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Noncommutative Instantons Revisited

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Abstract: We find a new gauge in which U(1) noncommutative instantons are explicitly non-singular on noncommutative $\mathbf{R}^4$. We also present a pedagogical introduction into noncommutative gauge theories.

1. Introduction

Recently there has been a revival of interest in noncommutative gauge theories [1, 2]. They are interesting as examples of field theories which have as their planar limit large N gauge theories [3, 4]; certain supersymmetric versions of noncommutative gauge theories arise as the $\alpha' \to 0$ limit of theories on Dp-branes in the presence of background B-field [6, 36]; the related theories arise in Matrix compactifications with C-field turned on [5]; finally, noncommutativity is in some sense an intrinsic feature of the open string field theory [9, 7, 10]. A lot of progress has been recently achieved in the analysis of the classical solutions of the noncommutative gauge theory. The first explicit solutions and their moduli were analysed in [11] where instantons in the four dimensional noncommutative gauge theory (with self-dual noncommutativity) were constructed. These instantons play an important role in the construction of the discrete light cone quantization of the M-theory fivebrane [35, 34], and they also gave a hope of giving an interpretation in the physical gauge theory language of the torsion free sheaves which appear in various interpretations of D-brane states [12, 13], in particular those responsible for the enthropy of black holes realized via D5-D1 systems [37], and also entering the S-duality invariant partition functions of $\mathcal{N} = 4$ super-Yang-Mills theory [38]. In addition to the instantons (which are particles in 4+1 dimensional theory), which represent the D0-D4 system, the monopole-like solutions were found [23] in U(1) gauge theory in 3+1 dimensions. The latter turn out to have a string attached to them. Both the string and the monopole at its end are the noncommutative field theory realization of the D3-D1 system, where the D1 string ends on the D3 brane and bends at some specific angle towards the brane. One can also find the solutions describing the string itself [24, 25], both the BPS and in the non-BPS states; also the dimensionally reduced solutions in 2+1 dimensions [25, 28], describing the D0-D2 systems; finite length strings, corresponding to $U(2)$ monopoles [26].

This paper is devoted to the clarification of the issue of nonsingularity of the noncommutative U(1) instantons. We shall show that one can find a gauge in which the solutions are explicitly nonsingular, and well-defined over all of noncommutative $\mathbb{R}^4$. Compared to [11] we also relax the assumption on the noncommutativity. We shall only demand that the Poisson tensor θ^{ij} has non-negative Pfaffian: $\mathrm{Pf}(\theta) \neq 0$ (and of course, that the space is noncommutative, i.e. at least one of the eigen-values of $\theta^{ij} G_{jk} \theta^{kl}$ must be non-vanishing, G being the Euclidean metric on the space).

The paper is organized as follows. Section 2 contains a pedagogical introduction into noncommutative gauge theories. Section 3 constructs instantons in noncommutative gauge theory on $\mathbf{R}^4$ for any group $U(N)$. Section 4 presents explicit formulae for the $U(1)$ gauge group.

Note added. As this paper was ready for publication two related papers appeared. The paper [28] also discusses codimension four solitons in noncommutative gauge theory, using operators S, $S^{\dagger}$ (which we introduce later in Sect. 4). These, however, are non-BPS solutions (and the role of S and $S^{\dagger}$ is reversed), and do not obey instanton equations. The paper [33] overlaps with us in that it also uses the operators S, $S^{\dagger}$ for constructing instanton gauge fields. Also, some of the discussion of the relation of the torsion free sheaves on $\mathbf{C}^2$ to the noncommutative instantons is similar.

2. Noncommutative Geometry and Noncommutative Field Theory

2.1. A brief mathematical introduction. It has been widely appreciated by mathematicians (starting with the seminal works of Gelfand, Grothendieck, and von Neumann) that the geometrical properties of a space X are encoded in the properties of the commutative algebra $C(X)$ of the continuous functions $f : X \to \mathbf{C}$ with the ordinary rules of point-wise addition and multiplication: $(f+g)(x) = f(x)+g(x)$, $f \cdot g(x) = f(x)g(x)$.

More precisely, $C(X)$ knows only about the topology of X, but one can refine the definitions and look at the algebra $C^{\infty}(X)$ of the smooth functions or even at the DeRham complex $\Omega^{\cdot}(X)$ to decipher the geometry of X.

The algebra $A = C(X)$ is clearly associative, commutative and has a unit ($\mathbf{1}(x) = 1$). It also has an involution, which maps a function to its complex conjugate: $f^{\dagger}(x) = \overline{f(x)}$.

The points x of X can be viewed in two ways: as maximal ideals of A: $f \in I_x \Leftrightarrow f(x) = 0$; or as the irreducible (and therefore one-dimensional for A is commutative) representations of A: $R_x(f) = f(x)$, $R_x \approx \mathbf{C}$.

The vector bundles over X give rise to projective modules over A. Given a bundle E let us consider the space $\mathcal{E} = \Gamma(E)$ of its sections. If $f \in A$ and $\sigma \in \mathcal{E}$ then clearly $f\sigma \in \mathcal{E}$. This makes $\mathcal{E}$ a representation of A, i.e. a module. Not every module over A arises in this way. The vector bundles over topological spaces have the following remarkable property, which is the content of the Serre-Swan theorem: for every vector bundle E there exists another bundle E' such that the direct sum $E \oplus E'$ is a trivial bundle $X \times \mathbf{C}^N$ for sufficiently large N. When translated to the language of modules this property reads as follows: for the module $\mathcal{E}$ over A there exists another module $\mathcal{E}'$ such that $\mathcal{E} \oplus \mathcal{E}' = F_N = A^{\oplus N}$. We have denoted by $F_N = A \otimes_{\mathbf{C}} \mathbf{C}^N$ the free module over A of rank N. Unless otherwise stated the symbol $\otimes$ below will be used for tensor products over $\mathbf{C}$. The modules with this property are called *projective*. The reason for them to be called in such a way is that $\mathcal{E}$ is an image of the free module F_N under the projection which is identity on $\mathcal{E}$ and zero on $\mathcal{E}'$. In other words, for each projective module $\mathcal{E}$ there exists N and an operator $P \in \mathrm{Hom}(F_N, F_N)$, such that $P^2 = P$, and $\mathcal{E} = P \cdot F_N$.

Noncommutative geometry relaxes the condition that A must be commutative, and tries to develop a geometrical intuition about the noncommutative associative algebras with anti-holomorphic involution † ($\mathbf{C}^*$-algebras).

In particular, the notion of a vector bundle over X is replaced by the notion of the projective module over A. Now, when A is noncommutative, there are two kinds of modules: left and right ones. The left A-module is the vector space M_l with the operation of left multiplication by the elements of the algebra A: for $m \in M_l$ and $a \in A$ there must be an element $am \in M_l$, such that for a_1, a_2: $a_1(a_2 m) = (a_1 a_2)m$. The definition of the right A-module M_r is similar: for $m \in M_r$ and $a \in A$ there must be an element $ma \in M_r$, such that for a_1, a_2: $(ma_1)a_2 = m(a_1 a_2)$. The free module $F_N = A \oplus \ldots_{N\,\text{times}} \oplus A = A \otimes \mathbf{C}^N$ is both a left and right one. The projective A-modules are defined just as in the commutative case, except that for the left projective A-module $\mathcal{E}$ the module $\mathcal{E}'$, such that $\mathcal{E} \oplus \mathcal{E}' = F_N$, also must be left, and similarly for the right modules.

