\mathfrak{h}$-space V is an $\mathfrak{h}$-algebra homomorphism $A\to D_{\mathfrak{h},V}$.*

Proposition 4.2 (see [9]). *Let $\omega\in\mathbb{C}$ be arbitrary and $\mathcal{H}^\omega$ be the $\mathfrak{h}$-space with basis $\{e_k\}_{k=0}^\infty$ and weight decomposition $\mathcal{H}^\omega=\bigoplus_{k=0}^\infty\mathcal{H}^\omega_{\omega-2k}$, $\mathcal{H}^\omega_{\omega-2k}=M_{\mathfrak{h}^*}e_k$. Then there exists a dynamical representation $\pi^\omega:\mathcal{F}_R(M(2))\to D_{\mathfrak{h},\mathcal{H}^\omega}$, defined on the generators by*

$$\pi^\omega(\alpha(z))(ge_k)=A_k(\lambda,z)T_{-1}ge_k,\qquad \pi^\omega(\beta(z))(ge_k)=B_k(\lambda,z)T_1ge_{k+1},$$
$$\pi^\omega(\gamma(z))(ge_k)=C_k(\lambda,z)T_{-1}ge_{k-1},\qquad \pi^\omega(\delta(z))(ge_k)=D_k(\lambda,z)T_1ge_k, \tag{4.2}$$
$$\pi^\omega(\mu_r(f))(ge_k)=f(\lambda)ge_k,\qquad \pi^\omega(\mu_l(f))(ge_k)=f(\lambda-\omega+2k)ge_k,$$

where $g \in M_{\mathfrak{h}^*}$ *and*

$$
\begin{aligned}
A_k(\lambda, z) &= q^{2k} \frac{\theta(q^{-2(\lambda+1)-2k})\theta(zq^{\omega-2k+1})}{\theta(q^{-2(\lambda+1)})\theta(zq^{\omega+1})}, \\
B_k(\lambda, z) &= q^k \frac{\theta(q^2)\theta(zq^{-2(\lambda+1)+\omega-2k-1})}{\theta(q^{-2(\lambda+1)})\theta(zq^{\omega+1})}, \\
C_k(\lambda, z) &= q^{-(k-1)} \frac{\theta(q^{2k})\theta(q^{2(\omega-k+1)})\theta(zq^{2(\lambda+1)-\omega+2k-1})}{\theta(q^2)\theta(q^{2(\lambda+1)})\theta(zq^{\omega+1})}, \ C_0(\lambda, z) = 0, \\
D_k(\lambda, z) &= \frac{\theta(q^{-2(\lambda+1-\omega+k)})\theta(zq^{-\omega+2k+1})}{\theta(q^{-2(\lambda+1)})\theta(zq^{\omega+1})}.
\end{aligned}
$$

Remark 4.3. Using the addition formula (1.1) we obtain

$$
\pi^{\omega}(\det(z)) = q^{\omega} \frac{\theta(zq^{1-\omega})}{\theta(zq^{1+\omega})} id,
$$

so $\det(z)$ acts as a scalar. Note that this scalar is 1 if $\omega = 0$.

The action of a matrix element in the dynamical representation defined above can be calculated in terms of elliptic hypergeometric series.

Lemma 4.4. *For the dynamical representation of Proposition 4.2 we have*

$$
\pi^{\omega}(t_{kj}^N(\mu, z))(ge_m) = \tau_{kjm}^{N\omega}(\lambda, z)(T_{N-2j}g)e_{m+k-j},
$$

