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# Ising Models with Four Spin Interaction at Criticality

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**Abstract:** We consider two bidimensional Ising models coupled by an interaction quartic in the spins, like in the spin representation of the Eight vertex or the Ashkin-Teller model. By Renormalization Group methods we write a convergent perturbative expansion for the specific heat and for the energy-energy correlation up to the critical temperature. A form of nonuniversality is proved, in the sense that the critical behaviour is described in terms of critical indices which are non trivial functions of the coupling. The logarithmic singularity of the specific heat of the Ising model is removed or changed in a power law (with a non universal critical index) depending on the sign of the interaction.

## 1. Main Results

*1.1.* Much of our understanding about phase transitions and critical behaviour of classical spin systems on a 2D lattice is based on some remarkable exact solutions. Onsager [O] solved the *Ising model*, in which the spins take two values and only nearest-neighbor two spin interactions are considered. Lieb [Li] and Baxter [B] solved respectively the *Six vertex* and *Eight vertex* models; in their original formulation such models are vertex models (to each site of a bidimensional lattice is associated a vertex with four arrows) but via a suitable identification of the parameters they can be written as *two* Ising models coupled by a four spin interaction [W]. The *critical exponents* describing the behaviour of the system close to the critical point can be exactly computed; it is remarkable that the critical indices in the Ising or in the vertex models are *different*.

The exact solutions provide indeed a lot of detailed information about such integrable models; however even very small and apparently harmless modifications of them completely destroy their integrability. On the other hand one can hope that many relevant properties of the integrable models are quite "robust" under perturbations. It is believed that a *universality property* holds for the Ising model, in the sense that by adding to it, for instance, a next to nearest neighbor or a four spin interaction the critical indices remain unchanged. A universality property is believed to hold also for the Eight vertex model; Kadanoff [K] by "operator algebra and scaling theory" found evidence that the Eight vertex model is in the same class of universality of the *Ashkin-Teller* model [AT], which is not integrable. Other evidence for such a conclusion was found in [PB] (using second order renormalization group arguments) and [LP, N] (by a heuristic mapping of both models into the massive Luttinger model describing interacting fermions in the continuum).

The natural method to relate non-integrable models to integrable ones is given by the Renormalization Group (RG); this was known for a long time but the main open problem in this context was to implement such a method in a rigorous way. While RG methods were generally applied to the spin variables, it was realized in recent times that it can be convenient to do this in the *fermionic representation* of spin models. The fermionic representation of the Ising model was done in [SML, H, Ka, MW, S, ID] and it was shown that the correlations can be written as Grassmann integrals formally describing *non-interacting fermions* on a lattice in $d = 1+1$. In the same way Ising models with quartic interactions can be written as Grassmann integrals formally describing *interacting* non-relativistic fermions. The rigorous analysis of Grassmann integrals for non-relativistic fermions via RG methods is quite well developed, starting from [G] and [BG1] (see also [BG] or [GM] for extensive reviews) and one can apply such methods to classical 2D spin systems (such methods were already applied to a closely related problem, the XYZ Heisenberg spin chain [BM]; the relation between the Eight vertex and XYZ model is well known, [Su, Ba]). Fermionic RG methods for classical spin models have been applied first in [PS] to the Ising model with a small next to nearest neighbor or four spin interaction. A form of *universality* was established in the sense that the interaction does not change certain critical indices; the fermionic interaction is, in this case, *irrelevant* in the RG sense and the fixed point of the RG transformation is the free one.

The aim of the present paper is to study two Ising models coupled by an interaction quartic in the spins, such that both the Eight vertex and the Ashkin-Teller models are included: the system is, in general, non-integrable. The specific heat and the energy-energy correlation are written as Grassmann integrals and studied by RG methods. In such cases the fermionic interaction is *marginal* and the RG transformation has a line of fixed points. The critical behaviour is different with respect to the case of the Ising model, and it is described in terms of critical indices which are *analytic non-trivial functions of* $\lambda$. In agreement with [K] we find that the behaviour of the system is quite independent from the details of the quartic interaction. In our analysis no use is made of the Six or Eight vertex model exact solutions; we use instead some properties which can be deduced from the solution [ML] of the (massless) Luttinger model following a strategy first outlined (in purely fermionic models) in [BG1].

Our analysis establishes as a mathematically rigorous statement the statement in [K, LP, N, PB] that the Gaussian boson model, the massive Luttinger model, the Eight vertex and the Ashkin-Teller models are in the same class of universality.

*1.2.* We consider two Ising models coupled by a four spin interaction bilinear in the energy densities of the two sublattices. Given $\Lambda \in Z^2$ a square lattice with side $M$ and periodic boundary condition, we call $\mathbf{x} = (x, x_0)$ a site of $\Lambda$. If $\sigma^{(1)}_{\mathbf{x}} = \pm 1$ and $\sigma^{(2)}_{\mathbf{x}} = \pm 1$, we write the following Hamiltonian

$$H_\Lambda(\sigma^{(1)}, \sigma^{(2)}) = H_I(\sigma^{(1)}) + H_I(\sigma^{(2)}) + V(\sigma^{(1)}, \sigma^{(2)}) \equiv \sum_{x,x_0=1}^{M} H_{\Lambda,\mathbf{x}}(\sigma^{(1)}, \sigma^{(2)}), \tag{1.1}$$

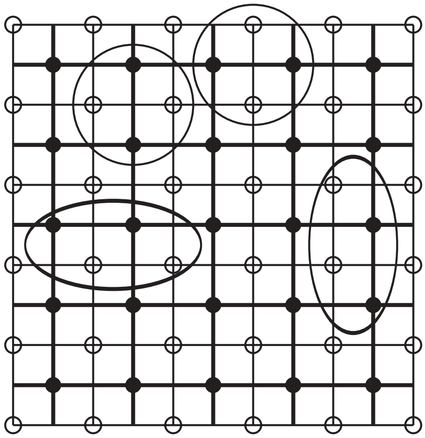

**Fig. 1.** The spins involved in the interaction of the models in Eq. (1.1). The heavy dots and lines or the light dots and lines mark the Ising lattices and the nearest neighbors Ising couplings. The ellipses symbolize the Ashkin–Teller four spin interactions and the circles the Eight vertex four spin interactions couplings

where, if $\alpha = 1, 2$

$$H_I(\sigma^{(\alpha)}) = -J \sum_{x,x_0=1}^{M} [\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x+1,x_0} + \sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x,x_0+1}] \tag{1.2}$$

is the Ising model hamiltonian and $V(\sigma^{(1)}, \sigma^{(2)})$ is the interaction between the Ising systems

$$\begin{aligned} V(\sigma^{(1)}, \sigma^{(2)}) = -\lambda \sum_{x,x_0=1}^{M} \Big\{ & a\,[\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x+1,x_0} + \sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1}] \\ & + b[\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1} + \sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x-1,x_0+1}\sigma^{(2)}_{x,x_0+1}]\Big\}. \end{aligned} \tag{1.3}$$

If $b = 0$ the Hamiltonian (1.1) coincides with the Hamiltonian of the spin representation [F] of the *Ashkin-Teller* model [AT]; if $a = 0$ it coincides with the spin representation [W] of the *Eight vertex* model.

For a given observable $O(\mathbf{x})$ localized near $\mathbf{x}$ we define the correlation

$$< O(\mathbf{x})O(\mathbf{y}) >_\Lambda = \frac{1}{Z_\Lambda} \sum_{\substack{\sigma^{(1)}_{\mathbf{x}},\sigma^{(2)}_{\mathbf{x}}=\pm 1 \\ \mathbf{x}\in\Lambda_M}} O(\mathbf{x})O(\mathbf{y})e^{-H_\Lambda(\sigma^{(1)},\sigma^{(2)})}, \tag{1.4}$$

where $Z_\Lambda = \sum_{\substack{\sigma^{(1)}_{\mathbf{x}},\sigma^{(2)}_{\mathbf{x}}=\pm 1 \\ \mathbf{x}\in\Lambda}} e^{-H_\Lambda(\sigma^{(1)},\sigma^{(2)})}$ is the the *partition function*. The *truncated correlation* of the observable $O(\mathbf{x})$ is

$$< O(\mathbf{x})O(\mathbf{y}) >_{\Lambda,T} = < O(\mathbf{x})O(\mathbf{y}) >_\Lambda - < O(\mathbf{x}) >_\Lambda < O(\mathbf{y}) >_\Lambda , \tag{1.5}$$

and the *energy-energy* truncated correlation function is given by (1.5) with $O(\mathbf{x}) = H_{\Lambda,\mathbf{x}}(\sigma^{(1)}, \sigma^{(2)})$; the *specific heat* $C_v^\lambda$ is

$$C_v^\lambda = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \sum_{\mathbf{x},\mathbf{y}\in\Lambda} < H_{\Lambda,\mathbf{x}}(\sigma^{(1)}, \sigma^{(2)}) H_{\Lambda,\mathbf{y}}(\sigma^{(1)}, \sigma^{(2)}) >_{\Lambda,T} \quad . \tag{1.6}$$

If $\lambda = 0$ the model reduces to two independent *Ising models* and close to the critical temperature (equal for both) it is

$$C_v^0 \simeq -C_1 \log |\frac{J_c}{J} - 1| + C_2 \ , \tag{1.7}$$

where $C_1, C_2$ are positive constants and $\tanh J_c = \sqrt{2} - 1$, see [MW] Eq. (3.58). The truncated correlation of the observable $O(\mathbf{x}) = H_{I,\mathbf{x}}(\sigma^{(\alpha)})$ for $\lambda = 0$ has the property $| < O(\mathbf{x}) O(\mathbf{y}) >_T | \le C e^{-A|t-t_c||\mathbf{x}-\mathbf{y}|}$ with $A, C$ suitable constants.

We expect that the interaction changes the value of the critical temperature (*i.e.* of $J_c$) by quantities $O(\lambda)$. However it is convenient to keep the critical singularity at a $\lambda$-independent value; we shall show that this can be done by choosing properly the molecular energy parameter $J$ as a function of $\lambda$. Therefore we consider the model (1.1) with $J_r$ replacing $J$, and we shall choose $J_r = J + O(\lambda)$ so that the critical coupling is precisely in correspondence of $\tanh^{-1}(\sqrt{2} - 1)$.

Denoting by $N$ an *arbitrary* positive integer, fixing $a + b \neq 0$ and with the notations $t \equiv \tanh J$, $\tanh J_r \stackrel{def}{=} \tanh J + \nu(\lambda)$ and $t_c \equiv \sqrt{2} - 1$, we shall rigorously derive the following result.

**Theorem.** *Assume $a = 0$ or $b = 0$. There are $C, C_N, C_1, C_2, \tau, \tilde{Z}_1$, positive $\lambda$–independent constants, such that for $\lambda$ small enough one can uniquely define $\nu'(\lambda)$, analytic in $\lambda$, so that the model in Eq.* (1.1), (1.3) *and with coupling $J_r = J + \nu'(\lambda)$ is critical at $t = t_c$. This means that, for $|t - t_c| > 0$,*

$$\lim_{|\Lambda|\to\infty} \langle H_{\Lambda,\mathbf{x}}(\sigma^{(1)}, \sigma^{(2)}) H_{\Lambda,\mathbf{y}}(\sigma^{(1)}, \sigma^{(2)}) \rangle_{\Lambda,T} = \Omega^a(\mathbf{x}, \mathbf{y}) + \Omega^b(\mathbf{x}, \mathbf{y}), \tag{1.8}$$

*and the bounds*

$$\begin{aligned} |\Omega^a(\mathbf{x}, \mathbf{y})| &\le \frac{1}{|\mathbf{x} - \mathbf{y}|^{2+2\eta_1}} \frac{C_N}{1 + (\Delta|\mathbf{x} - \mathbf{y}|)^N}, \\ |\Omega^b(\mathbf{x}, \mathbf{y})| &\le \frac{1}{|\mathbf{x} - \mathbf{y}|^{2+\tau}} \frac{C_N}{1 + (\Delta|\mathbf{x} - \mathbf{y}|)^N} \end{aligned} \tag{1.9}$$

*hold, with "correlation length" $\Delta^{-1}$ and "critical indices" $\eta_1, \eta_2$ given by*

$$\begin{aligned} &\Delta = |t - t_c|^{1+\eta_2}, \quad \eta_1(\lambda) = -a_1(a + b)\lambda + O(\lambda^2) \\ &\eta_2(\lambda) = -a_2(a + b)\lambda + O(\lambda^2) \end{aligned} \tag{1.10}$$

*with $a_1 > 0, a_2 > 0$ constants and $\eta_1, \eta_2$ analytic in $\lambda$. Furthermore if $1 \le |\mathbf{x}| \le \Delta^{-1}$ the correlation is asymptotic to $\Omega^a$ in the sense that $\Omega^b$ is neglegible because*

$$\Omega^a(\mathbf{x}, \mathbf{y}) = \frac{1}{\tilde{Z}_1^2} \frac{1 + A(\mathbf{x} - \mathbf{y})}{(\mathbf{x} - \mathbf{y})^{2+2\eta_1}}, \qquad |A(\mathbf{x})| \le C \left[ |\lambda| + (\Delta|\mathbf{x}|)^{\frac{1}{2}} \right]. \tag{1.11}$$

*Finally the specific heat* $C_v^\lambda$ (1.6) *verifies*

$$C_1\frac{1}{2\eta_1}[1-\Delta^{2\eta_1}]\le C_v^\lambda\le C_2\frac{1}{2\eta_1}[1-\Delta^{2\eta_1}], \tag{1.12}$$

*where* $C_1, C_2$ *are positive constants.*

*1.3.* The above result says that the interaction changes the value of the critical temperature and it qualitatively modifies the critical behaviour of the specific heat and of the energy-energy correlations. As $t$ gets closer and closer to the critical temperature the logarithmic singularity of the specific heat in the Ising model is changed by the four spin interaction into a power law singularity with non-universal critical indices if $\lambda\,(a+b)>0$; if $\lambda\,(a+b)<0$ the specific heat is instead continuous, but higher derivatives of the free energy are singular, as one can check from the proof of the Theorem.

Moreover one can distinguish two different regimes in the asymptotic behaviour of the energy- energy correlation function, discriminated by an intrinsic correlation length $\xi$ of order $|t-t_c|^{-1-\eta_2}$ with $\eta_2=O(\lambda)$. If $1\ll|\mathbf{x}-\mathbf{y}|\ll\xi$, the bound for the correlation function is power-like while if $\xi\ll|\mathbf{x}-\mathbf{y}|$, there is a faster than any power decay with rate of order $\xi^{-1}$. The splitting (1.8) and (1.9) might suggest that the fast decay is modulated by a power $|\mathbf{x}-\mathbf{y}|^{-2-2\eta_1}$ but it does not prove that because the first of (1.9) is an inequality rather than an asymptotic expression.

We do not study the free energy directly at $t=t_c$, therefore in order to show that $t=t_c$ is a critical point we must study some thermodynamic property like the specific heat by evaluating it at $t\neq t_c$ and $M=\infty$ and then verify that it has a singular behavior as $t\to t_c$. Moreover (1.11) holds uniformly for all $|t-t_c|>0$, hence we can draw the remarkable consequence that *assuming continuity for* $t\to t_c$ *, at fixed* $|\mathbf{x}-\mathbf{y}|$*, of the correlations in* (1.8) we obtain at $t=t_c$ a power law behaviour with critical index $\eta_1$. We cannot exclude a discontinuity at $t=t_c$ of the correlation in (1.8), not even at fixed $\mathbf{x}-\mathbf{y}$, because, as it is the case in various models which can be studied up to the critical point, the case $t$ precisely equal to $t_c$ cannot be discussed at the moment with our techniques in spite of the uniformity of our bounds as $t\to t_c$. In the case of the Eight vertex model our results are in agreement with the exact solution in [B] (see also [W]).

For definiteness we have chosen $V(\sigma^{(1)},\sigma^{(2)})$ of the form (1.3) but the proof of the Theorem does not depend on the details of the interaction but only on a few general properties; one needs essentially that the interaction is short ranged and it is invariant under the same symmetry transformations which leave invariant the "free" hamiltonian $H_I(\sigma^{(1)})+H_I(\sigma^{(1)})$. We will describe briefly how the proof of the theorem can be generalized in Appendix O.

*1.4.* The paper is organized in the following way. We begin to study the analyticity properties of the partition function. The starting point is the well known representation, due to [H, Ka, MW, S], of the Ising model partition function in terms of Grassmann integrals with a formal action which is quadratic. Also the partition function of the model (1.1) can be written in terms of Grassmann integrals, with a formal action which is however *non-quadratic*. By a suitable linear transformations, see §2, the Grassmann integrals can be written in a form which strongly resembles the partition function of a system of two interacting *Dirac fermions* on a lattice in $d=1+1$; one fermion (called *massive*) has an $O(1)$ mass, while the other (*light* fermion) has a mass $O(t-t_c)$ *i.e.* vanishing at criticality.

In §3 we "integrate out" the massive fermions, thus obtaining an effective theory in terms of the light fermions only. The integration of the light fields is much more difficult, as their mass is almost zero, and we perform a multiscale analysis based on Renormalization Group ideas, see §4; the result of such analysis is an integration procedure (or a resummmation prescription) for the partition functions which is written as a series in a number of functions which are called *running coupling constants* carrying a *scale label* $h = 0, 1, \dots$: for each scale there are only a few such running couplings. Contrary to the naive expansion in powers of $\lambda$ (which cannot be convergent at $t = t_c$), such expansion is well defined arbitrarily close to the critical temperature if the running coupling constants are small enough.

The running coupling constants verify a recursive relation expressing the running couplings on a given scale $h$ as a function of the ones on the previous scales $h' < h$: the latter function is usually called the *Beta function* and it is defined as long as its arguments are small enough. In §5 we show that the running coupling constants are indeed small, if $\nu'$ is chosen properly and $\lambda$ is small enough. In order to prove this one has to use two key results. The first is the exploitation of a number of symmetry cancellations to prove that a number of running coupling constants are exactly vanishing; such symmetries, which are manifest in the original spin variables, become quite involved in the fermionic representation. The second one is the decomposition of the beta function in the sum of many terms, in which only one of them is really crucial, while the others would have a small effect in the absence of the first one. One recognizes that such a crucial contribution to the Beta function of our model coincides with the Beta function of the Luttinger model: the latter Beta function was proved to be zero, as a consequence of its exact solution [ML], in [BGPS, GS, BM1] (see [BeM1] for a simplified proof). This means that the apparently largest contribution to the Beta function is essentially zero, if $\nu'(\lambda)$ is properly chosen. Note also that, despite the vanishing of the Luttinger model the Beta function is believed to be a consequence of suitable Ward identities, to convert such an argument on a rigorous proof seems at the moment quite difficult, see [BeM1], hence the only rigorous proof of such a key result is the one in [BGPS, GS, BM1].

Finally in §5 we define an expansion for the correlation functions and the specific heat; it is similar to the one for the partition function, with the main difference that one has to introduce new terms in the action associated with the external fields introduced to express via functional integrals the correlation functions.

The proof establishes rigorously a relationship between spin models with quartic interactions like the model (1.1) and the massive Luttinger model: in agreement with what was conjectured in [LP, N, PB]. Our results extend a previous paper [M1] (where Eq. (1.12) must replace Eq. (1.16) of [M1] which was incorrect). The analysis of ref. [M1] was restricted to the case $|t - t_c| \geq e^{-\frac{a}{\lambda^2}}$, where $a$ is a suitable constant. The paper is self contained aside from a few technical lemmata proved in full detail in [BM].

A very important open problem is to obtain by such fermionic RG methods the asymptotic behaviour of the spin-spin correlation function; its fermionic representation is much more involved than the one for the specific heat or for energy-energy correlations which are the only correlations considered here. One can study also the cases in which the parameters $J$ of the two Ising model hamiltonian are different so that there are two critical temperatures; new fermionic effective marginal interactions appear in such a case and universality will be probably found. Another possible extension is the analysis of *four* coupled Ising models; in this last case interacting *spinning* $d = 1$ fermions appear in the fermionic description, which are known to have a behaviour quite different from the spinless one (like in the $d = 1$ *Hubbard* model).

## 2. Fermionic Representation

*2.1.* The partition function $Z_I^{(\alpha)}$ of the Ising model with Hamiltonian $H_I(\sigma^{(\alpha)})$ in (1.3) can be written as a Grassmann integral; this is a classical result, mainly due to [Ka, H, MW, S] and rederived recently in §3 of [PS] to which we refer for a detailed proof. It is

$$Z_I^{(\alpha)} = (-1)^S \frac{(\cosh J)^B 2^S}{2} \sum_{\varepsilon,\varepsilon'=\pm} \int \prod_{\mathbf{x}\in\Lambda_M} dH_{\mathbf{x}}^{(\alpha)} d\overline{H}_{\mathbf{x}}^{(\alpha)} dV_{\mathbf{x}}^{(\alpha)} d\overline{V}_{\mathbf{x}}^{(\alpha)} (-1)^{\delta_\gamma} e^{S_{J;\gamma}^{(\alpha)}}, \quad (2.1)$$

where $\alpha = 1, 2$ denotes the lattice, $\gamma \stackrel{def}{=} (\varepsilon, \varepsilon')$ and $\delta_\gamma$ is $\delta_{+,+} \stackrel{def}{=} 1$, $\delta_{+,-} = \delta_{-,+} = \delta_{-,-} \stackrel{def}{=} 2$, $\Lambda_M = \Lambda$, $B$ is the total number of bonds and $S$ is the total number of sites,

$$\begin{aligned} S_{J,\gamma}^{(\alpha)} = \tanh J \sum_{\mathbf{x}\in\Lambda_M} [\overline{H}_{x,x_0}^{(\alpha)} H_{x+1,x_0}^{(\alpha)} + \overline{V}_{x,x_0}^{(\alpha)} V_{x,x_0+1}^{(\alpha)}] \\ + \sum_{\mathbf{x}\in\Lambda_M} [\overline{H}_{x,x_0}^{(\alpha)} H_{x,x_0}^{(\alpha)} + \overline{V}_{x,x_0}^{(\alpha)} V_{x,x_0}^{(\alpha)} + \overline{V}_{x,x_0}^{(\alpha)} \overline{H}_{x,x_0}^{(\alpha)} + V_{x,x_0}^{(\alpha)} \overline{H}_{x,x_0}^{(\alpha)} \\ + H_{x,x_0}^{(\alpha)} \overline{V}_{x,x_0}^{(\alpha)} + V_{x,x_0}^{(\alpha)} H_{x,x_0}^{(\alpha)}], \end{aligned} \quad (2.2)$$

where $H_{\mathbf{x}}^{(\alpha)}, \overline{H}_{\mathbf{x}}^{(\alpha)}, V_{\mathbf{x}}^{(\alpha)}, \overline{V}_{\mathbf{x}}^{(\alpha)}$ are *Grassmann variables* verifying different boundary conditions depending on the label $\gamma = (\varepsilon, \varepsilon')$ which is not affixed explicitly, to simplify the notations, *i.e.*

$$\begin{array}{ll} \overline{H}_{x,x_0+M}^{(\alpha)} = \varepsilon \overline{H}_{x,x_0}^{(\alpha)} & \overline{H}_{x+M,x_0}^{(\alpha)} = \varepsilon' \overline{H}_{x,x_0}^{(\alpha)} \\ H_{x,x_0+M}^{(\alpha)} = \varepsilon H_{x,x_0}^{(\alpha)} & H_{x+M,x_0}^{(\alpha)} = \varepsilon' H_{x,x_0}^{(\alpha)} \end{array} \quad \varepsilon, \varepsilon' = \pm, \quad (2.3)$$

and identical definitions are set for the variables $V^{(\alpha)}, \overline{V}^{(\alpha)}$. We call $D_\gamma$, for $\gamma = \varepsilon, \varepsilon'$ the set of $\mathbf{k}$'s such that

$$k = \frac{2\pi n_1}{M} + \frac{(\varepsilon' - 1)\pi}{2M} \qquad k_0 = \frac{2\pi n_0}{M} + \frac{(\varepsilon - 1)\pi}{2M} \quad (2.4)$$

and $-[M/2] \le n_0 \le [(M-1)/2]$, $-[M/2] \le n_1 \le [(M-1)/2]$, $n_0, n_1 \in Z$. We can write if $\mathbf{k} = (k_0, k)$,

$$H_{\mathbf{x}}^{(\alpha)} = \frac{1}{M^2} \sum_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}} H_{\mathbf{k}}^{(\alpha)} e^{-i\mathbf{k}\mathbf{x}} \quad \overline{H}_{\mathbf{x}}^{(\alpha)} = \frac{1}{M^2} \sum_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}} \overline{H}_{\mathbf{k}}^{(\alpha)} e^{-i\mathbf{k}\mathbf{x}}, \quad (2.5)$$

and similar expressions hold for $V_{\mathbf{x}}^{(\alpha)}, \overline{V}_{\mathbf{x}}^{(\alpha)}$.

The integration $\int \prod_x dH_x^{(\alpha)} d\overline{H}_x^{(\alpha)}$ or $\int \prod_x dV_x^{(\alpha)} d\overline{V}_x^{(\alpha)}$ is defined as a linear functional on the Grassmann algebra in the standard way: we recall it in Appendix A below.

It will be convenient to use auxiliary models in which $J$ is allowed to depend on $\alpha$ and on the bonds: *i.e.* we can imagine replacing the coupling $J$ of each bond $b$ joining the nearest neighbors $\mathbf{x}, \mathbf{y}$ on the lattice $\alpha$ by $J_b = J_{\mathbf{x},\mathbf{y}}^{(\alpha)}$. If $J$ is not constant but it depends on the bonds, one expresses the partition function $Z_I^{(\alpha)}(J_{\mathbf{x},\mathbf{x}'}^{(\alpha)})$ by a formula similar to Eq. (2.1) in which $S_{J,\gamma}^{(\alpha)}$, with $\gamma = (\varepsilon, \varepsilon')$, becomes

$$S_{J^{(\alpha)},\gamma}^{(\alpha)} = \sum_{\mathbf{x}} \tanh J_{x,x_0;x+1,x_0}^{(\alpha)} \overline{H}_{x,x_0}^{(\alpha)} H_{x+1,x_0}^{(\alpha)} + \tanh J_{x,x_0;x,x_0+1}^{(\alpha)} \overline{V}_{x,x_0}^{(\alpha)} V_{x,x_0+1}^{(\alpha)}$$

$$+\sum_{\mathbf{x}}[\overline{H}^{(\alpha)}_{x,x_0}H^{(\alpha)}_{x,x_0}+\overline{V}^{(\alpha)}_{x,x_0}V^{(\alpha)}_{x,x_0}+\overline{V}^{(\alpha)}_{x,x_0}\overline{H}^{(\alpha)}_{x,x_0}+V^{(\alpha)}_{x,x_0}\overline{H}^{(\alpha)}_{x,x_0}$$
$$+H^{(\alpha)}_{x,x_0}\overline{V}^{(\alpha)}_{x,x_0}+V^{(\alpha)}_{x,x_0}H^{(\alpha)}_{x,x_0}], \tag{2.6}$$

and the factor $(\cosh J)^B$ is replaced by $\prod_b \cosh J^{(\alpha)}_b$.

