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The Camassa-Holm Hierarchy, N-Dimensional Integrable Systems, and Algebro-Geometric Solution on a Symplectic Submanifold

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Abstract: This paper shows that the Camassa-Holm (CH) spectral problem yields two different integrable hierarchies of nonlinear evolution equations (NLEEs), one is of negative order CH hierachy while the other one is of positive order CH hierarchy. The two CH hierarchies possess the zero curvature representations through solving a key matrix equation. We see that the well-known CH equation is included in the negative order CH hierarchy while the Dym type equation is included in the positive order CH hierarchy. Furthermore, under two constraint conditions between the potentials and the eigenfunctions, the CH spectral problem is cast in:

1. a new Neumann-like N-dimensional system when it is restricted into a symplectic submanifold of $\mathbb{R}^{2N}$ which is proven to be integrable by using the Dirac-Poisson bracket and the r-matrix process; and
2. a new Bargmann-like N-dimensional system when it is considered in the whole $\mathbb{R}^{2N}$ which is proven to be integrable by using the standard Poisson bracket and the r-matrix process.

In the paper, we present two 4×4 instead of $N \times N$ r-matrix structures. One is for the Neumann-like system (not the peaked CH system) related to the negative order CH hierarchy, while the other one is for the Bargmann-like system (not the peaked CH system, either) related to the positive order hierarchy. The whole CH hierarchy (an integro-differential hierarchy, both positive and negative order) is shown to have the parametric solutions which obey the corresponding constraint relation. In particular, the CH equation, constrained to a symplectic submanifold in $\mathbb{R}^{2N}$, and the Dym type equation have the parametric solutions. Moreover, we see that the kind of parametric solution of the CH equation is not gauge equivalent to the peakons. Solving the parametric representation of the solution on the symplectic submanifold gives a class of a new algebro-geometric solution of the CH equation.

1. Introduction

The shallow water equation derived by Camassa-Holm (CH) in 1993 [7] is a new integrable system. This equation possesses the bi-Hamiltonian structure, Lax pair and peakon solutions, and retains higher order terms of derivatives in a small amplitude expansion of incompressible Euler's equations for unidirectional motion of waves at the free surface under the influence of gravity. In 1995 Calogero [8] extended the class of mechanical system of this type. Later, Ragnisco and Bruschi [20] and Suris [22], showed that the CH equation yields the dynamics of the peakons in terms of an N-dimensional completely integrable Hamiltonian system. Such a dynamical system has Lax pair and an $N \times N$ r-matrix structure [20].

Recently, the algebro-geometric solution on the CH equation and the CH hierarchy attracted much more attention. This kind of solution for most classical integrable PDEs can be obtained by using the inverse spectral transform theory, see Dubrovin 1981 [12], Ablowitz and Segur 1981 [1], Novikov et al. 1984 [17], Newell 1985 [16]. This is done usually by adopting the spectral technique associated with the corresponding PDE. Alber and Fedorov [4, 5] studied the stationary and the time-dependent quasi-periodic solution for the CH equation and Dym type equation using the methods of trace formula [3] and Abel mapping and functional analysis on the Riemann surfaces. Later, Alber, Camassa, Fedorov, Holm and Marsden [2] considered the trace formula under the nonstandard Abel-Jacobi equations and, by introducing new parameters, presented the so-called weak finite-gap piecewise-smooth solutions of the integrable CH equation and Dym type equations. Very recently, Gesztesy and Holden [14] discussed the algebro-geometric solutions for the CH hierarchy using the polynomial recursion formalism and the trace formula, and connected a Riccati equation to the Lax pair of the CH equation.

The present paper provides another approach to algebro-geometric solutions of the CH equation which is constrained to some symplectic submanifold. Our approach differs from the ones pursued in Refs. [2–5, 14] and we will outline the differences next. Based on the nonlinearization technique [9], we constrain the CH hierarchy to some symplectic submanifold and use the constraint between the potentials and the eigenfunctions first to give the parametric solution and then to give the algebro-geometric solution of the CH equation on the symplectic submanifold.

The main results of this paper are twofold.

- First, we extend the CH equation to the negative order CH hierarchy, which is a hierarchy of integrable integro-differential equations, through constructing the inverse recursion operator. This hierarchy is proven to have Lax pair through solving a key matrix equation. The CH spectral problem associated with this hierarchy is constrained to a symplectic submanifold and naturally gives a constraint between the spectral function and the potential. Under this constraint, the CH spectral problem (linear problem) is nonlinearized as a new N-dimensional canonical Hamiltonian system of Neumann type. This N-dimensional Neumann-like system is not the peaked dynamical system of the CH equation because the peakons do not come from the CH spectral problem. The Neumann-like CH system is shown integrable by using the so-called Dirac-Poisson brackets on the symplectic submanifold in $\mathbb{R}^{2N}$ and r-matrix process. Here we present a 4×4 r-matrix structure for the Neumann-like system, which is available to get the algebro-geometric solution of the CH equation on this symplectic submanifold. The negative order CH hierarchy is proven to have the parametric solution through employing the Neumann-like constraint relation. This parametric solution does not contain the peakons [7], and vice versa. Furthermore, solving the parametric representation of solution on the symplectic submanifold gives an algebro-geometric solution for the CH equation. We point out that our algebro-geometric solution (see Eq. (3.104) and Remarks 3 and 4) is different from the ones in Refs. [2–5, 14], and simpler in form.

- Second, based on the negative case, we naturally give the positive order CH hierarchy by considering the recursion operator. This hierarchy is shown integrable also by solving the same key matrix equation. The CH spectral problem, related to this positive order CH hierarchy, yields a new integrable N-dimensional system of Bargmann type (instead of Neumann type) by using the standard Poisson bracket and r-matrix procedure in $\mathbb{R}^{2N}$. A 4×4 r-matrix structure is also presented for the Bargmann-like system (not peaked CH system, either), which is available to get the parametric solution of a Dym type equation contained in the positive order CH hierarchy. This hierarchy also possesses the parametric solution using the Bargmann constraint relation.

Roughly speaking, our method works in the following steps (also see [19]):

- Start from the spectral problem.
- Find some constraint condition between the potentials and the eigenfunctions. Here, for the negative CH hierarchy, we restrict it to a symplectic manifold in $\mathbb{R}^{2N}$, but for the positive CH hierarchy, we will have the constraint condition in the whole $\mathbb{R}^{2N}$.
- Prove the constrained SPECTRAL PROBLEM is finite-dimensional integrable. Usually we use a Lax matrix and r-matrix procedure.
- Verify the above constrained potential(s) is (are) a parametric solution of the hierarchy.
- Solve the parametric representation of a solution in an explicit form, then give the algebro-geometric solutions of the equations on the symplectic manifold. In this process, we separate the variables of the Jacobi-Hamiltonian system [21], then construct the actional variables and angle-coordinates on the symplectic submanifold, and the residues at two infinity points for some composed Riemann-Theta functions give the algebro-geometric solutions.

The paper is organized as follows. The next section gives a general structure of the zero curvature representations of the all vector fields for a given isospectral problem. The key point is to construct a key matrix equation. In Sect. 3, we present the negative order CH hierarchy based on the inverse recursion operator. The well-known CH equation is included in the negative order hierarchy, and the CH spectral problem yields a new Neumann-like system which is constrained to a symplectic manifold. This system has canonical form and is integrable by using the Dirac-Poisson bracket and the r-matrix process. Here we obtain a 4×4 r-matrix structure for the Neumann-like CH system. Furthermore, the whole negative order CH hierarchy constrained on the symplectic submanifold has a parametric solution. In particular, the CH equation has a parametric solution on the submanifold. Finally we give an algebro-geometric solution of the CH equation on the submanifold. In Sect. 4, we deal with the positive order integrable CH hierarchy and give a new Bargmann-like integrable system. By the use of a similar process as in Sect. 3, the CH spectral problem is nonlinearized to be an integrable system under a Bargmann constraint. This integrable Bargmann system also has an r-matrix structure of 4×4. Moreover, the positive order CH hierarchy is also shown to have the parametric solution which obeys the Bargmann constraint relation. In particular, the Dym type equation in the positive order CH hierarchy has the parametric solution.

Let us now give some symbols and convention in this paper as follows:

$$f^{(k)} = \begin{cases} \frac{\partial^k}{\partial x^k} f = f_{kx}, \quad k = 0, 1, 2, \ldots, \\ \underbrace{\int \ldots \int}_{-k} f \mathrm{d}x, \ k = -1, -2, \ldots, \end{cases}$$

$f_t = \frac{\partial f}{\partial t}$, $f_{kxt} = \frac{\partial^{k+1} f}{\partial t \partial x^k}$ $(k = 0, 1, 2, \ldots)$, $\partial = \frac{\partial}{\partial x}$, ∂^{-1} is the inverse of ∂, i.e. $\partial\partial^{-1} = \partial^{-1}\partial = 1$, $\partial^k f$ means the operator $\partial^k f$ acts on some function g, i.e.

$$\partial^k f \cdot g = \partial^k (fg) = \begin{cases} \frac{\partial^k}{\partial x^k}(fg) = (fg)_{kx}, \ k = 0, 1, 2 \ldots, \\ \underbrace{\int \ldots \int}_{-k} fg \mathrm{d}x, \quad k = -1, -2, \ldots. \end{cases}$$

In the following the function m stands for potential, λ is assumed to be a spectral parameter, and the domain of the spatial variable x is Ω which becomes equal to $(-\infty, +\infty)$ or $(0, T)$, while the domain of the time variable t_k is the positive time axis $\mathbb{R}^+ = \{t_k | \ t_k \in \mathbb{R}, \ t_k \geq 0, \ k = 0, \pm 1, \pm 2, \ldots\}$. In the case $\Omega = (-\infty, +\infty)$, the decaying condition at infinity and in the case $\Omega = (0, T)$, the periodicity condition for the potential function is imposed.

$(\mathbb{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q)$ stands for the standard symplectic structure in Euclid space $\mathbb{R}^{2N} = \{(p, q) | \ p = (p_1, \ldots, p_N), q = (q_1, \ldots, q_N)\}$, p_j, q_j $(j = 1, \ldots, N)$ are N pairs of canonical coordinates, $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^N$; in $(\mathbb{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q)$, the Poisson bracket of two Hamiltonian functions F, H is defined by [6]

$$\{F, H\} = \sum_{j=1}^{N} \left(\frac{\partial F}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial H}{\partial q_j} \right) = \left\langle \frac{\partial F}{\partial q}, \frac{\partial H}{\partial p} \right\rangle - \left\langle \frac{\partial F}{\partial p}, \frac{\partial H}{\partial q} \right\rangle. \tag{1.1}$$

$\lambda_1, \ldots, \lambda_N$ are assumed to be N distinct spectral parameters, $\Lambda = diag(\lambda_1, \ldots, \lambda_N)$, and $I_{2\times 2} = diag(1, 1)$. Denote all infinitely times differentiable functions on real field $\mathbb{R}$ and all integers by $C^\infty(\mathbb{R})$ and by $\mathbb{Z}$, respectively.

2. The Camassa-Holm (CH) Spectral Problem and Zero Curvature Representation

Let us consider the Camassa-Holm (CH) spectral problem [7]:

$$\psi_{xx} = \frac{1}{4}\psi - \frac{1}{2} m\lambda\psi \tag{2.1}$$

with the potential function m.

Equation (2.1) is apparently equivalent to

$$y_x = Uy, \ \ U = U(m, \lambda) = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} - \frac{1}{2}m\lambda & 0 \end{pmatrix}, \tag{2.2}$$

where $y = (y_1, y_2)^T = (\psi, \psi_x)^T$. It is easy to see Eq. (2.2)'s spectral gradient

$$\nabla\lambda \equiv \frac{\delta\lambda}{\delta m} = \lambda y_1^2 \tag{2.3}$$

satisfies the following Lenard eigenvalue problem

$$K \cdot \nabla\lambda = \lambda J \cdot \nabla\lambda \tag{2.4}$$

with the pair of Lenard's operators

$$K = -\partial^3 + \partial, \quad J = \partial m + m\partial. \tag{2.5}$$

They yield the recursion operator

$$\mathcal{L} = J^{-1}K = (\partial m + m\partial)^{-1}(\partial - \partial^3), \tag{2.6}$$

which also has the product form $\mathcal{L} = \frac{1}{2}m^{-\frac{1}{2}}\partial^{-1}m^{-\frac{1}{2}}(\partial - \partial^3)$.

Apparently, the Gateaux derivative matrix $U_*(\xi)$ of the spectral matrix U in the direction $\xi \in C^\infty(\mathbb{R})$ at point m is

$$U_*(\xi) \triangleq \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} U(m+\epsilon\xi) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2}\lambda\xi & 0 \end{pmatrix} \tag{2.7}$$

which is obviously an injective homomorphism.

