Commun. Math. Phys. 245, 215–247 (2004)
Digital Object Identifier (DOI) 10.1007/s00220-003-1001-7

Communications in
Mathematical Physics

Existence and Asymptotic Behavior of Multi-Dimensional Quantum Hydrodynamic Model for Semiconductors

Hailiang Li[1,2,⋆], **Pierangelo Marcati**[3]

[1] Department of Mathematics, Capital Normal University, Beijing 100037, P.R. China
[2] Institute of Mathematics, University of Vienna, Austria
[3] Dipartimento di Matematica, Università dell'Aquila, 67100 L'Aquila, Italy. E-mail: marcati@univaq.it

Received: 26 November 2002 / Accepted: 21 August 2003
Published online: 29 January 2004 –

Abstract: This paper is devoted to the study of the existence and the time-asymptotic of multi-dimensional quantum hydrodynamic equations for the electron particle density, the current density and the electrostatic potential in spatial periodic domain. The equations are formally analogous to classical hydrodynamics but differ in the momentum equation, which is forced by an additional nonlinear dispersion term, (due to the quantum Bohm potential) and are used in the modelling of quantum effects on semiconductor devices.

We prove the local-in-time existence of the solutions, in the case of the *general, nonconvex* pressure-density relation and *large and regular* initial data. Furthermore we propose a "subsonic" type stability condition related to one of the classical hydrodynamical equations. When this condition is satisfied, the local-in-time solutions exist globally in-time and converge time exponentially toward the corresponding steady-state. Since for this problem classical methods like, for instance, the Friedrichs theory for symmetric hyperbolic systems cannot be used, we investigate via an iterative procedure an extended system, which incorporates the one under investigation as a special case. In particular the dispersive terms appear in the form of a fourth-order wave type equation.

1. Introduction and Main Results

Quantum hydrodynamic models become important and necessary to model and simulate electron transport, affected by extremely high electric fields, in ultra-small sub-micron semiconductor devices, such as resonant tunnelling diodes, where quantum effects (like particle tunnelling through potential barriers and build-up in quantum wells [10, 21]) take place and dominate the process. Such kinds of quantum mechanical phenomena cannot

⋆ *Current address:* Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560–0043, Japan. E-mail: lihl@math.sci.osaka-u.ac.jp

be simulated by classical hydrodynamical models. The advantage of the macroscopic quantum hydrodynamical models relies on the facts that they are not only able to describe directly the dynamics of the physical observable and simulate the main characters of quantum effects, but are also numerically less expensive than those microscopic models like the Schrödinger and Wigner-Boltzmann equations. Moreover, even in the process of the semiclassical (or zero dispersion) limit, the macroscopic quantum quantities like density, momentum, and temperature converge in some sense to those of Newtonian fluid-dynamical quantities [13]. Similar macroscopic quantum models are also used in other physical area such as superfluid [26] and superconductivity [5].

The idea to derive quantum fluid-type equations goes back to Madelung in 1927 [27, 24], where the relation between the (linear) Schrödinger equation and quantum fluid equation was described in view of the nonlinear geometric optic (WKB)–ansatz of the wave function for irrotational flow away from vacuum. This in fact gives a way to derive quantum fluid type equations, i.e., to make use of the WKB–expansion and derive the equations for (macroscopic) density and momentum from the single-state Schrödinger equation, or those with temperature involved from the mixed-state Schrödinger equation [14, 18, 13]. Another practicable way to derive quantum hydrodynamic equations is to take advantage of the kinetic structure behind the Schrödinger Hamiltonian through Wigner transformation [37]. In fact, the action of the Wigner transformation on the wave function describes the equivalence between the (linear) Schrödinger equation and Wigner-Boltzmann equation [31], the quantum kinetic transport equation. The application of the moment method to the Wigner-Boltzmann (or Wigner-Poisson) equation, yields the macroscopic quantities density, momentum and temperature, whose time-evolutions obey the quantum hydrodynamic equations [10, 11]. This is done in analogy with derivation of the first three moment equations, in the moment expansion for the Wigner (distribution) function of the Wigner-Boltzmann equation, under appropriate closure conditions [15] near the "quantum Maxwellian". For further references on the quantum modelling of semiconductor devices, we refer to [32, 10, 14, 18, 11] and the references quoted therein.

We are interested in the mathematical analysis of the quantum hydrodynamic model for semiconductors. In the present paper we consider the initial value problem (IVP) of the quantum hydrodynamic model for semiconductors where an additional relaxation term is involved in the linear momentum equation to model the interaction between the electron and crystal lattice. The re-scaled multi-dimensional quantum hydrodynamic models for semiconductors (QHD) then is given by

$$\partial_t \rho + \nabla\cdot(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P = \rho\nabla V + \tfrac{1}{4}\varepsilon^2\nabla\cdot\left(\rho\nabla^2\log\rho\right) - \tfrac{\rho\mathbf{u}}{\tau}, \tag{1.2}$$

$$\lambda^2\Delta V = \rho - \mathcal{C}, \tag{1.3}$$

$$\rho(x,0) = \rho_1(x),\ \mathbf{u}(x,0) = \mathbf{u}_1(x), \tag{1.4}$$

where $\rho > 0$, $\mathbf{u}$, $J = \rho\mathbf{u}$ denote the density, velocity and momentum respectively. $\varepsilon > 0$ the scaled Planck constant, $\tau > 0$ is the (scaled) momentum relaxation time, $\lambda > 0$ the (re-scaled) Debye length, and $\mathcal{C} = \mathcal{C}(x) > 0$ the doping profile simulating the semiconductor device under consideration [18, 32]. The pressure $P = P(\rho)$, like in classical fluid dynamics, often satisfies the γ-law expression

$$P(\rho) = \frac{T}{\gamma}\rho^\gamma, \quad \rho \geq 0, \quad \gamma \geq 1$$

with the temperature $T > 0$ [10, 17]. Notice that the particle temperature is $T(\rho) = T\rho^{\gamma-1}$. Moreover, the nonlinear dispersive term

$$\frac{1}{4}\varepsilon^2\nabla\cdot\left(\rho\nabla^2\log\rho\right) = \frac{1}{2}\varepsilon^2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)$$

is produced by the gradient of the quantum Bohm potential

$$Q(\rho) = \frac{1}{2}\varepsilon^2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}},$$

which requires the strict positivity of density for the classical solution.

Recently, many efforts have been made on the existence of (steady-state or time-dependent) solutions of QHD (1.1)–(1.3). The existence and uniqueness of (classical) steady-state solutions to the QHD (1.1)–(1.3) for current density $J = 0$ (thermal equilibrium) has been studied in one dimensional and multi-dimensional bounded domains for density and electrostatic potential boundary conditions [1, 12]. The thermal equilibrium state of the bipolar isothermic model in a bounded domain was considered in [36]. The stationary QHD (1.1)–(1.3) for $J > 0$ (non-thermal equilibrium) has been considered in [9, 17, 38] for general monotone pressure functions, but, with different boundary conditions, i.e., Dirichlet data for the velocity potential S [17] or by using nonlinear boundary conditions [9, 38]. The existence of the one-dimensional steady-state solutions to (1.1)–(1.3) subject to boundary conditions on the density and the electrostatic potential has been proved in [16], for the case of a linear pressure function $P(\rho) = \rho$, and in [19] for general pressure functions $P(\rho)$. The local in-time existence of the classical solution was obtained in the one-dimensional bounded domain [20] (subject to boundary conditions on the density and the electrostatic potential). In this case additional boundedness restrictions on initial velocity were required to keep the strict positivity of density. The case of large initial data and the strictly convex pressure function in $\mathbb{R}^n$ has been investigated by [25] . In both of these cases, the classical solutions exist globally in time for initial data which are small perturbations of stationary states [20, 25] (which are time exponentially stable).

In the present paper we consider the initial value problem (1.1)–(1.4) for a *general, nonconvex* pressure function in multi-dimension, and we focus on the *local* existence of the classical solutions $(\rho, \mathbf{u}, V)$ of IVP (1.1)–(1.4) for regular *large* initial data, and their time-asymptotic convergence to an asymptotic state under small perturbation. We give a general framework to show the local in-time existence of classical solutions for a general (nonconvex) pressure density function and for regular large initial data. Then, we propose a (generic) "subsonic" condition to prove the global existence of the classical solutions in the "subsonic" region and investigate their large time behavior.

It is convenient to make use of the variable transformation $\rho = \psi^2$ in (1.1)–(1.4). Then, we derive the corresponding IVP for $(\psi, \mathbf{u}, V)$:

$$2\psi\cdot\partial_t\psi + \nabla\cdot(\psi^2\mathbf{u}) = 0, \tag{1.5}$$

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla h(\psi^2) + \frac{\mathbf{u}}{\tau} = \nabla V + \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\psi}{\psi}\right), \tag{1.6}$$

$$\Delta V = \psi^2 - \mathcal{C}, \tag{1.7}$$

$$\psi(x,0) = \psi_1(x) := \sqrt{\rho_1(x)},\ \mathbf{u}(x,0) = \mathbf{u}_1(x), \tag{1.8}$$

with $\rho h'(\rho) = P'(\rho)$. Note here the two problems (1.1)–(1.4) and (1.5)–(1.8) are equivalent for classical solutions. For simplicity in this paper we consider the initial value problem (1.5)–(1.8) on the multi-dimensional torus $\mathbb{T}^n$, with $\mathbb{T} = [0, L]$ and $L > 0$ representing the period length. The doping profile $\mathcal{C}$ is therefore assumed to be a periodic function and in the present paper is set to be a positive constant for mathematical simplicity. Because of the periodicity in the space variables, the solution of the Poisson equation is not unique since each combination of one solution and a constant is another solution. It is natural to consider the Poisson equation (1.7) in homogeneous Sobolev space. And by choosing an appropriate reference value of voltage, we can consider the Poisson equation (1.7) for V satisfying

$$\int_{\mathbb{T}^n} V(x,t)dx = 0, \qquad t \geq 0.$$

In analogy, the right hand side term of Eq. (1.7) is required to belong to the homogeneous Sobolev space, i.e.,

$$\int_{\mathbb{T}^n} (\psi^2 - \mathcal{C})(x,t)dx = 0, \quad t \geq 0.$$

This can be guaranteed due to the conservation (neutrality) of density (1.5) and neutrality assumption on the initial datum

$$\int_{\mathbb{T}^n} (\psi_1^2 - \mathcal{C})(x)dx = 0. \tag{1.9}$$

In the present paper we consider the problem (1.5)–(1.8) for irrotational (quantum) flow. We describe some ideas to prove both the local and the global existence and we investigate the large time behavior in the "subsonic" regime. The general situation for rotational flow is more complicated and it is expected to be investigated in a forthcoming paper.

The first result is the following local existence theorem:

Theorem 1.1. *Suppose* $P(\rho) \in C^5(0,+\infty)$. *Assume* $(\psi_1, \mathbf{u}_1) \in H^6(\mathbb{T}^n) \times H^5(\mathbb{T}^n)$ $(n = 2, 3)$ *satisfying* (1.9), $\nabla \times \mathbf{u}_1 = 0$, *and* $\min_{x\in[0,1]} \psi_1(x) > 0$. *Then, there exists* $T_{**} > 0$, *such that there exists a unique solution* $(\psi, \mathbf{u}, V)$ *to the IVP* (1.5)–(1.8), *with* $\psi > 0$, *which satisfies*

$$\psi \in C^i([0, T_{**}]; H^{6-2i}(\mathbb{T}^n)) \bigcap C^3([0, T_{**}]; L^2(\mathbb{T}^n)), \ \ i = 0, 1, 2;$$
$$\mathbf{u} \in C^i([0, T_{**}]; H^{5-2i}(\mathbb{T}^n)), \ i = 0, 1, 2; \quad V \in C^1([0, T_{**}]; \dot{H}^4(\mathbb{T}^n)).$$

Remark 1.2. The irrotationality assumption on the velocity vector fields $\mathbf{u}$ is consistent with Eq. (1.6), namely it keeps this property as long as it is true initially. This can be justified via standard arguments as used in the case of ideal fluids in classical hydrodynamics based on Kelvin's theorem and Stokes's theorem, see for instance [23] for details.

The proof of the above local-in-time existence is based on the construction of approximate solutions and the application of compactness arguments. The main difficulties are given by the following facts. The former arises since the general pressure $P(\rho)$ can be non-convex (even zero), then the left part of (1.5)–(1.7) (or (1.1)–(1.3) resp.) may not be hyperbolic anymore and we cannot apply the theory of quasilinear symmetric hyperbolic systems like [25] to obtain the local existence. The latter is given by the nonlinear dispersion term in (1.6), which requires the density ψ (or ρ resp.) to be strictly positive, for regular solutions. Hence we have to establish the local-in-time existence of solutions in a less traditional way.

Indeed we are going to construct approximate solutions and to prove the local in-time existence of classical solutions $(\mathbf{v}, \varphi, \psi, \mathbf{u}, V)$ for an extended system, which incorporates our problem, constructed in a suitable way based on (1.5)–(1.8). Note that in this new system, there are two additional equations for the variable $\mathbf{v}$, the artificial "velocity" (a sort of Lagrangian type velocity), and the artificial "density" $\varphi > 0$. The key point is that the local in-time existence of classical solutions for this extended system for the unknown $(\mathbf{v}, \varphi, \psi, \mathbf{u}, V)$ will be equivalent to the original one given by (1.5)–(1.8), when $\mathbf{v} = \mathbf{u}$ and $\psi = \varphi$ (see Sect. 3 for a proof in detail).

In order to extend the local-in-time solution globally in time, we will need uniform a-priori estimates, that can be proved by assuming the initial data close to the time-asymptotic (stationary) state $(\bar{\psi}, \bar{\mathbf{u}}, \bar{V})$. Actually it will be possible to extend globally, the local-in-time solutions, in the "subsonic" region (in the sense defined by (1.10) or (1.12) below); namely we will prove the global existence of the local-in-time solution when it starts near a stationary state $(\bar{\psi}, \bar{\mathbf{u}}, \bar{V})$ located in the so called "subsonic" region (this notion to be provided later in a more precise fashion).

The well-posedness of the stationary state $(\bar{\psi}, \bar{\mathbf{u}}, \bar{V})$ of the boundary value problem (1.5)–(1.7) subject to density and electrostatic potential boundary conditions was established in one dimension [19] for a general (nonconvex) pressure function $P(\rho)$, and was obtained for multi-dimensional irrotational flow [17] for a monotone enthalpy function where an additional boundary condition was imposed for the Fermi potential. The argument [17, 19] could be applied also here to obtain the existence of the stationary solution with periodic boundary conditions. However, since here we are focusing our attention only on the global existence, for simplicity we will bound ourselves to consider only the very special stationary state $(\bar{\psi}, \bar{\mathbf{u}}, \bar{V}) = (\sqrt{\mathcal{C}}, 0, 0)$ and study the situation when the initial data are assumed in a small neighborhood of the stationary solution $(\sqrt{\mathcal{C}}, 0, 0)$ to (1.5)–(1.7). Here note that the same argument can be applied to treat the more general case, see item (1) of Remark 1.4 and Theorem 1.5 below for an explanation in detail.