The manifolds can be mapped one to another by means of smooth maps: $g : X_1 \to X_2$. The algebras of smooth functions are mapped in the opposite way: $g^* : C^\infty(X_2) \to C^\infty(X_1)$, $g^*(f)(x_1) = f(g(x_1))$. The induced map of the algebras is the algebra homomorphism:

$$g^*(f_1 f_2) = g^*(f_1)g^*(f_2),\ g^*(f_1 + f_2) = g^*(f_1) + g^*(f_2).$$

Naturally, the smooth maps between two manifolds are replaced by the homomorphisms of the corresponding algebras. In particular, the maps of the manifold to itself form the associative algebra $Hom(A, A)$. The diffeomorphisms would correspond to the invertible homomorphisms, i.e. automorphisms $Aut(A)$. Among those there are internal ones, generated by the invertible elements of the algebra:

$$a \mapsto g^{-1} a g.$$

The infinitesimal diffeomorphisms of the ordinary manifolds are generated by the vector fields $V^i \partial_i$, which differentiate functions,

$$f \mapsto f + \varepsilon V^i \partial_i f.$$

In the noncommutative setup the vector field is replaced by the derivation of the algebra $V \in Der(A)$:

$$a \mapsto a + \varepsilon V(a), \quad V(a) \in A$$

and the condition that $V(a)$ generates an infinitesimal homomorphism reads as:

$$V(ab) = V(a)b + aV(b),$$

which is just the definition of the derivation. Among various derivations there are internal ones, generated by the elements of the algebra itself:

$$V_c(a) = [a, c] := ac - ca, \quad c \in A.$$

These infinitesimal diffeomorphisms are absent in the commutative setup, but they have close relatives in the case of the Poisson manifold X.

2.2. Flat noncommutative space. The basic example of the noncommutative algebra which will be studied here is the enveloping algebra of the Heisenberg algebra. Consider the Euclidean space $\mathbf{R}^d$ with coordinates x^i, $i = 1, \ldots, d$. Suppose a constant antisymmetric matrix θ^{ij} is fixed. It defines a Poisson bi-vector field $\theta^{ij}\partial_i\partial_j$ and therefore the noncommutative associative product on $\mathbf{R}^d$. The coordinate functions x^i on the deformed noncommutative manifold will obey the following commutation relations:

$$[x^i, x^j] = i\theta^{ij}. \tag{1}$$

We shall call the algebra $\mathcal{A}_\theta$ (over $\mathbf{C}$) generated by the x^i satisfying (1), together with convergence conditions on the allowed expressions of the x^i – the noncommutative space-time. The algebra $\mathcal{A}_\theta$ has an involution $a \mapsto a^\dagger$ which acts as a complex conjugation on the central elements $(\lambda \cdot \mathbf{1})^\dagger = \bar{\lambda} \cdot \mathbf{1}$, $\lambda \in \mathbf{C}$ and preserves x^i: $(x^i)^\dagger = x^i$. The elements of $\mathcal{A}_\theta$ can be identified with ordinary complex-valued functions on $\mathbf{R}^d$, with the product of two functions f and g given by the Moyal formula (or star product):

$$f \star g\,(x) = \exp\left[\frac{i}{2}\theta^{ij}\frac{\partial}{\partial x_1^i}\frac{\partial}{\partial x_2^j}\right] f(x_1)g(x_2)|_{x_1=x_2=x}. \tag{2}$$

Fock space formalism. By an orthogonal change of coordinates we can map the Poisson tensor θ_{ij} onto its canonical form:

$$x^i \mapsto z_a, \bar{z}_a, \quad a = 1, \ldots, r\,; \quad y_b, \quad b = 1, \ldots, d-2r,$$

so that:

$$[y_a, y_b] = [y_b, z_a] = [y_b, \bar{z}_a] = 0, \quad [z_a, \bar{z}_b] = -2\theta_a\delta_{ab}, \quad \theta_a > 0 \tag{3}$$

$$ds^2 = dx_i^2 + dy_b^2 = dz_a d\bar{z}_a + dy_b^2.$$

Since $z(\bar{z})$ satisfy (up to a constant) the commutation relations of creation (annihilation) operators we can identify functions $f(x, y)$ with the functions of the y_a valued in the space of operators acting in the Fock space $\mathcal{H}_r$ of r creation and annihilation operators:

$$\begin{aligned}
\mathcal{H}_r &= \bigoplus_{\vec{n}} \mathbf{C}\,|n_1, \ldots, n_r\rangle, \\
c_a &= \frac{1}{\sqrt{2\theta_a}}\bar{z}_a, \quad c_a^\dagger = \frac{1}{\sqrt{2\theta_a}}z_a, \quad [c_a, c_b^\dagger] = \delta_{ab}, \\
c_a|\vec{n}\rangle &= \sqrt{n_a}|\vec{n} - 1_a\rangle, \quad c_a^\dagger|\vec{n}\rangle = \sqrt{n_a+1}|\vec{n} + 1_a\rangle.
\end{aligned} \tag{4}$$

Let $\hat{n}_a = c_a^\dagger c_a$ be the a^{th} number operator.

The Hilbert space $\mathcal{H}_r$ is an example of a left projective module over the algebra $\mathcal{A}_\theta$. Indeed, consider the element $P_0 = |\vec{0}\rangle\langle\vec{0}| \sim \exp - \sum_a \frac{z_a\bar{z}_a}{\theta_a}$. It obeys $P_0^2 = P_0$, i.e. it is a projector. Consider the rank one free module $F_1 = \mathcal{A}_\theta$ and let us consider its left sub-module, spanned by the elements of the form: $f \star P_0$. As a module it is clearly isomorphic to $\mathcal{H}_r$, the isomorphism being: $|\vec{n}\rangle \mapsto |\vec{n}\rangle\langle\vec{0}|$. It is a projective module, the complementary module being $\mathcal{A}_\theta(1 - P_0) \subset \mathcal{A}_\theta$.

The procedure that maps ordinary commutative functions onto operators in the Fock space acted on by $z_a, \bar{z}_a$ is called Weyl ordering and is defined by:

$$f(x) \mapsto \hat{f}(z_a, \bar{z}_a) = \int f(x) \frac{\mathrm{d}^{2r}x\ \mathrm{d}^{2r}p}{(2\pi)^{2r}}\ e^{i(\bar{p}_a z_a + p_a \bar{z}_a - p\cdot x)}. \tag{5}$$

It is easy to see that

$$\text{if} \quad f \mapsto \hat{f}, \quad g \mapsto \hat{g} \quad \text{then} \quad f \star g \mapsto \hat{f}\hat{g}. \tag{6}$$

Symmetries of the flat noncommutative space. The algebra (1) has an obvious symmetry: $x^i \mapsto x^i + \varepsilon^i$, for $\varepsilon^i \in \mathbf{R}$. For invertible Poisson structure θ it is an example of the internal automorphism of the algebra:

$$a \mapsto e^{i\theta_{ij}\varepsilon^i x^j} a e^{-i\theta_{ij}\varepsilon^i x^j}. \tag{7}$$

In addition, there are rotational symmetries which we shall not need.

