

Periodic Monopoles with Singularities and $\mathcal{N} = 2$ Super-QCD

Sergey A. Cherkis¹, Anton Kapustin²

¹ TEP, UCLA Physics Department, Los Angeles, CA 90095-1547, USA.

E-mail: cherkis@physics.ucla.edu

² Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA. E-mail: kapustin@ias.edu

Received: 9 March 2001 / Accepted: 15 January 2002

Published online: 20 January 2003 – © Springer-Verlag 2003

Abstract: We study solutions of the Bogomolny equation on $\mathbb{R}^2 \times \mathbb{S}^1$ with prescribed singularities. We show that the Nahm transform establishes a one-to-one correspondence between such solutions and solutions of the Hitchin equations on a punctured cylinder with the eigenvalues of the Higgs field growing at infinity in a particular manner. The moduli spaces of solutions have natural hyperkähler metrics of a novel kind. We show that these metrics describe the quantum Coulomb branch of certain $\mathcal{N} = 2$ $d = 4$ supersymmetric gauge theories on $\mathbb{R}^3 \times \mathbb{S}^1$. The Coulomb branches of the corresponding uncompactified theories have been previously determined by E. Witten using the M-theory fivebrane. We show that the Seiberg-Witten curves of these theories are identical to the spectral curves associated to solutions of the Bogomolny equation on $\mathbb{R}^2 \times \mathbb{S}^1$. In particular, this allows us to rederive Witten's results without recourse to the M-theory fivebrane.

Contents

1. Introduction	2
2. Periodic Monopoles with Singularities	4
2.1 Periodic $U(2)$ monopoles with singularities	4
2.2 Periodic $SO(3)$ monopoles with singularities	6
2.3 Periodic $U(m)$ monopoles with singularities	8
3. $N = 2$ Gauge Theories Compactified on a Circle	10
3.1 The geometry of the Coulomb branch	10
3.2 String theory picture	11
3.3 Periodic $U(m)$ monopoles and $N = 2$ Gauge theories	13
4. Nahm Transform	13
4.1 Direct Nahm transform	13
4.2 Reformulation of the Nahm transform	15
4.3 Monopole spectral data	16

4.4	Hitchin spectral data	16
4.5	Coincidence of the spectral data	17
4.6	Index computation	17
5.	Boundary Conditions for Hitchin Data	18
5.1	General remarks	18
5.2	The case $n_- < k = n_+$	19
5.3	The case $n_- < k > n_+$	23
5.4	The case $n_- < k < n_+$	24
5.5	The case $n_- = k = n_+$	24
6.	Inverse Nahm Transform	26
6.1	Construction of the monopole fields	26
6.2	Asymptotic behavior of the monopole fields	27
7.	Nahm Transform for Periodic $U(m)$ Monopoles	29
8.	Concluding Remarks	31
A.	Singularities of the Monopole Fields	32
	References	34

1. Introduction

Let X be an oriented Riemannian 3-manifold, E a unitary vector bundle on X , A a connection on E , and ϕ a Hermitian section of $\text{End}(E)$. The Bogomolny equation is a nonlinear differential equation

$$F_A = *d\phi. \quad (1)$$

In the case $X \cong \mathbb{R}^3$ with the standard metric, solutions of this equation have been extensively studied from many different viewpoints (see [1] and references therein).

In [5] we studied solutions of the Bogomolny equation for $\text{rank}(E) = 2$ and $X = \mathbb{R}^2 \times \mathbb{S}^1$ with a standard metric. The norm of the Higgs field was assumed to grow logarithmically at infinity. Such solutions were called periodic monopoles. They are topologically classified by a positive integer, the monopole charge. Using the Nahm transform, we showed that there is a one-to-one correspondence between periodic monopoles with charge k and solutions of the rank k Hitchin equations on a cylinder with a particular asymptotic behavior. The Hitchin equations are the dimensional reduction of the self-duality equation to two dimensions.