Theorem 1.3. *Let $P(\rho) \in C^5(0, +\infty)$ satisfying*

$$A_0 =: \frac{\pi^2}{L^2}\varepsilon^2 + P'(\mathcal{C}) > 0, \tag{1.10}$$

where $L > 0$ is the space period length. Let us assume $(\psi_1 - \sqrt{\mathcal{C}}, \mathbf{u}_1) \in H^6(\mathbb{T}^n) \times H^5(\mathbb{T}^n)$ $(n = 2, 3)$, the condition (1.9) and moreover $\nabla \times \mathbf{u}_1 = 0$. There exists $\eta > 0$ such that, if $\|\psi_1 - \sqrt{\mathcal{C}}\|_{H^6(\mathbb{T}^n)} + \|\mathbf{u}_1\|_{H^5(\mathbb{T}^n)} \le \eta$, the solution $(\psi, \mathbf{u}, V)$ of the IVP (1.5)–(1.8) exists globally in time and moreover one has

$$\|(\psi - \sqrt{\mathcal{C}})(t)\|^2_{H^6(\mathbb{T}^n)} + \|\mathbf{u}(t)\|^2_{H^5(\mathbb{T}^n)} + \|V(t)\|^2_{H^4(\mathbb{T}^n)} \le C\delta_0 e^{-\Lambda_0 t},$$

for all $t \ge 0$, where $C > 0$, $\Lambda_0 > 0$ are suitable constants, and

$$\delta_0 = \|\psi_1 - \sqrt{\mathcal{C}}\|^2_{H^6(\mathbb{T}^n)} + \|\mathbf{u}_1\|^2_{H^5(\mathbb{T}^n)}. \tag{1.11}$$

Remark 1.4. (1) Although in Theorem 1.3 we choose the special stationary state $(\sqrt{C}, 0, 0)$, we claim that the method used here can be applied to prove the time-asymptotic convergence toward any stationary state of (1.5)–(1.7) on the multi-dimensional torus $\mathbb{T}^n$, say $(\bar\psi, \bar{\mathbf{u}}, \bar V)$, with $\nabla\times\bar{\mathbf{u}} = 0$. Their well-posedness can be obtained by applying the arguments of [17], with suitable modifications. In this case, the corresponding "subsonic" condition has to be changed in the following way:

$$\frac{\pi^2}{L^2}\varepsilon^2 + P'(\bar\psi^2) > |\bar{\mathbf{u}}|^2. \tag{1.12}$$

(2) It is known that classical solutions of the hydrodynamical model for semiconductors (without dispersion term) for large initial data may blow up in finite time to form singularities [3]. Analogous results on the existence of the L^∞ solution and one or two dimensional transonic solutions for the hydrodynamical model for semiconductors was proven [6, 7]. However when dispersive regularity is involved in (1.10) or (1.12), it may prevent the formation of singularities, and classical solutions exist globally in time even in the transonic or supersonic region, in the classical sense [2, 4].
(3) Note here that the conditions (1.10) and (1.12) are exactly the subsonic conditions in the classical sense [2], when the re-scaled Planck constant ε goes to zero. If $\varepsilon > 0$ and $P'(\rho) > 0$, the "sound" speed $\tilde c(\bar\rho) = \sqrt{\pi^2\varepsilon^2/L^2 + P'(\bar\rho)}$ is bigger than the sound speed $c(\rho) = \sqrt{P'(\bar\rho)}$ for the classical hydrodynamic equations. □

Theorems 1.1–1.3 can be extended to the multi-dimensional torus $\mathbb{T}^n$, $n \geq 2$, for the IVP (1.5)–(1.8) with smooth initial data. Indeed, we have

Theorem 1.5. *Let* $P \in C^m(0,\infty)$*, with* $m \geq s-1$ *and* $s > \left[\frac{n}{2}\right] + 5$*. Let us assume that* $(\psi_1, \mathbf{u}_1) \in H^s(\mathbb{T}^n) \times H^{s-1}(\mathbb{T}^n)$*,* $\nabla\times\mathbf{u}_1 = 0$*, and* $\min_{x\in[0,1]}\psi_1(x) > 0$*, then, there exists* $T' > 0$ *such that a solution* $(\psi, \mathbf{u}, V)(t) \in H^s(\mathbb{T}^n)\times H^{s-1}(\mathbb{T}^n)\times H^{s-2}(\mathbb{T}^n)$ *of the IVP* (1.5)–(1.8)*, with* $\psi > 0$*, exists on* $[0, T']$.

Moreover, assume that $(\bar\psi, \bar{\mathbf{u}}, \bar V)$*, with* $\nabla\times\bar{\mathbf{u}} = 0$ *and* $\bar\psi > 0$*, is a classical stationary state of* (1.5)–(1.7) *with small oscillation and satisfies* (1.12)*. Then, if* $\|\psi_1 - \bar\psi\|_{H^s(\mathbb{T}^n)} + \|\mathbf{u}_1 - \bar{\mathbf{u}}\|_{H^{s-1}(\mathbb{T}^n)}$ *is sufficiently small, the solution* $(\psi, \mathbf{u}, V)(t)$ *of IVP* (1.5)–(1.8) *exists globally in time and satisfies*

$$\|(\psi-\bar\psi)(t)\|^2_{H^s(\mathbb{T}^n)} + \|(\mathbf{u}-\bar{\mathbf{u}})(t)\|^2_{H^{s-1}(\mathbb{T}^n)} + \|(V-\bar V)(t)\|^2_{H^{s-2}(\mathbb{T}^n)} \leq C\delta_1 e^{-\Lambda_2 t},$$

with $\Lambda_2 > 0$ *and*

$$\delta_1 = \|(\psi_1 - \bar\psi)\|^2_{H^s(\mathbb{T}^n)} + \|(\mathbf{u}_1 - \bar{\mathbf{u}})\|^2_{H^{s-1}(\mathbb{T}^n)}.$$

Remark 1.6. Once we prove the local existence (resp. global existence) of solutions $(\psi, \mathbf{u}, V)$ of IVP (1.5)–(1.8), we can obtain the local existence (resp. global existence) of solutions $(\rho, \mathbf{u}, V)$ of IVP (1.1)–(1.4) by setting $\rho = \psi^2$. □

This paper is organized in the following way. In Sect. 2, we present preliminary results on the divergence equation, Poisson equation, and a fourth order semilinear wave type equation on $\mathbb{T}^n$, then we list some known calculus inequalities. We prove Theorem 1.1 in Sect. 3. After the construction of our extended system in Sect. 3.1, we show the construction of the approximate solutions, we derive the uniform estimates, and we prove Theorem 1.1 in Sect. 3.2. Section 4 is concerned with the proof of Theorem 1.3. After the reformulation of original problem in Sect. 4.1, we establish the a-priori estimates on the local solutions in Sect. 4.2, and prove the global existence and the large time behavior in the remaining part.

Notation. C always denotes the generic positive constant. $L^2(\mathbb{T}^n)$ is the space of square integral functions on $\mathbb{T}^n$ with the norm $\|\cdot\|$. $H^k(\mathbb{T}^n)$ with integer $k \geq 1$ denotes the usual Sobolev space of function f, satisfying $\partial_x^i f \in L^2$ $(0 \leq i \leq k)$, with norm

$$\|f\|_k = \sqrt{\sum_{0\leq|l|\leq m} \|D^l f\|^2},$$

here and after $D^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$ for $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\partial_j = \partial_{x_j}$, $j = 1, 2, ..., n$, for abbreviation. In particular, $\|\cdot\|_0 = \|\cdot\|$. $\dot{H}^k(\mathbb{T}^n)$ denotes the subspace of function in $H^k(\Omega)$ satisfying

$$\int_\Omega u(x)\,dx = 0.$$

Let $T > 0$ and let $\mathcal{B}$ be a Banach space. $C^k(0, T; \mathcal{B})$ ($C^k([0, T]; \mathcal{B})$ resp.) denotes the space of $\mathcal{B}$-valued k-times continuously differentiable functions on $(0, T)$ (or $[0, T]$ resp.), $L^2([0, T];\ \mathcal{B})$ the space of $\mathcal{B}$-valued L^2-functions on $[0, T]$, and $H^k([0, T]; \mathcal{B})$ the spaces of functions f, such that $\partial_t^i f \in L^2([0, T]; \mathcal{B})$, $1 \leq i \leq k,\ 1 \leq p \leq \infty$.

2. Preliminaries

In this section, we prove the existence and uniqueness of solutions of the divergence equation on $\mathbb{T}^n$ and we recall a known result on the multi-dimensional Poisson equation with periodic boundary conditions. Then, we turn to prove the well-posedness for an abstract second order semi-linear evolution equation. Finally, some calculus inequalities are listed without proof.

First, we have the following theorem on the divergence operator and Laplace operator on $\mathbb{T}^n$:

Theorem 2.1. *Let $f \in \dot{H}^s(\mathbb{T}^n)$, $s \geq 0$. There exists a unique solution $u \in \left(H^{s+1}(\mathbb{T}^n)\right)^n$ satisfying*

$$\nabla\cdot \mathbf{u} = f, \quad \nabla\times\mathbf{u} = 0, \quad \int_{\mathbb{T}^n} (\mathbf{u} - \hat{u})\,dx = 0, \tag{2.1}$$

and

$$\|(\mathbf{u} - \hat{u})\|_{H^{s+1}(\mathbb{T}^n)} \leq c_1\|f\|_{\dot{H}^s(\mathbb{T}^n)}, \tag{2.2}$$

where $c_1 > 0$ is a suitable constant and $\hat{u}$ a vector in $\mathbb{R}^n$.

Theorem 2.2. *Let $f \in \dot{H}^s(\mathbb{T}^n)$, $s \geq 0$. There exists a unique solution $u \in \dot{H}^{s+2}(\mathbb{T}^n)$ to the Poisson equation*

$$\Delta u = f$$

satisfying

$$\|u\|_{\dot{H}^{s+2}(\mathbb{T}^n)} \leq c_2\|f\|_{\dot{H}^s(\mathbb{T}^n)} \tag{2.3}$$

with $c_2 > 0$.

The proofs of Theorems 2.1–2.2 can be completed with the help of the Fourier series expansion of the functions $\mathbf{u}$, u and f. Here we omit the details. □

Based on Theorem 2.2, we obtain the initial potential V_1 through (1.7) in view of the initial density:

$$\Delta V_1 = \psi_1^2 - \mathcal{C}, \quad \int_{\mathbb{T}^n} V_1(x)dx = 0. \tag{2.4}$$

By (1.9) and $\psi_1 - \sqrt{\mathcal{C}} \in H^3$, we obtain that $V_1 \in \dot{H}^5$ and satisfies

$$\|V_1\|_{\dot{H}^5(\mathbb{T}^n)} \leq c_3\|\psi_1^2 - \mathcal{C}\|_{\dot{H}^3(\mathbb{T}^n)} \leq c_4\|\psi_1 - \sqrt{\mathcal{C}}\|_{H^3(\mathbb{T}^n)}, \tag{2.5}$$

with $c_3, c_4 > 0$ constants.

Finally, let us consider the abstract initial value problem in the periodic Hilbert space $L^2(\mathbb{T}^n)$:

$$u'' + \frac{1}{\tau}u' + Au + \mathcal{L}u' = F(t), \tag{2.6}$$

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2.7}$$

Hereafter u' denotes $\frac{du}{dt}$. The operator A is defined by

$$Au = \nu_0\Delta^2 u + \nu_1 u, \tag{2.8}$$

where Δ is the Laplacian operator on $\mathbb{R}^n$, and $\nu_0, \nu_1 > 0$ are given constants. The domain of the linear operator A is $D(A) = H^4(\mathbb{T}^n)$. Related to the operator A, we define a continuous and symmetric bilinear form $a(u, v)$ on $H^2(\mathbb{T}^n)$,

$$a(u, v) = \int_{\mathbb{T}^n} (\nu_0 \Delta u \Delta v + \nu_1 uv)dx, \quad \forall\, u, v \in H^2(\mathbb{T}^n), \tag{2.9}$$

which is coercive, i.e.,

$$\exists\, \nu > 0, \quad a(u, u) \geq \nu\|u\|_{H^2(\mathbb{T}^n)}, \quad \forall\, u \in H^2(\mathbb{T}^n). \tag{2.10}$$

This means that there is a complete orthogonal family $\{r_l\}_{l\in N}$ of $L^2(\mathbb{T}^n)$ and a family $\{\mu_l\}_{l\in N}$ consisting of the eigenvectors and eigenvalues of operator A

$$\begin{aligned} &Ar_l = \mu_l r_l, \quad l = 1, 2, \cdots, \\ &0 < \mu_1 \leq \mu_2, \cdots, \quad \mu_l \to \infty \text{ as } l \to \infty. \end{aligned} \tag{2.11}$$

The family $\{r_l\}_{l\in N}$ is also orthogonal for $a(u, v)$ on $H^2(\mathbb{T}^n)$, i.e.,

$$a(r_l, r_j) =< Ar_l, r_j >= \mu_l(r_l, r_j) = \mu_l\delta_{lj}, \quad \forall\, l, j,$$

where δ_{lj} denotes the Kronecker symbol.

Related to $\mathcal{L}u$ and $F(t)$, we have

$$< \mathcal{L}u, v >= \int_{\mathbb{T}^n} (b(x, t)\cdot\nabla u)v dx, \quad u,\ v \in H^2(\mathbb{T}^n), \tag{2.12}$$

$$< F(t), v >= \int_{\mathbb{T}^n} f(x, t)v dx, \quad v \in H^2(\mathbb{T}^n), \tag{2.13}$$

where $b : \mathbb{T} \times [0, T] \to \mathbb{R}^n$ and $f : \mathbb{T} \times [0, T] \to \mathbb{R}$ are measurable functions.

By applying the Faedo-Galerkin method [35, 39], we can obtain the existence of solutions to (2.6)–(2.7) in a standard way.

Theorem 2.3. *Let* $T > 0$*,* $n = 2, 3$*, and assume that*

$$F \in H^1([0, T]; L^2(\mathbb{T}^n)), \quad b \in L^2([0, T]; H^3(\mathbb{T}^n)) \bigcap H^1([0, T]; H^2(\mathbb{T}^n)). \quad (2.14)$$

Then, if $u_0 \in H^4(\mathbb{T}^n)$ *and* $u_1 \in H^2(\mathbb{T}^n)$*, the solution to* (2.6)–(2.7) *exists and satisfies*

$$u \in C^i([0, T]; H^{4-2j}(\mathbb{T}^n)),\ j = 0, 1, \quad u'' \in L^\infty([0, T]; L^2(\mathbb{T}^n)). \quad (2.15)$$

Moreover, assume that

$$F',\ F \in L^2([0, T]; H^2(\mathbb{T}^n)), \quad (2.16)$$

then, if $u_0 \in H^6(\mathbb{T}^n)$ *and* $u_1 \in H^4(\mathbb{T}^n)$*, it follows*

$$u \in C^i([0, T]; H^{6-2j}(\mathbb{T}^n)),\ j = 0, 1, 2, \quad u''' \in L^\infty([0, T]; L^2(\mathbb{T}^n)). \quad (2.17)$$

Proof. The statement (2.17) follows from (2.15), if we consider the same type of problem for new unknown $v = \mathrm{D}^2 u$. The statement (2.15) can be proved by applying the Faedo-Galerkin method. We omit the details here since everything is quite standard. For general stability theory of abstract second order equations, the reader can refer to [29, 30]. □

Remark 2.4. Note that if (2.14) is replaced by

$$F \in C^1([0, T]; L^2(\mathbb{T}^n)), \quad b \in C^i[0, T]; H^{3-i}(\mathbb{T}^n)),\ i = 0, 1, \quad (2.18)$$

then in (2.15) it follows

$$u'' \in C([0, T]; L^2(\mathbb{T}^n)).$$

Furthermore, when (2.16) is replaced by

$$F \in C^1([0, T]; H^2(\mathbb{T}^n)), \quad (2.19)$$

it also holds in (2.17) that

$$u''' \in C([0, T]; L^2(\mathbb{T}^n)).$$

Finally, we list below the Moser-type calculus inequalities [22, 28, 34]:

Lemma 2.5. *Let* $f, g \in L^\infty(\mathbb{T}^n) \bigcap H^s(\mathbb{T}^n)$*. Then, it follows*

$$\|D^\alpha(fg)\| \le C\|g\|_{L^\infty}\|D^\alpha f\| + C\|f\|_{L^\infty}\|D^\alpha g\|, \quad (2.20)$$

$$\|D^\alpha(fg) - fD^\alpha g\| \le C\|g\|_{L^\infty}\|D^\alpha f\| + C\|f\|_{L^\infty}\|D^{\alpha-1} g\|, \quad (2.21)$$

for $1 \le |\alpha| \le s$.

3. Local Existence

This section is concerned with the proof of Theorem 1.1. We construct the new extended system based on (1.5)–(1.8) in Sect. 3.1, then we build up the approximate solutions, derive the uniform estimates, and prove Theorem 1.1 in Sect. 3.2. For simplicity, we set $\tau = 1$.