2.3. Gauge theory on noncommutative space. In an ordinary gauge theory with gauge group G the gauge fields are connections in some principal G-bundle. The matter fields are the sections of the vector bundles with the structure group G. Sections of the noncommutative vector bundles are elements of the projective modules over the algebra $\mathcal{A}_\theta$.

In the ordinary gauge theory the gauge field arises through the operation of covariant differentiation of the sections of a vector bundle. In the noncommutative setup the situation is similar. Suppose M is a projective module over $\mathcal{A}$. The connection ∇ is the operator

$$\nabla : \mathbf{R}^d \times M \to M, \quad \nabla_\varepsilon(m) \in M, \qquad \varepsilon \in \mathbf{R}^d,\ m \in M,$$

where $\mathbf{R}^d$ denotes the commutative vector space, the Lie algebra of the automorphism group generated by (7). The connection is required to obey the Leibnitz rule:

$$\nabla_\varepsilon(am_\mathbf{l}) = \varepsilon^i(\partial_i a)m_\mathbf{l} + a\nabla_\varepsilon m_\mathbf{l}, \tag{8}$$
$$\nabla_\varepsilon(m_\mathbf{r}a) = m_\mathbf{r}\varepsilon^i(\partial_i a) + (\nabla_\varepsilon m_\mathbf{r})a. \tag{9}$$

Here, (8) is the condition for the left modules, and (9) is the condition for the right modules. As usual, one defines the curvature $F_{ij} = [\nabla_i, \nabla_j]$ - the operator $\Lambda^2\mathbf{R}^d \times M \to M$ which commutes with the multiplication by $a \in \mathcal{A}_\theta$. In other words, $F_{ij} \in \mathrm{End}_\mathcal{A}(M)$. In ordinary gauge theories the gauge fields come with gauge transformations. In the noncommutative case the gauge transformations, just like the gauge fields, depend on the module they act in. For the module M the group of gauge transformations $\mathcal{G}_M$ consists of the invertible endomorphisms of M which commute with the action of $\mathcal{A}$ on M:

$$\mathcal{G}_M = \mathrm{GL}_\mathcal{A}(M).$$

All the discussion above can be specified to the case where the module has a Hermitian inner product, with values in $\mathcal{A}$.

Fock module and connections there. Recall that the algebra $\mathcal{A}_\theta$ for $d = 2r$ and non-degenerate θ has an important irreducible representation, the left module $\mathcal{H}_r$. Let us now ask, what kind of connections does the module $\mathcal{H}_r$ have?

By definition, we are looking for a collection of operators $\nabla_i : \mathcal{H}_r \to \mathcal{H}_r$, $i = 1, \ldots, 2r$, such that:

$$[\nabla_i, a] = \partial_i a$$

for any $a \in \mathcal{A}$. Using the fact that $\partial_i a = i\theta_{ij}[x^j, a]$ and the irreducibility of $\mathcal{H}_r$ we conclude that:

$$\nabla_i = i\theta_{ij}x^j + \kappa_i, \qquad \kappa_i \in \mathbf{C}. \tag{10}$$

If we insist on unitarity of ∇, then $i\kappa_i \in \mathbf{R}$. Thus, the space of all gauge fields suitable for acting in the Fock module is rather thin, and is isomorphic to the vector space $\mathbf{R}^d$ (which is canonically dual to the Lie algebra of the automorphisms of $\mathcal{A}_\theta$). The gauge group for the Fock module, again due to its irreducibility is simply the group $U(1)$, which multiplies all the vectors in $\mathcal{G}_r$ by a single phase. In particular, it preserves κ_i's, so they are gauge invariant. It remains to find out what is the curvature of the gauge field given by (10). The straightforward computation of the commutators gives:

$$F_{ij} = i\theta_{ij}, \tag{11}$$

i.e. all connections in the Fock module have the constant curvature.

Free modules and connections there. If the right (left) module M is free, i.e. it is a sum of several copies of the algebra $\mathcal{A}_\theta$ itself, then the connection ∇_i can be written as

$$\nabla_i = \partial_i + A_i,$$

where A_i is the operator of the left (right) multiplication by the matrix with $\mathcal{A}_\theta$-valued entries:

$$\nabla_i m_\mathbf{l} = \partial_i m_\mathbf{l} + m_\mathbf{l} A_i, \ \nabla_i m_\mathbf{r} = \partial_i m_\mathbf{r} + A_i m_\mathbf{r}. \tag{12}$$

In the same operator sense the curvature obeys the standard identity:

$$F_{ij} = \partial_i A_j - \partial_j A_i + A_i A_j - A_j A_i \ .$$

Given a module M over some algebra A one can multiply it by a free module $A^{\oplus N}$ to make it a module over an algebra $\mathrm{Mat}_{N\times N}(A)$ of matrices with elements from A. In the non-abelian gauge theory over A we are interested in projective modules over $\mathrm{Mat}_{N\times N}(A)$. If the algebra A (or perhaps its subalgebra) has a trace, Tr, then the algebra $\mathrm{Mat}_{N\times N}(A)$ has a trace given by the composition of a usual matrix trace and Tr.

It is a peculiar property of the noncommutative algebras that the algebras A and $\mathrm{Mat}_{N\times N}(A)$ have much in common. These algebras are called Morita equivalent and under some additional conditions the gauge theories over A and over $\mathrm{Mat}_{N\times N}(A)$ are also equivalent. This phenomenon is responsible for the similarity between the "abelian noncommutative" and "non-abelian commutative" theories.

If we represent ∂_i as $i\theta_{ij}[x^j, \cdot]$ then the expression for the covariant derivative becomes:

$$\nabla_i m_\mathbf{l} = i\theta_{ij}x^j m_\mathbf{r} + m_\mathbf{r} D_i, \quad \nabla_i m_\mathbf{r} = -m_\mathbf{r} i\theta_{ij}x^j + D_i^\dagger m_\mathbf{r}, \tag{13}$$

where

$$D_i = -i\theta_{ij}x^j + A_i. \tag{14}$$

3. Instantons in Noncommutative Gauge Theories

We would like to study the non-perturbative objects in noncommutative gauge theory.

Specifically we shall be interested in four dimensional instantons. They either appear as instantons themselves in the Euclidean version of the four dimensional theory (theory on Euclidean D3-brane), as solitonic particles in the theory on D4-brane, i.e. in 4+1 dimensions, or as instanton strings in the theory on D5-brane (and are related to little strings [21]). They also show up as "freckles" in the gauge theory/sigma model correspondence [14].

The theory depends on the dimensionful parameters θ_α which enter the commutation relation between the coordinates of the space: $[x, x] \sim i\theta$.

We treat only the bosonic fields, but these could be a part of a supersymmetric multiplet, with $\mathcal{N} = 2$ supersymmetry or higher. Such field theories arise on the world volume of D3-branes in the presence of a background constant B-field along the D3-brane.

A D3-brane can be surrounded by other branes as well. For example, in the Euclidean setup, a D-instanton could approach the D3-brane. In fact, unless the D-instanton is dissolved inside the brane, the combined system breaks supersymmetry [36]. The D3-D(-1) system can be rather simply described in terms of a noncommutative $U(1)$ gauge theory – the latter has instanton-like solutions [11]. It is the purpose of this note to explore these solutions in greater detail.