For any given C^∞-function G, we construct the following matrix equation with respect to $V = V(G)$:

$$V_x - [U, V] = U_*(K \cdot G - \lambda J \cdot G). \tag{2.8}$$

Theorem 1. *For the CH spectral problem (2.2) and an arbitrary C^∞-function G, the matrix equation (2.8) has the following solution:*

$$V = V(G) = \lambda \begin{pmatrix} -\frac{1}{2}G_x & -G \\ \frac{1}{2}G_{xx} - \frac{1}{4}G + \frac{1}{2}m\lambda G & \frac{1}{2}G_x \end{pmatrix}. \tag{2.9}$$

Proof. Directly substituting Eqs. (2.9), (2.5) and (2.7) into Eq. (2.8), we can complete the proof of this theorem. □

Theorem 2. *Let $G_0 \in Ker\ J = \{G \in C^\infty(\mathbb{R}) \mid JG = 0\}$ and $G_{-1} \in Ker\ K = \{G \in C^\infty(\mathbb{R}) \mid KG = 0\}$. We define Lenard's sequences as follows:*

$$G_j = \begin{cases} \mathcal{L}^j \cdot G_0, & j \geq 0, \\ \mathcal{L}^{j+1} \cdot G_{-1}, & j < 0, \end{cases} \quad j \in \mathbb{Z} \tag{2.10}$$

Then,

1. *all the vector fields $X_k = J \cdot G_k,\ k \in \mathbb{Z}$ satisfy the following commutator representation:*

$$V_{k,x} - [U, V_k] = U_*(X_k),\ \forall k \in \mathbb{Z}; \tag{2.11}$$

2. *the following hierarchy of nonlinear evolution equations*

$$m_{t_k} = X_k = J \cdot G_k, \ \forall k \in \mathbb{Z}, \tag{2.12}$$

possesses the zero curvature representation

$$U_{t_k} - V_{k,x} + [U, V_k] = 0, \ \forall k \in \mathbb{Z}, \tag{2.13}$$

where

$$V_k = \sum V_j \lambda^{k-j-1}, \ \sum = \begin{cases} \sum_{j=0}^{k-1}, & k > 0, \\ 0, & k = 0, \\ -\sum_{j=k}^{-1}, & k < 0, \end{cases} \tag{2.14}$$

and $V_j = V(G_j)$ *is given by Eq. (2.9) with* $G = G_j$.

Proof. 1. For $k = 0$, it is obvious. For $k < 0$, we have

$$\begin{aligned} V_{k,x} - [U, V_k] &= -\sum_{j=k}^{-1} \left(V_{j,x} - [U, V_j]\right) \lambda^{k-j-1} \\ &= -\sum_{j=k}^{-1} U_* \left(K \cdot G_j - \lambda K \cdot G_{j-1}\right) \lambda^{k-j-1} \\ &= U_* \left(\sum_{j=k}^{-1} K \cdot G_{j-1} \lambda^{k-j} - K \cdot G_j \lambda^{k-j-1} \right) \\ &= U_* \left(K \cdot G_{k-1} - K \cdot G_{-1} \lambda^k \right) \\ &= U_*(K \cdot G_{k-1}) \\ &= U_*(X_k). \end{aligned}$$

For the case of $k > 0$, it is proved similarly.

2. Noticing $U_{t_k} = U_*(m_{t_k})$, we obtain

$$U_{t_k} - V_{k,x} + [U, V_k] = U_*(m_{t_k} - X_k).$$

The injectiveness of U_* implies item 2 holds. □

3. Negative Order CH Hierarchy, Integrable Neumann-like System and Algebro-Geometric Solution

3.1. Negative order CH hierarchy. Let us first give the negative order hierarchy of the CH spectral problem (2.2) by considering the kernel element of Lenard's operator K. The kernel of operator K has the following three seed functions:

$$G_{-1}^1 = 1, \tag{3.1}$$

$$G_{-1}^2 = e^x, \tag{3.2}$$

$$G_{-1}^3 = e^{-x}, \tag{3.3}$$

where all possible linear combinations form the whole kernel of K. Let $G_{-1} \in Ker\ K$, then

$$G_{-1} = \sum_{l=1}^{3} a_l G^l_{-1}, \tag{3.4}$$

where $a_l = a_l(t_n),\ l = 1, 2, 3$, are three arbitrarily given C^∞-functions with respect to the time variables t_n ($n < 0, n \in \mathbb{Z}$), but independent of the spacial variable x. Therefore, G_{-1} directly generates an isospectral ($\lambda_{t_k} = 0,\ k < 0, k \in \mathbb{Z}$) hierarchy of nonlinear evolution equations for the CH spectral problem (2.2),

$$m_{t_k} = J\mathcal{L}^{k+1} \cdot G_{-1}, \quad k < 0,\ k \in \mathbb{Z}, \tag{3.5}$$

which is called the negative order CH hierarchy because of $k < 0$. In Eq. (3.5), the operator J is defined by Eq. (2.5) and $\mathcal{L}^{-1}$ is given by

$$\mathcal{L}^{-1} = K^{-1}J = \partial^{-1}e^x\partial^{-1}e^{-2x}\partial^{-1}e^x(\partial m + m\partial). \tag{3.6}$$

Here,

$$K^{-1} = \partial^{-1}e^x\partial^{-1}e^{-2x}\partial^{-1}e^x. \tag{3.7}$$

With setting $m = u - u_{xx}$, we obtain another form of $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1} = u + e^x\partial^{-1}e^{-2x}\partial^{-1}e^x\left(u\partial + 2u_x + \partial^{-1}m\right)\partial. \tag{3.8}$$

By Theorem 2, the negative CH hierarchy (3.5) has the zero curvature representation

$$U_{t_k} - V_{k,x} + [U, V_k] = 0,\ k < 0,\ k \in \mathbb{Z}, \tag{3.9}$$

$$V_k = -\sum_{j=k}^{-1} V_j\lambda^{k-j-1}, \tag{3.10}$$

i.e.

$$\begin{cases} y_x = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} - \frac{1}{2}m\lambda & 0 \end{pmatrix} y, \\ y_{t_k} = -\sum_{j=k}^{-1} \begin{pmatrix} -\frac{1}{2}G_{j,x} & -G_j \\ \frac{1}{2}G_{j,xx} - \frac{1}{4}G_j + \frac{1}{2}m\lambda G_j & \frac{1}{2}G_{j,x} \end{pmatrix} \lambda^{k-j} y, \\ \qquad k = -1, -2, \ldots, \end{cases} \tag{3.11}$$

where $G_j = \mathcal{L}^{j+1} \cdot G_{-1},\ j < 0,\ j \in \mathbb{Z}$. Thus, all nonlinear equations in the negative CH hierarchy (3.5) are integrable.

Let us now give some special reductions of Eq. (3.5).

- In the case of $a_1 = -1, a_2 = a_3 = 0$, i.e. $G_{-1} = -G^1_{-1} = -1$, because $G_{-2} = \mathcal{L}^{-1} \cdot G_{-1} = -u$, the second equation of Eq. (3.5) reads

$$m_{t_{-2}} = -(\partial m + m\partial) \cdot u, \tag{3.12}$$

i.e. (here noticing $m = u - u_{xx}$)

$$u_{t_{-2}} - u_{xx,t_{-2}} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{3.13}$$

which is exactly the Camassa-Holm equation [7]. According to Eq. (3.11), the CH equation (3.13) possesses the following zero curvature representation:

$$\begin{cases} y_x = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} - \frac{1}{2}m\lambda & 0 \end{pmatrix} y, \\ y_{t_{-2}} = \begin{pmatrix} -\frac{1}{2}u_x & -u - \lambda^{-1} \\ \frac{1}{2}mu\lambda + \frac{1}{4}u - \frac{1}{4}\lambda^{-1} & \frac{1}{2}u_x \end{pmatrix} y, \end{cases} \tag{3.14}$$

which is equivalent to

$$\begin{cases} \psi_{xx} = \frac{1}{4}\psi - \frac{1}{2}m\lambda\psi, \\ \psi_{t_{-2}} = \frac{1}{2}u_x\psi - u\psi_x - \lambda^{-1}\psi_x. \end{cases} \tag{3.15}$$

Equation (3.15) coincides with the one in Ref. [7].

- In the cases of $a_1 = 0, a_2 = 1, a_3 = 0$ and $a_1 = 0, a_2 = 0, a_3 = 1$, i.e. $G_{-1} = e^x, e^{-x}$, we can write them in a uniform expression:

$$G_{-1} = e^{\epsilon x}, \;\; \epsilon = \pm 1.$$

The first equation of Eq. (3.5) reads

$$m_{t_{-1}} = (m_x + 2\epsilon m)e^{\epsilon x}, \tag{3.16}$$

which is a linear PDE.

Because $G_{-2} = \mathcal{L}^{-1} \cdot G_{-1} = \left(u + \epsilon u^{(-1)}\right) e^{\epsilon x}$, the second equation of Eq. (3.5) reads

$$m_{t_{-2}} = \left(m_x \left(u + \epsilon u^{(-1)}\right) + 2m\left(u_x + 2\epsilon u + u^{(-1)}\right)\right) e^{\epsilon x}, \tag{3.17}$$

where $m = u - u_{xx}$. This equation has the following zero curvature representation:

$$\begin{cases} y_x = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} - \frac{1}{2}m\lambda & 0 \end{pmatrix} y, \\ y_{t_{-2}} = V_{-2} y, \end{cases} \tag{3.18}$$

where

$$\begin{aligned} V_{-2} &= -V(G_{-2})\lambda^{-1} - V(G_{-1})\lambda^{-2} \\ &= e^{\epsilon x} \begin{pmatrix} \frac{1}{2}\left(u_x + 2\epsilon u + u^{(-1)}\right) + \epsilon\lambda^{-1} & u + \epsilon u^{(-1)}) + \lambda^{-1} \\ \frac{3}{2}\epsilon u_x + \frac{7}{4}u + \frac{3}{4}\epsilon u^{(-1)} + \frac{1}{2}m\left(u + \epsilon u^{(-1)}\right)\lambda + \frac{1}{4}\lambda^{-1} & -\frac{1}{2}\left(u_x + 2\epsilon u + u^{(-1)}\right) + \epsilon\lambda^{-1} \end{pmatrix}. \end{aligned}$$

Equation (3.18) can be changed to the following Lax form:

$$\begin{cases} \psi_{xx} = \frac{1}{4}\psi - \frac{1}{2}m\lambda\psi, \\ \psi_{t_{-2}} = \left(u + \epsilon u^{(-1)} + \lambda^{-1}\right) e^{\epsilon x}\psi_x - \frac{1}{2}\left(u_x + 2\epsilon u + u^{(-1)} + \lambda^{-1}\right) e^{\epsilon x}\psi. \end{cases} \tag{3.19}$$

Both of the two cases: $\epsilon = \pm 1$ for Eq. (3.17) are integrable.

3.2. r-matrix structure for the Neumann-like CH system. Consider the following matrix (called "negative" Lax matrix)

$$L_{-}(\lambda) = \begin{pmatrix} A_{-}(\lambda) & B_{-}(\lambda) \\ C_{-}(\lambda) & -A_{-}(\lambda), \end{pmatrix}, \tag{3.20}$$

where

$$A_{-}(\lambda) = -\langle p, q\rangle \lambda^{-1} + \sum_{j=1}^{N} \frac{p_j q_j}{\lambda - \lambda_j}, \tag{3.21}$$

$$B_{-}(\lambda) = \lambda^{-2} + \langle q, q\rangle \lambda^{-1} - \sum_{j=1}^{N} \frac{q_j^2}{\lambda - \lambda_j}, \tag{3.22}$$

$$C_{-}(\lambda) = \frac{1}{4}\lambda^{-2} - \langle p, p\rangle \lambda^{-1} + \sum_{j=1}^{N} \frac{p_j^2}{\lambda - \lambda_j}. \tag{3.23}$$

We calculate the determinant of $L_{-}(\lambda)$:

$$\begin{aligned} \frac{1}{2}\lambda^2 \det L_{-}(\lambda) &= -\frac{1}{4}\lambda^2 \mathrm{Tr} L_{-}^2(\lambda) = -\frac{1}{2}\lambda^2 \left(A_{-}^2(\lambda) + B_{-}(\lambda) C_{-}(\lambda)\right) \\ &= \sum_{j=-2}^{1} H_j \lambda^j + \sum_{j=1}^{N} \frac{E_j^{-}}{\lambda - \lambda_j}, \end{aligned} \tag{3.24}$$

where Tr stands for the trace of a matrix, and

$$\begin{aligned} H_{-2} &= -\frac{1}{8}, \\ H_{-1} &= \frac{1}{2}\langle p, p\rangle - \frac{1}{8}\langle q, q\rangle, \\ H_0 &= \langle p, q\rangle \langle \Lambda p, q\rangle - \langle p, q\rangle^2, \\ H_1 &= -\langle p, q\rangle \left\langle \Lambda^{-1} p, q\right\rangle + \langle p, q\rangle^2, \end{aligned} \tag{3.25}$$

$$\begin{aligned} E_j^{-} &= \langle p, q\rangle \lambda_j p_j q_j - \frac{1}{2}\left(\langle q, q\rangle \lambda_j + 1\right) p_j^2 \\ &\quad -\frac{1}{2}\left(\langle p, p\rangle \lambda_j - \frac{1}{4}\right) q_j^2 + \frac{1}{2}\lambda_j^2 \Gamma_j^{-}, \quad j = 1, 2, \ldots, N, \\ \Gamma_j^{-} &= \sum_{l \neq j, l=1}^{N} \frac{(p_j q_l - p_l q_j)^2}{\lambda_j - \lambda_l}. \end{aligned} \tag{3.26}$$

Let

$$F_k = \sum_{j=1}^{N} \lambda_j^{k+1} E_j^{-}, \quad k = -1, -2, \ldots, \tag{3.27}$$

then it reads

$$F_k = \frac{1}{2}\left\langle \Lambda^{k+1} p, p\right\rangle - \frac{1}{8}\left\langle \Lambda^{k+1} q, q\right\rangle$$
$$+\frac{1}{2}\sum_{j=k}^{-2}\left(\left\langle \Lambda^{j+2} q, q\right\rangle\left\langle \Lambda^{k-j} p, p\right\rangle - \left\langle \Lambda^{j+2} p, q\right\rangle\left\langle \Lambda^{k-j} p, q\right\rangle\right), \quad (3.28)$$
$$k = -1, -2, -3, \ldots.$$

Obviously, $F_{-1} = H_{-1}$.