3.1. Construction of the extended system. We construct the extended system in this subsection. For irrotational flow, the velocity field can be represented as the gradient field of a phase function S:

$$\mathbf{u} = \nabla S. \tag{3.1}$$

In analogy, the continuous equation (1.6) for the irrotational velocity vector field $\mathbf{u}$ is changed into

$$\partial_t \mathbf{u} + \frac{1}{2}\nabla(|\mathbf{u}|^2) + \nabla h(\psi^2) + \mathbf{u} = \nabla V + \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\psi}{\psi}\right), \tag{3.2}$$

which, together with the initial data $\mathbf{u}(x,0) = \mathbf{u}_1(x)$, provides the time-decay of mean velocity on $\mathbb{T}^n$:

$$\int_{\mathbb{T}^n} \mathbf{u}(x,t)dx = \bar{\mathbf{u}}(t) =: e^{-t}\int_{\mathbb{T}^n} \mathbf{u}_1(x)dx, \quad t \geq 0. \tag{3.3}$$

For $\psi > 0$ Eq. (1.5) becomes

$$2\partial_t\psi + 2\mathbf{u}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{u} = 0. \tag{3.4}$$

We want to explain the main steps that we will use in the next subsection to implement an iterative procedure. Once we know $\mathbf{u}$ and ψ based on Eq. (3.4) and the previous observation, we introduce two new equations for the artificial "velocity" $\mathbf{v}$ and artificial "density" $\varphi > 0$,

$$\nabla\cdot\mathbf{v} = -\frac{2(\partial_t\psi + \mathbf{u}\cdot\nabla\psi)}{\varphi}, \quad \nabla\times\mathbf{v} = 0, \quad \int_{\mathbb{T}^n} \mathbf{v}(x,t)dx = \bar{\mathbf{u}}(t), \tag{3.5}$$

$$\partial_t\varphi + \frac{1}{2}\varphi\nabla\cdot\mathbf{v} + \mathbf{u}\cdot\nabla\psi = 0, \quad \varphi(x,0) = \psi_1(x) > 0. \tag{3.6}$$

Clearly to re-initialize the procedure, we have to determine ψ and $\mathbf{u}$ as long as we know φ and $\mathbf{v}$ (we will propose the corresponding equations, used in the next subsection, for ψ and $\mathbf{u}$ below). By a simple combination of Eqs. (3.5)–(3.6), we obtain

$$\partial_t[\varphi - \psi](x,t) = 0, \quad \forall\, x \in \mathbb{T}^n,$$

which implies

$$[\varphi - \psi](x,t) = 0 \text{ for } (x,t) \in \mathbb{T}^n \times (0,\infty), \quad \text{if } [\varphi - \psi](x,0) = 0. \tag{3.7}$$

By applying to (3.6) a standard argument in the theory of O.D.E. namely by multiplying Eq. (3.6) by the function $\exp\{\frac{1}{2}\int_0^t \nabla\cdot\mathbf{v}(x,s)ds\}$ and by integrating the resulting equation with respect to time, we can represent φ for $(x,t) \in \mathbb{T}^n \times [0,+\infty)$ by the identity

$$\varphi(x,t) = \psi_1(x)e^{-\frac{1}{2}\int_0^t \nabla\cdot\mathbf{v}(x,s)ds} - \int_0^t \mathbf{u}\cdot\nabla\psi(x,s)e^{-\frac{1}{2}\int_s^t \nabla\cdot\mathbf{v}(x,\xi)d\xi}ds. \tag{3.8}$$

This means that for short time (smooth) solutions (if they exist) satisfy

$$\varphi(x,t) > 0, \quad \text{if} \quad \psi_1(x) > 0, \quad x \in \mathbb{T}^n.$$

Based on Eq. (3.4) and Eq. (3.2), we show how to reconstruct the density ψ. Here we use the following second order evolutional problem:

$$\psi_{tt} + \psi_t + \frac{1}{4}\varepsilon^2\Delta^2\psi - \frac{1}{4}\varepsilon^2\frac{|\Delta\psi|^2}{\varphi} - \frac{1}{2\varphi}\Delta P(\psi^2) + \frac{1}{2}\psi\Delta V + \nabla\psi\cdot\nabla V$$
$$+ (\mathbf{u}+\mathbf{v})\cdot\nabla\psi_t - \frac{1}{2}\nabla\psi\cdot\nabla(|\mathbf{v}|^2) - \frac{1}{2}\psi\nabla\mathbf{v}:\nabla\mathbf{v} + \mathbf{v}\cdot\nabla(\mathbf{u}\cdot\nabla\psi)$$
$$- \frac{1}{\varphi}(\psi_t + \mathbf{u}\cdot\nabla\psi)(\mathbf{v}\cdot\nabla\psi) - \frac{\psi_t}{\varphi}(\psi_t + \mathbf{u}\cdot\nabla\psi) = 0, \quad (3.9)$$

with initial data

$$\psi(x,0) = \psi_1,\ \psi_t(x,0) = \psi_0 =: -\frac{1}{2}\psi_1\nabla\cdot\mathbf{u}_1 - \mathbf{u}\cdot\nabla\psi_1, \quad (3.10)$$

where $\mathbf{v} = (v^1, v^2, ..., v^n)$ and

$$\nabla\mathbf{v}:\nabla\mathbf{v} = \sum_{i,j}|\partial_j v^i|^2.$$

Indeed, let us multiply (3.2) by ψ^2, take divergence of the resulting equation, then use (3.4), the irrotationality assumption of velocity vector fields plus the relation

$$\nabla\cdot\left[\psi^2\nabla\left(\frac{\Delta\psi}{\psi}\right)\right] = \psi\left[\Delta^2\psi - \frac{|\Delta\psi|^2}{\psi}\right],$$

replace the nonlinear term $\frac{1}{4\psi}\nabla\cdot(\psi^2\nabla(|\mathbf{u}|^2))$ by

$$\frac{1}{2}\nabla\psi\cdot\nabla(|\mathbf{v}|^2) + \frac{1}{2}\psi\nabla\mathbf{v}:\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\psi_t - (\mathbf{v}\cdot\nabla)(\mathbf{u}\cdot\nabla\psi) + \frac{1}{\psi}(\psi_t + \mathbf{u}\cdot\nabla\psi)(\mathbf{v}\cdot\nabla\psi),$$

and finally replace $\frac{1}{\psi}$ in the resulting equation by $\frac{1}{\varphi}$; we get Eq. (3.9).

Similarly we can construct from (3.2) the equation for reconstructing the velocity $\mathbf{u}$,

$$\partial_t\mathbf{u} + \mathbf{u} + \frac{1}{2}\nabla(|\mathbf{v}|^2) + \nabla h(\psi^2) = \nabla V + \frac{\varepsilon^2}{2}\left(\frac{\nabla\Delta\psi}{\varphi} - \frac{\Delta\psi\nabla\psi}{\varphi^2}\right), \quad \mathbf{u}(x,0) = \mathbf{u}_1(x). \quad (3.11)$$

Here we have used the identity

$$\nabla\left(\frac{\Delta\psi}{\psi}\right) = \left(\frac{\nabla\Delta\psi}{\psi} - \frac{\Delta\psi\nabla\psi}{\psi^2}\right), \quad (3.12)$$

and we replaced $\frac{1}{\psi}$ and $|\mathbf{u}|^2$ by $\frac{1}{\varphi}$ and $|\mathbf{v}|^2$ respectively in (3.2).

Finally, from (1.7) the reconstruction of V is done directly by using the Poisson equation on $\mathbb{T}^n$ and involves only ψ:

$$\Delta V = \psi^2 - \mathcal{C} - \frac{1}{L^n}\int_{\mathbb{T}^n}(\psi^2 - \mathcal{C})(x,t)dx, \quad \int_{\mathbb{T}^n} V(x,t)dx = 0. \quad (3.13)$$

So far, we have constructed an extended coupled and closed system for the new unknown $U = (\mathbf{v}, \varphi, \psi, \mathbf{u}, V)$, which consists of two O.D.E.s (3.6) for φ and (3.11) for

$\mathbf{u}$, a second order evolutional equation (3.9) for ψ, a divergence equation (3.5) for $\mathbf{v}$, and an elliptic equation (3.13) for V. The most important fact (which we will be able to show later on) is to note that this extended system for $U = (\mathbf{v}, \varphi, \psi, \mathbf{u}, V)$ is equivalent to the original equations (1.5)–(1.7) of $(\psi, \mathbf{u}, V)$, as far as we look for classical solutions, when $\mathbf{u} = \mathbf{v}$ and $\psi = \varphi > 0$.

3.2. Iteration scheme and local existence. Now, we consider the corresponding problem for an approximate solution $\{U^i\}_{i=1}^{\infty}$ with $U^p = (\mathbf{v}_p, \varphi_p, \psi_p, \mathbf{u}_p, V_p)$ based on the extended system constructed in Subsect. (3.1). The iteration scheme for the approximate solution $U^{p+1} = (\mathbf{v}_{p+1}, \varphi_{p+1}, \psi_{p+1}, \mathbf{u}_{p+1}, V_{p+1})$, $p \geq 1$, is defined by solving the following problems on $\mathbb{T}^n$:

$$\nabla\cdot \mathbf{v}_{p+1} = r_p(t), \quad \nabla\times\mathbf{v}_{p+1} = 0, \ \int_{\mathbb{T}^n} \mathbf{v}_{p+1}(x,t)\, dx = \bar{\mathbf{u}}(t), \tag{3.14}$$

$$\begin{cases} \varphi'_{p+1} + \frac{1}{2}(\nabla\cdot \mathbf{v}_p)\varphi_{p+1} + \mathbf{u}_p\cdot\nabla\psi_p = 0, \quad t > 0, \\ \varphi_{p+1}(x,0) = \psi_1(x), \end{cases} \tag{3.15}$$

$$\begin{cases} \psi''_{p+1} + \psi'_{p+1} + \nu\Delta^2\psi_{p+1} + \nu\psi_{p+1} + k_p(t)\cdot\nabla\psi'_{p+1} = h_p(t), \quad t > 0, \\ \psi_{p+1}(x,0) = \psi_1(x),\ \psi'_{p+1}(x,0) = \psi_0 =: -\frac{1}{2}\psi_1\nabla\cdot \mathbf{u}_1 - \mathbf{u}\cdot\nabla\psi_1, \end{cases} \tag{3.16}$$

$$\begin{cases} \mathbf{u}'_{p+1} + \mathbf{u}_{p+1} = g_p(t), \quad t > 0, \\ \mathbf{u}_{p+1}(0) = \mathbf{u}_1,\ \nabla\times\mathbf{u}_1 = 0, \end{cases} \tag{3.17}$$

$$\Delta V_{p+1} = q_p(t), \quad \int_{\mathbb{T}^n} V_{p+1}(x,t)dx = 0, \tag{3.18}$$

where $\nu = \frac{1}{4}\varepsilon^2$, and

$$r_p(t) = r_p(x,t) = -\frac{2(\psi'_p + \mathbf{u}_p\cdot\nabla\psi_p)}{\varphi_p} + \frac{1}{L^n}\int_{\mathbb{T}^n} \frac{2(\psi'_p + \mathbf{u}_p\cdot\nabla\psi_p)}{\varphi_p}(x,t)dx, \tag{3.19}$$

$$k_p(t) = k_p(x,t) = \mathbf{u}_p(x,t) + \mathbf{v}_p(x,t), \tag{3.20}$$

$$\begin{aligned} h_p(t) = h_p(x,t) =& \frac{|\psi'_p|^2}{\varphi_p} + \frac{\psi'_p}{\varphi_p}\mathbf{u}_p\cdot\nabla\psi_p + \frac{\varepsilon^2}{4}\frac{|\Delta\psi_p|^2}{\varphi_p} - \frac{1}{2}\psi_p\Delta V_p - \nabla\psi_p\cdot\nabla V_p \\ &+ \frac{1}{2}\frac{\Delta P(\psi_p^2)}{\varphi_p} + \nu\psi_p + \frac{1}{2}\nabla\psi_p\cdot\nabla(|\mathbf{v}_p|^2) + \frac{1}{2}\psi_p\sum_{j,i}|\partial_j v_p^i|^2 \\ &- \mathbf{v}_p\cdot\nabla\left(\mathbf{u}_p\cdot\nabla\psi_p\right) + \frac{1}{\varphi_p}\left(\psi'_p + \mathbf{u}_p\cdot\nabla\psi_p\right)\mathbf{v}_p\cdot\nabla\psi_p, \end{aligned} \tag{3.21}$$

$$g_p(t) = g_p(x,t) = \nabla V_p - \frac{1}{2}\nabla\left(\left|\mathbf{v}_p\right|^2\right) - \nabla h(\psi_p^2) + \frac{1}{2}\varepsilon^2\left(\frac{\nabla\Delta\psi_p}{\varphi_p} - \frac{(\Delta\psi_p)\nabla\psi_p}{\varphi_p^2}\right), \tag{3.22}$$

$$q_p(t) = q_p(x,t) = \psi_p^2 - \mathcal{C} - \frac{1}{L^n}\int_{\mathbb{T}^n}(\psi_p^2 - \mathcal{C})(x,t)dx, \tag{3.23}$$

where $\mathbf{u}_p = (u_p^1, u_p^2, \cdots, u_p^n)$ and $\mathbf{v}_p = (v_p^1, v_p^2, \cdots, v_p^n)$.

Let us emphasize that here the functions $r_p(0), k_p(0), h_p(0), g_p(0), q_p(0)$ depend only upon the initial data $(\psi_1, \mathbf{u}_1)$ and moreover they are periodic in the space variables.

The main result in this section is the following concerning "a-priori estimates".

Lemma 3.1. *Let us assume that* $P \in C^5(0, \infty)$ *and* $(\psi_1, \mathbf{u}_1) \in H^6 \times H^5$, $\nabla\times\mathbf{u}_1 = 0$, *such that*

$$\psi^* = \max_{x\in\mathbb{T}^n} \psi_1(x), \quad \psi_* =: \min_{x\in\mathbb{T}^n} \psi_1(x) > 0. \tag{3.24}$$

Then, there exist a positive time T_* *and a sequence* $\{U^p\}_{p=1}^{\infty}$ *of approximate solutions, which solve the system* (3.14)–(3.18) *for* $t \in [0, T_*]$ *and satisfy*

$$\begin{cases} \mathbf{v}_p \in C^j([0, T_*]; H^{4-j}(\mathbb{T}^n)) \bigcap C^2([0, T_*]; H^1(\mathbb{T}^n)), \quad j = 0, 1, \\ \varphi_p \in C^1([0, T_*]; H^3(\mathbb{T}^n)) \bigcap C^2([0, T_*]; H^2(\mathbb{T}^n)) \bigcap C^3([0, T_*]; L^2(\mathbb{T}^n)), \\ \psi_p \in C^l([0, T_*]; H^{6-2l}(\mathbb{T}^n)) \bigcap C^3([0, T_*]; L^2(\mathbb{T}^n)), \quad l = 0, 1, 2, \\ \mathbf{u}_p \in C^1([0, T_*]; H^3(\mathbb{T}^n)) \bigcap C^2([0, T_*]; H^1(\mathbb{T}^n)), \\ V_p \in C([0, T_*]; \dot{H}^4(\mathbb{T}^n)) \bigcap C^1([0, T_*]; \dot{H}^4(\mathbb{T}^n)). \end{cases} \tag{3.25}$$

Moreover, there is a positive constant M_* *so that for all* $t \in [0, T_*]$, *we have*

$$\begin{cases} \|(\mathbf{u}_p, \mathbf{u}'_p)(t)\|_3^2 + \|(\mathbf{u}''_p, \mathbf{v}''_p)(t)\|_1^2 + \|\mathbf{v}_p(t)\|_4^2 + \|\mathbf{v}'_p(t)\|_3^2 + \|(V_p, V'_p)(t)\|_4^2 \le M_*, \\ \|(\psi_p, \psi'_p, \psi''_p, \psi'''_p)(t)\|^2_{H^6\times H^4\times H^2\times L^2} + \|(\varphi_p, \varphi'_p, \varphi''_p, \varphi'''_p)(t)\|^2_{H^3\times H^3\times H^2\times L^2} \le M_*, \end{cases} \tag{3.26}$$

uniformly with respect to $p \ge 1$.

Proof. Step 1: Estimates for $p=1$. Obviously, $U^1 = (\mathbf{u}_1(x), \psi_1(x), \psi_1(x), \mathbf{u}_1(x), V_1(x))$ satisfies (3.25)–(3.26) for the time interval [0, 1] with M_* replaced by some constant $B_1 > 0$ and V_1 determined by (2.4).

We start the iterative process with $U^1 = (\mathbf{u}_1, \psi_1, \psi_1, \mathbf{u}_1, V_1)$; then by solving the problems (3.14)–(3.18) for $p = 1$, we can prove the (local in time) existence of a solution $U^2 = (\mathbf{v}_2, \psi_2, \varphi_2, \mathbf{u}_2, V_2)$ which also satisfies (3.25)–(3.26) for a time interval (which without loss of generality is chosen to be [0, 1] since we focus on local in-time existence of solutions) and with M_* replaced by another constant $B_2 > 0$. In fact, for $U^1 = (\mathbf{u}_1, \psi_1, \psi_1, \mathbf{u}_1, V_1)$ the functions r_1, k_1, h_1, g_1, q_1 depend only on the initial data $(\psi_1, \mathbf{u}_1)$, i.e.,

$$r_1(x,t) = \tilde{r}_1(x),\ k_1(x,t) = \tilde{k}_1(x),\ h_1(x,t) = \tilde{h}_1(x),$$
$$g_1(x,t) = \tilde{g}_1(x),\ q_1(x,t) = \tilde{q}_1(x),$$

and

$$\|\tilde{r}_1\|_2^2 + \|\tilde{k}_1\|_3^2 + \|\tilde{h}_1\|_3^2 + \|\tilde{g}_1\|_3^2 + \|\tilde{q}_1\|_2^2 \le N a_0 I_0^4 e^{N\|\mathbf{u}_1\|_3}. \tag{3.27}$$

From now on, $N > 0$ denotes a generic constant independent of U^p, $p \ge 1$,

$$a_0 = \frac{(1+\psi^*)^m}{\psi_*^m}, \qquad \text{for a integer } m \ge 10, \tag{3.28}$$

and

$$I_0 = \|(\psi_1 - \sqrt{\mathcal{C}})\|^2 + \|\nabla\psi_1\|_5^2 + \|\mathbf{u}_1\|_5^2. \tag{3.29}$$

The system (3.14)–(3.18) with $p = 1$ is linear on the unknown $U^2 = (\mathbf{v}_2, \psi_2, \varphi_2, \mathbf{u}_2, V_2)$, therefore it can be solved based on the estimates (3.27) for the corresponding right-hand side terms as follows. Namely, by Theorem 2.1, we obtain the existence of the solution $\mathbf{v}_2$ to the divergence equation (3.14), with $r_1(x,t)$ replaced by $\tilde{r}_1(x)$, satisfying

$$\mathbf{v}_2 \in C^j([0,1]; H^{4-j}(\mathbb{T}^n)) \bigcap C^2([0,1]; H^1(\mathbb{T}^n)), \quad j = 0, 1.$$

Then by making use of the theory of the linear O.D.E. system, we prove the existence of the solution $\mathbf{u}_2$ of (3.17) for $g_1(x,t) = \tilde{g}_1(x)$ and then φ_2 of (3.15):

$$\mathbf{u}_2 \in C^1([0,1]; H^3(\mathbb{T}^n)) \bigcap C^2([0,1]; H^1(\mathbb{T}^n)),$$
$$\varphi_2 \in C^1([0,1]; H^3(\mathbb{T}^n)) \bigcap C^2([0,1]; H^2(\mathbb{T}^n)) \bigcap C^3([0,1]; L^2(\mathbb{T}^n)).$$

By applying Theorem 2.3 to (3.16), with $b(x,t) = 2\mathbf{u}_1(x)$ in (2.12) and $f(x,t) = \tilde{h}_1(x)$ in (2.13), we obtain the existence of a solution ψ_2 satisfying

$$\psi_2 \in C^j([0,1]; H^{6-2j}(\mathbb{T}^n)) \bigcap C^3([0,1]; L^2(\mathbb{T}^n)), \ j = 0, 1, 2.$$

Finally, the existence of a solution V_2 satisfying

$$V_2 \in C^1([0,1]; \dot{H}^4(\mathbb{T}^n))$$

follows from the application of Theorem 2.2 to Eq. (3.18) on $\mathbb{T}^n$, with $q_1(x,t)$ replaced by $\tilde{q}_1(x)$.