*2.2.* The partition function of the model (1.1) with $J_r$ replacing $J$ is

$$Z_{2I}=\sum_{\substack{\sigma^{(1)}_{\mathbf{x}}=\pm1\\ \mathbf{x}\in\Lambda_M}}\sum_{\substack{\sigma^{(2)}_{\mathbf{x}}=\pm1\\ \mathbf{x}=\Lambda_M}} e^{-H_I(\sigma^{(1)})}e^{-H_I(\sigma^{(2)})}e^{-V(\sigma^{(1)},\sigma^{(2)})}. \tag{2.7}$$

Setting $\widehat{\lambda a}\stackrel{def}{=}\tanh(\lambda\, a)$, $\widehat{\lambda b}\stackrel{def}{=}\tanh(\lambda\, b)$ we see that $Z_{2I}$ becomes $(\cosh\lambda a\ \cosh\lambda b)^{2S}$ times $\widehat{Z}_{2I}$ with

$$\begin{aligned}\widehat{Z}_{2I}=&\sum_{\substack{\sigma^{(1)}=\pm1\\ \mathbf{x}\in\Lambda_M}}\sum_{\substack{\sigma^{(2)}=\pm1\\ \mathbf{x}\in\Lambda_M}} e^{-H_I(\sigma^{(1)})}e^{-H_I(\sigma^{(2)})}\\
&\cdot\prod_{\mathbf{x}\in\Lambda_M}[1+\widehat{\lambda a}\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x+1,x_0}]\\
&\cdot\prod_{\mathbf{x}\in\Lambda_M}[1+\widehat{\lambda a}\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1}]\\
&\cdot\prod_{\mathbf{x}\in\Lambda_M}[1+\widehat{\lambda b}\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1}]\\
&\cdot\prod_{\mathbf{x}\in\Lambda_M}[1+\widehat{\lambda b}\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x-1,x_0+1}\sigma^{(2)}_{x,x_0+1}],\end{aligned} \tag{2.8}$$

where $H_I(\sigma^{(\alpha)})$ are defined as in (1.3) with $J_r$ replacing $J$. Note that

$$\sum_{\substack{\sigma^{(\alpha)}_{\mathbf{x}}=\pm1\\ \mathbf{x}\in\Lambda}}\sigma^{(\alpha)}_{\mathbf{x}}\sigma^{(\alpha)}_{\mathbf{x}'}e^{-H_I(\sigma^{(\alpha)})}=\frac{\partial}{\partial J^{(\alpha)}_{\mathbf{x},\mathbf{x}'}}Z^{(\alpha)}_I(\{J\}^{(\alpha)}_{\mathbf{x},\mathbf{x}'})|_{\{J^{(\alpha)}_{\mathbf{x},\mathbf{x}'}\}=\{J_r\}}, \tag{2.9}$$

where $\mathbf{x}$, $\mathbf{x}'$ are nearest neighbors on the lattice $\alpha$, and from (2.6) (and remembering that $a=0$ or $b=0$) this derivative gives an extra factor $\tanh J_r+\text{sech}^2 J_r^{(\alpha)}\overline{H}^{(\alpha)}_{x,x_0}H^{(\alpha)}_{x+1,x_0}$ in (2.1). We can therefore write $\widehat{Z}_{2I}$, hence $Z_{2I}$, as a Grassmann integral over the variables $H$, $V$, $\overline{H}$, $\overline{V}$. The algebra is straightforward and we reproduce it in Appendix B, and the result is that we can express $\widehat{Z}_{2I}$ as a sum of sixteen partition functions labeled by $\gamma_1,\gamma_2=(\varepsilon^{(1)},\varepsilon'^{(1)}),(\varepsilon^{(2)},\varepsilon'^{(2)})$ (corresponding to choosing each $\varepsilon$ and $\varepsilon'$ as $\pm$)

$$\widehat{Z}_{2I}=(\cosh\lambda a\ \cosh\lambda b)^{2S}\sum_{\gamma_1,\gamma_2}(-1)^{\delta_{\gamma_1}+\delta_{\gamma_2}}\widehat{Z}^{\gamma_1,\gamma_2}_{2I}, \tag{2.10}$$

each of which is given by a functional integral

$$\widehat{Z}^{\gamma_1,\gamma_2}_{2I}=\frac{(\cosh J_r)^{2B}2^{2S}}{4}\int\prod_{\alpha=1}^{2}\Big([\prod_{\mathbf{x}\in\Lambda_M}dH^{(\alpha)}_{\mathbf{x}}d\overline{H}^{(\alpha)}_{\mathbf{x}}dV^{(\alpha)}_{\mathbf{x}}d\overline{V}^{(\alpha)}_{\mathbf{x}}]\,e^{S^{(\alpha)}_{J,\gamma_\alpha}}\Big)\,e^{-\mathcal{V}}, \tag{2.11}$$

where $\mathcal{V}$ is an expression containing linear or bilinear terms in $\overline{H}^{(\alpha)}_{\mathbf{x}} H^{(\alpha)}_{x+1,x_0}$ or $\overline{V}^{(\alpha)}_{\mathbf{x}}$ $V^{(\alpha)}_{x,x_0+1}$, see (7.4). It is convenient to rewrite (2.11) in a form closer to an expression more familiar in the theory of fermionic ground states: our aim in fact is to reduce our critical point problem to a rather standard problem on the ground state of Fermi systems.

We shall consider for simplicity the partition function $\widehat{Z}^{-,-,-,-}_{2I} \stackrel{def}{=} \widehat{Z}^{-}_{2I}$, *i.e.* the partition function in which all Grassmann variables verify antiperiodic boundary conditions (see (2.3)). The other fifteen partition functions in (2.10) admit similar expressions. Furthermore it will appear that for $|t - t_c| > 0$ the logarithm of $Z^{\gamma_1,\gamma_2}_{2I}$ divided by its expression for $\lambda = 0$ is insensitive to boundary conditions up to corrections which are exponentially small in the size $M$ of the system in the thermodynamic limit in which $M \to \infty$ (see Appendix G) so that it will turn out that it is sufficient to study just one of the sixteen partition functions and $\widehat{Z}^{-,-,-,-}_{2I}$ is chosen here (arbitrarily). It is convenient to perform the following change of variables [ID], $\alpha = 1, 2$:

$$\begin{array}{ll} \overline{H}^{(\alpha)}_{\mathbf{x}} + i H^{(\alpha)}_{\mathbf{x}} = e^{i\frac{\pi}{4}} \psi^{(\alpha)}_{\mathbf{x}} - e^{i\frac{\pi}{4}} \chi^{(\alpha)}_{\mathbf{x}} & \overline{H}^{(\alpha)}_{\mathbf{x}} - i H^{(\alpha)}_{\mathbf{x}} = e^{-i\frac{\pi}{4}} \overline{\psi}^{(\alpha)}_{\mathbf{x}} - e^{-i\frac{\pi}{4}} \overline{\chi}^{(\alpha)}_{\mathbf{x}} \\ \overline{V}^{(\alpha)}_{\mathbf{x}} + i V^{(\alpha)}_{\mathbf{x}} = \psi^{(\alpha)}_{\mathbf{x}} + \chi^{(\alpha)}_{\mathbf{x}} & \overline{V}^{(\alpha)}_{\mathbf{x}} - i V^{(\alpha)}_{\mathbf{x}} = \overline{\psi}^{(\alpha)}_{\mathbf{x}} + \overline{\chi}^{(\alpha)}_{\mathbf{x}} \end{array} \tag{2.12}$$

which replaces the $H, V, \overline{H}, \overline{V}$ variables with "Majorana variables" $\psi^{(\alpha)}, \chi^{\alpha)}$. Subsequently we replace the Majorana variables with Dirac variables by setting

$$\psi^{\mp}_{1,\mathbf{x}} = \frac{1}{\sqrt{2}}(\psi^{(1)}_{\mathbf{x}} \pm i\psi^{(2)}_{\mathbf{x}}), \qquad \psi^{\mp}_{-1,\mathbf{x}} = \frac{1}{\sqrt{2}}(\overline{\psi}^{(1)}_{\mathbf{x}} \pm i\overline{\psi}^{(2)}_{\mathbf{x}})\ , \tag{2.13}$$

$$\chi^{\mp}_{1,\mathbf{x}} = \frac{1}{\sqrt{2}}(\chi^{(1)}_{\mathbf{x}} \pm i\chi^{(2)}_{\mathbf{x}}), \qquad \chi^{\mp}_{-1,\mathbf{x}} = \frac{1}{\sqrt{2}}(\overline{\chi}^{(1)}_{\mathbf{x}} \pm i\overline{\chi}^{(2)}_{\mathbf{x}})\ . \tag{2.14}$$

The final expression, see Appendix C for the algebra, is

$$\widehat{Z}^{-}_{2I} \stackrel{def}{=} \widehat{Z}^{-,-,-,-}_{2I} = \mathcal{N} \int P(d\psi) P(d\chi) e^{\mathcal{Q}(\chi,\psi) - \mathcal{V}(\chi,\psi)}, \tag{2.15}$$

where $\mathcal{N}$ is a suitable constant and, if $\phi$ denotes either $\psi$ or $\chi$,

$$\begin{aligned} &P(d\phi) = \mathcal{N}_\phi^{-1} \prod_{\mathbf{k}\in D_{-,-}} \prod_{\omega=\pm1} d\phi^{+}_{\mathbf{k},\omega} d\phi^{-}_{\mathbf{k},\omega} \exp\left[\frac{t}{2M^2} \sum_{\mathbf{k}\in D_{-,-}} -\xi^{(+)\mathbf{T}}_{\mathbf{k}} A_\phi(\mathbf{k}) \xi_{\mathbf{k}}\right], \\ &A_\phi(\mathbf{k}) = \begin{pmatrix} i\sin k + \sin k_0 & -im_\phi(\mathbf{k}) \\ im_\phi(\mathbf{k}) & i\sin k - \sin k_0 \end{pmatrix}, \ \xi^{\mathbf{T}}{}_{\mathbf{k}} = (\phi^{-}_{\mathbf{k},1}, \phi^{-}_{\mathbf{k},-1}) \\ &\xi^{+,\mathbf{T}}{}_{\mathbf{k}} = (\phi^{+}_{\mathbf{k},1}, \phi^{+}_{\mathbf{k},-1}) \end{aligned} \tag{2.16}$$

with $N_\phi$ a normalization constant, $m_\phi$ defined, *differently* for $\phi = \psi$ (choose $+$) and for $\phi = \chi$ (choose $-$), by

$$\frac{t}{2} m_\phi(\mathbf{k}) = (t - (\pm\sqrt{2} - 1)) + \frac{t}{2}(\cos k_0 + \cos k - 2). \tag{2.17}$$

*Remark.* Note that we are interested in $t$ close to $t_c = \sqrt{2} - 1$ hence, for $t \to t_c$, $m_\chi$ is bounded away from 0 and therefore $m_\chi^{-1}(\mathbf{0})$ defines a length scale which stays finite in this limit while $m_\psi(\mathbf{0}) \to 0$ and the corresponding length scale diverges. Note also that (2.15) for $\widehat{Z}^{+,+,+,+}_{2I}$ at $t = t_c$ is meaningless, as in that case $\mathcal{N} = 0$ (as $\mathcal{N}_\psi = 0$); hence the assumption $|t - t_c| > 0$.

Finally $\mathcal{Q}(\chi,\psi)$ and $\mathcal{V}(\chi,\psi)$ are obtained respectively from (7.10) and (7.5) in Appendix C through the change of variables (2.12), (2.13) and (2.14). The final expressions for them are rather intricate and we just extract from them a few properties which will be important in the following. Introducing the discrete derivatives of $\phi=\psi,\chi$ as

$$\partial_1\phi_{\mathbf{x}} \stackrel{def}{=} \phi_{x+1,x_0}-\phi_{\mathbf{x}}, \qquad \partial_0\phi_x \stackrel{def}{=} \phi_{x,x_0+1}-\phi_{\mathbf{x}}. \tag{2.18}$$

It turns out, see Appendix D, that $\mathcal{Q}$ and $\mathcal{V}$ are given by a sum of terms of the forms

$$\sum_{\mathbf{x}} A^{a;\sigma_1,\sigma_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}, \qquad \text{or} \qquad \sum_{\mathbf{x}} A^{b;\sigma_1,\sigma_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2} A^{b;\sigma_1',\sigma_2'}_{\mathbf{x}';\phi'',\omega_1';\phi''',\omega_2'}, \tag{2.19}$$

where $\mathbf{x}'=\mathbf{x}$ or $\mathbf{x}'=(x-1,x_0+1)$ with $\phi,\phi',\phi'',\phi'''\in\{\psi,\chi\}$, $\sigma=\pm$ and

1) If $\omega_1=\omega_2$ then for a suitable numerical coefficient $a_{\sigma_1,\sigma_2,\omega,c,n}$ it is, for $n=1,2$ and $c=a,b$,

$$A^{c;\sigma_1,\sigma_2}_{\mathbf{x};\phi,\omega;\phi',\omega} = a_{\sigma_1,\sigma_2,\omega,c,n}\phi^{\sigma_1}_{\omega,\mathbf{x}}\partial_{x_n}\phi'^{\sigma_2}_{\omega,\mathbf{x}} \qquad \text{with} \tag{2.20}$$

1a) If $n=1$ $\partial_{x_n}=\partial_{x_0}$ and $a_{\sigma_1,\sigma_2,\omega,c,1}$ is imaginary;
1b) If $n=2$ $\partial_{x_n}=\partial_x$ and $a_{\sigma_1,\sigma_2,\omega,c,2}$ is real.

2) If $\omega_1=-\omega_2$ then for suitable real numerical coefficients $b_{\sigma_1,\sigma_2,\omega,c,m}$, $c_{\sigma_1,\sigma_2,\omega,c,m}$ it is

$$\begin{aligned} &2a)\quad A^{c,\sigma_1,\sigma_2}_{\mathbf{x};\phi,\omega;\phi',-\omega} = ib_{\sigma_1,\sigma_2,\omega,c,m}\partial_{x_m}\phi^{\sigma_1}_{\omega,\mathbf{x}}\partial_{x_m}\phi'^{\sigma_2}_{-\omega,\mathbf{x}},\\ &\qquad\ \partial_{x_m}=\partial_{x_0} \text{ if } m=1,\ \partial_{x_m}=\partial_x \text{ if } m=2,\\ &2b)\quad A^{c,\sigma_1,\sigma_2}_{\mathbf{x};\phi,\omega;\phi',-\omega} = ic_{\sigma_1,\sigma_2,\omega,c,l}\phi^{\sigma_1}_{\omega,\mathbf{x}_l}\phi'^{\sigma_2}_{-\omega,\mathbf{x}_l}, \end{aligned} \tag{2.21}$$

with $l=1,2,3$ and $\mathbf{x}_l=\mathbf{x}$, $\mathbf{x}_l=(x+1,x_0)$, $\mathbf{x}_l=(x,x_0+1)$ for $l=1,2,3$ respectively.

*2.3.* The value of $\int P(d\phi)Q(\phi)$, where $Q(\phi)$ is any monomial on the $\phi=\psi,\chi$ variables, is given by the anticommutative Wick rule with propagator $\int P(d\phi)\,\phi^-_{\mathbf{x},\omega}\phi^+_{\mathbf{y},\omega'} = g^{(\phi)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})$ given by

$$g^{(\phi)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) = \frac{2}{tM^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}[A^{-1}_\phi(\mathbf{k})]_{\omega,\omega'}\,. \tag{2.22}$$

If we set $Q_\phi(\mathbf{k})=\det A_\phi(\mathbf{k})=-\sin^2 k_0-\sin^2 k-[m_\phi(\mathbf{k})]^2$, then

$$A^{-1}_\phi(\mathbf{k}) = \frac{1}{Q_\phi(\mathbf{k})}\begin{pmatrix} -\sin k_0+i\sin k & im_\phi(\mathbf{k})\\ -im_\phi(\mathbf{k}) & \sin k_0+i\sin k\end{pmatrix}\,. \tag{2.23}$$

The following bounds hold for the propagators, for any $N>1$ and for a suitable constant $C_N$

$$|g^{(\phi)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})| \le \frac{1}{1+|\mathbf{d}(\mathbf{x}-\mathbf{y})|}\frac{C_N}{1+|m_\phi(\mathbf{0})\mathbf{d}(\mathbf{x}-\mathbf{y})|^N}, \tag{2.24}$$

$$|g^{(\phi)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})| \le \frac{|m_\phi(\mathbf{0})|\log[1+(|m_\phi(\mathbf{0})||\mathbf{d}(\mathbf{x}-\mathbf{y})|)^{-1}]C_N}{1+|m_\phi(\mathbf{0})\mathbf{d}(\mathbf{x}-\mathbf{y})|^N}, \tag{2.25}$$

where $\mathbf{d}$ is a *distance* between $\mathbf{x}, \mathbf{y}$ which takes into account the antiperiodicity of the boundary conditions that we are considering, namely

$$\mathbf{d}(\mathbf{x}-\mathbf{y}) = \left(\frac{M}{\pi}\sin\left(\frac{\pi(x-y)}{M}\right), \frac{M}{\pi}\sin\left(\frac{\pi(x_0-y_0)}{M}\right)\right). \tag{2.26}$$

Note that the following parity properties hold:

$$g^{(\phi)}_{\omega,\omega}(\mathbf{x}) = -g^{(\phi)}_{\omega,\omega}(-\mathbf{x}), \quad g^{(\phi)}_{\omega,-\omega}(\mathbf{x}) = g^{(\phi)}_{\omega,-\omega}(-\mathbf{x}). \tag{2.27}$$

*Remark.* After the change of variables (2.12), (2.13) and (2.14) we have achieved writing $\widehat{Z}^{-}_{2I}$ as (2.15), which can be naturally seen as the partition function of a system of two kinds of bidimensional Dirac fermions on a lattice. The remark following (2.17) says that the $\chi$-fields mass is $O(1)$ while the $\psi$-fields mass is vanishing when $t = t_c$; hence the $\chi$-fields will be called massive fields and the $\psi$-fields will be called light fields. In contrast with this interpretation note, however, that the interaction $\mathcal{V}$ has a quite non-standard form; it is not invariant under global gauge transformations and is not given by products of density operators, unlike in the usual fermionic models.

## 3. Integration of Massive Fermions

*3.1.* Considering (2.15) we proceed to perform the Grassmann integration over the massive $\chi$ fields and to reduce the double integration over $\psi$, $\chi$ to an integration of a (more involved) new exponential $e^{-\mathcal{V}^{(1)}(\psi)}$ over the light fields $\psi$ alone,

$$\widehat{Z}^{-}_{2I} = \mathcal{N}\int P(d\psi)\int P(d\chi)e^{Q(\chi,\psi)}e^{-\mathcal{V}(\psi,\chi)} = \int \overline{P}(d\psi)e^{M^2\mathcal{N}^{(1)}-\mathcal{V}^{(1)}(\psi)}, \tag{3.1}$$

where $\mathcal{N}^{(1)}$ is a constant such that the *effective potential* $\mathcal{V}^{(1)}(\psi)$ vanishes at $\psi = 0$ and $\overline{P}$ is suitably defined. Indeed we prove the following result.

*3.2.* **Lemma 1.** *Assume $a = 0$ or $b = 0$. There exists $\varepsilon$ and $C$ such that, for $|\lambda|, |\nu| \le \varepsilon$,*

$$\mathcal{V}^{(1)} = \sum_{n\ge 1}\sum_{\underline{\alpha},\underline{\omega},\underline{\sigma}}\sum_{\mathbf{x}_1,..,\mathbf{x}_{2n}} W_{2n,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1,..,\mathbf{x}_{2n})\partial^{\alpha_1}\psi^{\sigma_1}_{\mathbf{x}_1,\omega_1}\ldots\partial^{\alpha_{2n}}\psi^{\sigma_{2n}}_{\mathbf{x}_{2n},\omega_{2n}},$$
$$|\widehat{W}_{2n,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{k}_1,\ldots\mathbf{k}_{n-1})| \le M^2C^n\varepsilon^{n/2}, \qquad n\ge 2. \tag{3.2}$$

*The addends in* (3.2) *with $n = 2$ can be written, for $l_1 = 2(\widehat{\lambda a}+\widehat{\lambda b})\mathrm{sech}^4 J_r + O(\varepsilon^2)$ real, as*

$$l_1\sum_{\mathbf{x}}\psi^{+}_{1,\mathbf{x}}\psi^{+}_{-1,\mathbf{x}}\psi^{-}_{-1,\mathbf{x}}\psi^{-}_{1,\mathbf{x}} + \sum_{\mathbf{x}_1,..,\mathbf{x}_4}\sum_{\alpha_1+..\alpha_4\ge 1,\underline{\sigma}} W_{4,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1,..,\mathbf{x}_4)$$
$$\times\,\partial^{\alpha_1}\psi^{\sigma_1}_{\mathbf{x}_1,\omega_1}\partial^{\alpha_2}\psi^{\sigma_2}_{\mathbf{x}_2,\omega_2}\partial^{\alpha_3}\psi^{\sigma_3}_{\mathbf{x}_3,\omega_3}\partial^{\alpha_4}\psi^{\sigma_4}_{\mathbf{x}_4,\omega_4}. \tag{3.3}$$

*The addend with $n = 1$ can be written, for $\nu_1 = \nu + O(\varepsilon)$, $a_1, a_2 = \nu/2 + O(\varepsilon)$, as*

$$\sum_{\omega}\sum_{\mathbf{x}}[-i\omega\nu_1\psi^{+}_{\mathbf{x},\omega}\psi^{-}_{\mathbf{x},-\omega} + \psi^{+}_{\mathbf{x},\omega}(i\omega a_1\partial_0 - a_2\partial_1)\psi^{-}_{\mathbf{x},\omega}]$$
$$+\sum_{\mathbf{x}_1,\mathbf{x}_2}\sum_{\{\omega\}}\sum_{\alpha_1+\alpha_2\ge 2,\sigma_1,\sigma_2} W_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1,\mathbf{x}_2)\partial^{\alpha_1}\psi^{\sigma_1}_{\mathbf{x}_1,\omega_1}\partial^{\alpha_2}\psi^{\sigma_2}_{\mathbf{x}_2,\omega_2} \tag{3.4}$$

*with* $\nu_1, a_1, a_2$ *real and* $|\widehat{W}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{k}_1)| \le M^2 C|\varepsilon|$. *Finally making use of a general notation for later reference, the Grassmann integration* $\overline{P}(d\psi)$ *is* $P_{Z_1,m_1,\widetilde{C}_1}(d\psi)$, *where*

$$P_{Z_1,m_1,\widetilde{C}_1}(d\psi) = \mathcal{N}^{-1} \prod_{\substack{\mathbf{k}\in D_{-,-}\\ \widetilde{C}_1(\mathbf{k})^{-1}>0}} \prod_{\omega=\pm 1} d\psi^+_{\mathbf{k},\omega} d\psi^-_{\mathbf{k},\omega} \exp\left[ -\frac{tZ_1\widetilde{C}_1(\mathbf{k})}{M^2} \sum_{\substack{\mathbf{k}\in D_{-,-}\\ \widetilde{C}_1(\mathbf{k})^{-1}>0}} \psi^+_{\mathbf{k},\omega} T^{(1)}_{\omega,\omega'}(\mathbf{k}) \psi^-_{\mathbf{k},\omega'} \right],$$

$$T^{(1)}(\mathbf{k}) \stackrel{def}{=} \frac{1}{C_0 + \mu_{0,0}(\mathbf{k})} \times \begin{pmatrix} \widetilde{Z}_1(i\sin k + \sin k_0) + \mu_{1,1}(\mathbf{k})Z_1^{-1} & -im_1 - i\mu_{1,2}(\mathbf{k})Z_1^{-1} \\ im_1 + i\mu_{1,2}(\mathbf{k})Z_1^{-1} & \widetilde{Z}_1(i\sin k - \sin k_0) + \mu_{2,2}(\mathbf{k})Z_1^{-1} \end{pmatrix} \quad (3.5)$$

*with* $C_0 = (t+1+\sqrt{2})^2$, $\widetilde{C}_1(\mathbf{k}) \equiv 1$, $m_1 = C_0(t-t_c)$, $Z_1 = 1$, $\widetilde{Z}_1 = \frac{t}{2}[(2t+2\sqrt{2}t) + (2\sqrt{2}+3+t^2)]$, $\mu_{i,j}(\mathbf{k})$ *analytic functions in* $\mathbf{k}$ *of size* $O(\mathbf{k}^2)$ *with* $\mu_{i,i}(\mathbf{k})$, $i = 1, 2$, *odd and* $\mu_{1,2}(\mathbf{k})$ *even and real; moreover* $C_0 + \mu_{0,0} \ge 1$ *and* $\det T^{(1)}(\mathbf{k})$ *is bounded above and below by two constants times* $-2t(1-t^2)(\cos k_0 + \cos k_1 - 2) + m_1^2$.

The proof of the above proposition is a repetition of standard arguments, see for instance [BGPS] or [BM]: the key is the Gram-Hadamard inequality applied along the lines of Lesniewski, [Le]. For completeness the details are in Appendix E and F.

The result says that the integration of the massive fermions has the "*only*" effect over the remaining (non trivial) $\psi$–integration of modifying the propagator of the light $\psi$ fields by a few trivial factors of $O(1)$ (analytically dependent on $\lambda$ for $\lambda$ small).

The only difficulty and novelty is that a detailed analysis of the bilinear and quartic terms in $\mathcal{V}^{(1)}$ is necessary. In fact we have to show that the quadratic part can be written as in (3.4), saying that *there are no terms of the form* $\psi^\sigma_{\mathbf{x},\omega}\psi^{-\sigma}_{\mathbf{x},\omega}$, *or* $\psi^\sigma_{\mathbf{x},\omega}\psi^\sigma_{\mathbf{x},-\omega}$ *or* $\psi^\sigma_{\mathbf{x},\omega}\partial\psi^\sigma_{\mathbf{x},-\omega}$; despite the fact that such terms are absent in $\mathcal{V}$, they *could* be generated by the integration of the $\chi$ variables. This is not the case, as a consequence of symmetry properties verified by the model (1.1), as it will be shown in Appendix F.

## 4. Renormalization Group for Light Fermions

*4.1. Multiscale analysis.* We continue the analysis of $\widehat{Z}^-_{2I}$ (2.15); after the integration over the $\chi$–fields we have to compute the Grassmann integral over the $\psi$–fields given by the r.h.s. of (3.1). The problem is quite different from the one treated in Sect. 3 because the $\psi$–field has propagator, (3.5), with "mass" $O(t-t_c)$ which can be arbitrarily close to 0, and we need estimates that are uniform in this quantity. Therefore we shall proceed via a *multiscale analysis* following the techniques developed to study the ground state of one–dimensional Fermi systems in [BG], [BGPS] and [BM].

We introduce a *scaling parameter* $\gamma > 1$ which will be used to define a geometrically growing sequence of length scales $1, \gamma, \gamma^2, \gamma^3, \dots$, *i.e.* of geometrically decreasing momentum scales $\gamma^h$, $h = 0, -1, -2, \dots$. Let $\chi(\mathbf{k}) \in C^\infty$ be a non-negative function such that

$$\chi(\mathbf{k}) = \chi(-\mathbf{k}) = \begin{cases} 1 & \text{if } |\mathbf{k}| < 1/\gamma\ , \\ 0 & \text{if } |\mathbf{k}| > 1\ , \end{cases}, \quad \text{where } |\mathbf{k}| = \sqrt{\sin k_0^2 + \sin k^2}\ , \qquad (4.1)$$

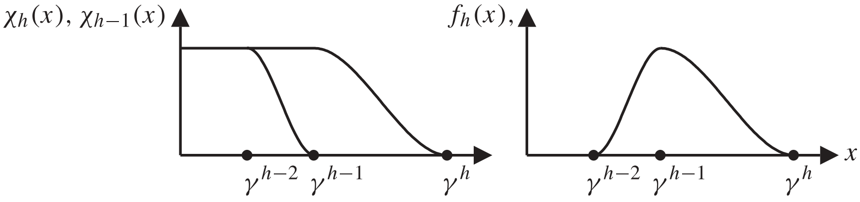


**Fig. 2.** *The function* $\chi(\gamma^{-h}x)$, $\chi(\gamma^{-(h-1)}x)$, $f(\gamma^{-h}x)$

and for $h \leq 0$ integer define $f_h(\mathbf{k}) \stackrel{def}{=} \chi(\gamma^{-h}\mathbf{k}) - \chi(\gamma^{-h+1}\mathbf{k})$ so that, for $h' < 0$, it is $\chi(\mathbf{k}) = \sum_{h=h'+1}^{0} f_h(\mathbf{k}) + \chi(\gamma^{-h'}\mathbf{k})$ .