Now, we consider the following symplectic submanifold in $\mathbb{R}^{2N}$

$$\mathbb{M} = \left\{ (q, p) \in \mathbb{R}^{2N} \,\middle|\, F \equiv \frac{1}{2}\left(\langle \Lambda q, q\rangle - 1\right) = 0,\ G \equiv \langle \Lambda q, p\rangle = 0 \right\} \quad (3.29)$$

and introduce the Dirac bracket on $\mathbb{M}$

$$\{f, g\}_D = \{f, g\} + \frac{1}{\left\langle \Lambda^2 q, q\right\rangle}\left(\{f, F\}\{G, g\} - \{f, G\}\{F, g\}\right) \quad (3.30)$$

which is easily proven to be bilinear, skew-symmetric and satisfy the Jacobi identity.

In particular, the Hamiltonian system $(H_{-1})_D$: $q_x = \{q, H_{-1}\}_D$, $p_x = \{p, H_{-1}\}_D$ on $\mathbb{M}$ reads

$$(H_{-1})_D: \quad \begin{cases} q_x = p, \\ p_x = \frac{1}{4}q - \frac{1+4\langle \Lambda p, p\rangle}{4\langle \Lambda^2 q, q\rangle}\Lambda q, \\ \langle \Lambda q, p\rangle = 0, \quad \langle \Lambda q, q\rangle = 1. \end{cases} \quad (3.31)$$

We call this a Neumann-like system on $\mathbb{M}$. Let

$$m = \frac{1 + 4\langle \Lambda p, p\rangle}{2\left\langle \Lambda^2 q, q\right\rangle}, \quad (3.32)$$

$$y_1 = q_j,\ y_2 = p, \quad \lambda = \lambda_j, \quad j = 1, \ldots, N. \quad (3.33)$$

Then, the Neumann-like flow $(H_{-1})_D$ on $\mathbb{M}$ exactly becomes

$$y_x = U(m, \lambda) y, \quad y = (y_1, y_2)^T, \quad (3.34)$$

which is nothing else but the CH spectral problem (2.2) with the potential function m. Therefore, we can call the canonical Hamiltonian system (3.31) **the Neumann-like CH system on** $\mathbb{M}$.

A long but direct computation leads to the following key equalities:

$$\{A_-(\lambda), A_-(\mu)\}_D = \{B_-(\lambda), B_-(\mu)\}_D = 0,$$
$$\{C_-(\lambda), C_-(\mu)\}_D = \frac{1 + 4\langle \Lambda p, p\rangle}{\left\langle \Lambda^2 q, q\right\rangle}\left(\frac{\lambda}{\mu}A_-(\lambda) - \frac{\mu}{\lambda}A_-(\mu)\right)$$
$$+\frac{4\lambda\mu}{\left\langle \Lambda^2 q, q\right\rangle}\left(C_-(\lambda)A_-(\mu) - C_-(\mu)A_-(\lambda)\right),$$

$$\{A_-(\lambda), B_-(\mu)\}_D = \frac{2}{\mu-\lambda}(B_-(\mu) - B_-(\lambda)) + \frac{2}{\lambda}B_-(\mu) + \frac{2}{\mu}B_-(\lambda)$$
$$-\frac{2\lambda\mu}{\langle \Lambda^2 q, q\rangle}B_-(\lambda)B_-(\mu),$$
$$\{A_-(\lambda), C_-(\mu)\}_D = \frac{2}{\mu-\lambda}(C_-(\lambda) - C_-(\mu)) - \frac{2}{\lambda}C_-(\mu) - \frac{2}{\mu}C_-(\lambda)$$
$$+\frac{2\lambda\mu}{\langle \Lambda^2 q, q\rangle}B_-(\lambda)C_-(\mu) - \frac{(1+4\langle \Lambda p, p\rangle)\lambda}{2\langle \Lambda^2 q, q\rangle \mu}B_-(\lambda),$$
$$\{B_-(\lambda), C_-(\mu)\}_D = \frac{4}{\mu-\lambda}(A_-(\mu) - A_-(\lambda)) + \frac{4}{\lambda}A_-(\mu) + \frac{4}{\mu}A_-(\lambda)$$
$$-\frac{4\lambda\mu}{\langle \Lambda^2 q, q\rangle}B_-(\lambda)A_-(\mu).$$

Let $L_1^-(\lambda) = L_-(\lambda) \otimes I_{2\times 2}$, $L_2^-(\mu) = I_{2\times 2} \otimes L_-(\mu)$, where $L_-(\lambda)$, $L_-(\mu)$ are given through Eq. (3.20). In the following, we search for a general 4×4 r-matrix structure $r_{12}^-(\lambda,\mu)$ such that the fundamental Dirac-Poisson bracket:

$$\left\{L_-(\lambda) \overset{\otimes}{,} L_-(\mu)\right\}_D = \left[r_{12}^-(\lambda,\mu), L_1^-(\lambda)\right] - \left[r_{21}^-(\mu,\lambda), L_2^-(\mu)\right] \tag{3.35}$$

holds, where the entries of the 4×4 matrix $\left\{L_-(\lambda) \overset{\otimes}{,} L_-(\mu)\right\}_D$ are

$$\left\{L_-(\lambda) \overset{\otimes}{,} L_-(\mu)\right\}_{D_{kl,mn}} = \left\{L_-(\lambda)_{km}, L_-(\mu)_{ln}\right\}_D, \quad k,l,m,n = 1,2,$$

and $r_{21}^-(\lambda,\mu) = P r_{12}^-(\lambda,\mu) P$, with

$$P = \frac{1}{2}\left(I_{2\times 2} + \sum_{j=1}^{3} \sigma_j \otimes \sigma_j\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where σ_j' are the Pauli matrices.

Theorem 3.

$$r_{12}^-(\lambda,\mu) = \frac{2\lambda}{\mu(\mu-\lambda)}P + S^- \tag{3.36}$$

is an r-matrix structure satisfying Eq. (3.35), where

$$S^- = \frac{\lambda(1+4\langle \Lambda p, p\rangle)}{2\mu\langle \Lambda^2 q, q\rangle}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$+\frac{2\lambda\mu}{\langle \Lambda^2 q, q\rangle}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} -B_-(\lambda) & 0 \\ 2A_-(\mu) & B_-(\lambda) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2\lambda\mu}{\langle \Lambda^2 q, q\rangle}B_-(\lambda) & 0 \\ \frac{\lambda(1+4\langle \Lambda p, p\rangle)}{2\mu\langle \Lambda^2 q, q\rangle} & 0 & \frac{4\lambda\mu}{\langle \Lambda^2 q, q\rangle}A_-(\mu) & \frac{2\lambda\mu}{\langle \Lambda^2 q, q\rangle}B_-(\lambda) \end{pmatrix}.$$

Apparently, our r-matrix structure (3.36) is 4×4 and is different from the one in Ref. [20].

3.3. Integrability. Because there is an r-matrix structure satisfying Eq. (3.35),

$$\left\{L_-^2(\lambda) \overset{\otimes}{,} L_-^2(\mu)\right\}_D = \left[\bar{r}_{12}^-(\lambda,\mu), L_1^-(\lambda)\right] - \left[\bar{r}_{21}^-(\mu,\lambda), L_2^-(\mu)\right], \tag{3.37}$$

where

$$\bar{r}_{ij}^-(\lambda,\mu) = \sum_{k=0}^{1}\sum_{l=0}^{1}\left(L_1^-\right)^{1-k}(\lambda)\left(L_2^-\right)^{1-l}(\mu)\, r_{ij}^-(\lambda,\mu)\left(L_1^-\right)^{k}(\lambda)\left(L_2^-\right)^{l}(\mu),$$
$$ij = 12, 21.$$

Thus,

$$4\left\{\mathrm{Tr}L_-^2(\lambda), \mathrm{Tr}L_-^2(\mu)\right\}_D = \mathrm{Tr}\left\{L_-^2(\lambda) \overset{\otimes}{,} L_-^2(\mu)\right\}_D = 0. \tag{3.38}$$

So, by Eq. (3.24) we immediately obtain the following theorem.

Theorem 4. *The following equalities*

$$\{E_i^-, E_j^-\}_D = 0,\ \{H_l, E_j^-\}_D = 0,\ \{F_k, E_j^-\}_D = 0, \tag{3.39}$$
$$i, j = 1, 2, \dots, N,\ l = -2, -1, 0, 1,\ k = -1, -2, \dots,$$

hold. Hence, the Hamiltonian systems $(H_l)_D$ *and* $(F_k)_D$ *on* $\mathbb{M}$

$$(H_l)_D: \quad q_x = \{q, H_l\}_D,\ p_x = \{p, H_l\}_D,\ \ l = -2, -1, 0, 1, \tag{3.40}$$
$$(F_k)_D: \quad q_{t_k} = \{q, F_k\}_D,\ p_{t_k} = \{p, F_k\}_D, \quad k = -1, -2, \dots, \tag{3.41}$$

are completely integrable.

In particular, we obtain the following results.

Corollary 1. *The Hamiltonian system* $(H_{-1})_D$ *defined by Eq. (3.31) is completely integrable.*

Corollary 2. *All composition functions* $f(H_l, F_k)$, $f \in C^\infty(\mathbb{R})$, $k = -1, -2, \dots$, *are completely integrable Hamiltonians on* $\mathbb{M}$.

3.4. Parametric solution of the negative order CH hierarchy restricted onto $\mathbb{M}$. In the following, we consider the relation between constraint and nonlinear equations in the negative order CH hierarchy (3.5). Let us start from the following setting:

$$G_{-1}^1 = \sum_{j=1}^{N}\nabla\lambda_j, \tag{3.42}$$

where $G_{-1}^1 = 1$, and $\nabla\lambda_j = \lambda_j q_j^2$ is the functional gradient of the CH spectral problem (2.2) corresponding to the spectral parameter λ_j $(j = 1, \dots, N)$.

Apparently Eq. (3.42) reads

$$\langle\Lambda q, q\rangle = 1. \tag{3.43}$$

After we do one time derivative with respect to x, we get

$$\langle\Lambda p, q\rangle = 0, \quad p = q_x. \tag{3.44}$$

This equality together with Eq. (3.43) forms the symplectic submanifold $\mathbb{M}$ we need. Apparently, derivating Eq. (3.44) leads to the constraint relation (3.32).

Remark 1. Because $\mathbb{M}$ defined by Eq. (3.29) is not the usual tangent bundle, i.e. $\mathbb{M} \neq TS^{N-1} = \left\{ (q, p) \in \mathbb{R}^{2N} \middle| \ \tilde{F} \equiv \langle q, q \rangle - 1 = 0, \tilde{G} \equiv \langle q, p \rangle = 0 \right\}$ and Eq. (3.32) is not the usual Neumann constraint on TS^{N-1}, Eq. (3.31) is therefore a new kind of Neumann system. In Subsect. 3.3 we have proven its integrability.

Since the Hamiltonian flows $(H_{-1})_D$ and $(F_k)_D$ on $\mathbb{M}$ are completely integrable and their Poisson brackets $\{H_{-1}, F_k\}_D = 0$ $(k = -1, -2, \ldots)$, their phase flows $g^x_{H_{-1}}, \ g^{t_k}_{F_k}$ commute [6]. Thus, we can define their compatible solution as follows:

$$\begin{pmatrix} q(x, t_k) \\ p(x, t_k) \end{pmatrix} = g^x_{H_{-1}} g^{t_k}_{F_k} \begin{pmatrix} q(x^0, t^0_k) \\ p(x^0, t^0_k) \end{pmatrix}, \quad k = -1, -2, \ldots, \tag{3.45}$$

where $x^0, \ t^0_k$ are the initial values of phase flows $g^x_{H_{-1}}, \ g^{t_k}_{F_k}$.

