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Ultraviolet Finite Quantum Field Theory on Quantum Spacetime

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Dedicated to Rudolf Haag on the occasion of his 80th birthday

Abstract: We discuss a formulation of quantum field theory on quantum space time where the perturbation expansion of the S-matrix is term by term ultraviolet finite.

The characteristic feature of our approach is a quantum version of the Wick product at coinciding points: the differences of coordinates $q_j - q_k$ are not set equal to zero, which would violate the commutation relation between their components. We show that the optimal degree of approximate coincidence can be defined by the evaluation of a conditional expectation which replaces each function of $q_j - q_k$ by its expectation value in optimally localized states, while leaving the mean coordinates $\frac{1}{n}(q_1 + \cdots + q_n)$ invariant.

The resulting procedure is to a large extent unique, and is invariant under translations and rotations, but violates Lorentz invariance. Indeed, optimal localization refers to a specific Lorentz frame, where the electric and magnetic parts of the commutator of the coordinates have to coincide [11].

Employing an adiabatic switching, we show that the S-matrix is term by term finite. The matrix elements of the transfer matrix are determined, at each order in the perturbative expansion, by kernels with Gaussian decay in the Planck scale. The adiabatic limit and the large scale limit of this theory will be studied elsewhere.

1. Introduction

Spacetime quantization was proposed earlier than renormalization theory as a possible way of regularizing quantum field theory [19]. Recently, a deeper motivation was given [10, 11]: the concurrence of the principles of quantum mechanics and of classical general relativity leads to spacetime uncertainty relations; the natural geometric background that implements those relations is a noncommutative model of spacetime. More precisely, in order to give localization in spacetime an operational meaning, *the energy transfer*

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associated to the localization of an event by the Heisenberg uncertainty principle should be limited so that the generated gravitational field does not trap the event itself inside an horizon; otherwise the observation would be prevented. This principle implies spacetime uncertainty relations which in a weaker form can be written as

$$\Delta q^0(\Delta q^1 + \Delta q^2 + \Delta q^3) \gtrsim \lambda_P^2,$$
$$\Delta q^1 \Delta q^2 + \Delta q^2 \Delta q^3 + \Delta q^3 \Delta q^1 \gtrsim \lambda_P^2,$$

where λ_P is the Planck length $\sqrt{G\hbar c^{-3}} \simeq 1.6 \times 10^{-33}$cm. It is possible to implement exactly these relations by appropriate commutation relations between the components of the spacetime coordinates q^μ [11, 10].

$$[q^\mu, q^\nu] = i\lambda_P^2 Q^{\mu\nu}, \tag{1.1}$$
$$[q^\mu, Q^{\nu\rho}] = 0, \tag{1.2}$$
$$Q_{\mu\nu} Q^{\mu\nu} = 0, \tag{1.3}$$
$$\left(\frac{1}{2} Q_{\mu\nu} (*Q)^{\mu\nu}\right)^2 = I, \tag{1.4}$$

where $*Q$ is the Hodge dual of Q. These relations are covariant under the full Poincaré group. The irreducible representations of the spacetime commutation relations (1) take the familiar form (in absolute units, where $\lambda_P = 1$)

$$[q^\mu, q^\nu] = i\sigma^{\mu\nu} I, \tag{1.5}$$

where σ is a real antisymmetric matrix in the manifold Σ defined by the conditions (1.3, 1.4) with $Q^{\mu\nu} = \sigma^{\mu\nu} I$. They evidently break Lorentz covariance. Interest in the relations (1.5) was more recently raised by the occurrence of closely related forms of noncommutativity also in string theory [8, 18]. There exists, however, an essentially unique, fully covariant representation where the pairwise commuting, selfadjoint operators $Q^{\mu\nu}$ have the full manifold Σ as their joint spectrum. The generalized Weyl correspondence

$$\mathcal{W}(g \otimes f) = g(Q) f(q)$$

extends to any symbol $F \in \mathcal{C}_0(\Sigma \times \mathbb{R}^4)$, $\check{F}(\sigma, \cdot) \in L^1(\mathbb{R}^4)$, where $\check{F}(\sigma, \cdot)$ is the inverse Fourier transform of $F(\sigma, \cdot)$, for σ fixed. In the above equation, $g(Q)$ is to be understood in the sense of the joint functional calculus of the $Q^{\mu\nu}$s, and

$$f(q) = \int_{\mathbb{R}^4} dk \ \check{f}(k) e^{ikq},$$

where $\check{f}(k) = (2\pi)^{-4} \int dx \ f(x) e^{-ikx}$ and $kq = k_\mu q^\mu$, $kx = k_\mu x^\mu$. The above correspondence induces a generalized twisted product

$$(F_1 \star F_2)(\sigma, \cdot) = F_1(\sigma, \cdot) \star_\sigma F_2(\sigma, \cdot),$$

on the symbols, by $\mathcal{W}(F_1 \star F_2) = \mathcal{W}(F_1)\mathcal{W}(F_2)$. Moreover, $\mathcal{W}(\overline{F}) = \mathcal{W}(F)^*$. We denote by $\mathcal{E}$ the enveloping C*-algebra of the resulting algebra; it is isomorphic to $\mathcal{C}_0(\Sigma, \mathcal{K})$, the C*-algebra of the continuous functions taking values in the algebra $\mathcal{K}$ of

the compact operators on the separable, infinite dimensional Hilbert space, and vanishing at infinity. The Poincaré group acts on the symbols in $\mathcal{E}$ by

$$(\tau_{(a,\Lambda)}F)(\sigma, x) = \det\,\Lambda\; F(\Lambda^{-1}\sigma\Lambda^{-1t}, \Lambda^{-1}(x-a)).$$

The manifold Σ is the orbit of the standard symplectic matrix $\sigma_0^{\mu\nu} = \left(\begin{smallmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right)$, under the action $\sigma \mapsto \Lambda\sigma\Lambda^t$ of the full Lorentz group.

By definition, the states of $\mathcal{E}$ with *optimal localization* (both in space and in time) minimize $\sum_\mu (\Delta q_\mu)^2$. This characterization is evidently invariant under rotations and translations, but not under Lorentz boosts. It can be shown (see [11]) that the optimally localized states are of the form

$$\langle \omega_a, F\rangle = \int_{\Sigma_1} \mu(d\sigma)(\eta_a F)(\sigma), \quad F \in \mathcal{E},$$

where μ is any probability measure on the distinguished subset Σ_1 of Σ, the orbit of σ_0 under the action of $O(\mathbb{R}^3)$, and $\eta_a : \mathcal{E} \to \mathcal{C}(\Sigma_1)$ is the localization map with localization centre $a \in \mathbb{R}^4$,

$$(\eta_a F)(\sigma) = \int_{\mathbb{R}^4} dk\; \check{F}(\sigma, k) \exp\left\{-\frac{1}{2}\sum_{\mu=0}^{3} {k_\mu}^2\right\} e^{ika}. \tag{1.6}$$

In what follows, we will need only the localization map with localization centre $a = 0$, and in order to simplify the notation we will denote it by η. However, the results below also hold for a general $a \in \mathbb{R}^4$.

It is convenient to introduce the enveloping C*-algebra $\mathcal{E}_1$ generated by the restrictions $\gamma F = F\restriction_{\Sigma_1}$ of the symbols to Σ_1. Then the localization map η is the composition $\eta = \eta^{(1)} \circ \gamma$ of the restriction map $\gamma : \mathcal{E} \to \mathcal{E}_1$, with a positive map $\eta^{(1)}$ from $\mathcal{E}_1$ to $\mathcal{C}(\Sigma_1)$, which is a conditional expectation in the sense that[1] $\eta^{(1)}(zF) = z\eta^{(1)}(F)$, $z \in \mathcal{C}(\Sigma_1)$, $F \in \mathcal{E}_1$. η will also denote the normal extension to the multiplier algebra $M(\mathcal{E})$. Then,

$$\left\langle \eta, e^{ikq}\right\rangle = e^{-\frac{1}{2}\sum_\mu {k_\mu}^2}, \tag{1.7}$$

as a constant function of $\sigma \in \Sigma_1$.

By analogy with the definition of $f(q)$, the evaluation of an ordinary quantum field ϕ on the quantum spacetime is given by

$$\phi(q) = \int_{\mathbb{R}^4} dk\; e^{ikq} \otimes \check{\phi}(k)$$

and is to be interpreted as a map from states on $\mathcal{E}$ to smeared field operators,

$$\omega \mapsto \phi(\omega) = \langle \omega \otimes \mathrm{id},\; \phi(q)\rangle = \int_{\mathbb{R}^4} dx\; \phi(x)\psi_\omega(x),$$

where the r.h.s. is a quantum field on the ordinary spacetime, smeared with the test function ψ_ω defined by $\check{\psi}_\omega(k) = \langle \omega,\; e^{ikq}\rangle$. If products of fields are evaluated in a state, the r.h.s. will in general involve nonlocal expressions.

[1] Note that, while $\mathcal{C}(\Sigma_1)$ is not a subalgebra of $\mathcal{E}_1$, it is a subalgebra (actually, the centre) of the multiplier algebra $M(\mathcal{E}_1)$.