In this paper we study solutions of the Bogomolny equation on $\mathbb{R}^2 \times \mathbb{S}^1$ with n points deleted. The behavior of A , ϕ in the neighborhood of a deleted point is that of a Dirac monopole minimally embedded in the nonabelian gauge group. The eigenvalues of the Higgs field ϕ are allowed to grow logarithmically at infinity. Solutions of this kind will be called periodic monopoles with n singularities. We mostly deal with the case $\text{rank}(E) = 2$, but we also sketch how our results can be generalized to higher rank. For $\text{rank}(E) = 2$ periodic monopoles with n singularities are topologically classified by a single integer k which satisfies $2k \geq n$. We call this integer the monopole charge.

There are several reasons to study periodic monopoles with singularities. First of all, their moduli spaces carry natural hyperkähler metrics of a novel kind. For example, for $k = 2$ and $n = 4$ the centered moduli space is a smooth four-dimensional hyperkähler manifold with a distinguished complex structure. As a complex manifold it is isomorphic to a blow-up of $(\mathbb{C} \times S)/\mathbb{Z}_2$ at four points, where S is an elliptic curve, and \mathbb{Z}_2 acts by reflection on \mathbb{C} and by a nontrivial element of $\text{Aut}(S)$ on S . The four blown-up points

are the fixed points of the \mathbb{Z}_2 action. This complex manifold is a noncompact analogue of the Kummer surface. It can be argued that the natural hyperkähler metric on it is complete, nondegenerate, and asymptotically locally flat. For a fascinating introduction to noncompact approximations to K3 metrics see [15].

From the physical point of view, these moduli spaces are interesting because they provide exact low-energy effective actions for $N = 2$ $d = 4$ gauge theories compactified on a circle. “Exact” here means that both perturbative and non-perturbative quantum corrections are included. While the effective action of $N = 2$ $d = 4$ gauge theories on \mathbb{R}^4 can be computed by a variety of methods [27, 31, 21, 2], the analogous problem on $\mathbb{R}^3 \times \mathbb{S}^1$ remained intractable so far. The main reason is the necessity to sum over an infinite number of instanton contributions, including virtual BPS monopoles wrapping \mathbb{S}^1 . In our previous paper [5] we explained how the moduli space of periodic monopoles can be used to solve this problem in the case of $N = 2$ Yang-Mills theory without matter. Periodic monopoles with singularities allow one to solve $N = 2$ gauge theories with gauge group $SU(k)$ and matter in the fundamental representation.

Finally, studying periodic monopoles with singularities provides a new example of the Nahm transform, which is a differential-geometric analogue of the Fourier-Mukai transform. The Bogomolny equation is a reduction of the self-duality equation to three dimensions. In general, the Nahm transform maps a solution of the former equation into a solution of some different system of equations, which is also a reduction of the self-duality equation. The precise form of this new system of equations depends on the boundary conditions imposed on the Bogomolny equation. For example, the Nahm transform takes monopoles on \mathbb{R}^3 with finite energy to solutions of the so-called Nahm equations, which are the reduction of the self-duality equation to one dimension. Periodic monopoles without singularities are mapped to solutions of the Hitchin equations on a cylinder [5]. We will see that Nahm transform establishes a one-to-one correspondence between periodic monopoles with singularities and solutions of the Hitchin equations on a cylinder with singularities.

The singularities of the Hitchin data on the cylinder are so-called tame singularities. Solutions of the Hitchin equations on compact curves with such singularities were previously studied by C. Simpson [29] and others. Since the Hitchin equations are conformally-invariant, one may also wish to compactify the cylinder to a \mathbb{P}^1 by adding two points at infinity. For general k and n the singularities of the Hitchin data at the two added points are not tame. Nevertheless, it appears that the Hitchin-Kobayashi correspondence, if properly understood, continues to hold in this situation. It would be very interesting to understand this issue in detail.