Note that, if $h \leq 0$, $f_h(\mathbf{k}) = 0$ for $|\mathbf{k}| < \gamma^{h-2}$ or $|\mathbf{k}| > \gamma^h$, and $f_h(\mathbf{k}) = 1$, if $|\mathbf{k}| = \gamma^{h-1}$. Furthermore with our boundary conditions $\varepsilon = \varepsilon' = -$, see (2.4), the momenta $\mathbf{k} = (k_0, k)$ are such that $|\mathbf{k}| > k_M \stackrel{def}{=} \frac{\sqrt{2}}{\pi M}$. Therefore if we define the "minimum" momentum scale larger than $k_M$ (*i.e.* $h_M = \min\{h : \gamma^h > k_M\}$) it will be for all such $\mathbf{k}$:

$$1 = \sum_{h=h_M}^{1} f_h(\mathbf{k}) \qquad f_1 = 1 - \chi(\mathbf{k}) \, , \tag{4.2}$$

which can be visualized as in Fig. 2.

Note that the fact that $h_M$ is finite plays essentially no role in the subsequent analysis; note also that we are making a multiscale decomposition around $k = k_0 = 0$ as it is the only pole of the propagator corresponding to $P_{Z_1,m_1,\widetilde{C}_1}(d\psi)$.

The purpose is to perform the integration over the light fermion fields in a iterative way. The iteration steps will be labeled by scale values $h = 1, 0, -1, \ldots, h_M$. The number of iterations will be $-h_M + 2$ and after each iteration we shall be left with a "simpler" Grassmann integration to perform: it will be an integration with respect to a field $\psi^{(\leq h)}$, $h = 0, -1, \ldots, h_M$ of

$$\int P_{Z_h,m_h,C_h}(d\psi^{(\leq h)})\, e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) - M^2 E_h} \, , \quad \mathcal{V}^{(h)}(0) = 0 \, , \tag{4.3}$$

where the quantities $P_{Z_h,m_h,C_h}(d\psi)$, $Z_h$, $m_h$, $C_h(\mathbf{k})$, $\mathcal{V}^{(\leq h)}(\psi)$, $E_h$ have to be defined recursively and the result of the last iteration will be $e^{-M^2 E_{-1+h_M}} \equiv \widehat{Z}_{2I}$, *i.e.* the value of the partition function.

The $P_{Z_h,m_h,C_h}(d\psi)$ integration is defined by (3.5) in which we replace $Z_1$, $m_1$, $\widetilde{C}_1(\mathbf{k})$ by other quantities

$$Z_h, m_h, C_h(\mathbf{k}) \text{ with } C_h(\mathbf{k})^{-1} = \sum_{j=h_M}^{h} f_j(\mathbf{k}), \tag{4.4}$$

keeping $\widetilde{Z}_1$ fixed to the value in (3.5) and $Z_h$, $m_h$ recursively defined as discussed below; moreover

$$\mathcal{V}^{(h)}(\psi^{(\leq h)}) = \sum_{n=1}^{\infty} \sum_{\substack{\mathbf{x}_1,\ldots,\mathbf{x}_{2n}, \\ \underline{\sigma},\underline{\omega},\underline{\alpha}}} \prod_{i=1}^{2n} \partial^{\alpha_i} \psi^{(\leq h)\sigma_i}_{\mathbf{x}_i,\omega_i} W^{(h)}_{2n,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1, \ldots, \mathbf{x}_{2n}) \, . \tag{4.5}$$

*4.2. The localization operator.* The effective potential $\mathcal{V}^{(h)}$ will be rather involved: to define it recursively it will be convenient to identify in it a part that can be called "*irrelevant*" and the rest. Here the word irrelevant does not mean "negligible": it identifies a part of $\mathcal{V}^{(h)}$ which can be expressed as a (convergent) power series in terms of a number of parameters $v_{h'}$, $h' > h$, which we call *running coupling constants*. The latter are also defined recursively and they can be isolated from the effective potential $\mathcal{V}^{(h)}$ by acting on it with a "*localization operator*" $\mathcal{L}$ which extracts from the sum of monomials in the fields in (3.1) the terms of degree $2n = 2, 4$ in the fields and from each of them it extracts the "local part": for $h \leq 0$ it acts on the kernels $W$ by simplifying them as follows:

1) If $2n = 4$, then we define

$$\mathcal{L}\widehat{W}^{(h)}_{4,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \widehat{W}^{(h)}_{4,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++}) \;, \tag{4.6}$$

where $\bar{\mathbf{k}}_{++} = (\frac{\pi}{M}, \frac{\pi}{M})$ is the smallest momentum allowed by the boundary conditions that we are using (see (2.4)).

2) If $2n = 2$ and $\bar{\mathbf{k}}_{\eta\eta'} = \left(\eta\frac{\pi}{M}, \eta'\frac{\pi}{M}\right)$, then

$$\mathcal{L}\widehat{W}^{(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{k}) = \frac{1}{4}\sum_{\eta,\eta'=\pm1} \widehat{W}^{(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'})\left[1+\frac{M}{\pi}a_M(\eta\sin k+\eta'\sin k_0)\right] \;, \tag{4.7}$$

where $a_M\frac{M}{\pi}\sin\frac{\pi}{M} = 1$.

3) In all other cases $\mathcal{L}\widehat{W}^{h}_{2n,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{k}_1,\ldots,\mathbf{k}_{2n-1}) = 0$.

*Remark.* Note that in the limit $M \to \infty$ (4.7) becomes simply

$$\mathcal{L}\widehat{W}^{(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{k}) = [\widehat{W}^{(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{0}) + \sin k_0\partial_{k_0}\widehat{W}^{(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{0}) + \sin k\partial_k\widehat{W}^{a(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{0})], \tag{4.8}$$

hence $\mathcal{L}\widehat{W}^{(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{k})$ has to be understood as a discrete version of the Taylor expansion up to order 1. Since $a_M = 1 + O(M^{-2})$ this property would be true also if $a_M = 1$; however the choice (4.7) shares with (4.8) another important property, that is $\mathcal{L}^2\widehat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{k}) = \mathcal{L}\widehat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{k})$, see [BM].

*4.3. Relevant, marginal and irrelevant operators.* By (4.6),(4.7) and the symmetry relations in Appendix F we can write $\mathcal{L}\mathcal{V}^{(h)}$ as:

$$\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\leq h)}) = (s_h+\gamma^h n_h)F_m^{(\leq h)} + l_h F_\lambda^{(\leq h)} + z_h F_\zeta^{(\leq h)} + a_h F_\alpha^{(\leq h)} \;, \tag{4.9}$$

where $s_h, n_h, l_h, z_h, a_h$ are real and, if $|\lambda|, |\nu| \leq \varepsilon$, $s_1 = O(m_1\lambda)$, $z_1, a_1 = O(\lambda)$, $l_1 = 2(\widehat{\lambda a} + \widehat{\lambda b})\,\mathrm{sech}^4 J_r + O(\lambda^2)$, $\gamma n_1 = \nu + O(\lambda)$; moreover

$$F_m^{(\leq h)} = \frac{1}{M^2}\sum_{\mathbf{k}\in\mathcal{D}_M}\sum_{\omega=\pm1} i\omega\widehat{\psi}^{(\leq h)+}_{\mathbf{k},\omega}\widehat{\psi}^{(\leq h)-}_{\mathbf{k},-\omega} \;,$$

$$F_\lambda^{(\leq h)} = \frac{1}{M^8}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_4\in\mathcal{D}_M} \widehat{\psi}^{(\leq h)+}_{\mathbf{k}_1,+1}\widehat{\psi}^{(\leq h)+}_{\mathbf{k}_2,-1}\widehat{\psi}^{(\leq h)-}_{\mathbf{k}_3,-1}\widehat{\psi}^{(\leq h)-}_{\mathbf{k}_4,+1}\delta(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4) \;,$$

$$F_\alpha^{(\leq h)} = \frac{1}{M^2}\sum_{\mathbf{k}\in\mathcal{D}_M}\sum_{\omega=\pm1} i\sin k\widehat{\psi}^{(\leq h)+}_{\mathbf{k},\omega}\widehat{\psi}^{(\leq h)-}_{\mathbf{k},\omega} \;,$$

$$F_\zeta^{(\le h)} = \frac{1}{M^2}\sum_{\mathbf{k}\in\mathcal{D}_M}\sum_{\omega=\pm1}\omega\sin k_0 \widehat{\psi}^{(\le h)+}_{\mathbf{k},\omega}\widehat{\psi}^{(\le h)-}_{\mathbf{k},\omega}\ , \tag{4.10}$$

where $\delta(\mathbf{k}) = 0$ if $\mathbf{k} \neq \mathbf{0}$ and $\delta(\mathbf{0}) = 1$. Applying the operations $\mathcal{L}$ to the kernels of the effective potential generates the sum in (4.9), *i.e.* a linear combination of the Grassmann monomials in (4.10) which, in the renormalization group language are called "*relevant*" operators (the first) and "*marginal*" operators (the three others); while applying the operations $1-\mathcal{L}$ generates a sum of (infinitely many, in the limit $M \to \infty$) monomials called *irrelevant operators*.

Note that one can repeat the analysis in Appendix F to conclude that many terms, which could be *a priori* present in (4.9) are indeed absent. Hence the constants $n_h, s_h, l_h, z_h, a_h$ are real and many possible marginal interactions (like $\sum_{\mathbf{k}} \sin k \widehat{\psi}^{(\le h)+}_{\mathbf{k},\omega}\widehat{\psi}^{(\le h)-}_{\mathbf{k},-\omega}$ or $\sum_{\mathbf{k}} \widehat{\psi}^{(\le h)+}_{\mathbf{k},\omega}\widehat{\psi}^{(\le h)-}_{\mathbf{k},\omega}$) are excluded. This remark is crucial in order to analyze the flow of the running coupling constant, see the next section: it shows that the number of relevant or marginal operators is far smaller than *a priori* one might expect, due to the symmetries in the hamiltonian.

Note also that we have written the coefficient of $F_\sigma^{(\le h)}$ as $s_h + \gamma^h n_h$ according to a rule which will be specified in (4.17), (4.18) below and for the reasons explained in the subsequent remark.

*4.4. Renormalization.* We have set all definitions needed to define the recursive procedure leading to the definition of the running couplings and of the effective potentials.

Suppose that $Z_k, m_k, C_k, \mathcal{V}^{(k)}$ in (4.3) have been defined for $k = 1, 0, \dots h+1$. Then we can write $\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})$ as $\mathcal{L}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) + (1-\mathcal{L})\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})$ and we split from $\mathcal{L}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})$ in (4.9) the three terms quadratic in $\psi^{(\le h)}$ given by $Z_h s_h F_\sigma^{(\le h)} + Z_h z_h(F_\zeta^{(\le h)} + F_\alpha^{(\le h)})$.

Since such terms are quadratic we can imagine to include them in the "the free integration" $P_{Z_h,m_h,C_h}(d\psi^{(\le h)})$ by simply replacing the integration symbol $P_{Z_h,m_h,C_h}(d\psi^{(\le h)})$ by a new Grassmann integration symbol $P_{\hat{Z}_{h-1},m_{h-1},C_h}(d\psi^{(\le h)})$ obtained from $P_{Z_h,m_h,C_h}$ $(d\psi^{(\le h)})$ via the substitutions of $Z_h, m_h(\mathbf{k})$ with

$$\begin{aligned}\hat{Z}_{h-1}(\mathbf{k}) &= Z_h[1 + t^{-1}C_h^{-1}\widetilde{Z}_1^{-1}(C_0 + \mu_{0,0}(\mathbf{k}))z_h],\\ m_{h-1}(\mathbf{k}) &= \frac{Z_h}{\hat{Z}_{h-1}(\mathbf{k})}[m_h(\mathbf{k}) + C_h^{-1}(\mathbf{k})t^{-1}(C_0 + \mu_{0,0}(\mathbf{k}))s_h]\ ;\end{aligned} \tag{4.11}$$

and correspondingly by replacing $\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})$ by $\widetilde{\mathcal{V}}^{(h)} = \mathcal{V}^{(h)} - Z_h s_h F_\sigma^{(\le h)} - Z_h z_h(F_\zeta^{(\le h)} + F_\alpha^{(\le h)})$. This means that the subtracted terms are imagined included in $P_{\hat{Z}_{h-1},m_{h-1},C_h}$ as an algebraic check confirms.

If $\exp(-M^2 t_h)$ is a suitable constant factor fixing normalization of the two integrations we get

$$\begin{aligned}&\int P_{Z_h,m_h,C_h}(d\psi^{(\le h)})e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})}\\ &\quad = e^{-M^2 t_h}\int P_{\hat{Z}_{h-1},m_{h-1},C_h}(d\psi^{(\le h)})\, e^{-\widetilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})}\ ,\end{aligned} \tag{4.12}$$

and we try to express the *r.h.s.* as a double integral by writing $\psi^{(\le h)} = \psi^{(\le h-1)} + \psi^{(h)}$.

We shall call $m_h(\mathbf{0}) \equiv m_h$ and $\hat{Z}_h(\mathbf{0}) \equiv Z_h$. The r.h.s of (4.12) can be written, as an algebraic check will confirm, as

$$e^{-M^2 t_h} \int P_{Z_{h-1},m_{h-1},C_{h-1}}(d\psi^{(\le h-1)}) \int P_{Z_{h-1},m_{h-1},\tilde{f}_h^{-1}}(d\psi^{(h)})\, e^{-\widetilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})} , \tag{4.13}$$

where we have set

$$Z_{h-1} = Z_h(1 + z_h t^{-1} \widetilde{Z}_1^{-1} C_0) , \quad \widetilde{f}_h(\mathbf{k}) = Z_{h-1}\left[\frac{C_h^{-1}(\mathbf{k})}{\hat{Z}_{h-1}(\mathbf{k})} - \frac{C_{h-1}^{-1}(\mathbf{k})}{Z_{h-1}}\right] . \tag{4.14}$$

Note that $\widetilde{f}_h(\mathbf{k})$ has the same support of $f_h(\mathbf{k})$. The *single scale* propagator is

$$\begin{aligned}
&\int P_{Z_{h-1},m_{h-1},\widetilde{f}_h^{-1}}(d\psi^{(h)})\, \psi^{(h)-}_{\mathbf{x},\omega} \psi^{(h)+}_{\mathbf{y},\omega'} = \frac{g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})}{Z_{h-1}} , \\
&g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) \stackrel{def}{=} \frac{1}{tM^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\, \widetilde{f}_h(\mathbf{k})[T_h^{-1}(\mathbf{k})]_{\omega,\omega'} ,
\end{aligned} \tag{4.15}$$

and $T_h(\mathbf{k})$ is defined by performing in (3.5) the replacement indicated in (4.4).

If $|\widetilde{Z}_1^{-1} C_0 z_h| \le \frac{1}{2}$, $|C_0 s_h| \le |m_h/2|$ and $\sup_{k\ge h} |\frac{Z_k}{Z_{k-1}}| \le e^{c_0|\lambda|}$, the large distance behavior of $g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})$ and of its (discrete) derivatives can be established in detail and one finds that it is characterized by a *single lengh scale*, namely $\gamma^{-h}$. The analysis leads to naively expected results that will be exploited in the following and it is performed in Appendix H.

We can now specify according to which rule the splitting $\widehat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'}) = s_h + \gamma^h n_h$ in (4.9) will be done. We write

$$\begin{aligned}
&g^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y}) = \widehat{g}^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y}) + \widetilde{g}^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y}), \\
&\widehat{g}^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y}) \stackrel{def}{=} \frac{1}{tM^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\, \widetilde{f}_h(\mathbf{k}) \frac{i m_h(\mathbf{k})}{\widetilde{Z}_1^2 \sin^2 k_0 + \widetilde{Z}_1^2 \sin k^2 + m_h^2(\mathbf{k})},
\end{aligned} \tag{4.16}$$

and $\widetilde{g}^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})$ is $g^{(h)}_{\omega,-\omega} - \widehat{g}^{(h)}_{\omega,-\omega}$ and it does not vanish for $m_h = 0$. We write

$$\widehat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}} = \widehat{W}^{a(h)}_{2,\underline{\sigma},\underline{\omega}} + \widehat{W}^{b(h)}_{2,\underline{\sigma},\underline{\omega}} \tag{4.17}$$

with $\widehat{W}^{a(h)}_{2,\underline{\sigma},\underline{\omega}}$ given by definition by a sum of terms containing at least a propagator $\widehat{g}^{(k)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})$, $k > h$ and we set

$$s_h = \delta_{\omega,-\omega}\left[\frac{1}{4}\sum_{\eta,\eta'=\pm1} \widehat{W}^{a(h)}_{2,\underline{\sigma},\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'})\right] \qquad \gamma^h n_h = \delta_{\omega,-\omega}\left[\frac{1}{4}\sum_{\eta,\eta'=\pm1} \widehat{W}^{b(h)}_{2,\underline{\sigma},\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'})\right]. \tag{4.18}$$

Such definitions imply that $W^{a(h)}_{2,\underline{\sigma},\underline{\omega}}$ is vanishing at $t = t_c$ for all $h$.

*Remark.* In a theory of fermions if there is no mass term in the action then no mass terms are generated by the Renormalization Group iterations, by local Gauge invariance. In our spin model this is not true, as the interaction is not Gauge invariant; hence even if $t = t_c$ (or $m_1 = 0$) a mass term in the Renormalization Group iterations can be generated. Hence we collect all the relevant terms which are vanishing if $t = t_c$, in $s_h$, which we include in the fermionic integration; the "mass" has a non trivial flow producing at the end the critical index of the correlation length. The remaining terms are left in the effective interaction; they constitute the running coupling constant $\nu_h$ whose flow is controlled by the counterterm $\nu$.

We now *rescale* the kernels $W^{(h)}_{2n,\underline{\sigma},\underline{\alpha},\underline{\omega}}$ in $\widetilde{\mathcal{V}}^{(h)}$, see (4.4), by a factor $\sqrt{Z_h}/\sqrt{Z_{h-1}}$ so that the effective potential $\widetilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})$ can be rewritten as

$$\widetilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) = \widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})\ ; \tag{4.19}$$

and as a consequence, see (4.9),

$$\begin{aligned} &\mathcal{L}\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{\le h}) = \gamma^h \nu_h Z_{h-1} F_\sigma^{(\le h)} + \delta_h Z_{h-1} F_\alpha^{(\le h)} + \lambda_h (Z_{h-1})^2 F_\lambda^{(\le h)}\ , \\ &\nu_h \stackrel{def}{=} \frac{Z_h}{Z_{h-1}} n_h\ , \quad \delta_h \stackrel{def}{=} \frac{Z_h}{Z_{h-1}}(a_h - z_h)\ , \quad \lambda_h \stackrel{def}{=} (\frac{Z_h}{Z_{h-1}})^2 l_h\ . \end{aligned} \tag{4.20}$$

We will call $v_h \stackrel{def}{=} (\lambda_h, \delta_h, \nu_h)$ the *running coupling constants* and $Z_h, m_h$ the *renormalization constants*.

If we now define $\mathcal{V}^{(h-1)}$, $\widetilde{E}_h$ by

$$e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)}) - M^2 \widetilde{E}_h} = \int P_{Z_{h-1}, m_{h-1}, \widetilde{f}_h^{-1}}(d\psi^{(h)})\, e^{-\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})}\ , \tag{4.21}$$

with $\widetilde{E}_h$ such that $\mathcal{V}^{(h-1)}(0) = 0$, we see that $\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)})$ is of the form (4.4) and $E_{h-1} = E_h + t_h + \widetilde{E}_h$: this is checked by decomposing $\psi^{(\le h)} = \psi^{(\le h-1)} + \psi^{(h)}$ and by means of the relation (which is, essentially, a definition of truncated expectations),

$$M^2 \widetilde{E}_h + \mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)}) = \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{n+1} \mathcal{E}_h^{T,n}(\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}))\ , \tag{4.22}$$

where $\mathcal{E}_h^{T,n}$ denotes the *truncated expectation of order n* with propagator $Z_{h-1}^{-1} g^{(h)}_{\omega,\omega'}$, see (4.15).

The above procedure allows us to write the kernels $W^{(h)}_{2n,\underline{\sigma},\underline{\omega},\underline{\alpha}}$ and $\tilde{E}_h$ by a convergent expansion in the running coupling constants and the renormalization constants at higher scales; more exactly we will prove in Appendix I the following proposition.

*4.5.* **Lemma 2.** *Suppose that $\varepsilon_h < \bar{\varepsilon}$, then if $\bar{\varepsilon}$ is small enough and if for some constant $c_1$,*

$$\max_{h'>h} |v_{h'}| \le \varepsilon_h, \quad \sup_{h'>h} \left|\frac{m_{h'}}{m_{h'-1}}\right| \le e^{c_1 \varepsilon_h},\ \sup_{h'>h} \left|\frac{Z_{h'}}{Z_{h'-1}}\right| \le e^{c_1 \varepsilon_h^2}, \tag{4.23}$$

*then for a suitable M–independent constant $c_0$ the kernels in* (4.4) *satisfy*

$$|\widehat{W}^{(h)}_{2n,\underline{\sigma},\underline{\omega},\underline{\alpha}}(\mathbf{k}_1,...,\mathbf{k}_{2n-1})| \le \gamma^{-hD_k(P_{v_0})}\,(c_0\,\varepsilon_h)^{max(1,n-1)} \tag{4.24}$$

*where* $D_k(P_{v_0}) = -2+n+k$ *and* $k=\sum_{i=1}^{2n}\alpha_i$. *Moreover*

$$(|n_h|+|z_h|+|a_h|+|l_h|) \le c_0\varepsilon_h\ ,\ \ |s_h| \le |m_h|\,c_0\,\varepsilon_h,\ |\widetilde{E}_{h+1}| \le \gamma^{2h}\,c_0\,\varepsilon_h. \tag{4.25}$$

## 5. The Flow of the Running Coupling Constants

*5.1.* By the result in Sect. (4.5) it follows that the kernels $W^{(h)}_{2n,\underline{\sigma},\underline{\omega},\underline{\alpha}}$ in (4.4) are bounded as soon as the condition (4.23) on the running coupling constants $v_{h'}$ and the renormalization constants $Z_{h'}, m_{h'}, h' > h$ are verified. Such quantities verify a set of recursive equations called *Beta function relations* of the form

$$\begin{aligned} \nu_{h-1} &= \gamma\nu_h + \beta^h_\nu(a_h,\nu_h;...;a_1,\nu_1), & a_{h-1} &= a_h + \beta^h_a(a_h,\nu_h;...;a_1,\nu_1),\\ \frac{m_{h-1}}{m_h} &= 1+\beta^h_m(a_h,\nu_h;...;a_1,\nu_1), & \frac{Z_{h-1}}{Z_h} &= 1+\beta^h_z(a_h,\nu_h;...;a_1,\nu_1), \end{aligned} \tag{5.1}$$

where $a_h = (\lambda_h,\delta_h)$. By explicit calculation of the lower order non-zero terms one finds, for $h \le 0$,

$$\begin{aligned} \beta^h_z(a_h,\nu_h;...;a_1,\nu_1) &= b_1\lambda_h^2 + O(\varepsilon_h^3)\ , \quad b_1>0\ ,\\ \beta^h_m(a_h,\nu_h;...;a_1,\nu_1) &= a_2\lambda_h + O(\varepsilon_h^2)\ , \quad a_2>0\ . \end{aligned} \tag{5.2}$$

It is possible to prove the following proposition, see Appendix L.

*5.2.* **Lemma 3.** *There are positive constants $c_i$* 1, . . . , 6, *such that for M large,* $|t-t_c|>0$ *and small enough and* $\lambda$ *small enough one can uniquely define* $\nu(\lambda)$ *such that there exists an integer* $h^*\le 0$ *such that for* $h^*-1\le h\le 0$,

$$\begin{aligned} &|\lambda_h-\lambda| \le c_1|\lambda|^{3/2}\ , \qquad |\delta_h|\le c_1|\lambda|\ , \qquad |\nu_h|\le c_6\,|\lambda|\,(\gamma^{-\frac12(h-h^*)}+\gamma^{\kappa h}),\\ &\gamma^{-\lambda c_2 h} < \frac{m_h}{m_1} < \gamma^{-\lambda c_3 h}\ , \qquad \gamma^{-c_4\lambda^2 h} < Z_h < \gamma^{-c_5\lambda^2 h}. \end{aligned} \tag{5.3}$$

*The scale $h^*$ is such that* $\gamma^{k-1}\ge 4|m_k|$ *for* $1\ge k\ge h^*$ *while* $\gamma^{h^*-2}\le 4|m_{h^*}|$; *it verifies*

$$\frac{\log_\gamma|m_1|}{1-\lambda c_2} \le h^* < \frac{\log_\gamma|m_1|}{1-\lambda c_2}+1\ . \tag{5.4}$$

*5.3. Remark.* $h^*$ is the scale at which the mass $m_h$ and the momentum scale $\gamma^h$ become of the same order (at the first steps $|m_h| << \gamma^h$ close to criticality). As $m_h$ has a non trivial flow, such scale depends on $\lambda$, see (5.4). It is sufficient to study the flow up to $h^*$ because the integration of $\psi^{(\le h^*)}$ can be performed in a single step as $m_{h^*}$ acts as an infrared cut-off on the momentum scale $\gamma^{h^*}$; see §5.2. For scales greater than $h^*$ (5.3) says that it is possible to choose the counterterm $\nu$ so that $\nu_h$ stays bounded, $\lambda_h,\delta_h$ remains close to their initial value, while $m_h, Z_h$ have a non trivial anomalous flow.

The main point in order to prove the above proposition is that the functions $\beta_a^h(a_h, \nu_h; ...; a_1, \nu_1)$ can be written as the sum of two terms; only one of them is really crucial while the other has little effect on the flow, if the counterterm $\nu$ is chosen properly. In particular we write $\beta_a^h(a_h, \nu_h; ...; a_1, \nu_1)$ in the following way:

$$\beta_a^h(a_h, \nu_h; ...; a_1, \nu_1) = \beta_{a,L}^h(a_h; ...; a_1) + r_a^h(a_h, \nu_h; \dots ; a_1, \nu_1), \tag{5.5}$$

where $\beta_{a,L}^h(a_h; ...; a_1)$ is obtained from $\beta_a^h(a_h, \nu_h; ...; a_1, \nu_1)$ by setting $\nu_k = m_k = 0$ for all $k \geq h$ and substituing, for all $k \geq h$, each propagator $g_{\omega,\omega}^{(k)}(\mathbf{x}-\mathbf{y})$ given by (4.16) with $g_{L;\omega,\omega}^{(k)}(\mathbf{x}-\mathbf{y})$ given by (7.63), and each propagator $g_{\omega,-\omega}^{(k)}(\mathbf{x}-\mathbf{y})$ with zero. By the estimates of the propagator in Appendix H and proceeding as in Appendix I it follows that, if (4.23) holds and $\bar{\varepsilon}$ is small enough, for $h \geq h^*$,

$$|r_\lambda^h| + |r_\delta^h| \leq C\overline{\lambda}_h^2 \, (\overline{\nu}_h + \gamma^{-\frac{1}{2}(h-h^*)} + \gamma^{\kappa h}), \tag{5.6}$$

where $0 < \kappa < 1$ is a constant and $\overline{\lambda}_h = \sup_{k\geq h} |\lambda_k|$, $\overline{\nu}_h = \sup_{k\geq h} |\nu_k|$.