Theorem 5. *Let $q(x, t_k), \ p(x, t_k)$ be a solution of the compatible Hamiltonian systems $(H_{-1})_D$ and $(F_k)_D$ on $\mathbb{M}$. Then*

$$m = \frac{1 + 4\,\langle \Lambda p(x, t_k), p(x, t_k) \rangle}{2\left\langle \Lambda^2 q(x, t_k), q(x, t_k) \right\rangle} \tag{3.46}$$

satisfies the negative order CH hierarchy

$$m_{t_k} = J\mathcal{L}^{k+1} \cdot 1, \quad k = -1, -2, \ldots, \tag{3.47}$$

where the operators J and $\mathcal{L}^{-1}$ are given by Eqs. (2.5) and (3.6), respectively.

Proof. On one hand, the recursion operator $\mathcal{L}$ acts on Eq. (3.42) and results in the following:

$$\begin{aligned} J\mathcal{L}^{k+1} \cdot G^1_{-1} &= J \cdot \left\langle \Lambda^{k+2} q, q \right\rangle \\ &= m_x \left\langle \Lambda^{k+2} q, q \right\rangle + 4m \left\langle \Lambda^{k+2} q, p \right\rangle \\ &= \frac{2(1 + 4\,\langle \Lambda p, p \rangle)}{\left\langle \Lambda^2 q, q \right\rangle^2} \left(\left\langle \Lambda^2 q, q \right\rangle \left\langle \Lambda^{k+2} q, p \right\rangle - \left\langle \Lambda^2 q, p \right\rangle \left\langle \Lambda^{k+2} q, q \right\rangle \right). \end{aligned} \tag{3.48}$$

In this procedure, Eqs. (2.4) and (3.31) are used.

On the other hand, we directly make the derivative of Eq. (3.46) with respect to t_k. Then we obtain

$$m_{t_k} = \frac{4\left\langle \Lambda^2 q, q \right\rangle \left\langle \Lambda^2 p, p_{t_k} \right\rangle - (1 + 4\,\langle \Lambda p, p \rangle) \left\langle \Lambda^2 q, q_{t_k} \right\rangle}{\left\langle \Lambda^2 q, q \right\rangle^2}, \tag{3.49}$$

where $q = q(x, t_k), \ p = p(x, t_k)$. But,

$$q_{t_k} = \{q, F_k\}_D, \quad p_{t_k} = \{p, F_k\}_D, \quad k = -1, -2, \ldots, \tag{3.50}$$

where F_k are defined by Eq. (3.28), i.e.

$$q_{t_k} = \sum_{j=k}^{-1} \left(\left\langle \Lambda^{j+2} q, q \right\rangle \Lambda^{k-j} p - \left\langle \Lambda^{j+2} q, p \right\rangle \Lambda^{k-j} q \right), \tag{3.51}$$

$$\begin{aligned} p_{t_k} &= \frac{1 + 4\langle \Lambda p, p \rangle}{4\left\langle \Lambda^2 q, q \right\rangle} \left(\left\langle \Lambda^2 q, q \right\rangle \Lambda^{k+1} q - \left\langle \Lambda^{k+2} q, q \right\rangle \Lambda q \right) \\ &\quad + \sum_{j=k}^{-1} \left(\left\langle \Lambda^{j+2} q, p \right\rangle \Lambda^{k-j} p - \left\langle \Lambda^{j+2} p, p \right\rangle \Lambda^{k-j} q \right). \end{aligned} \tag{3.52}$$

Therefore after substituting them into Eq. (3.49) and calculating it, we have

$$m_{t_k} = \frac{2(1 + 4\langle \Lambda p, p \rangle)}{\left\langle \Lambda^2 q, q \right\rangle^2} \left(\left\langle \Lambda^2 q, q \right\rangle \left\langle \Lambda^{k+2} q, p \right\rangle - \left\langle \Lambda^2 q, p \right\rangle \left\langle \Lambda^{k+2} q, q \right\rangle \right)$$

which completes the proof. □

Lemma 1. *Let $q,\ p$ satisfy the integrable Hamiltonian system $(H_{-1})_D$. Then on the symplectic submanifold $\mathbb{M}$, we have*

1.

$$\langle q, q \rangle - 4\langle p, p \rangle = 0. \tag{3.53}$$

2.

$$u = \langle q(x, t_k), q(x, t_k) \rangle, \quad k, = -1, -2, \ldots, \tag{3.54}$$

satisfies the equation $m = u - u_{xx}$, where m is given by Eq. (3.46), and $q(x, t_k)$, $p(x, t_k)$ is a solution of the compatible integrable Hamiltonian systems $(H_{-1})_D$ and $(F_k)_D$ on $\mathbb{M}$.

Proof.

$$\begin{aligned} (\langle q, q \rangle - 4\langle p, p \rangle)_x &= 2\langle q, p \rangle - 8\langle p, p_x \rangle \\ &= 0 \end{aligned}$$

completes the proof of the first equality.

As for the second one, we have

$$\begin{aligned} u - u_{xx} &= \langle q, q \rangle - 2\langle p, p \rangle - 2\left\langle q, \frac{1}{4} q - \frac{1}{2} m \Lambda q \right\rangle \\ &= \frac{1}{2}\langle q, q \rangle - 2\langle p, p \rangle + m \\ &= m. \end{aligned}$$

In particular, with $k = -2$, we obtain the following corollary. □

Corollary 3. *Let $q(x, t_{-2})$, $p(x, t_{-2})$ be a solution of the compatible integrable Hamiltonian systems $(H_{-1})_D$ and $(F_{-2})_D$ on $\mathbb{M}$. Then*

$$m = m(x, t_{-2}) = -\frac{1 + 4\langle \Lambda p(x, t_{-2}), p(x, t_{-2})\rangle}{2\left\langle \Lambda^2 q(x, t_{-2}), q(x, t_{-2})\right\rangle}, \tag{3.55}$$

$$u = u(x, t_{-2}) = \langle q(x, t_{-2}), q(x, t_{-2})\rangle, \tag{3.56}$$

satisfy the CH equation (3.12). Therefore, $u = \langle q(x, t_{-2}), q(x, t_{-2})\rangle$, is a solution of the CH equation (3.13) on $\mathbb{M}$. Here H_{-1} and F_{-2} are given by

$$H_{-1} = \frac{1}{2}\langle p, p\rangle - \frac{1}{8}\langle q, q\rangle,$$
$$F_{-2} = \frac{1}{2}\left\langle \Lambda^{-1} p, p\right\rangle - \frac{1}{8}\left\langle \Lambda^{-1} q, q\right\rangle + \frac{1}{2}\left(\langle q, q\rangle\langle p, p\rangle - \langle q, p\rangle^2\right).$$

Proof. Via some direct calculations, we obtain

$$m_{t_{-2}} = -\frac{2(1 + 4\langle \Lambda p, p\rangle)}{\left\langle \Lambda^2 q, q\right\rangle^2}\left(\left\langle \Lambda^2 q, q\right\rangle\langle q, p\rangle - \left\langle \Lambda^2 q, p\right\rangle\langle q, q\rangle\right).$$

And Lemma 1 gives

$$\begin{aligned}
-J \cdot u &= -J\mathcal{L}^{-1} \cdot G_{-1}^1 \\
&= J \cdot < q, q > \\
&= -\frac{2(1 + 4\langle \Lambda p, p\rangle)}{\left\langle \Lambda^2 q, q\right\rangle^2}\left(\left\langle \Lambda^2 q, q\right\rangle\langle q, p\rangle - \left\langle \Lambda^2 q, p\right\rangle\langle q, q\rangle\right) \\
&= m_{t_{-2}},
\end{aligned}$$

which completes the proof. □

By Theorem 5, the constraint relation given by Eq. (3.32) is actually a solution of the negative order CH hierarchy (3.5). Thus, we also **call the system $(H_{-1})_D$ (i.e. Eq. (3.31)) a negative order restricted CH flow (Neumann-like) of the spectral problem (2.2) on the symplectic submanifold $\mathbb{M}$. All Hamiltonian systems $(F_k)_D$, $k < 0, k \in \mathbb{Z}$ are therefore called the negative order integrable restricted flows (Neumann-type) on $\mathbb{M}$.**

Remark 2. Of course, we can also consider the integrable Bargmann-like CH system associated with the positive order CH hierarchy (4.1). Please see Sect. 4. A systematic approach to generate new integrable negative order hierarchies can be seen in Ref. [18].

3.5. Comparing parametric solution with peakons. Let us now compare the Neumann-like CH system $(H_{-1})_D$ with the peakons dynamical system.

Let P_j, Q_j $(j = 1, 2, \ldots, N)$ be peakons dynamical variables of the CH equation (3.13), then with [7]

$$u(x, t) = \sum_{j=1}^{N} P_j(t)e^{-|x - Q_j(t)|}, \tag{3.57}$$

$$m(x, t) = \sum_{j=1}^{N} P_j(t)\delta(x - Q_j(t)), \tag{3.58}$$

where $t = t_{-2}$ and $\delta(x)$ is the δ-function, the CH equation (3.13) yields a canonical peaked Hamiltonian system

$$\begin{cases} \dot{Q}_j(t) = \frac{\partial H}{\partial P_j} = \sum_{k=1}^{N} P_k(t) e^{-|Q_k(t)-Q_j(t)|}, \\ \dot{P}_j(t) = -\frac{\partial H}{\partial Q_j} = -P_j \sum_{k=1}^{N} P_k(t) sgn\left(Q_k(t) - Q_j(t)\right) e^{-|Q_k(t)-Q_j(t)|}, \end{cases} \tag{3.59}$$

with

$$H(t) = \frac{1}{2} \sum_{i,j=1}^{N} P_i(t) P_j(t) e^{-|Q_i(t)-Q_j(t)|}. \tag{3.60}$$

In Eq. (3.59), "sgn" means the signal function. The same meaning is used in the following.

A natural question is: what is the relationship between the peaked Hamiltonian system (3.59) and the Neumann-like systems (3.31) and (F_{-2})? Apparently, the peaked system (3.59) does not include the systems (3.31) and (F_{-2}) because the system (3.59) is only concerned about the time part.

By Corollary 3, we know that

$$u = u(x,t) = \langle q(x,t), q(x,t) \rangle = \sum_{j=1}^{N} q_j^2(x,t), \tag{3.61}$$

is a solution of the CH equation (3.13) on $\mathbb{M}$, where we set $t_{-2} = t$, and $q_j(x,t),\ p_j = \frac{\partial q_j(x,t)}{\partial x}$ satisfies the two integrable commuted systems $(H_{-1})_D,\ (F_{-2})_D$ on the symplectic submanifold $\mathbb{M}$.

Assume $P_j(t),\ Q_j(t)$ are the solutions of the peaked system (3.59); we make the following transformation (when $P_j(t) < 0$, we use $\sqrt{P_j(t)} = i\sqrt{-P_j(t)}$):

$$q_j(x,t) = \sqrt{P_j(t)} e^{-\frac{1}{2}|x-Q_j(t)|}, \tag{3.62}$$

which implies

$$p_j(x,t) = \frac{d}{dx} q_j(x,t) = -\frac{1}{2} sgn(x - Q_j(t)) q_j. \tag{3.63}$$

Hence, we obtain

$$\frac{d}{dx} p_j(x,t) = \frac{d^2}{dx^2} q_j(x,t) = -\delta(x - Q_j(t)) q_j + \frac{1}{4} q_j. \tag{3.64}$$

However, on $\mathbb{M}$ we have the constraints $< \Lambda q, q >= 1,\ < \Lambda q, p >= 0$ which implies

$$\sum_{j=1}^{N} \lambda_j q_j^2 \delta(x - Q_j) = \frac{1}{2}, \quad \forall x \in \mathbb{R}. \tag{3.65}$$

This equality is obviously not true! So, the CH peakon system (3.59) does not coincide with the nonlinearized CH spectral problem (3.31).

Let us now furthermore compute the derivative with respect to t. Inserting the peakon system (3.59), we get

$$\begin{aligned}\dot{q}_j(x,t) &= \frac{d}{dt} q_j(x,t) \\ &= \frac{1}{2} q_j \sum_{k=1}^{N} P_k(t) \left[sgn(x - Q_j(t)) - sgn\left(Q_k(t) - Q_j(t)\right)\right] e^{-|Q_k(t)-Q_j(t)|}.\end{aligned} \tag{3.66}$$

On the other hand, from the Neumann-like system $(F_{-2})_D$ we have

$$\begin{aligned}\dot{q}_j(x,t) &= \{q_j, F_{-2}\}_D \\ &= \lambda_j^{-1} p_j - \langle q, p\rangle q_j \\ &= \frac{1}{2} q_j \left(\sum_{k=1}^{N} P_k(t) sgn(x - Q_k(t)) e^{-|x-Q_k(t)|} - \lambda_j^{-1} sgn(x - Q_j(t)) \right).\end{aligned} \tag{3.67}$$

Apparently, (3.66) = (3.67) iff when $x = Q_j(x,t)$, i.e. for other x, they do not equal. Thus, the peakon system (3.59) is not the Neumann-like Hamiltonian system $(F_{-2})_D$, either.