There is one special situation ($2k = n$) where all four singularities of the Hitchin data on \mathbb{P}^1 are tame. In this case the rank of the Hitchin data is k . In particular, if we want the moduli space of the Hitchin data to be the noncompact Kummer surface mentioned above, we have to set $k = 2$ and consider the rank-two Hitchin equations on \mathbb{P}^1 with four tame singularities. This moduli space can be reinterpreted as the centered moduli space of two periodic monopoles with four Dirac-type singularities.

Monopoles on \mathbb{R}^3 with Dirac-type singularities have been studied in [6, 7 and 8]. Their moduli spaces are asymptotically locally flat hyperkähler manifolds which can be used to solve $N = 4$ $d = 3$ gauge theories with matter. The present work can be viewed as an extension of both [6 and 5].

In this paper we explain the relation between periodic monopoles with singularities and $N = 2$ gauge theories, and study the Nahm transform. The properties of the moduli space will be explored in a forthcoming publication [9].

2. Periodic Monopoles with Singularities

2.1. Periodic $U(2)$ monopoles with singularities. In this section we give the precise definition of a periodic monopole with singularities. Let X be $(\mathbb{R}^2 \times \mathbb{S}^1) \setminus \{p_1, \dots, p_n\}$, where $p_i, i = 1, \dots, n$, are distinct points. We will parametrize \mathbb{S}^1 by $\chi \in \mathbb{R}/(2\pi\mathbb{Z})$ and $z \in \mathbb{C} \cong \mathbb{R}^2$. Consider a $U(2)$ bundle E on X . Its topological type is completely determined by n integers e_1, \dots, e_n , the values of the first Chern class of E on small 2-spheres surrounding the points p_1, \dots, p_n . We will assume that $e_i = \pm 1$ for all i .

Let us set $\phi_0(r) = -1/(2r)$. Let us define a $U(1)$ connection $A_0(x)$ on a line bundle on $\mathbb{R}^3 \setminus \{0\}$ by $dA_0 = *d\phi_0$. The first Chern class of this line bundle evaluated on a 2-sphere enclosing the origin is one.

Let $\phi_\infty(z, \chi) = \frac{\log|z|}{2\pi}$ be a function on $\mathcal{M} = (\mathbb{R}^2 \times \mathbb{S}^1) \setminus \{z = 0\}$. We cover \mathcal{M} with two coordinate patches, $U_0 = \{\arg z \neq \pi\}$, and $U_1 = \{\arg z \neq 0\}$, where $\arg z$ is assumed to take values in the interval $(-\pi, \pi]$. Let L be a unitary line bundle on \mathcal{M} with the following transition function between U_0 and U_1 :

$$g(z, \chi) = \begin{cases} 1, & \text{Im } z < 0 \\ e^{-i\chi}, & \text{Im } z > 0. \end{cases}$$

The first Chern class of this line bundle evaluated on any 2-torus of the form $|z| = \text{const}$ is one. We define a unitary connection on L by

$$A_\infty = \begin{cases} \frac{\arg z}{2\pi} d\chi, & \arg z \neq \pi, \\ \frac{\arg(-z) - \pi}{2\pi} d\chi, & \arg z \neq 0. \end{cases} \quad (2)$$

The connection A_∞ satisfies $dA_\infty = *d\phi_\infty$.

A periodic monopole on E is a solution of the Bogomolny equation such that the connection and the Higgs field behave as

$$\phi(x) \sim g_i(x) \begin{pmatrix} e_i \phi_0(r_i) & 0 \\ 0 & 0 \end{pmatrix} g_i(x)^{-1} + O(1), \quad (3)$$

$$d_A \phi(x) \sim g_i(x) \begin{pmatrix} e_i d\phi_0(r_i) & 0 \\ 0 & 0 \end{pmatrix} g_i(x)^{-1} + O(1), \quad (4)$$

$$A(x) \sim g_i(x) \begin{pmatrix} e_i A_0(x - x_i) & 0 \\ 0 & 0 \end{pmatrix} g_i(x)^{-1} + i g_i(x) d g_i(x)^{-1} + O(1), \quad (5)$$