As in ordinary quantum field theory, due to the singular properties of fields, products of fields are not *a priori* well-defined. On the ordinary Minkowski spacetime, well-defined products of fields are given by the so-called Wick products. They may be defined by bringing the positive and negative frequency parts of the fields in the product into "normal order", which in momentum space corresponds to putting all destruction operators to the right. Another definition, which, contrary to normal-ordering may also be applied on curved spacetimes, is given in terms of the formal evaluation on the diagonal of a suitably subtracted product; one has, for instance, at second order $:\phi(x)^2: = \lim_{x\to y}\,(\,\phi(x)\phi(y) - (\Omega, \phi(x)\,\phi(y)\Omega)\,)$. The two constructions, while equivalent on the ordinary Minkowski spacetime, lead to inequivalent generalizations on the quantum spacetime.

In [11], for instance, an interaction Lagrangian was given in terms of the *usual* normal ordering of positive and negative frequency parts of $\phi(q)^n$,

$$\mathcal{L}_I(x) = :(\phi \star \cdots \star \phi)(x):. \tag{1.8}$$

Another possibility will be investigated in [2]. There, we consider products of fields at different points as they arise in the context of the Yang-Feldman equation, $\phi(q + x_1)\cdots\phi(q + x_n)$, $x_i \in \mathbb{R}^4$. We then define the so-called quasiplanar Wick products by allowing only terms which are local in a certain sense to be subtracted, and show that they are well-defined on the diagonal, i.e. in the limit of coinciding points where $x_i = x_j$.

In this paper we consider yet another approach. The evaluation on the diagonal is replaced by a suitable generalization compatible with the uncertainty relations, leading to a regularized nonlocal effective interaction. The idea is that a product of fields at different points, $\phi(q_1)\cdots\phi(q_n)$, may be defined by interpreting $q_1, \ldots, q_n$ as *mutually independent quantum coordinates*, that is, by defining

$$q_j{}^\mu = I \otimes \cdots \otimes I \otimes q^\mu \otimes I \otimes \cdots \otimes I \quad (n \text{ factors, } q^\mu \text{ in the } j^{\text{th}} \text{ slot}), \tag{1.9}$$

and

$$\phi(q_1)\cdots\phi(q_n) = \int dk_1 \cdots dk_n\; \check{\phi}(k_1)\cdots\check{\phi}(k_n) e^{i(k_1q_1+\cdots+k_nq_n)}.$$

Now, the different spacetime components of each variable $q_j - q_k$, $j \neq k$, no longer commute with one another, hence the limit $q_j - q_k \longrightarrow 0$ loses its natural meaning. We can, however, identify the central elements, i.e. take $[q_j^\mu, q_j^\nu] = i\,Q^{\mu\nu}$ for all j. This amounts to taking the tensor products in (1.9) not over the complex numbers, but over the centre $Z = \mathcal{C}_0(\Sigma)$ of (the multiplier algebra of) $\mathcal{E}$. The limit $q_j - q_k \longrightarrow 0$ will then be replaced by a *quantum diagonal map* which on each function of $q_j - q_k$ evaluates a state minimizing the square Euclidean length, while leaving the mean coordinates invariant (cf. [15, 7]).

As a consequence of taking the tensor product in (1.9) over Z, the mean coordinates $\frac{1}{n}\sum_j q_j$ commute with the relative coordinates $q_j - q_k$ (in the strong sense), e.g. for $n = 2$,

$$\big[(q_1 + q_2)^\mu, (q_1 - q_2)^\nu\big] = [q^\mu, q^\nu] \otimes I - I \otimes [q^\mu, q^\nu] = 0. \tag{1.10}$$

This fact turns out to be crucial for the construction of the quantum diagonal map and provides an additional motivation for taking the tensor product over Z.

Actually, the tensor product $\mathcal{E} \otimes_Z \mathcal{E}$ of Z-moduli can be defined as the completion relative to the maximal C*-seminorm of the quotient of the algebraic tensor product over $\mathbb{C}$ modulo the two sided ideal generated by the multiples in $\mathcal{E} \odot \mathcal{E}$ of $I \odot z - z \odot I$, where z varies in Z. It may be equivalently described as the fibrewise tensor product of bundles of C*-algebras (continuous fields of C*-algebras, in the terminology of [9]) over Σ.

Another motivation to use a Z-module tensor product is to view the components $Q^{\mu\nu}$ of the coordinates' commutator as universal data which are the same for the different variables corresponding to independent events. The $Q^{\mu\nu}$s are thus treated as a point independent geometric background, which, however, is translation invariant and Lorentz covariant.

Since the C*-algebra $\mathcal{E}$ describes the regular representations of (1), i.e. integrable to a representation of the Weyl relations

$$e^{ikq} e^{ihq} = e^{-\frac{i}{2} k_\mu h_\nu Q^{\mu\nu}} e^{i(k+h)q},$$

the uniqueness theorem of von Neumann [16], applied to each fibre at $\sigma \in \Sigma$, ensures that commutativity implies tensor factorization over Z. This fact will allow us to obtain the desired map as a conditional expectation.

Furthermore, we will use the tensor product of $n+1$ copies of the basic algebra as an auxiliary algebra, where the mean coordinates are (affiliated to the algebra) in the first factor, and where the algebra to which the difference variables $q_j - q_k$ are affiliated is identified with a subalgebra of the auxiliary algebra, associated to the factors from slot 2 to $n+1$. The desired quantum diagonal map

$$E^{(n)} : \mathcal{E} \otimes_Z \cdots \otimes_Z \mathcal{E} \longrightarrow \mathcal{E}_1$$

is then obtained by evaluating $\gamma \otimes_Z \eta^{n \otimes_Z}$ on such tensor products, where η is the localization map (1.6) with localization centre $a = 0$, and where $\gamma : \mathcal{E} \to \mathcal{E}_1$ is the restriction map. It turns out that the application of $E^{(n)}$ to functions whose symbols do not depend on Q explicitly, yields expressions which in turn are independent of Q.

The quantum diagonal map replaces the ordinary evaluation at coinciding points. Contrary to the ordinary case, it yields a well-defined expression when applied to a product of fields,

$$\begin{aligned} \phi^{(n)}(q) &= E^{(n)}(\phi(q_1) \cdots \phi(q_n)) \\ &= \int d^4k_1 \cdots d^4k_n \; r_n(k_1, \ldots, k_n) \, \check{\phi}(k_1) \cdots \check{\phi}(k_n) \, e^{i(k_1 + \cdots + k_n)q}, \end{aligned}$$

since a nonlocal regularizing kernel r_n appears. We conclude that contrary to the ordinary case, no infinite counterterms have to be subtracted and $\phi^{(n)}(q)$ may be used directly to define the interaction in the quantum theory. Regarding the combinatorics, it is, however, convenient to additionally apply ordinary normal ordering, and to define a *quantum Wick power* as

$$\begin{aligned} :\phi^n(q):_Q &= E^{(n)}(:\phi(q_1) \cdots \phi(q_n):) \\ &= \int d^4k_1 \cdots d^4k_n \; r_n(k_1, \ldots, k_n) :\check{\phi}(k_1) \cdots \check{\phi}(k_n): e^{i(k_1 + \cdots + k_n)q}. \end{aligned}$$

The quantum Wick power $:\phi(q)^n:_Q$ as well as $\phi^{(n)}(q)$ may be understood as functions of q, not explicitly depending on Q, taking values in the field operators. In other words, $:\phi(q)^n:_Q$ and $\phi^{(n)}(q)$ formally are elements of $\mathcal{E}_1 \otimes \mathfrak{F}$, where $\mathfrak{F}$ is the field algebra.

Once products of fields are given a precise meaning, one may apply an appropriate perturbative setup. Since sharp localization in time is compatible with the spacetime uncertainty relations (at the cost of complete delocalization in space), *one* possibility is, for instance, to follow the standard approach to perturbation theory in the interaction representation, involving integrations at sharp fixed times [12]. If the Lagrangian is symmetric, the resulting S-matrix is formally unitary by construction (at least before renormalization).

In [11], such an approach was proposed, based on the interaction Lagrangian (1.8). Unfortunately, the resulting perturbation theory is not free of ultraviolet divergences. This fact was first observed in [13] where, however, instead of the interaction picture used in [11], the theory was defined in terms of modified Feynman rules which may be formally derived from a path integral formulation. As first observed in [14] the resulting theory violates unitarity, a defect which may be traced back to the problem of time ordering on a (space/time)-noncommutative theory, as discussed in [1]: the time ordering naturally defined in the interaction picture formulation (cf. [11, Eq. (6.15)] and subsequent comments, as well as Sect. 4 of the present paper) does not violate unitarity. As a consequence, the formulation of the theory in terms of modified Feynman rules is not equivalent to the one discussed here.

Another inequivalent approach, which, however, yields a unitary perturbation theory was proposed in [1]. This approach is based on the Yang-Feldman equation and will be discussed elsewhere [2].