So, by the above analysis, we conclude: **the two solutions (3.57) and (3.61) of the CH equation (3.13) are not gauge equivalent**. In the next subsection we will concretely solve Eq. (3.61) on $\mathbb{M}$ in the form of Riemann-Theta functions.

3.6. Algebro-geometric solution of the CH equation on $\mathbb{M}$. Now, let us re-consider the Hamiltonian system $(H_{-1})_D$ on $\mathbb{M}$ under the substitution of $\lambda \to \lambda^{-1}$, $\lambda_j \to \lambda_j^{-1}$ (here we choose non-zero λ, λ_j). Then, the Lax matrix (3.20) has the following simple form:

$$L_{-}(\lambda) = -\lambda^3 L_{\mathrm{CH}}(\lambda), \tag{3.68}$$

where

$$\begin{aligned}L_{\mathrm{CH}}(\lambda) &= \begin{pmatrix} 0 & -\lambda^{-1} \\ -\frac{1}{4}\lambda^{-1} & 0 \end{pmatrix} + \sum_{j=1}^{N} \frac{\lambda^{-1}}{\lambda - \lambda_j} \begin{pmatrix} p_j q_j & -q_j^2 \\ p_j^2 & p_j q_j \end{pmatrix} \\ &\equiv \begin{pmatrix} A_{\mathrm{CH}}(\lambda) & B_{\mathrm{CH}}(\lambda) \\ C_{\mathrm{CH}}(\lambda) & -A_{\mathrm{CH}}(\lambda) \end{pmatrix},\end{aligned} \tag{3.69}$$

and the symplectic submanifold $\mathbb{M}$ becomes

$$M = \left\{ (q,p) \in \mathbb{R}^{2N} \,\middle|\, F \equiv \frac{1}{2}\left(\left\langle \Lambda^{-1} q, q \right\rangle - 1 \right) = 0,\ G \equiv \left\langle \Lambda^{-1} q, p \right\rangle = 0 \right\}, \tag{3.70}$$

where $\Lambda^{-1} = diag(\lambda_1^{-1}, \ldots, \lambda_N^{-1})$.

A direct calculation yields the following theorem.

Theorem 6. *On the symplectic submanifold M the Hamiltonian system $(H_{-1})_D$ defined by Eq. (3.31) has the Lax representation:*

$$\frac{\partial}{\partial x} L_{\mathrm{CH}} = [M_{\mathrm{CH}}, L_{\mathrm{CH}}], \tag{3.71}$$

where

$$M_{\mathrm{CH}} = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} - \frac{1+4\langle \Lambda p, p\rangle}{4\langle \Lambda^2 q, q\rangle}\lambda^{-1} & 0 \end{pmatrix}. \tag{3.72}$$

Notice. On M the Hamiltonian system $(H_{-1})_D$ is $2N-2$-dimensional, that is, there only exist $2N-2$ independent dynamical variables in all $2N$ variables $q_1, \ldots, q_N$; $p_1, \ldots, p_N$. Without loss of generality, we assume the Hamiltonian system $(H_{-1})_D$ has the independent dynamical variables $q_1, \ldots, q_{N-1};\ p_1, \ldots, p_{N-1}$ $(N > 1)$ on M. Then, on M we have

$$q_N^2 = \lambda_N - \sum_{j=1}^{N-1} \frac{\lambda_N}{\lambda_j} q_j^2, \tag{3.73}$$

$$p_N = -\sum_{j=1}^{N-1} \frac{\lambda_N}{\lambda_j q_N} q_j p_j, \tag{3.74}$$

where q_N in the latter is given by the former in terms of $q_1, \ldots, q_{N-1}$.

We will concretely give the expression $u = \langle q(x,t), q(x,t)\rangle$, $t = t_{-2}$ in an explicit form. By Eq. (3.73), rewrite the entry $B_{\mathrm{CH}}(\lambda)$ in the Lax matrix (3.69) as

$$\begin{aligned} B_{\mathrm{CH}}(\lambda) &= -\frac{1}{\lambda - \lambda_N}\left(1 + \sum_{j=1}^{N-1} \frac{(\lambda_j - \lambda_N)\lambda_j^{-1}}{\lambda - \lambda_j} q_j^2\right) \\ &\equiv -\frac{Q(\lambda)}{K(\lambda)}, \end{aligned} \tag{3.75}$$

where

$$Q(\lambda) = \prod_{j=1}^{N-1}\left(\lambda - \lambda_j\right) + \sum_{j=1}^{N-1} \frac{\lambda_j - \lambda_N}{\lambda_j} q_j^2 \prod_{k=1, k\neq j}^{N-1} (\lambda - \lambda_k), \tag{3.76}$$

$$K(\lambda) = \prod_{\alpha=1}^{N} (\lambda - \lambda_\alpha). \tag{3.77}$$

Apparently, $Q(\lambda)$ is a $N-1$ $(N > 1)$ order polynomial of λ. Choosing its $N-1$ distinct zero points $\mu_1, \ldots, \mu_{N-1}$, we have

$$Q(\lambda) = \prod_{j=1}^{N-1}\left(\lambda - \mu_j\right), \tag{3.78}$$

$$\langle q, q\rangle = \sum_{\alpha=1}^{N} \lambda_\alpha - \sum_{j=1}^{N-1} \mu_j. \tag{3.79}$$

Additionally, choosing $\lambda = \lambda_j$ in Eqs. (3.76) and (3.78) leads to an explicit form of q_j in terms of μ_l:

$$q_j^2 = \alpha_{jN} \prod_{l=1}^{N-1} (\lambda_j - \mu_l), \quad \alpha_{jN} = \frac{\lambda_j}{\prod_{k=1}^{N} (\lambda_j - \lambda_k)}, \tag{3.80}$$

which is similar to the result in Ref. [2]. By Eq. (3.79), we get an identity about μ_l:

$$\sum_{j=1}^{N} \alpha_{jN} \prod_{l=1}^{N-1} (\lambda_j - \mu_l) = \sum_{\alpha=1}^{N} \lambda_\alpha - \sum_{j=1}^{N-1} \mu_j.$$

Remark 3. The dynamical variable p_j corresponding to q_j is

$$p_j = \frac{dq_j}{dx} = -\frac{\alpha_{jN}}{2q_j} \sum_{k=1}^{N-1} \frac{d\mu_k}{dx} \prod_{l=1, l\neq k}^{N-1} (\lambda_j - \mu_l), \tag{3.81}$$

therefore,

$$p_j^2 = \frac{\alpha_{jN}}{4 \prod_{l=1}^{N-1} (\lambda_j - \mu_l)} \left(\sum_{k=1}^{N-1} \frac{d\mu_k}{dx} \prod_{l=1, l\neq k}^{N-1} (\lambda_j - \mu_l) \right)^2. \tag{3.82}$$

Substituting Eqs. (3.79) and (3.82) into the Hamiltonian $H_{-1} = \frac{1}{2}\langle p, p\rangle - \frac{1}{8}\langle q, q\rangle$ directly gives an expression in terms of μ_l:

$$H_{-1} = \frac{1}{8} \sum_{j=1}^{N} \frac{\alpha_{jN}}{\prod_{l=1}^{N-1} (\lambda_j - \mu_l)} \left(\sum_{k=1}^{N-1} \frac{d\mu_k}{dx} \prod_{l=1, l\neq k}^{N-1} (\lambda_j - \mu_l) \right)^2 + \frac{1}{8} \sum_{j=1}^{N-1} \mu_j - \frac{1}{8} \sum_{k=1}^{N} \lambda_k. \tag{3.83}$$

This is evidently different from the Hamiltonian function in Ref. [2] (see there Sect. 3). Here our H_{-1} comes from the nonlinearized CH spectral problemn, i.e. it is composing of a Neumann-like system on M, which is shown integrable in subsection 3.3 by r-matrix process. It is because of this difference that our parametric solution does not include the peakons (also see last subsection) and we will in the following procedure present a class of new algebro-geometric solution for the CH equation constrained on the symplectic submanifold M.

Let

$$\pi_j = A_{\mathrm{CH}}\left(\mu_j\right), \quad j = 1, \ldots, N-1, \tag{3.84}$$

then it is easy to prove the following proposition.

Proposition 1.

$$\{\mu_i, \mu_j\}_D = \{\pi_i, \pi_j\}_D = 0, \quad \{\pi_j, \mu_i\}_D = \delta_{ij}, \quad i, j = 1, 2, \ldots, N-1, \tag{3.85}$$

i.e. π_j, μ_j are conjugated, and thus they are the variables which can be separated [21].

Write

$$\begin{aligned}
-\det L_{\mathrm{CH}}(\lambda) &= A_{\mathrm{CH}}^2(\lambda) + B_{\mathrm{CH}}(\lambda)\, C_{\mathrm{CH}}(\lambda) \\
&= \frac{1}{\lambda^2}\left(\frac{1}{4} + \sum_{\alpha=1}^{N} \frac{E_\alpha}{\lambda - \lambda_\alpha}\right) \\
&= \frac{1}{\lambda(\lambda - \lambda_N)}\left(\frac{1}{4} + \sum_{\alpha=1}^{N-1} \frac{(\lambda_\alpha - \lambda_N)\lambda_\alpha^{-1}}{\lambda - \lambda_\alpha} E_\alpha\right) \\
&\equiv \frac{P(\lambda)}{\lambda K(\lambda)},
\end{aligned} \tag{3.86}$$

where E_α is defined by

$$E_\alpha = -p_\alpha^2 + \frac{1}{4}q_\alpha^2 - \Gamma_\alpha, \quad \Gamma_\alpha = \sum_{k=1, k\neq\alpha}^{N} \frac{(p_\alpha q_k - q_\alpha p_k)^2}{\lambda_\alpha - \lambda_k}, \tag{3.87}$$

$$P(\lambda) = \frac{1}{4}\prod_{j=1}^{N-1}\left(\lambda - \lambda_j\right) + \sum_{j=1}^{N-1} \frac{\lambda_j - \lambda_N}{\lambda_j} q_j^2 \prod_{k=1, k\neq j}^{N-1} (\lambda - \lambda_k), \tag{3.88}$$

and obviously $P(\lambda)$ is an $N-1$ order polynomial of λ whose first term's coefficient is $\frac{1}{4}$. Then we have

$$\pi_j^2 = \frac{P\left(\mu_j\right)}{\mu_j K\left(\mu_j\right)}, \;\; j = 1, \ldots, N-1. \tag{3.89}$$

Notice. On M we always have $\sum_{\alpha=1}^{N} \lambda_\alpha^{-1} E_\alpha = \frac{1}{4}$. Therefore, we assume $E_1, \ldots, E_{N-1}$ are independent. Then,

$$E_N = \frac{1}{4}\lambda_N - \sum_{\alpha=1}^{N-1} \frac{\lambda_N}{\lambda_\alpha} E_\alpha.$$

Actually in Eq. (3.86) we already used this fact.