where $\tau_{kjm}^{N\omega}(\lambda, z)$ *is given by*

$$
\begin{aligned}
&\tau_{kjm}^{N\omega}(\lambda, z) \\
&\quad = (-1)^{N-k}\theta(q^2)^{k-j} q^{\frac{3}{2}k(k-1)+N(N+1)+\frac{5}{2}j(j+1)+2N(\lambda-k-2j)+m(k-j)+3jk-2k\lambda} \\
&\qquad \times \frac{(q^{-2(\lambda+1)}, q^{2(m+k-j+1)}, q^{2(N-k-j+1)}, q^{2(\omega-m-k+1)}, zq^{2(\lambda+N-2j+m+2)-\omega-1})_j}{(q^2, q^{2(\lambda+N-k-2j+2)}, q^{2(\lambda-j+2)})_j} \\
&\qquad \times \frac{(zq^{-2(\lambda-2j+m+k)+\omega-1})_k}{(q^{-2(\lambda+N-2j)})_k} \frac{(q^{-2(\lambda+N-2j-\omega+m)}, zq^{2(m+k)-\omega+1})_{N-k-j}}{(q^{2(\lambda-j+1)})_{N-k-j}} \frac{1}{(zq^{\omega+1})_N} \\
&\qquad \times {}_{10}\omega_9[q^{2(\lambda+N-2j-k+1)}; q^{-2k}, q^{-2j}, q^{2(\lambda-j+1)}, q^{2(\lambda+N-2j-\omega+m+1)}, \\
&\qquad\quad q^{2(\lambda+N+2+m-2j)}, zq^{2(N-m-k)+\omega+1}, z^{-1}q^{-2(m+k-1)+\omega-1}].
\end{aligned}
$$

Note that if $k + j \geq N$, one of the elliptic Pochhammer symbols in the denominator of (1.2), $(q^{2(N-k-j+1)})_l$, equals zero. This singularity is only apparent because of $(q^{2(N-k-j+1)})_j$ in front of the ${}_{10}\omega_9$.

Lemma 4.4 is a generalization of [16, §6] for the quantum $SU(2)$ group and of [15, Prop. 4.5] for the dynamical quantum $SU(2)$ group.

Proof. From Proposition 4.2 and Theorem 3.4 it follows

$$\begin{aligned}\pi^{\omega}(t^N_{kj}(\mu,z))(ge_m) = &\sum_{l=\max(0,k+j-N)}^{\min(k,j)} \begin{bmatrix} k \\ l \end{bmatrix} \begin{bmatrix} N-k \\ j-l \end{bmatrix} \frac{(q^{2(\lambda+N-k-2j+l+2)})_l}{(q^{2(\lambda+N-2j+2)})_l} \\ &\times \frac{(q^{2(\lambda+l-j+2)})_{j-l}}{(q^{2(\lambda+N-2j-k+2l+2)})_{j-l}} \\ &\times \prod_{n=0}^{j-l-1} C_{m+k-l-n}(\lambda-j+l+1+n, q^{2(n+N-k-j+l)}z) \\ &\times \prod_{n=0}^{N-k-j+l-1} D_{m+k-l}(\lambda+N-k-2j+2l-1-n, q^{2n}z) \\ &\times \prod_{n=0}^{j-l-1} A_{m+k-l}(\lambda+n+N-k-2j+l+1, q^{2(N-l+n)}z) \\ &\times \prod_{n=0}^{k-l-1} B_{m+n}(\lambda+N-2j-1-n, q^{2(n+N-k)}z) \\ &\times (T_{N-2j}g)e_{m+k-j}. \end{aligned} \tag{4.3}$$

This gives the required form of the lemma, and it remains to show that we can identify $\tau^{N\omega}_{kjm}(\lambda,z)$ with an elliptic hypergeometric series. From the explicit expressions of Proposition 4.2 we see that we can rewrite the four products in terms of elliptic factorials:

$$\begin{aligned}&\prod_{n=0}^{j-l-1} A_{m+k-l}(\lambda+n+N-k-2j+l+1, q^{2(N-l+n)}z) \\ &\quad = (-1)^l z^{-l} q^{2l(l-N)-l(\omega+l)} \frac{(q^{2(\lambda+N-2j+m+2)}, zq^{2(N-m-k)+\omega+1}, q^{2(\lambda+N-2j-k+2)})_l}{(q^{-2(N-1)-\omega-1}/z)_l (q^{2(\lambda+N-2j-k+2)})_{2l}},\end{aligned}$$