Instead, we will again apply the standard approach to perturbation theory in the interaction representation, this time employing the quantum Wick products.

The interaction Hamiltonian on the quantum spacetime is then given by

$$\mathcal{H}_I(t) = \lambda \int_{q^0=t} d^3q \, :\phi(q)^n:_Q$$

as a constant operator–valued function of Σ_1 (i.e. $\mathcal{H}_I(t)$ is formally in $\mathcal{C}(\Sigma_1)\otimes\widehat{\mathcal{F}}$). While in [11] one still had to handle the dependence of the Hamiltonian on σ, in the approach adopted here, $\mathcal{H}_I(t)$ is a *constant* function of $\sigma \in \Sigma_1$. As a consequence, our procedure leads to a unique prescription for the interaction Hamiltonian on quantum spacetime.

The resulting effective non-local Hamiltonian is

$$H_I(t) = \lambda \int d^3\boldsymbol{x} \; \mathcal{L}_{\text{eff}}(t, \boldsymbol{x}), \tag{1.11}$$

where $\mathcal{L}_{\text{eff}}$ is the effective nonlocal interaction Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{eff}}(x) = c_n \int_{\mathbb{R}^{4n}} da_1 \cdots da_n \, &:\phi(x+a_1)\cdots\phi(x+a_n):\\ &\exp\left\{-\frac{1}{2}\sum_{j,\mu} a_j^{\mu 2}\right\} \delta^{(4)}\left(\frac{1}{n}\sum_{j=1}^{n} a_j\right).\end{aligned} \tag{1.12}$$

It will be shown in Sect. 3 that the corresponding perturbation theory is free of ultraviolet divergences. The ultraviolet regularization arises as a point–split regularization by convolution with Gaussian kernels, and we will show that, by insertion of an adiabatic switch, the perturbation series is order by order finite, and each term is a well defined,

closed operator with a common core. The only remaining source of divergences is then given by possible infinite volume effects arising in the adiabatic limit, which will be discussed elsewhere.

The ultraviolet finiteness of the theory presented in this paper is in accordance with the expectation that noncommutativity of spacetime may regularize the theory. Other examples for ultraviolet finite theories on noncommutative spaces were discussed in [6], for instance compact spacetimes, corresponding to finite dimensional algebras.

It is noteworthy that the transition matrix elements will vanish as Gaussian functions of the energies and momentum transfers expressed in Planck units.

While in the high energy limit the transition amplitudes vanish rapidly as a result of the quantum delocalization of the interaction, in the low energy limit one would expect that the corrections to the ordinary theory on Minkowski space vanish. This is clearly possible only after a finite renormalisation; the structure of the needed counterterms and the dependence upon the Planck length of the renormalisation constants will be studied elsewhere.

Note that in the limit where the Planck length can be neglected, the renormalized theory on quantum spacetime should coincide with the ordinary renormalized theory on Minkowski space. At the physical values of the Planck length, the effect of the quantum nature of spacetime should manifest itself as quadratic or higher order corrections, since gravitation is not explicitly taken into account, but manifests itself only through the commutator of the coordinates.

A weak point of the approach to quantum field theory on quantum spacetime presented here is that, while, as was first shown in [11], the prescription leading to (1.8) does not alter the *free* Hamiltonian, the prescription discussed here would indeed change it, replacing it by a deformed operator which would no longer be the zero component of a Lorentz vector. We therefore treat, in this paper, the interaction on a different footing than the unperturbed Hamiltonian which we identify with that of the usual free theory[2]. As a consequence, Lorentz invariance is violated in an essential way, since optimally localized states are defined relative to a particular Lorentz frame. However, spacetime translation and space rotation invariance are preserved. Moreover, the evaluation of optimally localized states on the difference variables $q_j - q_k$ automatically restricts the joint eigenvalues $\sigma^{\mu\nu}$ of $Q^{\mu\nu}$ to Σ_1, the basis of Σ, where the electric and magnetic parts of σ are equal or opposite. This gives an a posteriori motivation for a similar choice, made in [11], which was motivated by simplicity and by the need of preserving space rotation invariance.

2. The Quantum Diagonal Map

According to the previous discussion,

$$q_j{}^{\mu} = \underbrace{I \otimes_Z \cdots \otimes_Z I \otimes_Z q^{\mu} \otimes_Z I \otimes_Z \cdots \otimes_Z I}_{n \text{ factors}}, \quad q^{\mu} \text{ in the } j^{\text{th}} \text{ slot},$$

[2] There exists an alternative approach, based on the action principle, which avoids this unsatisfactory feature. It will be discussed in a forthcoming publication [3].

fulfill the relations (for any $i, j = 1, \cdots, n$)

$$[q_i{}^\mu, q_j{}^\nu] = i\lambda_P^2 \delta_{ij} \mathcal{Q}^{\mu\nu}, \tag{2.1}$$

$$[q_j{}^\mu, \mathcal{Q}^{\nu\rho}] = 0, \tag{2.2}$$

$$\mathcal{Q}_{\mu\nu}\mathcal{Q}^{\mu\nu} = 0, \tag{2.3}$$

$$\left(\frac{1}{2}\mathcal{Q}_{\mu\nu}(*\mathcal{Q})^{\mu\nu}\right)^2 = I. \tag{2.4}$$

The correspondence

$$\mathcal{W}^{(n)}(g \otimes f) = g(\mathcal{Q}) f(q_1, \ldots, q_n), \quad g \in \mathcal{C}_0(\Sigma),\ f \in \mathcal{C}_0(\mathbb{R}^{4n}),\ \check{f} \in L^1(\mathbb{R}^{4n}),$$

extends to the generalized symbols $F = F(\sigma, x_1, \ldots, x_n)$ as usual, where

$$f(q_1, \ldots, q_n) = \int dk_1 \ldots, k_n\ \check{f}(k_1, \ldots, k_n) e^{i(k_1 q_1 + \cdots + k_n q_n)}.$$

It induces a product and an involution on the generalized n-symbols, and the enveloping C*-algebra of the resulting algebra is precisely $\mathcal{E}^{(n)} = \mathcal{E} \otimes_Z \cdots \otimes_Z \mathcal{E}$.

Remark 2.1. Note that, since $\mathcal{K} \otimes \mathcal{K} \sim \mathcal{K}$ as C*-algebras, $\mathcal{E} \otimes_Z \cdots \otimes_Z \mathcal{E} \sim \mathcal{E} \sim \mathcal{C}(\Sigma, \mathcal{K})$. Closed 2-sided ideals J in $\mathcal{E} \otimes_Z \cdots \otimes_Z \mathcal{E}$ are then in a 1-1 correspondence with closed ideals in Z (the kernel of the restriction to Z of the canonical extension to $M(\mathcal{E})$ of the projection map mod J), hence are in a 1-1 correspondence with the closed subsets of Σ.

Let us now introduce the coordinates of the mean event, denoted mean coordinates for short,

$$\bar{q} = \frac{1}{n}(q_1 + \cdots + q_n)$$

as well as the separations

$$q_{ij} = q_i - q_j.$$

Then

$$q_i = \bar{q} + \frac{1}{n}\sum_j q_{ij}. \tag{2.5}$$

Since the commutator $[q_j{}^\mu, q_j{}^\nu] = i\mathcal{Q}^{\mu\nu}$ does not depend on j, the following strong commutation relations hold:

$$e^{ik_\mu \bar{q}^\mu} e^{ik'_\mu \bar{q}^\mu} = e^{-i\frac{1}{2n}(\mathcal{Q}^{\mu\nu}k_\mu k'_\nu)} e^{i(k+k')_\mu \bar{q}^\mu}, \tag{2.6}$$

$$e^{ik_\mu \bar{q}^\mu} e^{ik'_\mu q_{ij}^\mu} = e^{ik'_\mu q_{ij}^\mu} e^{ik_\mu \bar{q}^\mu}. \tag{2.7}$$

We have the following factorization. Let $\tilde{q}$ be coordinates with characteristic length $1/\sqrt{n}$, i.e. $[\tilde{q}^\mu, \tilde{q}^\nu] = \frac{i}{n}\mathcal{Q}^{\mu\nu}$. Define

$$\bar{\boldsymbol{q}}^\mu := \tilde{q}^\mu \otimes_Z I^{n\otimes_Z}, \qquad \boldsymbol{q}_{ij}{}^\mu = I \otimes_Z q_{ij}{}^\mu, \tag{2.8}$$

and

$$\boldsymbol{q}_i := \bar{\boldsymbol{q}} + \frac{1}{n}\sum_j \boldsymbol{q}_{ij}. \tag{2.9}$$

We immediately check that the above elements also fulfill the relations (2.1–2.4) in the regular form, where $[\boldsymbol{q}_j{}^\mu, \boldsymbol{q}_j{}^\nu] = i\,Q^{\mu\nu}$. By von Neumann uniqueness (at each fixed σ; see [11]), there exists a faithful *-homomorphism

$$\beta^{(n)} : \mathcal{E}(n) \mapsto M(\mathcal{E}^{(n+1)})$$

such that

$$\beta^{(n)}(q_i) = \boldsymbol{q}_i.$$

This follows from the fact that regularity implies that the map $q_i^\mu \mapsto \boldsymbol{q}_i^\mu$ determines a *-homomorphism $\beta_i : \mathcal{E} \to M(\mathcal{E}^{(n+1)})$ (whose canonical extension to $M(\mathcal{E})$ will still be denoted by β_i); the ranges of β_i and β_j commute for $i \neq j$ and $\beta_i \restriction_Z$ is an isomorphism independent of i. By the universal properties of the tensor product and its uniqueness for nuclear C*-algebras (as $\mathcal{E}$), there is a *-homomorphism $\beta^{(n)}$ of $\mathcal{E}^{(n)}$ to $M(\mathcal{E}^{(n+1)})$, s.t. $\beta(A_1 \otimes_Z \cdots \otimes_Z A_n) = \beta_1(A_1) \cdots \beta_n(A_n)$, $A_j \in \mathcal{E}$. By assumption, $\beta^{(n)}$ is faithful on Z, hence, by Remark 2.1 on p. 228, $\beta^{(n)}$ is faithful.