More generally, one can have a stack of k Euclidean D3-branes with N D(-1)-branes inside. This configuration will be described by charge N instantons in $U(k)$ gauge theory.

Let us work in four Euclidean space-time dimensions, $\mu = 1, 2, 3, 4$. As we said above, we shall look at the purely bosonic Yang-Mills theory on the space-time $\mathcal{A}_\theta$ with the coordinates functions x^μ obeying the Heisenberg commutation relations:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}. \tag{15}$$

We assume that the metric on the space-time is Euclidean:

$$ds^2 = \sum_\mu (dx^\mu)^2. \tag{16}$$

The action describing our gauge theory is given by:

$$S = -\frac{1}{4g_{\mathrm{YM}}^2}\mathrm{Tr}F \wedge \star F + \frac{\theta}{8\pi^2}\mathrm{Tr}F \wedge F, \tag{17}$$

where g_{YM}^2 is the Yang-Mills coupling constant, and

$$F = F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad F_{\mu\nu} = \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu. \tag{18}$$

The covariant derivatives ∇_μ act in some module $\mathcal{E}$ over the algebra $\mathcal{A}_\theta$ of functions on the noncommutative $\mathbf{R}^4$.

3.1. Instantons. The equations of motion following from (17) are:

$$\nabla_\mu F_{\mu\nu} = 0. \tag{19}$$

In general these equations are as hard to solve as the equations of motion of the ordinary non-abelian Yang-Mills theory. However, just like in the commutative case, there are special solutions, which are simpler to analyze and which play a crucial role in the analysis of the quantum theory. These are the so-called (anti-)instantons. The (anti-)instantons solve the first order equation:

$$F_{\mu\nu} = \pm \frac{1}{2} \varepsilon_{\mu\nu\kappa\lambda} F_{\kappa\lambda}. \tag{20}$$

These equations are easier to solve. The solutions are classified by the instanton charge:

$$N = -\frac{1}{8\pi^2} \mathrm{Tr} F \wedge F. \tag{21}$$

3.2. ADHM construction. In the commutative case all solutions to (20) with the finite action (17) are obtained via the so-called Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction. If we are concerned with the instantons in the $U(k)$ gauge group, then the ADHM data consists of

1. the pair of the two complex vector spaces V and W of dimensions N and k respectively;
2. the operators: $B_1, B_2 \in \mathrm{Hom}(V, V)$, and $I \in \mathrm{Hom}(W, V)$, $J \in \mathrm{Hom}(V, W)$;
3. the dual gauge group $G_N = U(N)$, which acts on the data above as follows:

$$B_\alpha \mapsto g^{-1} B_\alpha g, \;\; I \mapsto g^{-1} I, \;\; J \mapsto J g; \tag{22}$$

4. Hyperkähler quotient [15] with respect to the group (22). It means that one takes the set $X_{k,N} = \mu_r^{-1}(0) \cap \mu_c^{-1}(0)$ of the common zeroes of the three moment maps:

$$\begin{aligned} \mu_r &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + I I^\dagger - J^\dagger J, \\ \mu_c &= [B_1, B_2] + I J, \\ \bar{\mu}_c &= [B_2^\dagger, B_1^\dagger] + J^\dagger I^\dagger, \end{aligned} \tag{23}$$

and quotients it by the action of G_N.

The claim of ADHM is that the points in the space $\mathcal{M}_{k,N} = X_{k,N}^\circ / G_N$ parameterize the solutions to (20) (for $\theta = 0$) up to the gauge transformations. Here $X_{k,N}^\circ \subset X_{k,N}$ is the open dense subset of $X_{k,N}$ which consists of the solutions to $\vec{\mu} = 0$ such that their stabilizer in G_N is trivial. The explicit formula for the gauge field A_μ is also known. Define the Dirac-like operator:

$$\mathcal{D}^+ = \begin{pmatrix} -B_2 + z_2 & B_1 - z_1 & I \\ B_1^\dagger - \bar{z}_1 & B_2^\dagger - \bar{z}_2 & J^\dagger \end{pmatrix} : V \otimes \mathbf{C}^2 \oplus W \to V \otimes \mathbf{C}^2. \tag{24}$$

Here z_1, z_2 denote the complex coordinates on the space-time:

$$z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4, \quad \bar{z}_1 = x_1 - ix_2, \quad \bar{z}_2 = x_3 - ix_4.$$

The kernel of the operator (24) is the x-dependent vector space $\mathcal{E}_x \subset V \otimes \mathbf{C}^2 \oplus W$. For generic x, $\mathcal{E}_x$ is isomorphic to W. Let us denote by $\Psi = \Psi(x)$ this isomorphism. In plain words, Ψ is the fundamental solution to the equation:

$$\mathcal{D}^+ \Psi = 0, \qquad \Psi : W \to V \otimes \mathbf{C}^2 \oplus W. \tag{25}$$

If the rank of Ψ is x-independent (this property holds for generic points in $\mathcal{M}$), then one can normalize:

$$\Psi^\dagger \Psi = \mathrm{Id}_k, \tag{26}$$

which fixes Ψ uniquely up to an x-dependent $U(k)$ transformation $\Psi(x) \mapsto \Psi(x)g(x)$, $g(x) \in U(k)$. Given Ψ the anti-self-dual gauge field is constructed simply as follows:

$$\nabla_\mu = \partial_\mu + A_\mu, \qquad A_\mu = \Psi^\dagger(x) \frac{\partial}{\partial x^\mu} \Psi(x). \tag{27}$$

The space of (B_0, B_1, I, J) for which $\Psi(x)$ has maximal rank for all x is an open dense subset $\mathcal{M}_{N,k} = X^\circ_{N,k}/G_N$ in $\mathcal{M}$. The rest of the points in $X_{N,k}/G_N$ describes the so-called *point-like* instantons. Namely, $\Psi(x)$ has maximal rank for all x but some finite set $\{x_1, \ldots, x_l\}$, $l \leq k$. Equation (26) holds for $x \neq x_i$, $i = 1, \ldots, l$, where the left hand side of (26) simply vanishes.

The noncommutative deformation of the gauge theory leads to the noncommutative deformation of the ADHM construction. It turns out to be very simple yet surprising. The same data $V, W, B, I, J, \ldots$ is used. The deformed ADHM equations are simply

$$\mu_r = \zeta_r, \; \mu_c = \zeta_c, \tag{28}$$

where we have introduced the following notations. The Poisson tensor θ^{ij} entering the commutation relation $[x^i, x^j] = i\theta^{ij}$ can be decomposed into the self-dual and anti-self-dual parts $\theta^\pm$. If we look at the commutation relations of the complex coordinates $z_1, z_2, \bar{z}_1, \bar{z}_2$, then the self-dual part of θ enters the following commutators:

$$[z_1, z_2] = -\zeta_c \qquad [z_1, \bar{z}_1] + [z_2, \bar{z}_2] = -\zeta_r. \tag{29}$$

It turns out that as long as $|\zeta| = \zeta_r^2 + \zeta_c \bar{\zeta}_c > 0$ one needs not distinguish between $\widetilde{X}_{N,k}$ and $\widetilde{X}^\circ_{N,k}$, in other words the stabilizer of any point in $\widetilde{X}_{N,k} = \mu_r^{-1}(-\zeta_r) \cap \mu_c^{-1}(-\zeta_c)$ is trivial. Then the resolved moduli space is $\widetilde{\mathcal{M}}_{N,k} = \widetilde{X}_{N,k}/G_N$.