$$\begin{aligned}&\prod_{n=0}^{k-l-1} B_{m+n}(\lambda+N-2j-1-n, q^{2(n+N-k)}z) \\ &\quad = (-1)^l q^{2l(m+k-l)+l(l+1)+\frac{1}{2}(k-l)(2m+k-l-1)} \\ &\qquad \times \theta(q^2)^k \frac{(q^{2(\lambda+N-2j-k+1)}, q^{-2(N-1)-\omega-1}/z)_l (zq^{-2(\lambda+k+m-2j)+\omega-1})_k}{(q^{2(\lambda+m-2j+1)-\omega+1})_l (q^{-2(\lambda+N-2j)}, zq^{2(N-k)+\omega+1})_k},\end{aligned}$$

$$\begin{aligned}&\prod_{n=0}^{j-l-1} C_{m+k-l-n}(\lambda-j+l+1+n, q^{2(n+N-k-j+l)}z) \\ &\quad = (-1)^l \theta(q^2)^{-j} q^{j(l+1-m-k)-l(m+k+2)+\frac{1}{2}(j-l)(j-l-1)} \\ &\qquad \times \frac{(q^{2(\lambda-j+2)}, zq^{2(N-k-j)+\omega+1})_l}{(q^{-2(m+k)}, q^{2(\omega-m-k+1)}, zq^{2(\lambda+N-2j+m+2)-\omega-1})_l} \\ &\qquad \times \frac{(q^{2(\omega-m-k+1)}, zq^{2(\lambda+N-2j+M+2)-\omega-1}, q^{2(m+k-j+1)})_j}{(q^{2(\lambda-j+2)}, zq^{2(N-k-j)+\omega+1})_j},\end{aligned}$$

$$
\begin{aligned}
&\prod_{n=0}^{N-k-j+l-1} D_{m+k-l}(\lambda+N-k-2j+2l-1-n, q^{2n}z)\\
&\quad = (-1)^{N-j-k+l} z^l\\
&\qquad \times q^{(N-j-k)(2\lambda+N-3j-k+2l+2)+l(\omega+l)} \frac{(q^{-2(\lambda+N-2j-\omega+m)}, zq^{2(m+k)-\omega+1})_{N-k-j}}{(q^{2(\lambda-j+1)}, zq^{\omega+1})_{N-k-j}}\\
&\qquad \times \frac{(q^{2(\lambda+N-2j-\omega+m+1)}, q^{2(\lambda-j+1)}, q^{-2(m+k-1)+\omega-1}/z)_l}{(q^{2(\lambda+N-2j-k+1)})_{2l}(zq^{2(N-k-j)+\omega+1})_l},
\end{aligned}
\tag{4.4}
$$

where we use elementary transformation formulas, e.g. $(aq^{-4l})_l = (-1)^l (aq^{-4l})^l q^{l(l-1)} (q^2/a)_{2l}/(q^2/a)_l$ for the elliptic factorials.

Furthermore for the elliptic binomials and the other factor in (4.3) we have

$$
\begin{aligned}
&\begin{bmatrix} k \\ l \end{bmatrix} = (-1)^l q^{2l(k-l+1)+l(l-1)} \frac{(q^{-2k})_l}{(q^2)_l},\\
&\begin{bmatrix} N-k \\ j-l \end{bmatrix} = (-1)^l q^{2l(j-l+1)+l(l-1)} \frac{(q^{-2j})_l (q^{2(N-k-j+1)})_j}{(q^{2(N-k-j+1)})_l (q^2)_j},\\
&\frac{(q^{2(\lambda+N-k-2j+l+2)})_l}{(q^{2(\lambda+N-2j+2)})_l} \frac{(q^{2(\lambda+l-j+2)})_{j-l}}{(q^{2(\lambda+N-2j-k+2l+2)})_{j-l}} = (-1)^j q^{2j(\lambda-j+2)+j(j-1)}\\
&\quad \times \frac{[(q^{2(\lambda+N-k-2j+2)})_{2l}]^2 (q^{-2(\lambda+1)})_j}{(q^{2(\lambda+N-k-2j+2)}, q^{2(\lambda+N-2j+2)}, q^{2(\lambda-j+2)}, q^{2(\lambda+N-k-j+2)})_l (q^{2(\lambda+N-2j-k+2)})_j}.
\end{aligned}
\tag{4.5}
$$