Explicitly,

$$\beta^{(n)}\left(g(\mathcal{Q})f(q_1, \ldots, q_n)\right) = g(Q)f(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n),$$

where, of course,

$$f(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) = \int dk_1 \cdots dk_n \; \check{f}(k_1, \ldots, k_n) e^{ik_1\boldsymbol{q}_1} \cdots e^{ik_n\boldsymbol{q}_n}.$$

Definition 2.2. *The quantum diagonal map $E^{(n)} : \mathcal{E}^{(n)} \to \mathcal{E}_1$ is defined as*

$$E^{(n)} = \left(\gamma \otimes_Z \underbrace{\eta \otimes_Z \cdots \otimes_Z \eta}_{n \text{ factors}}\right) \circ \beta^{(n)},$$

where η, γ are the localization map and the restriction to Σ_1 (projection of $\mathcal{E}$ onto $\mathcal{E}_1$), respectively. Note that the generators $\tilde{q}^\mu$ of the algebra in which $E^{(n)}$ takes values have characteristic length $1/\sqrt{n}$.

To motivate this choice, let us recall that the difference variables $q_{ij}{}^\mu/\sqrt{2}$ fulfill the commutation relations (1.1–1.4), and a short computation yields

$$\left\langle \underbrace{\eta \otimes_Z \cdots \otimes_Z \eta}_{n \text{ factors}}, e^{ik_\mu q_{ij}{}^\mu/\sqrt{2}} \right\rangle = \left\langle \eta, e^{ik_\mu q^\mu} \right\rangle$$

(as constant functions of $\sigma \in \Sigma_1$; compare with (1.7)). In other words, $\eta^{n\otimes_Z}$ minimizes the Euclidean separation $\sum_\mu (q_i{}^\mu - q_j{}^\mu)^2$. $E^{(n)}$ will also denote the normal extension of the above map to the multiplier algebra $M(\mathcal{E}^{(n)})$.

Note also that, had we used η_a instead of $\eta = \eta_0$, we would have defined the same map $E^{(n)}$, since the separations $q_i - q_j$ are invariant under translations.

Proposition 2.3. *Let $f \in \mathcal{C}_0(\mathbb{R}^{4n})$, $\check{f} \in L^1(\mathbb{R}^{4n})$. The explicit form of the quantum diagonal map on f is given by*

$$E^{(n)}(f(q_1,\dots,q_n)) = \int_{\mathbb{R}^{4n}} dk_1\cdots dk_n \check{f}(k_1,\dots,k_n) r_n(k_1,\dots,k_n) e^{i(\sum_i k_i)\tilde{q}},$$

where

$$r_n(k_1,\dots,k_n) = \exp\left\{-\frac{1}{2}\sum_{\mu=0}^{3}\left(\sum_{j=1}^{n} k_{j\mu}^2 - \frac{1}{n}\sum_{j,l=1}^{n} k_{j\mu}k_{l\mu}\right)\right\}. \tag{2.10}$$

Equivalently,

$$E^{(n)}\left(f(q_1,\dots,q_n)\right) = h(\tilde{q}),$$

where

$$h(x) = c_n \int_{\mathbb{R}^{4n}} da_1\cdots da_n\ f(x+a_1,\dots,x+a_n)\hat{r}_n(a_1,\dots,a_n),$$

with $c_n = n^2(2\pi)^{-8(n-1)}$ and, with $|a|^2 = \sum_{\mu=0}^{3} a_\mu a_\mu$,

$$\hat{r}_n(a_1,\dots,a_n) = \exp\left(-\frac{1}{2}|a_1|^2 - \cdots - \frac{1}{2}|a_n|^2\right)\delta^{(4)}\left(\frac{1}{n}\sum_{j=1}^{n} a_j\right).$$

In particular, $E^{(n)}\left(f(q_1,\dots,q_n)\right)$ is a constant function of $\sigma \in \Sigma_1$.

Proof. A simple computation yields, by the definition (2.9) of $\boldsymbol{q}_i$,

$$\exp\left\{i\sum_j k_j \boldsymbol{q}_j\right\} = \exp\left\{i\left(\sum_j k_j\right)\bar{q}\right\} \otimes_Z \exp\left\{i\sum_j\left(k_j - \frac{1}{n}\sum_l k_l\right)q_j\right\}.$$

By the above and (2.6), (2.7), we have

$$\begin{aligned}
&\left\langle \gamma \otimes_Z \eta^{n\otimes_Z},\ f(\boldsymbol{q}_1,\dots,\boldsymbol{q}_1)\right\rangle \\
&= \int dk_1\cdots dk_n \check{f}(k_1,\dots,k_n)\exp\left\{i\left(\sum_j k_j\right)\tilde{q}\right\} \\
&\quad\times \prod_{j=1}^{n}\left\langle \eta,\ \exp\left\{i\left(k_j - \frac{1}{n}\sum_l k_l\right)q_j\right\}\right\rangle
\end{aligned}$$

as a constant function of σ; (2.10) then follows by a straightforward computation. Standard computations provide the configuration space kernel. □

The quantum diagonal map takes a particular simple form if evaluated in optimally localized states. Indeed, let $\tilde{\eta}_a$ denote the localization map η_a applied to the mean position coordinates $\tilde{q}$. Then a simple calculation yields the formula

$$\tilde{\eta}_a \circ E^{(n)} = \eta_a^{\otimes_Z^n}.$$

Since the function $a \to \eta_a(f)$ may be understood as the best commutative analogue of an element f of the noncommutative algebra, this formula provides an additional justification of the present approach. (Cf. also the discussion in [11] and [6].)

3. A Class of Ultraviolet Finite Theories on the Quantum Spacetime

The uncertainty relations (2) are compatible with sharp localization in time, at the cost of total delocalization in space. Consistently, the centre valued map

$$g(Q)f(q) \mapsto g(Q) \int d^3\boldsymbol{x}\ f(t,\boldsymbol{x})$$

extends to a positive partial trace $\int_{q^0=t} d^3q$ (see [11] for details), which commutes with the restriction γ to Σ_1.

For a fixed choice of a frame of reference, we formulate a traditional perturbative setup in the spirit of [12, 4]. Consider for simplicity the $\lambda\phi^n$ interaction; then the formal interaction Hamiltonian will be defined as

$$\mathcal{H}_I(t) = \lambda \int_{q^0=t} d^3q\ \mathcal{L}_I(q),$$

where the interaction Lagrangian $\mathcal{L}_I(q)$ may be either the n^{th} *quantum Wick power*, defined by evaluating the quantum diagonal map on a normally ordered product of fields[3],

$$\begin{aligned}:\phi^n(q):_Q &= E^{(n)}(:\phi(q_1)\cdots\phi(q_n):)\\ &= \int d^4k_1\cdots d^4k_n\ r_n(k_1,\ldots,k_n) :\check{\phi}(k_1)\cdots\check{\phi}(k_n): e^{i(k_1+\cdots+k_n)q},\end{aligned}$$

or the *regularized product*, defined by evaluating the quantum diagonal map on a product of fields as it stands, without application of normal order,

$$\begin{aligned}&\phi^{(n)}(q)\\ &= E^{(n)}(\phi(q_1)\cdots\phi(q_n))\\ &= \int d^4k_1\cdots d^4k_n\ r_n(k_1,\ldots,k_n)\ \check{\phi}(k_1)\cdots\check{\phi}(k_n)\, e^{i(k_1+\cdots+k_n)q}\\ &= \int dk e^{ikq} \int dy e^{-iky} \int d^4x_1\cdots d^4x_n\ \hat{r}_n(y-x_1,\ldots,y-x_n)\,\phi(x_1)\cdots\phi(x_n).\end{aligned}$$

Clearly, the first definition yields a well-defined expression, but, contrary to the ordinary case, normal ordering is not necessary due to the regulating kernel r_n which renders the second product well-defined as well[4]. In fact, both the quantum Wick power $:\phi(q)^n:_Q$ as well as $\phi^{(n)}(q)$ may be understood as functions of q, not explicitly depending on Q, taking values in the field operators. In other words, $:\phi(q)^n:_Q$ and $\phi^{(n)}(q)$ formally are elements of $\mathcal{E}_1 \otimes \mathfrak{F}$, where $\mathfrak{F}$ is the field algebra[5].