Now, we choose the generating function

$$\begin{aligned}
W &= \sum_{j=1}^{N-1} W_j\left(\mu_j, \{E_\alpha\}_{\alpha=1}^{N-1}\right) \\
&= \sum_{j=1}^{N-1} \int_{\mu_0}^{\mu_j} \sqrt{\frac{P(\lambda)}{\lambda K(\lambda)}}\mathrm{d}\lambda,
\end{aligned} \tag{3.90}$$

where μ_0 is an arbitrarily given constant. Let us view E_α $(\alpha = 1, \ldots, N-1)$ as action variables, then angle-coordinates Q_α are chosen as

$$Q_\alpha = \frac{\partial W}{\partial E_\alpha}, \quad \alpha = 1, \ldots, N-1,$$

i.e.

$$Q_\alpha = \sum_{k=1}^{N-1} \int_{\mu_0}^{\mu_k} \frac{K(\lambda)}{2\sqrt{\lambda K(\lambda) P(\lambda)}} \cdot \frac{\lambda_\alpha - \lambda_N}{\lambda_\alpha(\lambda - \lambda_\alpha)(\lambda - \lambda_N)} \mathrm{d}\lambda$$
$$\equiv \frac{\lambda_\alpha - \lambda_N}{\lambda_\alpha} \sum_{k=1}^{N-1} \int_{\mu_0}^{\mu_k} \tilde{\omega}_\alpha, \tag{3.91}$$

where

$$\tilde{\omega}_\alpha = \frac{\prod_{k\neq\alpha,k=1}^{N-1} (\lambda - \lambda_k)}{2\sqrt{\lambda K(\lambda) P(\lambda)}} \mathrm{d}\lambda, \quad \alpha = 1, \ldots, N-1.$$

Therefore, on the symplectic submanifold $\left(M^{2N-2}, dE_\alpha \wedge dQ_\alpha\right)$ the Hamiltonian function

$$H_{-1} = \frac{1}{2}\langle p, p\rangle - \frac{1}{8}\langle q, q\rangle = -\frac{1}{2}\sum_{\alpha=1}^{N} E_\alpha$$
$$= -\frac{1}{8}\lambda_N - \frac{1}{2}\sum_{\alpha=1}^{N-1} \frac{\lambda_\alpha - \lambda_N}{\lambda_\alpha} E_\alpha \tag{3.92}$$

produces a linearized x-flow of the CH equation

$$\begin{cases} Q_{\alpha,x} = \frac{\partial H_{-1}}{\partial E_\alpha} = -\frac{\lambda_\alpha - \lambda_N}{2\lambda_\alpha}; \\ E_{\alpha,x} = 0, \end{cases} \tag{3.93}$$

as well, the Hamiltonian function

$$F_{-2} = \frac{1}{2}\langle \Lambda p, p\rangle - \frac{1}{8}\langle \Lambda q, q\rangle + \frac{1}{2}\left(\langle q, q\rangle\langle p, p\rangle - \langle q, p\rangle^2\right) = -\frac{1}{2}\sum_{\alpha=1}^{N} \lambda_\alpha E_\alpha$$
$$= -\frac{1}{8}\lambda_N^2 - \frac{1}{2}\sum_{\alpha=1}^{N-1} \frac{\lambda_\alpha^2 - \lambda_N^2}{\lambda_\alpha} E_\alpha \tag{3.94}$$

yields a linearized t-flow of the CH equation

$$\begin{cases} Q_{\alpha,t} = \frac{\partial F_{-2}}{\partial E_\alpha} = -\frac{\lambda_\alpha^2 - \lambda_N^2}{2\lambda_\alpha}, \\ E_{\alpha,t} = 0. \end{cases} \tag{3.95}$$

The above two flows imply

$$Q_\alpha = \frac{\lambda_N - \lambda_\alpha}{2\lambda_\alpha}\Big[x + (\lambda_N + \lambda_\alpha)t - 2Q_\alpha^0\Big], \tag{3.96}$$
$$E_\alpha = \text{constant}, \ \ \alpha = 1, \ldots, N-1, \tag{3.97}$$

where Q_α^0 is an arbitrarily chosen constant. Therefore we have

$$\sum_{k=1}^{N-1} \int_{\mu_0}^{\mu_k} \tilde{\omega}_\alpha = -\frac{1}{2}\Big[x + (\lambda_N + \lambda_\alpha)t\Big] + Q_\alpha^0. \tag{3.98}$$

Choose a basic system of closed paths α_i, β_i $(i = 1, \dots, N-1)$ of Riemann surface $\bar{\Gamma}$: $\mu^2 = \lambda P(\lambda) K(\lambda)$ with $N-1$ handles. $\tilde{\omega}_j$ $(j = 1, \dots, N-1)$ are exactly $N-1$ linearly independent holomorphic differentials of the first kind on the Riemann surface $\bar{\Gamma}$. Let $\tilde{\omega}_j$ be normalized as $\omega_j = \sum_{l=1}^{N-1} r_{j,l} \tilde{\omega}_l$, i.e. ω_j satisfy

$$\oint_{\alpha_i} \omega_j = \delta_{ij}, \quad \oint_{\beta_i} \omega_j = B_{ij},$$

where $B = \left(B_{ij}\right)_{(N-1)\times(N-1)}$ is symmetric and the imaginary part $\mathrm{Im}B$ of B is a positive definite matrix.

By the Riemann Theorem [15] we know: μ_k satisfy

$$\sum_{k=1}^{N-1} \int_{\mu_0}^{\mu_k} \omega_j = \phi_j,$$

$$\phi_j = \phi_j(x,t) \stackrel{\mathrm{d}}{=} \sum_{l=1}^{N-1} r_{j,l} \left(Q_l^0 - \frac{1}{2}\Big[x + (\lambda_N + \lambda_l)t\Big] \right),$$
$$j = 1, \dots, N-1,$$

iff μ_k are the zero points of the Riemann-Theta function $\tilde{\Theta}(P) = \Theta(A(P) - \phi - \mathbb{K})$ which has exactly $N-1$ zero points, where

$$A(P) = \left(\int_{P_0}^{P} \omega_1, \cdots, \int_{P_0}^{P} \omega_{N-1} \right)^T,$$
$$\phi = \phi(x,t) = (\phi_1(x,t), \cdots, \phi_{N-1}(x,t))^T,$$

$\mathbb{K} = (K_1, \dots, K_{N-1})^T \in \mathbb{C}^{N-1}$ is the Riemann constant vector, P_0 is an arbitrarily given point on the Riemann surface $\bar{\Gamma}$ (Θ-function and the related properties are seen in the Appendix).

Because of [10],

$$\frac{1}{2\pi i} \oint_{\gamma} \lambda d \ln \tilde{\Theta}(P) = C_1\left(\bar{\Gamma}\right), \tag{3.99}$$

where the constant $C_1\left(\bar{\Gamma}\right)$ has nothing to do with ϕ; γ is the boundary of a simple connected domain obtained through cutting the Riemann surface $\bar{\Gamma}$ along closed paths α_i, β_i. Thus, we have a key equality

$$\sum_{k=1}^{N-1} \mu_k = C_1\left(\bar{\Gamma}\right) - \mathrm{Res}_{\lambda=\infty_1} \lambda d \ln \tilde{\Theta}(P) - \mathrm{Res}_{\lambda=\infty_2} \lambda d \ln \tilde{\Theta}(P), \tag{3.100}$$

where $\mathrm{Res}_{\lambda=\infty_k}$ $(k = 1, 2)$ mean the residue at points ∞_k:

$$\infty_1 \stackrel{\mathrm{d}}{=} \left(0, \sqrt{z^{-1}P\left(z^{-1}\right)K\left(z^{-1}\right)}\bigg|_{z=0}\right),$$
$$\infty_2 \stackrel{\mathrm{d}}{=} \left(0, -\sqrt{z^{-1}P\left(z^{-1}\right)K\left(z^{-1}\right)}\bigg|_{z=0}\right).$$

Now, we need to calculate the above two residues.

Lemma 2.

$$\mathrm{Res}_{\lambda=\infty_1}\lambda d\ln\tilde{\Theta}(P) = -2\frac{\partial}{\partial x}\ln\Theta(\phi+\mathbb{K}+\eta_1), \tag{3.101}$$

$$\mathrm{Res}_{\lambda=\infty_2}\lambda d\ln\tilde{\Theta}(P) = 2\frac{\partial}{\partial x}\ln\Theta(\phi+\mathbb{K}+\eta_2), \tag{3.102}$$

where η_1, η_2 are two different $N-1$ dimensional constant vectors.

Proof. Consider the following smooth superelliptic curve $\bar{\Gamma}$: $\mu^2 = \lambda P(\lambda) K(\lambda)$. Because $\lambda P(\lambda) K(\lambda)$ is a $2N^{\text{th}}$ order polynomial with respect to λ, ∞ is not a branch point, i.e. on $\bar{\Gamma}$ there are two infinity points ∞_1 and ∞_2. All points P on $\bar{\Gamma}$ are denoted by $(\lambda, \pm\mu)$. On $\bar{\Gamma}$ we choose a group of basis of normalized closed paths $\alpha_1, \cdots, \alpha_{N-1}; \beta_1, \cdots, \beta_{N-1}$. They are mutually independent, and their intersection number are

$$\alpha_i \circ \alpha_j = \beta_i \circ \beta_j = 0, \quad \alpha_i \circ \beta_j = \delta_{ij}.$$

It is easy to see that $\tilde{\omega}_j$ $(j = 1, \ldots, N-1)$ are $N-1$ linearly independent holomorphic differential forms on $\bar{\Gamma}$.

Let us now come to calculate the residues of $\lambda d\ln\tilde{\Theta}(P)$ at the two infinity points: ∞_1, ∞_2. At ∞_1, the j^{th} variable I_j of $\tilde{\Theta}(z)$ produces the following result through multiplying by -1 (please note that the local coordinate at ∞_1 and ∞_2 is $z = \lambda^{-1}$):

$$\begin{aligned}
-I_j &= \phi_j + K_j + \eta_{1,j} - \sum_{l=1}^{N-1} r_{j,l}\int_0^z \tilde{\omega}_l \\
&= \phi_j + K_j + \eta_{1,j} + \sum_{l=1}^{N-1} r_{j,l}\int_0^z \left.\frac{\prod_{\alpha\neq l,\alpha=1}^{N-1}(\lambda-\lambda_\alpha)}{2\sqrt{\lambda P(\lambda) K(\lambda)}}\right|_{\lambda=z^{-1}} z^{-2}\mathrm{d}z \\
&= \phi_j + K_j + \eta_{1,j} + \sum_{l=1}^{N-1} r_{j,l}\int_0^z \frac{z^{-N} - (\sum_{j=1}^{N-1}\lambda_j - \lambda_l)z^{-N+1} + \cdots}{\sqrt{z^{-2N} + \cdots}}\mathrm{d}z \\
&= \phi_j + K_j + \eta_{1,j} + \sum_{l=1}^{N-1} r_{j,l}\int_0^z \frac{1+O(z)}{\sqrt{1+O(z)}}\mathrm{d}z \\
&= \phi_j + K_j + \eta_{1,j} + \sum_{l=1}^{N-1} r_{j,l} z + O\left(z^2\right),
\end{aligned}$$

where $\eta_{1,j} = \int_{\infty_1}^{P_0}\omega_j,\ j = 1, \cdots, N-1.$

Because

$$\frac{\partial\Theta}{\partial x} = \frac{1}{2}\sum_{j=1}^{N-1}\sum_{l=1}^{N-1}\frac{\partial\Theta}{\partial I_j} r_{j,l}$$

and $\tilde{\Theta}(z)$ has the expansion formula

$$\tilde{\Theta}(z) = \Theta(\phi+\mathbb{K}+\eta_1) - \sum_{j=1}^{N-1}\sum_{l=1}^{N-1}\frac{\partial\Theta}{\partial I_j} r_{j,l} z + O\left(z^2\right),$$

where $\eta_1 = (\eta_{1,1}, \dots, \eta_{1,N-1})^T$. Therefore,

$$\tilde{\Theta}(z) = \Theta(\phi + \mathbb{K} + \eta_1) - 2\frac{\partial \Theta}{\partial x} z + O\left(z^2\right).$$

So, we obtain the following residue:

$$\begin{aligned}
\mathrm{Res}_{\lambda=\infty_1} \lambda d \ln \tilde{\Theta}(P) &= \mathrm{Res}_{z=0} z^{-1} d \ln \tilde{\Theta}(z) = \mathrm{Res}_{z=0} \frac{1}{z} \frac{\tilde{\Theta}_z(z)}{\tilde{\Theta}(z)} \\
&= \mathrm{Res}_{z=0} \frac{1}{z} \frac{-2\Theta_x + O(z)}{\Theta - 2\Theta_x z + O\left(z^2\right)} \\
&= \mathrm{Res}_{z=0} \frac{1}{z} \frac{-2\Theta_x + O(z)}{\Theta\left(1 - 2\Theta^{-1}\Theta_x z + O\left(z^2\right)\right)} \\
&= \frac{-2\Theta_x}{\Theta} = -2\frac{\partial}{\partial x} \ln \Theta(\phi + \mathbb{K} + \eta_1).
\end{aligned}$$

In a similar way, we can obtain the second residue formula.

So, by Eq. (3.79) and this lemma, we immediately have

$$\langle q(x,t), q(x,t) \rangle = \sum_{\alpha=1}^{N} \lambda_\alpha - C_1\left(\bar{\Gamma}\right) + 2\frac{\partial}{\partial x}\left(\ln \frac{\Theta(\phi + \mathbb{K} + \eta_2)}{\Theta(\phi + \mathbb{K} + \eta_1)}\right), \tag{3.103}$$

where the j^{th} component of $\eta_i \ (i = 1, 2)$ is $\eta_{i,j} = \int_{\infty_i}^{P_0} \omega_j$.