Then, substituting (4.4) and (4.5) into (4.3) gives the required result. □

Analogously we can compute

Lemma 4.5. *For the dynamical representations of Proposition 4.2 we have*

$$
\pi^\omega((t_{kj}^N(\mu, z))^*)(ge_m) = \tilde{\tau}_{kjm}^{N\omega}(\lambda, z)(T_{-N+2j}g)e_{m+j-k},
$$

where $\tilde{\tau}_{kjm}^{N\omega}(\lambda, z)$ *is given by*

$$
\begin{aligned}
&\tilde{\tau}_{kjm}^{N\omega}(\lambda, z)\\
&\quad = q^{2j^2+\frac{1}{2}k(k-1)-\frac{1}{2}j(j-1)+k(1-m-j)+2m(N-j-k)+mj+2(N-k)(\lambda-k+1)-(N-k-j)(N-k-j-1)}\\
&\qquad \times (-1)^{N-j}\theta(q^2)^{j-k} \frac{(q^{2(N-k-j+1)}, q^{-2(\lambda-N+2j+1)}, q^{-2(\lambda+2j-k+m)+\omega+1}/\overline{z})_j}{(q^2)_j (q^{2(\lambda-k+2)}, q^{-2(\lambda+2j-N)}, q^{-2(N-k-1)+\omega+1}/\overline{z})_j}\\
&\qquad \times \frac{(q^{2(m+j-k+1)}, q^{2(\omega-m-j+1)}, q^{-2(N-3-\lambda+k-m-j)-\omega-1}/\overline{z})_k}{(q^{2(\lambda-k+2)}, q^{-2(N-1)+\omega+1}/\overline{z})_k}\\
&\qquad \times \frac{(q^{-2(\lambda+j+m+1-k)}, q^{-2(N-k+m-1)+\omega+1}/\overline{z})_{N-j-k}}{(q^{2(\lambda+2-N+j)}, q^{-2(N-j-k)+\omega+1}/\overline{z})_{N-j-k}}\\
&\qquad \times {}_{10}\omega_9[q^{2(\lambda-k+1)}; q^{-2k}, q^{-2j}, q^{2(\lambda+j-N+1)}, q^{2(\lambda-k-\omega+m+j+1)},\\
&\qquad\quad q^{2(\lambda+j+m-k+2)}, \overline{z}q^{2(N-m-j)+\omega-1}, q^{-2(m+j-1)+\omega+1}/\overline{z}].
\end{aligned}
$$

Lemmas 4.4 and 4.5 can be used to convert the relations (4.1) to bi-orthogonality relations for elliptic hypergeometric series. The resulting bi-orthogonality relations of Theorem 4.6 and 4.8 have been obtained previously by Frenkel and Turaev [11] and Spiridonov and Zhedanov [25] (see also Remark 4.9).