near the i^{th} singularity (here r_i is the distance to the i^{th} singularity and $g_i(x)$ is a $U(2)$ -valued function), while at infinity their behavior is given by

$$2\pi\phi(x) \sim g(x) \text{diag} \left(2\pi\ell_1\phi_\infty + v_1 + \text{Re} \frac{\mu_1}{z}, 2\pi\ell_2\phi_\infty + v_2 + \text{Re} \frac{\mu_2}{z} \right) g(x)^{-1} + O\left(\frac{1}{|z|^2}\right), \quad (6)$$

$$2\pi d_A \phi(x) \sim g(x) \text{diag} \left(2\pi \ell_1 d\phi_\infty - \text{Re} \frac{\mu_1 dz}{z^2}, 2\pi \ell_2 d\phi_\infty - \text{Re} \frac{\mu_2 dz}{z^2} \right) g(x)^{-1} + O\left(\frac{1}{|z|^3}\right), \quad (7)$$

$$2\pi A(x) \sim g(x) \text{diag} \left(2\pi \ell_1 A_\infty(x) + \left[b_1 + \text{Im} \frac{\mu_1}{z} \right] d\chi + \alpha_1 d \arg z, \quad (8)$$

$$2\pi \ell_2 A_\infty(x) + \left[b_2 + \text{Im} \frac{\mu_2}{z} \right] d\chi + \alpha_2 d \arg z \right) g(x)^{-1} + 2\pi i g(x) dg(x)^{-1} + O\left(\frac{1}{|z|^2}\right). \quad (9)$$

Here $g(x)$ is a $U(2)$ -valued function, $\ell_i, v_i, b_i, \alpha_i \in \mathbb{R}, i = 1, 2, \mu_i \in \mathbb{C}, i = 1, 2$. The integers e_i will be referred to as the abelian charges of the periodic monopole. We can assume that $\ell_1 \geq \ell_2$ without loss of generality. If $\ell_1 = \ell_2$, we will require in addition that $v_1 > v_2$. Physically, this means that for large $|z|$ the $U(2)$ gauge symmetry is broken to $U(1) \times U(1)$ by the Higgs field.

The numbers α_1 and α_2 are not gauge-invariant: a gauge transformation may shift them by $2\pi m, m \in \mathbb{Z}$. Therefore we prefer to regard $\alpha_{1,2}$ as taking values in $\mathbb{R}/(2\pi\mathbb{Z})$.

From these formulas it is easy to see that $\ell_1 + \ell_2$ is the value of the first Chern class of E on any sufficiently large 2-torus enclosing all the singularities. Since this 2-torus is homologous to the union of n small 2-spheres surrounding the singularities, it follows that

$$\ell_1 + \ell_2 = \sum e_i.$$

Both ℓ_1 and ℓ_2 are integers. Indeed, the eigenvalues of the Higgs field outside a sufficiently large compact region are distinct, and therefore we can define a line subbundle of E associated with the largest eigenvalue of the Higgs field. The value of its Chern class on a large 2-torus is ℓ_1 , therefore ℓ_1 must be an integer. Hence ℓ_2 is also an integer.

There are also relations between the continuous parameters appearing in (3) and (6). If we denote by (z_i, χ_i) the coordinates of the point $p_i, i = 1, \dots, n$, then these relations read:

$$\mu_1 + \mu_2 = - \sum_{i=1}^n e_i z_i, \quad (10)$$

$$\alpha_1 + \alpha_2 = \sum_{i=1}^n e_i \chi_i. \quad (11)$$

The derivation of these relations is presented in the next subsection.

We define the nonabelian charge of a monopole to be

$$k = \frac{1}{2}(\ell_1 - \ell_2 + n).$$

It is easy to see that k is a positive integer; in fact, since $\ell_1 \geq \ell_2$, it satisfies

$$2k \geq n.$$

The asymptotic behavior of the Higgs field is completely fixed once we specify $e_i, i = 1, \dots, n$, and k .