So, the CH equation (3.13) has the following explicit solution, called the **algebro-geometric solution**:

$$u(x,t) = R + 2\frac{\partial}{\partial x}\left(\ln \frac{\Theta(\phi + \mathbb{K} + \eta_2)}{\Theta(\phi + \mathbb{K} + \eta_1)}\right), \tag{3.104}$$

where $R = \sum_{\alpha=1}^{N} \lambda_\alpha - C_1\left(\bar{\Gamma}\right)$ is a constant. □

Theorem 7. *The algebro-geometric solution of the* CH *equation* (3.13) *can be given through Eq. (3.104).*

Remark 4. Here the algebro-geometric solution (3.104) is smooth and occurs in the x-direction (spacial variable) derivative, and apparently differs from the piecewise smooth algebro-geometric solution in the t-direction derivatives given in Ref. [2]. It is also different from the results in Ref. [3–5, 14], because we are studying the CH equation constrained on M. In the paper, we do not need to calculate each $q_j \ (j = 1, \dots, N-1)$ but the sum $\sum_{j=1}^{N-1} q_j^2$, which we know from Eqs. (3.79) and (3.56). From the above subsection's comparison and these comments, therefore we think Eq. (3.104) is a class of new solution of the CH equation (3.13). Apparently, Eq. (3.104) is simpler in form than in Ref. [2].

4. Positive Order CH Hierarchy, Integrable Bargmann System, and Parametric Solution

4.1. Positive order CH hierarchy. Let us now give the positive order hierarchy of Eq. (2.2) through employing the kernel element of Lenard's operator J.

Obviously, $G_0 = cm^{-\frac{1}{2}}$ form all kernel elements of J, where $c = c(t_n) \in C^\infty(\mathbb{R})$ is an arbitrarily given function with respect to the time variables t_n ($n \geq 0, n \in \mathbb{Z}$), but independent of the spacial variable x. G_0 and the recursion operator (2.6) yield the following hierarchy of the CH spectral problem (2.2):

$$m_{t_k} = cJ\mathcal{L}^k \cdot m^{-\frac{1}{2}}, \quad k = 0, 1, 2, \ldots, \tag{4.1}$$

where the operators J and $\mathcal{L}$ are defined by Eqs. (2.5) and (2.6), respectively, and $\mathcal{L}$ and J have a further product form

$$\mathcal{L} = J^{-1}K = \frac{1}{2}m^{-\frac{1}{2}}\partial^{-1}m^{-\frac{1}{2}}(\partial - \partial^3), \tag{4.2}$$

$$J^{-1} = \frac{1}{2}m^{-\frac{1}{2}}\partial^{-1}m^{-\frac{1}{2}}. \tag{4.3}$$

Because Eq. (4.1) is related to the Camassa-Holm spectral problem (2.2) for the case of $k \geq 0,\ k \in \mathbb{Z}$, it is called **the positive order Camassa-Holm (CH) hierarchy**. Equation (4.1) has the following representative equations:

$$m_{t_0} = 0, \quad \text{trivial case}, \tag{4.4}$$

$$m_{t_1} = -c\left(m^{-\frac{1}{2}}\right)_{xxx} + c\left(m^{-\frac{1}{2}}\right)_x. \tag{4.5}$$

Apparently, with $c = -1$, Eq. (4.5) becomes a Dym type equation

$$m_{t_1} = \left(m^{-\frac{1}{2}}\right)_{xxx} - \left(m^{-\frac{1}{2}}\right)_x, \tag{4.6}$$

which has an extra term $-\left(m^{-\frac{1}{2}}\right)_x$ more than the usual Harry-Dym equation $m_{t_1} = \left(m^{-\frac{1}{2}}\right)_{xxx}$. Therefore, Eq. (4.1) gives an extended Dym hierarchy corresponding to the isospectral case: $\lambda_{t_k} = 0$.

By Theorem 2, the whole positive order CH hierarchy (4.1) has the zero curvature representation

$$U_{t_k} - V_{k,x} + [U, V_k] = 0,\ k > 0,\ k \in \mathbb{Z}, \tag{4.7}$$

$$V_k = \sum_{j=0}^{k-1} V_j \lambda^{k-j-1}, \tag{4.8}$$

where U is given by Eq. (2.2), and $V_j = V(G_j)$ is given by Eq. (2.9) with $G = G_j = \mathcal{L}^j \cdot G_0 = c\mathcal{L}^j \cdot m^{-\frac{1}{2}},\ j > 0,\ j \in \mathbb{Z}$. Thus, all nonlinear equations in the positive CH hierarchy (4.1) are integrable. Therefore we obtain the following theorem:

Theorem 8. *The positive order CH hierarchy (4.1) possesses the Lax pair*

$$\begin{cases} y_x = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} - \frac{1}{2}m\lambda & 0 \end{pmatrix} y, \\ y_{t_k} = \sum_{j=0}^{k-1} \begin{pmatrix} -\frac{1}{2}G_{j,x} & -G_j \\ \frac{1}{2}G_{j,xx} - \frac{1}{4}G_j + \frac{1}{2}m\lambda G_j & \frac{1}{2}G_{j,x} \end{pmatrix} \lambda^{k-j} y \\ \qquad k = 0, 1, 2, \ldots. \end{cases} \tag{4.9}$$

Equation (4.9) can be reduced to the following Lax pair:

$$\begin{cases} \psi_{xx} = \frac{1}{4}\psi - \frac{1}{2}m\lambda\psi \\ \psi_{t_k} = \sum_{j=0}^{k-1} \left(\frac{1}{2}G_{j,x}\lambda^{k-j}\psi - G_j\lambda^{k-j}\psi_x \right), \ k = 0, 1, \ldots. \end{cases} \tag{4.10}$$

In particular, the Dym type equation (4.6) has the Lax pair

$$\begin{cases} \psi_{xx} = \frac{1}{4}\psi - \frac{1}{2}m\lambda\psi \\ \psi_{t_1} = -\frac{1}{2}\left(m^{-\frac{1}{2}}\right)_x \lambda\psi + m^{-\frac{1}{2}}\lambda\psi_x. \end{cases} \tag{4.11}$$

Remark 5. This Lax pair coincides with the one obtained by the usual method of finite power expansion with respect to the spectral parameter λ. However, we here present the positive hierarchy (4.1) mainly by Lenard's operators pair satisfying Eq. (2.4). Because it contains the spectral gradient $\nabla\lambda$ in Eq. (2.4), this procedure of generating evolution equations from a given spectral problem is called the spectral gradient method.

Using this method, how to determine a pair of Lenard's operators associated with the given spectral problem mainly depends on the concrete forms of spectral problems and spectral gradients, and some computational techniques. From this method, we have already derived the negative order CH hierarchy and the positive order CH hierarchy.

4.2. r-matrix structure and integrability for the Bargmann CH system. Consider the following matrix (called "positive" Lax matrix):

$$L_+(\lambda) = \begin{pmatrix} A_+(\lambda) & B_+(\lambda) \\ C_+(\lambda) & -A_+(\lambda) \end{pmatrix}, \tag{4.12}$$

where

$$A_+(\lambda) = \sum_{j=1}^{N} \frac{\lambda_j p_j q_j}{\lambda - \lambda_j}, \tag{4.13}$$

$$B_+(\lambda) = -\sum_{j=1}^{N} \frac{\lambda_j q_j^2}{\lambda - \lambda_j}, \tag{4.14}$$

$$C_+(\lambda) = \frac{1}{2\langle \Lambda q, q\rangle} + \sum_{j=1}^{N} \frac{\lambda_j p_j^2}{\lambda - \lambda_j}. \tag{4.15}$$

We calculate the determinant of $L_+(\lambda)$:

$$-\frac{1}{2}\det L_+(\lambda) = \frac{1}{4}\mathrm{Tr}L_+^2(\lambda) = \frac{1}{2}\left(A_+^2(\lambda) + B_+(\lambda)\, C_+(\lambda)\right) = \sum_{j=1}^{N} \frac{E_j^+}{\lambda - \lambda_j}, \tag{4.16}$$

where Tr stands for the trace of a matrix, and

$$E_j^+ = -\frac{1}{4\langle \Lambda q, q\rangle}\lambda_j q_j^2 - \frac{1}{2}\Gamma_j^+, \quad j = 1, 2, \ldots, N, \tag{4.17}$$

$$\Gamma_j^+ = \sum_{l\neq j, l=1}^{N} \frac{\lambda_j\lambda_l(p_jq_l - p_lq_j)^2}{\lambda_j - \lambda_l}. \tag{4.18}$$

Let

$$F_k = \sum_{j=1}^{N} \lambda_j^k E_j^+, \quad k = 0, 1, 2, \ldots, \tag{4.19}$$

then it reads

$$F_k = -\frac{\langle \Lambda^{k+1} q, q\rangle}{4\langle \Lambda q, q\rangle} + \frac{1}{2}\sum_{j=0}^{k-1}\left(\left\langle \Lambda^{j+1}q, p\right\rangle\left\langle \Lambda^{k-j}q, p\right\rangle - \left\langle \Lambda^{j+1}q, q\right\rangle\left\langle \Lambda^{k-j}p, p\right\rangle\right), \tag{4.20}$$

where $k = 0, 1, 2, \ldots$. A long but direct computation leads to the following key equalities:

$$\begin{aligned}
&\{A_+(\lambda), A_+(\mu)\} = \{B_+(\lambda), B_+(\mu)\} = 0,\\
&\{C_+(\lambda), C_+(\mu)\} = \frac{2}{\langle \Lambda q, q\rangle^2}\left(\lambda A_+(\lambda) - \mu A_+(\mu)\right),\\
&\{A_+(\lambda), B_+(\mu)\} = \frac{2}{\mu - \lambda}(\mu B_+(\mu) - \lambda B_+(\lambda)),\\
&\{A_+(\lambda), C_+(\mu)\} = \frac{2}{\mu - \lambda}(\lambda C_+(\lambda) - \mu C_+(\mu)) - \frac{1}{\langle \Lambda q, q\rangle^2}\lambda B_+(\lambda),\\
&\{B_+(\lambda), C_+(\mu)\} = \frac{4}{\mu - \lambda}(\mu A_+(\mu) - \lambda A_+(\lambda)).
\end{aligned}$$

Let $L_1^+(\lambda) = L_+(\lambda) \otimes I_{2\times 2}$, $L_2^+(\mu) = I_{2\times 2} \otimes L_+(\mu)$, where $L_+(\lambda)$, $L_+(\mu)$ are given through Eq. (4.12). In the following, we search for a 4×4 r-matrix structure $r_{12}^+(\lambda, \mu)$ such that the fundamental Poisson bracket [13]:

$$\left\{L_+(\lambda) \overset{\otimes}{,} L_+(\mu)\right\} = \left[r_{12}^+(\lambda, \mu), L_1^+(\lambda)\right] - \left[r_{21}^+(\mu, \lambda), L_2^+(\mu)\right] \tag{4.21}$$

holds, where the entries of the 4×4 matrix $\left\{L_+(\lambda) \overset{\otimes}{,} L_+(\mu)\right\}$ are

$$\left\{L_+(\lambda) \overset{\otimes}{,} L_+(\mu)\right\}_{kl,mn} = \left\{L_+(\lambda)_{km}, L_+(\mu)_{ln}\right\}, \quad k, l, m, n = 1, 2,$$

and $r_{21}(\lambda,\mu) = Pr_{12}(\lambda,\mu)P$, with

$$P = \frac{1}{2}\left(I_{2\times 2} + \sum_{j=1}^{3}\sigma_j \otimes \sigma_j\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where σ_j' are the Pauli matrices.

Theorem 9.

$$r_{12}^{+}(\lambda,\mu) = \frac{2\lambda}{\mu(\mu-\lambda)}P + \frac{\lambda}{\langle \Lambda q, q\rangle^2}S^{+} \tag{4.22}$$

is an r-matrix structure satisfying Eq. (4.21), where

$$S^{+} = \sigma_{-} \otimes \sigma_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In fact, the r-matrix satisfying Eq. (4.21) is not unique [19]. Obviously, this r-matrix structure is also of 4×4 and different from the one in Ref. [20].

Because there is an r-matrix structure satisfying Eq. (4.21),

$$\left\{L_{+}^{2}(\lambda) \overset{\otimes}{,} L_{+}^{2}(\mu)\right\} = \left[\bar{r}_{12}^{+}(\lambda,\mu), L_1^{+}(\lambda)\right] - \left[\bar{r}_{21}^{+}(\mu,\lambda), L_2^{+}(\mu)\right], \tag{4.23}$$

where

$$\begin{aligned}\bar{r}_{ij}^{+}(\lambda,\mu) = \sum_{k=0}^{1}\sum_{l=0}^{1}&\left(L_1^{+}\right)^{1-k}(\lambda)\left(L_2^{+}\right)^{1-l}(\mu)\,r_{ij}(\lambda,\mu)\left(L_1^{+}\right)^{k}(\lambda)\left(L_2^{+}\right)^{l}(\mu),\\ ij &= 12, 21.\end{aligned}$$

Thus,

$$4\left\{\mathrm{Tr}L_{+}^{2}(\lambda), \mathrm{Tr}L_{+}^{2}(\mu)\right\} = \mathrm{Tr}\left\{L_{+}^{2}(\lambda) \overset{\otimes}{,} L_{+}^{2}(\mu)\right\} = 0. \tag{4.24}$$

So, by Eq. (4.16) we immediately obtain the following theorem.