Theorem 4.6. *A bi-orthogonality relation for the elliptic hypergeometric series is given by*

$$\begin{aligned}\delta_{kl}h_k = \sum_{j=0}^{N} w_j \ {}_{10}\omega_9[&q^{2(\Lambda-2l-j+1)}; q^{-2j}, q^{-2l}, q^{2(\Lambda-l-N+1)}, q^{2(\Lambda-l-\omega+M+1)},\\ &q^{2(\Lambda-l+M+2)}, zq^{2(N-M-j-l)+\omega-1}, q^{-2(M+j+l-2)+\omega-1}/z]\\ \times\, {}_{10}\omega_9[&q^{2(\Lambda-2k-j+1)}; q^{-2j}, q^{-2k}, q^{2(\Lambda-k-N+1)}, q^{2(\Lambda-k-\omega+M+1)},\\ &q^{2(\Lambda-k+M+2)}, zq^{2(N-M-j-k)+\omega-3}, q^{-2(M+j+k-2)+\omega+1}/z],\end{aligned}$$

where the quadratic norm h_k and the weight function w_j are given by

$$\begin{aligned}h_k = &\frac{(q^2, q^{-2(\Lambda+M+1)}, q^{-2(\Lambda-\omega+M)}, q^{-2(\Lambda-N)})_k}{(q^{2(M+1)}, q^{-2N}, q^{-2(\omega-M)}, q^{-2\Lambda})_k} \frac{(q^{-2(\Lambda+1)})_{2k}}{(q^{-2\Lambda})_{2k}}\\ &\times \frac{(zq^{2M-\omega-1}, q^{2(M-N)-\omega+5}/z)_k}{(q^{-2(\Lambda+M)+\omega+1}/z, zq^{2(N-\Lambda-M)+\omega-5})_k}\\ &\times \frac{(q^{-2(\Lambda-\omega+2M+1)}, q^{-2\Lambda})_N}{(q^{-2(\Lambda+M+1)}, q^{-2(\Lambda-\omega+M)})_N} \frac{(zq^{\omega-1}, zq^{-\omega-3})_N}{(zq^{2M-\omega-1}, zq^{-2M)+\omega-3})_N},\end{aligned}$$

and $w_j =: w_1(j,k)w_2(j,l)$ with

$$\begin{aligned}&w_1(j,k)\\ &= q^{2j-2k} \frac{\theta(q^{2(\Lambda-\omega+2M-N+1+2j)})}{\theta(q^{2(\Lambda-\omega+2M-N+1)})} \frac{(q^{2(\Lambda-\omega+2M-N+1)})_j}{(q^{2(\Lambda-\omega+2M+2)})_j}\\ &\quad\times \frac{(zq^{2(\Lambda-k+M)-\omega-1})_j}{(q^{-2(N-2-M-k)-\omega+1}/z)_j}\\ &\quad\times \frac{(q^{2(M+k+1)}, q^{-2(N-k)}, q^{-2(\Lambda-k+1)}, q^{-2(\omega-M-k)})_j}{(q^2, q^{2(\Lambda-N+M+2)}, q^{2(M+1)}, q^{-2(\Lambda-2k+1)}, q^{-2N}, q^{-2(\omega-M)}, q^{2(\Lambda-N-\omega+M+1)})_j},\\ &w_2(j,l) = \frac{(q^{2(M+l+1)}, q^{-2(N-l)}, q^{-2(\Lambda-l+1)}, q^{-2(\omega-M-l)})_j}{(q^{-2(\Lambda-2l+1)})_j}\\ &\qquad\qquad \times \frac{(q^{-2(N-3-\Lambda+l-M)-\omega-1}/z)_j}{(zq^{2(M+l)-\omega-1})_j}.\end{aligned}$$

Remark 4.7. These relations are bi-orthogonality relations since there is a shift in the spectral parameter z. Omitting all other parameters the bi-orthogonality relations are in fact relations of the form

$$\delta_{kl}h_k = \sum_j w_j P_l(j, q^2 z) P_k(j, z).$$

Proof. Applying the dynamical representation π^ω of Proposition 4.2 to (4.1a) gives

$$\delta_{kl} e_m = \sum_{j=\max(0,k-m)}^{N} \tilde{\tau}^{N\omega}_{j,l,m+j-k}(\lambda, z) \frac{\Gamma_j(\lambda-\omega-N+2m+2j-2k+2l)}{\Gamma_k(\lambda-N+2l)} \times \left[\prod_{i=0}^{N-1} q^{-\omega} \frac{\theta_p(q^{-2i+1+\omega}/\overline{z})}{\theta_p(q^{-2i+1\omega}/\overline{z})}\right] \tau^{N\omega}_{jkm}(\lambda-N+2l, q^{-2(N-2)}/\overline{z}) e_{m-k+l}.$$