Let n_{\pm} be the total number of singularities with $e_i = \pm 1$. By definition, $n_+ + n_- = n$. The integers ℓ_1, ℓ_2 which determine the behavior of the Higgs field at infinity can be expressed in terms of k, n_+ , and n_- :

$$\ell_1 = k - n_-, \quad \ell_2 = n_+ - k.$$

2.2. Periodic $SO(3)$ monopoles with singularities. A closely related problem is that of $SO(3)$ monopoles with singularities on $\mathbb{R}^2 \times \mathbb{S}^1$. These are solutions of the Bogomolny equation with traceless A and ϕ . The behavior of A, ϕ near the singularities is given by

$$\phi(x) \sim g_i(x) \begin{pmatrix} \frac{1}{2}\phi_0(r_i) & 0 \\ 0 & -\frac{1}{2}\phi_0(r_i) \end{pmatrix} g_i(x)^{-1} + O(1), \quad (12)$$

$$d_A\phi(x) \sim g_i(x) \begin{pmatrix} \frac{1}{2}d\phi_0(r_i) & 0 \\ 0 & -\frac{1}{2}d\phi_0(r_i) \end{pmatrix} g_i(x)^{-1} + O(1), \quad (13)$$

$$A(x) \sim g_i(x) \begin{pmatrix} \frac{1}{2}A_0(x - x_i) & 0 \\ 0 & -\frac{1}{2}A_0(x - x_i) \end{pmatrix} g_i(x)^{-1} + i g_i(x) d g_i(x)^{-1} + O(1), \quad (14)$$

where $g_i(x)$ are again $U(2)$ -valued functions. The behavior at infinity is given by

$$4\pi\phi(x) \sim g(x) \text{diag} \left(2\pi k_{\infty} \phi_{\infty} + v + \text{Re} \frac{\mu}{z}, -2\pi k_{\infty} \phi_{\infty} - v - \text{Re} \frac{\mu}{z} \right) g(x)^{-1} + O\left(\frac{1}{|z|^2}\right), \quad (15)$$

$$4\pi d_A\phi(x) \sim g(x) \text{diag} \left(2\pi k_{\infty} d\phi_{\infty} - \text{Re} \frac{\mu dz}{z^2}, -2\pi k_{\infty} d\phi_{\infty} + \text{Re} \frac{\mu dz}{z^2} \right) g(x)^{-1} + O\left(\frac{1}{|z|^3}\right), \quad (16)$$

$$4\pi A(x) \sim g(x) \text{diag} \left(2\pi k_{\infty} A_{\infty}(x) + \left[b + \text{Im} \frac{\mu}{z} \right] d\chi + \alpha d \arg z, -2\pi k_{\infty} A_{\infty}(x) - \left[b + \text{Im} \frac{\mu}{z} \right] d\chi - \alpha d \arg z \right) g(x)^{-1} + 4\pi i g(x) d g(x)^{-1} + O\left(\frac{1}{|z|^2}\right), \quad (17)$$

where $g(x)$ is a $U(2)$ -valued function, $k_{\infty}, v, b, \alpha \in \mathbb{R}, \mu \in \mathbb{C}$. We may assume that $k_{\infty} \geq 0$ without loss of generality. The number k_{∞} is, in fact, an integer, as it measures the value of the first Chern class of the eigenbundle corresponding to the positive eigenvalue of ϕ on a large 2-torus.

The second Stiefel-Whitney class of this $SO(3)$ bundle evaluated on a small 2-sphere surrounding the i^{th} singularity is 1, therefore it cannot be lifted to an $SU(2)$ bundle. The Stiefel-Whitney class of the bundle evaluated on a large 2-torus is $k_{\infty} \bmod 2$. Since the large 2-torus is homologous to the sum of the small 2-spheres surrounding the singularities, it follows that $k_{\infty} = n \bmod 2$, where n is the total number of singularities. We will define the nonabelian charge of an $SO(3)$ monopole to be $(k_{\infty} + n)/2$. In view of the above, the nonabelian charge is greater than or equal to $n/2$.