Moreover, based on the estimates (3.27), we conclude there is a constant $B_2 > 0$, such that U^2 satisfies

$$\begin{cases} \|(\mathbf{u}_2, \mathbf{u}_2')(t)\|_3^2 + \|(\mathbf{u}_2'', \mathbf{v}_2'')(t)\|_1^2 + \|\mathbf{v}_2(t)\|_4^2 + \|\mathbf{v}_2'(t)\|_3^2 + \|(V_2, V_2')(t)\|_4^2 \le B_2, \\ \|(\varphi_2, \varphi_2', \varphi_2'', \varphi_2''')(t)\|^2_{H^6\times H^3\times H^2\times L^2} + \|(\psi_2, \psi_2', \psi_2'', \psi_2''')(t)\|^2_{H^6\times H^4\times H^2\times L^2} \le B_2, \end{cases}$$

for all $t \in [0,1]$.

Step 2: Estimates for $p \ge 2$. Now, assume that $\{U^i\}_{i=1}^p$ ($p \ge 2$) exist in the time interval $[0,1]$, solve the system (3.14)–(3.18), and satisfy (3.25)–(3.26), with M_* replaced by the max B_p ($\ge \max_{1\le j\le p-1}\{B_j\}$). For given U^p, the system (3.17)–(3.18) is linear in $U^{p+1} = (\mathbf{v}_{p+1}, \varphi_{p+1}, \psi_{p+1}, \mathbf{u}_{p+1}, V_{p+1})$. As before, we apply Theorem 2.1 to Eq. (3.14) for $\mathbf{v}_{p+1}$, the theory of linear O.D.E. systems to Eq. (3.15) for φ_{p+1} and Eq. (3.17) for $\mathbf{u}_{p+1}$, Theorem 2.3 to wave type equation (3.16) for ψ_{p+1} with $f(x,t) = h_p(t)$ and $b(x,t) = k_p(t)$, and Theorem 2.2 to the Poisson equation (3.18) for V_{p+1}. Therefore we obtain the existence of $U^{p+1} = (\mathbf{v}_{p+1}, \psi_{p+1}, \varphi_{p+1}, \mathbf{u}_{p+1}, V_{p+1})$ on the time interval $[0,1]$ and moreover it follows:

$$\begin{cases} \mathbf{v}_{p+1} \in C^j([0,1]; H^{4-j}(\mathbb{T}^n)) \bigcap C^2([0,1]; H^1(\mathbb{T}^n)), \ j = 0, 1, \\ \varphi_{p+1} \in C^1([0,1]; H^3(\mathbb{T}^n)) \bigcap C^2([0,1]; H^2(\mathbb{T}^n)) \bigcap C^3([0,1]; L^2(\mathbb{T}^n)), \\ \psi_{p+1} \in C^j([0,1]; H^{6-2j}(\mathbb{T}^n)) \bigcap C^3([0,1]; L^2(\mathbb{T}^n)), \ j = 0, 1, 2, \\ \mathbf{u}_{p+1} \in C^1([0,1]; H^3(\mathbb{T}^n)) \bigcap C^2([0,1]; H^1(\mathbb{T}^n)), \\ V_{p+1} \in C([0,1]; \dot{H}^4(\mathbb{T}^n)) \bigcap C^1([0,1]; \dot{H}^4(\mathbb{T}^n)). \end{cases}$$

Now our goal is to deduce uniform bounds for U^{j+1}, $1 \le j \le p$, for some time interval. Let us first estimate the L^2 norms of the initial value of $\psi_{p+1}, \psi'_{p+1}, \psi''_{p+1}$, where the initial value Ψ_0 of ψ''_{p+1} is obtained through $(3.16)_1$ at $t = 0$, where ψ_{p+1} and ψ'_{p+1} are replaced by the initial data ψ_1, ψ_0:

$$\Psi_0 = -\psi_0 - \nu\Delta^2\psi_0 - \nu\psi_1 - 2\mathbf{u}_1 \cdot \nabla\psi_0 + \tilde{h}(0), \tag{3.30}$$

and $\tilde{h}(0) = h_p(0)$ depending only on $(\psi_1, \mathbf{u}_1)$. Hence these initial values will depend only on $(\psi_1, \mathbf{u}_1)$ and are periodic functions of the space variables. Obviously, there is a constant $M_2 > 0$, such that the initial values of $\psi_{p+1}, \psi'_{p+1}, \psi''_{p+1}$ for $p \ge 1$ are bounded by

$$M_2 I_0 \ge \max\left\{ \|\psi_1\|_2^2,\ \|\psi_0\|_2^2,\ \|\Psi_0\|_2^2,\ \|\mathbf{u}_1\|_3^2 \right\}. \tag{3.31}$$

Here, we recall that I_0 is defined by means of (3.29).

Denote by

$$M_0 = 40 M_2 I_0 \cdot \max\{1, \nu^{-1}\}, \tag{3.32}$$
$$M_1 = 3Na_0^2(I_0 + 1 + M_0)^7 \cdot \max\{1, \nu^{-2}\}, \tag{3.33}$$

and choose

$$T_* = \min\left\{1,\ \frac{\psi_*}{4M_0},\ \frac{M_2 I_0}{NM_3},\ \frac{\ln 2}{NM_4},\ \frac{2M_2 I_0}{NM_5},\ \frac{2M_2 I_0}{NM_6}\right\}, \tag{3.34}$$

where

$$\begin{aligned} &M_3 = 5a_0^2(I_0 + 1 + M_0 + M_1)^6,\ \ M_4 = 2a_0^3(I_0 + 1 + M_0 + M_1)^8, \\ &M_5 = a_0^2(I_0 + 1 + M_0 + M_1)^7,\ \ M_6 = a_0^5(I_0 + 1 + M_0 + M_1)^{14}. \end{aligned} \tag{3.35}$$

As before $N \ge M_2$ denotes a generic constant independent of U^p, $p \ge 1$, and a_0 is defined by (3.28).

Step 2.1: We claim that if the solution $\{U^j\}_{j=1}^p$, $(p \ge 2)$, to the problems (3.14)–(3.18) satisfies

$$\begin{cases} \|\mathbf{u}_j(t)\|_3^2 + \|(\psi_j, \psi'_j)(t)\|_4^2 + \|\psi''_j(t)\|_2^2 \le M_0, \\ \|\mathbf{v}_j(t)\|_4^2 + \|D\Delta\psi_j(t)\|_1^2 \le a_0 M_1, \end{cases} \tag{3.36}$$

for all $1 \le j \le p$ and $t \in [0, T_*]$, then this is also true for U^{p+1}, namely

$$\begin{cases} \|\mathbf{u}_{p+1}(t)\|_3^2 + \|(\psi_{p+1}, \psi'_{p+1})(t)\|_4^2 + \|\psi''_{p+1}(t)\|_2^2 \le M_0, \\ \|\mathbf{v}_{p+1}(t)\|_4^2 + \|D\Delta\psi_{p+1}(t)\|_1^2 \le a_0 M_1, \end{cases} \tag{3.37}$$

for all $t \in [0, T_*]$. Here M_0 and M_1 are given by (3.32) and (3.33).

We prove (3.37) in the following Steps 2.2–2.4, namely, we first obtain the uniform bounds for V_{j+1} $(1 \le j \le p)$ based on (3.36), then we estimate uniform bounds of $\varphi_{j+1}, \mathbf{v}_{j+1}, \mathbf{u}_{j+1} (1 \le j \le p)$ and their time derivatives in Sobolev space and prove that $\mathbf{v}_{p+1}, \mathbf{u}_{p+1}$ satisfy (3.37), and finally we estimate ψ_{j+1} $(1 \le j \le p)$. Meanwhile, related to this, we can get uniform estimates on the time derivatives of $\mathbf{u}_{p+1}, \mathbf{v}_{p+1}$ and on ψ'''_{p+1}.

Step 2.2: Estimate on V_{j+1}. Based on (3.36) we derive the estimates on V_{j+1} ($1 \le j \le p$) by solving the Poisson equation (3.18) on $\mathbb{T}^n$ for V_{j+1}, $1 \le j \le p$. Since it always holds

$$\int_{\mathbb{T}^n} q_j(x,t)dx = 0, \quad 1 \le j \le p, \quad t \in [0, T_*],$$

by using Theorem 2.2 there exists a unique solution V_{j+1} of Eq. (3.18) satisfying

$$\|V_{j+1}(t)\|_4^2 \le N\|q_j(t)\|_2^2 \le N\|\psi_j(t)\|_2^4 \le NM_0^2, \qquad t \in [0, T_*],\ 1 \le j \le p, \tag{3.38}$$

$$\|V'_{j+1}(t)\|_4^2 \le N\|q'_j(t)\|_2^2 \le N\|(\psi'_j, \psi_j)(t)\|_2^4 \le NM_0^2, \ \ t \in [0, T_*],\ 1 \le j \le p. \tag{3.39}$$

Thus, we conclude that $V_{p+1} \in C^1([0, T_*]; \dot{H}^4(\mathbb{T}^n))$ is uniformly bounded so long as (3.36) is true.

Step 2.3: Estimates on $\varphi_j, \mathbf{v}_j, \mathbf{u}_j$. We estimate $\varphi_j, \mathbf{v}_j, \mathbf{u}_j$, $1 \le j \le p$ for $(x,t) \in \mathbb{T}^n \times [0, T_*]$ based on (3.36). For $(x,t) \in \mathbb{T}^n \times [0, T_*]$ by using the same ideas as in deriving (3.8) it follows for φ_{j+1} from (3.15) that

$$\begin{cases} \varphi_{j+1}(x,t) = \left(\psi_1(x) - \int_0^t e^{\frac{1}{2}\int_0^s \nabla\cdot\mathbf{v}_j(x,\xi)d\xi}\mathbf{u}_j \cdot\nabla\psi_j(x,s)ds\right) e^{-\frac{1}{2}\int_0^t \nabla\cdot\mathbf{v}_j(x,s)ds}, \\ \varphi_{j+1} \in C^1([0,1]; H^3(\mathbb{T}^n)) \bigcap C^2([0,1]; H^2(\mathbb{T}^n)) \bigcap C^3([0,1]; L^2(\mathbb{T}^n)), \end{cases} \tag{3.40}$$

which satisfies for all $(x,t) \in \mathbb{T}^n \times [0, T_*]$,

$$\frac{1}{4}\psi_* \le \frac{1}{2}\psi_* e^{-N(1+M_1)T_*} \le \varphi_{j+1}(x,t) \le (\psi^* + \psi_*)e^{N(1+M_1)T_*} \le 2(\psi^* + \psi_*). \tag{3.41}$$

Moreover the L^2 norm of φ_{j+1}, with $1 \le j \le p$, and its derivatives are bounded for all $t \in [0, T_*]$, through those of $\mathbf{v}_j, \mathbf{u}_j$ and through the initial data by

$$\left\|\varphi_{j+1}(t)\right\|_3^2 \le Ne^{N(1+M_1)T_*}(\|\psi_1\|_3^2 + T_*(\|\mathbf{u}_j(t)\|_3^2 \cdot \|\psi_j(t)\|_4^2)) \le NI_0, \tag{3.42}$$

and

$$\begin{aligned} \left\|\varphi'_{j+1}(t)\right\|_3^2 \le& N\left(I_0 + \|\mathbf{v}_j\|_4^2 + \|\mathbf{u}_j(t)\|_3^2 + \|\psi_j(t)\|_4^2\right)^2 \\ \le& Na_0\left(I_0 + 1 + M_0 + M_1\right)^2, \end{aligned} \tag{3.43}$$

$$\begin{aligned}\left\|\varphi''_{j+1}(t)\right\|_2^2 \leq& NM_0\left(\|\varphi'_{j+1}(t)\|_2^2+\|\mathbf{u}'_j(t)\|_2^2+M_0\right)+N\|\varphi_{j+1}(t)\|_3^2\cdot\|\mathbf{v}'_j(t)\|_3^2\\ \leq& Na_0^2\,(I_0+1+M_0+M_1)^3+N(I_0+M_0)\|(\mathbf{u}'_j,\mathbf{v}'_j)(t)\|_3^2, \qquad (3.44)\end{aligned}$$

$$\begin{aligned}\left\|\varphi'''_{j+1}(t)\right\|^2 \leq& N(I_0+M_0)\left(\|\varphi''_{j+1}(t)\|^2+\|(\mathbf{u}'_j,\mathbf{u}''_j)(t)\|^2+\|\mathbf{v}''_j(t)\|_1^2+M_0\right)\\ &+N\left(\|\varphi'_{j+1}(t)\|_3^4+\|\mathbf{v}'_j(t)\|_3^4\right)\\ \leq& N\left(\|\mathbf{v}'_j(t)\|_3^4+(I_0+M_0)^2\|(\mathbf{u}'_j,\mathbf{v}'_j)(t)\|_3^2\right)\\ &+N(I_0+M_0)\left(\|\mathbf{u}''_j(t)\|^2+\|\mathbf{v}''_j(t)\|_1^2\right)\\ &+Na_0^2\,(I_0+1+M_0+M_1)^4\,. \qquad (3.45)\end{aligned}$$

Let us consider the divergence equation (3.14) for $\mathbf{v}_{j+1}$, with $1 \leq j \leq p$. Since one has

$$\int_{\mathbb{T}^n} r_j(x,t)\,dx = 0, \quad 1 \leq j \leq p, \quad t \in [0, T_*],$$

the application of Theorem 2.1 yields the existence of a unique solution $\mathbf{v}_{j+1}$ of Eq. (3.14) for $t \in [0, T_*]$, which, in view of (3.40)–(3.44) and (3.36), satisfies the following bounds:

$$\begin{aligned}\|\mathbf{v}_{j+1}(t)\|_4^2 \leq N\|r_j(t)\|_3^2 \leq& Na_0\|\varphi_j(t)\|_3^2\left(\|\psi'_j(t)\|_3^2+\|\psi_j(t)\|_4^2+\|\mathbf{u}_j(t)\|_3^2\right)\\ \leq& Na_0\,(I_0+1+M_0)^3 \qquad (3.46)\\ \leq& \frac{1}{3}M_1, \qquad t \in [0, T_*],\ 1 \leq j \leq p, \qquad (3.47)\end{aligned}$$

$$\begin{aligned}\|\mathbf{v}'_{j+1}(t)\|_3^2 \leq& N\|r'_j(t)\|_2^2\\ \leq& Na_0\|\varphi_j\|_2^2\left(\|\psi''_j(t)\|_2^2+M_0\|(\psi'_j,\mathbf{u}'_j)(t)\|_2^2\right)\\ &+Na_0I_0\|\varphi'_j(t)\|_3^2\left(M_0\|\mathbf{u}_j\|_2^2+\|\psi'_j(t)\|_2^2\right)\\ \leq& Na_0^2\,(I_0+1+M_0+M_1)^5\\ &+Na_0(I_0+M_0)^2\|\mathbf{u}'_j(t)\|_2^2, \qquad t \in [0, T_*], 1 \leq j \leq p, \qquad (3.48)\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{v}''_{j+1}(t)\|_1^2 \leq& N\|r''_j(t)\|^2\\ \leq& Na_0\left(\|\psi'''_j(t)\|^2+M_0\|(\mathbf{u}''_j,\mathbf{u}'_j)(t)\|^2+M_0\|\psi''_j(t)\|_2^2\right)\\ &+Na_0\|\varphi''_j(t)\|^2\left(\|\psi'_j(t)\|_2^2+M_0\|\mathbf{u}_j(t)\|_2^2\right)+Na_0(1+M_0)^2\|\varphi'_j(t)\|_3^4\\ &+Na_0\|\varphi'_j(t)\|_3^2\left(\|\psi''_j(t)\|^2+\|\psi_j(t)\|_4^2+\|(\mathbf{u}_j,\mathbf{u}'_j)(t)\|^2\right)\end{aligned}$$

$$
\begin{aligned}
&\leq N a_0 \left(\|\psi_j'''(t)\|_3^2 + M_0 \|\mathbf{u}_j''(t)\|^2\right) + N a_0^3 \,(I_0+1+M_0+M_1)^6 \\
&\quad + N a_0 \,(I_0+1+M_0+M_1)^3 \left(\|\mathbf{u}_j'(t)\|_3^2 + \|\mathbf{v}_j'(t)\|_3^2\right) \\
&\leq N a_0 \left(\|\psi_j'''(t)\|_3^2 + M_0 \|\mathbf{u}_j''(t)\|^2\right) + N a_0^3 \,(I_0+1+M_0+M_1)^8 \\
&\quad + N a_0^2 \,(I_0+1+M_0+M_1)^3 \,\|\mathbf{u}_j'(t)\|_3^2 \\
&\quad + N a_0^2 (I_0+1+M_0)^5 \|\mathbf{u}_{j-1}'(t)\|_2^2, \qquad t \in [0,T_*],\ 2 \leq j \leq p, \qquad (3.49)
\end{aligned}
$$

where we have already used (3.48) for $\mathbf{v}_j'$.