Replacing $\lambda+2l$ by Λ, $m-k$ by M and z by $\overline{z}$ we obtain

$$\delta_{kl} = \sum_{j=\max(0,-M)}^{N} \frac{\Gamma_j(\Lambda-\omega+2M+2j-N)}{\Gamma_k(\Lambda-N)} \prod_{i=0}^{N} q^{-\omega} \frac{\theta_p(q^{-2i+1+\omega}/z)}{\theta_p(q^{-2i+1\omega}/z)} \times \tilde{\tau}^{N\omega}_{j,l,M+j}(\Lambda-2l, \overline{z}) \tau^{N\omega}_{j,k,M+k}(\Lambda-N, q^{-2(N-2)}/z).$$

Using Lemmas 4.4 and 4.5 and elementary relations for the elliptic factorials proves the theorem. □

Theorem 4.8. *The dual bi-orthogonality relation for the elliptic hypergeometric series is given by*

$$\delta_{kl} = \sum_j \frac{w_1(l,j) w_2(k,j)}{(h_j)} {}_{10}\omega_9[q^{2(\Lambda-2j-l+1)}; q^{-2l}, q^{-2j}, q^{2(\Lambda-N-j+1)}, q^{2(\Lambda-j-\omega+M+1)}, q^{2(\Lambda+M-j+2)}, q^{-2(M+j+l-2)+\omega+1}/z, zq^{-2(M+j-N+l+1)+\omega-1}] \times {}_{10}\omega_9[q^{2(\Lambda-2j-k+1)}; q^{-2k}, q^{-2j}, q^{2(\Lambda-N-j+1)}, q^{2(\Lambda-j-\omega+M+1)}, q^{2(\Lambda+M-j+2)}, q^{-2(M+j+k-1)+\omega+1}/z, zq^{-2(M+j-N+k)+\omega-1}],$$

where w_1, w_2 and h_j are as in Theorem 4.6.

Proof. These dual bi-orthogonality relations can be computed from (4.1b) by applying the dynamical representation. Since the bi-orthogonal system in Theorem 4.6 is known to be self-dual [25], we can also obtain the dual relations from Theorem 4.6. □

Remark 4.9. In [11] an elliptic analogue of Bailey's transformation formula is proved. Let $bcdefg = a^3 q^{2(n+2)}$ and $\lambda = a^2q^2/bcd$. Then

$$ {}_{10}\omega_9[a; b, c, d, e, f, g, q^{-2n}] = \frac{(aq^2, aq^2/ef, \lambda q^2/e, \lambda q^2/f)_n}{(aq^2/e, aq^2/f, \lambda q^2/ef, \lambda q^2)_n} {}_{10}\omega_9[\lambda; \lambda b/a, \lambda c/a, \lambda d/a, e, f, g, q^{-2n}]. \tag{4.6}$$

We can relate the bi-orthogonality relations of Theorem 4.6 and 4.8 to the ones given in [25]. To obtain this relation explicitly we have to apply the elliptic analogue of Bailey's transformation formula (4.6) twice to both ${}_{10}\omega_9$-functions in our bi-orthogonality relations in different ways. Finally, let us emphasize that we do not need Bailey's transformation formula to obtain the bi-orthogonality relations of Theorem 4.6 and 4.8 in the symmetric form given.

Remark 4.10. Using the dynamical representation of Proposition 4.2 we can obtain the transformation formula (4.6) from the unitarity property of the corepresentations stated in Proposition 3.5.

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