In the following, however, we will base our investigation on an interaction given by a quantum Wick power. For one thing, it simplifies the combinatorics, and in view of

[3] We recall that a monomial $A = a^\sharp(\psi_1)\cdots a^\sharp(\psi_n)$ in the creation and destruction operators ($a^\sharp = a, a^\dagger$) is called normally ordered or Wick ordered and denoted $:A:$, if all creation operators stand left of the destruction operators.

[4] That this is true may be either checked by expanding the product of fields in normally ordered products and check that all integrals are finite, or by employing the method of wavefront sets to show that $\mathrm{Diag} \circ (\hat{r}_n \times (\phi\ldots\phi))$, where $\times$ denotes the ordinary convolution, is well-defined.

[5] In more rigorous mathematical terms, $:\phi(q)^n:_Q$ and $\phi^{(n)}(q)$ are affine maps from states on $\mathcal{E}$ to quadratic forms.

the adiabatic limit, which we hope to study in a later paper, normal ordering may even be necessary.

The resulting Hamiltonian $\mathcal{H}_I(t)$ is formally affiliated to $\mathcal{C}(\Sigma_1) \otimes \widetilde{\mathfrak{F}}$, where $\widetilde{\mathfrak{F}}$ is the free Bose field algebra on the ordinary spacetime. Roughly speaking, $\mathcal{H}_I(t)$ is a function from Σ_1 to (formal) field operators, i.e. to quadratic forms.

In order to retrieve the Hamiltonian of the equivalent theory on the ordinary spacetime, one has to integrate over some probability measure μ over Σ_1, defining $H_I(t) = \int d\mu \ \mathcal{H}_I(t)$. Since, however, $\mathcal{H}_I(t)$ is a constant function of σ, the choice of μ is irrelevant. This fact should be contrasted with the case considered in [11], where the non-irrelevant choice of a particular measure — though the most reasonable in that context — was to some extent arbitrary, as well as the special rôle played by Σ_1.

The resulting Hamiltonian $H_I(t)$ for the equivalent theory on the ordinary spacetime can then be put in the form (1.11), where the effective nonlocal Lagrangian is given by

$$\mathcal{L}_{\text{eff}}(x) = \int dk \ \check{\mathcal{L}}_{\text{eff}}(k) e^{ikx},$$

$$\check{\mathcal{L}}_{\text{eff}}(k) = \int d^4k_1 \cdots d^4k_n \ r_n(k_1, \ldots, k_n) {:}\check{\phi}(k_1) \cdots \check{\phi}(k_n){:} \, \delta^{(4)}\left(k - \sum_{j=1}^{n} k_j\right).$$

Note that in the perturbation series ((3.1) here below) the time ordering of products $H_I(t_1) \cdots H_I(t_N)$ will refer to the variables $t_1, \ldots, t_N$ rather than to the integration variables in (1.12).

The fundamental result of this section is that the finite 4-volume theory yields a finite perturbation series. More precisely, we turn the coupling constant λ into a smooth function of x vanishing at infinity sufficiently fast, of the form $\lambda(t, \boldsymbol{x}) = \lambda'(t)\lambda''(\boldsymbol{x})$, and we show that the corresponding Dyson series is well defined at all orders. Well-known methods from ordinary quantum field theory are employed.

Proposition 3.1. *For any Schwartz function λ of the form $\lambda(t, \boldsymbol{x}) = \lambda'(t)\lambda''(\boldsymbol{x})$, $\lambda' \in \mathcal{S}(\mathbb{R})$, $\lambda'' \in \mathcal{S}(\mathbb{R}^3)$, the formal series*

$$S[\lambda] = T \exp\left\{-i \int d^4x \ \lambda(x) \mathcal{L}_{\mathit{eff}}(x)\right\} = I + \sum_{N=1}^{\infty} (-i)^N S^{(N)}[\lambda] \tag{3.1}$$

is finite at all orders. More precisely, it is possible — by Wick reduction — to put the N^{th} order contribution in the form of a finite sum of closable operators with common core $D_{\mathcal{S}}$ (the subspace of the Fock space consisting of the vectors with finitely many particles and with Schwartz n-particle components for each n). By construction, $S[\lambda]$ is unitary.

Remark 3.2. While the existence of the adiabatic limit $\lambda \to 1$ is questionable due to the breakdown of Lorentz covariance, the infinite volume limit $\lambda'' \to 1$ (with λ' fixed) of the Gell-Mann–Low formula for $S[\lambda' \otimes \lambda'']/\langle S[\lambda' \otimes \lambda'']\rangle_0$ exists as a quadratic form. Indeed, the only terms in the perturbation expansion of $S[\lambda' \otimes \lambda'']$ which are divergent in the limit $\lambda'' \to 1$ (with λ' fixed) are precisely those containing vacuum–vacuum parts. The actual behaviour of the full adiabatic limit will be investigated elsewhere.

Proof. We shall follow standard conventions (see e.g. [17]): in particular, $a(g)$, $a^\dagger(g)$ are the destruction and creation operators on the symmetric Fock space $\exp(L^2(\mathbb{R}^3))$, $g \in L^2(\mathbb{R}^3)$, and

$$
\begin{aligned}
a(g) &= \int_{\mathbb{R}^3} d\boldsymbol{k}\; a(\boldsymbol{k})\overline{g(-\boldsymbol{k})}, \qquad a^\dagger(g) = \int_{\mathbb{R}^3} d\boldsymbol{k}\; a^\dagger(\boldsymbol{k}) g(\boldsymbol{k}), \\
\phi(t,\boldsymbol{x}) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{d\boldsymbol{k}}{\sqrt{2\omega(\boldsymbol{k})}} \left\{ e^{ik_\mu x^\mu} a^\dagger(\boldsymbol{k}) + e^{-ik_\mu x^\mu} a(\boldsymbol{k}) \right\}
\end{aligned}
\tag{3.2}
$$

as quadratic forms on $D_{\mathcal{S}} \times D_{\mathcal{S}}$, where $\omega(\boldsymbol{k}) = \sqrt{|\boldsymbol{k}|^2 + m^2}$, and $k = (\omega(\boldsymbol{k}), \boldsymbol{k})$.

The cutoff Lagrangian $H_I^{(\lambda)}(t)$ is given by

$$
H_I^{(\lambda)}(t) = \int_{\mathbb{R}^3} d\boldsymbol{x}\; \lambda(t,\boldsymbol{x}) \mathcal{L}_{\text{eff}}(t,\boldsymbol{x}) = \lambda'(t) \int_{\mathbb{R}^3} d\boldsymbol{x}\; \lambda''(\boldsymbol{x}) \mathcal{L}_{\text{eff}}(t,\boldsymbol{x}).
$$

We introduce the following compact notations:

$$
\underline{a}^\mu = (a_1^\mu, \dots, a_n^\mu) \in \mathbb{R}^n, \tag{3.3}
$$

$$
\underline{\boldsymbol{a}} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) \in \mathbb{R}^{3n}, \tag{3.4}
$$

$$
\underline{a} = (\underline{a}^0, \underline{\boldsymbol{a}}) = (a_1, \dots, a_n) \in \mathbb{R}^{4n}. \tag{3.5}
$$

The translation of all the 4-vectors in $\underline{a}$ by the same 4 vector x will be denoted by

$$
\underline{a} - x = (a_1 - x, a_2 - x, \cdots, a_n - x), \quad x, a_j \in \mathbb{R}^4. \tag{3.6}
$$

The symbol $\cdot$ will denote the canonical Euclidean scalar product in $\mathbb{R}^n$, $\mathbb{R}^{3n}$, $\mathbb{R}^{4n}$, depending on the context; then

$$
\underline{a}\; \underline{b} = \underline{a}_\mu \cdot \underline{b}^\mu = \underline{a}^0 \cdot \underline{b}^0 - \underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}}, \tag{3.7}
$$

$$
|\underline{a}|^2 = \underline{a} \cdot \underline{a}. \tag{3.8}
$$

Moreover,

$$
d\underline{a} = d\underline{a}^0 d\underline{\boldsymbol{a}} = \prod_{\mu=0}^{3} d\underline{a}^\mu = \prod_{\mu=0}^{3} \prod_{j=1}^{n} da_j^\mu. \tag{3.9}
$$

Finally, for any function $g = g(x)$ of $\mathbb{R}^4$, we write

$$
g^{(n)}(\underline{x}) = g(x_1) \cdots g(x_n), \tag{3.10}
$$

and, in particular,

$$
:\phi^{(n)}(\underline{x}): = :\phi(x_1) \dots \phi(x_n):. \tag{3.11}
$$

Standard computations yield

$$
\begin{aligned}
S_N[\lambda] = \int_{\mathbb{R}^{4nN}} d\underline{a}_1 \cdots d\underline{a}_N \left(\prod_{1 \le M < N} \theta\left(\kappa^0 (\underline{a}_{M+1}^0 - \underline{a}_M^0) \right) \right) \\
G_\lambda(\underline{a}_1) \cdots G_\lambda(\underline{a}_N)\; :\phi^{(n)}(\underline{a}_1): \cdots :\phi^{(n)}(\underline{a}_N):,
\end{aligned}
$$

where $\kappa^0(\underline{a}^0)$ is the time component of the mean point

$$\kappa(\underline{a}) = (\kappa^0(\underline{a}^0), \boldsymbol{\kappa}(\underline{\boldsymbol{a}})) = \frac{1}{n}\sum_{j=1}^{n} a_j \in \mathbb{R}^4 \tag{3.12}$$

of $\underline{a} = (a_1, \ldots, a_n)$, and

$$G_\lambda(\underline{a}) = \lambda(\kappa(\underline{a}))e^{-\frac{1}{2}|\underline{a}-\kappa(\underline{a})|^2} \tag{3.13}$$

is a Schwartz function.