For the functions U^i $(1 \leq j \leq p)$ satisfying (3.36), it is easy to verify that g_j, g_j' $(1 \leq j \leq p)$ belong to $H^3(\mathbb{T}^n)$ and $H^1(\mathbb{T}^n)$. By (3.36), (3.38)–(3.43) and (3.47)–(3.48), we can obtain the L^2 norm of g_j, $\mathbf{u}_j'$, $\mathbf{v}_{j+1}'$ $(1 \leq j \leq p)$ and those of their derivatives as follows. We observe that

$$
\begin{aligned}
\|g_j(t)\|_3^2 &\leq N a_0 \left(\|\psi_j(t)\|_6^2 + \|\varphi_j(t)\|_3^2\right)^5 + N \left(\|\nabla V_j(t)\|_3^2 + \|\mathbf{v}_j(t)\|_4^4\right) \\
&\leq N a_0^2 \,(I_0+1+M_0+M_1)^6, \qquad t \in [0,T_*],\ 1 \leq j \leq p. \qquad (3.50)
\end{aligned}
$$

Then from (3.17) and (3.36) one has

$$
\begin{aligned}
\|\mathbf{u}_j'(t)\|_3^2 &\leq N \left(\|\mathbf{u}_j\|_3^2 + \|g_{j-1}(t)\|_3^2\right) \\
&\leq N a_0^2 \,(I_0+1+M_0+M_1)^6, \qquad t \in [0,T_*],\ 1 \leq j \leq p. \qquad (3.51)
\end{aligned}
$$

And we can estimate $\mathbf{v}_{j+1}'$ in view of (3.48) as follows:

$$
\begin{aligned}
\|\mathbf{v}_{j+1}'(t)\|_3^2 &\leq N a_0^2 \,(I_0+1+M_0+M_1)^5 + N a_0 \,(I_0+M_0)^2 \,\|\mathbf{u}_j'(t)\|_2^2 \\
&\leq N a_0^3 \,(I_0+1+M_0+M_1)^8, \qquad t \in [0,T_*],\ 1 \leq j \leq p. \qquad (3.52)
\end{aligned}
$$

By differentiating (3.22) with respect to t, and using (3.36), (3.39), (3.42)–(3.43), (3.47) and (3.52), we obtain

$$
\begin{aligned}
\|g_j'(t)\|_1^2 &\leq N a_0 \left(\|(\psi_j', \psi_j)(t)\|_4^2 + \|\varphi_j(t)\|_3^2\right)^3 \\
&\quad + N a_0 \|\varphi_j'(t)\|_3^2 \left(\|\psi_j(t)\|_4^2 + \|\varphi_j(t)\|_3^2\right)^4 \\
&\quad + N \left(\|\nabla V_j'(t)\|_1^2 + \|\mathbf{v}_j(t)\|_3^2 \cdot \|\mathbf{v}_j'(t)\|_3^2\right) \\
&\leq N a_0^4 \,(I_0+1+M_0+M_1)^{11}, \qquad t \in [0,T_*],\ 1 \leq j \leq p. \qquad (3.53)
\end{aligned}
$$

Hence, we obtain, after differentiating (3.17) with respect to time, that

$$
\begin{aligned}
\|\mathbf{u}_j''(t)\|_1^2 &\leq N(\|\mathbf{u}_j'\|_1^2 + \|g_{j-1}'(t)\|_1^2) \\
&\leq N a_0^4 \,(I_0+1+M_0+M_1)^{11}, \qquad t \in [0,T_*],\ 2 \leq j \leq p, \qquad (3.54)
\end{aligned}
$$

and from (3.49) that

$$\begin{aligned}\|\mathbf{v}''_{j+1}(t)\|_1^2 \le & Na_0 \left(\|\psi'''_j(t)\|_3^2 + M_0\|\mathbf{u}''_j(t)\|^2\right) + Na_0^3 (I_0 + 1 + M_0 + M_1)^8 \\ & + Na_0^2 (I_0 + 1 + M_0 + M_1)^3 \|\mathbf{u}'_j(t)\|_3^2 \\ & + Na_0^2 (I_0 + 1 + M_0)^5 \|\mathbf{u}'_{j-1}(t)\|_2^2 \\ \le & Na_0^5 (I_0 + 1 + M_0 + M_1)^{12} \\ & + Na_0 \|\psi'''_j(t)\|_3^2, \qquad t \in [0, T_*],\ 1 \le j \le p. \end{aligned} \tag{3.55}$$

By the previous estimates, it is easy to obtain the estimates for $\mathbf{u}_{p+1}$. In fact, by taking the inner product between $D^\alpha(3.17)_1$ $(0 \le |\alpha| \le 3)$ and $2D^\alpha \mathbf{u}_{p+1}$ over $\mathbb{T}^n$, we obtain

$$\frac{d}{dt}\|D^\alpha \mathbf{u}_{p+1}\|^2 + \|D^\alpha \mathbf{u}_{p+1}\|^2 \le \|D^\alpha g_p(t)\|^2. \tag{3.56}$$

Hence by summing (3.56) with respect to $|\alpha| = 0, 1, 2, 3$, and integrating it over $[0, t]$, and by the Gronwall lemma, we have

$$\begin{aligned}\|\mathbf{u}_{p+1}(t)\|_3^2 \le & \|\mathbf{u}_1\|_3^2 + \int_0^t \|g_p(s)\|_3^2 e^{-(t-s)} ds \\ \le & M_2 I_0 + T_* N a_0^2 (I_0 + 1 + M_0 + M_1)^6 \le \frac{2}{5} M_0, \quad t \in [0, T_*], \end{aligned} \tag{3.57}$$

with T_* defined by (3.34). With the help of (3.50), (3.53) and (3.57), the corresponding H^3 and H^1 norms of $\mathbf{u}'_{p+1}$ and $\mathbf{u}''_{p+1}$ are bounded, similarly to (3.51) and (3.54), by

$$\|\mathbf{u}'_{p+1}(t)\|_3^2 \le N\left(\|\mathbf{u}_p(t)\|_3^2 + \|g_p(t)\|_3^2\right) \le Na_0^2 (I_0 + 1 + M_0 + M_1)^6, \tag{3.58}$$

$$\|\mathbf{u}''_{p+1}(t)\|_1^2 \le N(\|u'_p(t)\|_1^2 + \|g'_p(t)\|_1^2) \le Na_0^4 (I_0 + 1 + M_0 + M_1)^{11}, \tag{3.59}$$

for $t \in [0, T_*]$.

In addition, with the help of previous estimates on $\mathbf{v}$, $\mathbf{u}$ (i.e., (3.51), (3.52), (3.54), and (3.55)), we obtain from (3.44)–(3.45) that

$$\begin{aligned}\left\|\varphi''_{j+1}(t)\right\|_2^2 \le & Na_0^2 (I_0 + 1 + M_0 + M_1)^3 + N(I_0 + M_0)\|(\mathbf{u}'_j, \mathbf{v}'_j)(t)\|_3^2 \\ \le & Na_0^3 (I_0 + 1 + M_0 + M_1)^9, \end{aligned} \tag{3.60}$$

and

$$\begin{aligned}\left\|\varphi'''_{j+1}(t)\right\|^2 \le & N\left(\|\mathbf{v}'_j(t)\|_3^4 + (I_0 + M_0)^2\|(\mathbf{u}'_j, \mathbf{v}'_j)(t)\|_3^2\right) \\ & + N(I_0 + M_0)\left(\|\mathbf{u}''_j(t)\|^2 + \|\mathbf{v}''_j(t)\|_1^2\right) \\ & + Na_0^2 (I_0 + 1 + M_0 + M_1)^4 \\ \le & Na_0^6 (I_0 + 1 + M_0 + M_1)^{16} + Na_0(I_0 + M_0)\|\psi'''_j(t)\|_3^2. \end{aligned} \tag{3.61}$$

So far, we have proved that $\mathbf{v}_{p+1}$ and $\mathbf{u}_{p+1}$ satisfy (3.37) (i.e., (3.47) and (3.57)) as long as (3.36) holds, and the time derivatives of them (i.e., (3.52), (3.58), and (3.59)) are also bounded uniformly in Sobolev space, with the exception of (3.55) for $\mathbf{v}''_{j+1}$ relative to ψ'''_{j+1} $(1 \le j \le p)$. Furthermore, from (3.42)–(3.43) and (3.60)–(3.61) we conclude that φ_{p+1} and its time derivatives are uniformly bounded in Sobolev space, with the exception of φ'''_{j+1}, i.e., (3.61), relative to ψ'''_{j+1} $(1 \le j \le p)$.

Step 2.4: Estimates on ψ_{j+1}, $\mathbf{v}''_{j+1}$, φ'''_{j+1}. We estimate ψ_{j+1} and then $\mathbf{v}''_{j+1}$ and φ'''_{j+1}, $1 \le j \le p$, for $(x,t) \in \mathbb{T}^n \times [0, T_*]$. By (3.36), (3.46), (3.51), and (3.52), it is easy to verify the upper bounds of k_j, k'_j in $H^3(\mathbb{T}^n)$, for all $t \in [0, T_*]$, namely

$$\|k_j(t)\|_3^2 \le N(\|\mathbf{u}_j(t)\|_3^2 + \|\mathbf{v}_j(t)\|_3^2) \le Na_0(I_0 + 1 + M_0)^3, \quad 1 \le j \le p, \tag{3.62}$$

and

$$\|k'_j(t)\|_3^2 \le N(\|\mathbf{u}'_j(t)\|_3^2 + \|\mathbf{v}'_j(t)\|_3^2) \le Na_0^3\,(I_0 + 1 + M_0 + M_1)^8\,,\ 1 \le j \le p. \tag{3.63}$$

With the help of (3.36), (3.38)–(3.39), (3.42), (3.43), (3.46), (3.51) and (3.52), we obtain, from (3.21), the following bounds on $h_p(t)$, $h'_p(t)$:

$$\begin{aligned}\|h_j(t)\|_2^2 \le & Na_0 \left(\|\varphi_j(t)\|_3^2 + \|\psi_j(t)\|_4^2 + \|\psi'_j(t)\|_2^2 + \|\mathbf{u}_j(t)\|_3^2\right)^4 \\ & + Na_0\|\mathbf{v}_j(t)\|_4^2 \left(\|\psi_j(t)\|_4^2 + \|(\psi'_j, \varphi_j)(t)\|_2^2 + \|\mathbf{u}_j(t)\|_3^2\right)^3 \\ & + N\|\psi_j(t)\|_4^2 \left(\|V_j(t)\|_4^2 + \|\mathbf{v}_j(t)\|_4^4\right) \\ \le & Na_0^2(I_0 + 1 + M_0)^7, \qquad 1 \le j \le p,\ t \in [0, T_*],\end{aligned} \tag{3.64}$$

and

$$\begin{aligned}\|h'_j(t)\|_2^2 \le & Na_0 \left(\|(\varphi'_j, \varphi_j)(t)\|_3^2 + \|(\psi'_j, \psi_j)(t)\|_4^2 + \|\psi''_j(t)\|_2^2 + \|\mathbf{u}_j(t)\|_3^2\right)^5 \\ & + Na_0\|\mathbf{u}'_j(t)\|_3^2 \left(1 + \|\mathbf{v}_j(t)\|_4^2\right) \left(\|\varphi_j(t)\|_2^2 + \|(\psi'_j, \psi_j)(t)\|_4^2\right)^4 \\ & + Na_0\|\mathbf{v}_j(t)\|_4^2 \left(\|(\psi'_j, \psi_j)(t)\|_4^2 + \|(\psi''_j, \varphi'_j, \varphi_j)(t)\|_2^2 + \|\mathbf{u}_j(t)\|_3^2\right)^4 \\ & + Na_0\|\mathbf{v}'_j(t)\|_4^2 \left(\|\psi_j(t)\|_4^2 + \|(\psi'_j, \varphi_j)(t)\|_2^2 + \|\mathbf{u}_j(t)\|_4^2\right)^5 \\ & + N\|\psi_j(t)\|_4^2 \left(\|V'_j(t)\|_4^2 + \|\mathbf{v}'_j(t)\|_3^2 \cdot \|\mathbf{v}_j(t)\|_4^2\right) \\ & + N\|\psi'_j(t)\|_4^2 \left(\|V_j(t)\|_4^2 + \|\mathbf{v}_j(t)\|_4^4\right) \\ \le & Na_0^5(I_0 + 1 + M_0 + M_1)^{14}, \qquad 2 \le j \le p,\ t \in [0, T_*].\end{aligned} \tag{3.65}$$

To obtain the bounds on the L^2 norm of ψ_{p+1} and its derivatives, we first take the inner product between Eq. $(3.16)_1$ and $2\psi'_{p+1}$ and then we integrate by parts. By using Lemma 2.5, we have

$$\begin{aligned}&\frac{d}{dt}(\|\psi'_{p+1}(t)\|^2 + \nu\|\psi_{p+1}(t)\|^2 + \nu\|\Delta\psi_{p+1}(t)\|^2) \\ &\quad \le |\nabla\cdot k_p(t)|_{L^\infty}\|\psi'_{p+1}\|^2 + \|h_p(t)\|^2 \\ &\quad \le N(1 + \|k_p(t)\|_3^2)\|\psi'_{p+1}(t)\|^2 + \|h_p(t)\|^2.\end{aligned} \tag{3.66}$$

Take the inner product between Eq. $D^\alpha(3.16)_1$ and $2D^\alpha \psi'_{p+1}$ with $1 \le |\alpha| \le 2$ and integrate it by parts over $\mathbb{T}^n$. It follows

$$\begin{aligned} &\frac{d}{dt}(\|D^\alpha \psi'_{p+1}(t)\|^2 + \nu\|D^\alpha \psi_{p+1}(t)\|^2 + \nu\|\Delta D^\alpha \psi_{p+1}(t)\|^2) \\ &\quad \le |\nabla\cdot k_p(t)|_{L^\infty}\|D^\alpha \psi'_{p+1}(t)\|^2 + \|D^\alpha h_p(t)\|^2 + N\int_{\mathbb{T}^n}|H_\alpha(\psi'_{p+1}, k_p)|^2 dx \\ &\quad \le N(1+\|k_p(t)\|_3^2)\|D\psi'_{p+1}(t)\|^2 + \|D^\alpha h_p(t)\|^2 + N\int_{\mathbb{T}^n}|H_\alpha(\psi'_{p+1}, k_p)|^2 dx, \end{aligned} \tag{3.67}$$

where

$$H_\alpha(\psi, k) = D^\alpha\,(k\cdot\nabla\psi) - k\cdot\nabla(D^\alpha\psi).$$

By Lemma 2.5, (3.62), we get

$$\int_{\mathbb{T}^n}|H_\alpha(\psi,k)|^2 dx \le \begin{cases} N(1+\|k(t)\|_3^2)\|D\psi\|^2, & |\alpha| = 1, \\ N(1+\|k(t)\|_3^2)(\|D\psi\|^2 + \|D^\alpha\psi\|^2), & |\alpha| = 2. \end{cases} \tag{3.68}$$

By substituting (3.68) into (3.67) and taking summation of these differential inequalities with respect to $|\alpha| = 0, 1, 2,$ we have

$$\begin{aligned} &\frac{d}{dt}\left(\|\psi'_{p+1}(t)\|_2^2 + \nu\|\psi_{p+1}(t)\|_2^2 + \nu\|\Delta\psi_{p+1}(t)\|_2^2\right) \\ &\quad \le N(1+\|k_p(t)\|_3^2)\left(\|\psi'_{p+1}(t)\|_2^2 + \nu\|\psi_{p+1}(t)\|_2^2 + \nu\|\Delta\psi_{p+1}(t)\|_2^2\right) \\ &\qquad + \|h_p(t)\|_2^2. \end{aligned} \tag{3.69}$$

By applying the Gronwall inequality and by using (3.62), (3.64), we obtain

$$\begin{aligned} &\|\psi'_{p+1}(t)\|_2^2 + \|\psi_{p+1}(t)\|_2^2 + \|\Delta\psi_{p+1}(t)\|_2^2 \\ &\quad \le \max\{1,\ \nu^{-1}\}\cdot(\|\psi_0\|_2^2 + \|\psi_1\|_4^2 + T_* N M_5)e^{T_* N a_0 (1+M_0+M_1)^3} \\ &\quad \le 2(2M_2 I_0 + T_* M_5)\cdot\max\{1,\ \nu^{-1}\} \\ &\quad \le 8M_2 I_0 = \frac{1}{5}M_0, \quad t\in[0,T_*],\ \ p\ge 1, \end{aligned} \tag{3.70}$$

where we recall that M_0, T_* and M_5 are defined by (3.32), (3.34), and (3.35) respectively.