On the other hand it was proved, following the strategy outlined in [BG], in [BGPS, GS, BM1] (see also [BeM1] for a simplified proof) that, with the latter definition of $\overline{\lambda}_h$ and for $h^* \leq h \leq 0$, the following result holds

*5.4.* **Lemma 4.** *There exist constants $\overline{\varepsilon}_0$ and $\eta' > 0$, such that, if $|a_h| \equiv |(\delta_h, \lambda_h)| \leq \overline{\varepsilon}_0$, if the label $a$ is $a = \lambda, \delta$ and if $h \leq 0$,*

$$|\beta_{a,L}^h(a_h, \dots , a_h)| \leq C\overline{\lambda}_h^2 \gamma^{\eta' h} \,. \tag{5.7}$$

The proof of the above statement is based on a Renormalization Group analysis of the *Luttinger model* (see for instance [BGM] for the definition of this model). Proceeding as in §4 one gets an expansion for the correlation functions in terms of running coupling constants $\lambda_h^{(L)}, \delta_h^{(L)}$ verifying

$$\begin{aligned} \lambda_{h-1}^{(L)} &= \lambda_h^{(L)} + \bar{\beta}_{\lambda,L}^h(a_h^{(L)}, \dots , a_1^{(L)}) \,, \\ \delta_{h-1}^{(L)} &= \delta_h^{(L)} + \bar{\beta}_{\delta,L}^h(a_h^{(L)}, \dots , a_1^{(L)}) \,, \end{aligned} \tag{5.8}$$

where $\bar{\beta}_{\lambda,L}^h, \bar{\beta}_{\lambda,L}^h$ are the same as the ones in (5.2) up to $O((\overline{\lambda}_h^{(L)})^2\gamma^{\eta' h})$ terms for a suitable $\eta' > 0$. The proof of (5.7) is done by comparing the expression for the correlation functions obtained by the exact solution in [ML] with their expression as series in terms of running coupling constants, see [BGPS, GS, BM1] and [BeM1].

Hence by (5.6), (5.7) and some properties of the functions $\beta_i^h(a_h, \nu_h; ...; a_1, \nu_1)$ in (5.1) the above proposition on the flow follows, see Appendix L.

*5.5.* The propagator corresponding to the integration of all the scales between $h^*$ and $h_M$,

$$\frac{\overline{g}_{\omega,\omega'}^{(\leq h^*)}(\mathbf{x}-\mathbf{y})}{Z_{h^*-1}} \equiv \int P_{Z_{h^*-1},m_{h^*-1},C_{h^*}}(d\psi^{(\leq h^*)})\psi_{\mathbf{x},\omega}^{(\leq h^*)-}\psi_{\mathbf{y},\omega'}^{(\leq h^*)+} \,, \tag{5.9}$$

obeys the same bound as the propagator of the integration of a single scale greater than $h^*$, see (7.67) in Appendix H ; this property can used to perform the integration of all the scales $\leq h^*$ in a single step. We define

$$e^{-M^2\widetilde{E}_{\leq h^*}} = \int P_{Z_{h^*-1},m_{h^*-1},C_{h^*}}(d\psi^{(\leq h^*)})\, e^{-\widehat{\mathcal{V}}^{(h^*)}(\sqrt{Z_{h^*-1}}\psi^{(\leq h^*)})} \,, \tag{5.10}$$

and in Appendix I it is proved that:

*5.6. Suppose that* $\varepsilon_{h^*} < \bar\varepsilon$*, then if* $\bar\varepsilon$ *is small enough and* (4.23) *holds with* $h = h^* - 1$*, then*

$$|\widetilde E_{\le h^*}| \le \gamma^{2h^*} c_0\, \varepsilon_h. \tag{5.11}$$

Then by the statements in §4.5, §5.2, §5.6 for $\lambda$ small enough and $\nu$ suitably chosen we get a convergent expansion for the free energy

$$-\frac{1}{M^2}\log \hat Z^-_{2I} = \sum_{h=h^*+1}^{1} [\widetilde E_h + t_h] + \widetilde E_{\le h^*} + t_{\le h^*}\ . \tag{5.12}$$

The quantities $\widetilde E_h$, $\widetilde E_{\le h^*}$ are written by a convergent *tree expansion*, see Appendix I. Note that the fact that $h_M$ is finite, which is due to the fact that we are considering the addend with $\gamma^{(1)}, \gamma^{(2)} = -, -, -, -$ in (2.10), plays essentially no role in the analysis.

*Remark.* $\gamma^{h^*}$ is a momentum scale and, roughly speaking, for momenta bigger than $\gamma^{h^*}$ the theory is "essentially" a massless theory (up to $O(m_h\gamma^{-h})$ terms), while for momenta smaller than $\gamma^{h^*}$ it is a "massive" theory with mass $O(\gamma^{h^*})$ which can be integrated without multiscale decomposition.

## 6. Correlation Functions and the Specific Heat

*6.1.* In the preceding sections we have found a convergent expansion for the free energy; the latter is not interesting *per se* until we show that the free energy as a function of $t-t_c$ has some singularity at $t = t_c$. In order to show that $t = t_c$ is a critical point we can study some correlation functions or some thermodynamic property like the specific heat by evaluating them at $t \neq t_c$ and $M = \infty$ and then verify that they have a singular behavior as $t \to t_c$. We shall study, for this purpose, the energy-energy correlation function (1.8) and the specific heat. We start by considering the following expression:

$$\Omega_\Lambda(\mathbf{x},\mathbf{y}) =< [H_{I,\mathbf{x}}(\sigma^{(1)}) + H_{I,\mathbf{x}}(\sigma^{(2)})][H_{I,\mathbf{y}}(\sigma^{(1)}) + H_{I,\mathbf{y}}(\sigma^{(2)})] >_{\Lambda,T}, \tag{6.1}$$

where $H_I(\sigma^{(\alpha)}) = \sum_{\mathbf{x}} H_{I,\mathbf{x}}(\sigma^{(1)})$, and $H_I(\sigma^{(\alpha)})$ is the Ising model hamiltonian defined in the first of (1.3). By using (2.9) we get, for $\mathbf{x} \neq \mathbf{y}$

$$\Omega_\Lambda(\mathbf{x},\mathbf{y}) = \sum_{\gamma_1,\gamma_2} (-1)^{\delta_{\gamma_1}+\delta_{\gamma_2}} \frac{Z^{\gamma_1,\gamma_2}_{2I}}{Z_{2I}} \Omega_{\gamma_1,\gamma_2,\Lambda}(\mathbf{x}-\mathbf{y}), \tag{6.2}$$

where

$$\begin{aligned}\Omega_{\gamma_1,\gamma_2,\Lambda}(\mathbf{x}-\mathbf{y}) = \Bigg\langle \Bigg\{ \sum_{\alpha=1}^{2} \Bigg[ &\operatorname{sech}^2 J_r\, \partial_t S^{\alpha}_{\mathbf{x},\varepsilon^{(a)},\varepsilon'^{(\alpha)}} - \frac{\partial}{\partial J^{(\alpha)}_{x,x_0;x+1,x_0}} \mathcal{V}\Big|_{\{J_r\}} \\ &- \frac{\partial}{\partial J^{(\alpha)}_{x,x_0;x,x_0+1}} \mathcal{V}\Big|_{\{J_r\}} \Bigg]\Bigg\}; \Bigg\{ \sum_{\alpha=1}^{2} \Bigg[ \operatorname{sech}^2 J_r\, \partial_t S^{\alpha}_{\mathbf{y},\varepsilon^{(a)},\varepsilon'^{(\alpha)}} \\ &- \frac{\partial}{\partial J^{(\alpha)}_{r;y,y_0;y+1,y_0}} \mathcal{V}\Big|_{\{J_r\}} - \frac{\partial}{\partial J^{(\alpha)}_{y,y_0;y,y_0+1}} \mathcal{V}\Big|_{\{J_r\}} \Bigg]\Bigg\} \Bigg\rangle^T .\end{aligned} \tag{6.3}$$

If $A_1, \dots, A_n$ are functions of the field, we are using the symbol $< A_1; \dots; A_n >^T$ to denote the truncated expectation *w.r.t.* the fermionic integration $\frac{1}{Z_{2I}^{\gamma_1,\gamma_2}} \int [\prod_{\alpha=1}^2 P^{(\alpha)}_{\varepsilon(\alpha),\varepsilon'(\alpha)} (dH^{(\alpha)}, dV^{(\alpha)})] e^{-\mathcal{V}}$ of $\prod_{i=1}^n A_i$.

Let us consider first the following expression, which gives the dominant large distance contribution

$$\tilde{\Omega}_{\gamma_1,\gamma_2,\Lambda}(\mathbf{x}-\mathbf{y}) = \left\langle \sum_{\alpha=1}^{2} \left[\partial_t S^{\alpha}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}\right]; \left[ \sum_{\alpha=1}^{2} \left[\partial_t S^{\alpha}_{\mathbf{y},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}\right]\right] \right\rangle^T . \tag{6.4}$$

Performing the change of variable (2.12), (2.13) we see that the r.h.s. of (6.4) can be written as $\tilde{\Omega}_{\gamma_1,\gamma_2,\Lambda}(\mathbf{x}-\mathbf{y}) = \frac{\partial}{\partial\phi(\mathbf{x})}\frac{\partial}{\partial\phi(\mathbf{y})}\mathcal{S}_{\gamma_1,\gamma_2}(\phi)|_{\phi=0}$ where, with the notation of (3.1),

$$e^{\mathcal{S}_{\gamma_1,\gamma_2}(\phi)} = \int P(d\psi) \int P(d\chi) e^{\mathcal{Q}(\chi,\psi)-\mathcal{V}(\psi,\chi)} e^{\sum_{\mathbf{x}} \phi(\mathbf{x})\left[\partial_t S^1_{\mathbf{x},\varepsilon^1,\varepsilon'^1} + \partial_t S^2_{\mathbf{x},\varepsilon^2,\varepsilon'^2}\right]}. \tag{6.5}$$

This is a new expression similar but not identical to the ones studied to obtain analyticity of the free energy for $t \neq t_c$. We can study (6.5) in a similar way, by adapting the free energy analysis for the integration of $\mathcal{S}_{\gamma_1,\gamma_2}(\phi)$.

Consider $\mathcal{S}_{-,-,-,-}(\phi) \overset{def}{=} \mathcal{S}(\phi)$ and $\tilde{\Omega}_{-,-,-,-,\Lambda}(\mathbf{x}-\mathbf{y}) \overset{def}{=} \tilde{\Omega}_{\Lambda}(\mathbf{x}-\mathbf{y})$. One can proceed as in §3 in order to integrate the massive $\chi$ fields and finding, for $|\lambda| \leq \varepsilon$ and with the notations of (3.1),

$$e^{\mathcal{S}(\phi)} = e^{M^2\mathcal{N}} \int \overline{P}(d\psi) e^{-\mathcal{V}^{(1)}(\psi)+\mathcal{B}(\phi,\psi)}, \tag{6.6}$$

where $\mathcal{N}$ is a normalization constant and

$$\mathcal{B}(\psi,\phi) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\underline{\sigma},\underline{\alpha},\underline{\omega}}\sum_{\mathbf{x}_1}\cdots\sum_{\mathbf{x}_m}\sum_{\mathbf{y}_1}\cdots\sum_{\mathbf{y}_{2n}} \cdot$$
$$\cdot\, B_{m,2n,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1,\dots,\mathbf{x}_m;\mathbf{y}_1,\dots,\mathbf{y}_{2n})\left[\prod_{i=1}^{m}\phi(\mathbf{x}_i)\right]\left[\prod_{i=1}^{2n}\partial^{\alpha_i}\psi^{\sigma_i}_{\mathbf{y}_i,\omega_i}\right], \tag{6.7}$$

where for $n \geq 2$,

$$\sum_{\mathbf{y}_1,\dots,\mathbf{y}_{2n}} |B_{m,2n,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1,\dots,\mathbf{x}_m;\mathbf{y}_1,\dots,\mathbf{y}_{2n})| \leq C^n \varepsilon^{\frac{n}{2}}, \tag{6.8}$$

and for $n = 1$

$$\sum_{\mathbf{x}} i\omega Z_1^{(1)}\phi(\mathbf{x})\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega} + \sum_{\mathbf{y}_1,\mathbf{y}_2}\sum_{\mathbf{x}}\sum_{\{\sigma,\omega\}}\sum_{\alpha_1+\alpha_2\geq 1} B_{1,2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)\phi(\mathbf{x})$$
$$\times\, \partial^{\alpha_1}\psi^{\sigma_1}_{\mathbf{y}_1,\omega_1}\partial^{\alpha_2}\psi^{\sigma_2}_{\mathbf{y}_2,\omega_2} + \widetilde{B}(\phi,\psi), \tag{6.9}$$

where $Z_1^{(1)}$ is an $O(1)$ constant, $\sum_{\mathbf{y}_1,\mathbf{y}_2} |B_{1,2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)| \leq C$ and $\widetilde{B}(\phi,\psi)$ contains the terms with $m \geq 2$. All kernels $B_{m,2n,\underline{\sigma},\underline{\alpha},\underline{\omega}}$ and $Z_1^{(1)}$ are analytic in $\lambda$, as follows by proceeding as in Appendix E.

*The symmetry considerations in Appendix F apply here as well and imply that the only possible local terms with $n = m = 1$ are of the form $\phi(\mathbf{x})\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},-1}$.*

*6.2.* We shall evaluate the integral, over the light fermions, in the r.h.s. of (6.6) in a way which is very close to that used for the integration of the r.h.s. of (3.1). We introduce the scale decomposition described in §4 and we perform iteratively the integration of the single scale fields, starting from the field of scale 1. After integrating the fields $\psi^{(1)}, ...\psi^{(h+1)}$, $0 \geq h \geq h^*$, we find

$$e^{\mathcal{S}(\phi)} = e^{-M^2 E_h + S^{(h+1)}(\phi)} \int P_{Z_h,m_h,C_h}(d\psi^{\leq h}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) + \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)},\phi)}\ , \tag{6.10}$$

where $P_{Z_h,m_h,C_h}(d\psi^{(\leq h)})$ and $\mathcal{V}^h$ are given by (4.4), respectively, while $S^{(h+1)}\ (\phi)$, which denotes the sum over all terms dependent on $\phi$ but independent of the $\psi$ field, and $\mathcal{B}^{(h)}(\psi^{(\leq h)}, \phi)$, which denotes the sum over all terms containing at least one $\phi$ field and two $\psi$ fields, can be represented in the form

$$S^{(h+1)}(\phi) = \sum_{m=1}^{\infty} \sum_{\mathbf{x}_1} \cdots \sum_{\mathbf{x}_m} S_m^{(h+1)}(\mathbf{x}_1, \dots, \mathbf{x}_m) \Big[ \prod_{i=1}^{m} \phi(\mathbf{x}_i) \Big], \tag{6.11}$$

$$\begin{aligned} \mathcal{B}^{(h)}(\psi^{(\leq h)}, \phi) = & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\underline{\alpha},\underline{\omega}} \sum_{\mathbf{x}_1} \cdots \sum_{\mathbf{x}_m} \sum_{\mathbf{y}_1} \cdots \sum_{\mathbf{y}_{2n}} \cdot \\ & \cdot B^{(h)}_{m,2n,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1, \dots, \mathbf{x}_m; \mathbf{y}_1, \dots, \mathbf{y}_{2n}) \cdot \\ & \cdot \Big[ \prod_{i=1}^{m} \phi(\mathbf{x}_i) \Big] \Big[ \prod_{i=1}^{2n} \partial^{\alpha_i} \psi^{(\leq h)\sigma_i}_{\mathbf{y}_i,\omega_i} \Big] . \end{aligned} \tag{6.12}$$

Since the field $\phi$ is equivalent, from the point of view of dimensional considerations, to two $\psi$ fields, the only terms in the r.h.s. of (6.12) which are not irrelevant are those with $m = 1$ and $n = 1$, which are marginal. Repeating the symmetry considerations in Appendix F we can conclude that the only local terms with $n = m = 1$ and $\alpha_1 = \alpha_2 = 0$ have the form $\phi(\mathbf{x})\psi^{(\leq h)+}_{\mathbf{x},\omega}\psi^{(\leq h)-}_{\mathbf{x},-\omega}$. Hence we extend the definition of the localization operator $\mathcal{L}$, so that its action on $\mathcal{B}^{(h)}(\psi^{(\leq h)}, \phi)$ is defined by its action on the kernels $B^{(h)}_{m,2n,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1, \dots, \mathbf{x}_m; \mathbf{y}_1, \dots, \mathbf{y}_{2n})$:

1) if $m = 1, n = 1, \alpha_1 = \alpha_2 = 0$, then

$$\mathcal{L}B^{(h)}_{1,2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1; \mathbf{y}_1, \mathbf{y}_2) = \delta(\mathbf{y}_1 - \mathbf{x}_1)\delta(\mathbf{y}_2 - \mathbf{x}_1) \int d\mathbf{z}_1 d\mathbf{z}_2 B^{(h)}_{1,2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1; \mathbf{z}_1, \mathbf{z}_2), \tag{6.13}$$

2) in all the other cases,

$$\mathcal{L}B^{(h)}_{m,2n,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{x}_1, ...\mathbf{x}_m; \mathbf{y}_1, ..., \mathbf{y}_{2n}) = 0\ . \tag{6.14}$$

Hence, by the symmetry reasons discussed in Appendix F,

$$\mathcal{L}\mathcal{B}^{(h)}(\psi^{(\leq h)}, \phi) = \frac{Z^{(1)}_h}{Z_h} F_1^{(\leq h)}\ , \tag{6.15}$$

where $Z_h^{(1)}$ is a real number and

$$F_1^{(\le h)} = \sum_{\mathbf{x}} \phi(\mathbf{x}) i [\psi^{(\le h)+}_{\mathbf{x},1} \psi^{(\le h)-}_{\mathbf{x},-1} - \psi^{(\le h)+}_{\mathbf{x},-1} \psi^{(\le h)-}_{\mathbf{x},1}] \,. \tag{6.16}$$

In the expansion for the energy-energy correlation function there is then a *renormalization constant* more, namely $Z_h^{(1)}$.

With the notation of §4 we can write the integral in the r.h.s. of (6.10)

$$\begin{aligned} e^{-M^2 t_h} &\int P_{\widetilde{Z}_{h-1}, m_{h-1}, C_h}(d\psi^{(\le h)}) e^{-\widetilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) + \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}, \phi)} \\ &= e^{-M^2 t_h} \int P_{Z_{h-1}, m_{h-1}, C_{h-1}}(d\psi^{(\le h-1)}) \cdot \\ &\quad \cdot \int P_{Z_{h-1}, m_{h-1}, \tilde{f}_h^{-1}}(d\psi^{(h)}) e^{-\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}) + \widehat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}, \phi)} \,, \end{aligned} \tag{6.17}$$

where $\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})$ is defined as in (4.19) and $\widehat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}, \phi) = \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}, \phi)$; moreover $\mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)}, \phi)$ and $S^{(h)}(\phi)$ are then defined through the relation analogue of (4.21), that is

$$\begin{aligned} &e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)}) + \mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)}, \phi) - M^2 \widetilde{E}_h + \widetilde{S}^{(h)}(\phi)} \\ &\quad = \int P_{Z_{h-1}, m_{h-1}, \tilde{f}_h^{-1}}(d\psi^{(h)}) e^{-\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}) + \widehat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}, \phi)} \,. \end{aligned} \tag{6.18}$$

As in §5.5, the fields of scale between $h^*$ and $h_M$ are integrated in a single step without any multiscale decomposition. Hence we define, in analogy to (5.10),

$$\begin{aligned} e^{\widetilde{S}^{(h^*)}(\phi) - M^2 \widetilde{E}_{h^*}} \stackrel{def}{=} &\int P_{Z_{h^*-1}, m_{h^*-1}, C_{h^*}}(d\psi^{(\le h^*)}) \\ &\times e^{-\widehat{\mathcal{V}}^{(h^*)}(\sqrt{Z_{h^*-1}}\psi^{(\le h^*)}) + \widehat{\mathcal{B}}^{(h^*)}(\sqrt{Z_{h^*-1}}\psi^{(\le h^*)}, \phi)} \,. \end{aligned} \tag{6.19}$$

It follows that

$$S(\phi) = -M^2 E_M + S^{(h^*)}(\phi) = -M^2 E_M + \sum_{h=h^*}^{1} \widetilde{S}^{(h)}(\phi) \,; \tag{6.20}$$

hence, if $\widetilde{S}_2^{(h)}(\mathbf{x}, \mathbf{y}) = \frac{\partial}{\partial \phi(\mathbf{x})} \frac{\partial}{\partial \phi(\mathbf{y})} \widetilde{S}^{(h)}(\phi)|_{\phi=0}$,

$$\tilde{\Omega}_\Lambda(\mathbf{x}, \mathbf{y}) = S_2^{(h^*)}(\mathbf{x}, \mathbf{y}) = \sum_{h=h^*}^{1} \widetilde{S}_2^{(h)}(\mathbf{x}, \mathbf{y}) \,. \tag{6.21}$$

*6.3.* It is shown in Appendix M that $\tilde{\Omega}_\Lambda(\mathbf{x}, \mathbf{y}) = \Omega^\alpha_\Lambda(\mathbf{x}, \mathbf{y}) + \Omega^\beta_\Lambda(\mathbf{x}, \mathbf{y})$, where

$$\begin{aligned} \Omega^\alpha_\Lambda(\mathbf{x}, \mathbf{y}) = &\sum_{h,h'=h^*}^{1} \sum_{\omega=\pm 1} \left\{ \frac{(Z^{(1)}_{h\vee h'})^2}{Z_{h-1} Z_{h'-1}} [g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y}) g^{(h')}_{-\omega,-\omega}(\mathbf{y}-\mathbf{x}) - \right. \\ &\left. g^{(h)}_{+1,-1}(\mathbf{x}-\mathbf{y}) g^{(h')}_{-1,+1}(\mathbf{y}-\mathbf{x})] \right\} + \sum_{h=h^*}^{1} \left( \frac{Z_h^{(1)}}{Z_h} \right)^2 G_\Lambda^{(h),a}(\mathbf{x}, \mathbf{y}), \end{aligned} \tag{6.22}$$

where $h \vee h' = \max\{h, h'\}$ and $g^{(h^*)}_{\omega_1,\omega_2}(\mathbf{x})$ has to be understood as $g^{(\leq h^*)}_{\omega_1,\omega_2}(\mathbf{x})$; moreover for all $N > 0$ there exists a constant $C_N$ such that

$$|\partial_x^{m_1}\partial_{x_0}^{m_0} G^{(h),\alpha}_\Lambda(\mathbf{x},\mathbf{y})| \leq \gamma^{(2+m_0+m_1)h}|\lambda_1|\frac{C_N}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N} . \tag{6.23}$$

For $\Omega^\beta_\Lambda(\mathbf{x},\mathbf{y})$ the following bound holds:

$$|\partial_x^{m_1}\partial_{x_0}^{m_0}\Omega^\beta_\Lambda(\mathbf{x},\mathbf{y})| \leq \sum_{h=h^*}^{1}\left(\frac{Z^{(1)}_h}{Z_h}\right)^2 \gamma^{(2+m_0+m_1+\tau)h}\frac{C_N}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N} , \tag{6.24}$$

where $0 < \tau < 1$ is a constant. A similar bound, by dimensional reasons, holds for $\mathrm{sech}^4\tilde{\Omega}_\Lambda(\mathbf{x},\mathbf{y}) - \Omega_{-,-,-,-,\Lambda}(\mathbf{x},\mathbf{y})$. It is shown in Appendix M that

$$\frac{Z^{(1)}_{h-1}}{Z^{(1)}_h} = 1 + z^{(1)}_h = 1 + a_1\lambda_h + O(\mu_h^2) , \tag{6.25}$$

so that there exist two constants $c_1, c_2$ such that $\gamma^{-\lambda_1 c_1 h} < \frac{Z^{(1)}_h}{Z_h} < \gamma^{-\lambda_1 c_2 h}$. If we define

$$\eta = \log_\gamma(1 + z_{[h^*/2]}) , \tag{6.26}$$

with $C_0\widetilde{Z}_1^{-1}z_h = \frac{Z_{h-1}}{Z_h} - 1$, we can check that $|\frac{\gamma^{-\eta h}}{Z_h} - 1| \leq C\lambda_1^2$ and, from (5.2), $\eta = b_1\lambda_1^2 + O(\lambda^3)$. In a similar way, if we define $\widetilde{\eta}_1 = \log_\gamma(1 + z^{(1)}_{[h^*/2]})$, it holds $|\frac{Z^{(1)}_1\gamma^{-\widetilde{\eta}_1 h}}{Z^{(1)}_h} - 1| \leq C|\lambda_1|$ and $\widetilde{\eta}_1 = a_1\lambda_1 + O(\lambda^2)$.

Note also that, by reasoning as in Appendix G, for $\mathbf{x}, \mathbf{y}$ and $t - t_c$ fixed

$$\lim_{M\to\infty}[\Omega_{\gamma_1,\gamma_2,\Lambda}(\mathbf{x},\mathbf{y}) - \Omega_{-,-,-,-,\Lambda}(\mathbf{x},\mathbf{y})] = 0. \tag{6.27}$$

and the limit is reached exponentially fast. Then (6.2) is equal to the limiting value of $\Omega_{-,-,-,-,\Lambda}(\mathbf{x},\mathbf{y})$. In order to prove the first inequality in (1.9), we write, if $m_0+m_1 = n$ and $\eta_1 = \eta - \widetilde{\eta}_1$,

$$\left|\sum_{h=h^*}^{1}\left(\frac{Z^{(1)}_h}{Z_h}\right)^2 \partial_x^{m_1}\partial_{x_0}^{m_0}G^{(h),\alpha}_\Lambda(\mathbf{x},0)\right|$$
$$\leq C_{N,n}\sum_{h=h^*}^{0}\frac{\gamma^{(2+2\eta_1+n)h}}{[1+(\gamma^h|\mathbf{d}(\mathbf{x})|)^N]} \leq \frac{C_{N,n}}{|\mathbf{d}(\mathbf{x})|^{2+2\eta_1+n}}H_{N,2+2\eta_1+n}(|\mathbf{d}(\mathbf{x})|), \tag{6.28}$$

where $\eta_1 = \eta - \widetilde{\eta}_1$,

$$H_{N,\alpha}(r) = \sum_{h=h^*}^{0}\frac{(\gamma^h r)^\alpha}{1+(\gamma^h r)^N} . \tag{6.29}$$

On the other hand one sees that, if $\alpha \geq 1/2$ and $N-\alpha \geq 1$, there exists a constant $C_{N,\alpha}$,

$$H_{N,\alpha}(r) \leq \frac{C_{N,\alpha}}{1+(\Delta r)^{N-\alpha}} , \quad \Delta = \gamma^{h^*}, \tag{6.30}$$

and this implies the first inequality in (1.9). Proceeding in the same way by using (6.24) one can prove the second inequality in (1.9). Moreover by writing the propagators in the first two sums in the r.h.s. of (6.22) as in (7.62), (4.16) and using (7.63),(7.65),(7.66),(4.16) it follows (1.11).