With $P(I_n)$ the set of all subsets of $I_n = \{1, \cdots, n\}$ (including the trivial subsets), we shall write $J_\bullet$ for any choice of N elements of $P(I_n)$, namely $J_\bullet = \{J_1, \ldots, J_N\}$ with $J_M \in P(I_n)$, $M = 1, \cdots, N$. Then, by (3.2), we get

$$\begin{aligned} S_N[\lambda] = \sum_{J_\bullet} \int_{\mathbb{R}^{3nN}} d\underline{\boldsymbol{k}}_1 \cdots d\underline{\boldsymbol{k}}_N \frac{K_\lambda^{J_\bullet}(\underline{\boldsymbol{k}}_1, \ldots, \underline{\boldsymbol{k}}_N)}{\sqrt{\omega^{(n)}(\underline{\boldsymbol{k}}_1)\cdots\omega^{(n)}(\underline{\boldsymbol{k}}_N)}} \\ \times \left(\prod_{u_1 \in J_1} a^\dagger(\boldsymbol{k}_{u_1})\right)\left(\prod_{v_1 \in I_n \setminus J_1} a(\boldsymbol{k}_{v_1})\right)\cdots \\ \cdots \left(\prod_{u_N \in J_N} a^\dagger(\boldsymbol{k}_{u_N})\right)\left(\prod_{v_N \in I_n \setminus J_N} a(\boldsymbol{k}_{v_N})\right), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} &K_\lambda^{J_\bullet}(\underline{\boldsymbol{k}}_1, \ldots, \underline{\boldsymbol{k}}_N) \\ &= c_{n,N} \int_{\mathbb{R}^{4nN}} d\underline{a}_1 \cdots d\underline{a}_N \left(\prod_{1 \le M < N} \theta\left(\kappa^0(\underline{a}^0_{M+1} - \underline{a}^0_M)\right)\right) G_\lambda(\underline{a}_1)\cdots G_\lambda(\underline{a}_N) \\ &\quad \times \exp\left\{ i \sum_{M=1}^{N} \underline{k}_{M\mu} \cdot \left(U_M \underline{a}_M\right)^\mu \right\}. \end{aligned} \tag{3.15}$$

Here U_M is a diagonal $n \times n$ matrix, with diagonal entries

$$U_{M\,uu} = \begin{cases} 1, & u \in J_M, \\ -1, & u \in I_n \setminus J_n, \end{cases} \tag{3.16}$$

and

$$\underline{\tilde{k}} = (\tilde{k}_1, \ldots, \tilde{k}_M).$$

By the second Wick theorem, each term in the above sum may be written as a sum of terms with contractions, where the surviving creation and destruction operators appear in normal order (creation on the left), and the contracted pairs are replaced by $\delta^{(3)}$ distributions. Since each integration variable $\boldsymbol{k}_{M\,j}$ appears as the argument of precisely one operator $a^\sharp$, $(a^\sharp = a, a^\dagger)$, then the arising product of δs is well defined by the following elementary

Remark 3.3. The map

$$f \mapsto \int da_1 \cdots da_d f(a_1, \cdots, a_d) \prod_{j=1}^{s} \delta(l_j(a_1, \ldots, a_d)), \quad s \leq d, \tag{3.17}$$

on the Schwartz functions, with l^j a real linear functional on $\mathbb{R}^d$, $j = 1, \ldots, s$, is a well defined distribution if and only if the functionals $l^1, \ldots, l^s$ are linearly independent. By performing s integrations, the above distribution always takes the form

$$\int db_1 \cdots db_{d-s} \; f\,(a_1(b_1, \cdots, b_{d-s}), \ldots, a_d(b_1, \cdots, b_{d-s}))\,,$$

where the linear maps $a_j = a_j(b_1, \cdots, b_{d-s})$, $j = 1, d$, provide a linear injection of $\mathbb{R}^{d-s}$ into $\mathbb{R}^d$.

After complete Wick reduction, the generic term will be of the form

$$\int_{\mathbb{R}^{6k}} d\boldsymbol{p}_1 \cdots d\boldsymbol{p}_{r+s} W(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{r+s}) a^\dagger(\boldsymbol{p}_1) \cdots a^\dagger(\boldsymbol{p}_r) a(\boldsymbol{p}_{r+1}) \cdots a(\boldsymbol{p}_{r+s}),$$

where

$$W(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_r, \boldsymbol{p}_{r+1}, \ldots, \boldsymbol{p}_{r+s}) = \frac{K_\lambda^{J_\bullet}(\underline{\boldsymbol{k}}_1(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{r+s}), \ldots, \underline{\boldsymbol{k}}_N(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{r+s})}{\sqrt{\omega^{(n)}(\underline{\boldsymbol{k}}_1(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{r+s})) \cdots \omega^{(n)}(\underline{\boldsymbol{k}}_N(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{r+s}))}}$$

for a suitable set of linear maps (see Remark 3.3)

$$\underline{\boldsymbol{k}}_1 = \underline{\boldsymbol{k}}_1(\boldsymbol{p}_1 \cdots \boldsymbol{p}_{r+s}), \ldots, \underline{\boldsymbol{k}}_N = \underline{\boldsymbol{k}}_N(\boldsymbol{p}_1 \cdots \boldsymbol{p}_{r+s})$$

with $r + s \leq nN$.[6] The proof is complete by [17, Theorem X.44], if we show that the function W is in $L^2(\mathbb{R}^{3(r+s)})$. To this end, it is enough to prove that $|K_\lambda^{J_\bullet}|$ is bounded by a Schwartz function. Due to the form of λ, the integrations over the space and time variables in (3.15) factorize, and $K_\lambda^{J_\bullet}$ is the product of a function with Gaussian decay with the Fourier transform — evaluated at some point continuously depending on the $\underline{\boldsymbol{k}}_M$'s — of an L^1 function; the latter is then a bounded, continuous function. Indeed, we will show in Appendix 4 that $K_\lambda^{J_\bullet}$ is a Schwartz function, by carrying out explicitly the above mentioned computations. □

Note that the kernels $W(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_r, \boldsymbol{p}_{r+1}, \ldots, \boldsymbol{p}_{r+s})$ appearing in the above proof are transition amplitudes of scattering processes with s incoming and r outgoing particles. They decay as Gaussian functions of the energies and momentum transfers expressed in Planck units.

[6] Of course, the case $r + s = nN$ corresponds to the term with no contractions; in that case, the above mentioned linear maps reduce to renaming the integration variables.

4. Time-Ordering and Diagrammatics

We complement our discussion with some formal remarks aiming to clarify the relation between the approach followed here and the usual formulation of the perturbation theory in terms of Feynman diagrams. We have already shown in [1] that the perturbative setup dealt with in [11] (as well as the approach based on the Yang-Feldman equation) is inequivalent to the by now standard setup in terms of the modified Feynman rules [13]. In this section we will make explicit that in the framework of the regularized interaction proposed in this paper, Feynman propagators are no longer available at all.

In this discussion we ignore all problems which may arise in the adiabatic limit, and use the formal (i.e. defined with constant coupling constant λ) time dependent interaction Hamiltonian $H_I(t)$ throughout,

$$H_I(t) = \lambda \int_{\mathbb{R}^{4n}} dx_1 \cdots dx_n \; w_t(x_1, \ldots, x_n) {:}\phi(x_1) \cdots \phi(x_n){:},$$

where the integral kernel w_t is of the form

$$w_t(x_1, \ldots, x_n) = w'(x_1, \ldots, x_n)\, \delta^{(1)}\left(t - \frac{1}{n}\sum_j {x_j}^0\right),$$

and the Gaussian kernel w' does not depend on t. Introducing the time ordered kernels

$$\mathcal{T}_{t_1,\ldots,t_N} = \sum_{\pi \in \mathbb{P}_N} \theta\left(t_{\pi(1)} - t_{\pi(2)}\right) \ldots \theta\left(t_{\pi(N-1)} - t_{\pi(N)}\right) w_{t_{\pi(1)}} \otimes \cdots \otimes w_{t_{\pi(n)}},$$

formal computations yield the formal Dyson expansion (see [12])

$$S = I + \sum_{N=1}^{\infty} \frac{(-i\lambda)^N}{N!} \int dt_1 \ldots dt_N \int d^{4n}\underline{x}_1 \ldots d^{4n}\underline{x}_N,$$
$$\mathcal{T}_{t_1,\ldots,t_N}(\underline{x}_1, \ldots \underline{x}_N) {:}\phi^n(\underline{x}_1){:} \ldots {:}\phi^n(\underline{x}_N){:},$$

where we use the notations (3.3–3.10).