Let us take the inner product between Eq. $D^\alpha\partial_t(3.16)_1$ and $2D^\alpha\psi''_{p+1}$, with $0 \le |\alpha| \le 2$, and integrate by parts over $\mathbb{T}^n$, then by summing the resulting differential inequality with respect to α, by (3.68) and by the following estimates:

$$\int_{\mathbb{T}^n}|D^\alpha\left(k'_p\cdot\nabla\psi'_{p+1}\right)|^2 \le \begin{cases} N\nu\|k'_p(t)\|_2^2\left(\|\psi'_{p+1}(t)\|^2 + \|\Delta\psi'_{p+1}(t)\|^2\right), & \alpha = 0, \\ N\nu\|k'_p(t)\|_2^2\left(\|D\psi'_{p+1}\|_1^2 + \|\Delta\psi'_{p+1}(t)\|^2\right), & |\alpha| = 1, \\ N\nu\|k'_p(t)\|_2^2\left(\|D\psi'_{p+1}\|^2 + \|\Delta\psi'_{p+1}\|_1^2\right), & |\alpha| = 2, \end{cases}$$

we obtain, in analogy to (3.66), (3.69), that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\psi''_{p+1}(t)\|_2^2+\nu\|\psi'_{p+1}(t)\|_2^2+\nu\|\Delta\psi'_{p+1}(t)\|_2^2\right)\\
&\quad\le N(1+\|k_p(t)\|_3^2)\left(\|D\psi''_{p+1}(t)\|_1^2+\nu\|D\psi'_{p+1}(t)\|_1^2+\nu\|\Delta D\psi'_{p+1}(t)\|_1^2\right)\\
&\qquad+\|h'_p(t)\|_2^2+N\sum_{0\le|\alpha|\le 2}\int_{\mathbb{T}^n}(|D^\alpha\left(k'_p\cdot\nabla\psi'_{p+1}\right)|^2+|H_\alpha(\psi''_{p+1},k_p)|^2)\,dx\\
&\quad\le NB_1\left(\|\psi''_{p+1}\|_2^2+\nu\|\psi'_{p+1}(t)\|_2^2+\nu\|\Delta\psi'_{p+1}(t)\|_2^2\right)+\|h'_p(t)\|_2^2, \qquad (3.71)
\end{aligned}$$

where

$$B_1=a_0^3(I_0+1+M_0+M_1)^8.$$

By applying the Gronwall inequality to (3.71), it follows

$$\begin{aligned}
&\|\psi''_{p+1}(t)\|_2^2+\|\psi'_{p+1}(t)\|_2^2+\|\Delta\psi'_{p+1}(t)\|_2^2\\
&\quad\le\max\{1,\nu^{-1}\}\cdot(\|\Psi_0\|_2^2+\|\psi_0\|_4^2+T_*NM_6)e^{T_*Na_0^3(I_0+1+M_0+M_1)^8}\\
&\quad\le 2(2M_2I_0+T_*NM_6)\cdot\max\{1,\nu^{-1}\}\\
&\quad\le 8M_2I_0=\frac{1}{5}M_0, \qquad t\in[0,T_*],\ p\ge 1, \qquad (3.72)
\end{aligned}$$

where we recall that M_0, T_* and M_6 are defined by (3.32), (3.34), and (3.35) respectively.

To estimate the L^2 bounds of $D^5\psi_{p+1}$ and $D^6\psi_{p+1}$, it is sufficient to estimate those of $\Delta^2D\psi_{p+1}$ and $\Delta^2D^2\psi_{p+1}$. By differentiating Eq. $(3.16)_1$ twice with respect to x and by taking the inner product with $\Delta^2D\psi_{p+1}$ and $\Delta^2D^2\psi_{p+1}$ over $\mathbb{T}^n$, and using the estimates (3.62), (3.64), (3.70), and (3.72), one has

$$\begin{aligned}
\|\Delta^2D\psi_{p+1}(t)\|^2\le&\frac{N}{\nu^2}\left(\|\psi''_{p+1}(t)\|_1^2+\|\psi'_{p+1}(t)\|_1^2+\|\psi_{p+1}(t)\|_1^2\right)\\
&+\frac{N}{\nu^2}\|D(k_p\cdot\nabla\psi'_{p+1})(t)\|^2+\frac{N}{\nu^2}\|h_p(t)\|_1^2\\
\le&\frac{N}{\nu^2}a_0^2(I_0+1+M_0)^7\le\frac{1}{3}M_1, \qquad t\in[0,T_*],\ p\ge 1, \qquad (3.73)\\
\|\Delta^2D^2\psi_{p+1}(t)\|^2\le&\frac{N}{\nu^2}\left(\|\psi''_{p+1}(t)\|_2^2+\|\psi'_{p+1}(t)\|_2^2+\|\psi_{p+1}(t)\|_2^2\right)\\
&+\frac{N}{\nu^2}\|D^2(k_p\cdot\nabla\psi'_{p+1})(t)\|^2+\frac{N}{\nu^2}\|h_p(t)\|_2^2\\
\le&\frac{N}{\nu^2}a_0^2(I_0+1+M_0)^7\le\frac{1}{3}M_1, \qquad t\in[0,T_*],\ p\ge 1, \qquad (3.74)
\end{aligned}$$

where we recall M_1 and T_* are defined by (3.33) and (3.34) respectively.

We now need to show the L^2 norm of ψ'''_{j+1} and $\mathbf{v}''_{j+1}$ for $1\le j\le p$. By taking the inner product between $\partial_t(3.16)_1$ and ψ'''_{p+1} and using the above estimates, we obtain

$$\begin{aligned}
\|\psi'''_{p+1}(t)\|^2\le&N\left(\|\psi''_{p+1}(t)\|^2[1+\|k_p(t)\|_2^2]+\|h'_p(t)\|^2\right)\\
&+N\|\psi'_{p+1}(t)\|_4^2\left(1+\|k'_p(t)\|_2^2\right)\\
\le&Na_0^5(I_0+1+M_0+M_1)^{14}, \quad t\in[0,T_*],\ 1\le j\le p, \qquad (3.75)
\end{aligned}$$

which gives from (3.55) that

$$\|\mathbf{v}_{j+1}''(t)\|_1^2 \le Na_0^5\,(I_0+1+M_0+M_1)^{12} + Na_0\|\psi_j'''(t)\|_3^2$$
$$\le Na_0^6(I_0+1+M_0+M_1)^{14}, \quad t\in[0,T_*],\ 1\le j\le p, \tag{3.76}$$

and from (3.61) that

$$\left\|\varphi_{j+1}'''(t)\right\|^2 \le Na_0^6\,(I_0+1+M_0+M_1)^{16} + Na_0(I_0+M_0)\|\psi_j'''(t)\|_3^2$$
$$\le Na_0^6\,(I_0+1+M_0+M_1)^{16}, \quad t\in[0,T_*],\ 1\le j\le p. \tag{3.77}$$

Step 3: End of proof. By the previous estimates (3.38)–(3.39) on V_{p+1}, (3.47), (3.52), and (3.76) on $\mathbf{v}_{p+1}$, (3.42)–(3.43), (3.60), and (3.77) on φ_{p+1}, (3.57)–(3.59) on $\mathbf{u}_{p+1}$, and (3.70) and (3.72)–(3.75) on ψ_{p+1}, we conclude that the approximate solution $U^{p+1} = (\mathbf{v}_{p+1}, \varphi_{p+1}, \psi_{p+1}, \mathbf{u}_{p+1}, V_{p+1})$ is uniformly bounded in the time interval $[0,T_*]$ and it satisfies (3.37) for each $p\ge 1$ as long as U^p satisfies (3.36) with M_0, M_1, and T_* defined by (3.32), (3.33), and (3.34) respectively, which are independent of U^{p+1}, $p\ge 1$. By repeating the procedure used above, we can construct the approximate solution $\{U^i\}_{i=1}^\infty$, which solves (3.25)–(3.26) on $[0,T_*]$, with T_* defined by (3.34) and the constant $M_*>0$ chosen by

$$M_* = \max\left\{M_0,\ M_1,\ Na_0^6(I_0+1+M_0+M_1)^{16}\right\}. \tag{3.78}$$

Let us recall here that M_0, M_1 and a_0 are defined by (3.32), (3.33) and (3.28) respectively and $N>0$ is a generic constant independent of U^{p+1}, $p\ge 1$. Therefore, the proof of Lemma 3.1 is completed. □

Proof of Theorem 1.1. By means of Lemma 3.1, we obtain an approximate solution sequence $\{U^p\}_{p=1}^\infty$ satisfying (3.25)–(3.26). Therefore, the proof of Theorem 1.1 is completed if we show that the whole sequence converges. Indeed, based on Lemma 3.1, we can obtain the estimates of the difference $Y^{p+1} =: U^{p+1}-U^p$, $p\ge 1$, of the approximate solution sequence $\{U^p\}_{p=1}^\infty$. Let us denote $Y^{p+1} = (\bar{\mathbf{v}}_{p+1}, \bar\varphi_{p+1}, \bar\psi_{p+1}, \bar{\mathbf{u}}_{p+1}, \bar V_{p+1})$ by

$$\bar{\mathbf{v}}_{p+1} = \mathbf{v}_{p+1}-\mathbf{v}_p, \quad \bar\varphi_{p+1} = \varphi_{p+1}-\varphi_p,$$
$$\bar\psi_{p+1} = \psi_{p+1}-\psi_p, \quad \bar{\mathbf{u}}_{p+1} = \mathbf{u}_{p+1}-\mathbf{u}_p, \quad \bar V_{p+1} = V_{p+1}-V_p.$$

We can obtain for $p\ge 4$,

$$\|\bar{\mathbf{v}}_{p+1}(t)\|_4^2 + \|(\bar V_{p+1}, \bar V_{p+1}')(t)\|_3^2 \le N_*\left(\|(\bar\psi_p, \bar\psi_p')(t)\|_4^2 + \|(\bar\varphi_p, \bar{\mathbf{u}}_p)(t)\|_3^2\right),$$
$$\|(\bar{\mathbf{v}}_{p+1}', \bar{\mathbf{u}}_{p+1}', \bar\varphi_{p+1}')(t)\|_3^2 \le N_*\sum_{j=0}^{2}\|(\bar\psi_{p-j}, \bar\psi_{p-j}')(t)\|_4^2$$
$$+N_*\sum_{j=0}^{2}\left(\|\bar\psi_{p-j}''(t)\|_2^2 + \|(\bar\varphi_{p-j}, \bar{\mathbf{u}}_{p-j})(t)\|_3^2\right),$$

$$\sum_{5\le|\alpha|\le 6}\|D^\alpha\bar{\psi}_{p+1}(t)\|^2 \le N_*\|\bar{\psi}''_{p-j}(t)\|_2^2 + N_*\sum_{j=0}^{2}\|(\bar{\psi}_{p+1-j},\bar{\psi}'_{p+1-j})(t)\|_4^2$$
$$+N_*\sum_{j=0}^{1}\left(\|\bar{\psi}''_{p-j}(t)\|_2^2+\|(\bar{\varphi}_{p-j},\bar{\mathbf{u}}_{p-j})(t)\|_3^2\right).$$

Here N_* denotes a constant dependent on M_*. By using the previous estimates, Lemma 3.1, and an argument similar to the one used to get (3.42), (3.57), (3.70), and (3.72), we show, after a tedious computation, that there exists $0 < T_{**} \le T_*$, such that the difference $Y^{p+1} = U^{p+1} - U^p$, $p \ge 1$, of the approximate solution sequence satisfies the following estimates

$$\sum_{p=1}^{\infty}\left(\|(\bar{\mathbf{u}}_{p+1},\bar{\varphi}_{p+1})\|^2_{C^1([0,T_{**}];H^3)}+\|\bar{V}_{p+1}\|^2_{C^1([0,T_{**}];\dot{H}^4)}\right)\le C_*, \tag{3.79}$$

$$\sum_{p=1}^{\infty}\left(\|\bar{\psi}_{p+1}\|^2_{C^i([0,T_{**}];H^{6-2i})}+\|\bar{\mathbf{v}}_{p+1}\|^2_{C([0,T_{**}];H^4)}+\|\bar{\mathbf{v}}'_{p+1}\|^2_{C([0,T_{**}];H^3)}\right)\le C_*, \tag{3.80}$$

where $i = 0, 1, 2$, and $C_* = C_*(N, M_*)$ denote a positive constant depending on N and M_*. Then by applying the Ascoli-Arzela Theorem (to the time variable) and the Rellich-Kondrachev Theorem (to the spatial variables) [33], we prove, in a standard way (see for instance [28]), that there exists a (unique) $U = (\mathbf{v}, \varphi, \psi, \mathbf{u}, V)$, such that as $p \to \infty$,

$$\begin{cases} \mathbf{v}_p \to \mathbf{v} & \text{strongly in } C^i([0,T_{**}];H^{4-i-\sigma}(\mathbb{T}^n)),\\ \varphi_p \to \varphi & \text{strongly in } C^1([0,T_{**}];H^{3-\sigma}(\mathbb{T}^n))\bigcap C^2([0,T_{**}];H^{2-\sigma}(\mathbb{T}^n)),\\ \psi_p \to \psi & \text{strongly in } C^i([0,T_{**}];H^{6-2i-\sigma}(\mathbb{T}^n))\bigcap C^2([0,T_{**}];H^{2-\sigma}(\mathbb{T}^n)),\\ \mathbf{u}_p \to \mathbf{u} & \text{strongly in } C^i([0,T_{**}];H^{3-\sigma}(\mathbb{T}^n)),\\ V_p \to V & \text{strongly in } C^i([0,T_{**}];\dot{H}^{4-\sigma}(\mathbb{T}^n)), \end{cases} \tag{3.81}$$

holds with $i = 0, 1$, and $\sigma > 0$. Moreover, by (3.41) one has

$$\varphi(x,t) \ge \frac{1}{4}\psi_* > 0, \quad (x,t)\in\mathbb{T}^n\times[0,T_{**}]. \tag{3.82}$$

If we take $\sigma \ll 1$ in (3.81) and we pass into the limit as $p \to \infty$ in (3.14)–(3.18), we obtain the (short time) existence and uniqueness of the classical solution of the system (3.5), (3.6), (3.9), (3.11), and (3.13) constructed in Sect. 3.1.