Finally first note that the specific heat $C_v^\lambda$ differs, by trivial dimensional arguments, from the sum $\sum_{\mathbf{x}} \Omega_{\underline{\varepsilon},\Lambda}(\mathbf{x},\mathbf{0})$ by terms which are $O(\lambda)$. By (6.22),(6.23),(6.24) it holds

$$\sum_{\mathbf{x}} |\Omega^a_\Lambda(\mathbf{x},\mathbf{0})| \le C \sum_{h=h^*}^{1} \left(\frac{Z_h^{(1)}}{Z_h}\right)^2 \le C_2 \sum_{h=h^*}^{1} \gamma^{2\eta_1 h} \le C_2 \frac{1-\gamma^{2\eta_1 h^*}}{2\eta_1} . \tag{6.31}$$

On the other hand, by (6.22),(6.23),(6.24),

$$\left|\sum_{\mathbf{x}} \Omega_\Lambda(\mathbf{x},\mathbf{0}) - \sum_{h,h'=h^*}^{1} \sum_{\omega=\pm1} \sum_{\mathbf{x}} \frac{(Z^{(1)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}} g^{(h)}_{L,\omega,\omega}(\mathbf{x}) g^{(h')}_{L,-\omega,-\omega}(-\mathbf{x})\right|$$
$$\le C \sum_{h=h^*}^{1} \left(\frac{Z_h^{(1)}}{Z_h}\right)^2 \left[|\lambda| + \gamma^{\tau h} + \frac{|m_h|}{\gamma^h}\right], \tag{6.32}$$

so the first of the two inequalities in (1.12) follows.

*Remark.* It is interesting to see how the results in [PS, Spe] can be recovered by our analysis. We can consider the hamiltonian (1.1) with interaction given for instance by

$$V = -\lambda \sum_{x,x_0} \sum_{\alpha=1}^{2} [\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x+1,x_0}\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x+1,x_0} + \sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x,x_0+1}\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x,x_0+1}] \tag{6.33}$$

describing two independent Ising models with a quartic interaction. We will briefly explain in Appendix N that all the above analysis can be repeated in such a case and, due to the special form of (6.33), formulas (1.8)-(1.12) hold with $\eta_1 = \eta_2 = 0$, *i.e.* there is universality.

## 7. Appendices

*7.1. Appendix A: Grassmann integration.* Grassmann variables $\overline{H}^{(\alpha)}_{\mathbf{x}}$, $H^{(\alpha)}_{\mathbf{x}}$, $\overline{V}^{(\alpha)}_{\mathbf{x}}$, $V^{(\alpha)}_{\mathbf{x}}$, $\mathbf{x} \in \Lambda$ are such that all functions of them are polynomials. The *Grassmann integration* $\int \prod_{\mathbf{x}\in\Lambda} dH^{(\alpha)}_{\mathbf{x}} d\overline{H}^{(\alpha)}_{\mathbf{x}}$ of a monomial $Q(H^{(\alpha)}, \overline{H}^{(\alpha)})$ in the variables $H^{(\alpha)}_{\mathbf{x}}$, $\overline{H}^{(\alpha)}_{\mathbf{x}}$, $\mathbf{x} \in \Lambda_M$, is defined to be zero, except in the case $Q(H^{(\alpha)}, \overline{H}^{(\alpha)}) = \prod_{\mathbf{x}} H^{(\alpha)}_{\mathbf{x}} \overline{H}^{(\alpha)}_{\mathbf{x}}$, up to a permutation of the variables. In this case the value of the functional is determined, by using the anticommuting properties of the variables, by the condition

$$\int \left[\prod_{\mathbf{x}\in\Lambda_M} d\overline{H}^{(\alpha)}_{\mathbf{x}} dH^{(\alpha)}_{\mathbf{x}}\right] \prod_{\mathbf{x}\in\Lambda_M} H^{(\alpha)}_{\mathbf{x}} \overline{H}^{(\alpha)}_{\mathbf{x}} = 1 . \tag{7.1}$$

In a similar way the Grassmann integration for $V^{(\alpha)}$, $\overline{V}^{(\alpha)}$ is defined likewise exchanging $H, \overline{H}$ with $V, \overline{V}$.

*7.2. Appendix B: Expression of the partition function as a Grassmann integral.* If $a = 0$ or $b = 0$ we can write $\widehat{Z}_{2I}$, see (2.8), by making use of (2.9), as in (2.10) with

$$\begin{aligned}
\widehat{Z}_{2I}^{\gamma_1,\gamma_2} =& (\cosh J_r)^{2B} 2^{2S} \frac{1}{4} \\
&\cdot \int \prod_{\alpha=1}^{2} [\prod_{\mathbf{x}\in\Lambda_M} dH_{\mathbf{x}}^{(\alpha)} d\overline{H}_{\mathbf{x}}^{(\alpha)} dV_{\mathbf{x}}^{(\alpha)} d\overline{V}_{\mathbf{x}}^{(\alpha)}] e^{S^{(1)}_{J_r;\varepsilon(1),\varepsilon'(1)}} e^{S^{(2)}_{J_r;\varepsilon(2),\varepsilon'(2)}} \\
&\cdot \prod_{\mathbf{x}\in\Lambda_M} [1 + \widehat{\lambda a}(\tanh J_r + \mathrm{sech}^2 J_r \overline{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0})(\tanh J_r + \mathrm{sech}^2 J_r \overline{H}^{(2)}_{x,x_0} H^{(2)}_{x+1,x_0})] \\
&\cdot \prod_{\mathbf{x}\in\Lambda_M} [1 + \widehat{\lambda a}(\tanh J_r + \mathrm{sech}^2 J_r \overline{V}^{(1)}_{x,x_0} V^{(1)}_{x,x_0+1})(\tanh J_r + \mathrm{sech}^2 J_r \overline{V}^{(2)}_{x,x_0} V^{(2)}_{x,x_0+1})] \\
&\cdot \prod_{\mathbf{x}\in\Lambda_M} [1 + \widehat{\lambda b}(\tanh J_r + \mathrm{sech}^2 J_r \overline{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0})(\tanh J_r + \mathrm{sech}^2 J_r \overline{V}^{(2)}_{x,x_0} V^{(2)}_{x,x_0+1})] \\
&\cdot \prod_{\mathbf{x}\in\Lambda_M} [1 + \widehat{\lambda b}(\tanh J_r + \mathrm{sech}^2 J_r \overline{V}^{(1)}_{x,x_0} V^{(1)}_{x,x_0+1})(\tanh J_r + \mathrm{sech}^2 J_r \overline{H}^{(2)}_{x-1,x_0+1} H^{(2)}_{x,x_0+1})].
\end{aligned} \tag{7.2}$$

By writing $\tanh J_r \stackrel{def}{=} \tanh J + \nu(\lambda)$, we have $e^{S^{(\alpha)}_{J_r;\varepsilon,\varepsilon'}} = e^{S^{(\alpha)}_{J;\varepsilon,\varepsilon'}} e^{S^{(\alpha),\nu}_{\varepsilon,\varepsilon'}}$, where $S^{(\alpha)}_{J;\varepsilon,\varepsilon'}$ is given by (2.2) and

$$S^{(\alpha),\nu}_{\varepsilon,\varepsilon'} = \nu \sum_{\mathbf{x}\in\Lambda_M} [\overline{H}^{(\alpha)}_{x,x_0} H^{(\alpha)}_{x+1,x_0} + \overline{V}^{(\alpha)}_{x,x_0} V^{(\alpha)}_{x,x_0+1}] \,. \tag{7.3}$$

We can check that $\widehat{Z}_{2I}$ can be written as in (2.11) with

$$\mathcal{V} = \mathcal{V}_a + \mathcal{V}_b - \sum_{\alpha=1}^{2} S^{(\alpha),\nu}_{\varepsilon(\alpha),\varepsilon'(\alpha)}, \tag{7.4}$$

and, if $f_i = \log(1 + \widehat{\lambda[i]} \tanh^2 J_r)$ and $[i] = a, b$

$$\begin{aligned}
-\mathcal{V}_a =& \sum_{\mathbf{x}\in\Lambda_M} [f_a + \widetilde{\lambda}_a [\overline{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0} + \overline{H}^{(2)}_{x,x_0} H^{(2)}_{x+1,x_0}] \\
&+ \lambda_a \overline{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0} \overline{H}^{(2)}_{\mathbf{x}} H^{(2)}_{x+1,x_0}] + \sum_{\mathbf{x}\in\Lambda_M} [f_a + \widetilde{\lambda}_a [\overline{V}^{(1)}_{x,x_0} V^{(1)}_{x,x_0+1} \\
&+ \overline{V}^{(2)}_{x,x_0} V^{(2)}_{x,x_0+1}] + \lambda_a \overline{V}^{(1)}_{\mathbf{x}} V^{(1)}_{x,x_0+1} \overline{V}^{(2)}_{x,x_0} V^{(2)}_{x,x_0+1}] \\
-\mathcal{V}_b =& \sum_{\mathbf{x}\in\Lambda_M} [f_b + \widetilde{\lambda}_b [\overline{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0} + \overline{V}^{(2)}_{x,x_0} V^{(2)}_{x,x_0+1}] + \lambda_b \overline{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0} \overline{V}^{(2)}_{\mathbf{x}} V^{(2)}_{x,x_0+1}] \\
&+ \sum_{\mathbf{x}\in\Lambda_M} [f_b + \widetilde{\lambda}_b [\overline{V}^{(1)}_{x,x_0} V^{(1)}_{x,x_0+1} + \overline{H}^{(2)}_{x-1,x_0+1} H^{(2)}_{x,x_0+1}] \\
&+ \lambda_b \overline{V}^{(1)}_{x,x_0} V^{(1)}_{x,x_0+1} \overline{H}^{(2)}_{x-1,x_0+1} H^{(2)}_{x,x_0+1}],
\end{aligned} \tag{7.5}$$

where

$$\begin{aligned}
&\widetilde{\lambda}_i (1 + \widehat{\lambda[i]} \tanh^2 J_r) = \widehat{\lambda[i]} \mathrm{sech}^2 J_r \tanh J_r, \\
&(1 + \widehat{\lambda[i]} \tanh^2 J_r)(\lambda_i + (\widetilde{\lambda}_i)^2) = \widehat{\lambda[i]} \mathrm{sech}^4 J_r.
\end{aligned} \tag{7.6}$$

For small $\lambda$ it is $\widetilde{\lambda}_i = \widehat{\lambda[i]}(\tanh J_r \mathrm{sech}^2 J_r + O(\lambda))$, $\lambda_i = \widehat{\lambda[i]}(\mathrm{sech}^4 J_r + O(\lambda))$.

*7.3. Appendix C: Change from Majorana to Dirac Grassmann variables.* If $S^{(\alpha)}_{J;\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}} = \sum_{\mathbf{x}} S^{(\alpha)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}$ we get, from the change of variables (2.12),

$$S^{(\alpha)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}} = S^{(\alpha,\psi)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}} + S^{(\alpha,\chi)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}} + Q^{(\alpha)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}},$$

where

$$\begin{aligned} S^{(\alpha,\psi)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}} = & \frac{t}{4}[\psi^{(\alpha)}_{\mathbf{x}}(\partial_1 - i\partial_0)\psi^{(\alpha)}_{\mathbf{x}} + \overline{\psi}^{(\alpha)}_{\mathbf{x}}(\partial_1 + i\partial_0)\overline{\psi}^{(\alpha)}_{\mathbf{x}}] + \frac{t}{4}[-i\overline{\psi}^{(\alpha)}_{\mathbf{x}}(\partial_1\psi^{(\alpha)}_{x} \\ & +\partial_0\psi^{(\alpha)}_{\mathbf{x}}) + i\psi^{(\alpha)}_{\mathbf{x}}(\partial_1\overline{\psi}^{(\alpha)}_{\mathbf{x}} + \partial_0\overline{\psi}^{(\alpha)}_{\mathbf{x}})] + i(\sqrt{2}-1-t)\overline{\psi}^{(\alpha)}_{\mathbf{x}}\psi^{(\alpha)}_{\mathbf{x}} \end{aligned} \tag{7.7}$$

with the definitions

$$\partial_1\psi^{(\alpha)}_{\mathbf{x}} = \psi^{(\alpha)}_{x+1,x_0} - \psi^{(\alpha)}_{\mathbf{x}} \qquad \partial_0\psi^{(\alpha)}_{x} = \psi^{(\alpha)}_{x,x_0+1} - \psi^{(\alpha)}_{\mathbf{x}}. \tag{7.8}$$

Moreover

$$\begin{aligned} S^{(\alpha,\chi)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}} = & \frac{t}{4}[\chi^{(\alpha)}_{\mathbf{x}}(\partial_1 - i\partial_0)\chi^{(\alpha)}_{\mathbf{x}} + \overline{\chi}^{(\alpha)}_{\mathbf{x}}(\partial_1 + i\partial_0)\overline{\chi}^{(\alpha)}_{\mathbf{x}}] \\ & +\frac{t}{4}[-i\overline{\chi}^{(\alpha)}_{\mathbf{x}}(\partial_1\chi^{(\alpha)}_{x} + \partial_0\chi^{(\alpha)}_{\mathbf{x}}) \\ & +i\chi^{(\alpha)}_{\mathbf{x}}(\partial_1\overline{\chi}^{(\alpha)}_{\mathbf{x}} + \partial_0\overline{\chi}^{(\alpha)}_{\mathbf{x}})] - i(\sqrt{2}+1+t)\overline{\chi}^{(\alpha)}_{\mathbf{x}}\chi^{(\alpha)}_{\mathbf{x}}, \end{aligned} \tag{7.9}$$

and finally $Q(\chi,\psi) = \sum_{\mathbf{x}}[Q^{(1)}_{\mathbf{x},-,-} + Q^{(2)}_{\mathbf{x},-,-}]$ with

$$\begin{aligned} Q^{(\alpha)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}} = & \frac{t}{4}\{-\psi^{(\alpha)}_{\mathbf{x}}(\partial_1\chi^{(\alpha)}_{\mathbf{x}} + i\partial_0\chi^{(\alpha)}_{\mathbf{x}}) - \overline{\psi}^{(\alpha)}_{\mathbf{x}}(\partial_1\overline{\chi}^{(\alpha)}_{\mathbf{x}} - i\partial_0\overline{\chi}^{(\alpha)}_{\mathbf{x}}) \\ & -\chi^{(\alpha)}_{\mathbf{x}}(\partial_1\psi^{(\alpha)}_{\mathbf{x}} + i\partial_0\psi^{(\alpha)}_{\mathbf{x}}) - \overline{\chi}^{(\alpha)}_{\mathbf{x}}(\partial_1\overline{\psi}^{(\alpha)}_{\mathbf{x}} - i\partial_0\overline{\psi}^{(\alpha)}_{\mathbf{x}}) + i\overline{\psi}^{(\alpha)}_{\mathbf{x}}(\partial_1\chi^{(\alpha)}_{\mathbf{x}} \\ & -\partial_0\chi^{(\alpha)}_{\mathbf{x}}) + i\psi^{(\alpha)}_{\mathbf{x}}(-\partial_1\overline{\chi}^{(\alpha)}_{\mathbf{x}} + \partial_0\overline{\chi}^{(\alpha)}_{\mathbf{x}}) + i\overline{\chi}^{(\alpha)}_{\mathbf{x}}(\partial_1\psi^{(\alpha)}_{\mathbf{x}} - \partial_0\psi^{(\alpha)}_{\mathbf{x}}) \\ & +i\chi^{(\alpha)}_{\mathbf{x}}(-\partial_1\overline{\psi}^{(\alpha)}_{\mathbf{x}} + \partial_0\overline{\psi}^{(\alpha)}_{\mathbf{x}})\}\ . \end{aligned} \tag{7.10}$$

Moreover

$$\overline{H}^{(\alpha)}_{x,x_0}H^{(\alpha)}_{x+1,x_0} + \overline{V}^{(\alpha)}_{x,x_0}V^{(\alpha)}_{x,x_0+1} = \partial_t S^{\alpha}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}} \tag{7.11}$$

so that $S^{(\alpha),\nu}_{\varepsilon^{\alpha},\varepsilon'^{\alpha}} = \nu\sum_{\mathbf{x}\in\Lambda}\partial_t S^{\alpha}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}$.

Let us define

$$\begin{aligned} P^{(\alpha)}_{-,-}(d\psi) = & \mathcal{N}_{\psi}^{-1}[\prod_{\mathbf{k}\in D_{-,-}} d\overline{\psi}^{(\alpha)}_{\mathbf{k}} d\psi^{(\alpha)}_{\mathbf{k}}]\exp[\frac{t}{4M^2}\sum_{\mathbf{k}\in D_{-,-}}[\psi^{(\alpha)}_{\mathbf{k}}\psi^{(\alpha)}_{-\mathbf{k}}(i\sin k + \sin k_0) \\ & +\overline{\psi}^{(\alpha)}_{\mathbf{k}}\overline{\psi}^{(\alpha)}_{-\mathbf{k}}(i\sin k - \sin k_0) - i2m_{\psi}(\mathbf{k})\overline{\psi}^{(\alpha)}_{\mathbf{k}}\psi^{(\alpha)}_{-\mathbf{k}}]], \end{aligned} \tag{7.12}$$

where $\mathcal{N}_{\psi}$ is a normalization constant and $\frac{t}{2}m_{\psi}(\mathbf{k}) = (-\sqrt{2}+1+t) + \frac{t}{2}(\cos k_0 + \cos k - 2)$. Defining in the same way $P^{(\alpha)}_{-,-}(d\chi)$, with the only difference that $m_{\chi}(\mathbf{k}) = -(\sqrt{2}+1+t) - \frac{t}{2}(\cos k_0 + \cos k - 2)\frac{2}{t}$ replaces $m_{\psi}(\mathbf{k})$, we can rewrite $\widehat{Z}^{----}_{2I}$ as

$$\widehat{Z}^{-,-,-,-}_{2I} = (\cosh J_r)^{2B}2^{2S}\frac{1}{4}$$

$$\int \left[\prod_{\alpha=1}^{2} P^{(\alpha)}_{-,-}(d\psi) P^{(\alpha)}_{-,-}(d\chi)\right] e^{Q(\chi,\psi)-\mathcal{V}(\chi,\psi)}. \tag{7.13}$$

We can perform the following change of variables:

$$\begin{aligned} \psi^-_{1,\mathbf{k}} &= \frac{1}{\sqrt{2}}(\psi^{(1)}_{\mathbf{k}} + i\psi^{(2)}_{\mathbf{k}}), \quad \psi^+_{1,-\mathbf{k}} = \frac{1}{\sqrt{2}}(\psi^{(1)}_{\mathbf{k}} - i\psi^{(2)}_{\mathbf{k}}), \\ \psi^-_{-1,\mathbf{k}} &= \frac{1}{\sqrt{2}}(\overline{\psi}^{(1)}_{\mathbf{k}} + i\overline{\psi}^{(2)}_{\mathbf{k}}), \quad \psi^+_{-1,-\mathbf{k}} = \frac{1}{\sqrt{2}}(\overline{\psi}^{(1)}_{\mathbf{k}} - i\overline{\psi}^{(2)}_{\mathbf{k}}) \end{aligned} \tag{7.14}$$

which in coordinate space is (2.13) if $\psi^\sigma_{\omega,\mathbf{x}} = \frac{1}{M}\sum_{\mathbf{k}} e^{i\sigma\mathbf{k}\mathbf{x}}\psi^\sigma_{\omega,\mathbf{k}}$, $\sigma = \pm$. By this change of variables $P_{-,-}(d\psi^{(1)})P_{-,-}(d\psi^{(2)}) \equiv P(d\psi)$ and $P^{(1)}_{-,-}(d\chi^{(1)})P^{(2)}_{-,-}(d\chi^{(2)}) = P(d\chi)$, where $P(d\psi)$, $P(d\chi)$ given by (2.16).

In the physical language, the change of variables (2.13) means that one is describing the system in terms of *Dirac fermions* instead of in terms of *Majorana fermions*.

*7.4. Appendix D: The interaction in fermionic Grassmann variable.* Note that

$$\overline{V}^{(\alpha)}_{x,x_0} V^{(\alpha)}_{x,x_0+1} = Q^{1(\alpha)}_{\mathbf{x}} + Q^{2(\alpha)}_{\mathbf{x}} + Q^{3(\alpha)}_{\mathbf{x}}, \tag{7.15}$$

where

$$\begin{aligned} Q^{1(\alpha)}_{\mathbf{x}} &= \frac{1}{4i}[\psi^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0+1} - \overline{\psi}^{(\alpha)}_{x,x_0}\overline{\psi}^{(\alpha)}_{x,x_0+1} + \overline{\psi}^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0+1} - \psi^{(\alpha)}_{x,x_0}\overline{\psi}^{(\alpha)}_{x,x_0+1}], \\ Q^{2(\alpha)}_{\mathbf{x}} &= \frac{1}{4i}[\chi^{(\alpha)}_{x,x_0}\chi^{(\alpha)}_{x,x_0+1} - \overline{\chi}^{(\alpha)}_{x,x_0}\overline{\chi}^{(\alpha)}_{x,x_0+1} + \overline{\chi}^{(\alpha)}_{x,x_0}\chi^{(\alpha)}_{x,x_0+1} - \chi^{(\alpha)}_{x,x_0}\overline{\chi}^{(\alpha)}_{x,x_0+1}], \\ Q^{3(\alpha)}_{\mathbf{x}} &= \frac{1}{4i}[\psi^{(\alpha)}_{x,x_0}\chi^{(\alpha)}_{x,x_0+1} - \overline{\psi}^{(\alpha)}_{x,x_0}\overline{\chi}^{(\alpha)}_{x,x_0+1} - \psi^{(\alpha)}_{x,x_0}\overline{\chi}^{(\alpha)}_{x,x_0+1} + \overline{\psi}^{(\alpha)}_{x,x_0}\chi^{(\alpha)}_{x,x_0+1} \\ &\quad + \chi^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0+1} - \overline{\chi}^{(\alpha)}_{x,x_0}\overline{\psi}^{(\alpha)}_{x,x_0+1} - \chi^{(\alpha)}_{x,x_0}\overline{\psi}^{(\alpha)}_{x,x_0+1} + \overline{\chi}^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0+1}]. \end{aligned} \tag{7.16}$$

A similar expression holds for $\overline{H}^{(\alpha)}_{x,x_0} H^{(\alpha)}_{x+1,x_0}$.

The above expressions can be naturally expressed in terms of (discrete) derivatives of the fields. In fact by looking for instance to the first of (7.16) one finds

$$\psi^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0+1} - \overline{\psi}^{(\alpha)}_{x,x_0}\overline{\psi}^{(\alpha)}_{x,x_0+1} = \psi^{(\alpha)}_{x,x_0}\partial_{x_0}\psi^{(\alpha)}_{x,x_0} - \overline{\psi}^{(\alpha)}_{x,x_0}\partial_{x_0}\overline{\psi}^{(\alpha)}_{x,x_0} \tag{7.17}$$

and

$$\overline{\psi}^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0+1} - \psi^{(\alpha)}_{x,x_0}\overline{\psi}^{(\alpha)}_{x,x_0+1} = -\partial_{x_0}\overline{\psi}^{(\alpha)}_{x,x_0}\partial_{x_0}\psi^{(\alpha)}_{x,x_0} + \overline{\psi}^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0} + \overline{\psi}^{(\alpha)}_{x,x_0+1}\psi^{(\alpha)}_{x,x_0+1}. \tag{7.18}$$

From (7.5) we see that $\mathcal{V}$ is the sum of expressions linear or bilinear in $\overline{H}^{(\alpha)}_{\mathbf{x}} H^{(\alpha)}_{x+1,x_0}$ or $\overline{V}^{(\alpha)}_{\mathbf{x}} V^{(\alpha)}_{x,x_0+1}$; moreover the change of variables (2.13) and (2.14) replaces a $\psi$, $\chi$ field with a $\psi^\pm_1$, $\chi^\pm_1$ field, and a $\overline{\psi}$, $\overline{\chi}$ field with a $\psi^\pm_{-1}$, $\chi^\pm_{-1}$ field; hence we see that $\mathcal{V}$ is a sum of terms of the form (2.19). Analogous considerations hold for $Q$.

*7.5. Appendix E: The integration of the $\chi$ fields.* We start from the definition of truncated expectation:

$$\mathcal{E}^T_\chi(X;n) = \frac{\partial^n}{\partial\lambda^n}\log\int P(d\chi)e^{\lambda X(\chi)}\Big|_{\lambda=0} \tag{7.19}$$

so that, calling

$$\overline{\mathcal{V}}(\chi,\psi) = -\mathcal{Q}(\chi,\psi) + \mathcal{V}(\chi,\psi) \tag{7.20}$$

we obtain

$$M^2\mathcal{N}^{(1)} - \mathcal{V}^{(1)}(\psi) = \log\int P(d\chi)e^{-\overline{\mathcal{V}}(\chi,\psi)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\mathcal{E}^T_\chi(\overline{\mathcal{V}}(.,\psi;n)). \tag{7.21}$$

We label each one of the monomials (whose number will be called $\widetilde{C}_0$) in $\overline{\mathcal{V}}$ by an index $v_i$, so that each monomial can be written as

$$\sum_{\mathbf{x}_{v_i}} v(\mathbf{x}_{v_i})\prod_{f\in\widetilde{P}_{v_i}}\partial^{\alpha(f)}\psi^{\varepsilon(f)}_{\omega(f),\mathbf{x}(f)}\prod_{f\in P_{v_i}}\partial^{\alpha(f)}\chi^{\varepsilon(f)}_{\omega(f),\mathbf{x}(f)}, \tag{7.22}$$

where $\mathbf{x}_{v_i}$ is the total set of coordinates associated to $v_i$, and $P_{v_i}$ and $\widetilde{P}_{v_i}$ are the set of indices labeling the $\chi$ or $\psi$-fields; $v(\mathbf{x}_{v_i})$ are short ranged functions (products of Kronecker deltas, see (7.5)). We can write

$$\mathcal{V}^{(1)}(\psi) = \sum_{\widetilde{P}_{v_0}\neq 0}\mathcal{V}^{(1)}(\widetilde{P}_{v_0}), \tag{7.23}$$

$$\mathcal{V}^{(1)}(\widetilde{P}_{v_0}) = \sum_{\mathbf{x}_{v_0}}\Big[\prod_{f\in\widetilde{P}_{v_0}}\partial^{\alpha(f)}\psi^{\varepsilon(f)}_{\omega(f),\mathbf{x}(f)}\Big]K_{\widetilde{P}_{v_0}}(\mathbf{x}_{v_0}), \tag{7.24}$$

$$K_{\widetilde{P}_{v_0}}(\mathbf{x}_{v_0}) = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{v_1,..,v_n}\mathcal{E}^T_\chi[\widetilde{\chi}(P_{v_1}),\dots,\widetilde{\chi}(P_{v_n})]\prod_{i=1}^{n}v_i(\mathbf{x}_{v_i}), \tag{7.25}$$

where $\widetilde{\chi}(P_v) = \prod_{f\in P_v}\partial^{\alpha(f)}\chi^{\varepsilon(f)}_{\omega(f),\mathbf{x}(f)}$, $\sum_{v_1,\dots v_n}1 \le \widetilde{C}_0^n$, $\widetilde{P}_{v_0} = \bigcup_i\widetilde{P}_{v_i}$ and $\mathbf{x}_{v_0} = \bigcup_i\mathbf{x}_{v_i}$. We use now the well known expression for $\mathcal{E}^T_\chi$ (see for instance [Le])

$$\mathcal{E}^T_\chi(\widetilde{\chi}(P_1),...,\widetilde{\chi}(P_s)) = \sum_T\prod_{l\in T}g^{(\chi)}_{\omega^-,\omega^+}(\mathbf{x}_l-\mathbf{y}_l)\int dP_T(\mathbf{t})\det G^T(\mathbf{t}), \tag{7.26}$$

where:

a) $P$ is a set of indices, and $\widetilde{\chi}(P) = \prod_{f\in P}\partial^{\alpha(f)}\chi^{\varepsilon(f)}_{\mathbf{x}(f),\omega(f)}$.

b) $T$ is a set of lines forming an *anchored tree* between the cluster of points $P_1,..,P_s$ *i.e.* $T$ is a set of lines which becomes a tree if one identifies all the points in the same clusters.

c) $\mathbf{t} = \{t_{i,i'}\in[0,1], 1\le i,i'\le s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'} = \mathbf{u}_i\cdot\mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i\in\mathbb{R}^s$ of unit norm.

d) $G^T(\mathbf{t})$ is a $(n-s+1)\times(n-s+1)$ matrix, whose elements are given by $G^T_{ij,i'j'} = t_{i,i'}g_{\omega^-,\omega^+}(\mathbf{x}_{ij}-\mathbf{y}_{i'j'})$ with $(f^-_{ij},f^+_{i'j'})$ not belonging to $T$.