By integrating over the time variables t_j, we obtain

$$S = I + \sum_{N=1}^{\infty} \frac{(-i\lambda)^N}{N!} \int d^{4n}\underline{x}_1 \ldots d^{4n}\underline{x}_N$$
$$w(\underline{x}_1) \cdots w(\underline{x}_N) AT {:}\phi^n(\underline{x}_1){:} \ldots {:}\phi^n(\underline{x}_N){:}.$$

Here AT is the ordering of the Wick monomials with respect to the average time, i.e. the time of the mean position of each $\underline{x}_j$: more precisely, with

$$\tau(\underline{x}) = \kappa^0(\underline{x}^0) = \frac{x_1^0 + \cdots + x_n^0}{n} \tag{4.1}$$

the time component of the four vector $(x_1 + \cdots + x_n)/n$, we have

$$\begin{aligned} &AT {:}\phi^n(\underline{x}_1){:} \cdots {:}\phi^n(\underline{x}_N){:} \\ &\equiv \sum_{\pi \in \mathbb{P}_n} \theta\left(\tau(\underline{x}_{\pi(1)}) - \tau(\underline{x}_{\pi(2)})\right) \cdots \theta\left(\tau(\underline{x}_{\pi(N-1)}) - \tau(\underline{x}_{\pi(N)})\right) \\ &\quad {:}\phi^n(\underline{x}_{\pi(1)}){:} \cdots {:}\phi^n(\underline{x}_{\pi(N)}){:}, \end{aligned}$$

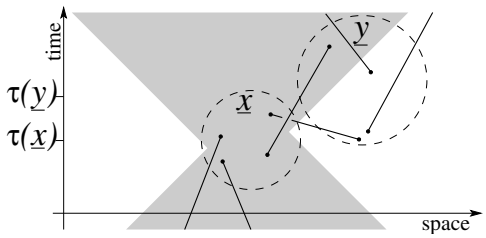

Fig. 4.1. Multi-vertices are represented by the points within a circle; $\tau(\underline{x})$ is the average time of the multi-vertex $\underline{x}$, and the shadowed area represents its causal completion

where $\underline{x}_j = (\underline{x}_{j,1}, \ldots, \underline{x}_{j,n})$ can be thought of as a "fat vertex", i.e. a multi-vertex actually consisting of n Minkowski vectors.

The time ordering thus defined, as well as the one discussed in [11], arises from ordering the variables $t_1, \ldots, t_N$ in products $H_I(t_1) \cdots H_I(t_N)$.

However, in both approaches causality is violated at the Planck length scale (cf. also [5]). A typical manifestation of the violation of causality is the following: suppose that

$$AT{:}\phi^n(\underline{x}){::}\phi^n(\underline{y}){:} = {:}\phi^n(\underline{x}){::}\phi^n(\underline{y}){:},$$

i.e. $\tau(\underline{x}) < \tau(\underline{y})$, then it is not forbidden that $x_i^0 > y_{i'}^0$ for some $i, i' \in \{1, \ldots, n\}$. In other words, a single field $\phi(x_i)$ belonging to the first Wick monomial can be subsequent in time to some field $\phi(y_{i'})$.

Note however, that the above picture is consistent with the request that the large scale limit should reproduce the ordinary (non-regularized) theory with ${:}\phi^n{:}$ interaction; in particular, as $\lambda_P \to 0$, the multi-vertices shrink to their mean positions, and the above average time-ordering reduces to the usual one.

We conclude our discussion by showing that it is not possible to absorb the time ordering in Feynman propagators, $i\Delta_F(x-y) = (\Omega, T[\phi(x), \phi(y)]\Omega)$, as one usually does in the framework of ordinary local theories. In fact, in the approach followed here, we cannot construct Feynman propagators *at all*, while in the original Hamiltonian approach based on (1.8) as well as in the approach based on the Yang-Feldman equation it is still possible to rewrite the time ordering in terms of Feynman propagators *together with* advanced and retarded propagators [1].

To see what happens in the approach adopted here, consider as an example the second order contribution to the Dyson series with two internal contractions in quantum ϕ^4-interaction, as depicted in Fig. 4.1.

Here, the two multi-vertices are sets of four distinguishable points, $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$, and at most one line originates from each of these points. A kernel w' belongs to each multi-vertex. The internal contractions lead to positive and negative frequency parts of the commutator function, $\Delta_{\pm}(x_i - y_{i'})$, where $i, i' \in \{1, \ldots, 4\}$. , while the time ordering is done with respect to the time components of the mean coordinates of each vertex, leading to the Heaviside functions $\theta\left(\pm\left((x_1^0 + \cdots + x_4^0) - (y_1^0 + \cdots + y_4^0)\right)\right)$. We may thus conclude that the distributions arising in the fish graph will not yield Feynman propagators at all.

This is consistent with the fact that the usual interpretation of Feynman graphs as pictorial representations of scattering processes ("first A is annihilated, then B is created") should break down on quantum spacetime, as it needs sharply localized events, which are no longer available.

Appendix

In this appendix, we explicitly compute the kernel $K_\lambda^{J\bullet}$, for future reference.

Recall that, for any $\boldsymbol{k} \in \mathbb{R}^3$, we write $\tilde{k} = (\omega(\boldsymbol{k}), \boldsymbol{k})$, and $\underline{\tilde{k}} = (\tilde{k}_1, \dots, \tilde{k}_n)$. Let $f \in \mathcal{S}(\mathbb{R}^{3nN})$ be any Schwartz function, and $\lambda(t, \boldsymbol{x}) = \lambda'(t)\lambda''(\boldsymbol{x})$. By Eq. (3.15),

$$\langle K_\lambda^{J\bullet}, f\rangle = \int_{\mathbb{R}^{3nN}} d\underline{\boldsymbol{k}}_1 \cdots d\underline{\boldsymbol{k}}_N \; F_\lambda^{J\bullet}(\underline{\tilde{k}}_1, \dots, \underline{\tilde{k}}_N) f(\underline{\boldsymbol{k}}_1, \dots, \underline{\boldsymbol{k}}_N),$$

where $F_\lambda^{J\bullet} \in \mathcal{C}(\mathbb{R}^{4nN})$ is given by

$$\begin{aligned} &F_\lambda^{J\bullet}(\underline{k}_1, \dots, \underline{k}_N) \\ &= c_{n,N} \int_{\mathbb{R}^{4nN}} d\underline{a}_1 \cdots d\underline{a}_N \left(\prod_{1 \le M < N} \theta(\kappa^0(\underline{a}^0_{M+1} - \underline{a}_M)) \right) G_\lambda(\underline{a}_1) \cdots G_\lambda(\underline{a}_N) \\ &\quad \times \exp\left\{ i \sum_{M=1}^N \underline{k}_{M\mu} \cdot \left(U_M \underline{a}_M\right)^\mu \right\}; \end{aligned}$$

the orthogonal matrix U_M is given by (3.16).

We now set

$$\underline{\xi} = n^{-1/2}(1, 1, \dots, 1) \in \mathbf{R}^n.$$

Then, if P is the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}\underline{\xi}$, we set $P' = (P, \boldsymbol{P})$ as a projection onto $\mathbb{R}^{4n}$, $P' = P \oplus P \oplus P \oplus P$; in the same spirit, $\underline{\xi}' = (\underline{\xi}, \underline{\boldsymbol{\xi}}) \in \mathbb{R}^{4n}$. Finally, $U'_M = (U_M, \boldsymbol{U_M})$, and $I' = (I, \boldsymbol{I})$ is the identity. We finally need the maps $V_{\underline{\xi}}(\underline{a}) = \underline{\xi} \cdot \underline{a}$, $V_{\underline{e}_n}(\underline{a}) = \underline{e}_n \cdot \underline{a}$ from $\mathbb{R}^n$ to $\mathbb{R}$, with the corresponding maps $V'_{\underline{\xi}} = (V_{\underline{\xi}}, \boldsymbol{V}_{\underline{\xi}})$, $V'_{\underline{e}_n} = (V_{\underline{e}_n}, \boldsymbol{V}_{\underline{e}_n})$ from $\mathbb{R}^{4n}$ to $\mathbb{R}^4$.