By making an orthogonal rotation on the coordinates x^μ we can map the algebra $\mathcal{A}_\theta$ onto the sum of two copies of the Heisenberg algebra. These two algebras can have different values of "Planck constants". Their sum is the norm of the self-dual part of θ, i.e. $|\zeta|$, and their difference is the norm of anti-self-dual part of θ:

$$[z_1, \bar{z}_1] = -\zeta_1, \quad [z_2, \bar{z}_2] = -\zeta_2, \tag{30}$$

where $\zeta_1 + \zeta_2 = |\theta^+|$, $\zeta_1 - \zeta_2 = |\theta^-|$. By the additional reflection of the coordinates, if necessary, one can make both ζ_1 and ζ_2 positive (however, one should be careful, since if the odd number of reflections was made, then the orientation of the space was changed and the notions of the instantons and anti-instantons are exchanged as well).

The next step in the ADHM construction was the definition of the isomorphism Ψ between the fixed vector space W and the fiber $\mathcal{E}_x$ of the gauge bundle, defined as the kernel of the operator $\mathcal{D}^+$. In the noncommutative setup one can also define the operator $\mathcal{D}^+$ by the same formula (24). It is a map between two free modules over $\mathcal{A}_\theta$:

$$\mathcal{D}_x^+ : \left(V \otimes \mathbf{C}^2 \oplus W\right) \bigotimes \mathcal{A}_\theta \to \left(V \otimes \mathbf{C}^2\right) \bigotimes \mathcal{A}_\theta \tag{31}$$

which commutes with the right action of $\mathcal{A}_\theta$ on the free modules. Clearly,

$$\mathcal{E} = \mathrm{Ker}\mathcal{D}^+$$

is a right module over $\mathcal{A}_\theta$, for if $\mathcal{D}^+ s = 0$, then $\mathcal{D}^+(s \cdot a) = 0$, for any $a \in \mathcal{A}_\theta$.

$\mathcal{E}$ is also a projective module, for the following reason. Consider the operator $\mathcal{D}^+\mathcal{D}$. It is a map from the free module $V \otimes \mathbf{C}^2 \bigotimes \mathcal{A}_\theta$ to itself. Thanks to (28) this map actually equals $\Delta \otimes \mathrm{Id}_{\mathbf{C}^2}$, where Δ is the following map from the free module $V \otimes \mathcal{A}_\theta$ to itself:

$$\Delta = (B_1 - z_1)(B_1^\dagger - \bar{z}_1) + (B_2 - z_2)(B_2^\dagger - \bar{z}_2) + II^\dagger. \tag{32}$$

We claim that Δ has no kernel, i.e. no solutions to the equation $\Delta\, v = 0,\ v \in V \otimes \mathcal{A}_\theta$. Recall the Fock space representation $\mathcal{H}$ of the algebra $\mathcal{A}_\theta$. The coordinates $z_\alpha, \bar{z}_\alpha$, obeying (30), with $\zeta_1, \zeta_2 > 0$, are represented as follows:

$$z_1 = \sqrt{\zeta_1}\, c_1^\dagger,\ \bar{z}_1 = \sqrt{\zeta_1}\, c_1, \quad z_2 = \sqrt{\zeta_2}\, c_2^\dagger,\ \bar{z}_2 = \sqrt{\zeta_2}\, c_2, \tag{33}$$

where $c_{1,2}$ are the annihilation operators and $c_{1,2}^\dagger$ are the creation operators for the two-oscillators Fock space

$$\mathcal{H} = \bigoplus_{n_1, n_2 \geq 0} \mathbf{C}\, |n_1, n_2\rangle.$$

Let us assume the opposite, namely that there exists a vector $v \in V \otimes \mathcal{A}_\theta$ such that $\Delta v = 0$. Let us act by this vector on an arbitrary state $|n_1, n_2\rangle$ in $\mathcal{H}$. The result is the vector $\nu_{\vec{n}} \in V \otimes \mathcal{H}$ which must be annihilated by the operator Δ, acting in $V \otimes \mathcal{H}$ via (33). By taking the Hermitian inner product of the equation $\Delta \nu_{\vec{n}} = 0$ with the conjugate vector $\nu_{\vec{n}}^\dagger$ we immediately derive the following three equations:

$$\begin{aligned} (B_2^\dagger - \bar{z}_2)\nu_{\vec{n}} &= 0, \\ (B_1^\dagger - \bar{z}_1)\nu_{\vec{n}} &= 0, \\ I^\dagger \nu_{\vec{n}} &= 0. \end{aligned} \tag{34}$$

Using (28) we can also represent Δ_x in the form:

$$\Delta = (B_1^\dagger - \bar{z}_1)(B_1 - z_1) + (B_2^\dagger - \bar{z}_2)(B_2 - z_2) + J^\dagger J. \tag{35}$$

From this representation another triple of equations follows:

$$\begin{aligned} (B_2 - z_2)\nu_{\vec{n}} &= 0, \\ (B_1 - z_1)\nu_{\vec{n}} &= 0, \\ J \nu_{\vec{n}} &= 0. \end{aligned} \tag{36}$$

Let us denote by e_i, $i = 1, \ldots, N$ some orthonormal basis in V. We can expand $\nu_{\bar{n}}$ in this basis as follows:

$$\nu_{\bar{n}} = \sum_{i=1}^{N} e_i \otimes v_{\bar{n}}^i, \qquad v_{\bar{n}}^i \in \mathcal{H}.$$

Equations (34), (36) imply:

$$(B_\alpha)^i_j v^j_{\bar{n}} = z_\alpha v^i_{\bar{n}}, \quad (B^\dagger_\alpha)^i_j v^j_{\bar{n}} = \bar{z}_\alpha v^i_{\bar{n}}, \qquad \alpha = 1, 2, \tag{37}$$

in other words the matrices B_α, $B_\alpha^\dagger$ form a finite-dimensional representation of the Heisenberg algebra which is impossible if either ζ_1 or $\zeta_2 \neq 0$. Hence $\nu_{\bar{n}} = 0$, for any $\bar{n} = (n_1, n_2)$ which implies that $v = 0$.