Next, we claim the local in-time classical solution $(\mathbf{v}, \varphi, \psi, \mathbf{u}, V)$, with initial data $(\mathbf{v}, \varphi, \psi, \mathbf{u}, V)(x, 0) = (\mathbf{u}_1, \psi_1, \psi_1, \mathbf{u}_1, V_1)(x)$ also satisfies

$$\psi = \varphi, \quad \mathbf{u} = \mathbf{v}, \tag{3.83}$$

and then solves the IVP (1.5)–(1.8). Indeed by passing into the limit in $(3.18)_1$, we have

$$\varphi_t + \mathbf{u}\cdot\nabla\psi + \frac{1}{2}\varphi\nabla\cdot\mathbf{v} = 0, \tag{3.84}$$

which yields

$$\frac{2(\varphi_t + \mathbf{u}\cdot\nabla\psi)}{\varphi} = -\nabla\cdot\mathbf{v}, \tag{3.85}$$

$$\int_{\mathbb{T}^n} \frac{2(\varphi_t + \mathbf{u}\cdot\nabla\psi)}{\varphi}(x,t)dx = -\int_{\mathbb{T}^n} \nabla\cdot\mathbf{v}(x,t)dx = 0. \tag{3.86}$$

Let us note here that $\varphi > 0$. Then by taking the limiting equation of $\mathbf{v}$ (passing into the limit in (3.14) and (3.19))

$$\nabla\cdot\mathbf{v} = -\frac{2(\psi_t + \mathbf{u}\cdot\nabla\psi)}{\varphi} + \frac{1}{L^n}\int_{\mathbb{T}^n} \frac{2(\psi_t + \mathbf{u}\cdot\nabla\psi)}{\varphi}(x,t)dx, \tag{3.87}$$

and using (3.85), (3.86), one has

$$\frac{(\varphi-\psi)_t(x,t)}{\varphi} - \frac{1}{L^n}\int_{\mathbb{T}^n} \frac{(\varphi-\psi)_t}{\varphi}(x,t)dx = 0, \quad \forall\, x\in\mathbb{T}^n,\ t\geq 0. \tag{3.88}$$

Since by a straightforward computation we obtain $(\varphi-\psi)_t(x,0) = 0$ from (3.85) and (3.87) with $t = 0$, then, from (3.88), we conclude that

$$(\varphi-\psi)_t(x,t) = \varphi(x,t)f(t), \quad t\geq 0,$$

for any $f \in C^2([0,T_{**}])$, with $f(0) = 0$. In particular we can choose

$$f(t) = 0, \quad t\geq 0,$$

hence by (3.82) and the fact

$$\varphi(x,0) = \psi(x,0) = \psi_1(x) \quad \Rightarrow \quad (\varphi-\psi)(x,0) = 0,$$

we obtain

$$\psi(x,t) = \varphi(x,t) \geq \frac{1}{4}\psi_* > 0, \qquad t\in[0,T_{**}],\ x\in\mathbb{T}^n, \tag{3.89}$$

$$\psi_t + \mathbf{u}\cdot\nabla\psi + \frac{1}{2}\psi\nabla\cdot\mathbf{v} = 0, \qquad t\in[0,T_{**}],\ x\in\mathbb{T}^n. \tag{3.90}$$

By passing into the limit $p\to\infty$ in (3.17) we recover the equation for $\mathbf{u}$, i.e., (3.11). By using (3.89) and (3.12), from (3.11), one has

$$\partial_t\mathbf{u} + \frac{1}{2}\nabla(|\mathbf{v}|^2) + \nabla h(\psi^2) + \mathbf{u} = \nabla V + \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\psi}{\psi}\right). \tag{3.91}$$

This equation, together with the fact $\nabla\times\mathbf{u}_1(x) = 0$, implies

$$\nabla\times\mathbf{u} = 0, \qquad \forall\, x\in\mathbb{T}^n,\ t\geq 0. \tag{3.92}$$

Similarly, by passing into limit in (3.16) we recover Eq. (3.9) for ψ, hence recombining the various terms, with the help of (3.89) and (3.90), we get

$$\psi_{tt} + \psi_t + \mathbf{u}\cdot\nabla\psi_t + \frac{1}{2}\psi_t(\nabla\cdot\mathbf{v}) - \frac{1}{4\psi}\nabla\cdot(\psi^2\nabla(|\mathbf{v}|^2)) - \frac{1}{2\psi}\Delta P(\psi^2)$$
$$+\frac{1}{2\psi}\nabla\cdot(\psi^2\nabla V) + \frac{1}{4\psi}\varepsilon^2\nabla\cdot\left(\psi^2\nabla\left(\frac{\Delta\psi}{\psi}\right)\right) = 0. \tag{3.93}$$

From (3.90) we have $\psi_t = -\mathbf{u}\cdot\nabla\psi - \frac{1}{2}\psi\nabla\cdot\mathbf{v}$, then by substituting it into (3.93) and by representing $\mathbf{u}_t$ by (3.91), it follows

$$\nabla\cdot(\mathbf{u}-\mathbf{v})_t + \nabla\cdot(\mathbf{u}-\mathbf{v}) = 0.$$

By integrating previously the above equation with respect to time on $[0, T_{**}]$, since $\nabla\cdot(\mathbf{u}-\mathbf{v})(x,0) = 0$ and

$$\int_{\mathbb{T}^n}\mathbf{u}(x,t)dx = \int_{\mathbb{T}^n}\mathbf{v}(x,t)dx = \bar{\mathbf{u}}(t),$$

we get the conclusion, by applying Theorem 2.1 where we choose $f = 0$, namely we have $\hat{u} = 0$, that

$$\mathbf{u}(x,t) = \mathbf{v}(x,t), \quad t \in [0, T_{**}],\ x \in \mathbb{T}^n, \tag{3.94}$$

for the irrotational flow. Thus, by (3.91) and (3.94), we recover the equation for $\mathbf{u}$ which is exactly Eq. (3.2) (and then Eq. (1.6) for the irrotational flow). Multiplying (3.90) by ψ and by using (3.94) we recover the equation for ψ (which is exactly Eq. (1.5))

$$\partial_t(\psi^2) + \nabla\cdot(\psi^2\mathbf{u}) = 0. \tag{3.95}$$

From (3.95) the conservation (neutrality) of the density

$$\int_{\mathbb{T}^n}(\psi^2 - \mathcal{C})(x,t)dx = \int_{\mathbb{T}^n}(\psi_1^2 - \mathcal{C})(x)dx = 0, \quad t > 0 \tag{3.96}$$

follows. Therefore passing into the limit as $p \to \infty$, by (3.18) and by Theorem 2.2 one has that $V \in C^1([0, T_{**}]; \dot{H}^4)$ is the unique solution of the periodic boundary problem of the Poisson equation:

$$\Delta V = \psi^2 - \mathcal{C}, \quad \int_{\mathbb{T}^n} V dx = 0.$$

Therefore $(\psi, \mathbf{u}, V)$ with $\psi \geq \frac{1}{2}\psi_* > 0$ is the unique local (in time) solution of IVP (1.5)–(1.8). By a straightforward computation once more, we get

$$\psi \in C^i([0, T_{**}]; H^{6-2i}(\mathbb{T}^n)) \bigcap C^3([0, T_{**}]; L^2(\mathbb{T}^n)), \ \ i = 0, 1, 2;$$
$$\mathbf{u} \in C^i([0, T_{**}]; H^{5-2i}(\mathbb{T}^n)),\ i = 0, 1, 2; \quad V \in C^1([0, T_{**}]; \dot{H}^4(\mathbb{T}^n)).$$

The proof of Theorem 1.1 is completed. □

4. Global Existence and Large Time Behavior

We prove here uniform a-priori estimates for the local classical solutions $(\psi, \mathbf{u}, V)$ of IVP (1.5)–(1.8) for any fixed $T > 0$, when $(\psi, \mathbf{u}, V)$ is close to the steady state $(\sqrt{\mathcal{C}}, 0, 0)$.

4.1. Reformulation of original problem. In this subsection, we reformulate the original problem (1.5)–(1.8) into an equivalent one for classical solutions. For simplicity, we still set $\tau = 1$.

Set

$$w = \psi - \sqrt{\mathcal{C}}.$$

By using (1.5), (1.7) and (3.9), we have the following systems for $(w, \mathbf{u}, V)$:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{u} = f_1(x, t), \tag{4.1}$$

$$w_{tt} + w_t + \frac{1}{4}\varepsilon^2 \Delta^2 w + \mathcal{C} w = f_2(x, t) + f_3(x, t), \tag{4.2}$$

$$\Delta V = (2\sqrt{\mathcal{C}} + w)w, \tag{4.3}$$

and the corresponding initial values are

$$w(x, 0) = w_1(x), \;\; w_t(x, 0) = w_2(x), \;\; \mathbf{u}(x, 0) = \mathbf{u}_1(x), \tag{4.4}$$

with

$$w_1(x) =: \psi_1 - \sqrt{\mathcal{C}}, \;\; w_2(x) =: \mathbf{u}_1 \cdot \nabla(\sqrt{\mathcal{C}} + w_1) - \frac{1}{2}(\sqrt{\mathcal{C}} + w_1)\nabla \cdot \mathbf{u}_1. \tag{4.5}$$

Here

$$f_1(x, t) = \nabla V - \nabla(h((\sqrt{\mathcal{C}} + w)^2) - h(\mathcal{C})) + \frac{1}{2}\varepsilon^2 \nabla \left(\frac{\Delta w}{w + \sqrt{\mathcal{C}}} \right), \tag{4.6}$$

$$f_2(x, t) = -2\mathbf{u} \cdot \nabla w_t + P'(\mathcal{C})\Delta w, \tag{4.7}$$

$$\begin{aligned} f_3(x, t) = &- \frac{w_t^2}{w + \sqrt{\mathcal{C}}} - \frac{1}{2} w^2 (3\sqrt{\mathcal{C}} + w) - \nabla w \cdot \nabla V + \frac{\varepsilon^2}{4} \frac{|\Delta w|^2}{(\sqrt{\mathcal{C}} + w)} \\ &+ (P'((\sqrt{\mathcal{C}} + w)^2) - P'(\mathcal{C}))\Delta w + \frac{(P'((\sqrt{\mathcal{C}} + w)^2)(\sqrt{\mathcal{C}} + w))'}{\sqrt{\mathcal{C}} + w} |\nabla w|^2 \\ &+ \frac{1}{2(\sqrt{\mathcal{C}} + w)} \nabla^2 \cdot \Big([\sqrt{\mathcal{C}} + w]^2 \mathbf{u} \otimes \mathbf{u} \Big) + 2\mathbf{u} \cdot \nabla w_t. \end{aligned} \tag{4.8}$$

The derivatives of w and $\mathbf{u}$ satisfy:

$$2w_t + 2\mathbf{u} \cdot \nabla(\sqrt{\mathcal{C}} + w) + (\sqrt{\mathcal{C}} + w)\nabla \cdot \mathbf{u} = 0. \tag{4.9}$$

4.2. The a-priori estimates. For all $T > 0$, define a suitable function space for the unknown $(w, \mathbf{u}, V)$ of the IVP (4.2)–(4.4) in the following way:

$$X(T) = \{(w, \mathbf{u}, V) \in H^6(\mathbb{T}^n) \times H^5(\mathbb{T}^n) \times \dot{H}^4(\mathbb{T}^n), \quad 0 \le t \le T\}$$

with norm

$$M(0, T) = \max_{0 \le t \le T} \{\|w(t)\|_{H^6(\mathbb{T}^n)} + \|\mathbf{u}(t)\|_{H^5(\mathbb{T}^n)} + \|V(t)\|_{\dot{H}^4(\mathbb{T}^n)}\},$$

and assume that

$$\delta_T = \max_{0\le t\le T}(\|w(t)\|_{H^6(\mathbb{T}^n)} + \|\mathbf{u}(t)\|_{H^5(\mathbb{T}^n)}) \ll 1. \tag{4.10}$$

Under the assumption (4.10), it follows immediately

$$-\frac{1}{2}\sqrt{\mathcal{C}} \le w \le \frac{1}{2}\sqrt{\mathcal{C}}. \tag{4.11}$$

Lemma 4.1. *Let* $(w, \mathbf{u}, V) \in X(T)$*, let the multi-index* α *satisfy* $0 \le |\alpha| \le 4$*, then the following inequality holds*

$$|\nabla V|^2 + \|V\|_5^2 \le C\|w\|_3^2, \quad |V_t|^2 + |\nabla V_t|^2 + \|V_t\|_4^2 \le C\|D^\alpha w_t\|_2^2, \tag{4.12}$$

$$\|D^\alpha \mathbf{u}\|^2 \le C\|D^\alpha \mathbf{u}_1\|^2 e^{-t} + C\|D^\alpha(\nabla V, w_t, w, \nabla w, \Delta w)\|^2, \tag{4.13}$$

$$\|D^\alpha f_3\|^2 \le C\|D^\alpha \mathbf{u}_1\|^2 e^{-t} + C\delta_T\|D^\alpha(\nabla V, w_t, w, \nabla w, \Delta w)\|^2, \tag{4.14}$$

provided that $\delta_T \ll 1$.

Proof. The estimates (4.12) follows from Theorem 2.3, since the integral of the right-hand side term of (4.3) equals zero due to the conservation of density and (1.9). By (4.12) and by (4.10), we have

$$|\nabla V| + |V_t| + |\nabla V_t| + ||(\nabla V, \nabla V_t)|| \le C\delta_T. \tag{4.15}$$

In order to estimate (4.13) we take the inner product between (4.1) and $\mathbf{u}$ on $\mathbb{T}^n$, then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \|\mathbf{u}\|^2 &= -\frac{1}{2}\int_{\mathbb{T}^n} \mathbf{u}\cdot\nabla(|\mathbf{u}|^2)dx + \int_{\mathbb{T}^n} f_1\cdot\mathbf{u}\,dx \\ &\le \left(\frac{1}{4} + C\delta_T\right)\|\mathbf{u}\|^2 + C\|\nabla\cdot\mathbf{u}\|^2 + C\|(w, \nabla V, \Delta w)\|^2. \end{aligned}\tag{4.16}$$

By replacing $\nabla\cdot\mathbf{u}$ in (4.16) by (4.9) and by (4.12), one has

$$\frac{d}{dt}\|\mathbf{u}\|^2 + (3/2 - C\delta_T)\|\mathbf{u}\|^2 \le C\|(\nabla w, w_t, \Delta w)\|^2 dx. \tag{4.17}$$

By applying the Gronwall Lemma, by taking δ_T small enough such that $1 - C\delta_T \le 1/2$, we get (4.13) for $\alpha = 0$.

In order to get higher order estimates, we set $\hat{u} = D^\alpha\mathbf{u}$. It satisfies the equation[1]

$$\hat{u}_t + (\mathbf{u}\cdot\nabla)\hat{u} + \hat{u} = f_5 + \nabla f_6, \tag{4.18}$$

where

$$f_5(x,t) = \nabla(D^\alpha V) - D^\alpha\nabla h(\sqrt{\mathcal{C}} + w) - [D^\alpha((\mathbf{u}\cdot\nabla)\mathbf{u}) - (\mathbf{u}\cdot\nabla)D^\alpha\mathbf{u}], \tag{4.19}$$

$$f_6(x,t) = \frac{1}{2}\varepsilon^2 D^\alpha\left(\frac{\Delta w}{\sqrt{\mathcal{C}} + w}\right). \tag{4.20}$$

[1] For the proof of the case $|\alpha| = 4$, we can assume that the solutions $(w, \mathbf{u}, V)$ have high order regularity to have enough smooth derivatives, since the a-priori estimates (4.24) and (4.31) below are still valid for these solutions when smoothed by Friedrich's mollifier under assumptions similar to (4.10). We omit all the details here.

Let us take the inner product between (4.18) and $\hat{u}$ and integrate by parts over $\mathbb{T}^n$. Then, it follows

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\hat{u}\|^2 + \left(\frac{3}{4} - \frac{1}{2}\nabla\cdot\mathbf{u}\right)\|\hat{u}\|^2 \\
\leq &- C\|f_5\|^2 + \frac{1}{2}\varepsilon^2\int_{\mathbb{T}^n}|D^\alpha\nabla\cdot\mathbf{u}|\left|D^\alpha((\sqrt{\mathcal{C}}+w)^{-1}\Delta w)\right|dx \\
\leq &C\|D^\alpha(\nabla V, w, \nabla w, \Delta w)\|^2 + C\delta_T\|\hat{u}\|^2 + \frac{1}{4}\|\nabla\cdot(D^\alpha\mathbf{u})\|^2. \qquad (4.21)
\end{aligned}
$$

By Lemma 2.5 and by (4.9), one has

$$
\begin{aligned}
\|\nabla\cdot(D^\alpha\mathbf{u})\|^2 \leq &C\|D^\alpha((\sqrt{\mathcal{C}}+w)^{-1}w_t)\|^2 + C\|D^\alpha((\sqrt{\mathcal{C}}+w)^{-1}(\mathbf{u}\cdot\nabla)w)\|^2 \\
\leq &C\|D^\alpha(w_t, w, \nabla w)\|^2 + C\delta_T\|D^\alpha\mathbf{u}\|^2. \qquad (4.22)
\end{aligned}
$$

By substituting (4.22) into (4.21) and by using the Gronwall inequality, one obtains (4.13) for $1 \leq |\alpha| \leq 4$, provided that δ_T is small enough.

Finally, we estimate (4.14), with the help of Lemma 2.5, (4.10)–(4.13), (4.9), as

$$
\begin{aligned}
\|D^\alpha f_3\|^2 \leq &C\delta_T\|D^\alpha(\nabla V, w, w_t, \nabla w, \Delta w, \mathbf{u}, D^\alpha\nabla^2 w)\|^2 + C\delta_T\sum_{l,j}\|D^\alpha\partial_l u_j\|^2 \\
\leq &C\delta_T\|D^\alpha(\nabla V, w, w_t, \nabla w, \Delta w, \mathbf{u}, \nabla\cdot\mathbf{u})\|^2 \\
\leq &C\|D^\alpha\mathbf{u}_1\|^2 e^{-t} + C\delta_T\|D^\alpha(\nabla V, w, w_t, \nabla w, \Delta w)\|^2. \qquad (4.23)
\end{aligned}
$$

Thus, the proof of Lemma 4.1 is complete. □

We have the following basic estimates:

Lemma 4.2. *Let $(w, \mathbf{u}, V) \in X(T)$, then there exists $\beta_1 > 0$, such that*

$$
\|(w, \nabla w, \Delta w, w_t)(t)\|^2 + \|\mathbf{u}(t)\|_1^2 + \|V(t)\|_2^2 \leq C(||w_1||_2^2 + ||\mathbf{u}_1||_1^2)e^{-\beta_1 t}, \qquad (4.24)
$$

provided that δ_T is small enough.

