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Hitchin Systems – Symplectic Hecke Correspondence and Two-Dimensional Version

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Abstract: The aim of this paper is two-fold. First, we define symplectic maps between Hitchin systems related to holomorphic bundles of different degrees. We call these maps the Symplectic Hecke Correspondence (SHC) of the corresponding Higgs bundles. They are constructed by means of the modification of the underlying holomorphic bundles. SHC allows to construct Bäcklund transformations in the Hitchin systems defined over Riemann curves with marked points. We apply the general scheme to the elliptic Calogero-Moser (CM) system and construct SHC to an integrable $\mathrm{SL}(N, \mathbb{C})$ Euler-Arnold top (the elliptic $\mathrm{SL}(N, \mathbb{C})$-rotator). Next, we propose a generalization of the Hitchin approach to 2d integrable theories related to the Higgs bundles of infinite rank. The main example is an integrable two-dimensional version of the two-body elliptic CM system. The previous construction allows us to define SHC between the two-dimensional elliptic CM system and the Landau-Lifshitz equation.

Contents

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1. Introduction

Nowadays many examples of integrable one-dimensional and two-dimensional models are known. The problem of listing all of them, up to some equivalence, was solved for some particular forms of two-dimensional models [1]. The recently developed concept of duality for one-dimensional models [2] can shed light on the classification problem in analogy with string theory. In spite of this progress we are still far from understanding the structure of this universe. Therefore, the classification of integrable systems, apart from solving any individual equation, continues to be an actual task. We will consider integrable systems that have the Lax or Zakharov-Shabat representations. In these cases the gauge transformations of the accompanying linear equations lead essentially to the same systems, though their equations of motion differ in a significant way. For example, the non-linear Schrödinger model is gauge equivalent to the isotropic Heisenberg magnet [3]. In such a manner the integrable system should be classified up to gauge equivalence, though it is not the only equivalence principle in their possible classifications. The crucial and delicate point of this approach is the exact definition of allowed gauge transformations, and it will be discussed here.

We restrict ourselves to Hitchin systems [4] and their two-dimensional generalizations that we will construct. The Hitchin construction establishes relations between finite dimensional integrable systems and the moduli space of holomorphic vector bundles over Riemann curves. The phase space of the integrable system is the cotangent bundle to the moduli space and the dual variables Φ are called the Higgs fields. The pair (E, Φ), where E is a holomorphic bundle, is called the Higgs bundle. The Lax representation arises immediately in this scheme as the equation of motion and the Lax operator is just the Higgs field defined on shell. The C^∞ gauge transformations of the Lax pair define the equivalent holomorphic bundles. The different gauge fixing conditions give equivalent integrable systems.

We consider the generalization of the Hitchin systems based on the quasi-parabolic Higgs bundles [5], where the Higgs fields are allowed to have the first order poles at the marked points on the base curve. The gauge transformations preserve the flag structures that arise at the marked points. The corresponding integrable systems were considered in [6–9]. We loosen the smoothness condition of the gauge transformations and allow them to have a simple zero or a pole at one of the marked points. This type of gauge transformations (the upper and lower symplectic Hecke correspondence (SHC)) is suggested by the geometric Langlands program. SHC changes the degree of the underlying bundles on ± 1. We assume, that HC is consistent with flag structures on the source and target bundles. It allows to choose a canonical form of the modifications. HC can be lifted as the symplectic correspondence (SHC) to the Higgs bundles. In this way SHC define a map of Hitchin systems related to bundles of different degrees. One can consider an arbitrary chain of consecutive SHC attributed to different marked points. If the resulting transformation preserves the degree of bundle, then it defines the Bäcklund transformations of the Hitchin system related to the initial bundle, or the integrable discrete time map [10]. Our construction is similar to the scheme proposed by Arinkin and Lysenko [11] in the investigations of the flat $\mathrm{SL}(2, \mathbb{C})$- bundles over rational curves and the geometric structure of the Bäcklund transformations in the Painléve 6 system [12].

As an example, we consider a trivial holomorphic $\mathrm{SL}(N, \mathbb{C})$-bundle E^{CM} ($\deg(E^{CM}) = 0$) over an elliptic curve with a marked point. The corresponding quasi-parabolic Higgs bundle leads to the elliptic N-body Calogero-Moser system (CM system). The upper SHC defines a map of the Higgs bundle related to E^{CM} to the Higgs bundle (E^{rot}, Φ^{rot}) with $\deg(E^{rot}) = 1$. SHC is generated by the N^{th} order matrix Ξ with theta-functions depending on coordinates of the particles as the matrix elements. The system (E^{rot}, Φ^{rot}) is the integrable $\mathrm{SL}(N, \mathbb{C})$-Euler-Arnold top ($\mathrm{SL}(N, \mathbb{C})$-elliptic rotator). The Lax pair for this top was proposed earlier [13]. The consecutive upper and lower SHC define the Bäcklund transformations of both systems. A construction of this type was suggested in [14] for studying the Bäcklund transformations of the Ruijsenaars model. Another way to find a Bäcklund transformation is achieved by applying N consecutive upper modifications, since they lead to equivalent Higgs bundles.

In the second part of the paper we try to gain insight into the interrelation between integrable theories in dimension one and two. It is known that some one-dimensional integrable systems can be extended to the two-dimensional case without sacrificing the integrability. For example, the Toda field theory comes from the corresponding Toda lattice. To understand this connection we apply the Hitchin construction to two-dimensional systems. For this purpose we consider infinite rank bundles over the Riemann curves with marked points. The transition group of the bundles is the central extended loop group $\hat{L}(\mathrm{GL}(N, \mathbb{C}))$. If the central charge vanishes the theory in essence becomes one-dimensional. In the two-dimensional situation the Higgs field is a $\mathrm{gl}(N, \mathbb{C})$ connection on a circle S^1. In addition, we put coadjoint orbits of $\hat{L}(\mathrm{GL}(N, \mathbb{C}))$ at the marked points and in this way introduce the quasi-parabolic structure on the Higgs bundle of infinite rank. The monodromy of the Higgs field is a generating function for the infinite number of conservation laws. The equations of motion on the reduced phase space are the Zakharov-Shabat equations. A similar class of Hitchin type systems from a different point of view was introduced recently by Krichever [15]. We consider in detail the case of a $\hat{L}(\mathrm{SL}(2,\mathbb{C}))$-bundle over an elliptic curve with n marked points. The Higgs bundle corresponds to the two-dimensional version of the elliptic Gaudin system. For the 1 marked point case we come to the 2d two-body elliptic CM theory. The upper SHC is working in the two-dimensional situation as well. It leads to the map of the 2-body elliptic CM field theory to the Landau-Lifshitz equation.[1] To summarize we consider here the following diagram:

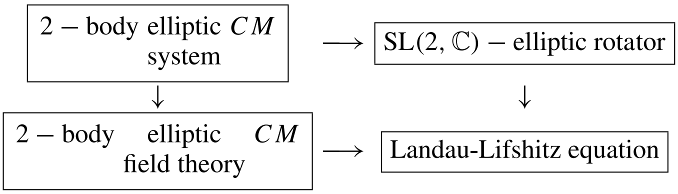

Fig. 1. Interrelation in integrable theories

In fact, the upper SHC can be applied to the $\mathrm{SL}(N,\mathbb{C})$ case. The quadratic Hamiltonian of the N-body elliptic CM field theory was constructed in [15], but the $\mathrm{SL}(N,\mathbb{C})$ generalization of the Landau-Lifshitz equation is unknown.

It should be mentioned that the quantum version of $\mathrm{SL}(N,\mathbb{C})$ SHC appeared in a different context long ago [16]. It was defined as a twist transformation of the quantum R-matrices, and Hasegawa [17] constructed such types of twists that transform the dynamical elliptic R-matrix of Felder [18] to the non-dynamical R-matrix of Belavin [19]. It was proved [20] that the dynamical R-matrix corresponds to the elliptic Ruijsenaars system [21]. The later is the relativistic deformation of the elliptic CM system. In this way the Hasegawa twist is the quantization of SHC we have constructed, since the elliptic CM system and the elliptic Ruijsenaars system are governed by the same R-matrix [22].

2. Hitchin Systems in the Čech Description

In this section we consider vector bundles with structure group $G = \mathrm{GL}(N,\mathbb{C})$, or any simple complex Lie group.

2.1. The moduli space of holomorphic quasi-parabolic bundles in the Čech description. Let E be a trivial rank r holomorphic vector bundle over a Riemann curve Σ_n with n marked points. Consider a covering of Σ_n by open disks $\mathcal{U}_a,\ a = 1, 2 \ldots$. Some of them may contain one marked point w_α. The holomorphic structure on E can be described by the differential d''. On $\mathcal{U}_a$ it can be represented as

$$d'' = \bar{\partial}_a + \bar{A}_a,\ \ \bar{A}_a = h_a^{-1}\bar{\partial}_a h_a,\ \ \bar{\partial}_a = \frac{\partial}{\partial \bar{z}_a},$$

[1] The equivalence of these models was pointed out by A. Shabat.

where z_a is a local coordinate on $\mathcal{U}_a$, and h_a is a C^∞ G-valued function on $\mathcal{U}_a$. It is a section of the local sheaf $\Omega^0_{C^\infty}(\Sigma_n, \text{Aut } E)$.

The transition functions $g_{ab} = h_a h_b^{-1}$ are defined on the intersections $\mathcal{U}_{ab} = \mathcal{U}_a \cap \mathcal{U}_b$. They are holomorphic since $\bar{A}_a = \bar{A}_b$ on $\mathcal{U}_{ab}$

$$g_{ab} \in \Omega^0_{hol}(\mathcal{U}_{ab}, \text{Aut } E).$$

The transformation $h_a \to f_a h_a$ by a function holomorphic on $\mathcal{U}_a$ ($f_a \in \Omega^0_{hol}(\mathcal{U}_a, \text{Aut } E)$) does not change $\bar{A}_a$. Similarly, the transformation $h_b \to f_b h_b$ by $f_b \in \Omega^0_{hol}(\mathcal{U}_b, \text{Aut } E)$ does not change $\bar{A}_b$. Then the holomorphic structures described by the transition functions g_{ab} and $f_a g_{ab} f_b^{-1}$ are equivalent. Globally we have the collection of transition maps

$$\mathcal{L}^C_\Sigma = \{g_{ab}(z_a) = h_a(z_a) h_b^{-1}(z_b(z_a)),\ z_a \in \mathcal{U}_{ab},\ a, b = 1, 2 \ldots, \}. \tag{2.1}$$

They define holomorphic structures on E or $P = \text{Aut}E$ depending on the choice of the representations.

The definition of the holomorphic structures by the transition functions works as well in the case if $\deg(E) \neq 0$ ($G = \text{GL}(N, \mathbb{C})$). They should satisfy the cocycle condition

$$g_{ab}(z) g_{bc}(z) g_{ca}(z) = \text{Id},\ \ z \in \mathcal{U}_a \cap \mathcal{U}_b \cap \mathcal{U}_c, \tag{2.2}$$

and

$$g_{ab} = g_{ba}^{-1}. \tag{2.3}$$

The degree of the bundle E is defined as the degree of the linear bundle $L = \det g$.

We choose an open subset of stable holomorphic structures $\mathcal{L}^{C,st}_\Sigma$ in $\mathcal{L}^C_\Sigma$. The gauge group $\mathcal{G}^{hol}_\Sigma$ acts as the automorphisms of $\mathcal{L}^{C,st}_\Sigma$,

$$g_{ab} \to f_a g_{ab} f_b^{-1},\ f_a = f(z_a),\ f_b = f_b(z_b(z_a)),\ f \in \mathcal{G}^{hol}_\Sigma. \tag{2.4}$$

We prescribe the local behavior of the gauge transformations $\mathcal{G}^{hol}_\Sigma$ at the marked points. Let

$$P_1, \ldots, P_\alpha, \ldots, P_n$$

be parabolic subgroups of G attributed to the marked points. Then we assume that

$$f_a = \begin{cases} \tilde{f}^{(0)}_\alpha + z_\alpha f^{(1)}_\alpha + \ldots,\ \tilde{f}^{(0)}_\alpha \in P_\alpha,\ \text{if } z_\alpha = z - w_\alpha,\ w_\alpha \text{ is a marked point,} \\ f^{(0)}_a + z_a f^{(1)}_a + \ldots,\ f^{(0)}_a \in G \text{ if } a \neq \alpha,\ (\mathcal{U}_a \text{ does not contain a marked point}). \end{cases} \tag{2.5}$$

It follows from (2.4) that the left action of the gauge group at the marked points preserves the flags

$$E_\alpha(s) \sim P_\alpha \setminus G,\ \ E_\alpha = Fl_1(\alpha) \supset \cdots \supset Fl_{s_\alpha}(\alpha) \supset Fl_{s_\alpha+1}(\alpha) = 0. \tag{2.6}$$

The moduli space of the stable holomorphic bundles $\mathcal{M}_n(\Sigma, G)$ with the quasi-parabolic structure at the marked points is defined in Ref.[23] as the factor space under this action

$$\mathcal{M}_n = \mathcal{G}^{hol}_\Sigma \backslash \mathcal{L}^{C,st}_\Sigma. \tag{2.7}$$

For $G = \text{GL}(N, \mathbb{C})$ we have a disjoint union of components labeled by the corresponding degrees $d = c_1(\det E) : \mathcal{M}_n(\Sigma, G) = \bigsqcup \mathcal{M}^{(d)}_n$.

The tangent space to $\mathcal{M}_n(\Sigma, G)$ is isomorphic to $h^1(\Sigma, \mathrm{End}E)$. Its dimension can be extracted from the Riemann-Roch theorem and for curves without marked points ($n = 0$)

$$\dim h^0(\Sigma, \mathrm{End}E) - \dim h^1(\Sigma, \mathrm{End}E) = (1-g)\dim G.$$

For stable bundles $h^0(\Sigma, \mathrm{End}E) = 1$ and

$$\dim \mathcal{M}_0(\Sigma, G) = (g-1)N^2 + 1$$

for $\mathrm{GL}(N, \mathbb{C})$, and

$$\dim \mathcal{M}_0(\Sigma, G) = (g-1)\dim G$$

for simple groups. For elliptic curves one has

$$\dim h^1(\Sigma, \mathrm{End}E) = \dim h^0(\Sigma, \mathrm{End}E),$$

and

$$\dim \mathcal{M}_0^d = \text{g.c.d.}(N, d). \tag{2.8}$$

In this case the structure of the moduli space for the trivial bundles (i.e. with $\deg(E) = 0$ and, for example, for bundles with $\deg(E) = 1$ are different. We use this fact below.

For the quasi-parabolic bundles we have

$$\dim \mathcal{M}_n^d = \dim \mathcal{M}_0^d + \sum_{\alpha=1}^{n} f_\alpha, \tag{2.9}$$

where f_α is the dimension of the flag variety E_α. In particular, for $G = \mathrm{GL}(N, \mathbb{C})$, we get

$$f_\alpha = \frac{1}{2}\left(N^2 - \sum_{i=1}^{s_\alpha} m_i^2(\alpha)\right), \quad m_i(\alpha) = \dim Fl_i(\alpha) - \dim Fl_{i+1}(\alpha). \tag{2.10}$$

The space $\mathcal{L}_\Sigma^C$ is a sort of a lattice 2d gauge theory. Consider the skeleton of the covering $\{\mathcal{U}_a,\ a = 1, \ldots\}$. It is an oriented graph whose vertices V_a are some fixed inner points in $\mathcal{U}_a$ and edges L_{ab} connect those V_a and V_b for whose $U_{ab} \neq \emptyset$. We choose an orientation of the graph, saying that $a > b$ on the edge L_{ab} and put the holomorphic function $z_b(z_a)$ which defines the holomorphic map from $\mathcal{U}_a$ to $\mathcal{U}_b$. Then the space $\mathcal{L}_\Sigma^C$ can be defined by the following data. To each edge $L_{ab},\ a > b$ we attach a matrix valued function $g_{ab} \in G$ along with $z_b(z_a)$. The gauge fields f_a are living on the vertices V_a and the gauge transformation is given by (2.4).

2.2. Hitchin systems. The Hitchin systems in the Čech description can be constructed in the following way [24]. We start from the cotangent bundle $T^*\mathcal{L}_{\Sigma_n}^C$ to the holomorphic structures on $P = \mathrm{Aut}E$ (2.1). Now

$$T^*\mathcal{L}_{\Sigma_n}^C = \{\eta_{ab}, g_{ab} |\ \eta_{ab} \in \Omega_{hol}^{(1,0)}(\mathcal{U}_{ab}, (\mathrm{End}E)^*),\ g_{ab} \in \Omega_{hol}^0(\mathcal{U}_{ab}, P)\}. \tag{2.11}$$

The one forms η_{ab} are called the Higgs fields. This bundle can be endowed with a symplectic structure by means of the Cartan-Maurer one-forms on $\Omega_{hol}^0(\mathcal{U}_{ab}, P)$.

Let $\Gamma_a^b(\beta\gamma)$ be an oriented edge in $\mathcal{U}_{ab}$ with the end points in the triple intersections $\beta \in \mathcal{U}_{abc} = \mathcal{U}_a \cap \mathcal{U}_b \cap \mathcal{U}_c$, $\gamma \in \mathcal{U}_{abd}$. The fields η_{ab}, g_{ab} are attributed to the edge

$\Gamma_a^b(\beta\gamma)$. If we change the orientation $\Gamma_a^b(\beta\gamma) \to \Gamma_b^a(\gamma\beta)$ the fields should be replaced on $g_{ba} = g_{ab}^{-1}$ (see (2.3)) and

$$\eta_{ab}(z_a) = g_{ab}(z_a)\eta_{ba}(z_b(z_a))g_{ab}^{-1}(z_a). \tag{2.12}$$

For this reason the integral

$$\int_{\Gamma_a^b(\beta\gamma)} \mathrm{tr}\left(\eta_{ab}(z_a)Dg_{ab}g_{ab}^{-1}(z_a)\right) \tag{2.13}$$

is independent of the orientation.