Thus the Hermitian operator Δ is invertible. It allows to prove the following theorem: each vector ψ in the free module $(V \otimes \mathbf{C}^2 \oplus W) \otimes \mathcal{A}_\theta$ can be decomposed as a sum of two orthogonal vectors:

$$\psi = \Psi_\psi \oplus \mathcal{D}\chi_\psi, \qquad \mathcal{D}^+\Psi_\psi = 0, \quad \chi_\psi \in (V \otimes \mathbf{C}^2) \otimes \mathcal{A}_\theta, \tag{38}$$

where the orthogonality is understood in the sense of the following $\mathcal{A}_\theta$-valued Hermitian product:

$$\langle \psi_1, \psi_2 \rangle = \mathrm{Tr}_{V\otimes\mathbf{C}^2\oplus W} \quad \left(\psi_1^\dagger \psi_2\right).$$

The component Ψ_ψ is annihilated by $\mathcal{D}^+$, that is $\Psi_\psi \in \mathcal{E}$. The image of $\mathcal{D}$ is another right module over $\mathcal{A}$ (being the image of the free module $(V \otimes \mathbf{C}^2) \otimes \mathcal{A}_\theta$):

$$\mathcal{E}' = \mathcal{D}(V \otimes \mathbf{C}^2 \otimes \mathcal{A}_\theta)$$

and their sum is a free module:

$$\mathcal{E} \oplus \mathcal{E}' = \mathcal{F} := \left(V \otimes \mathbf{C}^2 \oplus W\right) \otimes \mathcal{A}_\theta,$$

hence $\mathcal{E}$ is projective. It remains to give the expressions for Ψ_ψ, χ_ψ:

$$\chi_\psi = \frac{1}{\mathcal{D}^+\mathcal{D}}\mathcal{D}^+\psi, \quad \Psi_\psi = \Pi\psi, \quad \Pi = \left(1 - \mathcal{D}\frac{1}{\mathcal{D}^+\mathcal{D}}\mathcal{D}^+\right). \tag{39}$$

The noncommutative instanton is a connection in the module $\mathcal{E}$ which is obtained simply by projecting the trivial connection on the free module $\mathcal{F}$ down to $\mathcal{E}$. To get the covariant derivative of a section $s \in \mathcal{E}$ we view this section as a section of $\mathcal{F}$, differentiate it using the ordinary derivatives on $\mathcal{A}_\theta$ and project the result down to $\mathcal{E}$ again:

$$\nabla s = \Pi\, \mathrm{d}s. \tag{40}$$

The curvature is defined through ∇^2, as usual:

$$\nabla\nabla s = F \cdot s = \mathrm{d}\Pi \wedge \mathrm{d}\Pi \cdot s, \tag{41}$$

where we used the following relations:

$$\Pi^2 = \Pi, \quad \Pi s = s. \tag{42}$$

Let us now show explicitly that the curvature (41) is anti-self-dual, i.e.

$$[\nabla_\mu, \nabla_\nu] + \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}[\nabla_\lambda, \nabla_\rho] = 0. \tag{43}$$

First we prove the following lemma: for any $s \in \mathcal{E}$:

$$\mathrm{d}\Pi \wedge \mathrm{d}\Pi s = \Pi \mathrm{d}\mathcal{D}\frac{1}{\mathcal{D}^+\mathcal{D}}\mathrm{d}\mathcal{D}^+ s. \tag{44}$$

Indeed,

$$\begin{aligned}
\mathrm{d}\Pi \wedge \mathrm{d}\Pi &= \mathrm{d}\left(\mathcal{D}\frac{1}{\mathcal{D}^+\mathcal{D}}\mathcal{D}^+\right) \wedge \mathrm{d}\left(\mathcal{D}\frac{1}{\mathcal{D}^+\mathcal{D}}\mathcal{D}^+\right),\\
\mathrm{d}\left(\mathcal{D}\frac{1}{\mathcal{D}^+\mathcal{D}}\mathcal{D}^+\right) &= \Pi \mathrm{d}\mathcal{D}\frac{1}{\mathcal{D}^+\mathcal{D}}\mathcal{D}^+ + \mathcal{D}\frac{1}{\mathcal{D}^+\mathcal{D}}\mathrm{d}\mathcal{D}^+\Pi,\\
\mathcal{D}^+\Pi &= 0,
\end{aligned}$$

hence

$$\begin{aligned}
&\mathrm{d}\left(\mathcal{D}\frac{1}{\mathcal{D}^+\mathcal{D}}\mathcal{D}^+\right) \wedge \mathrm{d}\left(\mathcal{D}\frac{1}{\mathcal{D}^+\mathcal{D}}\mathcal{D}^+\right)\\
&= \Pi \mathrm{d}\mathcal{D}\frac{1}{\mathcal{D}^+\mathcal{D}}\mathrm{d}\mathcal{D}^+\Pi + \mathcal{D}\frac{1}{\mathcal{D}^+\mathcal{D}}\mathrm{d}\mathcal{D}^+\,\Pi\,\mathrm{d}\mathcal{D}\frac{1}{\mathcal{D}^+\mathcal{D}}\mathcal{D}^+,
\end{aligned}$$

and the second term vanishes when acting on $s \in \mathcal{E}$, while the first term gives exactly what Eq. (44) states.

Now we can compute the curvature more or less explicitly:

$$F \cdot s = 2\Pi \begin{pmatrix} \frac{1}{\Delta}f_3 & \frac{1}{\Delta}f_+ & 0\\ \frac{1}{\Delta}f_- & -\frac{1}{\Delta}f_3 & 0\\ 0 & 0 & 0 \end{pmatrix} \cdot s, \tag{45}$$

where f_3, f_+, f_- are the basic anti-self-dual two-forms on $\mathbf{R}^4$:

$$f_3 = \tfrac{1}{2}\left(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 - \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2\right),\ f_+ = \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_2,\ f_- = \mathrm{d}\bar{z}_1 \wedge \mathrm{d}z_2. \tag{46}$$

Thus we have constructed the nonsingular anti-self-dual gauge fields over $\mathcal{A}_\theta$. The interesting feature of the construction is that it produces the non-trivial modules over the algebra $\mathcal{A}_\theta$, which are projective for any point in the moduli space. This feature is lacking in the $\zeta \to 0$, where it is spoiled by the point-like instantons. This feature is also lacking if the deformed ADHM equations are used for construction of gauge bundles directly over a commutative space. In this case it turns out that one can construct a torsion free sheaf over $\mathbf{C}^2$, which sometimes can be identified with a holomorphic bundle. However, generically this sheaf will not be locally free. It can be made locally free by blowing up sufficiently many points on $\mathbf{C}^2$, thereby effectively changing the topology of the space [31]. The topology change is rather mysterious if we recall that it is purely gauge theory we are dealing with. However, in the supersymmetric case this gauge theory is an $\alpha' \to 0$ limit of the theory on a stack of Euclidean D3-branes. One could think that the topology changes reflect the changes of topology of D3-branes embedded into flat ambient space. This is indeed the case for monopole solutions, e.g. [16, 43, 18, 19]. It is not completely unimaginable possibility, but so far it has not been justified (besides the fact that the DBI solutions [36, 41] are ill-defined without a blowup of the space). What makes this unlikely is the fact that the instanton backgrounds have no worldvolume scalars turned on.

At any rate, the noncommutative instantons constructed above are well-defined and nonsingular without any topology change.

Also note that we have constructed instantons for an arbitrary noncommutativity tensor $\theta_{\mu\nu}$, the only requirement being the positivity of the Pfaffian $\mathrm{Pf}(\theta) \propto \zeta_1\zeta_2 > 0$ (for $\mathrm{Pf}(\theta) < 0$ our formulae define anti-instantons).