With these notations, we have

$$\kappa^0(\underline{a}^0) = \frac{\underline{\xi} \cdot \underline{a}^0}{\sqrt{n}} = \frac{V_{\underline{\xi}}(\underline{a}^0)}{\sqrt{n}},$$

$$\kappa(\underline{a}) = \frac{V'_{\underline{\xi}}(\underline{a}^0)}{\sqrt{n}}, \quad \boldsymbol{\kappa}(\underline{\boldsymbol{a}}) = \frac{\boldsymbol{V}_{\underline{\xi}}(\underline{\boldsymbol{a}})}{\sqrt{n}}.$$

Moreover, the function G_λ defined by (3.13) may be written as

$$G_\lambda(\underline{a}) = \lambda\left(\frac{V'_{\underline{\xi}} \cdot \underline{a}}{\sqrt{n}}\right) \exp\left\{-\frac{1}{2}|(I' - P')\underline{a}|^2\right\},$$

and

$$\begin{aligned} &F_\lambda^{J\bullet}(\underline{k}_1, \dots, \underline{k}_N) \\ &= c_{n,N} \int_{\mathbb{R}^{4nN}} d\underline{a}_1 \cdots d\underline{a}_N \left(\prod_{1 \le M < N} \theta(\underline{\xi} \cdot (\underline{a}^0_{M+1} - \underline{a}_M)) \right) \left(\prod_{M=1}^N \lambda\left(\frac{V'\underline{\xi} \cdot \underline{a}_M}{\sqrt{n}}\right) \right. \\ &\quad \times \exp\left\{ -\frac{1}{2} \sum_{M=1}^N |(I' - P')\underline{a}_M|^2 \right\} \exp\left\{ i \sum_{M=1}^N \underline{k}_{M\mu} \cdot \left(U_M \underline{a}_M\right)^\mu \right\}. \end{aligned}$$

We may now take a $n \times n$ real orthogonal matrix R, such that

$$R\underline{\xi} = \underline{e}_n = (0, 0, \ldots, 0, 1) \in \mathbf{R}^n.$$

Then, $E = R^t P R$ is the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}\underline{e}_n$, and $E' = (E, \boldsymbol{E})$, as usual. Furthermore, $V'_{\underline{e}_n}(R\underline{a}) = V'_{\underline{\xi}}(\underline{a})$. With the change of integration variables $\underline{\boldsymbol{b}} = R\underline{\boldsymbol{a}}$, we obtain

$$\begin{aligned}
&F_\lambda^{J_\bullet}(\underline{k}_1, \ldots, \underline{k}_N) \\
&= c_{n,N} \int_{\mathbb{R}^{4nN}} d\underline{b}_1 \cdots d\underline{b}_N \left(\prod_{1 \le M < N} \theta(\underline{e}'_n \cdot (\underline{b}^0_{M+1} - \underline{b}_M)) \right) \left(\prod_{M=1}^N \lambda \left(\frac{V'_{\underline{e}_n}(\underline{b}_M)}{\sqrt{n}} \right) \right) \\
&\quad \times \exp\left\{ -\frac{1}{2} \sum_{M=1}^N |(I' - E')\underline{b}_M|^2 \right\} \exp\left\{ i \sum_{M=1}^N \left(RU_M \underline{k}_{M_0} \right) \cdot \underline{b}^0_M \right\} \\
&\quad \times \exp\left\{ -i \sum_{M=1}^N \left(\boldsymbol{R}\boldsymbol{U}_M \underline{\boldsymbol{k}}_M \right) \cdot \underline{\boldsymbol{b}}_M \right\}.
\end{aligned}$$

The integration over the variables $\underline{b}_{Mn} \in \mathbb{R}^4$, $M = 1, \ldots, N$, is completely separated from the integration over the variables $\underline{b}^*_M = (b_{M1}, \ldots, b_{Mn-1}) \in \mathbb{R}^{4(n-1)}$. Hence

$$\begin{aligned}
&F_\lambda^{J_\bullet}(\underline{k}_1, \ldots, \underline{k}_N) \\
&= c_{n,N} \left(\int_{\mathbb{R}^{4(n-1)N}} d\underline{b}^*_1 \cdots d\underline{b}^*_N \exp\left\{ -\frac{1}{2} \sum_{M=1}^N |\underline{b}^*_M|^2 \right\} \right. \\
&\quad \times \exp\left\{ -i \sum_{M=1}^N \left(\boldsymbol{R}\boldsymbol{U}_M \underline{\boldsymbol{k}}_M \right)^* \cdot \underline{\boldsymbol{b}}^*_M \right\} \exp\left\{ i \sum_{M=1}^N \left(RU_M \underline{k}_{M_0} \right)^* \cdot \underline{b}^{0\,*}_M \right\} \Bigg) \\
&\quad \times \left(\int_{\mathbb{R}^{4N}} db_{1n} \cdots db_{Nn} \left(\prod_{1 \le M < N} \theta(\underline{e}_n \cdot (b_{M+1n}{}^0 - b_{Mn}{}^0)) \right) \left(\prod_{M=1}^N \lambda \left(\frac{b_{Mn}{}^0}{\sqrt{n}} \right) \right) \right. \\
&\quad \left. \left. \times \exp\left\{ i \sum_{M=1}^N (V'_{\underline{e}_n}(RU_M \underline{k}_M)_\mu \underline{b}_{Mn}{}^\mu \right\} \right).
\end{aligned}$$

The first factor is a Fourier transform of a Gaussian function. Renaming the integration variables $b_{Mn} = \beta_M = (\beta^0_M, \boldsymbol{\beta}_M) \in \mathbb{R}^4$,

$$\begin{aligned}
&F_\lambda^{J_\bullet}(\underline{k}_1, \ldots, \underline{k}_N) \\
&= \frac{c_{n,N}}{(2\pi)^{4(n-1)N}} \exp\left\{ -\frac{1}{2} \sum_{M=1}^N \left| (I' - P')U'_M \underline{k}_M \right|^2 \right\} \\
&\quad \times \left(\int_{\mathbb{R}^{4N}} d\beta_1 \cdots d\beta_N \left(\prod_{1 \le M < N} \theta(\beta^0_{M+1} - \beta^0_M) \right) \left(\prod_{M=1}^N \lambda \left(\frac{\beta_M}{\sqrt{n}} \right) \right) \right. \\
&\quad \left. \times \exp\left\{ i \sum_{M=1}^N \left(V'_{\underline{\xi}}(U_M \underline{k}_M) \right)_\mu \beta^\mu_M \right\} \right),
\end{aligned}$$

where we used

$$\left|\left(R'U'_M\underline{k}_M\right)^*\right| = \left|(I'-E')R'U'_M\underline{k}_M\right| = \left|(I'-P')U'_M\underline{k}_M\right|.$$

Due to the form of $\lambda = \lambda' \otimes \lambda''$, the integral in the above expression may be further factorized as an integral over the variables $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_N$, times an integral over the variables $\beta^0_1, \ldots, \beta^0_N$:

$$\begin{aligned}
&F^{J\bullet}_\lambda(\underline{k}_1, \ldots, \underline{k}_N)\\
&= \frac{c_{n,N}}{(2\pi)^{4(n-1)N}} \exp\left\{-\frac{1}{2}\sum_{M=1}^{N}\left|(I'-P')U'_M\underline{k}_M\right|^2\right\}\\
&\quad\times\left\{\int_{\mathbb{R}^{3N}} d\boldsymbol{\beta}_1\cdots d\boldsymbol{\beta}_N\ \left(\prod_{M=1}^{N}\lambda''\left(\frac{\boldsymbol{\beta}_M}{\sqrt{n}}\right)\right)\exp\left\{-i\left(\boldsymbol{V}_{\underline{\xi}}(\boldsymbol{U}_M\underline{\boldsymbol{k}}_M)\right)\cdot\boldsymbol{\beta}_M\right\}.\right\}\\
&\quad\times\left\{\int_{\mathbb{R}^N} d\beta^0_1\cdots d\beta^0_N\ \exp\left\{i\left(V_{\underline{\xi}}(U_M\underline{k}^0_M)\right)\beta^0_M\right\}\left(\prod_{M=1}^{N}\lambda'\left(\frac{\beta^0_M}{\sqrt{n}}\right)\right)\right.\\
&\quad\left.\times\left(\prod_{1\le M<N}\theta(\beta^0_{M+1}-\beta^0_M)\right)\right\}.
\end{aligned}$$

The integral over $\beta^0_1, \ldots, \beta^0_M$ in the above expression is the Fourier transform of an L^1 function, hence it vanishes at infinity. Moreover, it is the product of a Schwartz function in $\underline{\boldsymbol{k}}_1, \ldots, \underline{\boldsymbol{k}}_N$, times a $\mathcal{C}^\infty$ function of $\underline{k}_{10}, \ldots, \underline{k}_{N0}$, vanishing at infinity. As a consequence,

$$(\underline{\boldsymbol{k}}_1, \ldots, \underline{\boldsymbol{k}}_N) \mapsto K^{J\bullet}_\lambda(\underline{\boldsymbol{k}}_1, \cdots, \underline{\boldsymbol{k}}_N) = F^{J\bullet}_\lambda(\tilde{\underline{k}}_1, \cdots, \tilde{\underline{k}}_N)$$

is a Schwartz function.

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