We can put the data (2.11) on the graph $\{\Gamma_a^b\}$ corresponding to the covering $\{\mathcal{U}_a\}$. Taking into account (2.13) we define the symplectic structure

$$\omega^C = \sum_{\text{edges}} \int_{\Gamma_a^b(\beta\gamma)} D\mathrm{tr}\left(\eta_{ab}(z_a)Dg_{ab}g_{ab}^{-1}(z_a)\right). \tag{2.14}$$

Since η_{ab} and g_{ab} are both holomorphic in $\mathcal{U}_{ab}$, the integral is independent of the choice of the path Γ_a^b within $\mathcal{U}_{ab}$. It is worthwhile to note that the cocycle condition (2.2) does not yield the additional constraints.

The symplectic form is invariant under the gauge transformations (2.4) supplemented by

$$\eta_{ab} \to f_a\eta_{ab}f_a^{-1}. \tag{2.15}$$

The set of invariant commuting Hamiltonians on $T^*\mathcal{L}_\Sigma^C$ is

$$I_{j,k}^C = \sum_{\text{edges}} \int_{\Gamma_a^b(\beta\gamma)} \nu_{(j,k)}^C(z_a)\mathrm{tr}(\eta_{ab}^{d_j}(z_a)), \quad (k = 1,\dots,n_j), \tag{2.16}$$

where d_j are the orders of the basic invariant polynomials corresponding to G and $\nu_{j,k}^C$ are $(1-d_j, 0)$-differentials. They are related locally to the $(1-j, 1)$-differentials by $\nu_{j,k}^D = \bar{\partial}\nu_{j,k}^C$ and

$$n_j = h^1(\Sigma, \mathcal{T}^{\otimes(d_j-1)}) = (2d_j - 1)(g-1) + (d_j - 1)n, \quad (j = 1,\dots,r)$$

for the simple groups, and

$$n_j = \begin{cases} (2j-1)(g-1) + (j-1)n, & (j = 2,\dots,N) \\ g, & j = 1 \end{cases}$$

for $\mathrm{GL}(N,\mathbb{C})$. The total number of independent Hamiltonians is equal to

$$\sum_{j=1}^N n_j = \mathcal{M}_0^d + \frac{1}{2}r(r+1)n.$$

This number is greater than the dimension of the moduli space $\mathcal{M}_n^d$ (2.9). There are rn highest weight integrals, $(j = r)$, that become Casimir elements of coadjoint orbits after the symplectic reduction, that we will consider below.

Perform the symplectic reduction with respect to the gauge action (2.4), (2.15) of $\mathcal{G}^{hol}_{\Sigma_n}$ (2.5). The moment map is

$$\mu_{\mathcal{G}^{hol}_{\Sigma}}(\eta_{ab}, g_{ab}) : \ T^*\mathcal{L}^C_{\Sigma} \to \mathrm{Lie}^*(\mathcal{G}^{hol}_{\Sigma}).$$

Here the Lie coalgebra $\mathrm{Lie}^*(\mathcal{G}^{hol}_{\Sigma})$ is defined with respect to the pairing

$$\sum_{\mathrm{edges}} \int_{\Gamma^b_a(\beta\gamma)} \mathrm{tr}(\xi_a \epsilon_a), \quad \epsilon_a \in \mathrm{Lie}(\mathcal{G}^{hol}_{\Sigma}).$$

Then locally we have

$$\xi_a = \begin{cases} \left(z_a^{-1}\tilde{\xi}_a + z_a^{-2}\xi_a^{(-2)} + \ldots\right) dz_a, & \tilde{\xi}_a \in \mathrm{Lie}^*(P_\alpha), \ (\mathcal{U}_a \text{ contains a marked point } w_\alpha) \\ \left(z_a^{-1}\xi_a^{(-1)} + z_a^{-2}\xi_a^{(-2)} + \ldots\right) dz_a, & \xi_a^{(-1)} \in \mathrm{Lie}^*(G) (\mathcal{U}_a \text{ does not contain } \mathrm{w}_\alpha). \end{cases} \tag{2.17}$$

The canonical gauge transformations (2.4),(2.15) of the symplectic form (2.14) are generated by the Hamiltonian

$$\begin{aligned} F_{\epsilon^{hol}} &= \sum_{\mathrm{edges}} \int_{\Gamma^b_a(\beta\gamma)} \mathrm{tr}(\eta_{ab}(z_a)\epsilon^{hol}_a(z_a)) - \mathrm{tr}(\eta_{ab}(z_a) g_{ab}(z_a)\epsilon^{hol}_b(z_b(z_a)) g_{ab}(z_a)^{-1}) \\ &= \sum_a \int_{\Gamma_a} \sum_b \mathrm{tr}(\eta_{ab}(z_a)\epsilon^{hol}_a(z_a)), \end{aligned}$$

where Γ_a is an oriented contour around $\mathcal{U}_a$.

The non-zero moment is fixed in a special way at the neighborhoods of the marked points. Let $\tilde{G}_\alpha \subset P_\alpha$ be the maximal semi-simple subgroup of the parabolic group P_α defined at the marked point w_α. We drop for a moment the index α for simplicity. We choose an ordering in the Cartan subalgebra $\mathfrak{h}$ $\in$Lie(G), which is consistent with the embedding $P \subset G$. Let $\tilde{\mathfrak{h}} = \mathfrak{h} \cap \tilde{G}$ be the Cartan subalgebra in $\tilde{G}$. Consider the orthogonal decomposition of $\mathfrak{h}^*$,

$$\mathfrak{h}^* = \tilde{\mathfrak{h}}^* + \mathfrak{h}'^*.$$

We fix a vector $p^{(0)} \in \mathfrak{h}^*$ such that it is a generic element in $\mathfrak{h}'^*$ and

$$\langle p^{(0)}, \tilde{\mathfrak{h}}^* \rangle = 0, \tag{2.18}$$

where $\langle \ , \ \rangle$ is the Killing scalar product in $\mathfrak{h}^*$. Since $\mathfrak{h}'^* \subset$Lie$^*(P)$, we can take $\mu_{\mathcal{G}^{hol}_{\Sigma}}$ in the form

$$\mu_{\mathcal{G}^{hol}_{\Sigma}} = \mu_0 = \sum_{\alpha=1}^{n} p^{(0)}_\alpha z_\alpha^{-1} dz_\alpha, \ p^{(0)} \in \mathfrak{h}'^*, \tag{2.19}$$

where $z_\alpha = z - w_\alpha$ is a local coordinate in $\mathcal{U}_\alpha$. The moment equation $\mu_{\mathcal{G}^{hol}_{\Sigma}} = \mu_0$ can be read off from $F_{\epsilon^{hol}}$. It follows from the definition of $\mathrm{Lie}^*(\mathcal{G}^{0,hol}_{\Sigma})$ that η_{ab} is the boundary value of some holomorphic or meromorphic one-form H_a defined on $\mathcal{U}_a$ via

$$\eta_{ab}(z_a) = H_a(z_a), \quad \text{for } z_a \in \mathcal{U}_{ab}, \quad H_a \in \Omega^{(1,0)}_{hol}(\mathcal{U}_a, \mathrm{End}^*(E)), \tag{2.20}$$

where

$$H_a = \begin{cases} z_a^{-1} p_\alpha^{(0)} + H_a^{(0)} + z_a H_a^{(1)} + \dots, & \text{if } \mathcal{U}_a \text{ contains a marked point } w_\alpha \\ H_a^{(0)} + z_a H_a^{(1)} + \dots, & \text{if } \mathcal{U}_a \text{ does not contain a marked point.} \end{cases} \tag{2.21}$$

The gauge fixing means that the transition functions g_{ab} are elements of the moduli space $\mathcal{M}_n^d(\Sigma, E)$. The symplectic quotient

$$\mathcal{H}_n^d = \mathcal{G}_\Sigma^{hol} \backslash\backslash T^* \mathcal{L}_\Sigma^C = \mathcal{G}_\Sigma^{hol} \backslash \mu^{-1}(\mu_0) \tag{2.22}$$

is called the Higgs bundle with the quasi-parabolic structures. We set off the zero modes $g_{\alpha b}^{(0)}$ of the transition functions in the symplectic form on the reduced space (see (2.14))

$$\begin{aligned} \omega^C &= \textstyle\sum_{\text{edges}} \int_{\Gamma_a^b(\beta\gamma)} D\text{tr} \left(\eta_{ab}(z_a) D g_{ab} g_{ab}^{-1}(z_a) \right) \\ &\quad + 2\pi i \textstyle\sum_{\alpha=1}^n \sum_b D\text{tr} \left(p_\alpha^{(0)} D g_{\alpha b}^{(0)} (g_{\alpha b}^{(0)})^{-1} \right). \end{aligned} \tag{2.23}$$

The last sum defines the Kirillov-Kostant symplectic forms on the set of coadjoint orbits $\mathcal{O}(n) = (\mathcal{O}_1, \dots \mathcal{O}_\alpha, \dots, \mathcal{O}_n)$, where

$$\mathcal{O}_\alpha = \{ p_\alpha \in \text{Lie}^*(G) \mid p_\alpha = (g_\alpha^{(0)})^{-1} p_\alpha^{(0)} g_\alpha^{(0)} \}. \tag{2.24}$$

Note that $\dim(\mathcal{O}_\alpha) = 2 f_\alpha$ (2.10).

Remark 2.1. It is possible to construct another type of orbit $\mathcal{O}'_\alpha$ of the same dimension. There exist elements $p_\alpha^{'(0)}$ that belong to the complements of $\text{Lie}^*(\tilde{G}_\alpha)$ in $\text{Lie}^*(P_\alpha)$ such that the orbit

$$\mathcal{O}'_\alpha = \{ p_\alpha = (g_\alpha^{(0)})^{-1} p_\alpha^{'(0)} g_\alpha^{(0)} \}$$

is symplectomorphic to the cotangent bundles to the corresponding flags E_α (2.6) without the zero section $T^* E_\alpha \setminus O(E_\alpha)$, while $\mathcal{O}_\alpha$ (2.24) is a torsor over $\mathcal{O}'_\alpha$. Globally, $\mathcal{H}_n^d$ (2.22) is a torsor over $T^* \mathcal{M}_n^d$.

2.3. Standard description of the Hitchin system. The standard approach of the Hitchin systems [4] is based on the description of the holomorphic bundles in terms of the operator d''. The upstairs phase space has the form

$$T^* \mathcal{L}_{\Sigma_n}^D = \{ \Phi, d'' \mid \Phi \in \Omega_{C^\infty}^{(1,0)}(\Sigma_n, \text{End}^* E) \}, \tag{2.25}$$

where Φ is called the Higgs field. The symplectic form

$$\omega^D = \int_{\Sigma_n} \text{tr}(D\Phi \wedge D\bar{A}) \tag{2.26}$$

is invariant under the action of the gauge group

$$\begin{aligned} &\mathcal{G}_\Sigma^{C^\infty} = \{ f \in \Omega_{C^\infty}^0(\Sigma_n, \text{Aut } V) \}, \\ &\Phi \to f^{-1} \Phi f, \ \ \bar{A} \to f^{-1} \bar{\partial} f + f^{-1} \bar{A} f. \end{aligned} \tag{2.27}$$

The gauge invariant integrals take the evident form (compare with (2.16))

$$I^D_{j,k} = \int_{\Sigma_n} \nu^D_{(j,k)} \mathrm{tr}(\Phi^{d_j}), \quad (k = 1, \dots, n_j), \tag{2.28}$$

where $\nu^D_{j,k}$ are $(1-j, 1)$-differentials on Σ_n. The symplectic reduction with respect to this action leads to the moment map

$$\mu \; : \; T^*\mathcal{L}^D_{\Sigma_n} \to \mathrm{Lie}^*(\mathcal{G}^{C^\infty}_\Sigma) \;\; \mu = \bar\partial\Phi + [\bar A, \Phi].$$

The Higgs field Φ is related to η in a simple way,

$$\eta_{ab} = h_a^{-1}\Phi h_a|_{\mathcal{U}_{ab}},$$

and $\bar A_a = h_a^{-1}\bar\partial_a h_a$. The holomorpheity of η is equivalent to the equation $\mu(\Phi, \bar A) = 0$, and Φ has the same simple poles as H_a (2.20). For simplicity, we call η the Higgs field. The bundle E equipped with the one-form η is called the Higgs bundle.

2.4. Modified Čech description of the moduli space. We modify the Čech description of the moduli space of GL$(N, \mathbb{C})$-vector bundles in the following way. Consider a formal (or rather small) disk D embedded into Σ in such way that its center maps to the point w.

Consider first the case of $G = \mathrm{PGL}(N, \mathbb{C})$-bundles. The moduli space $\mathcal{M}^d_n$ is the quotient of the space $\mathcal{G}_{D^*}$ of G-valued functions g on the punctured disk D^* by the right action of the group $\mathcal{G}_{out}$ of G-valued holomorphic functions on the complement to w and by the left action of the group $\mathcal{G}_{int}$ of G-valued holomorphic functions on the disk:

$$\mathcal{M}^d_n = \mathcal{G}_{int}\backslash\mathcal{G}_{D^*}/\mathcal{G}_{out}, \quad g \to h_{int} g h_{out}.$$

We assume that these transformations preserve the quasi-parabolic structure of the vector bundle E.

Now consider GL$(N, \mathbb{C})$-bundles. The group GL$(N, \mathbb{C})$ is not semi-simple. One has an action of the Jacobian $Jac(\Sigma)$ on the moduli space of vector bundles by the tensor multiplication, and the quotient is equal to the space of PGL$(N, \mathbb{C})$-bundles. This follows from the exact sequence

$$1 \to \mathcal{O}^* \to \mathrm{GL}(N, \mathcal{O}) \to \mathrm{PGL}(N, \mathcal{O}) \to 1.$$

Hence locally the moduli space of vector bundles is the product of the Jacobian of the curve and moduli space of PGL$(N, \mathbb{C})$-bundles. We associate to the pair (g, L) the bundle which is equal to $\mathbb{C}^N \otimes L$ on the complement of a point, and the transition function on the punctured disk is g.

Assume for simplicity that there is only one marked point and it coincides with the center of D^*. Let z be the local coordinate on D^*. Then the gauge group $\mathcal{G}_{D^*}$ can be identified with the loop group $L(\mathrm{GL}(N, \mathbb{C}))$. A parabolic subgroup of $L(\mathrm{GL}(N, \mathbb{C}))$ has the form

$$\mathcal{G}_{int} \sim P \cdot \exp L^+(\mathrm{gl}(N, \mathbb{C})), \quad L^+(\mathrm{gl}(N, \mathbb{C})) = \sum_{j>0} \mathfrak{g}_j z^j, \mathfrak{g}_j \in \mathrm{gl}(N, \mathbb{C}),$$

where P is a parabolic subgroup in $\mathrm{GL}(N,\mathbb{C})$. The quotient $LF(s) = \mathcal{G}_{int}\backslash\mathcal{G}_{D^*}$ is the infinite-dimensional flag variety, corresponding to the finite-dimensional flag $E(s)$ (see (2.6)),

$$LFl(s) = \cdots \supset LFl_{r,k} \supset LFl_{r+1,k} \supset \cdots \supset LFl_{s,k} \supset LF_{0,k-1} \supset \cdots, \qquad (2.29)$$

$$LFl_{r,k} = z^k Fl_r + \sum_{j<k} Ez^j) \ \ (LFl_{s+1,k} = LFl_{0,k-1}).$$

The $\mathrm{GL}(N,\mathbb{C})$ Higgs bundle $\mathcal{H}_n^d$ (2.22) can be identified with the Hamiltonian quotient

$$\mathcal{G}_{in}\backslash\backslash T^*\mathcal{G}_{D^*} \times T^*Jac(\Sigma)//\mathcal{G}_{out}.$$

The cotangent bundle of $T^*\mathcal{G}_{D^*}$ is identified with the space of pairs (g,η), where η is a $\mathrm{Lie}^*(G)$-valued one-form. The canonical one-form is equal to $\mathrm{res}_w(\mathrm{tr}(\eta Dgg^{-1}))$. The second component $T^*Jac(\Sigma)$ is the pair (t, L), where L is a point of $Jac(\Sigma)$ and t is the corresponding co-vector. The canonical one-form is $\langle t, DL\rangle_{Jac}$ and the brackets denote the pairing between vectors and co-vectors on the Jacobian.

The group $\mathcal{G}_{out}$ acts as $(g,\eta) \to (gh_{out},\eta)$. The corresponding momentum constraint can be reformulated as the following condition: $g\eta g^{-1}$ is the restriction of some $\mathrm{Lie}^*(G)$-valued form on the complement to w. The group $\mathcal{G}_{int}$ acts as $(g,\eta) \to (h_{int}g, h_{int}\eta h_{int}^{-1})$. The momentum constraint means that η is holomorphic in $\mathcal{U}_w$ if w is a generic point, or it has the first order pole if w is a marked point.

3. Symplectic Hecke Correspondence

In this section we consider only $\mathrm{GL}(N,\mathbb{C})$-bundles.