If $s = 1$ the sum over $T$ is empty, but we can still use the above equation by interpreting the r.h.s. as 1 if $P_1$ is empty, and $\det G(P_1)$ otherwise.

We bound the determinant using the well known *Gram-Hadamard inequality*, stating that, if $M$ is a square matrix with elements $M_{ij}$ of the form $M_{ij} =< A_i, B_j >$, where $A_i, B_j$ are vectors in a Hilbert space with scalar product $< \cdot, \cdot >$, then

$$|\det M| \leq \prod_i ||A_i|| \cdot ||B_i||, \tag{7.27}$$

where $|| \cdot ||$ is the norm induced by the scalar product.

Let $\mathcal{H} = \mathbb{R}^s \otimes \mathcal{H}_0$, where $\mathcal{H}_0$ is the Hilbert space of complex four dimensional vectors $F(\mathbf{k}) = (F_1(\mathbf{k}), \dots, F_4(\mathbf{k}))$, $F_i(\mathbf{k})$ being a function on the set $\mathcal{D}_{-,-}$, with scalar product

$$< F, G >= \sum_{i=1}^{4} \frac{1}{M^2} \sum_{\mathbf{k}} F_i^*(\mathbf{k}) G_i(\mathbf{k}), \tag{7.28}$$

and one checks that

$$G^T_{ij,i'j'} = t_{i,i'} g^{(\chi)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'}) =< \mathbf{u}_i \otimes A_{\mathbf{x}(f^-_{ij}),\omega(f^-_{ij})}, \mathbf{u}_{i'} \otimes B_{\mathbf{x}(f^+_{i'j'}),\omega(f^+_{i'j'})} > , \tag{7.29}$$

where $\mathbf{u}_i \in \mathbb{R}^s$, $i = 1, \dots, s$, are the vectors such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$, and (with $Q(\mathbf{k})$ defined in (2.23))

$$\begin{aligned} A_{\mathbf{x},\omega}(\mathbf{k}) &= e^{i\mathbf{k}'\mathbf{x}} \frac{1}{\sqrt{-Q_\chi(\mathbf{k})}} \cdot \begin{cases} (-\sin k_0 + i \sin k, 0, -im_\chi(\mathbf{k}), 0), & \text{if } \omega = +1, \\ (0, im_\chi(\mathbf{k}), 0, m_\chi(\mathbf{k})), & \text{if } \omega = -1, \end{cases} \\ B_{\mathbf{x},\omega} &= e^{i\mathbf{k}'\mathbf{y}} \frac{1}{\sqrt{-Q_\chi(\mathbf{k})}} \cdot \begin{cases} (1, 1, 0, 0), & \text{if } \omega = +1, \\ (0, 0, 1, (\sin k_0 + i \sin k)/m_\chi(\mathbf{k})), & \text{if } \omega = -1. \end{cases} \end{aligned} \tag{7.30}$$

Hence from (7.27) we immediately find

$$|G^T_{ij,i'j'}| \leq C_1^n \tag{7.31}$$

where $C_1$ is an $O(1)$ constant. Finally we get

$$\sum_{\mathbf{x}_{v_0}} |K_{\widetilde{P}_{v_0}}(\mathbf{x}_{v_0})| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{v_1,..,v_n} \sum_{\mathbf{x}_{v_1},\dots,\mathbf{x}_{v_n}} C_1^n \sum_T [\prod_{l \in T} |g^{(\chi)}_{\omega^-,\omega^+}(\mathbf{x}_l - \mathbf{y}_l)|] \prod_{i=1}^{n} |v_i(\mathbf{x}_{v_i})|, \tag{7.32}$$

where we have used that $\int dP_T(\mathbf{t}) = 1$. The number of addends in $\sum_T$ is bounded by $n! C_2^n$. Finally $T$ and the $\bigcup_i \mathbf{x}_{v_i}$ form a tree connecting all points, so that using that the propagator is massive and that the interactions are short ranged $\sum_{\mathbf{x}_{v_1},\dots \mathbf{x}_{v_n}} \sum_T [\prod_{l \in T} |g^{(\chi)}_{\omega^-,\omega^+}(\mathbf{x}_l - \mathbf{y}_l)|] \prod_{i=1}^n |v_i(\mathbf{x}_{v_i})| \leq C_3^n |\varepsilon|^{\widetilde{n}} M^2$, where $\widetilde{n}$ is the number of couplings $O(\varepsilon)$.

Let us consider the case $|\widetilde{P}_{v_0}| \geq 4$. Note that if to $v_i$ are associated only terms from $\mathcal{V}(\psi, \chi)$, then $\widetilde{n} = n$. Let us consider now the case in which there are end-points associated to $Q(\psi, \chi)$, which have $O(1)$ coupling; there are at most $|\widetilde{P}_{v_0}|$ end-points associated with $Q(\psi,\chi)$. In fact in $Q(\psi,\chi)$ there are only terms of the form $\psi\chi$, so at most the number of them is equal to the number of $\psi$ fields. If we call $n_\lambda \le \widetilde{n}$ the number of vertices quartic in the fields it is clear that $n_\lambda \ge |\widetilde{P}_{v_0}|/2-1 \ge |\widetilde{P}_{v_0}|/4$; hence

$$\sum_{\mathbf{x}_{v_0}} |K_{\widetilde{P}_{v_0}}(\mathbf{x}_{v_0})| \le M^2 \sum_{\widetilde{n}=|\widetilde{P}_{v_0}|/4}^{\infty} C^{\widetilde{n}+|\widetilde{P}_{v_0}|}|\varepsilon|^{\widetilde{n}} \tag{7.33}$$

and the second of (3.2) holds for $|\widetilde{P}_{v_0}| \ge 4$.

Consider now the case $|\widetilde{P}_{v_0}| = 2$; in this case there are terms $O(1)$, obtained when all the $v_i$ are associated with elements of $Q(\psi,\chi)$. It is convenient to include all such terms in the gaussian integration, as they cannot be considered as perturbations (they are not $O(\varepsilon)$). Hence we define

$$\mathcal{N}\int \overline{P}(d\psi) = \int P(d\psi)\int P(d\chi)e^{Q(\psi,\chi)} \tag{7.34}$$

and, if $< X >_0 = \int \overline{P}(d\psi)X$, it holds $< \psi^-_{\mathbf{x},1}\psi^+_{\mathbf{y},1} >_0 = < \psi^{(1)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{y}} >_0$, $< \psi^-_{\mathbf{x},-1}\psi^+_{\mathbf{y},-1} >_0 = < \overline{\psi}^{(1)}_{\mathbf{x}}\overline{\psi}^{(1)}_{\mathbf{y}} >_0$ and $< \psi^-_{\mathbf{x},1}\psi^+_{\mathbf{y},-1} >_0 = < \psi^{(1)}_{\mathbf{x}}\overline{\psi}^{(1)}_{\mathbf{y}} >_0$. By using the explicit expressions for $< \psi^{(1)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{y}} >_0$, $< \overline{\psi}^{(1)}_{\mathbf{x}}\overline{\psi}^{(1)}_{\mathbf{y}} >_0$, $< \psi^{(1)}_{\mathbf{x}}\overline{\psi}^{(1)}_{\mathbf{y}} >_0$ in [MPW], (3.5) follows.

In order to obtain (3.3) we single out the local part of the terms quartic in the fields; the fact that $l_1 = 2\lambda(a+b)\mathrm{sech}^4 J + O(\varepsilon^2)$ can be checked by an explicit computation of all the contributions with coupling $O(\lambda)$ to $W_2$, noting that they can only be obtained contracting a term quartic in the $\chi$ fields with one of the addends of (7.10); each of such terms carries a derivative in the coordinate space, hence the Fourier transform of such terms is vanishing at zero momentum.

### *7.6. Appendix F: Symmetry cancellations in the effective potential.*

- There are no local terms in the r.h.s. of (3.2) of the form $\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},1}$; in fact by (2.13) $\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},1} = i\psi^{(1)}_{\mathbf{x}}\psi^{(2)}_{\mathbf{x}}$, but the system is invariant under the transformation

$$\begin{aligned}\psi^{(1)},\overline{\psi}^{(1)},\chi^{(1)},\overline{\chi}^{(1)} &\to -\psi^{(1)},-\overline{\psi}^{(1)},-\chi^{(1)},-\overline{\chi}^{(1)}\\ \psi^{(2)},\overline{\psi}^{(2)},\chi^{(2)},\overline{\chi}^{(2)} &\to \psi^{(2)},\overline{\psi}^{(2)},\chi^{(2)},\overline{\chi}^{(2)},\end{aligned} \tag{7.35}$$

  hence such terms cannot be present as they violate such symmetry.
- There are no terms in the r.h.s. of (3.2) of the form $\psi^-_{1,\mathbf{x}}\psi^-_{-1,\mathbf{y}}$ or $\psi^+_{1,\mathbf{x}}\psi^+_{-1,\mathbf{y}}$; in fact,

$$\psi^-_{1,\mathbf{x}}\psi^-_{-1,\mathbf{y}} = \frac{1}{2}[\psi^{(1)}_{\mathbf{x}}\overline{\psi}^{(1)}_{\mathbf{y}} - \psi^{(2)}_{\mathbf{x}}\overline{\psi}^{(2)}_{\mathbf{y}} + i\psi^{(1)}_{\mathbf{x}}\overline{\psi}^{(2)}_{\mathbf{y}} + i\psi^{(2)}_{\mathbf{x}}\overline{\psi}^{(1)}_{\mathbf{y}}] \tag{7.36}$$

  and the last two terms violate the symmetry (7.35). Moreover the first two terms are forbidden

  a) in the case $b = 0$ by the invariance under the symmetry

$$\psi^{(1)}_{x,x_0},\chi^{(1)}_{x,x_0},\psi^{(2)}_{x,x_0},\chi^{(2)}_{x,x_0} \to \psi^{(2)}_{x,x_0},\chi^{(2)}_{x,x_0},\psi^{(1)}_{x,x_0},\chi^{(1)}_{x,x_0}\ ; \tag{7.37}$$

  b) in the case $a = 0$ by the invariance under the symmetry

$$\psi^{(1)}_{x,x_0},\chi^{(1)}_{x,x_0},\psi^{(2)}_{x,x_0},\chi^{(2)}_{x,x_0} \to \psi^{(2)}_{x,x_0},\chi^{(2)}_{x,x_0},\psi^{(1)}_{x+1,x_0-1},\chi^{(1)}_{x+1,x_0-1}\ . \tag{7.38}$$

In fact consider in $\mathcal{V}^{(1)}$ (3.1) the terms of the form $\sum_{\mathbf{x},\mathbf{y}}[\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{y}} w^{(1)}(\mathbf{x},\mathbf{y}) + \psi^{(2)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{y}} w^{(2)}(\mathbf{x},\mathbf{y})]$; $w^{(1)}(\mathbf{x},\mathbf{y})$ is obtained by the truncated expectation $\mathcal{E}^T_\chi$ of a certain number of $(\mathcal{V}+Q)|_{\psi=0}$, of a term $\frac{\partial}{\partial\psi^{(1)}_{\mathbf{x}}}(\mathcal{V}+Q)|_{\psi=0}$ and of a term $\frac{\partial}{\partial\bar\psi^{(1)}_{\mathbf{y}}}(\mathcal{V}+Q)|_{\psi=0}$. If we perform in the truncated expectation the change of variable (7.37) or (7.38)we get that $(\mathcal{V}+Q)|_{\psi=0}$ is invariant while $\frac{\partial}{\partial\psi^{(1)}_{\mathbf{x}}}(\mathcal{V}+Q)_{\psi=0}$ is changed in $\frac{\partial}{\partial\psi^{(2)}_{\mathbf{x}}}(\mathcal{V}+Q)_{\psi=0}$ and $\frac{\partial}{\partial\bar\psi^{(1)}_{\mathbf{y}}}(\mathcal{V}+Q)_{\psi=0}$ is changed in $\frac{\partial}{\partial\bar\psi^{(2)}_{\mathbf{y}}}(\mathcal{V}+Q)_{\psi=0}$; this shows that $w^{(1)}(\mathbf{x},\mathbf{y}) = w^{(2)}(\mathbf{x},\mathbf{y})$.

A similar argument can be repeated for $\psi^+_{1,\mathbf{x}}\psi^+_{-1,\mathbf{y}}$.

- There are no terms in the r.h.s. of (3.2) of the form $\psi_{\omega,\mathbf{x}}\psi_{\omega,\mathbf{y}}$ or $\psi^+_{\omega,\mathbf{x}}\psi^+_{\omega,\mathbf{y}}$; in fact,

$$\psi^-_{1,\mathbf{x}}\psi^-_{1,\mathbf{x}'} = \frac{1}{2}\left[\psi^{(1)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{y}} - \psi^{(2)}_{\mathbf{x}}\psi^{(2)}_{\mathbf{y}} + i\psi^{(1)}_{\mathbf{x}}\psi^{(2)}_{\mathbf{y}} + i\psi^{(2)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{y}}\right] \tag{7.39}$$

  and we can proceed as in the previous case.
- The model is invariant under complex conjugation and the exchange

$$\psi^{(\alpha)}_{\mathbf{x}}, \overline{\psi}^{(\alpha)}_{\mathbf{x}} \to \overline{\psi}^{(\alpha)}_{\mathbf{x}}, \psi^{(\alpha)}_{\mathbf{x}} \quad \chi^{(\alpha)}_{\mathbf{x}}, \overline{\chi}^{(\alpha)}_{\mathbf{x}} \to \overline{\chi}^{(\alpha)}_{\mathbf{x}}, \chi^{(\alpha)}_{\mathbf{x}}; \tag{7.40}$$

  this follows from the fact that, from (2.12), $\overline{H}^{(\alpha)}$, $H^{(\alpha)}$, $\overline{V}^{(\alpha)}$, $V^{(\alpha)}$, written in terms of $\overline{\psi}^{(\alpha)}$, $\psi^{(\alpha)}$, $\overline{\chi}^{(\alpha)}$, $\chi^{(\alpha)}$, are invariant under such transformation. Hence the coefficient of the local part of the quartic (non-vanishing) terms is real; in fact $\psi^+_{1,\mathbf{x}}\psi_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}\psi_{-1,\mathbf{x}} \equiv \psi^{(1)}_{\mathbf{x}}\overline{\psi}^{(1)}_{\mathbf{x}}\psi^{(2)}_{\mathbf{x}}\overline{\psi}^{(2)}_{\mathbf{x}}$ times $\widehat{w}(0,0,0)$ must be equal, by the above invariance, to $\widehat{w}^*(0,0,0)\overline{\psi}^{(1)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{x}}\overline{\psi}^{(2)}_{\mathbf{x}}\psi^{(2)}_{\mathbf{x}}$, hence $\widehat{w}(0,0,0) = \widehat{w}^*(0,0,0)$. Finally the combination of local terms $\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},-1} + \psi^+_{\mathbf{x},-1}\psi^-_{\mathbf{x},1}$ is equal to $i[\psi^{(1)}_{\mathbf{x}}\overline{\psi}^{(2)}_{\mathbf{x}} - \psi^{(2)}_{\mathbf{x}}\overline{\psi}^{(1)}_{\mathbf{x}}]$ so it cannot be present as it violates the symmetry (7.35). On the other hand $\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},-1} - \psi^+_{\mathbf{x},-1}\psi^-_{\mathbf{x},1}$ is equal to $[\psi^{(1)}_{\mathbf{x}}\overline{\psi}^{(1)}_{\mathbf{x}} + \psi^{(2)}_{\mathbf{x}}\overline{\psi}^{(2)}_{\mathbf{x}}]$; hence the coefficient of the local part is imaginary and odd in $\omega$; in fact $\widehat{w}(0)[\psi^{(1)}_{\mathbf{x}}\overline{\psi}^{(1)}_{\mathbf{x}} + \psi^{(2)}_{\mathbf{x}}\overline{\psi}^{(2)}_{\mathbf{x}}]$ must be equal to $\widehat{w}^*(0)[\overline{\psi}^{(1)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{x}} + \overline{\psi}^{(2)}_{\mathbf{x}}\overline{\psi}^{(2)}_{\mathbf{x}}]$, by the invariance under complex conjugation and (7.40), hence $\widehat{w}(0) = -\widehat{w}^*(0)$.
- We consider now the addends with $n = 1$ in the r.h.s. of (3.2),

$$\sum_{\mathbf{x},\mathbf{y}} W_{\omega_1,\omega_2}(\mathbf{x},\mathbf{y})\psi^+_{\mathbf{x},\omega_1}\psi^-_{\mathbf{y},\omega_2}. \tag{7.41}$$

  We can represent $W_{\omega_1,\omega_2}(\mathbf{x},\mathbf{y})$ as the sum over *Feynman graphs* $g$ in the usual way (see for instance [GM]); the external lines are associated to the $\psi$ fields, and to the internal lines are associated the propagators $g^\chi_{\omega,\omega'}(\mathbf{x}-\mathbf{y})$; moreover the vertices associated to the interaction are linear or bilinear in $A^{c,\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$.

We show that

$$\sum_{\mathbf{x}} \sin\frac{\pi x}{M} W_{\omega,-\omega}(\mathbf{x},\mathbf{0}) = \sum_{\mathbf{x}} \sin\frac{\pi x_0}{M} W_{\omega,-\omega}(\mathbf{x},\mathbf{0}) = 0\ . \tag{7.42}$$

We can consider a single Feynman diagram value $\widehat{W}^g_{\omega,-\omega}(\mathbf{x},\mathbf{0})$ and we call

1) $n_a^\omega$ is the number in $g$ of terms $A^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ with $\omega_1 = \omega_2 = \omega$; $n_a^1 + n_a^{-1} = n_a$.
2) $n_b$ is the number of $A^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ with $\omega_1 = -\omega_2 = \omega$.
3) $n_+^\omega$ is the number of diagonal propagators $g^{(\chi)}_{\omega,\omega}$.
4) $n_-$ is the number of non diagonal propagators $g^{(\chi)}_{\omega,-\omega}$.

If we make the transformation $\mathbf{x}_i \to -\mathbf{x}_i$ in all the sums in $\sum_{\mathbf{x}} W^g_{\omega,-\omega}(\mathbf{x},0)\sin\frac{\pi x}{M}$ and we use (2.27), then in each Feynman graph each propagator $g^{(\chi)}_{\omega,\omega'}(\mathbf{x})$ is replaced by $(-1)^{\delta_{\omega,\omega'}} g^{(\chi)}_{\omega,\omega'}(\mathbf{x})$. Moreover the propagators $\partial g^{(\chi)}_{\omega,\omega'}(\mathbf{x})$ are replaced by $(-1)^{\delta_{\omega,\omega'}+1}$ $\widetilde{\partial} g^{(\chi)}_{\omega,\omega'}(\mathbf{x})$, where $\widetilde{\partial}_{x_0} f_{x,x_0} = f_{x,x_0} - f_{x,x_0-1}$ and $\widetilde{\partial}_x f_{x,x_0} = f_{x,x_0} - f_{x-1,x_0}$; finally $g^{(\chi)}_{\omega,\omega'}(\mathbf{x}+\mathbf{a})$ is replaced by $(-1)^{\delta_{\omega,\omega'}} g^{(\chi)}_{\omega,\omega'}(\mathbf{x}-\mathbf{a})$, if $\mathbf{a}$ is a constant vector.

On the other hand we could equivalently write the interaction (1.3) as

$$\begin{aligned} V(\sigma^{(1)},\sigma^{(2)}) = -\sum_{x,x_0=1}^{M} & \lambda a[\sigma^{(1)}_{x-1,x_0}\sigma^{(1)}_{x,x_0}\sigma^{(2)}_{x-1,x_0}\sigma^{(2)}_{x,x_0} + \sigma^{(1)}_{x,x_0-1}\sigma^{(1)}_{x,x_0}\sigma^{(2)}_{x,x_0-1}\sigma^{(2)}_{x,x_0}] \\ & +\lambda b[\sigma^{(1)}_{x-1,x_0}\sigma^{(1)}_{x,x_0}\sigma^{(2)}_{x,x_0-1}\sigma^{(2)}_{x,x_0} \\ & +\sigma^{(1)}_{x,x_0-1}\sigma^{(1)}_{x,x_0}\sigma^{(2)}_{x,x_0-1}\sigma^{(2)}_{x+1,x_0-1}]\}\ ; \end{aligned} \tag{7.43}$$

Equation (7.43) can be found from (1.3) making the change of variables $\mathbf{x} \to -\mathbf{x}$, and then making the transformation $\sigma^{(\alpha)}_{\mathbf{x}} \to \sigma^{(\alpha)}_{-\mathbf{x}}$. Starting from this expression and repeating the computations in §2, §3 we get an expression similar to (2.11), where $\mathcal{V}$ is now an expression linear or bilinear in $\overline{H}^{(\alpha)}_{x-1,x_0} H^{(\alpha)}_{x,x_0}$ or $\overline{V}^{(\alpha)}_{x,x_0-1} V^{(\alpha)}_{x,x_0}$.

From (2.12) it holds

$$\overline{V}^{(\alpha)}_{x,x_0-1} V^{(\alpha)}_{x,x_0} = \widetilde{Q}^{1(\alpha)}_{\mathbf{x}} + \widetilde{Q}^{2(\alpha)}_{\mathbf{x}} + \widetilde{Q}^{3(\alpha)}_{\mathbf{x}}, \tag{7.44}$$

where

$$\begin{aligned} \widetilde{Q}^{1(\alpha)}_{\mathbf{x}} &= \frac{1}{4i}[\psi^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0} - \overline{\psi}^{(\alpha)}_{x,x_0-1}\overline{\psi}^{(\alpha)}_{x,x_0} + \overline{\psi}^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0} - \psi^{(\alpha)}_{x,x_0-1}\overline{\psi}^{(\alpha)}_{x,x_0}], \\ \widetilde{Q}^{2(\alpha)}_{\mathbf{x}} &= \frac{1}{4i}[\chi^{(\alpha)}_{x,x_0-1}\chi^{(\alpha)}_{x,x_0} - \overline{\chi}^{(\alpha)}_{x,x_0-1}\overline{\chi}^{(\alpha)}_{x,x_0} + \overline{\chi}^{(\alpha)}_{x,x_0-1}\chi^{(\alpha)}_{x,x_0} - \chi^{(\alpha)}_{x,x_0-1}\overline{\chi}^{(\alpha)}_{x,x_0}], \\ \widetilde{Q}^{3(\alpha)}_{\mathbf{x}} &= \frac{1}{4i}[\psi^{(\alpha)}_{x,x_0-1}\chi^{(\alpha)}_{x,x_0} - \overline{\psi}^{(\alpha)}_{x,x_0-1}\overline{\chi}^{(\alpha)}_{x,x_0} - \psi^{(\alpha)}_{x,x_0-1}\overline{\chi}^{(\alpha)}_{x,x_0} + \overline{\psi}^{(\alpha)}_{x,x_0-1}\chi^{(\alpha)}_{x,x_0} \\ &\quad +\chi^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0} - \overline{\chi}^{(\alpha)}_{x,x_0-1}\overline{\psi}^{(\alpha)}_{x,x_0} - \chi^{(\alpha)}_{x,x_0-1}\overline{\psi}^{(\alpha)}_{x,x_0} + \overline{\chi}^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0}]. \end{aligned} \tag{7.45}$$

A similar expression hold for $\overline{H}^{(\alpha)}_{x-1,x_0} H^{(\alpha)}_{x,x_0}$. Note that, looking for instance to the first of (7.45), we get

$$\psi^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0} - \overline{\psi}^{(\alpha)}_{x,x_0-1}\overline{\psi}^{(\alpha)}_{x,x_0} = \psi^{(\alpha)}_{x,x_0}\widetilde{\partial}_{x_0}\psi^{(\alpha)}_{x,x_0} - \overline{\psi}^{(\alpha)}_{x,x_0}\widetilde{\partial}_{x_0}\overline{\psi}^{(\alpha)}_{x,x_0} \tag{7.46}$$

and

$$\begin{aligned} \overline{\psi}^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0} - \psi^{(\alpha)}_{x,x_0-1}\overline{\psi}^{(\alpha)}_{x,x_0} &= -\widetilde{\partial}_{x_0}\overline{\psi}^{(\alpha)}_{x,x_0}\widetilde{\partial}_{x_0}\psi^{(\alpha)}_{x,x_0} + \overline{\psi}^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0} \\ &\quad +\overline{\psi}^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0-1}\ . \end{aligned} \tag{7.47}$$

One verifies that $\mathcal{V}$ is a sum of the form $\sum_{\mathbf{x}} \widetilde{A}^{\sigma_1,\sigma_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ or $\sum_{\mathbf{x}} \widetilde{A}^{\sigma_1,\sigma_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2} \widetilde{A}^{\sigma'_1,\sigma'_2}_{\mathbf{x}';\phi'',\omega'_1;\phi''',\omega'_2}$ where $\mathbf{x}' = \mathbf{x}$ or $\mathbf{x}' = (x+1, x_0-1)$ and $\widetilde{A}^{\sigma_1,\sigma_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ is identical to $A^{\sigma_1,\sigma_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ up to the substitutions $\partial \to \widetilde{\partial}$, $x+1 \to x-1$ and $x_0+1 \to x_0-1$. Hence it holds

$$\sum_{\mathbf{x}} \sin \frac{\pi x}{M} W_{\omega,-\omega}(\mathbf{x},\mathbf{0}) = (-1)^{n_a+n_++1} \sum_{\mathbf{x}} \sin \frac{\pi x}{M} W_{\omega,-\omega}(\mathbf{x},\mathbf{0}) \ . \tag{7.48}$$

It holds $2n_a + 2n_b = 2(n_+ + n_-) + 2$ so that

$$\begin{aligned}\sum_{\mathbf{x}} \sin \frac{\pi x}{M} W_{\omega,-\omega}(\mathbf{x},\mathbf{0}) &= (-1)^{2n_a+n_b-n_-} \sum_{\mathbf{x}} \sin \frac{\pi x}{M} W_{\omega,-\omega}(\mathbf{x},\mathbf{0}) \\ &= (-1)^{n_b-n_-} \sum_{\mathbf{x}} \sin \frac{\pi x}{M} W_{\omega,-\omega}(\mathbf{x},\mathbf{0}). \end{aligned}\tag{7.49}$$

The number of fields with $\omega = 1$ is $2n^1_a + n_b$ and the number of external fields with $\omega = 1$ is then $2n^1_a + n_b - 2n^1_+ - n_- = 2(n^1_a - n^1_+) + n_b - n_-$ which implies that $n_b - n_-$ must be an odd number if the number of external fields with $\omega = 1$ is 1. Hence $\sum_{\mathbf{x}} \sin \frac{\pi x}{M} W_{\omega,-\omega}(\mathbf{x},\mathbf{0}) = (-1) \sum_{\mathbf{x}} \sin \frac{\pi x}{M} W_{\omega,-\omega}(\mathbf{x},\mathbf{0})$ so that $\sum_{\mathbf{x}} \sin \frac{\pi x}{M} W_{\omega,-\omega}(\mathbf{x},\mathbf{0}) = 0$.