The relation between the $U(2)$ and $SO(3)$ periodic monopoles is the following. If we decompose $U(2)$ monopole fields A and ϕ into a trace part and a trace-free part,

$$A = A_{tr} + A_{tf}, \quad \phi = \phi_{tr} + \phi_{tf},$$

then (A_{tr}, ϕ_{tr}) and (A_{tf}, ϕ_{tf}) separately satisfy the Bogomolny equation. It is easy to see that the behavior of (A_{tf}, ϕ_{tf}) near the singularities is described by (12), while their behavior at infinity is described by (15) with

$$k_\infty = \ell_1 - \ell_2, \quad v = v_1 - v_2, \quad b = b_1 - b_2, \quad \mu = \mu_1 - \mu_2, \quad \alpha = \alpha_1 - \alpha_2.$$

Furthermore, (A_{tr}, ϕ_{tr}) represents n periodic Dirac monopoles and therefore obeys [5]

$$\begin{aligned} \phi_{tr} &= \frac{1}{2\pi}(v_1 + v_2) + \frac{1}{2} \sum_i e_i V(x - x_i) \sim \frac{1}{2\pi}(v_1 + v_2) + \frac{1}{2}(\ell_1 + \ell_2) \frac{\log |z|}{2\pi} \\ &\quad - \frac{1}{4\pi} \sum_i e_i \operatorname{Re} \frac{z_i}{z} + O\left(\frac{1}{|z|^2}\right), \end{aligned} \quad (18)$$

$$\begin{aligned} A_{tr} &\sim \frac{1}{2}(\ell_1 + \ell_2) A_\infty(x) + \frac{1}{4\pi} \left[b_1 + b_2 - \sum_i e_i \operatorname{Im} \frac{z_i}{z} \right] d\chi + \frac{1}{2}(\alpha_1 + \alpha_2) \frac{d \arg z}{2\pi} \\ &\quad + O\left(\frac{1}{|z|^2}\right). \end{aligned} \quad (19)$$

Here the function $V(x)$ is given by

$$\frac{\log(4\pi) - \gamma}{2\pi} - \frac{1}{2} \sum'_{p=-\infty}^{\infty} \left[\frac{1}{\sqrt{|z|^2 + (\chi - 2\pi p)^2}} - \frac{1}{2\pi|p|} \right],$$

where the prime means that for $p = 0$ the second term in the square brackets must be omitted, and γ is Euler's constant. Equation (18) implies that μ_1 and μ_2 cannot be chosen arbitrarily, but must satisfy

$$\mu_1 + \mu_2 = - \sum_{i=1}^n e_i z_i.$$

There is another important relation constraining the parameters of the periodic monopole. It relates the asymptotic parameters $\alpha_i, i = 1, 2$, in Eq. (6) and the positions $\chi_1, \chi_2, \dots, \chi_n$ of the singularities along the \mathbb{S}^1 . Consider the holonomy of A_{tr} along a circle $|z| = R, \chi = \chi_0$. For $R \rightarrow \infty$ the holonomy tends to

$$\frac{1}{2}(\alpha_1 + \alpha_2 - (l_1 + l_2)\chi_0).$$

This follows from Eq. (6) and the choice of trivialisation specified by Eq. (2). On the other hand, it is clear from symmetry considerations that the curvature $F_{z\bar{z}}$ constructed from A_{tr} vanishes identically on the plane $\chi = \chi_0$ if χ_0 is the χ -coordinate of the center-of-mass of the singularities, i.e. if

$$\chi_0 = \frac{1}{n} \sum e_i \chi_i. \quad (20)$$

By Stokes' theorem, the limiting holonomy of A_{I_r} must vanish for this value of χ_0 . This implies that

$$\alpha_1 + \alpha_2 = \sum e_i \chi_i. \quad (21)$$

To summarize, to any $U(2)$ periodic monopole with n singularities one can associate a $U(1)$ periodic monopole with n singularities and an $SU(2)/\mathbb{Z}_2 = SO(3)$ periodic monopole with n singularities. The nonabelian charge of the $SO(3)$ monopole is equal to the nonabelian charge of the $U(2)$ monopole. Moreover, it is easy to see that the induced map on the moduli spaces of $U(2)$ and $SO(3)$ monopoles is an isometry. This basically follows from the fact that a periodic $U(1)$ monopole is completely determined by e_i , the location of the singularities, and the asymptotics of the Higgs field at infinity, and therefore it has no moduli.