3.1. Hecke correspondence. Let E and $\tilde{E}$ be two bundles over Σ of the same rank. Assume that there is a map $\Xi^+ : E \to \tilde{E}$ (more precisely a map of the sheaves of sections $\Gamma(E) \to \Gamma\tilde{E}$) such that it is an isomorphism on the complement to w and it has one-dimensional cokernel at $w \in \Sigma$:

$$0 \to E \stackrel{\Xi^+}{\to} \tilde{E} \to \mathbb{C}|_w \to 0. \qquad (3.1)$$

It is the so-called *upper modification* of the bundle E at the point w. On the complement to the point w consider the map

$$E \stackrel{\Xi^-}{\leftarrow} \tilde{E},$$

such that $\Xi^-\Xi^+ = \mathrm{Id}$. It defines *the lower modification* $\mathfrak{H}_w^-$ at the point w.

Definition 3.1. *The upper Hecke correspondence (HC) at the point $w \in \Sigma$ is an auto-correspondence $\mathfrak{H}_w^+$ on the moduli space of Higgs bundles $\mathcal{H}$ related to the upper modification Ξ^+ (3.1).*

HC $\mathfrak{H}_w^+$ has components placed only at $\mathcal{M}^{(d)} \times \mathcal{M}^{(d+1)}$. The lower HC is defined in a similar way. In this form the HC was used in the Hitchin systems in Ref. [15, 25].

Now consider two quasi-parabolic bundles E and $\tilde{E}$ with the flag structure at the marked points. While the flag $E_\alpha(s)$ at w_α corresponding to E has the form (2.6), for $\tilde{E}$ we declare the following flag structure:

$$\tilde{E}_\alpha(s) = \tilde{Fl}_1(\alpha) \supset \cdots \supset \tilde{Fl}_{s_\alpha}(\alpha) \supset \tilde{Fl}_{s_\alpha+1}(\alpha) = 0,$$

where $\tilde{Fl}_k \sim Fl_{k-1}/Fl_{s_\alpha}$ for $s_\alpha + 1 \geq k \geq 2$. We define $\tilde{E}$ in terms of the sheaves of sections $\Gamma(E)$. Let Ξ_α^+ be a map of the sheaves of sections $\Gamma(E) \to \Gamma(\tilde{E})$ such that it is an isomorphism on the complement to a marked point $w_\alpha \in \Sigma$. Let $\sigma \in \Gamma(E)$ and $\Xi_\alpha^+ : \sigma \to \tilde{\sigma} \in \Gamma(\tilde{E})$. If $\sigma|_{w_\alpha} \in Fl_{k-1}$, then $\tilde{\sigma}|_{w_\alpha} \in \tilde{Fl}_k$. The section σ can be singular of order one if its principle part belongs to Fl_{s_α}.

All together this means that Ξ_α^+ acts as the shift on the infinite flag (2.29) at the marked point

$$\Xi_\alpha^+(LFl_{r,k}) = LFl_{r-1,k}. \tag{3.2}$$

We call Ξ_α^+ the upper modification of the quasi-parabolic bundle E. The lower modification of the quasi-parabolic bundles acts in the opposite direction. It looks like the upper modification (3.1), but we temporally do not assume that Ξ_α^+ has a one-dimensional cokernel.

Definition 3.2. *The upper Hecke correspondence of the quasi-parabolic bundles at the marked point w_α is an auto-correspondence $\mathfrak{H}_w^+$ on $\mathcal{M}$ related to the the upper modification Ξ^+ (3.2).*

Let the flag E_α (2.6) have a one-dimensional subspace ($\dim(Fl_{s_\alpha}) = 1$). In this case the upper modification Ξ_α^+ can be fixed in the following way. Let $(e_1, \ldots, e_N)$ be a basis of local sections of E compatible with the flag structure

$$Fl_1 \to (e_1, \ldots, e_N), \ldots, Fl_{s_\alpha} \to (e_N).$$

It follows from Definition 3.2 that Ξ_α^+ can be gauge transformed to the canonical form

$$\Xi_\alpha^+ = \begin{pmatrix} 0 & \mathrm{Id}_{N-1} \\ z_\alpha & 0 \end{pmatrix}. \tag{3.3}$$

It is just the Coxeter transformation in the loop algebra $L(\mathrm{gl}(N, \mathbb{C}))$, that has been defined on the punctured disk $D_\alpha^* \subset \mathcal{U}_\alpha$ in Subsect. 2.4. The Coxeter transformation provides the upper modification $E^d \to \tilde{E}^{d+1}$. In fact, the sheaf of sections $\Gamma(\tilde{E})$ coincides with the sheaf of sections $\Gamma(E)$ with a singularity of the first order at w_α and the singular section lies in the kernel of Ξ^+ (see Ref.[11]). For the local basis of $\Gamma(E)$ we have $(e_N z^{-1}, e_1, \ldots, e_{N-1})$. In this way the HC of the quasi-parabolic bundles is described by the diagram (3.1).

In a similar way the lower modification can be transformed to the form

$$\Xi_\alpha^+ = \begin{pmatrix} 0 & z_\alpha^{-1} \\ \mathrm{Id}_{N-1} & 0 \end{pmatrix}. \tag{3.4}$$

3.2. Symplectic Hecke correspondence. We define a map of the Higgs bundles $f : (E, \eta) \to (\tilde{E}, \tilde{\eta})$ as the bundle map $f : E \to \tilde{E}$ such that

$$f\eta = \tilde{\eta} f. \tag{3.5}$$

Consider two Higgs bundles (E, η) and $(\tilde{E}, \tilde{\eta})$, where E is a quasi-parabolic bundle and $\tilde{E}$ is the upper modification Ξ_α^+ of E at $w_a \in \Sigma$. We call $(\tilde{E}, \tilde{\eta})$ the upper modification of (E, η) $\Xi_\alpha^+ \eta = \tilde{\eta} \Xi_\alpha^+$.

Definition 3.3. *The upper symplectic Hecke correspondence (SHC) $\mathfrak{S}_\alpha^+$ at a point w_α is an auto-correspondence on $T^*\mathcal{M}$ related to the upper modification Ξ_α^+ of the Higgs bundles.*

The lower SHC $\mathfrak{S}_a^-$ is defined in a similar way.

Let w_α be a marked point. The Higgs field η has the first order poles at w_α (2.21) and the residue $p_\alpha^{(0)}$ of η defines an orbit $\mathcal{O}_\alpha$.

Lemma 3.1. *The gauge transforms $\Xi_\alpha^\pm$ corresponding to $\mathfrak{H}_\alpha^\pm$:*
- *do not change singularity of the Higgs field at w_α;*
- *are symplectic;*
- *preserve the Hamiltonians (2.16).*

Proof. The choice of $p_\alpha^{(0)}$ (2.18) is consistent with the canonical forms (3.3), (3.4) of $\Xi_\alpha^\pm$ and their action does not change the order of the pole. The action is symplectic with respect to (2.23) since $\Xi_\alpha^\pm$ depends only on $p_\alpha^{(0)}$. The invariance of the Hamiltonians follows from (3.5). □

In particular, Lemma 3.1 means that $\Xi_\alpha^\pm$ preserves the whole Hitchin hierarchy defined by the set of Hamiltonians (2.16) and the symplectic form (2.23).

3.3. SHC and skew-conormal bundles. Here we consider the curves without marked points. The general case can be derived in a similar way and we drop it for the sake of simplicity.

For any smooth correspondence Z between equi-dimensional varieties X and Y we define a *skew-conormal bundle* $\mathcal{S}N^*Z$ of Z as follows. Let

$$\nu = (\nu_X, \nu_Y) \in T_z^*(X \times Y) = T_x^*X \oplus T_y^*Y$$

be a co-vector attached to a point $z = (x, y) \in Z \subset X \times Y$. It belongs to the fiber $\mathcal{S}N_z^*Z$ of the skew-conormal bundle $\mathcal{S}N^*Z$ at the point z iff for any vector $v = (v_X, v_Y)$ tangent to Z,

$$\nu_X(v_X) = \nu_Y(v_Y).$$

Note that for the conormal bundle one has the opposite sign: $\nu_X(v_X) = -\nu_Y(v_Y)$.

The total space of the skew-conormal bundle is a Lagrangian subvariety of the total space of the cotangent bundle $T^*(X \times Y)$ with respect to the symplectic form $\omega_X - \omega_Y$, where ω denotes the canonical symplectic form on the cotangent bundle. So, the skew-conormal bundle of a correspondence is rather close to the graph of a symplectic map between cotangent bundles.

Proposition 3.1. *The graph of the SHC $\mathfrak{S}_w$ is isomorphic to the skew-conormal bundle $\mathcal{S}N^*\mathfrak{H}_w$ of the usual Hecke correspondence $\mathfrak{H}_w$.*

Proof. As explained in Subsect. 2.4, a $\mathrm{GL}(N, \mathbb{C})$-bundle E is determined by the pair (g, L) in a neighborhood of a point $w \in \Sigma$. An upper HC of E corresponds to $(\tilde{g}, L)$, where $\tilde{g} = \Xi g$ and

$$\bar{\partial}\Xi = 0 \text{ in } \mathcal{U}_w, \quad \mathrm{ord}_w(\det(\Xi)) = 1. \tag{3.6}$$

Therefore, the skew-conormal bundle $\mathcal{S}N^*\mathfrak{H}_w$ of the HC $\mathcal{S}N^*\mathfrak{H}_w$ can be described by the data

$$(g, \tilde{g}; \eta, \tilde{\eta}; t, \tilde{t}, L), \;\; \tilde{g} = \Xi g,$$

where Ξ satisfies (3.6), and

$$\langle t, DL\rangle_{Jac} = \langle \tilde{t}, DL\rangle_{Jac}, \tag{3.7}$$

$$\mathrm{res}_w(\mathrm{Tr}(\tilde{\eta}D\tilde{g}\tilde{g}^{-1})) = \mathrm{res}_w(\mathrm{Tr}(\eta Dgg^{-1})) \tag{3.8}$$

for any variations of g and $\tilde{g}$, that preserve properties of $\Xi = g^{-1}\tilde{g}$. The first condition (3.7) means that $t = \tilde{t}$.

The condition (3.8) can be rewritten as

$$\mathrm{res}_w\left(\mathrm{tr}(\tilde{\eta}D\Xi\Xi^{-1} + (\Xi^{-1}\tilde{\eta}\Xi - \eta)g^{-1}D\, g)\right) = 0.$$

Since variations of g and Ξ are independent, both terms in the last expression must vanish separately:

$$\mathrm{res}_w(\mathrm{tr}(\tilde{\eta}D\Xi\Xi^{-1})) = 0, \tag{3.9}$$

$$\mathrm{res}_w\left(\mathrm{tr}(\Xi^{-1}\tilde{\eta}\Xi - \eta)g^{-1}D\, g)\right) = 0. \tag{3.10}$$

Consider first the case when w is not a marked point. Then we will demonstrate that (3.9) means that $\eta' = \Xi^{-1}\tilde{\eta}\Xi$ is holomorphic in w. Consider the value of Ξ at zero, this matrix has rank $N-1$. Denote by K its kernel and by I its image. An essential variation of Ξ corresponds to the variations of its image, so it is a map $I \to \mathbb{C}^N/I$. This variation corresponds to the right action: $D\,\Xi = \Xi\epsilon$. The singular part Ξ^{-1}_{sing} of Ξ^{-1} at w is an operator of rank 1. Its kernel equals I and its image equals K, so the singular part η'_{sing} of η' is a map from $\mathbb{C}^N/\mathrm{Ker}(\Xi^{-1}_{sing}) = \mathbb{C}^N/I$ to $\mathrm{Im}(\Xi^{-1}_{sing}) = I$. The first condition can be rewritten as:

$$0 = \mathrm{res}_w(\mathrm{tr}(\tilde{\eta}D\Xi\Xi^{-1})) = \mathrm{res}_w(\mathrm{tr}(\eta'\Xi^{-1}D\Xi)) = \mathrm{tr}(\eta'_{sing}\epsilon)$$

for any $\epsilon \in \mathrm{Hom}(I, \mathbb{C}^r/I)$. The space $\mathrm{Hom}(I, \mathbb{C}^r/I)$ is dual to $\mathrm{Hom}(\mathbb{C}^r/I, I)$, so η'_{sing} vanishes and η' is holomorphic.

Note that η' determines some Higgs field for g. Indeed, it is holomorphic in $\mathcal{U}_w$ and $g^{-1}\eta' g = \tilde{g}^{-1}\tilde{\eta}\tilde{g}$ is the restriction of some one-form on the complement to w. As the canonical one-form $\mathrm{tr}(\eta Dgg^{-1})$ is non-degenerate on $T^*\mathcal{M}^d_n$, from the second condition (3.10) we conclude that $\eta' = \eta$.

If w is a marked point then Ξ is fixed and it maps the Higgs field η into the Higgs field $\tilde{\eta}$ (Lemma 3.1). There is no variation of Ξ and we immediately have that again $\eta' = \eta$. □

3.4. Bäcklund transformation. Consider the Higgs bundles with the quasi-parabolic structures at the marked points. The gauge transformations $\Xi_\alpha^\pm$ related to the SHC $\mathfrak{S}_\alpha^\pm$ depend only on the marked point w_α. They define the maps of the Hitchin systems

$$\mathfrak{S}_\alpha^+ \sim \xi^\alpha : \; T^*\mathcal{M}^{(d)}(\Sigma_n, G) \to T^*\mathcal{M}^{(d+1)}(\Sigma_n, G), \tag{3.11}$$

$$\mathfrak{S}_\alpha^- \sim \xi_\beta : \; T^*\mathcal{M}^{(d)}(\Sigma_n, G) \to T^*\mathcal{M}^{(d-1)}(\Sigma_n, G). \tag{3.12}$$

Consider consecutive upper and lower modifications

$$\xi_{\alpha_2}^{\alpha_1} = \xi^{\alpha_1} \cdot \xi_{\alpha_2}. \tag{3.13}$$

Since $\deg(E)$ does not change it is a symplectic transform $T^*\mathcal{M}_n^d(\Sigma, E)$. In this way $\xi_{\alpha_2}^{\alpha_1}$ maps solutions of the Hitchin hierarchy into solutions.

Corollary 3.1. *The map (3.13) is* the Bäcklund transformation*, parameterized by a pair of marked points* $(w_{\alpha_1}, w_{\alpha_2})$.

We can generalize (3.13) as

$$\xi_{\alpha_{i_1};\ldots;\alpha_{i_s}}^{\alpha_{j_1};\ldots;\alpha_{j_s}} = \xi^{\alpha_{j_1}} \cdot \xi_{\alpha_{i_1}} \cdots .$$

Because the Bäcklund transformation is a canonical one, we can consider a discrete Hamiltonian system defined on the phase space $T^*\mathcal{M}_n^d(\Sigma, E)$. They pairwise commute and in terms of the angle variables generate a lattice in the Liouville torus [10, 26]. In our case the dimension of the Liouville torus is equal to $\dim \mathcal{M}_n^d$ (2.9), but the lattice we have constructed has in general a smaller dimension.

Note that when Σ_n is an elliptic curve, the Hitchin systems corresponding to $d = kN$ and $d = 0$ (d =deg(V)) are equivalent. Hence, in this case one can construct some Bäcklund transformations by applying the upper SHC N times.

4. Elliptic CM System – Elliptic SL$(N, \mathbb{C})$-Rotator Correspondence

4.1. Elliptic CM system. The elliptic CM system was first introduced in the quantum version [27]. It is defined on the phase space

$$\mathcal{R}^{CM} = \left(\mathbf{v} = (v_1, \ldots, v_N), \; \mathbf{u} = (u_1, \ldots, u_N), \quad \sum_j v_j = 0, \; \sum_j u_j = 0 \right), \tag{4.1}$$

with the canonical symplectic form

$$\omega^{CM} = (D\mathbf{v} \wedge D\mathbf{u}). \tag{4.2}$$

The second order with respect to the momenta $\mathbf{v}$ Hamiltonian is

$$H_2^{CM} = \frac{1}{2}\sum_{j=1}^N v_j^2 + \nu^2 \sum_{j>k} \wp(u_j - u_k; \tau).$$

It was established in [7, 28] that the elliptic CM system can be derived in the Hitchin approach. The Lax operator L^{CM} is the reduced Higgs field η over the elliptic curve

$$E_\tau = \mathbb{C}/\mathbb{L}, \;\; \mathbb{L} = \mathbb{Z} + \tau\mathbb{Z}$$

with a marked point $z = 0$. In this way the phase space $\mathcal{R}^{CM}$ is the space of pairs (quasi-parabolic SL_N-bundle V over E_τ, the Higgs field L^{CM} on this bundle (4.8)).

The bundle is determined by the transition functions (the multipliers)

$$Id_N: \ z \to z+1, \tag{4.3}$$

$$\mathbf{e}(\mathbf{u}) = \mathrm{diag}(\mathbf{e}(u_1), \ldots, \mathbf{e}(u_N)): \ z \to z+\tau,$$

where $\mathbf{e}$ is defined in (A.1). The Lax operator $L^{CM}(z)$ is the quasi-periodic one-form

$$L^{CM}(z+1) = L^{CM}(z), \ \ L^{CM}(z+\tau) = \mathbf{e}(-\mathbf{u})L^{CM}(z)\mathbf{e}(\mathbf{u}). \tag{4.4}$$

It is the N^{th} order matrix with the first order pole at $z=0$ and the residue

$$p^{(0)} = \mathrm{Res}_{z=0}(L^{CM}(z)) = L^{CM}_{-1} = \nu \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}. \tag{4.5}$$

This residue defines the minimal coadjoint orbit $\mathcal{O}$ (2.24) ($\dim(\mathcal{O}) = 2N-2$). These degrees of freedom are gauged away by the action of rest gauge symmetries generated by the constant diagonal matrices. For this reason the second term in (2.23) does not contribute in the symplectic form (4.2).