We consider now $W_{\omega,\omega}(\mathbf{x};\mathbf{0})$; we have already proved that $\sum_{\mathbf{x}} W_{\omega,\omega}(\mathbf{x};\mathbf{0}) = 0$. We want to show that

$$\sum_{\mathbf{x}} \frac{\sin \pi x_0}{M} W_{\omega,\omega}(\mathbf{x};\mathbf{0}) = i\omega\alpha; \quad \sum_{\mathbf{x}} \frac{\sin \pi x}{M} W_{\omega,\omega}(\mathbf{x};\mathbf{0}) = \beta$$

with $\alpha, \beta$ real. From (2.23) we see that $g_{\omega,-\omega}(\mathbf{x})$ is even in the exchange $\mathbf{x} \to -\mathbf{x}$ and imaginary. Moreover we can write

$$\widehat{g}_{\omega,\omega}(\mathbf{k}) = \frac{-i \sin k}{\sin^2 k + \sin^2 k_0 + m^2_\chi} + \frac{\omega \sin k_0}{\sin^2 k + \sin^2 k_0 + m^2_\chi} = \widehat{g}^1_{\omega,\omega}(\mathbf{k}) + \widehat{g}^2_{\omega,\omega}(\mathbf{k}) \tag{7.50}$$

with $g^1_{\omega,\omega}(\mathbf{x})$ real, odd in the exchange $x \to -x$ and even in $x_0 \to -x_0$; $g^2_{\omega,\omega}(\mathbf{x})$ is imaginary, even in the exchange $x \to -x$ and odd in $x_0 \to -x_0$. Remember that (see §2.4) the coefficient of $A^{\sigma_1,\sigma_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ is a) imaginary if $\omega_1 = \omega_2$ and $\alpha = 1$, $\partial_{x_\alpha} = \partial_{x_0}$; b) real if $\omega_1 = \omega_2$ and $\alpha = 2$, $\partial_{x_\alpha} = \partial_x$ ; c)imaginary if $\omega_1 = -\omega_2$. Given a Feynman diagram $g$ contributing to $i \sum_{\mathbf{x}} \sin \frac{\pi x}{M} W_{\omega,\omega}$, by parity it must be present a total odd number of $g^1_{\omega,\omega}(\mathbf{x})$ propagators and $\partial_x$ derivatives from the interaction. Moreover by parity the number of $g^2_{\omega,\omega}(\mathbf{x})$ and $\partial_{x_0}$ from the interaction must be even. Finally as the external lines have the same $\omega$ index, the sum of the number of non diagonal propagators $g_{\omega,-\omega}(\mathbf{x})$ plus the number of $A^{\sigma_1,\sigma_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ with $\omega_1 = -\omega_2$ must be even. Hence $\sum_{\mathbf{x}} \frac{\sin \pi x}{M} W_{\omega,\omega}(\mathbf{x};\mathbf{0})$ is real and $\omega$-independent. In the same way one sees that $\sum_{\mathbf{x}} \frac{\sin \pi x_0}{M} W_{\omega,\omega}(\mathbf{x};\mathbf{0}) = i\omega\alpha$.

*7.7. Appendix G: Independence from boundary conditions.* We show that, if $|t-t_c| > 0$,

$$\widetilde{Z}_{2I}^{\varepsilon(1),\varepsilon(2),\varepsilon(3),\varepsilon(4)} = Z_{2I}^{\varepsilon(1),\varepsilon(2),\varepsilon(3),\varepsilon(4)} (Z_{0,2I}^{\varepsilon(1),\varepsilon(2),\varepsilon(3),\varepsilon(4)})^{-1}, \tag{7.51}$$

where $Z_{0,2I}^{\varepsilon(1),\varepsilon(2),\varepsilon(3),\varepsilon(4)}$ is given by (7.2) with $\lambda = 0$, is exponentially insensitive to boundary conditions. In particular we show that for $|t-t_c| > 0$, $\lambda$ small enough

$$\left|\log \frac{\widetilde{Z}_{2I}^{\gamma(1),\gamma(2)}}{\widetilde{Z}_{2I}^{-,-,-,-}}\right| \le |\lambda| M^2 e^{-c_1|t-t_c|M} , \tag{7.52}$$

where $c_1 > 0$ is a suitable constant.

The above equation implies in particular that the partition function is non-vanishing; in fact, from (2.10) $Z_{2I}$ is $(\cosh \lambda a \cosh \lambda b)^{2S}$ times

$$\begin{aligned}&\sum_{\underline{\varepsilon}} (-1)^{\delta_{\gamma_1}+\delta_{\gamma_2}} \widetilde{Z}_{2I}^{\varepsilon(1),\varepsilon(2),\varepsilon(3),\varepsilon(4)} Z_{0,2I}^{\varepsilon(1),\varepsilon(2),\varepsilon(3),\varepsilon(4)} \\ &\quad = \widetilde{Z}_{2I}^{-,-,-,-} Z_{0,2I} + \widetilde{Z}_{2I}^{-,-,-,-} \sum_{\underline{\varepsilon}} (-1)^{\delta_{\gamma_1}+\delta_{\gamma_2}} \left[\frac{\widetilde{Z}_{2I}^{\varepsilon(1),\varepsilon(2),\varepsilon(3),\varepsilon(4)}}{\widetilde{Z}_{2I}^{-,-,-,-}} - 1\right] \\ &\quad \times Z_{0,2I}^{\varepsilon(1),\varepsilon(2),\varepsilon(3),\varepsilon(4)},\end{aligned} \tag{7.53}$$

where $Z_{0,2I} = Z_I^2$ and $Z_I = \sum_{\varepsilon(1),\varepsilon(2)} (-1)^{\delta_{\varepsilon(1),\varepsilon(2)}} Z_I^{\varepsilon(1),\varepsilon(2)}$ is the Ising model partition function.

We recall that in §4 of [MW] it was proved that the limit $M \to \infty$ of $|Z_I^{\varepsilon(1),\varepsilon(2)}|$ if $|t-t_c| > 0$ is exponentially independent from boundary conditions; moreover if $t-t_c < 0$ for any choice of $\varepsilon_1, \varepsilon_2$ the functions $Z_I^{\varepsilon(1),\varepsilon(2)}$ have a positive limit, while if $t-t_c > 0$ the limit of $Z_I^{+,+}$ is negative, and for the other choices the limit is a positive number.

Hence by (7.52) the second addend in (7.53) is bounded by $C|\widetilde{Z}_{2I}^{-,-,-,-} Z_{0,2I}||\lambda|M^2 e^{-c_1|t-t_c|M}$ so (7.53) is non-vanishing.

In order to prove (7.52) we can write, see (7.12) and (7.20),

$$\log \tilde{Z}_{2I}^{\varepsilon(1),\varepsilon'(1),\varepsilon(2),\varepsilon'(2)} = \int \left[\prod_{\alpha=1}^{2} P^{(\alpha)}_{\varepsilon_\alpha^{(\alpha)},\varepsilon_\alpha^{\prime(\alpha)}}(d\psi^{(\alpha)}) P^{(\alpha)}_{\varepsilon_\alpha^{(\alpha)},\varepsilon_\alpha^{\prime(\alpha)}}(d\chi^{(\alpha)})\right] e^{Q-\mathcal{V}}, \tag{7.54}$$

and proceeding as in §3 we see that $\log Z_{2I}^{\varepsilon(1),\varepsilon'(1),\varepsilon(2),\varepsilon'(2)}$ can be written as in (5.12). The terms $\tilde{E}_h$ are the sum of addends of the form $\sum_{\mathbf{x}_1,..,\mathbf{x}_n} W_{\underline{\varepsilon}}(\mathbf{x}_1,..,\mathbf{x}_n)$, with $\mathbf{x}_i$ varying in $[-\frac{M}{2}, \frac{M}{2}] \times [-\frac{M}{2}, \frac{M}{2}]$ and the $W$ are truncated expectations for which a formula like (7.26) holds. Note that $W(\mathbf{x}_1,..,\mathbf{x}_n)$ is periodic with period $M$ in any of its coordinates, for any $\underline{\varepsilon}$; this follows from the fact that there is an even number of $\psi$, $\chi$ fields associated to any $\mathbf{x}_i$, and from the form of $\mathcal{V}$. Moreover $W(\mathbf{x}_1,..,\mathbf{x}_n)$ is translation invariant, so that we can fix one variable to $(0,0)$, for instance $\mathbf{x}_1$; hence it holds

$$\sum_{\mathbf{x}_1,..,\mathbf{x}_n} W_{\underline{\varepsilon}}(\mathbf{x}_1,..,\mathbf{x}_n) = \sum_{\mathbf{x}_1,..,\mathbf{x}_n} W_{\underline{\varepsilon}}(\mathbf{0},\mathbf{x}_2..,\mathbf{x}_n) . \tag{7.55}$$

We can write $\sum_{\mathbf{x}_1,..,\mathbf{x}_n} W$ as $\sum^*_{\mathbf{x}_1,..,\mathbf{x}_n} W + \sum^{**}_{\mathbf{x}_1,..,\mathbf{x}_n} W$, where $\sum^*_{\mathbf{x}_1,..,\mathbf{x}_n}$ is over $\mathbf{x}_i$ varying in $[-\frac{M}{4}, \frac{M}{4}] \times [-\frac{M}{4}, \frac{M}{4}]$. Then $\sum^{**}_{\mathbf{x}_1,..,\mathbf{x}_n} W$ is $O(e^{-c_1|t-t_c|M)})$, as in $W$ there is surely a chain of propagators exponentially decaying connecting the point (0, 0) with a point outside $[-\frac{M}{4}, \frac{M}{4}] \times [-\frac{M}{4}, \frac{M}{4}]$. On the other hand in $\sum^*_{\mathbf{x}_1,..,\mathbf{x}_n} W$ we can use the Poisson summation formula, stating that

$$\frac{1}{M}\sum_{n=0}^{M-1} f\left(\frac{n2\pi}{M}+\frac{\alpha\pi}{M}\right)=\sum_{n\in Z}\widehat{f}(nM)(-1)^{\alpha n}, \tag{7.56}$$

where $f$ is a $2\pi$-periodic function and $\alpha = (0, 1)$. From (7.56) we find, if $g^{(i)}_{\Lambda,\varepsilon,\varepsilon'}(x, x_0)$, $i = \psi, \chi$ is the propagator corresponding to $P_{\varepsilon,\varepsilon'}(d\psi)$ or $P_{\varepsilon,\varepsilon'}(d\chi)$ (7.12),

$$\begin{aligned} g^{(i)}_{\Lambda,\varepsilon,\varepsilon'}(x-y, x_0-y_0) &= \sum_{n,n_0\in Z}(-1)^{n\delta_\varepsilon}(-1)^{n\delta_{\varepsilon'}}g^{(i)}(x-y+nM, x_0-y_0+n_0M)\\ &\equiv g^{(i)}(x-y,x_0-y_0)+\delta g^{'(i)}_{\varepsilon,\varepsilon'}(x-y,x_0-y_0), \end{aligned} \tag{7.57}$$

where $g^{(i)}(x, x_0) = \lim_{M\to\infty} g^{(i)}_{\Lambda,\varepsilon,\varepsilon'}(x, x_0)$ and $\delta_\varepsilon = 1$ if $\varepsilon = -$ and $\delta_\varepsilon = 0$ if $\varepsilon = +$. Note that the only dependence on boundary conditions in the r.h.s. of (7.57) is in $\delta g^{(i)}_{\varepsilon,\varepsilon'}(x - y, x_0 - y_0)$ and it holds, if $|x - y| \le \frac{M}{2}$, $|x_0 - y_0| \le \frac{M}{2}$,

$$|\delta g^{(i)}(x-y, x_0-y_0)| \le e^{-c_2|m_i|M}\ , \tag{7.58}$$

with a proper constant $c_2$. Hence all the terms in $\sum^*_{\mathbf{x}_1,..,\mathbf{x}_n} W$ with at least a $\delta g^{(i)}(x - y, x_0 - y_0)$ are exponentially bounded, and the part with only $g^{(i)}(x - y, x_0 - y_0)$ is independent from boundary conditions. By (7.56) it holds that also the terms $t_h$ are exponentially insensitive to boundary conditions.

*7.8. Appendix H:Asymptotic properties of the propagators on scale h.* If $|\widetilde{Z}_1^{-1}C_0 z_h| \le \frac{1}{2}$, $|C_0 s_h| \le |m_h/2|$ and $\sup_{k\ge h}|\frac{Z_k}{Z_{k-1}}| \le e^{|\lambda|}$, for $\lambda, t - t_c$ small enough, given the positive integers $N, n_0, n_1$ and if $n = n_0 + n_1$, it holds

$$|\partial^{n_0}_{x_0}\partial^{n_1}_{x} g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})| \le C_{N,n}\frac{\gamma^{h(1+n)}}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N}, \tag{7.59}$$

$$|\partial^{n_0}_{x_0}\partial^{n_1}_{x} g^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})| \le C_{N,n}|\frac{m_h}{\gamma^h}|\frac{\gamma^{h(1+n)}}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N}, \tag{7.60}$$

where $\partial_x$ denotes the discrete derivative. This follows immediately from the compact support properties of $\tilde{f}_h(\mathbf{k})$ and the fact that

$$\begin{aligned} &d_M(x-y)^{n_1}d_M(x_0-y_0)^{n_0}g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})\\ &\quad= e^{-i\pi(xM^{-1}n_1+x_0M^{-1}n_0)}(-i)^{n_0+n_1}\frac{1}{M^2}\sum_{\mathbf{k}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\partial^{n_1}_k\partial^{n_0}_{k_0}\\ &\qquad\times\left[\tilde{f}_h(\mathbf{k})[T_h^{-1}(\mathbf{k})]_{\omega,\omega'}\right]\ , \end{aligned} \tag{7.61}$$

where $T_h$ is the quadratic form associated to $P_{Z_{h-1},m_{h-1},C_h}(d\psi)$.

It will be useful to write

$$g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y}) = g^{(h)}_{L;\omega,\omega}(\mathbf{x}-\mathbf{y}) + \widehat{g}^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y}) + \widetilde{g}^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y}) \tag{7.62}$$

with

$$g^{(h)}_{L;\omega,\omega}(\mathbf{x}-\mathbf{y}) = \frac{1}{M^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{\widetilde{f}_h(\mathbf{k})}{-\widetilde{Z}_1\omega k_0 + i\widetilde{Z}_1 k}\ , \tag{7.63}$$

which is of course obeying the bound (7.59). The decomposition (7.62) is related to the following identity:

$$\begin{aligned}[T_h^{-1}(\mathbf{k}')]_{\omega,\omega} = &\frac{1}{-\omega k_0 + ik} + \left[\frac{1}{-\omega\sin k_0 + i\sin k} - \frac{1}{-\omega k_0 + ik}\right]\\ &+ \left[\frac{-\omega\sin k_0 + i\sin k}{\sin^2 k_0^2 + \sin^2 k + [m_{h-1}(\mathbf{k})]^2} - \frac{1}{-\omega\sin k_0 + i\sin k}\right]. \end{aligned}\tag{7.64}$$

From (7.64) one shows that

$$|\partial^{n_0}_{x_0}\partial^{n_1}_{x}\widetilde{g}^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})| \le C_{N,n}\frac{\gamma^{(2+n)h}}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y}))|^N}, \tag{7.65}$$

$$|\partial^{n_0}_{x_0}\partial^{n_1}_{x}\widehat{g}^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})| \le C_{N,n}|\frac{m_h}{\gamma^h}|^2\frac{\gamma^{h(1+n)}}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N}\ . \tag{7.66}$$

Analogously the decomposition (4.16) is such that $\widehat{g}^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})$ verifies (7.60) and $\widetilde{g}^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})$, verifying (7.65).

Finally note that, with the definition (5.9), it holds, given the positive integers $N, n_0, n_1$ and putting $n = n_0 + n_1$, that there exists a constant $C_{N,n}$ such that

$$|\partial^{n_0}_{x_0}\partial^{n_1}_{x} g^{(\le h^*)}_{\omega,\omega'}(\mathbf{x};\mathbf{y})| \le C_{N,n}\frac{\gamma^{h^*(1+n)}}{1+(\gamma^{h^*}|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N}\ . \tag{7.67}$$

*7.9. Appendix I: The integration of the $\psi$ fields.* It is possible to write $\mathcal{V}^{(h)}$ in terms of *Gallavotti-Nicolo trees*

We need some definitions and notations.

1) Let us consider the family of all trees which can be constructed by joining a point $r$, the *root*, with an ordered set of $n \ge 1$ points, the *endpoints* of the *unlabeled tree*, so that $r$ is not a branching point. $n$ will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. Then the number of unlabeled trees with $n$ end-points is bounded by $4^n$. We shall consider also the *labeled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

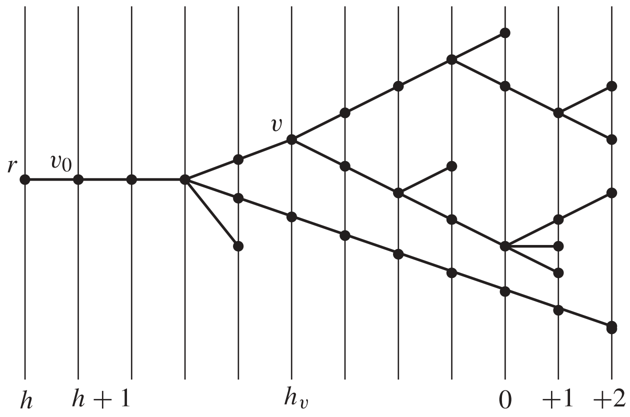


**Fig. 3.** A tree with its scale labels

2) We associate a label $h \leq 0$ with the root and we denote $\mathcal{T}_{h,n}$ the corresponding set of labeled trees with $n$ endpoints. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in $[h, 2]$, and we represent any tree $\tau \in \mathcal{T}_{h,n}$ so that, if $v$ is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of $v$, while the root is on the line with index $h$. There is the constraint that, if $v$ is an endpoint, $h_v > h+1$; if there is only one end-point its scale must be equal to $h+2$, for $h \leq 0$.
The tree will intersect in general the vertical lines in set of points different from the root, the endpoints and the non trivial vertices; these points will be called *trivial vertices*. The set of the *vertices* of $\tau$ will be the union of the endpoints, the trivial vertices and the non trivial vertices. Note that, if $v_1$ and $v_2$ are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.
Moreover, there is only one vertex immediately following the root, which will be denoted $v_0$ and can not be an endpoint; its scale is $h+1$.
3) With each endpoint $v$ of scale $h_v = +2$ we associate one of the contributions to $\mathcal{V}^{(1)}$ given by (3.2); with each endpoint $v$ of scale $h_v \leq 1$ one of the terms in $\mathcal{L}\mathcal{V}^{(h_v-1)}$ defined in (4.19). Moreover, we impose the constraint that, if $v$ is an endpoint and $h_v \leq 1$, $h_v = h_{v'} + 1$, if $v'$ is the non trivial vertex immediately preceding $v$.
4) If $v$ is not an endpoint, the *cluster* $L_v$ with frequency $h_v$ is the set of endpoints following the vertex $v$; if $v$ is an endpoint, it is itself a (*trivial*) cluster. The tree provides an organization of endpoints into a hierarchy of clusters.
5) We introduce a *field label* $f$ to distinguish the field variables appearing in the terms associated with the endpoints as in item 3); the set of field labels associated with the endpoint $v$ will be called $I_v$. Analogously, if $v$ is not an endpoint, we shall call $I_v$ the set of field labels associated with the endpoints following the vertex $v$; $\mathbf{x}(f)$, $\sigma(f)$ and $\omega(f)$ will denote the space-time point, the $\sigma$ index and the $\omega$ index, respectively, of the field variable with label $f$.
6) We associate with any vertex $v$ of the tree a subset $P_v$ of $I_v$, the *external fields* of $v$. These subsets must satisfy various constraints. First of all, if $v$ is not an endpoint and $v_1, \ldots, v_{s_v}$ are the $s_v$ vertices immediately following it, then $P_v \subset \cup_i P_{v_i}$; if $v$ is an endpoint, $P_v = I_v$. We shall denote $Q_{v_i}$ the intersection of $P_v$ and $P_{v_i}$; this definition implies that $P_v = \cup_i Q_{v_i}$. The subsets $P_{v_i} \backslash Q_{v_i}$, whose union will be made, by definition, of the *internal fields* of $v$, have to be non empty, if $s_v > 1$, that is if $v$ is a non-trivial vertex.

Given $\tau \in \mathcal{T}_{j,n}$, there are many possible choices of the subsets $P_v$, $v \in \tau$, compatible with the previous constraints; let us call $\mathbf{P}$ one of these choices. Given $\mathbf{P}$, we consider the family $\mathcal{G}_{\mathbf{P}}$ of all connected Feynman graphs, such that, for any $v \in \tau$, the internal fields of $v$ are paired by propagators of scale $h_v$, so that the following condition is satisfied: for any $v \in \tau$, the subgraph built by the propagators associated with all vertices $v' \geq v$ is connected. The sets $P_v$ have, in this picture, the role of the external legs of the subgraph associated with $v$. The graphs belonging to $\mathcal{G}_{\mathbf{P}}$ will be called *compatible with* $\mathbf{P}$ and we shall denote $\mathcal{P}_\tau$ the family of all choices of $\mathbf{P}$ such that $\mathcal{G}_{\mathbf{P}}$ is not empty.

As explained for instance in §3.2 of [BM] we can write, if $h \leq 0$,

$$\mathcal{V}^{(h)}\left(\sqrt{Z_h}\psi^{(\leq h)}\right) + M^2\widetilde{E}_{h+1} = \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{\mathbf{P}\in\mathcal{P}_\tau}\sqrt{Z_h}^{|P_{v_0}|}\sum_{\mathbf{x}_{v_0}}\widetilde{\psi}^{(\leq h)}(P_{v_0})K_{\tau,\mathbf{P}}^{(h+1)}(\mathbf{x}_{v_0}) \,, \tag{7.68}$$

where

$$\widetilde{\psi}^{(\leq h)}(P_v) = \prod_{f\in P_v}\psi^{(\leq h)}_{\mathbf{x}(f),\omega(f)} \tag{7.69}$$

and $K_{\tau,\mathbf{P}}^{(j+1)}(\mathbf{x}_{v_0})$ is a suitable function, which is obtained by summing the values of all the Feynman graphs compatible with $\mathbf{P}$, see item 6) above, and applying iteratively in the vertices of the tree, different from the endpoints and $v_0$, the $\mathcal{R}$-operation, starting from the vertices with higher scale.

In order to control, uniformly in $M$, the various terms in (7.68) one has to exploit the Gram-Hadamard inequality (see Appendix E) and to take into account the $\mathcal{R}$ operation acting on the vertices of the tree, as explained in full detail in [BM], §3. The result of this analysis, which applies essentially unchanged in the present case, is the following bound (see (3.105) of [BM]), if $k = \sum_i \alpha_i$,

$$\sum_{\mathbf{x}_{v_0}}|K_{\tau,\mathbf{P}}^{(h+1)}(\mathbf{x}_{v_0})| \leq C^n M^2 \varepsilon_h^n \gamma^{-hD_k(P_{v_0})} \,\cdot$$
$$\cdot \prod_{v\text{ not e.p.}}\left\{\frac{1}{s_v!}C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}\left(\frac{Z_{h_v}}{Z_{h_v-1}}\right)^{\frac{|P_v|}{2}}\gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)]}\right\}, \tag{7.70}$$

with $-2 + \frac{|P_v|}{2} + z(P_v) > 0$ and

$$z(P_v) = \begin{cases} 1 & \text{if } |P_v| = 4, \\ 2 & \text{if } |P_v| = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{7.71}$$

The above bound admits a simple dimensional interpretation. If we erase the $\mathcal{R}$ operation from all the vertices of the tree, then $z(P_v) = 0$ and (7.70) allow us to associate a factor $\gamma^{2-\frac{|P_v|}{2}}$ with any trivial or non-trivial vertex of the tree. This would allow us to control the sums over the scale labels and $\mathcal{P}_\tau$, provided that $|P_v|$ were larger than 4 in all vertices, which is however not true. The effect of the $\mathcal{R}$ operation is to improve the bound with the factor $\gamma^{-z(P_v)}$, so that there is a factor $\gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)]}$ smaller than 1 associated with all the vertices.

In order to perform the sums note that the number of unlabeled trees is $\le 4^n$; fix an unlabeled tree, the number of terms in the sum over the various labels of the tree is bounded by $C^n$, except the sums over the scale labels. In order to bound the sums over the scale labels and **P** we first use the inequality

$$\prod_{v\,\text{not e.p.}} \gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)]} \le \left[\prod_{\widetilde{v}} \gamma^{-2\alpha(h_{\widetilde{v}}-h_{\widetilde{v}'})}\right]\left[\prod_{v\,\text{not e.p.}} \gamma^{-2\alpha|P_v|}\right], \tag{7.72}$$

where $\widetilde{v}$ are the non-trivial vertices, and $\widetilde{v}'$ is the non trivial vertex immediately preceding $\widetilde{v}$ or the root. The factors $\gamma^{-2\alpha(h_{\widetilde{v}}-h_{\widetilde{v}'})}$ in the r.h.s. of (7.72) allow us to bound the sums over the scale labels by $C^n$; $\alpha$ is a suitable constant (one finds $\alpha = \frac{1}{40}$).

Finally the sum over **P** can be bounded by using the following combinatorial inequality, trivial for $\gamma$ large enough. Let $\{p_v,\ v \in \tau\}$ be a set of integers such that $p_v \le \sum_{i=1}^{s_v} p_{v_i}$ for all $v \in \tau$ which are not endpoints; then

$$\prod_{v\,\text{not e.p.}} \sum_{p_v} \gamma^{-\frac{p_v}{40}} \le C^n\ . \tag{7.73}$$

It follows that

$$\sum_{\substack{\mathbf{P}\\ |P_{v_0}|=2m}} \prod_{v\,\text{not e.p.}} \gamma^{-\frac{|P_v|}{40}} \le \prod_{v\,\text{not e.p.}} \sum_{p_v} \gamma^{-\frac{p_v}{40}} \le C^n\ . \tag{7.74}$$

*7.10. Appendix L: The flow of running coupling constants. Choice of the counterterm* $\nu$. Let us call $\mu_h = \sup_{k\ge h} \max\{|\lambda_k|, |\delta_k|\}$. Let us consider the first of Eqs. (5.1) for fixed values of $a_h$, $Z_{h-1}$ and $m_{h-1}(\mathbf{k})$, $\tilde{h} \le h \le 1$, if $\tilde{h}$ is a negative integer, satisfying the conditions

$$\mu_h \le \bar{\varepsilon}_1 \le \bar{\varepsilon}_0\ , \qquad\qquad a_0\gamma^{h-1} \ge 4|m_h|, \tag{7.75}$$

$$\gamma^{-c_0\mu_h} \le \frac{m_{h-1}}{m_h} \le \gamma^{+c_0\mu_h}\ , \qquad \gamma^{-c_0\mu_h^2} \le \frac{Z_{h-1}}{Z_h} \le \gamma^{+c_0\mu_h^2} \tag{7.76}$$

for some constant $c_0$.