Proof. Take the inner product between (4.2) and $w + 2w_t$ and integrate by parts over $\mathbb{T}^n$. Therefore one has

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^n}\left(\frac{1}{2}w^2 + ww_t + w_t^2 + \mathcal{C}w^2 + \frac{1}{4}\varepsilon^2|\Delta w|^2\right)dx \\
&+ \frac{1}{4}\varepsilon^2\|\Delta w\|^2 + \mathcal{C}\|w\|^2 + \|w_t\|^2 \\
&\quad = \int_{\mathbb{T}^n}(f_2 + f_3)(w + 2w_t)dx \\
&\quad \leq C\delta_T\|(w_t, w, \nabla w, \Delta w)\|^2 + C\|\mathbf{u}_1\|^2 e^{-t} \\
&\qquad + \frac{1}{4}\mathcal{C}\|w\|^2 + \frac{1}{4}\|w_t\|^2 + \int_{\mathbb{T}^n} f_2(w + 2w_t)dx. \qquad (4.25)
\end{aligned}
$$

By integration by parts and (4.9), the last term on the right hand side of (4.25) can be estimated by

$$
\begin{aligned}
\int_{\mathbb{T}^n} f_2(w+2w_t)dx =& \int_{\mathbb{T}^n} (2ww_t\nabla\cdot\mathbf{u} + 2w_t\mathbf{u}\cdot\nabla w + w_t^2\nabla\cdot\mathbf{u})dx \\
& - P'(\mathcal{C})\frac{d}{dt}\|\nabla w\|^2 - P'(\mathcal{C})\|\nabla w\|^2 \\
\leq& C\delta_T\|(w, w_t, \nabla w)\|^2 - P'(\mathcal{C})\frac{d}{dt}\|\nabla w\|^2 - P'(\mathcal{C})\|\nabla w\|^2.
\end{aligned} \tag{4.26}
$$

Since

$$
\|\nabla w\|^2 \leq \frac{L^2}{4\pi^2}\|\Delta w\|^2, \tag{4.27}
$$

it follows

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^n}\left(\frac{1}{2}w^2 + ww_t + w_t^2 + \mathcal{C}w^2 + \frac{1}{4}\varepsilon^2|\Delta w|^2 + P'(\mathcal{C})|\nabla w|^2\right)dx \\
&+\left(\frac{1}{4}A_0 - C\delta_T\right)\|\Delta w\|^2 + \left(\frac{3}{4}\mathcal{C} - C\delta_T\right)\|w\|^2 + \left(\frac{3}{4} - C\delta_T\right)\|w_t\|^2 \\
&\leq C\|\mathbf{u}_1\|^2 e^{-t},
\end{aligned} \tag{4.28}
$$

where A_0 is defined by the "subsonic" condition (1.10)

$$
A_0 = \frac{\pi^2}{L^2}\varepsilon^2 + P'(\mathcal{C}) > 0.
$$

Note that there are positive constants $\kappa_1,\ \beta_0$ such that

$$
\begin{aligned}
&||(w, w_t, \nabla w, \Delta w)||^2 \\
&\quad\leq \kappa_1\int_{\mathbb{T}^n}\left(\frac{1}{2}w^2 + ww_t + w_t^2 + \mathcal{C}w^2 + \frac{1}{4}\varepsilon^2|\Delta w|^2 + P'(\mathcal{C})|\nabla w|^2\right)dx \\
&\quad\leq \kappa_1\beta_0^{-1}\|(w_t, w, \Delta w)\|^2.
\end{aligned} \tag{4.29}
$$

Hence, by applying the Gronwall lemma to (4.28) and using (4.29), we get

$$
||(w, w_t, \nabla w, \Delta w)||^2 \leq C(||w_1||_2^2 + ||\mathbf{u}_1||_1^2)e^{-\beta_1 t} \tag{4.30}
$$

with $0 < \beta_1 < \min\{1,\ \kappa_2\beta_0\}$, provided that δ_T is sufficiently small to have

$$
\min\left\{\frac{1}{4}A_0 - C\delta_T,\ \frac{3}{4}\mathcal{C} - C\delta_T,\ \frac{3}{4} - C\delta_T\right\} =: \kappa_2 > 0.
$$

The combination of (4.30) and (4.12)–(4.13) with $\alpha = 0$ yields (4.24). □

In order to obtain higher order estimates, we differentiate (4.1)–(4.2) with respect to x; therefore by repeating the previous steps and by using Lemmas 4.1–4.2, we have

Lemma 4.3. *Let* $(w, \mathbf{u}, V) \in X(T)$*, then there exists* $\beta_4 > 0$*, such that the following inequality holds:*

$$\|(w, \Delta w, w_t)(t)\|_{|\alpha|}^2 + \|\mathbf{u}(t)\|_{1+|\alpha|}^2 + \|V(t)\|_4^2 \le C(||w_1||_{2+|\alpha|}^2 + ||\mathbf{u}_1||_{1+|\alpha|}^2)e^{-\beta_4 t} \tag{4.31}$$

for $1 \le |\alpha| \le 4$*, provided that* $\delta_T \ll 1$.

Proof. Let $\tilde{w} = D^\alpha w$, with $1 \le |\alpha| \le 4$. Then $\tilde{w}$ satisfies the equation

$$\tilde{w}_{tt} + \tilde{w}_t + \frac{1}{4}\varepsilon^2\Delta^2\tilde{w} + \mathcal{C}\tilde{w} = D^\alpha f_2(x,t) + D^\alpha f_3(x,t). \tag{4.32}$$

Let us take the inner product between (4.32) by $\tilde{w} + 2\tilde{w}_t$ and integrate it by parts over $\mathbb{T}^n$. By using (4.10), (4.11), and (4.14), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^n}\left(\frac{1}{2}\tilde{w}^2 + \tilde{w}\tilde{w}_t + \tilde{w}_t^2 + \mathcal{C}\tilde{w}^2 + \frac{1}{4}\varepsilon^2|\Delta\tilde{w}|^2\right)dx \\
&+ \frac{1}{4}\varepsilon^2\|\Delta\tilde{w}\|^2 + \mathcal{C}\|\tilde{w}\|^2 + \|\tilde{w}_t\|^2 \\
&\quad \le C\delta_T\|(\tilde{w}_t, \tilde{w}, \nabla\tilde{w}, \Delta\tilde{w}, \nabla V)\|^2 + \frac{1}{8}\mathcal{C}\|\tilde{w}\|^2 + \frac{1}{8}\|\tilde{w}_t\|^2 \\
&\qquad + C\|D^\alpha\mathbf{u}_1\|^2\exp\{-t\} + \int_{\mathbb{T}^n} D^\alpha f_2(\tilde{w} + 2\tilde{w}_t)dx.
\end{aligned} \tag{4.33}$$

By integrating by parts and by using (4.9), (4.13), the last term on the right hand side of (4.33) can be estimated as follows:

$$\begin{aligned}
\int_{\mathbb{T}^n} D^\alpha f_2(\tilde{w} + 2\tilde{w}_t)dx =& -2\int_{\mathbb{T}^n}[D^\alpha(\mathbf{u}\cdot\nabla w_t) - \mathbf{u}\cdot\nabla\tilde{w}_t](\tilde{w} + 2\tilde{w}_t)dx \\
&+ \int_{\mathbb{T}^n}(2\tilde{w}\tilde{w}_t\nabla\cdot\mathbf{u} + 2\tilde{w}_t\mathbf{u}\cdot\nabla\tilde{w} + \tilde{w}_t^2\nabla\cdot\mathbf{u})dx \\
&- P'(\mathcal{C})\frac{d}{dt}\|\nabla\tilde{w}\|^2 - P'(\mathcal{C})\|\nabla\tilde{w}\|^2 \\
\le& C\delta_T\|(D^\alpha\mathbf{u}, \nabla w_t, \tilde{w}, \nabla\tilde{w})\|^2 + \frac{1}{8}\mathcal{C}\|\tilde{w}\|^2 + \frac{1}{8}\|\tilde{w}_t\|^2 \\
&- P'(\mathcal{C})\frac{d}{dt}\|\nabla\tilde{w}\|^2 - P'(\mathcal{C})\|\nabla\tilde{w}\|^2 \\
\le& C\delta_T\|(w_t, w, \Delta w, D^\alpha\nabla V)\|^2 + C\|D^\alpha\mathbf{u}_1\|^2 e^{-t} \\
&+ \frac{1}{8}\mathcal{C}\|\tilde{w}\|^2 + \frac{1}{8}\|\tilde{w}_t\|^2 \\
&- P'(\mathcal{C})\frac{d}{dt}\|\nabla\tilde{w}\|^2 - P'(\mathcal{C})\|\nabla\tilde{w}\|^2,
\end{aligned} \tag{4.34}$$

where we used the Nirenberg type inequality

$$\|\nabla\tilde{w}\| \le C(\|\tilde{w}\|^2 + \|\Delta\tilde{w}\|^2). \tag{4.35}$$

By substituting (4.34) into (4.33), by using the Gronwall inequality, (4.24), (4.35), and an argument similar to the one for (4.29), we have, for $1 \le |\alpha| \le 4$, that

$$\|(\tilde{w}, \tilde{w}_t, \tilde{w}, \Delta\tilde{w})\|^2 \le C(\|w_1\|^2_{2+|\alpha|} + \|\mathbf{u}_1\|^2_{1+|\alpha|})e^{-\beta_2 t}, \tag{4.36}$$

where β_2 is a suitable positive constant. Finally we have

$$\begin{aligned}\|D^{\alpha+1}\mathbf{u}\|^2 \le& C\|\nabla\cdot(D^\alpha\mathbf{u})\|^2 \le C\|D^\alpha(w_t, w, \nabla w, \mathbf{u})\|^2\\ \le& C\|D^\alpha(w_t, w, \nabla w, \Delta w)\|^2 + C\|D^\alpha\mathbf{u}_1\|^2 e^{-t}\\ \le& C(\|w_1\|^2_{2+|\alpha|} + \|\mathbf{u}_1\|^2_{1+|\alpha|})e^{-\beta_3 t}\end{aligned} \tag{4.37}$$

with $\beta_3 = \min\{\beta_2,\ 1\}$.

The estimate (4.31) follows from (4.36)–(4.37) and Lemma 4.1. □

Hence by Lemmas 4.1–4.3, (4.35) and by the Sobolev embedding theorem, we get the following result.

Theorem 4.4. *Let $(w, \mathbf{u}, V) \in X(T)$, then the following inequality holds:*

$$\|w(t)\|^2_{H^6(\mathbb{T}^n)} + \|\mathbf{u}(t)\|^2_{H^5(\mathbb{T}^n)} + \|V(t)\|^2_{\dot{H}^4(\mathbb{T}^n)} \le C\delta_0 e^{-\beta_5 t}, \tag{4.38}$$

provided that $\delta_T \ll 1$. Here $\beta_5 = \min\{\beta_4,\ \beta_1\}$ and δ_0 is given by (1.11).

Proof of the Theorem 1.3. Based on Theorem 4.4, we can prove that (4.10) is true for the classical solution existing locally in time, as long as $\delta_0 = ||\psi_1 - \sqrt{C}||_6^2 + ||\mathbf{u}_1||_5^2$ is small enough (e.g. $C\delta_0 \ll 1$). Then via the classical continuity argument and the uniform a-priori bounds (4.38) we have the global existence, and the time-asymptotic behavior of our solutions. □

Acknowledgement. The authors thank the referees for useful comments on the presentation of this paper.
The authors thank Professor C. Dafermos for his interest and discussion. H.L. is supported by JSPS post-doctor fellowship and by the Wittgenstein Award 2000 of Peter A. Markowich, funded by the Austrian FWF. Part of the research was made when H.L. visited the Departmento di Matematica Pura e Applicata, University of L'Aquila; he is grateful for the hospitality of the department. P.M. is partially supported by RTN Grant HPRN-CT-2002-00282 (HYKE European Network) and MIUR-COFIN-2002.

References

1. Brezzi, F., Gasser, I., Markowich, P., Schmeiser, C.: Thermal equilibrium state of the quantum hydrodynamic model for semiconductor in one dimension. Appl. Math. Lett. **8**, 47–52 (1995)
2. Courant, R., Friedrichs, K.O.: *Supersonic flow and shock waves*. AMS, Vol. **21**, New York-Heidelberg: Springer-Verlag, 1976
3. Chen, G., Wang, D.: Formation of singularities in compressible Euler-Poisson fluids with heat diffusion and damping relaxation. J. Diff. Eqs. **144**, 44–65 (1998)
4. Dafermos, C.: *Hyperbolic conservation law in continous mechanics*. Grundlehren der mathematischen Wissenschaften Vol. **325**, Berlin: Springer, 2000
5. Feynman, R.: *Statistical mechanics*, A set of lectures. New York: W.A. Benjamin, 1972
6. Gamba, I.: Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductor. Commun. PDEs. **17**, 553–577 (1992)
7. Gamba, I., Morawetz, C.S., A viscous approximation for a 2-D steady semiconductor or transonic gas dynamic flow. Existence theorem for potential flow. Comm. Pure Appl. Math. **49**, 999–1049 (1996)
8. Gamba, I., Jüngel, A.: Asymptotic limits in quantum trajectory models. To appear in Commun. PDEs., 2001

9. Gamba, I., Jüngel, A.: Positive solutions to singular second and third order differential equations for quantum fluids. Arch. Rat. Mech. Anal. **156**, 183–203 (2001)
10. Gardner, C.: The quantum hydrodynamic model for semiconductors devices. SIAM J. Appl. Math. **54**, 409–427 (1994)
11. Gardner, C., Ringhofer, C.: Dispersive/hyperbolic models for transport in semiconductor devices. Accepted for publication in IMA Volumes in Mathematics and its Applications
12. Gasser, I., Jüngel, A.: The quantum hydrodynamic model for semiconductors in thermal equilibrium. Z. Angew. Math. Phys. **48**, 45–59 (1997)
13. Gasser, I., Lin, C.-K., Markowich, P.: A review of dispersive limits of the (non)linear Schrödinger-type equation. Taiwanese J. Math. **4**, 501–529 (2000)
14. Gasser, I., Markowich, P.: Quantum hydrodynamics, Wigner transforms and the classical limit. Asymptotic Anal. **14**, 97–116 (1997)
15. Gasser, I., Markowich, P.A., Ringhofer, C.: Closure conditions for classical and quantum moment hierarchies in the small temperature limit. Transp. Theory Stat. Phys. **25**, 409–423 (1996)
16. Gyi, M.T., Jüngel, A.: A quantum regularization of the one-dimensional hydrodynamic model for semiconductors. Adv. Diff. Eqs. **5**, 773–800 (2000)
17. Jüngel, A.: A steady-state potential flow Euler-Poisson system for charged quantum fluids. Comm. Math. Phys. **194**, 463–479 (1998)
18. Jüngel, A.: *Quasi-hydrodynamic semiconductor equations*. Progress in Nonlinear Differential Equations, Basel: Birkhäuser, 2001
19. Jüngel, A., Li,H.-L.: *Quantum Euler-Poisson system: Existence of stationary states*. Preprint 2001
20. Jüngel, A., Li, H.-L.: *Quantum Euler-Poisson system: Global existence and exponential decay*. Preprint 2002
21. Klusdahl, N., Kriman, A., Ferry, D., Ringhofer, C.: Self-consistent study of the resonant-tunneling diode. Phys. Rev. B. **39**, 7720–7735 (1989)
22. Kato, T.: *Quasi-linear equations of evolution, with applications to partial differential equations*. Lecture Notes in Math. **448**. Berlin: Springer, 1975, pp. 25–70
23. Landau, L.D., Lifshitz, E.M.: *Fluid dynamics*. Oxford: Pergamon Press, 1959
24. Landau, L.D., Lifshitz, E.M.: *Quantum mechanics: Non-relativistic theory*. New York: Pergamon Press, 1965
25. Li, H.-L., Lin, C.-K.: *Semiclassical limit and well-posedness of nonlinear Schrödinger-Poisson*. Preprint 2001
26. Loffredo, M., Morato, L.: On the creation of quantum vortex lines in rotating HeII. Il nouvo cimento **108B**, 205–215 (1993)
27. Madelung, E.: Quantentheorie in hydrodynamischer Form. Z. Physik **40**, 322 (1927)
28. Majda, A.: *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*. Berlin Heidelberg-New York: Springer-Verlag, 1984
29. Marcati, P.: Stability for second order abstract evolution equations. Nonl. Anal. TMA. **8**, 237–252 (1984)
30. Marcati, P.: Decay and stability for nonlinear hyperbolic equations. J. Diff. Eqs. **55**, 30–58 (1984)
31. Markowich, P.: On the equivalentce of the Schrödinger and the quantum Liouville equations. Math. Meth. Appl. Sci. **11**, 459–469 (1989)
32. Markowich, P.A., Ringhofer, C., Schmeiser, C.: *Semiconductor Equations*. Wien: Springer, 1990
33. Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Math. Pura. Appl. **146**, 65–96 (1987)
34. Taylor, M.E.: *Pseudodifferential operators and nonlinear PDE*. Progress in Mathematics Vol. **100**, Boston: Birkhauser 1991
35. Temam, R.: *Infinite-dimensional dynamical systems in mechanics and physics*. Appl. Math. Sci. **68**. Berlin-Heidelberg-New York: Springer-Verlag, 1988
36. Unterreiter, A.: The thermal equilibrium solution of a generic bipolar quantum hydrodynamic model. Commun. Math. Phys. **188**, 69–88 (1997)
37. Wigner, E.: On the quantum correction for thermodynamic equilibrium. Phys. Rev. **40**, 749–759 (1932)
38. Zhang, B., Jerome, W.: On a steady-state quantum hydrodynamic model for semiconductors. Nonlinear Anal. TMA **26**, 845–856 (1996)
39. Zeidler, E.: *Nonlinear functional analysis and its applications*. Vol. **II:** Nonlinear monotone operators. Berlin-Heidelberg-New York: Springer-Verlag, 1990

Communicated by P. Constantin