The column-vector $e_1 = (1, 1, \cdots, 1)$ is an eigen-vector e_1,

$$L^{CM}_{-1} e_1 = (N-1)\nu e_1. \tag{4.6}$$

There is also an $(N-1)$-dimensional eigen-subspace T_{N-1} corresponding to the degenerate eigen-value $-\nu$,

$$L^{CM}_{-1} e_{\mathbf{a}} = -\nu e_{\mathbf{a}}, \ \ e_{\mathbf{a}} = (a_1, \ldots, a_N), \ \left(\sum_n a_n = 0\right). \tag{4.7}$$

The quasi-periodicity (4.5) leads to the following form of L^{CM}:

$$L^{CM} = P + X, \text{ where } P = \mathrm{diag}(v_1, \ldots, v_N), \ \ X_{jk} = \nu\phi(u_j - u_k, z), \tag{4.8}$$

and ϕ is defined as (B.5).

The M^{CM}-operator corresponding to H_2^{CM} has the form

$$M^{CM} = -D + Y, \text{ where } D = \mathrm{diag}(Z_1, \ldots, Z_N), \ \ Y_{jk} = y(u_j - u_k, z), \tag{4.9}$$

$$Z_j = \sum_{k \neq j} \wp(u_j - u_k), \ \ y(u,z) = \frac{\partial \phi(u,z)}{\partial u}$$

4.2. The elliptic SL$(N, \mathbb{C})$*-rotator.* The elliptic SL$(N, \mathbb{C})$-rotator is an example of the Euler-Arnold top [20]. It is defined on a coadjoint orbit of SL$(N, \mathbb{C})$:

$$\mathcal{R}^{rot} = \{\mathbf{S} \in \mathrm{sl}(N, \mathbb{C}), \ \ \mathbf{S} = g^{-1}\mathbf{S}^{(0)}g\}, \tag{4.10}$$

where g is defined up to the left multiplication on the stationary subgroup G_0 of $\mathbf{S}^{(0)}$. The phase space $\mathcal{R}^{rot}$ is equipped with the Kirillov-Kostant symplectic form

$$\omega^{rot} = \mathrm{tr}(\mathbf{S}^{(0)}Dgg^{-1}Dgg^{-1}). \tag{4.11}$$

The Hamiltonian is defined as

$$H^{rot} = -\frac{1}{2}\mathrm{tr}(\mathbf{S}J(\mathbf{S})), \tag{4.12}$$

where J is a linear operator on Lie(SL$(N, \mathbb{C})$). The inverse operator is called the inertia tensor. The equation of motion takes the form

$$\partial_t\mathbf{S} = [J(\mathbf{S}), \mathbf{S}]. \tag{4.13}$$

We consider here a special form J, that provides the integrability of the system. Let

$$J(\mathbf{S}) = \mathbf{J}\cdot\mathbf{S} = \sum_{mn} J_{mn}S_{mn},$$

where $\mathbf{J}$ is a N^{th} order matrix,

$$\mathbf{J} = \{J_{mn}\} = \left\{\wp\begin{bmatrix} m \\ n \end{bmatrix}\right\}, \quad (m, n = 1, \dots, N), \quad (m, n \in \mathbb{Z} \bmod N, \ m + n\tau \notin \mathbb{L}), \tag{4.14}$$
$$\wp\begin{bmatrix} m \\ n \end{bmatrix} = \wp\left(\frac{m+n\tau}{N}; \tau\right).$$

We write down (4.13) in the basis of the sin-algebra $\mathbf{S} = S_{mn}E_{mn}$ (see (A.4)),

$$\partial_t S_{mn} = \frac{N}{\pi}\sum_{k,l} S_{k,l}S_{m-k,n-l}\wp\begin{bmatrix} k \\ l \end{bmatrix}\sin\frac{\pi}{N}(kn - ml). \tag{4.15}$$

The elliptic rotator is a Hitchin system [7]. We give a proof of this statement.

Lemma 4.1. *The elliptic* SL$(N, \mathbb{C})$*-rotator is a Hitchin system corresponding to the Higgs quasi-parabolic* GL$(N, \mathbb{C})$*-bundle E (deg(E)=1) over the elliptic curve E_τ with the marked point $z = 0$.*

Proof. It can be proved that (4.15) is equivalent to the Lax equation. The Lax matrices in the basis of the sin-algebra take the form

$$L^{rot} = \sum_{m,n} S_{mn}\varphi\begin{bmatrix} m \\ n \end{bmatrix}(z)E_{mn}, \quad \varphi\begin{bmatrix} m \\ n \end{bmatrix}(z) = \mathbf{e}\left(-\frac{nz}{N}\right)\phi\left(-\frac{m+n\tau}{N}; z\right), \tag{4.16}$$

$$M^{rot} = \sum_{m,n} S_{mn}f\begin{bmatrix} m \\ n \end{bmatrix}(z)E_{mn}, \quad f\begin{bmatrix} m \\ n \end{bmatrix}(z) = \mathbf{e}\left(-\frac{nz}{N}\right)\partial_u\phi(u; z)|_{u=-\frac{m+n\tau}{N}}. \tag{4.17}$$

They lead to the Lax equation for the matrix elements

$$\partial_t S_{mn}\varphi\begin{bmatrix} m \\ n \end{bmatrix}(z) = \sqrt{-1}\sum_{k,l} S_{m-k,n-l}S_{kl}\varphi\begin{bmatrix} m-k \\ n-l \end{bmatrix}(z)f\begin{bmatrix} k \\ l \end{bmatrix}(z)\sin\frac{\pi}{N}(nk-ml).$$

Using the Calogero functional equation (B.27) we rewrite it in the form (4.15). Since

$$\frac{1}{N}\mathrm{tr}(L^{rot})^2 = -2H^{rot} + \mathrm{tr}\mathbf{S}^2\wp(z),$$

H^{rot} is the Hitchin quadratic integral.

The Lax operator satisfies the Hitchin equation

$$\bar{\partial}L^{rot} = 0, \quad \mathrm{Res}L^{rot}|_{z=0} = 2\pi\sqrt{-1}\mathbf{S}$$

and is quasi-periodic

$$L^{rot}(z+1) = Q(\tau)L^{rot}(z)Q^{-1}(\tau), \tag{4.18}$$

$$L^{rot}(z+\tau) = \tilde{\Lambda}(z,\tau)L^{rot}(z)(\tilde{\Lambda}(z,\tau))^{-1}, \tag{4.19}$$

where $\tilde{\Lambda}(z,\tau) = -\mathbf{e}(\frac{-z-\frac{1}{2}\tau}{N})\Lambda$ and the matrices Q and Λ are defined in (A.2),(A.3). The transition functions

$$Q(\tau): \quad z \to z+1, \tag{4.20}$$

$$\tilde{\Lambda}(z,\tau): \quad z \to z+\tau \tag{4.21}$$

define the $\mathrm{GL}(N,\mathbb{C})$-bundle over E_τ with $\deg(V) = 1$. For these bundles we have $\dim(\mathcal{M}_0^1) = 1$ (2.8) and after the symplectic reduction we come to the coadjoint orbit $G_0 \setminus \mathrm{SL}(N,\mathbb{C})$ (4.10). The Kirillov-Kostant form (4.11) arises as the last terms in (2.23) attributed to the point $z = 0$. Thus, the phase space of the SL_N-rotator is the space of the Higgs fields L^{rot} on the bundle determined by multipliers Q, $\tilde{\Lambda}$ with the first order singularities at zero. □

4.3. A map $\mathcal{R}^{CM} \to \mathcal{R}^{rot}$. We construct a map from the phase space of the elliptic CM system $\mathcal{R}^{CM}$ into the phase space of the SL_N-rotator $\mathcal{R}^{rot}$. We assume here that the SL_N-rotator is living on the most degenerate orbit corresponding to L_{-1}^{CM} (4.5). The phase space of CM systems with spins is mapped into the general coadjoint orbits. This generalization is straightforward. In this way, for $N = 2$ we describe the upper horizontal arrow in Fig. 1.

The map is defined as the conjugation of L^{CM} by some matrix $\Xi(z)$:

$$L^{rot} = \Xi \times L^{CM} \times \Xi^{-1}. \tag{4.22}$$

It follows from comparing (4.4) with (4.18) and (4.19) that Ξ must intertwine the multipliers of bundles:

$$\Xi(z+1,\tau) = Q \times \Xi(z,\tau), \tag{4.23}$$

$$\Xi(z+\tau,\tau) = \tilde{\Lambda}(z,\tau) \times \Xi(z,\tau) \times \mathrm{diag}(\mathbf{e}(u_j)). \tag{4.24}$$

The matrix $\Xi(z)$ degenerates at $z = 0$, and the column-vector $(1,\cdots,1)$, in accordance with Lemma 3.1, should belong to the kernel of $\Xi(0)$. In this case, $\Xi \times L^{CM} \times \Xi^{-1}$ has a first order pole at $z = 0$.

Consider the following $(N \times N)$- matrix $\tilde{\Xi}(z, u_1, \dots, u_N; \tau)$:

$$\tilde{\Xi}_{ij}(z, u_1, \dots, u_N; \tau) = \theta\begin{bmatrix} \frac{i}{N} - \frac{1}{2} \\ \frac{N}{2} \end{bmatrix}(z - Nu_j, N\tau), \tag{4.25}$$

where $\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau)$ is the theta function with a characteristic (B.31). Sometimes we omit nonessential arguments of Ξ for brevity.

Lemma 4.2. *The matrix $\tilde{\Xi}$ is transformed under the translations $z \to z+1$, $z \to z+\tau$ and $u_j \to u_j + 1$, $u_j \to u_j + \tau$ as :*

$$\tilde{\Xi}(z+1, \tau) = -Q \times \tilde{\Xi}(z, \tau), \tag{4.26}$$

$$\tilde{\Xi}(z+\tau, \tau) = \tilde{\Lambda}(z, \tau) \times \tilde{\Xi}(z, \tau) \times \mathrm{diag}(\mathbf{e}(u_j)), \tag{4.27}$$

$$\tilde{\Lambda}(z, \tau) = -\mathbf{e}\left(-\frac{\tau}{2N} - \frac{z}{N}\right)\Lambda;$$

$$\tilde{\Xi}(u_j + 1, ; \tau) = \tilde{\Xi}(u_j; \tau) \times \mathrm{diag}(1, \cdots, (-1)^N, \cdots, 1), \tag{4.28}$$

$$\tilde{\Xi}(u_j + \tau; \tau) = \tilde{\Xi}(u_j; \tau) \times \mathrm{diag}\left(1, \cdots, (-1)^N \mathbf{e}\left(-\frac{N\tau}{2} + z - Nu_j\right) \cdots, 1\right). \tag{4.29}$$

Proof. The statement of the lemma follows from the properties of the theta functions with characteristics (B.33)–(B.35). □

Now we assume that $\sum u_j = 0$, so u_N is no more an independent variable, but it is equal to $-\sum_{j=1}^{N-1} u_j$.

The determinant formula of the Vandermonde type [17]

$$\det\left[\frac{\tilde{\Xi}_{ij}(z, u_1, \dots, u_N; \tau)}{\sqrt{-1}\eta(\tau)}\right] = \frac{\vartheta(z)}{\sqrt{-1}\eta(\tau)} \prod_{1 \le k < l \le N} \frac{\vartheta(u_l - u_k)}{\sqrt{-1}\eta(\tau)} \tag{4.30}$$

is used to show that the matrix $\tilde{\Xi}_{ij}(z)$ truly degenerates at $z = 0$. Here $\eta(\tau)$ is the Dedekind function.

Lemma 4.3. *The kernel of $\tilde{\Xi}$ at $z = 0$ is generated by the following column-vector:*

$$\left\{(-1)^l \prod_{j<k;\, j,k \neq l} \vartheta(u_k - u_j, \tau)\right\}, \quad l = 1, 2, \cdots, N.$$

Proof. We must prove that for any i the following expression

$$\sum_{l=1}^{N} (-1)^l \theta\begin{bmatrix} \frac{i}{N} - \frac{1}{2} \\ \frac{N}{2} \end{bmatrix}(z - Nu_l, N\tau) \prod_{j<k;\, j,k \neq l} \vartheta(u_k - u_j, \tau) \tag{4.31}$$

vanishes. First, the symmetric group S_N acts on $\mathbf{u}$ by permutation of $u_1, \dots, u_N$ and (4.31) is antisymmetric with respect to the S_N action. Hence it vanishes on the hyperplanes $u_i = u_j$. As a function on u_1, (4.31) has $2N$ zeroes: $N-2$ zeroes $u_1 = u_k$,

$k \neq 1, N$, $N-2$ zeroes $u_N = u_k$, $k \neq 1, N$ and four zeroes $u_1 = u_N$ (the last equation is $2u_1 = -\sum_{j=2}^{N-1} u_j$).

Second, (4.31) is quasiperiodic with respect to the shifts $u_1 \to u_1 + 1$, $u_1 \to u_1 + \tau$ with multiplicators 1 and $\mathbf{e}\left(-(N-1)\tau - (N-1)(u_1 - u_N)\right)$. Any quasiperiodic function with such multiplicators is either zero or has $2N-2$ zeroes. Since our expression vanishes in $2N$ points it vanishes identically. □

It follows from the previous lemmas that the matrix

$$\Xi(z) = \tilde{\Xi}(z) \times \operatorname{diag}\left((-1)^l \prod_{j<k;\, j,k \neq l} \vartheta(u_k - u_j, \tau) \right) \tag{4.32}$$

is the singular gauge transform from Lemma 2.1 that maps L^{CM} to L^{rot}. This transformation leads to the symplectic map

$$\mathcal{R}^{CM} \to \mathcal{R}^{rot}, \quad (\mathbf{v}, \mathbf{u}) \mapsto \mathbf{S}. \tag{4.33}$$

Consider in detail the case $N = 2$. Let

$$\mathbf{S} = S_a \sigma_a,$$

where σ_a denote the sigma matrices subject to the commutation relations

$$[\sigma_a, \sigma_b] = 2\sqrt{-1}\varepsilon_{abc}\sigma_c.$$

Then the transformation has the form

$$\begin{cases} S_1 = -\nu \dfrac{\theta_{10}(0)}{\vartheta'(0)} \dfrac{\theta_{10}(2u)}{\vartheta(2u)} - \nu \dfrac{\theta_{10}^2(0)}{\theta_{00}(0)\theta_{01}(0)} \dfrac{\theta_{00}(2u)\theta_{01}(2u)}{\vartheta^2(2u)}, \\ S_2 = -\nu \dfrac{\theta_{00}(0)}{\sqrt{-1}\vartheta'(0)} \dfrac{\theta_{00}(2u)}{\vartheta(2u)} - \nu \dfrac{\theta_{00}^2(0)}{\sqrt{-1}\theta_{10}(0)\theta_{01}(0)} \dfrac{\theta_{10}(2u)\theta_{01}(2u)}{\vartheta^2(2u)}, \\ S_3 = -\nu \dfrac{\theta_{01}(0)}{\vartheta'(0)} \dfrac{\theta_{01}(2u)}{\vartheta(2u)} - \nu \dfrac{\theta_{01}^2(0)}{\theta_{00}(0)\theta_{10}(0)} \dfrac{\theta_{00}(2u)\theta_{10}(2u)}{\vartheta^2(2u)}. \end{cases} \tag{4.34}$$

Formulae of this kind were obtained in [16].

4.4. Bäcklund transformations in the CM systems. We now use the map (4.33) to construct the Bäcklund transformation in the CM systems

$$\xi : (\mathbf{v}, \mathbf{u}) \to (\tilde{\mathbf{v}}, \tilde{\mathbf{u}}).$$

Let the Lax matrix depend on the new coordinates and momenta $L = L(\tilde{\mathbf{v}}, \tilde{\mathbf{u}})$. Consider the upper modification $\Xi(z)$ (4.32). To construct the Bäcklund transformation ξ, we map $(\mathbf{v}, \mathbf{u})$ and $(\tilde{\mathbf{v}}, \tilde{\mathbf{u}})$ to the same point $\mathbf{S} \in \mathcal{R}^{rot}$:

$$\begin{array}{ccccc} & & \boxed{L^{rot}(\mathbf{S})} & & \\ & \nearrow \Xi(z, \mathbf{u}) & & \nwarrow \Xi(z, \tilde{\mathbf{u}}) & \\ \boxed{L^{CM}(\mathbf{v}, \mathbf{u})} & & \xrightarrow{\xi} & & \boxed{L^{CM}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}})} \end{array}$$

In this way we reproduce implicitly the general formula (3.13) for the Bäcklund transformations. This transformation defines an integrable discrete time dynamics of a CM system. One example of this dicretization was proposed in [29]. It can be supposed to correspond to ξ.

Another way to construct new solutions from $(\mathbf{v}, \mathbf{u})$ is to act by N consecutive upper modifications

$$\Xi(N) = D_N \Xi_N \cdots \Xi_j \cdots \Xi_2 \cdot \Xi. \tag{4.35}$$

Here the matrices Ξ_j, $j = 2, \ldots, N$, satisfy the quasi-periodicity conditions

$$\Xi_j(z+\tau) = \tilde{\Lambda}^j \Xi_j(z) \tilde{\Lambda}^{1-j},$$

and D_N is an arbitrary diagonal matrix. We come back to the N-dimensional moduli space $\mathcal{M}^{(N)}$ (see (2.5)) and to the map

$$L^{CM}(\mathbf{v}, \mathbf{u}) \xrightarrow{\Xi(N)} L^{CM}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}).$$

If we break the chain (4.35) on a step $k < L$, then we obtain the map

$$L^{CM} \to L^{rot,k},$$

where $L^{rot,k}$ is the Lax operator for the elliptic rotator related to the holomorphic bundle of degree k. It satisfies the quasi-periodicity condition (4.18) and

$$L^{rot,k}(z+\tau) = \tilde{\Lambda}^j L^{rot,k}(z) \tilde{\Lambda}^{-j}$$

instead of (4.19).