We prove that, if $\bar{\varepsilon}_0$ is small enough, there exist some constants $\bar{\varepsilon}_1$, $\kappa$, $\gamma'$, $c_1$, $B$, and a family of intervals $I^{(\bar{h})}$, $\tilde{h} \le \bar{h} \le 0$, such that $\bar{\varepsilon}_1 \le \bar{\varepsilon}_0$, $0 < \kappa < 1$, $1 < \gamma' < \gamma$, $I^{(\bar{h})} \subset I^{(\bar{h}+1)}$, $|I^{(\bar{h})}| \le c_1\bar{\varepsilon}_1(\gamma')^{\bar{h}}$ and, if $\nu = \nu_1 \in I^{(\bar{h})}$,

$$|\nu_h| \le B\bar{\varepsilon}_1[\gamma^{-\frac{1}{2}(h-\bar{h})} + \gamma^{\kappa h}] \le \bar{\varepsilon}_0\ , \quad \bar{h} \le h \le 1. \tag{7.77}$$

In order to show this, note that if $|\nu_h| \le \bar{\varepsilon}_0$ for $\bar{h} \le h \le 1$ and $\bar{\varepsilon}_0$ is small enough, the r.h.s. of the first of (5.1) is well defined for $h = \bar{h}$ and we can write

$$\nu_{\bar{h}-1} = \gamma\nu_{\bar{h}} + b_{\bar{h}} + r_{\bar{h}}\ , \tag{7.78}$$

where $b_{\bar{h}} = c^{\nu}_{\bar{h}-1}\gamma^{\bar{h}-1}\lambda_{\bar{h}}$ and $r_{\bar{h}}$ collects all terms of second or higher order in $\bar{\varepsilon}_0$. In the tree expansion of $\beta^h_\nu$, there is no contribution from the trees with $n \ge 2$ endpoints, which are only of type $\nu$ or $\delta$, because of the support properties of the single scale propagators; hence by (7.75) $|r_{\bar h}| \le c_2 \mu_{\bar h} \bar\varepsilon_0$. Let us now fix a positive constant $c$, consider the intervals

$$J^{(h)} = \left[ -\frac{b_h}{\gamma - 1} - c\bar\varepsilon_1, -\frac{b_h}{\gamma-1} + c\bar\varepsilon_1 \right] . \tag{7.79}$$

By using (7.78) one can show by an inductive argument (see for instance §4.3 of [BM]) that there exists a decreasing family of intervals $I^{(\bar h)}$, $\tilde h \le \bar h \le 0$, such that, if $\nu = \nu_1 \in I^{(\bar h)}$, then the sequence $\nu_h$ is well defined for $h \ge \bar h$ and satisfies the bound $|\nu_h| \le \bar\varepsilon_0$.

In order to prove the bound (7.77) we note that, if we iterate the first of (5.1), we can write, if $\bar h \le h \le 0$ and $\nu_1 \in I^{(\bar h)}$,

$$\nu_h = \gamma^{-h+1} \left[ \nu_1 + \sum_{k=h+1}^{1} \gamma^{k-2} \beta_\nu^k(\nu_k, \ldots, \nu_1) \right] , \tag{7.80}$$

where now the functions $\beta_\nu^k$ are thought of as functions of $\nu_k, \ldots, \nu_1$ only.

If we put $h = \bar h$ in (7.80), we get the following identity:

$$\nu_1 = -\sum_{k=\bar h+1}^{1} \gamma^{k-2} \beta_\nu^k(\nu_k, \ldots, \nu_1) + \gamma^{\bar h - 1} \nu_{\bar h} . \tag{7.81}$$

Equations (7.80) and (7.81) are equivalent to

$$\nu_h = -\gamma^{-h} \sum_{k=\bar h+1}^{h} \gamma^{k-1} \beta_\nu^k(\nu_k, \ldots, \nu_1) + \gamma^{-(h-\bar h)} \nu_{\bar h} , \qquad \bar h < h \le 1 . \tag{7.82}$$

By construction, see §4.4, $\beta_k^\nu$ is given by the sum over trees with at least an end-point $\nu_k$, $k \ge h$ or at least a propagator $\widetilde g_{\omega,-\omega}$, see (4.16), or at least with an end-point at scale 2 to which is associated one of the terms in $\mathcal{R}\mathcal{V}^{(1)}$. Hence, we can write

$$\beta_\nu^h = \mu_h \sum_{k=h}^{1} \nu_k \widetilde\beta_{h,k}^\nu \gamma^{-2\kappa(k-h)} + \gamma^{\kappa h} \mu_h R_h^\nu , \tag{7.83}$$

where $|R_h^\nu|, |\widetilde\beta_{h,k}^\nu| \le C$ and $\kappa$ is a constant. The second term in (7.83) comes from the trees with at least a propagator $\widetilde g_{\omega,-\omega}$ or with an end-point at scale 2, and the first term from the trees with at least a $\nu_k$ end-point. The factor $\gamma^{-2\kappa(k-h)}$ in the r.h.s. of (7.83) follows from the simple remark that the bound over all the trees contributing to $\nu_h$, which have at least one endpoint of fixed scale $k > h$, can be improved by a factor $\gamma^{-\eta'(k-h)}$, with $\eta'$ positive but small enough. It is sufficient to use (7.72), which allows to extract such a factor from the r.h.s. before performing the sum over the scale indices, and to choose $\eta' = 2\kappa$, which is possible if $\kappa$ is small enough.

Let us now observe that the sequence $\nu_h$, $\bar h < h \le 1$, satisfying (7.77) can be obtained as the limit as $n \to \infty$ of the sequence $\{\nu_h^{(n)}\}$, $\bar h < h \le 1$, $n \ge 0$, parameterized by $\nu_{\bar h} \in J^{(\bar h+1)}$ and defined recursively in the following way:

$$\begin{aligned}&\nu_h^{(0)} = 0\,,\\&\nu_h^{(n)} = -\gamma^{-h} \sum_{k=\bar h+1}^{h} \gamma^{k-1} \beta_k^\nu(\nu_k^{(n-1)},\ldots,\nu_1^{(n-1)}) + \gamma^{-(h-\bar h)}\nu_{\bar h}\,,\quad n\geq 1\,.\end{aligned} \tag{7.84}$$

In fact, by induction one verifies that, if $\bar\varepsilon_1$ is small enough, $|\nu_h^{(n)}| \leq C\bar\varepsilon_1 \leq \bar\varepsilon_0$, so that (7.84) is meaningful, and $\max_{h^*<h\leq 1} |\nu_h^{(n)} - \nu_h^{(n-1)}| \leq (C\bar\varepsilon_1)^n$. In fact for $n = 1$ it is trivial and for $n > 1$ it follows by the fact that $\beta_k^\nu(\nu_k^{(n-1)},\ldots,\nu_1^{(n-1)}) - \beta_k^\nu(\nu_k^{(n-2)},\ldots,\nu_1^{(n-2)})$ can be written as a sum of terms in which there is at least one endpoint of type $\nu$, with a difference $\nu_{h'}^{(n-1)} - \nu_{h'}^{(n-2)}$, $h' \geq k$, in place of the corresponding running coupling, and one endpoint of type $\lambda$. Then $\nu_h^{(n)}$ converges as $n \to \infty$, for $\bar h < h \leq 1$, to a limit $\nu_h$, satisfying (7.77) and the bound $|\nu_h| \leq \bar\varepsilon_0$, if $\bar\varepsilon_1$ is small enough. Hence, if $\bar\varepsilon_1$ is small enough, by (7.83),

$$|\beta_k^\nu| \leq C\bar\varepsilon_1 \left[\sum_{m=k}^{1} |\nu_m|\gamma^{-2\kappa(m-k)} + \gamma^{\kappa k}\right]. \tag{7.85}$$

Hence

$$|\nu_h^{(n)}| \leq \bar c\bar\varepsilon_1\{\gamma^{-h} \sum_{k=\bar h+1}^{h} \gamma^k \left[\sum_{m=k}^{1} |\nu_m^{(n-1)}|\gamma^{-2\kappa(m-k)} + \gamma^{\kappa k}\right] + \gamma^{-(h-\bar h)}\}\,. \tag{7.86}$$

Let us now suppose that, for some constant $c_{n-1}$,

$$|\nu_m^{(n-1)}| \leq c_{n-1}\bar\varepsilon_1(\gamma^{\kappa m} + \gamma^{-\frac{1}{2}(m-\bar h)}) \leq \bar\varepsilon_0\,, \tag{7.87}$$

which is true for $n = 1$, since $\nu_m^{(0)} = 0$, if $\bar\varepsilon_1$ is small enough. One then checks that the same bound is verified by $\nu_m^{(n)}$, if $c_{n-1}$ is substituted with $c_n = \bar c(1 + c_4 c_{n-1}\bar\varepsilon_1)$, where $c_4$ is a suitable constant. Hence, we can prove the bound (7.77) for $\nu_h = \lim_{n\to\infty} \nu_h^{(n)}$, for $\bar\varepsilon_1$ small enough. □

*Proof of Lemma 3.* We shall proceed by induction. The second part of (5.1) and the above analysis imply that, if $\lambda$ is small enough, there exists an interval $I^{(0)}$, whose size is of order $\lambda$, such that, if $\nu \in I^{(0)}$, then the bound (7.77) is satisfied, together with $|\lambda_0 - \lambda| \leq C|\lambda|^2$. Let us now suppose that the solution of (5.1) is well defined for $\bar h \leq h \leq 0$ and satisfies the conditions (7.75),(7.77), for any $\nu$ belonging to an interval $I^{(\bar h)}$. Suppose also that there exists a constant $c_0$, such that

$$\mu_{\bar h} \leq c_0|\lambda|\,. \tag{7.88}$$

We want to prove that all these conditions are verified also if $\bar h$ is substituted with $\bar h - 1$, if $\lambda$ is small enough. The induction will be stopped as soon as the second condition in (7.75) is violated for some $\nu \in I^{(\bar h)}$. We shall put $\nu$ equal to one of these values, so defining $h^*$ as equal to $\bar h + 1$.

By using (5.5) we have

$$a_{\bar{h}-1} = a_{\bar{h}} + \beta_{\bar{h}}^{\alpha,L}(a_{\bar{h}}, \dots, a_{\bar{h}}) + \sum_{k=\bar{h}+1}^{1} D_{\bar{h},k}^{\alpha} + r_{\bar{h}}^{\alpha}(a_{\bar{h}}, \nu_{\bar{h}}; \dots; a_1, \nu_1; u) , \quad (7.89)$$

where

$$D_{h,k}^{\alpha} = \beta_h^{\alpha,L}(a_h, \dots, a_h, a_k, a_{k+1}, \dots, a_1) - \beta_h^{\alpha,L}(a_h, \dots, a_h, a_h, a_{k+1}, \dots, a_1) . \quad (7.90)$$

On the other hand, one checks that $D_{h,k}^{\alpha}$ admits a tree expansion similar to that of the functions $\beta_h^{\alpha,L}(a_h, \dots, a_1)$, with the property that all trees giving a non zero contribution must have an endpoint of scale $h$, associated with a difference $\lambda_k - \lambda_h$ or $\delta_k - \delta_h$. Hence, if $\kappa$ is the same constant in (7.83) and $h \le 0$,

$$|D_{h,k}^{\alpha}| \le C|\bar{\lambda}_h|\gamma^{-\kappa(k-h)}|a_k - a_h| . \quad (7.91)$$

Let us now suppose that $\bar{h} \le h \le 0$ and that there exists a constant $c_0$, such that

$$|a_{k-1} - a_k| \le c_0|\lambda|^{3/2}[\gamma^{-\frac{1}{2}(k-\bar{h})} + \gamma^{\vartheta k}] , \quad h < k \le 0, \quad (7.92)$$

where $\vartheta = \min\{\kappa/2, \eta'\}$. Equation(7.92) is certainly verified for $k = 0$, thanks to the second part of (5.1); we want to show that it is verified also if $h$ is substituted with $h-1$, if $\lambda_1$ is small enough.

By using (7.89), (5.6), (5.7) and (7.92), we get

$$\begin{aligned} |a_{h-1} - a_h| \le{}& C\bar{\lambda}_h^2\gamma^{\eta' h} + C|\bar{\lambda}_h|^2[\gamma^{-\frac{1}{2}(h-\bar{h})} + \gamma^{\vartheta h}] \\ &+ Cc_0|\bar{\lambda}_h|^{5/2}\sum_{k=h+1}^{1}\gamma^{-\kappa(k-h)}\sum_{h'=h+1}^{k}[\gamma^{-\frac{1}{2}(h'-h^*)} + \gamma^{\vartheta h'}] , \end{aligned} \quad (7.93)$$

which immediately implies (7.92) with $h \to h-1$ and (7.88) with $\bar{h} \to \bar{h}-1$. The bound (7.93) implies also the first of (5.3). Finally the second of (5.3) follows from (5.2). □

*Independence of* $\nu$ *from* $t - t_c$. We have shown that by choosing $\nu \in I_{h^*}$ then (5.3) holds; such $\nu$ are parametrized by $\nu_{h^*} \in J^{(h^*+1)}$. Assuming (7.75) and $\bar{h} = h_M$, one can proceed as above to show that there exists a sequence $\nu_h'$, $h_M < h \le 1$ such that (so that ${\nu'}_{h_M} = 0$)

$$\nu_h' = -\gamma^{-h}\sum_{k=h_M+1}^{h}\gamma^{k-1}\beta_k^{\nu}(\nu_k', \dots, \nu_1') . \quad (7.94)$$

If $\nu_h$, $h^* < h \le 1$ verify (7.82) with $\nu_{h^*} = 0$ it holds that

$$|\nu_h - \nu_h'| \le C\bar{\varepsilon}_1\gamma^{\kappa h^*} \quad h^* \le h \le 1 ; \quad (7.95)$$

this implies that one can choose $\nu = \nu'_1$ for any $h^*$. Equation (7.95)) is proved by induction assuming that it holds for any $k \ge h+1$ and subtracting (7.82) with $\bar{h} = h^*$ and $\nu_{h^*+1} = 0$ from (7.94), finding

$$\nu'_h - \nu_h = -\gamma^{-h} \sum_{k=h^*+1}^{h} \gamma^{k-1} [\beta^\nu_k(\nu'_k, \dots, \nu'_1) - \beta^\nu_k(\nu_k, \dots, \nu_1)] \\ -\gamma^{-h} \sum_{k=h_M+1}^{h^*} \gamma^{k-1} \beta^\nu_k(\nu'_k, \dots, \nu'_1) \, . \tag{7.96}$$

By using (7.83) and the inductive hypothesis, (7.95) follows.

*7.11. Appendix M: Physical observables.* The functionals $\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}, \phi)$ and $S^{(h)}(\phi)$ defined in (6.11),(6.12) can be written in terms of a tree expansion similar to the one introduced in Appendix I.

We introduce, for each $n \ge 0$ and each $m \ge 1$, a family $\mathcal{T}^m_{h,n}$ of trees, which are defined as in Appendix I, with some differences.

1) First of all, if $\tau \in \mathcal{T}^m_{h,n}$, the tree has $n+m$ (instead of $n$) endpoints. Moreover, among the $n+m$ endpoints, there are $n$ endpoints, which we call *normal endpoints*, which are associated with a contribution to the effective potential on scale $h_v - 1$. The $m$ remaining endpoints, which we call *special endpoints*, are associated with a local term of the form (6.15); we shall say that they are of type $Z^{(1)}$.
2) We associate with each vertex $v$ a new integer $l_v \in [0, m]$, which denotes the number of special endpoints following $v$, *i.e.* contained in $L_v$.

In order to study the expansion of the correlation function $\Omega_\Lambda(\mathbf{x}, \mathbf{0}) \equiv \Omega_\Lambda(\mathbf{x})$, which follows from (6.21), we have to consider the trees with two special endpoints, whose space-points we shall denote $\mathbf{x}$ and $\mathbf{y} = \mathbf{0}$; moreover, we shall denote by $h_\mathbf{x}$ and $h_\mathbf{y}$ the scales of the two special endpoints and by $h_{\mathbf{x},\mathbf{y}}$ the scale of the smallest cluster containing both special endpoints.

The decomposition $\tilde{\Omega}_\Lambda(\mathbf{x}, \mathbf{y}) = \Omega^\alpha_\Lambda(\mathbf{x}, \mathbf{y}) + \Omega^\beta_\Lambda(\mathbf{x}, \mathbf{y})$ is such that $\Omega^\alpha_\Lambda(\mathbf{x}, \mathbf{y})$ is given by the sum over trees belonging to $\mathcal{T}^2_{h,n}$ with endpoints $v$ to which are associated only terms in $\mathcal{L}\mathcal{V}^{(h_v-1)}$ or $\mathcal{L}\mathcal{B}^{(h_v-1)}$, and $\Omega^\beta_\Lambda(\mathbf{x}, \mathbf{y})$ is the sum over the remaining trees. The first two addends in (6.22) are the contribution from the trees with $n = 0$, while $(\frac{Z^{(1)}_h}{Z_h})^2 G^{(h),\alpha}_\Lambda(\mathbf{x})$ is given by the sum of trees with $n \ge 1$,

$$G^{(h),\alpha}_\Lambda(\mathbf{x}) = \sum_{n=1}^{\infty} \sum_{h_r=h^*-1}^{h-1} \sum_{\substack{\tau \in \mathcal{T}^2_{h_r,n,l} \\ h_{\mathbf{x},\mathbf{y}}=h}} \sum_{\substack{\mathbf{P} \in \mathcal{P}_\tau, \mathbf{r} \\ P_{v_0}=\emptyset}} G^{(h,h_r),\alpha}_\Lambda(\mathbf{x}, \tau, \mathbf{P}) \, , \tag{7.97}$$

and, as proved in full detail in §5 of [BM], the following bound holds, see (5.60) of [BM],

$$|G^{(h,h_r),\alpha}_\Lambda(\mathbf{x}, \tau, \mathbf{P})| \le (C\varepsilon_h)^n C_N (2n+1)^N \frac{\gamma^{2h}}{1 + [\gamma^h \mathbf{d}(\mathbf{x})]^N} \, .$$

$$\cdot\left(\frac{Z^{(1)}_{h_{\mathbf{x}}}Z_h}{Z_{h_{\mathbf{x}}-1}Z^{(1)}_h}\right)\left(\frac{Z^{(1)}_{h_{\mathbf{y}}}Z_h}{Z_{h_{\mathbf{y}}-1}Z^{(1)}_h}\right)\prod_{v\,\text{not e.p.}}\Big\{\frac{1}{s_v!}C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}$$
$$\cdot\Big(Z_{h_v}/Z_{h_v-1}\Big)^{|P_v|/2}\gamma^{-[-2+\frac{|P_v|}{2}+l_v+z(P_v,l_v)]}\,.\Big\}, \tag{7.98}$$

where $z(P_v,l_v)=1$ if $P_v=4, l_v=0$; $z(P_v,l_v)=2$ if $P_v=2, l_v=0$; $z(P_v,l_v)=1$ if $P_v=2, l_v=1$; $z(P_v,l_v)=0$ in all other cases.

We can now perform as in Appendix I the various sums in the r.h.s. of (7.97). There are some differences in the sum over the scale labels, but they can be easily treated. First of all, one has to take care of the factors $(Z^{(1)}_{h_{\mathbf{x}}}Z_h)/(Z_{h_{\mathbf{x}}-1}Z^{(1)}_h)$ and $(Z^{(1)}_{h_{\mathbf{y}}}Z_h)/(Z_{h_{\mathbf{y}}-1}Z^{(1)}_h)$, with the only effect of adding to the final bound a factor $\gamma^{C|\lambda|(h_v-h_{v'})}$ for each non-trivial vertex $v$ containing one of the special endpoints and strictly following the vertex $v_{\mathbf{x},\mathbf{y}}$; this has a negligible effect, thanks to a bound analogous to (7.72), valid in this case. The other difference is in the fact that, instead of fixing the scale of the root, we have now to fix the scale of $v_{\mathbf{x},\mathbf{y}}$. However, this has no effect, since we bound the sum over the scales with the sum over the differences $h_v-h_{v'}$.

The previous considerations are sufficient to get the bound (6.23) for $G^{(h)\alpha}_{\Lambda}(\mathbf{x})$. An expression similar to (7.97) holds also for $G^{(h)\beta}_{\Lambda}(\mathbf{x})$; the extra factor $\gamma^{\tau h}$ in the bound (6.24) (with respect to (6.23)) is due to the fact that the bound over all the trees which have at least one endpoint $v$ of fixed scale $h_v=2$ can be improved by a factor $\gamma^{\tau h}$. It is sufficient to use (7.72), which allows to extract such a factor from the r.h.s. before performing the sum over the scale indices.

Note also that from (6.15), (6.17) we get (6.25) , where $z^{(1)}_h$ is given by

$$z^{(1)}_h=\sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}^1_{h,n},\mathbf{P}\in\mathcal{P}_\tau,P_{v_0}=(f_1,f_2)} z^{(1)}_h(\tau,\mathbf{P})\ , \tag{7.99}$$

with

$$|z^{(1)}_h(\tau,\mathbf{P})|\le C^n\varepsilon_h^n\gamma^{-h[D_0(P_{v_0})+l_{v_0}]}\prod_{v\,\text{not e.p.}}\Big\{C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}$$
$$\cdot\frac{1}{s_v!}\Big(Z_{h_v}/Z_{h_v-1}\Big)^{|P_v|/2}\gamma^{-[-2+\frac{|P_v|}{2}+l_v+z(P_v,l_v)]}\Big\}\,. \tag{7.100}$$

Finally note that $\operatorname{sech}^4 J_r\tilde{\Omega}_\Lambda(\mathbf{x},\mathbf{y})-\Omega_{-,-,-,-,\Lambda}(\mathbf{x},\mathbf{y})$ is given by a sum of terms in which three or four external $\phi$ fields are present. Essentially by power counting one gets a bound similar to (7.98) in which $\gamma^{2h}$ is replaced by $\gamma^{3h}$ or $\gamma^{4h}$ depending if there are three or four external $\phi$ fields.

*7.12. Appendix N: Perturbations of a single Ising model.* If we consider the hamiltonian (1.1) with interaction given by (6.33) all the analysis in §2, §3 is still valid; the only place in which we have used the explicit form of $V$ is in Appendix F, but the symmetry cancellations exploited there hold also in the case of $V$ given by (6.33). The integration of the light fermions is done exactly as in §4 but now in (4.9) and (4.20) $F^{(\le h)}_{\lambda}=0$, *i.e.* the term and quartic in the field is missing in $\mathcal{L}\mathcal{V}^{(h)}$; the reason is that

$$\psi^{(\le h)+}_{\mathbf{x},1}\psi^{(\le h)-}_{\mathbf{x},1}\psi^{(\le h)+}_{\mathbf{x},-1}\psi^{(\le h)-}_{\mathbf{x},-1}=\bar\psi^{(\le h)(1)}_{\mathbf{x}}\psi^{(\le h)(1)}_{\mathbf{x}}\bar\psi^{(\le h)(2)}_{\mathbf{x}}\psi^{(\le h)(2)}_{\mathbf{x}}\,, \tag{7.101}$$

but such a term cannot be present as the (1) and (2) systems are independent. As a consequence, in (5.1) $\beta^h_m$, $\beta^h_\delta$, $\beta^h_z$ are all $O(\varepsilon_h\gamma^{\kappa h})$, if $\kappa$ is a constant, for the same considerations used in Appendix L: there is no contribution from trees with only end-points of type $\nu$ or $\delta$, because of the support properties of the single scale propagators. Hence $\beta^h_m$, $\beta^h_\delta$, $\beta^h_z$ are given by a sum of trees with at least an end-point of scale $h_v=2$ and by (7.72) the bound for them can be improved by a factor $\gamma^{\kappa h}$. Then, choosing $\nu$ properly, $\delta_h=O(\lambda)$, $m_h=m_0(1+O(\lambda))$, $Z_h=1+O(\lambda)$. For the same reasons the analysis in §6 still holds but $Z^{(1)}_h=1+O(\lambda)$ and at the end (1.8)–(1.12) hold with $\eta_1=\eta_2=0$.

*7.13. Appendix O: Extensions of the main Theorem.* It should be clear from the above analysis that the correlation function or the specific heat behaviour in (1.10) or (1.12) does not depend on the details of the interaction (1.3) but on a few general properties. In fact assume that $V$ verifies the following properties.

1) $V$ is symmetric under the exchange $\{\sigma^{(1)}_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda},\{\sigma^{(2)}_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda}\to\{\sigma^{(2)}_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda},\{\sigma^{(1)}_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda}$. This is true for the Ashkin-Teller Hamiltonian which is invariant under the operation $\sigma^{(1)}_{x,x_0},\sigma^{(2)}_{x,x_0}\to\sigma^{(2)}_{x,x_0},\sigma^{(1)}_{x,x_0}$, and for the Eight vertex model which is invariant under $\sigma^{(1)}_{x,x_0},\sigma^{(2)}_{x,x_0}\to\sigma^{(2)}_{x,x_0},\sigma^{(1)}_{x+1,x_0-1}$ for any $\mathbf{x}\in\Lambda$.
2) $V$ is given by the sum of monomials in the spin variables each one of the form

$$\lambda v(\mathbf{x}_1,..,\mathbf{x}_n)\prod_{i=1}^n\sigma^{(\alpha_i)}_{\mathbf{x}_i}\sigma^{(\alpha_i)}_{\mathbf{x}'_i} \tag{7.102}$$

with $\alpha_i=1,2$ , $\mathbf{x}_i,\mathbf{x}'_i$ nearest neighbor, $v(\mathbf{x}_1,..,\mathbf{x}_2)$ short ranged and $\lambda$ small. The above two properties ensure that the effective potential can be written in the form (3.1), with $\mathcal{V}$ given by a sum over short range monomials in the Grassmann variables $\psi,\chi$. Moreover the analysis in Appendix F can be repeated, as the symmetries which were true in the Ashkin Teller or in the Eight vertex model are true also here, and the marginal or relevant terms in the Renormalization group analysis are the same as in the Eight vertex or Ashkin Teller models. Note that the interaction in the Ashkin-Teller or the Eight-vertex model verify an extra symmetry, namely a symmetry in the exchange $x,x_0\to x_0,x$; such extra symmetry is however not used in our analysis. Finally:
3) $V$ is such that in $\mathcal{V}$ (3.1) there is a non vanishing local term of the form

$$[c\lambda+O(\lambda^2)]\psi^+_{1,\mathbf{x}}\psi^-_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}\psi^-_{-1,\mathbf{x}} \tag{7.103}$$

with $c\neq 0$ a constant.

If such conditions are verified, then a statement identical to the main Theorem follows.

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