3.3. The identificator Ψ. In the noncommutative case one can also try to construct the identifying map Ψ. It is to be thought as of the homomorphism of the modules over $\mathcal{A}$:

$$\Psi : W \otimes \mathcal{A}_\theta \to \mathcal{E}.$$

The normalization (26), if obeyed, would imply the unitary isomorphism between the free module $W \otimes \mathcal{A}_\theta$ and $\mathcal{E}$. We can write: $\Pi = \Psi\Psi^\dagger$ and the elements s of the module $\mathcal{E}$ can be cast in the form:

$$s = \Psi \cdot \sigma, \qquad \sigma \in W \otimes \mathcal{A}_\theta. \tag{47}$$

Then the covariant derivative can be written as:

$$\nabla s = \Pi d(\Psi \cdot \sigma) = \Psi\Psi^\dagger d\,(\Psi\sigma) = \Psi\,(d\sigma + A\sigma)\,, \tag{48}$$

where

$$A = \Psi^\dagger d\Psi \tag{49}$$

just like in the commutative case. Introducing the *background independent* operators $D_\mu = i\theta_{\mu\nu}x^\nu + A_\mu$, we can write:

$$D_\mu = i\Psi^\dagger\theta_{\mu\nu}x^\nu\Psi. \tag{50}$$

4. Abelian Instantons

Let us describe the case of $U(1)$ instantons in detail. In our notations above we have: $k = 1$. It is known, from [30], that for $\zeta_r > 0, \zeta_c = 0$ the solutions to the deformed ADHM equations have $J = 0$. Let us denote by V the complex Hermitian vector space of dimensionality N, where B_α, $\alpha = 1, 2$ act. Then I is identified with a vector in V. We can choose our units and coordinates in such a way that $\zeta_r = 2$, $\zeta_c = 0$.

4.1. Torsion free sheaves on $\mathbf{C}^2$. Let us recall at this point the algebraic-geometric interpretation of the space V and the triple (B_1, B_2, I). The space $\tilde{X}_{N,1}$ parameterizes the rank one torsion free sheaves on $\mathbf{C}^2$. In the case of $\mathbf{C}^2$ these are identified with the ideals $\mathcal{I}$ in the algebra $\mathbf{C}[z_1, z_2]$ of holomorphic functions on $\mathbf{C}^2$, such that $V = \mathbf{C}[z_1, z_2]/\mathcal{I}$ has dimension N. An ideal of the algebra $\mathcal{O} \approx \mathbf{C}[z_1, z_2]$ is a subspace $\mathcal{I} \subset \mathcal{O}$, which is invariant under the multiplication by the elements of $\mathcal{O}$, i.e. if $g \in \mathcal{I}$ then $fg \in \mathcal{I}$ for any $\mathcal{O}$.

An example of such an ideal is given by the space of functions of the form:

$$f(z_1, z_2) = z_1^N g(z_1, z_2) + z_2 h(z_1, z_2).$$

The operators B_α are simply the operations of multiplication of a function, representing an element of V by the coordinate function z_α, and the vector I is the image in V of the constant function $f = 1$. In the example above, following [31] we identify V with $\mathbf{C}[z_1]/z_1^N$, the operator $B_2 = 0$, and in the basis $e_i = \sqrt{(i-1)!}z_1^{N-i}$ the operator B_1 is represented by a Jordan-type block: $B_1 e_i = \sqrt{2(i-1)}e_{i-1}$, and $I = \sqrt{2N}e_N$.

Conversely, given a triple (B_1, B_2, I), such that the ADHM equations are obeyed the ideal $\mathcal{I}$ is reconstructed as follows. The polynomial $f \in \mathbf{C}[z_1, z_2]$ belongs to the ideal, $f \in \mathcal{I}$ if and only if $f(B_1, B_2)I = 0$. Then, from the ADHM equations it follows that by acting on the vector I with polynomials in B_1, B_2 one generates the whole of V. Hence $\mathbf{C}[z_1, z_2]/\mathcal{I} \approx V$ and has dimension N, as required.

4.2. Identificator Ψ *and projector* P. Let us now solve the equations for the identificator: $\mathcal{D}^\dagger \Psi = 0,\ \Psi^\dagger\Psi = 1$. We decompose:

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \\ \xi \end{pmatrix}, \tag{51}$$

where $\psi_\pm \in V \otimes \mathcal{A}_\theta$, $\xi \in \mathcal{A}_\theta$. The normalization (26) is now:

$$\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- + \xi^\dagger \xi = 1. \tag{52}$$

It is convenient to work with rescaled matrices B: $B_\alpha = \sqrt{\zeta_\alpha}\beta_\alpha$, $\alpha = 1, 2$. The equation $\mathcal{D}^\dagger \Psi = 0$ is solved by the substitution:

$$\psi_+ = -\sqrt{\zeta_2}(\beta_2^\dagger - c_2)v, \quad \psi_- = \sqrt{\zeta_1}(\beta_1^\dagger - c_1)v \tag{53}$$

provided

$$\hat{\Delta}v + I\xi = 0, \qquad \hat{\Delta} = \sum_\alpha \zeta_\alpha(\beta_\alpha - c_\alpha^\dagger)(\beta_\alpha^\dagger - c_\alpha) \tag{54}$$

Fredholm's alternative states that the solution ξ of (54) must have the property, that for any $\nu \in \mathcal{H}$, $\chi \in V$, such that

$$\hat{\Delta}(\nu \otimes \chi) = 0, \tag{55}$$

the equation

$$\left(\nu^\dagger \otimes \chi^\dagger\right) I\xi = 0 \tag{56}$$

holds. It is easy to describe the space of all $\nu \otimes \chi$ obeying (55): it is spanned by the vectors:

$$e^{\sum \beta_\alpha^\dagger c_\alpha^\dagger}|0, 0\rangle \otimes e_i, \quad i = 1, \dots, N, \tag{57}$$

where e_i is any basis in V. Let us introduce a Hermitian operator G in V:

$$G = \langle 0, 0|e^{\sum \beta_\alpha c_\alpha} I I^\dagger e^{\sum \beta_\alpha^\dagger c_\alpha^\dagger}|0, 0\rangle. \tag{58}$$

It is positive definite, which follows from the representation:

$$G = \langle 0, 0|e^{\sum \beta_\alpha c_\alpha} \sum \zeta_\alpha(\beta_\alpha^\dagger - c_\alpha)(\beta_\alpha - c_\alpha^\dagger) e^{\sum \beta_\alpha^\dagger c_\alpha^\dagger}|0, 0\rangle$$

and the fact that $\beta_\alpha - c_\alpha^\dagger$ has no kernel in $\mathcal{H} \otimes V$. Then define an element of the algebra $\mathcal{A}_\theta$

$$P = I^\dagger e^{\sum \beta_\alpha^\dagger c_\alpha^\dagger} |0,0\rangle G^{-1} \langle 0,0| e^{\sum \beta_\alpha c_\alpha} I, \tag{59}$$

which obeys $P^2 = P$, i.e. it is a projector. Moreover, it is a projection onto an N-dimensional subspace in $\mathcal{H}$, isomorphic to V.