5. Hitchin Systems of Infinite Rank

Here we generalize the derivation of finite-dimensional integrable systems in the form (2.25)–(2.28) on two-dimensional integrable field theories.

5.1. Holomorphic $\hat{L}(\mathrm{GL}(N,\mathbb{C}))$*-bundles.* Let $L(\mathrm{gl}(N,\mathbb{C}))$ be the loop algebra of C^∞-maps $L(\mathrm{gl}(N,\mathbb{C})) : S^1 \to \mathrm{gl}(N,\mathbb{C})$, and $\hat{L}(\mathrm{gl}(N,\mathbb{C}))$ be its central extension with the multiplication

$$(g, c) \times (g', c') = \left(gg', cc' \exp \mathcal{C}(g, g')\right), \tag{5.1}$$

where $\exp \mathcal{C}(g, g')$ is a 2-cocycle of $\hat{L}(\mathrm{GL}(N,\mathbb{C}))$ providing the associativity of the multiplication.

Consider a holomorphic vector bundle V of an infinite rank over a Riemann curve Σ_n with n marked points. The bundle is defined by the transition functions from $\hat{L}(\mathrm{GL}(N,\mathbb{C}))$. Its fibers are isomorphic to the Lie algebra $\hat{L}(\mathrm{gl}(N,\mathbb{C}))$. The holomorphic structure on V is defined by the operator

$$d'' : \Omega^{(0)}_{C^\infty}(\Sigma_n, \mathrm{End}\, V) \to \Omega^{(0,1)}_{C^\infty}(\Sigma_n, \mathrm{End}\, V).$$

It has two components $d'' = d''_{\bar{A}} + d''_{\lambda}$. The first component is

$$d''_{\bar{A}} : \Omega^{(0)}_{C^\infty}(\Sigma_n, L(\mathrm{gl}(N,\mathbb{C}))) \to \Omega^{(0,1)}_{C^\infty}(\Sigma_n, L(\mathrm{gl}(N,\mathbb{C}))).$$

Locally

$$d''_{\bar{A}} = \bar{\partial} + \bar{A}, \quad \bar{\partial} = \partial_{\bar{z}}, \quad \bar{A} = \bar{A}(x, z, \bar{z}), \quad x \in S^1.$$

The operator $d''_{\bar{A}}$ acts on a N-dimensional column vector $\vec{e}(x; z, \bar{z})$. The second component is defined by the connection d''_{λ} on a trivial linear bundle $\mathcal{L}$ on Σ_n, given by

$$d''_{\lambda} = \bar{\partial} + \lambda.$$

The field λ is a map from Σ_n to the central element of the Lie algebra $\hat{L}(\mathrm{gl}(N, \mathbb{C}))$. A local section σ of V is holomorphic if $d''\sigma = 0$. The sections allow to define the transition functions. We assume that $\bar{A}$ and λ are smooth at the marked points.

In addition we define n copies of the central extended loop groups located at the marked points

$$\hat{L}G_\alpha = (g_\alpha(x), c_\alpha), \ G_\alpha = \mathrm{GL}(N, \mathbb{C}), \ (\alpha = 1, \ldots, n), \ x \in S^1,$$

with the multiplication (5.1).

Thus, we have the set $\mathcal{R}$ of fields playing the role of the "coordinate space":

$$\mathcal{R} = \left\{\bar{A}, \lambda, (g_1, c_1), \ldots, (g_n, c_n)\right\}. \tag{5.2}$$

5.2. Gauge symmetries. Let $\mathcal{G}$ be the group of automorphisms of $\mathcal{R}$ (the gauge group),

$$\mathcal{G} = C^\infty \mathrm{Map}(\Sigma_n \to \hat{L}(\mathrm{GL}(N, \mathbb{C}))) = \{f(z, \bar{z}, x), s(z, \bar{z})\},$$

where $f(z, \bar{z}, x)$ takes values in $\mathrm{GL}(N, \mathbb{C})$, and $s(z, \bar{z})$ is the map to the central element of $\hat{L}(\mathrm{GL}(N, \mathbb{C}))$. The multiplication is pointwise with respect to Σ_n,

$$(f_1, s_1) \times (f_2, s_2) = (f_1 f_2, s_1 s_2 \exp \mathcal{C}(f_1, f_2)),$$

where $\exp \mathcal{C}(f_1, f_2)$ is a map from Σ_n to the 2-cocycle of $\hat{L}(\mathrm{GL}(N, \mathbb{C}))$.

Let $(f_\alpha = f_\alpha(x), s_\alpha)$ be the value of the gauge fields at the marked point w_α. The action of $\mathcal{G}$ on $\mathcal{R}$ takes the following form:

$$\bar{A} \to f^{-1}\bar{\partial} f + f^{-1}\bar{A} f, \tag{5.3}$$

$$\lambda \to \lambda + s^{-1}\bar{\partial} s + \oint \mathrm{tr}(\bar{A} f^{-1}\partial_x f) dx, \tag{5.4}$$

$$c_\alpha \to c_\alpha s_\alpha, \ g_\alpha \to g_\alpha f_\alpha. \tag{5.5}$$

The quotient space $\mathcal{N} = \mathcal{R}/\mathcal{G}$ is the moduli space of infinite rank holomorphic bundles over Riemann curves with marked points.

5.3. Phase space. The cotangent space to $\mathcal{R}$ has the following structure. Consider the analog of the Higgs field $\Phi \in \Omega^{(1,0)}_{C^\infty}(\Sigma_n, (\text{End } V)^*)$. It is a one-form Φ on Σ_n taking values in the Lie coalgebra $L^*(\text{gl}(N, \mathbb{C}))$. Let k be a scalar one-form on Σ_n, $k \in \Omega^{(1,0)}_{C^\infty}(\Sigma_n)$. It is dual to the field λ. At the marked points we have the Lie coalgebras $\text{Lie}^*(G_\alpha) \sim L(\text{gl}(N, \mathbb{C}))$ along with the central elements r_α, dual to c_α. Thus the cotangent bundle $T^*\mathcal{R}$ contains the fields

$$T^*\mathcal{R} = \left\{(\bar{A}, \Phi), (\lambda, k); (g_1, c_1; p_1, r_1), \ldots, (g_n, c_n; p_n, r_n)\right\}. \tag{5.6}$$

There is a canonical symplectic structure on $T^*\mathcal{R}$. For $F \in \Omega^{(1,0)}_{C^\infty}(\Sigma_n, (\text{End } V)^*)$ and $G \in \Omega^{(0,1)}_{C^\infty}(\Sigma_n, \hat{L}(\text{gl}(N, \mathbb{C})))$ defines the pairing

$$< F|G >= \int_{\Sigma_n} \oint \text{tr}(FG)dx.$$

Then

$$\omega =< D\Phi \wedge D\bar{A} > + \int_{\Sigma_n} Dk \wedge D\lambda + \sum_{\alpha=1}^{n} \omega_\alpha, \tag{5.7}$$

where ω_α is a canonical form on $T^*\hat{L}(G_\alpha)$. It is constructed in the canonical way by means of the Maurer-Cartan form on $\hat{L}(G_\alpha) = \{g_\alpha, c_\alpha\}$. The result is

$$\omega_\alpha = \oint_{S^1_\alpha} \text{tr}(D(p_\alpha g_\alpha^{-1})Dg_\alpha) + D(r_\alpha c_\alpha^{-1})Dc_\alpha + \frac{r_\alpha}{2} \oint_{S^1_\alpha} \text{tr}\left(g_\alpha^{-1} Dg_\alpha \partial_x (g_\alpha^{-1} Dg_\alpha)\right). \tag{5.8}$$

5.4. Symplectic reduction. Now consider the lift of $\mathcal{G}$ to the global canonical transformations of $T^*\mathcal{R}$. In addition to (5.3),(5.4),(5.5) we have the following action of $\mathcal{G}$:

$$\Phi \to f^{-1}k\partial_x f + f^{-1}\Phi f, \quad k \to k, \tag{5.9}$$

$$p_\alpha \to f_\alpha^{-1} p_\alpha f_\alpha + r_\alpha f_\alpha^{-1} \partial_x f_\alpha, \quad r_\alpha \to r_\alpha. \tag{5.10}$$

This transformation leads to the moment map from the phase space to the Lie coalgebra of the gauge group $\mu : T^*\mathcal{R} \to \text{Lie}^*(\mathcal{G})$. It takes the form

$$\mu = \left(\bar{\partial}\Phi - k\partial_x \bar{A} + [\bar{A}, \Phi] + \sum_{\alpha=1}^{n} p_\alpha \delta(z_\alpha),\ \bar{\partial}k + \sum_{\alpha=1}^{n} r_\alpha \delta(z_\alpha)\right). \tag{5.11}$$

We assume that $\mu = (0, 0)$. Therefore, we have the two holomorphity conditions

$$\bar{\partial}\Phi - k\partial_x \bar{A} + [\bar{A}, \Phi] + \sum_{\alpha=1}^{n} p_\alpha \delta(z_\alpha) = 0, \tag{5.12}$$

$$\bar{\partial}k + \sum_{\alpha=1}^{n} r_\alpha \delta(z_\alpha) = 0. \tag{5.13}$$

The constraint equation (5.13) means that the k-component of the Higgs field is a holomorphic one-form on Σ with first order poles at the marked points.

Let us fix a gauge

$$\bar{L} = f^{-1}\bar{\partial} f + f^{-1}\bar{A} f. \tag{5.14}$$

The same gauge action transform Φ as

$$L = kf^{-1}\partial_x f + f^{-1}\Phi f. \tag{5.15}$$

We preserve the same notations g_α, p_α for the gauge transformed variables. The moment constraint equation (5.12) has the same form in terms of $\bar{L}$ and L,

$$\bar{\partial} L - k\partial_x \bar{L} + [\bar{L}, L] + \sum_{\alpha=1}^{n} p_\alpha \delta(z_\alpha) = 0. \tag{5.16}$$

Solutions of this equation along with (5.13) define the reduced phase space

$$T^*\mathcal{R}//\mathcal{G} \sim T^*\mathcal{N}.$$

The symplectic form (5.7) on $T^*\mathcal{N}$ becomes

$$\omega =< \delta L|\delta \bar{L} > + \int_{\Sigma_n} \delta k \delta \lambda + \sum_{\alpha=1}^{n} \omega_\alpha. \tag{5.17}$$

5.5. Coadjoint orbits. Consider in detail the symplectic form ω (5.8) on $T^*\hat{L}(G) \sim \{(p, r); (g, c)\}$. We omit the subscript $_\alpha$ below. The following canonical transformation of ω by $(f, s) \in \hat{L}(G)$, where s is a central element,

$$g \to fg, \;\; p \to p, \;\; r \to r, \;\; c \to sc, \quad f \in L(G), \tag{5.18}$$

has not been used so far. The symplectic reduction with respect to this transformation leads to the coadjoint orbits of $\hat{L}(\mathrm{GL}(N, \mathbb{C}))$. In fact, the moment map

$$\mu : T^*\hat{L}(G) \to \mathrm{Lie}^*(\hat{L}(\mathrm{GL}(N, \mathbb{C})))$$

takes the form

$$\mu = (-gpg^{-1} + r\partial_x g g^{-1}, r).$$

Let us fix the moment $\mu = (p^{(0)}, r^{(0)})$. The result of the symplectic reduction of $T^*\hat{L}(G)$ is the coadjoint orbit

$$\mathcal{O}(p^{(0)}, r^{(0)}) = (p = -g^{-1}p^{(0)}g - r^{(0)}g^{-1}\partial_x g, \; r^{(0)}) = \mu^{-1}\left(T^*\hat{L}(\mathrm{SL}(N, \mathbb{C}))\right)/G_0,$$

where G_0 is the subgroup of $\hat{L}(\mathrm{GL}(N, \mathbb{C}))$ that preserves μ,

$$G^0 = \{g \in L(\mathrm{GL}(N, \mathbb{C}), \; s \text{ is arbitrary}) \mid p^{(0)} = -g^{-1}p^{(0)}g + r^{(0)}g^{-1}\partial_x g\}.$$

The symplectic form (5.8) being pushed forward on $\mathcal{O}$ takes the form

$$\omega = \oint \mathrm{tr}(D(pg^{-1})Dg) + \frac{r^{(0)}}{2} \oint \mathrm{tr}\left(g^{-1}DgD(g^{-1}\partial_x g)\right). \tag{5.19}$$

In what follows we will consider the collection of the orbits $\mathcal{O}_\alpha(p_\alpha^{(0)}, r_\alpha^{(0})$ at the marked points instead of the cotangent bundles $T^*\hat{L}(G_\alpha)$. In this way we come to the notion of the Higgs bundle of infinite rank $\hat{\mathcal{H}}_n^d$ (see (2.22) and (5.6))

$$\hat{\mathcal{H}}_n^d = \left\{(\bar{A}, \Phi), (\lambda, k), \mathcal{O}_1(p_1^{(0)}, r_1^{(0}), \ldots, \mathcal{O}_n(p_n^{(0)}, r_n^{(0)})\right\}. \tag{5.20}$$

5.6. Conservation laws I. The Higgs field Φ is transformed as a connection with respect to the circles S^1 (5.9). If the central charge $k \neq 0$, the standard Hitchin integrals (2.28) cease to be gauge invariant. Invariant integrals are generated by the traces of the monodromies of the Higgs field Φ. The generating function of Hamiltonians is given by

$$H(z) = \mathrm{tr}\left(P \exp \frac{1}{k} \oint_{S_1} \Phi\right), \tag{5.21}$$

where z is a local coordinate of an arbitrary point. At a marked point, Φ has a first order pole and

$$H(z) = \sum_{j=-1}^{+\infty} H_j z^j. \tag{5.22}$$

Since $H(z)$ is gauge invariant one can replace Φ by L in (5.21),

$$H(z) = \mathrm{tr}\left(P \exp \frac{1}{k} \oint_{S_1} L\right). \tag{5.23}$$

5.7. Equations of motion. Consider the equations of motion on the "upstairs" space $T^*\mathcal{R}$ (5.20). They are derived by means of the symplectic form ω (5.7), where ω_α is replaced by (5.19), and the Hamiltonians (5.21), (5.22). Let t_j be a time variable corresponding to the Hamiltonian H_j. Taking into account that H_j is a functional depending on the Higgs field and the central charge k only, we arrive at the following free system;

$$\partial_j \Phi = 0, \tag{5.24}$$

$$\partial_j \bar{A} = \frac{\delta H_j}{\delta \Phi}, \tag{5.25}$$

$$\partial_j k = 0, \quad \partial_j \lambda = \frac{\delta H_j}{\delta k}, \quad \partial_j p_\alpha = 0. \tag{5.26}$$

After the symplectic reduction we are led to the fields $\bar{L}$ (5.14) and L (5.15). For simplicity, we keep the same notation for the coadjoint orbits variables p_α, so they are transformed as in (5.10). Substituting (5.15) in (5.24) we obtain the Zakharov-Shabat equation

$$\partial_j L - k\partial_x M_j + [M_j, L] = 0, \quad (\partial_j = \partial_{t_j}), \tag{5.27}$$

where $M_j = \partial_j f f^{-1}$. The operator M_j can be restored partly from the second equation (5.25),

$$\bar{\partial} M_j - \partial_j \bar{L} + [M_j, \bar{L}] = \frac{\delta H_j}{\delta L}. \tag{5.28}$$

The last two equations along with the moment constraint equation (5.16) are the consistency conditions for the linear system

$$(k\partial_x + L)\Psi = 0, \tag{5.29}$$

$$(\bar{\partial} + \bar{L})\Psi = 0, \tag{5.30}$$

$$(\partial_j + M_j)\Psi = 0. \tag{5.31}$$

5.8. Conservation laws II. The matrix equation (5.29) allows to write down the conservation laws. Its generic solutions can be represented in the form

$$\Psi(x) = (I + R)\exp\left(-\frac{1}{k}\int_0^x \mathcal{S}dx'\right), \tag{5.32}$$

where I is the identity matrix, R is an off-diagonal periodic matrix and $\mathcal{S} = \mathrm{diag}(\mathcal{S}^1, \dots, \mathcal{S}^N)$ is a diagonal matrix. Equation (5.29) means that L can be gauge transformed to the diagonal form

$$(I + R)\mathcal{S} = k\partial_x(I + R) + L(I + R). \tag{5.33}$$

Consider this equation in neighborhood of a point on Σ_n with a local coordinate z. Assume, for simplicity, that it is a pole of the Lax operator and k is a constant. In particular, it follows from (5.13) that $r_\alpha^0 = 0$ and the coadjoint orbits have the form $\mathcal{O}_\alpha = \{p_\alpha = -gp_\alpha^0 g^{-1}\}$. Then substitute into (5.33) the series expansions

$$L(z) = L_{-1}z^{-1} + L_0 + L_1 z + \dots, \quad (L_{-1} = \mathrm{res}L = p),$$
$$\mathcal{S}(z) = \mathcal{S}_{-1}z^{-1} + \mathcal{S}_0 + \mathcal{S}_1 z + \dots,$$
$$(I + R)(z) = h + R_1 z + R_2 z^2 + \dots, \quad (\mathrm{diag}(R_m) \equiv 0).$$

It follows from (5.32) that the diagonal matrix elements $\mathcal{S}_j^m$ are the densities of the conservation laws

$$\log H_{j,l} = \oint \mathcal{S}_j^l dx.$$

We present a recurrence procedure to define the diagonal matrices $\mathcal{S}_j$. On the first step we find that

$$\mathcal{S}_{-1} = h^{-1}L_{-1}h = h^{-1}ph, \quad p = L_{-1} = \mathrm{Res}\, L_{z=0}. \tag{5.34}$$

In other words the diagonal matrix S_{-1} determines the orbit located at the point $z = 0$.