Theorem 10. *The following equalities*

$$\begin{aligned}&\{E_i^{+}, E_j^{+}\} = 0, \quad \{F_k, E_j^{+}\} = 0,\\ &i, j = 1, 2, \ldots, N, \quad k = 0, 1, 2, \ldots,\end{aligned} \tag{4.25}$$

hold. Hence, all Hamiltonian systems (F_k)

$$\begin{aligned}(F_k): \quad q_{t_k} = \{q, F_k\} = \frac{\partial F_k}{\partial p}, \quad p_{t_k} = \{p, F_k\} = -\frac{\partial F_k}{\partial q},\\ k = 0, 1, 2, \ldots,\end{aligned} \tag{4.26}$$

are completely integrable.

Furthermore, we find the following Hamiltonian function:

$$H^+ = \frac{1}{2}\langle p, p\rangle - \frac{1}{8}\langle q, q\rangle - \frac{1}{4\langle \Lambda q, q\rangle} \tag{4.27}$$

is involutive with E_j^+, F_k, i.e.

$$\begin{aligned} &\{H^+, E_j^+\} = 0, \ \ \{H^+, F_k\} = 0, \\ &j = 1, 2, \ldots, N, \ \ k = 0, 1, 2, \ldots. \end{aligned} \tag{4.28}$$

Here, E_j^+ are N independent functions.

Therefore, we obtain the following results.

Corollary 4. *The canonical Hamiltonian system* (H^+)*:*

$$(H^+): \ \begin{cases} q_x = \frac{\partial H^+}{\partial p} = p, \\ p_x = -\frac{\partial H^+}{\partial q} = \frac{1}{4}q - \frac{1}{2\langle \Lambda q, q\rangle^2}\Lambda q. \end{cases} \tag{4.29}$$

is completely integrable.

Corollary 5. *All composition functions* $f\left(H^+, F_k^+\right)$*,* $f \in C^\infty(\mathbb{R})$*,* $k = 0, 1, 2, \ldots$*, are completely integrable Hamiltonians.*

Let

$$m = \frac{1}{\langle \Lambda q, q\rangle^2}, \tag{4.30}$$

$$\psi = q_j, \quad \lambda = \lambda_j, \quad j = 1, \ldots, N. \tag{4.31}$$

Then, the integrable flow (H^+) defined by Eq. (4.29) also exactly becomes the CH spectral problem (2.1) with the potential function m.

Remark 6. Equation (4.30) is a Bargmann constraint in the whole space $\mathbb{R}^{2N}$. Therefore, *the Hamiltonian system* (H^+) *is of integrable Bargmann type*.

4.3. Parametric solution of the positive order CH hierarchy. In the following, we shall consider the relation between constraint and nonlinear equations in the positive order CH hierarchy (4.1). Let us start from the following setting

$$G_0 = -\sum_{j=1}^{N} \nabla\lambda_j, \tag{4.32}$$

where $G_0 = -m^{-\frac{1}{2}}$, and $\nabla\lambda_j = \lambda_j q_j^2$ is the functional gradient of the CH spectral problem (2.2) corresponding to the spectral parameter λ_j $(j = 1, \ldots, N)$.

Apparently Eq. (4.32) reads

$$m = \frac{1}{\langle \Lambda q, q\rangle^2} \tag{4.33}$$

which coincides with the constraint relation (4.30).

Since the Hamiltonian flows (H^+) and (F_k) are completely integrable and their Poisson brackets $\{H^+, F_k\} = 0$ $(k = 0, 1, 2, \ldots)$, their phase flows $g^x_{H^+},\ g^{t_k}_{F_k}$ commute [6]. Thus, we can define their compatible solution as follows:

$$\begin{pmatrix} q(x,t_k) \\ p(x,t_k) \end{pmatrix} = g^x_{H^+} g^{t_k}_{F_k} \begin{pmatrix} q(x^0,t_k^0) \\ p(x^0,t_k^0) \end{pmatrix}, \quad k = 0, 1, 2, \ldots, \tag{4.34}$$

where $x^0,\ t_k^0$ are the initial values of phase flows $g^x_{H^+},\ g^{t_k}_{F_k}$.

Theorem 11. *Let $q(x,t_k),\ p(x,t_k)$ be a solution of the compatible Hamiltonian systems (H^+) and (F_k). Then*

$$m = \frac{1}{\langle \Lambda q(x,t_k), q(x,t_k)\rangle^2} \tag{4.35}$$

satisfies the positive order CH equation

$$m_{t_k} = -J\mathcal{L}^k \cdot m^{-\frac{1}{2}}, \quad k = 0, 1, 2, \ldots, \tag{4.36}$$

where the operators J and $\mathcal{L}$ are given by Eqs. (2.5) and (2.6), respectively.

Proof. This proof is similar to the negative case. □

In particular, with $k = 1$, we obtain the following corollary.

Corollary 6. *Let $q(x,t_1),\ p(x,t_1)$ be a solution of the compatible integrable Hamiltonian systems (H^+) and (F_1). Then*

$$m = m(x,t_1) = \frac{1}{\langle \Lambda q(x,t_1), q(x,t_1)\rangle^2}, \tag{4.37}$$

is a solution of the Dym type equation (4.6). Here H^+ and F_1 are given by

$$\begin{aligned} H^+ &= \frac{1}{2}\langle p,p\rangle - \frac{1}{8}\langle q,q\rangle - \frac{1}{4\langle \Lambda q,q\rangle}, \\ F_1 &= -\frac{\langle \Lambda^2 q,q\rangle}{4\langle \Lambda q,q\rangle} + \frac{1}{2}\Big(\langle \Lambda q,p\rangle^2 - \langle \Lambda q,q\rangle\langle \Lambda p,p\rangle\Big). \end{aligned}$$

By Theorem 11, the Bargmann constraint given by Eq. (4.30) is actually a solution of the positive order CH hierarchy (4.1). Thus, we also **call the system (H^+) (i.e. Eq. (4.29)) a positive order constrained CH flow (i.e. Bargmann type) of the spectral problem (2.2). All Hamiltonian systems (F_k), $k \geq 0, k \in \mathbb{Z}$ are therefore called the positive order integrable Bargmann flows in the whole** $\mathbb{R}^{2N}$. In a further procedure, we can also discuss the algebro-geometric solutions for the positive order CH hierarchy.

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Appendix. Abel Mapping and the Θ-Function

1. If the genus of a Riemann surface is g, this surface is homomorphic to a sphere with g handles. Such a basic system of closed paths (or contours) $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ can be chosen such that the only intersections among them are those of α_i and β_i with the same number i. Let the Riemann surface be covered with charts (U_i, z_i), where z_i are local parameters in open domains U_i, the transition from z_i to z_j in intersections $U_i \cap U_j$ being holomorphic. If in any U_i a differential $\varphi_i(z_i)\,dz_i$ with meromorphic $\varphi_i(z_i)$ is given and in the common parts $U_i \cap U_j$, $\varphi_i(z_i)\,dz_i = \varphi_j(z_j)\,dz_j$, then we say that there is an Abel differential Ω on the whole surface with restrictions $\Omega|_{v_i} = \varphi_i(z_i)\,dz_i$. The Abel differential is of the first kind if all the $\varphi_i(z_i)$ are holomorphic. There are exactly g linearly independent differentials of the first kind $\omega_1, \ldots, \omega_g$. They are normed if $\int_{\alpha_i} \omega_j = \delta_{ij}$, which condition determines them uniquely. We shall always assume them normed. The numbers $\int_{\beta_i} \omega_j = B_{ij}$ are called β-periods. The matrix $B = (B_{ij})$ has the following properties: 1) $B_{ij} = B_{ji}$, 2) $\tau=$ Im B is a positive definite matrix.

We consider a g-dimensional vector $\mathcal{A}(P) = \left\{\int_{P_0}^{P} \omega_j\right\}$, where P_0 is a fixed point of the Riemann surface and P is an arbitrary point. This vector is not uniquely determined, but depends on the path of integration. If the latter is changed then a linear combination of α and β-periods with integer coefficients can be added: $(\mathcal{A}(P))_j \mapsto (\mathcal{A}(P))_j + \sum_1^g n_i\delta_{ij} + \sum_1^g m_i B_{ij}$, i.e. $\mathcal{A}(P) \mapsto \mathcal{A}(P) + \sum n_i\delta_i + \sum m_i B_i$, where δ_i is the vector with coordinates δ_{ij}, B_i is the vector with coordinates B_{ij}. Thus $\mathcal{A}(P)$ determines a mapping of the Riemann surface on the torus $\mathbb{J} = \mathbb{C}^g/\mathbb{T}$, where $\mathbb{T}$ is the lattice generated by $2g$ vectors $\{\delta_i, B_i\}$ (which are linearly independent over $\mathbb{R}$). This mapping is called the Abel mapping, and the torus $\mathbb{J}$ is the Jacobi manifold or the Jacobian or the Riemann surface. The Abel mapping extends by linearity to the divisors:

$$\mathcal{A}\left(\sum n_k P_k\right) = \sum n_k \mathcal{A}(P_k).$$

2. *Abel Theorem.* Those and only those divisors go to zero of the Jacobian by the Abel mapping which are principal. The latter means that they are divisors of zeros and poles of meromorphic functions on the surface. If P_k is a zero of the function, then $n_k > 0$ and n_k is the degree of this zero. If P_k is a pole, then $n_k < 0$ and $|n_k|$ is the degree of this pole.

Of special interest is the case of divisors of degree g with all $n_k = 1$, i.e. of non-ordered sets of g points $P_1, \ldots, P_g$ of the Riemann surface. All the sets of such kind form the symmetrical g^{th} power of the Riemann surface. The Abel mapping has the form

$$\mathcal{A}(P_1, \ldots, P_g) = \left\{\sum_{j=1}^{g} \int_{P_0}^{P} \omega_j\right\}, \; j = 1, \ldots, g.$$

The symmetrical g^{th} power is a complex manifold of the complex dimension g and it is mapped on the Jacobian which is a manifold of the same dimension. A problem of conversion arises (the Jacobi inverse problem). It can be solved with the help of the Θ-function.

3. For arbitrary $P \in \mathbb{C}^g$, let

$$\Theta(P) = \sum_{Z \in \mathbb{Z}^g} \exp\{\pi i (BZ, Z) + 2\pi i (P, Z)\},$$

$$(BZ, Z) = \sum_{i,j=1}^{g} B_{ij} z_i z_j, \ Z = (z_1, \dots, z_g)^T,$$

$$(P, Z) = \sum_{i=1}^{g} p_i z_i, \ P = (p_1, \dots, p_g)^T.$$

The series converges owing to the properties of the matrix B. The Θ-function has the properties

$$\begin{aligned}\Theta(-P) &= \Theta(P)\\ \Theta(P + \delta_k) &= \Theta(P)\\ \Theta(P + B_k) &= \Theta(P) \exp\{-\pi i (B_{kk} + 2p_k)\}.\end{aligned}$$

Note that the Θ-function is not defined on the Jacobian because of the latter property.

4. *Riemann Theorem.* There are constants $\mathbb{K} = \{k_i\}$, $i = 1, \dots, g$ (Riemann constants) determined by the Riemann surface such that the set of points $P_1, \dots, P_g$ is a solution of the system of equations

$$\sum_{i=1}^{g} \int_{P_0}^{P_i} \omega_j = l_j, \ \mathbb{L} = \{l_j\} \in \mathbb{J}, \ j = 1, \dots, g,$$

if and only if $P_1, \dots, P_g$ are the zeros of the function $\tilde{\Theta}(P) = \Theta(\mathcal{A}(P) - \mathbb{L} - \mathbb{K})$ (which has exactly g zeros). Note that while the function $\tilde{\Theta}(P)$ is not uniquely determined on the Riemann surface (it is multivalued) its zeros are multivalued, since distinct branches of $\tilde{\Theta}(P)$ differ by exponents.

5. We define now Abel differentials of the second and of the third kind. The Abel differential of the second kind, $\Omega_P^{(k)}$, $k = 1, 2, \dots$, has the only singularity at the point P which is a pole of the order $k + 1$. The differential can be represented at this point as dz_+^{-k} (holomorphic differential), z being the local parameter at this point. Such a differential is uniquely determined if it is normed: $\int_{\alpha_i} \Omega_P^{(k)} = 0, \ \forall i$.

The Abel differential of the third kind Ω_{PQ} has only singularities which are simple poles at the points P and Q with the residues $+1$ and -1, respectively. It is uniquely determined by the same condition.

6. *Proposition.* If z is a local parameter in a neighbourhood of the point P and $\omega_i = \varphi_i(z)\, dz$ is the Abel differential of the first kind, then

$$\frac{1}{2\pi i} \int_{\beta_i} \Omega_P^{(k)} = -\frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \varphi_i(z) \Big|_{z=0}, \ i = 1, \dots, g,$$

and

$$\frac{1}{2\pi i} \int_{\beta_i} \Omega_{PQ} = \int_Q^P \omega_i, \ \ i = 1, \dots g,$$

which is also seen in Ref. [11].

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