Dual gauge invariance. The normalization condition (26) is invariant under the action of the dual gauge group $G_N \approx U(N)$ on B_α, I. However, the projector P is invariant under the action of a larger group - the complexification $G_N^{\mathbf{C}} \approx GL_N(\mathbf{C})$:

$$(B_\alpha, I) \mapsto (g^{-1} B_\alpha g, g^{-1} I), \quad (B_\alpha^\dagger, I^\dagger) \mapsto (g^\dagger B_\alpha g^{\dagger,-1}, I^\dagger g^{\dagger,-1}). \tag{60}$$

This makes the computations of P possible even when the solution to the $\mu_r = \zeta_r$ part of the ADHM equations is not known. The moduli space $\widetilde{\mathcal{M}}_{N,k}$ can be described both in terms of the hyperkahler reduction as above, or in terms of the quotient of the space of stable points $Y_{N,k}^s \subset \mu_c^{-1}(0)$ by the action of $G_N^{\mathbf{C}}$ (see [29, 30] for related discussions). The stable points (B_1, B_2, I) are the ones where B_1 and B_2 commute, and generate all of V by acting on I: $\mathbf{C}[B_1, B_2] I = V$, i.e. precisely those triples which correspond to the codimension N ideals in $\mathbf{C}[z_1, z_2]$.

Instanton gauge field. Clearly, P annihilates ξ, thanks to (56). Let S be an operator in $\mathcal{H}$ which obeys the following relations:

$$SS^\dagger = 1, \quad S^\dagger S = 1 - P. \tag{61}$$

The existence of S is merely a reflection of the fact that as Hilbert spaces $\mathcal{H}_{\mathcal{I}} \approx \mathcal{H}$. So it just amounts to relabeling the orthonormal bases in $\mathcal{H}_{\mathcal{I}}$ and $\mathcal{H}$ to construct S.

Now, $\hat{\Delta}$ restricted at the subspace $S^\dagger \mathcal{H} \otimes I \subset \mathcal{H} \otimes V$, is invertible. We can now solve (54) as follows:

$$\xi = \Lambda^{-\frac{1}{2}} S^\dagger, v = -\frac{1}{\hat{\Delta}} I \xi, \tag{62}$$

where

$$\Lambda = 1 + I^\dagger \frac{1}{\hat{\Delta}} I. \tag{63}$$

Λ is not an element of $\mathcal{A}_\theta$, but Λ^{-1} and $\Lambda S^\dagger$ are. Finally, the gauge fields can be written as:

$$D_\alpha = \sqrt{\frac{1}{\zeta_\alpha}} S \Lambda^{-\frac{1}{2}} c_\alpha \Lambda^{\frac{1}{2}} S^\dagger, \qquad \bar{D}_{\bar{\alpha}} = -\sqrt{\frac{1}{\zeta_\alpha}} S \Lambda^{\frac{1}{2}} c_\alpha^\dagger \Lambda^{-\frac{1}{2}} S^\dagger. \tag{64}$$

Ideal meaning of P. We can explain the meaning of P in an invariant fashion. Consider the ideal $\mathcal{I}$ in $\mathbf{C}[z_1, z_2]$, corresponding to the triple (B_1, B_2, I) as explained above. Any polynomial $f \in \mathcal{I}$ defines a vector $f(\sqrt{\zeta_1} c_1^\dagger, \sqrt{\zeta_2} c_2^\dagger)|0, 0\rangle$ and their totality span a subspace $\mathcal{H}_{\mathcal{I}} \subset \mathcal{H}$ of codimension N. The operator P is simply an orthogonal projection onto the complement to $\mathcal{H}_{\mathcal{I}}$. The fact $\mathcal{I}$ is an ideal in $\mathbf{C}[z_1, z_2]$ implies that $c_\alpha^\dagger(\mathcal{H}_{\mathcal{I}}) \subset \mathcal{H}_{\mathcal{I}}$, hence:

$$c_\alpha^\dagger S^\dagger \eta = S^\dagger \eta'$$

for any $\eta \in \mathcal{A}_\theta$, and also $\Lambda^{-\frac{1}{2}} S^\dagger = S^\dagger \eta''$ for some $\eta', \eta'' \in \mathcal{A}_\theta$.

Notice that the expressions (64) are well-defined. For example, the $\bar{D}_{\bar{\alpha}}$ component contains a dangerous piece $\Lambda^{\frac{1}{2}} c_\alpha^\dagger \dots$ in it. However, in view of the previous remarks it is multiplied by $S^\dagger$ from the right and therefore well-defined indeed.

4.3. Charge one instanton. In this case: $I = \sqrt{2}$, one can take $B_\alpha = 0$, $\hat{\Delta} = \sum \zeta_\alpha n_\alpha$,

$$\Lambda = \frac{M+2}{M}$$

$M = \sum_\alpha \zeta_\alpha n_\alpha$, $\sum_\alpha \zeta_\alpha = 2$. Let us introduce the notation $N = n_1 + n_2$. For the pair $\bar{n} = (n_1, n_2)$ let $\rho_{\bar{n}} = \frac{1}{2} N(N-1) + n_1$. The map $\bar{n} \leftrightarrow \rho_{\bar{n}}$ is one-to-one. Let $S^\dagger |\rho_{\bar{n}}\rangle = |\rho_{\bar{n}} + 1\rangle$. Clearly, $SS^\dagger = 1$, $S^\dagger S = 1 - |0, 0\rangle\langle 0, 0|$.

The formulae (64) are explicitly non-singular. Let us demonstrate the anti-self-duality of the gauge field (64) in this case.

$$\sum_\alpha D_\alpha \bar{D}_{\bar{\alpha}} = -S \frac{1}{\zeta_\alpha}(n_\alpha + 1) \frac{M}{M+2} \frac{M+2+\zeta_\alpha}{M+\zeta_\alpha} S^\dagger,$$
$$\sum_\alpha \bar{D}_{\bar{\alpha}} D_\alpha = S \frac{1}{\zeta_\alpha} n_\alpha \frac{M-\zeta_\alpha}{M+2-\zeta_\alpha} \frac{M+2}{M} S^\dagger.$$

A simple calculation shows:

$$\sum_\alpha [D_\alpha, \bar{D}_{\bar{\alpha}}] = -\frac{2}{\zeta_1 \zeta_2} = -\left(\frac{1}{\zeta_1} + \frac{1}{\zeta_2}\right), \qquad [D_\alpha, D_\beta] = 0, \tag{65}$$

hence

$$\sum_\alpha F_{\alpha\bar{\alpha}} = 0 \tag{66}$$

as

$$i \sum_\alpha \theta_{\alpha\bar{\alpha}} = \frac{1}{\zeta_1} + \frac{1}{\zeta_2}.$$

This is a generalization of the charge one instanton constructed in [11], written in the explicitly non-singular gauge.

Remark on gauges. The gauge which was chosen in the examples considered in [11] and subsequently adopted in [32, 20] had $\xi = \xi^{\dagger}$. It was shown in [32] that this gauge does not actually lead to the canonically normalized identificator Ψ: one had $\Psi^{\dagger}\Psi = 1 - P$. Our paper showed that this gauge is in some sense an analogue of the 't Hooft singular gauge for commutative instantons: it leads to singular formulae, if the gauge field is considered to be well-defined globally over $\mathcal{A}_{\theta}$. However, as we showed above, there are gauges in which the gauge field is globally well-defined, non-singular, and anti-self-dual. They simply have $\xi \neq \xi^{\dagger}$.

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