In the general case we get the following equation:

$$\mathcal{S}_k + [h^{-1}R_{k+1}, \mathcal{S}_{-1}] = h^{-1}k\partial_x R_k + h^{-1}\sum_{l=1}^{k} L_{l-1}R_{k-l+l} - R_l\mathcal{S}_{k-l}h^{-1}L_k h.$$

Separating the diagonal and the off-diagonal parts allows us to express $\mathcal{S}_k$ and R_k in terms of the lower coefficients

$$\mathcal{S}_k = \left(h^{-1}k\partial_x R_k + h^{-1}\sum_{l=0}^{k-1} L_l R_{k-l} + h^{-1}\sum_{l=1}^{k} L_{l-1}R_{k-l+l} - R_l\mathcal{S}_{k-l}h^{-1}L_k h\right)_{diag}, \tag{5.35}$$

$$[h^{-1}R_k, \mathcal{S}_{-1}] = \left(h^{-1}k\partial_x R_{k-1} + h^{-1}\sum_{l=1}^{k} L_{l-1}R_{k-l+l} - R_l\mathcal{S}_{k-l}h^{-1}L_k h\right)_{nondiag}. \tag{5.36}$$

In particular,

$$\mathcal{S}_0 = (h^{-1}k\partial_x h + h^{-1}L_0 h)_{diag}, \tag{5.37}$$

$$\mathcal{S}_1 = \left(h^{-1}k\partial_x R_1 + h^{-1}L_1 h - h^{-1}R_1\mathcal{S}_0 + h^{-1}L_0 R_1\right)_{diag}, \tag{5.38}$$

where R_1 is defined by the equation

$$[h^{-1}R_1, \mathcal{S}_{-1}] = (h^{-1}k\partial_x h + h^{-1}L_0 h)_{nondiag}. \tag{5.39}$$

5.9. Hamiltonians in $SL(2,\mathbb{C})$ case. Let us perform the gauge transformation

$$f^{-1}Lf + kf^{-1}\partial_x f = L', \tag{5.40}$$

with f defined as follows:

$$f = \begin{pmatrix} \sqrt{L_{12}} & 0 \\ -\frac{L_{11}}{\sqrt{L_{12}}} - k\frac{\partial_x\sqrt{L_{12}}}{L_{12}} & \frac{1}{\sqrt{L_{12}}} \end{pmatrix}. \tag{5.41}$$

Then the Lax matrix L is transformed into

$$L' = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \tag{5.42}$$

where

$$T = L_{21}L_{12} + L_{11}^2 + k\frac{L_{11}\partial_x L_{12}}{L_{12}} - k\partial_x L_{11} - \frac{1}{2}k^2\frac{\partial_x^2 L_{12}}{L_{12}} + \frac{3}{4}k^2\frac{(\partial_x L_{12})^2}{L_{12}^2}. \tag{5.43}$$

The linear problem

$$\begin{cases} (xk\partial_x + L')\psi = 0, \\ (\partial_j + M_j')\psi = 0, \end{cases} \tag{5.44}$$

where ψ is the Bloch wave function $\psi = exp\{-i\oint\chi\}$, leads to the Riccati equation:

$$ik\partial_x\chi - \chi^2 + T = 0. \tag{5.45}$$

The decomposition of $\chi(z)$ provides densities of the conservation laws (see [30]):

$$\chi = \sum_{k=-1}^{\infty} z^k\chi_k, \tag{5.46}$$

$$H_k \sim \oint dx\,\chi_{k-1}. \tag{5.47}$$

The values of χ_k can be found from (5.45) using the expression (5.43) for $T(z) = \sum_{k=-2}^{\infty} z^k T_k$ in a neighborhood of zero. For $k = -2,\ -1$ and 0 we have:

$$\begin{cases} \chi_{-1} = \sqrt{T_{-2}} = \sqrt{h}, \\ 2\sqrt{h}\chi_0 = T_{-1} + ik\partial_x\chi_{-1} = T_{-1}, \\ 2\sqrt{h}\chi_1 = T_0 + ik\partial_x\chi - \chi_0^2. \end{cases} \tag{5.48}$$

In Subsects. 7.2, 7.3 below, explicit formulae for T_k are used for the computation of the Hamiltonians for the elliptic 2d Calogero-Moser and the elliptic Gaudin models.

6. $\hat{L}(\mathrm{SL}(N,\mathbb{C}))$-Bundles over elliptic Curves with Marked Points

6.1. General case. We apply the general construction to the $\hat{L}(\mathrm{SL}(N,\mathbb{C}))$-bundle over that elliptic curve E_τ with marked points $w_\alpha,\ \alpha = 1,\dots,n$. It is a two-dimensional generalization of the elliptic Gaudin model [8]. In particular, for one marked point $z = 0$ we come to the N-body elliptic CM field theory.

Let us construct solutions of the moment equations (5.11), taking for simplicity at the marked points the orbits with vanishing central charges

$$\mathcal{O}_\alpha = \left\{p^\alpha_{ij},\ \ r_\alpha = 0\right\}.$$

For elliptic curves one can fix the central charge as $k = 1$. For the stable bundles the gauge transformation (5.3) allows to diagonalize $\bar{A}$:

$$\bar{A}_{ij} = \delta_{ij}\frac{2\pi\sqrt{-1}}{\tau-\bar{\tau}}u_i. \tag{6.1}$$

Then the Lax operator L^G should satisfy (5.16). It takes the form:

$$\begin{aligned} L^G_{ij} &= -\frac{\delta_{ij}}{2\pi\sqrt{-1}}\left(\frac{v_i}{2} + \sum_\alpha p^\alpha_{ii}\left(2\pi\sqrt{-1}\frac{z-\bar{z}}{\tau-\bar{\tau}} + E_1(z-w_\alpha)\right)\right) \\ &\quad - \frac{1-\delta_{ij}}{2\pi\sqrt{-1}}\sum_\alpha p^\alpha_{ij}\mathbf{e}\left(\frac{z-w_\alpha-(\bar{z}-\bar{w}_\alpha)}{\tau-\bar{\tau}}u_{ij}\right)\phi(u_{ij}, z-w_\alpha),\ (u_{ij} = u_i - u_j). \end{aligned} \tag{6.2}$$

By the quasiperiodic gauge transform

$$f = \mathrm{diag}\{\mathbf{e}(\frac{z-\bar{z}}{\tau-\bar{\tau}}u_i)\}, \tag{6.3}$$

one comes to the holomorphic quasiperiodic Lax operator

$$l^G_{ij}(z) = -\frac{\delta_{ij}}{2\pi\sqrt{-1}}\left(\frac{v_i}{2} + \sum_\alpha p^\alpha_{ii}E_1(z-w_\alpha)\right) - \frac{1-\delta_{ij}}{2\pi\sqrt{-1}}\sum_\alpha p^\alpha_{ij}\phi(u_{ij}, z-w_\alpha). \tag{6.4}$$

Reducing the moment map equation to the diagonal gives the additional constraint

$$\frac{1}{2\pi\sqrt{-1}}\sum_\alpha p^\alpha_{ii} = \partial_x u_i. \tag{6.5}$$

6.2. $\hat{L}(SL(2,\mathbb{C}))$-bundles over elliptic curves with marked points. In this subsection we study 2-body elliptic Calogero field theory in detail.

The operator L. According to (6.4) the holomorphic Lax operator is

$$\begin{cases} l_{11}^G = -\frac{v}{4\pi\sqrt{-1}} - \sum_\alpha \frac{p_{11}^\alpha}{2\pi\sqrt{-1}} E_1(z - w_\alpha), \\ l_{12}^G = -\sum_\alpha \frac{p_{12}^\alpha}{2\pi\sqrt{-1}} \phi(2u, z - w_\alpha), \\ l_{21}^G = -\sum_\alpha \frac{p_{21}^\alpha}{2\pi\sqrt{-1}} \phi(-2u, z - w_\alpha), \end{cases} \tag{6.6}$$

with the additional constraint (6.5)

$$\frac{1}{2\pi\sqrt{-1}} \sum_\alpha p_{11}^\alpha = u_x. \tag{6.7}$$

We still have the freedom to fix the gauge with respect to the action of the diagonal subgroup. The corresponding moment map is (6.7).

For the one marked point $w_1 = 0$ the corresponding orbit is

$$p = 2\pi\sqrt{-1} \begin{pmatrix} u_x & -\nu \\ -\nu & -u_x \end{pmatrix}, \tag{6.8}$$

where ν =const. is the result of the gauge fixing. In this case the Lax operator is a 2d generalization of the Lax operator for the two-body CM model:

$$L_{2D}^{CM} = \begin{pmatrix} -\frac{1}{4\pi\sqrt{-1}} v - u_x E_1(z) & \nu\phi(2u, z) \\ \nu\phi(-2u, z) & \frac{1}{4\pi\sqrt{-1}} v + u_x E_1(z) \end{pmatrix}. \tag{6.9}$$

This operator is still periodic under the shift $z \to z + 1$ and

$$L_{2D}^{CM}(z + \tau) = \mathbf{e}(u) L_{2D}^{CM}(z) \mathbf{e}(-u) + \mathbf{e}(u) \partial_x \mathbf{e}(-u),$$

where $\mathbf{e}(u) = \mathrm{diag}(\exp u, \exp -u)$.

Hamiltonians for the 2d elliptic $sl(2, \mathbb{C})$ CM model. In this case the coefficients T_k are (see (5.43)–(5.48)):

$$\begin{cases} T_{-2}^{CM} = u_x^2 + \nu^2 = h \\ T_{-1}^{CM} = 2\frac{v}{4\pi\sqrt{-1}} u_x - \frac{\nu_x}{\nu} u_x + u_{xx} \\ T_0^{CM} = -\frac{v^2}{16\pi^2} + (2u_x^2 - \nu^2)\wp(2u) - \frac{v}{4\pi\sqrt{-1}} \frac{\nu_x}{\nu} + \frac{1}{4}(\frac{\nu_x}{\nu})^2 \end{cases}, \tag{6.10}$$

where h is the Casimir function, fixing the coadjoint orbit at the marked point. It can be chosen as a constant. Thus, we have

$$\nu^2 = h - u_x^2.$$

The next order Hamiltonian is quadratic

$$H_{-1}^{CM} = \oint \frac{v}{2\pi\sqrt{-1}} u_x - \frac{\nu_x}{\nu} u_x. \tag{6.11}$$

It can be written in the following way:

$$H_{-1}^{CM} = \oint \frac{v}{2\pi\sqrt{-1}} u_x + \frac{u_{xx} h}{\nu^2}. \tag{6.12}$$

Since $\{\oint dx \frac{u_{xx}}{\nu^2}, v(y)\} = 0$, the equations of motion are:

$$\begin{cases} u_t = \frac{1}{2\pi\sqrt{-1}} u_x, \\ v_t = \frac{1}{2\pi\sqrt{-1}} v_x. \end{cases} \tag{6.13}$$

Note that the L-M pair is simple in this case: $M = \frac{1}{2\pi\sqrt{-1}} L$.

The first nontrivial Hamiltonian H_0 is quadratic in the momenta field v. It is a two-dimensional generalization of the quadratic CM Hamiltonian

$$H_0^{CM} = \oint dx 2\sqrt{h}\chi_1 = \oint dx \left(T_0 - \frac{1}{4h} T_{-1}^2 \right). \tag{6.14}$$

A direct evaluation yields:

$$T_0^{CM} - \frac{1}{4h}(T_{-1}^{CM})^2 = -\frac{v^2}{16\pi^2}\left(1 - \frac{u_x^2}{h}\right) + (3u_x^2 - h)\wp(2u) - \frac{u_{xx}^2}{4\nu^2}. \tag{6.15}$$

The equations of motion produced by H_0^{CM} are:

$$u_t = -\frac{v}{8\pi^2}\left(1 - \frac{u_x^2}{h}\right), \tag{6.16}$$

$$v_t = \frac{1}{8\pi^2 h}\partial_x(v^2 u_x) - 2(3u_x^2 - h)\wp'(2u) + 6\partial_x(u_x\wp(2u)) + \frac{1}{2}\partial_x\left(\frac{u_{xxx}\nu - \nu_x u_{xx}}{\nu^3}\right).$$

It is reduced to the two-body elliptic CM system for the x-independent fields.

The L-M pair for the 2d elliptic $sl(2,\mathbb{C})$ CM model. The equations of motion (6.16) produced by the quadratic Hamiltonian H_0^{CM} can be represented in a form of the Zakharov-Shabat equation with the L matrix defined by (6.9) and the M matrix given as follows:

$$\begin{cases} M_{11} = -u_t E_1(z) - \frac{1}{4\pi\sqrt{-1}}\left(\frac{1}{8\pi^2 h}v^2 u_x + 6u_x\wp(2u) + \frac{u_{xxx}\nu - \nu_x u_{xx}}{2\nu^3}\right) \\ \qquad + \frac{u_x}{2\pi\sqrt{-1}}(E_2(2u) - E_2(z)), \\ M_{12} = -\frac{\nu}{2\pi\sqrt{-1}}\phi'(2u,z) + \left(\frac{\nu}{2\pi\sqrt{-1}}E_1(z) + \frac{v u_x \nu}{8\pi^2 h} - \frac{1}{4\pi\sqrt{-1}}\frac{u_{xx}}{\nu}\right)\phi(2u,z), \\ M_{21} = -\frac{\nu}{2\pi\sqrt{-1}}\phi'(-2u,z) + \left(\frac{\nu}{2\pi\sqrt{-1}}E_1(z) + \frac{v u_x \nu}{8\pi^2 h} + \frac{1}{4\pi\sqrt{-1}}\frac{u_{xx}}{\nu}\right)\phi(-2u,z). \end{cases} \tag{6.17}$$

See Appendix C for details of the proof. This construction completes the description of right vertical arrow in Fig.1.

2d CM - LL correspondence. The upper modification that produces the map of the elliptic CM system into the elliptic rotator (4.10), (4.12) works in the two-dimensional case as well.

The two-dimensional extension of the SL$(2,\mathbb{C})$-elliptic rotator is the Landau-Lifshitz (LL) equation

$$\partial_t \mathbf{S} = \frac{1}{2}[\mathbf{S}, J(\mathbf{S})] + \frac{1}{2}[\mathbf{S}, \partial_{xx}\mathbf{S}]. \tag{6.18}$$

This equation can be fitted in the Zakharov-Shabat form [32]. The Lax operator L^{LL} has the same form as for the SL(2, $\mathbb{C}$) elliptic rotator L^{rot} (4.16). For sl(2, $\mathbb{C}$) the basis of the sigma matrices coincides with the basis of the sin-algebra and L^{LL} takes the form

$$L = \sum_a u_a(z) S_a \sigma_a,$$

$$u_1 = \varphi \begin{bmatrix} 0 \\ 1 \end{bmatrix}(z),\ u_2 = \varphi \begin{bmatrix} 1 \\ 1 \end{bmatrix}(z),\ u_3 = \varphi \begin{bmatrix} 1 \\ 0 \end{bmatrix}(z).$$

The M^{LL} operator has a very simple extension

$$M^{LL} = M^{rot} - L^{rot} E_1(z) + \sum_a u_a(z) \mathrm{tr}(\sigma_a [\mathbf{S}, \partial_x \mathbf{S}]) \sigma_a.$$

It is easy to check that the Zakharov-Shabat equation leads to (6.18) if

$$\sum_a S_a^2 = 1.$$

Thereby we have defined the right vertical arrow in Fig.1.

Consider the upper modification Ξ_{2D} that has the same quasi-periodicity as Ξ but corresponds to the residue p (6.8) of L_{2D}^{CM} (6.9). Then the Lax operator for the LL system is the result of the upper modification

$$L^{LL} = \Xi_{2D} \partial_x \Xi_{2D}^{-1} + \Xi_{2D} L_{2D}^{CM} \Xi_{2D}^{-1}. \tag{6.19}$$

It means that we can pass from the CM fields $v(x,t), u(x,t)$ and the constant ν to the LL fields $\mathbf{S} = (S_1, S_2, S_3)$ with the orbit fixing condition

$$\sum_a S_a^2 = -\frac{1}{2\pi^2}(u_x^2 + \nu^2) = 1.$$

It completes the description of the diagram on Fig.1.

Relations with the Sinh-G equation and the nonlinear Schrödinger equation. It is known that the LL model is universal; it contains as a special limit the Sinh-Gordon and the Nonlinear Schrödinger models [3]. In this way they can be derived within the 2d CM system.

The scaling limit in the CM model is a combination of the trigonometric limit $Im\tau \to \infty$ with shifts of coordinates: $u = U + \frac{1}{2} Im\tau$ and renormalization of the coupling constant $\nu = \bar{\nu} e^{\frac{1}{2} Im\tau}$ [31]. This procedure applied to the 2d elliptic CM Hamiltonian yields the sinh-Gordon system:

$$H_{SG} = -\frac{v^2}{16\pi^2} - \bar{\nu}^2 (e^{2U} + e^{-2U}) + \frac{U_x^2}{4}. \tag{6.20}$$

The equations of motion are:

$$\begin{cases} U_t = -\frac{v}{8\pi^2}, \\ v_t = 2\bar{\nu}^2 (e^{2U} - e^{-2U}) + \frac{1}{2} U_{xx}. \end{cases} \tag{6.21}$$

The L-M pair is:

$$L^{SG} = \begin{pmatrix} -\frac{v}{4\pi\sqrt{-1}} - \frac{1}{2}U_x & \bar{v}(1 - e^{2U}Z) \\ \bar{v}(\frac{1}{Z} - e^{-2U}) & \frac{v}{4\pi\sqrt{-1}} + \frac{1}{2}U_x \end{pmatrix}, \tag{6.22}$$

$$M^{SG} = \begin{pmatrix} -\frac{U_t}{2} - \frac{1}{8\pi\sqrt{-1}}U_x & \frac{\bar{v}}{4\pi\sqrt{-1}}(1 + e^{2U}Z) \\ \frac{\bar{v}}{4\pi\sqrt{-1}}(e^{-2U} + \frac{1}{Z}) & \frac{U_t}{2} + \frac{1}{8\pi\sqrt{-1}}U_x \end{pmatrix}. \tag{6.23}$$

Let us consider $2d$ CM theory for $N = 2$ in the rational limit when the both periods of the basic spectral curve go to infinity. The upper modification (6.19) transforms this system in the Heisenberg magnet. Then using the non-singular gauge transform from Ref. [3] we come to the nonlinear Schrödinger equation.

6.3. Hamiltonians for the 2d elliptic Gaudin model. Using (B.28) we obtain the Hamiltonian:

$$\begin{aligned} H^G_{-1,a} &= 2\frac{v}{4\pi\sqrt{-1}}\frac{p^a_{11}}{2\pi\sqrt{-1}} + 2\sum_b \frac{p^a_{11}}{2\pi\sqrt{-1}}\frac{p^b_{11}}{2\pi\sqrt{-1}}E_1(z_a - z_b) \\ &\quad -\sum_{a\neq b}\frac{p^a_{12}p^b_{21}}{(2\pi\sqrt{-1})^2}\phi(2u, z_b - z_a) + \sum_{a\neq b}\frac{p^b_{12}p^a_{21}}{(2\pi\sqrt{-1})^2}\phi(2u, z_a - z_b) \\ &\quad -\frac{p^a_{11}}{2\pi\sqrt{-1}}\frac{\partial_x p^a_{12}}{p_{12}}. \end{aligned} \tag{6.24}$$

The last term makes the above Hamiltonian different from the one-dimensional version.

Let us consider the $\mathrm{sl}(2, \mathbb{C})$ case with two marked points on the elliptic curve.

We will use the following notations:

$$\begin{cases} p^1_{11} = 2\pi\sqrt{-1}\gamma_1, \quad p^2_{11} = 2\pi\sqrt{-1}\gamma_2, \\ p^1_{12} = -2\pi\sqrt{-1}\nu_+, \quad p^1_{21} = -2\pi\sqrt{-1}\nu_-, \\ p^2_{12} = -2\pi\sqrt{-1}\mu_+, \quad p^2_{21} = -2\pi\sqrt{-1}\mu_-. \end{cases} \tag{6.25}$$

The L matrix is:

$$\begin{cases} l^G_{11} = -\frac{v}{4\pi\sqrt{-1}} - \gamma_1 E_1(z - z_1) - \gamma_2 E_1(z - z_2), \\ l^G_{12} = \nu\phi(2u, z - z_1) + \mu_+\phi(2u, z - z_2), \\ l^G_{21} = \nu\phi(-2u, z - z_1) + \mu_-\phi(-2u, z - z_2). \end{cases} \tag{6.26}$$

The solution exists if

$$\gamma_1 + \gamma_2 = u_x. \tag{6.27}$$

The gauge fixing condition is chosen to be

$$\nu_+ = \nu_- = \nu. \tag{6.28}$$

We fix the Casimir elements $h_1 = \gamma_1^2 + \nu^2$ and $h_2 = \gamma_2^2 + \mu_+\mu_-$ to be constants: $h_1,\ h_2 \in \mathbb{C}$.

On the reduced phase space there are two independent fields besides u and v. Let them be for example ν and μ_+, then

$$\begin{cases} \gamma_1 = \sqrt{h_1 - \nu^2}, \\ \gamma_2 = u_x - \sqrt{h_1 - \nu^2}, \\ \mu_- = \frac{1}{\mu_+}(h_2 - (u_x - \sqrt{h_1 - \nu^2})^2). \end{cases} \tag{6.29}$$

However we are going to use all kinds of variables in order to make the formulae more transparent. The non-trivial brackets on the reduced phase space are:

$$\begin{gathered} \{v(x), u(y)\} = \delta(x-y),\ \ \{v(x), \gamma_1(y)\} = -\delta'(x-y),\ \ \{v(x), \nu(y)\} = \tfrac{\gamma_1}{\nu}\delta'(x-y), \\ \{\mu_+(x), \gamma_1(y)\} = -\tfrac{1}{2\pi\sqrt{-1}}\mu_+\delta(x-y),\ \ \{\mu_+(x), \mu_-(y)\} = -2\tfrac{1}{2\pi\sqrt{-1}}\gamma_2\delta(x-y), \\ \{\mu_+(x), \gamma_2(y)\} = \tfrac{1}{2\pi\sqrt{-1}}\mu_+\delta(x-y), \quad \{\mu_+(x), \nu(y)\} = \tfrac{1}{2\pi\sqrt{-1}}\tfrac{\gamma_1}{\nu}\mu_+\delta(x-y), \\ \{\nu(x), \mu_-(y)\} = \tfrac{1}{2\pi\sqrt{-1}}\tfrac{\gamma_1}{\nu}\mu_-\delta(x-y). \end{gathered} \tag{6.30}$$

The Hamiltonian is:

$$\begin{aligned} H_{-1}^G = \oint dx \Bigg(& 2\gamma_1 \frac{v}{4\pi\sqrt{-1}} - \gamma_1 \frac{\nu_x}{\nu} + \partial_x\gamma_1 + \nu\mu_+\phi(2u, z_1 - z_2) \\ & - \nu\mu_-\phi(2u, z_2 - z_1) + -2\gamma_1\gamma_2 E_1(z_1 - z_2)\Bigg). \end{aligned} \tag{6.31}$$

The equations of motion are:

$$\begin{cases} \partial_t u(x) = \dfrac{1}{2\pi\sqrt{-1}}\gamma_1(x), \\ \partial_t v(x) = \dfrac{1}{2\pi\sqrt{-1}}v_x - \partial_x(\dfrac{\gamma_1\mu_+}{\nu}\phi(2u, z_1 - z_2)) + \partial_x(\dfrac{\gamma_1\mu_-}{\nu}\phi(2u, z_2 - z_1)) \\ \qquad - 2\nu\mu_+\phi'(2u, z_1 - z_2) + 2\nu\mu_-\phi'(2u, z_2 - z_1) - \partial_x\left(\frac{\gamma_1\partial_x\gamma_1}{\nu^2}\right), \\ \partial_t\nu = -\frac{1}{2\pi\sqrt{-1}}\partial_x\left(\frac{\gamma_1^2}{\nu}\right) + \frac{\gamma_1}{2\pi\sqrt{-1}\nu}(\mu_+\phi(2u, z_1 - z_2) - \mu_-\phi(2u, z_2 - z_1)), \\ \partial_t\mu_+ = \dfrac{1}{2\pi\sqrt{-1}}\left(2\dfrac{v}{4\pi\sqrt{-1}}\mu_+ 2\mu_+(\gamma_2 - \gamma_1)E_1(z_1 - z_2)\right) \\ \qquad - \dfrac{2\nu\gamma_2}{2\pi\sqrt{-1}}\phi(2u, z_2 - z_1) \\ \qquad - \dfrac{\gamma_1\mu_+}{2\pi\sqrt{-1}\nu}(\mu_+\phi(2u, z_1 - z_2) - \mu_-\phi(2u, z_2 - z_1)). \end{cases} \tag{6.32}$$

The quadratic Hamiltonian is the direct generalization of (6.15)

$$H_0^G = \oint dx 2\sqrt{h_1}\chi_1 = \oint dx \left(T_0 - \frac{1}{4h_1}T_{-1}^2\right), \tag{6.33}$$

where

$$
\begin{aligned}
2\sqrt{h_1}\chi_1 = & -\frac{\nu^2}{16\pi^2}(1-\frac{\gamma_1^2}{h_1}) + (2u_x\gamma_1 - \nu^2)\wp(2u) - \frac{(\partial_x\gamma_1)^2}{4\nu^2} + \mu_+\mu_-(E_2(z_1-z_2) \\
& - E_2(2u)) + 4\eta_1\gamma_1\gamma_2 + \nu\mu_-\phi(2u, z_2-z_1)(E_1(z_1-z_2) - E_1(2u) \\
& + E_1(2u+z_2-z_1)) - \nu\mu_+\phi(2u, z_1-z_2)(E_1(z_1-z_2)\gamma_2^2 E_1^2(z_1-z_2) \\
& + E_1(2u) - E_1(2u+z_1-z_2)) + 2\gamma_2\frac{\nu}{4\pi\sqrt{-1}}E_1(z_1-z_2) \\
& - \gamma_2\frac{\nu_x}{\nu}E_1(z_1-z_2) + \gamma_1\frac{\mu_+}{\nu^2}\phi(2u, z_1-z_2) - \gamma_1\phi(2u, z_1-z_2) \\
& \times\left[\frac{\partial_x\mu_+}{\nu} + 2u_x\frac{\mu_+}{\nu}(E_1(z_1-z_2+2u) - E_1(2u))\right] \\
& - \frac{1}{4h_1}(\nu\mu_+\phi(2u, z_1-z_2) - \nu\mu_+\phi(2u, z_2-z_1) + 2\gamma_1\gamma_2 E_1(z_1-z_2))^2 \\
& - \frac{1}{2h_1}(\nu\mu_+\phi(2u, z_1-z_2) - \nu\mu_+\phi(2u, z_2-z_1) \\
& + 2\gamma_1\gamma_2 E_1(z_1-z_2))\,(2\gamma_1\frac{\nu}{4\pi\sqrt{-1}} - \gamma_1\frac{\nu_x}{\nu} + \partial_x\gamma_1). \qquad (6.34)
\end{aligned}
$$

7. Conclusion

Here we briefly summarize the results of our analysis and discuss some unsolved related problems. The following two subjects were investigated in the paper.

(i) We have constructed symplectic maps between Hitchin systems related to holomorphic bundles of different degrees. It allowed us to construct the Bäcklund transformations in the Hitchin systems defined over Riemann curves with marked points. We applied the general scheme to the elliptic CM systems and constructed the symplectic map to an integrable $SL(N, \mathbb{C})$ Euler-Arnold top (the elliptic $SL(N, \mathbb{C})$-rotator). The open problem is to write down the explicit expressions for the spin variables in terms of the CM phase space for an arbitrary N as was done for the case $N = 2$ (4.34). It should help to construct the Bäcklund transformations for the CM systems explicitly, and more generally, to construct the generating function for them. The later can be considered as the integrable discrete time mapping [10].

(ii) We have proposed a generalization of the Hitchin approach to 2d integrable theories related to holomorphic bundles of infinite rank. The main example is the integrable two-dimensional version of the two-body elliptic CM system. The upper modification allows to define the symplectic map to the Landau-Lifshitz equation and to find, in principle, the Bäcklund transformations in the field theories.

It will be extremely interesting to find the 2d generalization of the $SL(N, \mathbb{C})$-rotator for $N > 2$ (the matrix LL equation).

There is another point of view on the 2d generalizations of the Hitchin systems. One can try to define them starting from holomorphic bundles over complex surfaces, that are fibrations over Riemann curves. In this case the spectral parameter lives on the base of the fibration, while the space variable lives on the fibers. It will be interesting to analyze, for example, the known solutions of the LL equation from this point of view.

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8. Appendix

8.1. Appendix A. Sin-Algebra.

$$\mathbf{e}(z) = \exp(2\pi\sqrt{-1}z), \tag{A.1}$$

$$Q = \mathrm{diag}(\mathbf{e}(1/N), \dots, \mathbf{e}(m/N), \dots, 1), \tag{A.2}$$

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{A.3}$$

$$E_{mn} = \mathbf{e}(\frac{mn}{2N})Q^m\Lambda^n, \ (m = 0, \dots, N-1, \ n = 0, \dots, N-1, (\mathrm{mod} N) \ m^2 + n^2 \neq 0) \tag{A.4}$$

is the basis in $\mathrm{sl}(N, \mathbb{C})$. The commutation relations in this basis take the form

$$[E_{sk}, E_{nj}] = 2\sqrt{-1}\sin\frac{\pi}{N}(kn - sj)E_{s+n,k+j}, \tag{A.5}$$

$$\mathrm{tr}(E_{sk}E_{nj}) = \delta_{s,-n}\delta_{k,-j}N. \tag{A.6}$$

8.2. Appendix B. Elliptic functions. We summarize the main formulae for elliptic functions, borrowed mainly from [33 and 34]. We assume that $q = \exp 2\pi i\tau$, where τ is the modular parameter of the elliptic curve E_τ.

The basic element is the theta function:

$$\begin{aligned}\vartheta(z|\tau) &= q^{\frac{1}{8}}\sum_{n\in\mathbf{Z}}(-1)^n e^{\pi i(n(n+1)\tau+2nz)} \\ &= q^{\frac{1}{8}}e^{-\frac{i\pi}{4}}(e^{i\pi z} - e^{-i\pi z})\prod_{n=1}^{\infty}(1-q^n)(1-q^n e^{2i\pi z})(1-q^n e^{-2i\pi z}).\end{aligned} \tag{B.1}$$

The Eisenstein functions.

$$E_1(z|\tau) = \partial_z \log\vartheta(z|\tau), \quad E_1(z|\tau) \sim \frac{1}{z} - 2\eta_1 z, \tag{B.2}$$

where

$$\eta_1(\tau) = \zeta\left(\frac{1}{2}\right) = \frac{3}{\pi^2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty\,\prime}\frac{1}{(m\tau+n)^2} = \frac{24}{2\pi i}\frac{\eta'(\tau)}{\eta(\tau)}, \tag{B.3}$$

where

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n>0}(1-q^n)$$

is the Dedekind function,

$$E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log \vartheta(z|\tau), \quad E_2(z|\tau) \sim \frac{1}{z^2} + 2\eta_1. \tag{B.4}$$

The next important function is

$$\phi(u,z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}. \tag{B.5}$$

It has a pole at $z = 0$ and

$$\phi(u,z) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \wp(u)) + \ldots, \tag{B.6}$$

and

$$\phi(u,z)^{-1}\partial_u \phi(u,z) = E_1(u+z) - E_1(u). \tag{B.7}$$

The following formula plays an important role in checking the zero curvature equation:

$$\phi''(u,z) = \phi(u,z)(E_2(z) - E_1^2(z) + 2E_1(z)(E_1(u+z) - E_1(u)) + 2E_2(u) - 6\eta_1). \tag{B.8}$$

It follows from:

$$(E_1(z) + E_1(u) - E_1(z+u))^2 = E_2(u) + E_2(z) + E_2(u+z) - 6\eta_1. \tag{B.9}$$

Relations to the Weierstrass functions.

$$\zeta(z|\tau) = E_1(z|\tau) + 2\eta_1(\tau)z, \tag{B.10}$$

$$\wp(z|\tau) = E_2(z|\tau) - 2\eta_1(\tau), \tag{B.11}$$

$$\phi(u,z) = \exp(-2\eta_1 uz)\frac{\sigma(u+z)}{\sigma(u)\sigma(z)}, \tag{B.12}$$

$$\phi(u,z)\phi(-u,z) = \wp(z) - \wp(u) = E_2(z) - E_2(u). \tag{B.13}$$

Particular values.

$$E_1\left(\frac{1}{2}\right) = 0, \quad E_1\left(\frac{\tau}{2}\right) = E_1\left(\frac{1+\tau}{2}\right) = -\pi\sqrt{-1}. \tag{B.14}$$

Series representations.

$$\begin{aligned} E_1(z|\tau) &= -2\pi i\left(\frac{1}{2} + \sum_{n\neq 0}\frac{e^{2\pi i z}}{1-q^n}\right) \\ &= -2\pi i\left(\sum_{n<0}\frac{1}{1-q^n e^{2\pi i z}} + \sum_{n\geq 0}\frac{q^n e^{2\pi i z}}{1-q^n e^{2\pi i z}} + \frac{1}{2}\right), \end{aligned} \tag{B.15}$$

$$E_2(z|\tau) = -4\pi^2 \sum_{n\in\mathbf{Z}}\frac{q^n e^{2\pi i z}}{(1-q^n e^{2\pi i z})^2}, \tag{B.16}$$

$$\phi(u,z) = 2\pi i \sum_{n\in\mathbf{Z}}\frac{e^{-2\pi i n z}}{1-q^n e^{-2\pi i u}}. \tag{B.17}$$

Parity.

$$\vartheta(-z) = -\vartheta(z), \tag{B.18}$$

$$E_1(-z) = -E_1(z), \tag{B.19}$$

$$E_2(-z) = E_2(z), \tag{B.20}$$

$$\phi(u, z) = \phi(z, u) = -\phi(-u, -z). \tag{B.21}$$

Quasi-periodicity.

$$\vartheta(z+1) = -\vartheta(z), \quad \vartheta(z+\tau) = -q^{-\frac{1}{2}} e^{-2\pi i z} \vartheta(z), \tag{B.22}$$

$$E_1(z+1) = E_1(z), \quad E_1(z+\tau) = E_1(z) - 2\pi i, \tag{B.23}$$

$$E_2(z+1) = E_2(z), \quad E_2(z+\tau) = E_2(z), \tag{B.24}$$

$$\phi(u+1, z) = \phi(u, z), \quad \phi(u+\tau, z) = e^{-2\pi i z} \phi(u, z). \tag{B.25}$$

Addition formula.

$$\phi(u, z)\partial_v \phi(v, z) - \phi(v, z)\partial_u \phi(u, z) = (E_2(v) - E_2(u))\phi(u+v, z), \tag{B.26}$$

or

$$\phi(u, z)\partial_v \phi(v, z) - \phi(v, z)\partial_u \phi(u, z) = (\wp(v) - \wp(u))\phi(u+v, z). \tag{B.27}$$

The proof of (B.26) is based on (B.6),(B.21), and (B.25). In fact, $\phi(u, z)$ satisfies more a general relation which follows from the Fay three-section formula

$$\phi(u_1, z_1)\phi(u_2, z_2) - \phi(u_1+u_2, z_1)\phi(u_2, z_2-z_1) - \phi(u_1+u_2, z_2)\phi(u_1, z_1-z_2) = 0. \tag{B.28}$$

A particular case of this formula is

$$\phi(u_1, z)\phi(u_2, z) - \phi(u_1+u_2, z)(E_1(u_1) + E_1(u_2)) + \partial_z \phi(u_1+u_2, z) = 0. \tag{B.29}$$

Integrals.

$$\int_{E_\tau} E_1(z|\tau) dz d\bar{z} = 0. \tag{B.30}$$

Theta functions with characteristics. For $a, b \in \mathbb{Q}$ put :

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{j\in\mathbb{Z}} \mathbf{e}\left((j+a)^2\frac{\tau}{2} + (j+a)(z+b)\right). \tag{B.31}$$

In particular, the function ϑ (B.1) is the theta function with a characteristic

$$\vartheta(x,\tau) = \theta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(x,\tau). \tag{B.32}$$

One has

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z+1,\tau) = \mathbf{e}(a)\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau), \tag{B.33}$$

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z+a'\tau,\tau) = \mathbf{e}\left(-a'^2\frac{\tau}{2} - a'(z+b)\right)\theta\begin{bmatrix} a+a' \\ b \end{bmatrix}(z,\tau), \tag{B.34}$$

$$\theta\begin{bmatrix} a+j \\ b \end{bmatrix}(z,\tau) = \theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau), \quad j\in\mathbb{Z}. \tag{B.35}$$

For simplicity we denote $\theta\begin{bmatrix} a/2 \\ b/2 \end{bmatrix} = \theta_{ab}$.

The following identities are useful for the upper modification procedure in sl$(2,\mathbb{C})$ case:

$$\begin{array}{c} \theta_{01}(x,\tau)\theta_{00}(y,\tau) + \theta_{01}(y,\tau)\theta_{00}(x,\tau) = 2\theta_{01}(x+y,2\tau)\theta_{01}(x-y,2\tau), \\ \theta_{01}(x,\tau)\theta_{00}(y,\tau) - \theta_{01}(y,\tau)\theta_{00}(x,\tau) = 2\vartheta(x+y,2\tau)\vartheta(x-y,2\tau), \\ \theta_{00}(x,\tau)\theta_{00}(y,\tau) + \theta_{01}(y,\tau)\theta_{01}(x,\tau) = 2\theta_{00}(x+y,2\tau)\theta_{00}(x-y,2\tau), \\ \theta_{00}(x,\tau)\theta_{00}(y,\tau) - \theta_{01}(y,\tau)\theta_{01}(x,\tau) = 2\theta_{10}(x+y,2\tau)\theta_{10}(x-y,2\tau); \end{array} \tag{B.36}$$

$$\begin{array}{c} 2\vartheta(x,2\tau)\theta_{01}(y,2\tau) = \vartheta\left(\frac{x+y}{2},\tau\right)\theta_{10}\left(\frac{x-y}{2},\tau\right) + \theta_{10}\left(\frac{x+y}{2},\tau\right)\vartheta\left(\frac{x-y}{2},\tau\right), \\ 2\theta_{00}(x,2\tau)\theta_{10}(y,2\tau) = \vartheta\left(\frac{x+y}{2},\tau\right)\vartheta\left(\frac{x-y}{2},\tau\right) + \theta_{10}\left(\frac{x+y}{2},\tau\right)\theta_{10}\left(\frac{x-y}{2},\tau\right), \\ 2\theta_{00}(x,2\tau)\theta_{00}(y,2\tau) = \theta_{00}\left(\frac{x+y}{2},\tau\right)\theta_{00}\left(\frac{x-y}{2},\tau\right) + \theta_{01}\left(\frac{x+y}{2},\tau\right)\theta_{01}\left(\frac{x-y}{2},\tau\right), \\ 2\theta_{10}(x,2\tau)\theta_{10}(y,2\tau) = \theta_{00}\left(\frac{x+y}{2},\tau\right)\theta_{00}\left(\frac{x-y}{2},\tau\right) - \theta_{01}\left(\frac{x+y}{2},\tau\right)\theta_{01}\left(\frac{x-y}{2},\tau\right). \end{array} \tag{B.37}$$

8.3. Appendix C: 2d sl$(2,\mathbb{C})$ Calogero L-M pair. The Zakharov-Shabat equations in sl$(2,\mathbb{C})$ case are:

$$\begin{cases} 11: \partial_t L_{11} - \partial_x M_{11} = M_{21}L_{12} - M_{12}L_{21}, \\ 12: \partial_t L_{12} - \partial_x M_{12} = 2L_{11}M_{12} - 2L_{12}M_{11}, \\ 21: \partial_t L_{21} - \partial_x M_{21} = 2M_{11}L_{21} - 2L_{11}M_{21}. \end{cases} \tag{C.1}$$

Let the non-diagonal terms in the M matrix be of the form:

$$\begin{cases} M_{12} = c(x)\Phi'(2u,z) + (f^1_{12}(x)E_1(z) + f^0_{12}(x))\Phi(2u,z), \\ M_{21} = c(x)\Phi'(-2u,z) + (f^1_{21}(x)E_1(z) + f^0_{21}(x))\Phi(-2u,z). \end{cases} \tag{C.2}$$

Then from the diagonal part of (C.1) we conclude:

$$M_{11} = -u_t E_1(z) + \alpha(x) + \triangle M_{11}, \tag{C.3}$$

where

$$\alpha(x) = -\frac{1}{4\pi\sqrt{-1}}\left(\frac{1}{8\pi^2 h}\nu^2 u_x + 6u_x\wp(2u) + \frac{u_{xxx}\nu - \nu_x u_{xx}}{2\nu^3}\right), \tag{C.4}$$

and $\triangle M_{11}$ will be defined in the following. It is supposed to be dependent on $E_2(2u)$ in order to cancel terms proportional to $E_2(2u)$ and $E_2'(2u)$ in (C.1).

Using formula (B.8) in the non-diagonal part of (C.1), we arrive at some conditions equivalent to cancellations of the terms proportional to functions $\xi(2u, z) = E_1(2u+z) - E(2u)$, $E_1(z)$, $E_1^2(z)$, $E_1(z)\xi(-2u, z)$:

$$(12) \quad \begin{cases} E_1(z)\xi(2u,z): \quad f_{12}^1 = -c, \\ E_1^2(z): \quad f_{12}^1 = -c, \\ \xi(2u,z): \quad 2\nu u_t - c_x - 2u_x f_{12}^0 = -2c\frac{\nu}{4\pi\sqrt{-1}}, \\ E_1(z): \quad -\partial_x f_{12}^1 = -2\frac{\nu}{4\pi\sqrt{-1}} f_{12}^1 - 2u_x f_{12}^0 + 2u_t\nu, \end{cases} \tag{C.5}$$

$$(21) \quad \begin{cases} E_1(z)\xi(-2u,z): \quad f_{12}^1 = -c, \\ E_1^2(z): \quad f_{12}^1 = -c, \\ \xi(-2u,z): \quad -2\nu u_t - c_x + 2u_x f_{21}^0 = 2c\frac{\nu}{4\pi\sqrt{-1}}, \\ E_1(z): \quad -\partial_x f_{21}^1 = 2\frac{\nu}{4\pi\sqrt{-1}} f_{12}^1 + 2u_x f_{21}^0 - 2u_t\nu. \end{cases} \tag{C.6}$$

Thus

$$\begin{cases} f_{12}^1 = f_{21}^1 = -c, \\ 2u_x f_{12}^0 = 2\nu u_t - c_x + 2c\frac{\nu}{4\pi\sqrt{-1}}, \\ 2u_x f_{21}^0 = 2\nu u_t + c_x + 2c\frac{\nu}{4\pi\sqrt{-1}}, \end{cases} \tag{C.7}$$

$$\begin{cases} f_+ = f_{21}^0 + f_{12}^0 = \frac{2}{u_x}(\nu u_t + c\frac{\nu}{4\pi\sqrt{-1}}), \\ f_- = f_{21}^0 - f_{12}^0 = \frac{c_x}{u_x}. \end{cases} \tag{C.8}$$

The remaining parts of the non-diagonal equations are:

$$\begin{cases} \nu_t + 12cu_x\eta_1 - \partial_x f_{12}^0 - 2cu_x E_2(z) - 4cu_x E_2(2u) \\ \qquad = -2\frac{\nu}{4\pi\sqrt{-1}} f_{12}^0 - 2\nu\alpha - 2\nu\triangle M_{11}, \\ -\nu_t + 12cu_x\eta_1 + \partial_x f_{21}^0 - 2cu_x E_2(z) - 4cu_x E_2(2u) \\ \qquad = -2\frac{\nu}{4\pi\sqrt{-1}} f_{21}^0 - 2\nu\alpha - 2\nu\triangle M_{11}. \end{cases} \tag{C.9}$$

Subtracting the above equations we have:

$$2\frac{u_x}{\nu}\partial_x u_t + \partial_x f_+ = -2\frac{\nu}{4\pi\sqrt{-1}} f_-. \tag{C.10}$$

Substituting f_+ and f_- from (C.8) into (C.10) we arrive at the equation for c:

$$\frac{u_x}{\nu}\partial_x u_t + \partial_x\left(\frac{1}{u_x}\left(\nu u_t + c\frac{\nu}{4\pi\sqrt{-1}}\right)\right) = -\frac{\nu}{4\pi\sqrt{-1}}\frac{c_x}{u_x}. \tag{C.11}$$

Now some concrete equations of motion should be used. For H_{-1}^{CM} this equation yields $c \sim \sqrt{\frac{u_x}{\nu}}$. However the coefficient of the proportionality appears to be equal to zero. For H_0^{CM} (6.15) we have $c = -\frac{\nu}{2\pi\sqrt{-1}}$.

References

1. Sokolov, V.V., Shabat, A.B.: Classification of integrable evolution equations. Soviet Sci. Rev. **C4**, 221–280 (1984).
   Mikhailov, A.V., Shabat, A.B., Yamilov, R.I.: The symmetry approach to classification of nonlinear equations. Complete list of integrable systems. Uspekhi Mat. Nauk **42**, 3–53 (1987)
   Fokas, A.S.: Symmetries and integrability. Stud.Appl.Math. **77**, 253–299 (1987)
2. Ruijsenaars, S.N.M.: Action-Angle maps and Scattering Theory for Some Finite-Dimensional Integrable systems. Commun. Math. Phys. **115**, 127–165 (1988)
   Fock, V.: In: *Geometry and Integrable Models*. P. Pyatov, S. Solodukhin (eds.), Singapore: World Scientific, 1995, p. 20
   Gorsky, A.: Integrable many-body problems from the field theories. Theor. Math. Phys. **103**, 681–700 (1995), hep-th/9410228
   Nekrasov, N.: *On a Duality in Calogero-Moser-Sutherland Systems*. hep-th/9707111
   Fock, V., Gorsky, A., Nekrasov, N., Rubtsov, V.: *Duality in integrable systems and gauge theories*. hep-th/9906235
   Mironov, A.: *Seiberg-Witten theory and duality in integrable systems*. hep-th/0011093
3. Takhtajan, L., Zakharov, V.: The equivalence the nonlinear Schrodinger equation and the Heizenberg ferromagnet. Theor. Math. Phys. **38**, 26–35 (1979)
4. Hitchin, N.: Stable bundles and integrable systems. Duke Math. J. **54**, 91–114 (1987)
5. Simpson, S.T.: Harmonic bundles on non-compact curves. J. of AMS **3**, 713–770 (1990)
6. Beauville, A.: *Systèmes hamiltoniens complètement intégrables associés aux surfaces* $K3$. Problems in the theory of surfaces and their classification (Cortona, 1988), Sympos. Math., XXXII, London: Academic Press, 1991, pp. 25–31
7. Markman, E.: Spectral curves and integrable systems. Comp. Math. **93**, 255–290 (1994)
8. Nekrasov, N.: Holomorphic bundles and many-body systems. Commun. Math. Phys. **180**, 587-604 (1996) hep-th/9503157
9. Enriquez, B., Rubtsov, V.: Hitchin systems, higher Gaudin operators and R-matrices. Math. Res. Lett. **3**, 343–357 (1996)
10. Veselov, A.P.: Integrable maps. Russ. Math. Surv. **46**, 1–51 (1991)
11. Arinkin, D., Lysenko, S.: *Isomorphisms between moduli spaces of SL(2)-bundles with connections on* $\mathbb{P}^1/\{x_1, \dots, x_4\}$. Preprint
    On the moduli spaces of SL(2)-bundles with connections on $\mathbb{P}^1/\{x_1, \dots, x_4\}$. Preprint
12. Okamoto, K.: Studies in the painlevé equations I. Sixth Painlevé equation PVI. Annali Mat. Pura Appl. **146**, 337–387 (1987)
13. Reyman, A.G., Semenov-Tyan-Shanskii, M.A.: Lie algebras and Lax equations with spectral parameter on elliptic curve. Notes from Sci. Seminar LOMI Vol. **150**, 104–118 (1986) (in Russian)
14. Vakulenko, V.: *Note on the Ruijsenaars-Schneider model*. math.QA/9909079
15. Krichever, I.: *Vector bundles and Lax equations on algebraic curves*. hep-th/0108110
16. Baxter, R.J.: Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. I. Ann. Phys. **76**, 1–24 (1973)
    Jimbo, M., Miwa, T., Okado, M.: Local state probabilities of solvable lattice models: An A^1_{n-1} family. Nucl. Phys. **B300, [FS22]**, 74–108 (1988)
    Faddeev, L., Takhtajan, L.: The quantum method of the inverse problem and the Heisenberg XYZ model. Uspekhi Mat. Nauk **34:5**, 11–68 (1979) (English transl.)
17. Hasegawa, K.: *Ruijsenaars' commuting difference operators as commuting transfer matrices*. q-alg/9512029
18. Felder, G.: *Conformal field theory and integrable systems associated to elliptic curves*. Proc. of the ICM 94, Boston-Basel, Birkhaeuser 1994, pp. 1247–1255, hep-th/9609153
19. Belavin, A.A.: Dynamical symmetry of integrable system. Nucl. Phys. **180 [FS2]**, 189–200 (1981)
20. Arutyunov, G., Chekhov, L., Frolov, S.: Quantum dynamical R-matrices. Amer. Math. Soc. Transl. **191**(2), 1–33 (1999)
21. Ruijsenaars, S.N.M.: Complete integrability of relativistic Calogero-Moser systems and elliptic function identities. Commun. Math. Phys. **110**, 191 (1987)
22. Suris, B.Yu.: *Elliptic Ruijsenaars-Schneider and Calogero-Moser hierarchies are governed by the same r-matrix*. solv-int/9603011
23. Mehta, V.B., Seshardi, C.S.: Moduli of vector bundles on curves with parabolic structures. Math. Annalen **248**, 205–239 (1980)
24. Levin, A., Olshanetsky, M.: Double coset construction of moduli space of holomorphic bundles and Hitchin systems. Commun. Math. Phys. **188**, 449–466 (1997)
25. Enriquez, B., Rubtsov, V.: *Hecke-Tyurin parametrization of the Hitchin and KZB systems*. math.AG/9911087

26. Kuznetsov, V., Sklyanin, E.: On Bcklund transformations for many-body systems. J. Phys. A **31**, 2241–2251 (1998)
27. Calogero, F.: J. Math. Phys. **12**, 419 (1971)
Moser, J.: Adv. Math. **16**, 197–220 (1975)
28. Gorsky, A., Nekrasov, N.: *Elliptic Calogero-Moser system from two dimensional current algebra*. hep-th/9401021
29. Nijhoff, F.W., Pang, G.D.: Discrete-time Calogero-Moser model and lattice KP equations. In: *Symmetries and Integrability of Difference Equations*. (Estérel, PQ, 1994), CRM Proc. Lecture Notes, **9**, Providence, RI: Amer. Math. Soc., 1996, pp. 253–264
30. Dubrovin, B., Matveev, V., Novikov, S.: *Nonlinear equations of KdV type and finite gap linear operators and abelian manifolds*. UMN (1976), **XXXI**, 1 (187) (in Russian)
31. Inozemtsev, V.I.: The finite toda lattices. Commun. Math. Phys. **121**, 629–638 (1989)
32. Sklyanin, E.: *On complete integrability of the Landau-Lifshitz equation*. Preprint LOMI E-3-79, 1979
Borovik, A.E., Robuk, V.N.: Linear pseudopotentials and conservation laws for the Landau-Lifshitz equation. Theor. Math. Phys. **46**, 371–381 (1981)
33. Weil, A.: *Elliptic functions according to Eisenstein and Kronecker*. Berlin-Heidelberg: Springer-Verlag, 1976
34. Mumford, D.: *Tata Lectures on Theta I, II*. Boston: Birkhäuser, 1983, 1984

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