We just learned that the moduli space of periodic $U(2)$ monopoles depends only on the total number of singularities n and the nonabelian charge k , but not on the individual values of e_i . If we were only interested in the moduli space, we could have set all e_i to be one, for example. We prefer to keep e_i arbitrary, since the Nahm transform depends on the abelian charges in a nontrivial way.

2.3. Periodic $U(m)$ monopoles with singularities. One can similarly define periodic monopoles with singularities for other classical groups. As an example, we consider $U(m)$ periodic monopoles. The asymptotic behavior of the Higgs field and the connection at infinity is given by

$$\begin{aligned} 2\pi\phi(x) &\sim g(x)\text{diag}\left(2\pi\ell_1\phi_\infty(x) + v_1 + \text{Re}\frac{\mu_1}{z}, 2\pi\ell_2\phi_\infty(x) + v_2 + \text{Re}\frac{\mu_2}{z}, \right. \\ &\quad \left. \dots, 2\pi\ell_m\phi_\infty(x) + v_m + \text{Re}\frac{\mu_m}{z}\right)g(x)^{-1} + O\left(\frac{1}{|z|^2}\right), \quad (22) \\ 2\pi d_A\phi(x) &\sim g(x)\text{diag}\left(2\pi\ell_1 d\phi_\infty(x) - \text{Re}\frac{\mu_1 dz}{z^2}, 2\pi\ell_2 d\phi_\infty(x) - \text{Re}\frac{\mu_2 dz}{z^2}, \right. \\ &\quad \left. \dots, 2\pi\ell_m d\phi_\infty(x) - \text{Re}\frac{\mu_m dz}{z^2}\right)g(x)^{-1} + O\left(\frac{1}{|z|^3}\right), \\ 2\pi A(x) &\sim g(x)\text{diag}\left(2\pi\ell_1 A_\infty(x) + \left[b_1 + \text{Im}\frac{\mu_1}{z}\right]d\chi + \alpha_1 d\arg z, \right. \\ &\quad 2\pi\ell_2 A_\infty(x) + \left[b_2 + \text{Im}\frac{\mu_2}{z}\right]d\chi + \alpha_2 d\arg z, \dots, \\ &\quad \left. 2\pi\ell_m A_\infty(x) + \left[b_m + \text{Im}\frac{\mu_m}{z}\right]d\chi + \alpha_m d\arg z\right)g(x)^{-1} \\ &\quad + 2\pi i g(x)dg(x)^{-1} + O\left(\frac{1}{|z|^2}\right). \quad (23) \end{aligned}$$

We may assume that $\ell_1 \geq \ell_2 \geq \dots \geq \ell_m$. In addition we assume that if $\ell_i = \ell_{i+1}$, then $v_i > v_{i+1}$. The numbers ℓ_1, \dots, ℓ_m must be integers for $A(x)$ to be a well-defined connection. One way to see this is to note that since all of the eigenvalues of ϕ are

distinct for large enough $|z|$, there is a well-defined splitting of E into the eigenbundles of ϕ , and ℓ_1, \dots, ℓ_m are equal to the values of the first Chern class of these line bundles on a large 2-torus $|z| = \text{const}$.

The singularities at the points p_1, \dots, p_n are given by the Dirac monopole minimally embedded into the $U(m)$ gauge group:

$$\begin{aligned}\phi(x) &\sim g_i(x) \text{diag}(e_i \phi_0(r_i), 0, \dots, 0) g_i(x)^{-1} + O(1), \\ d_A \phi(x) &\sim g_i(x) \text{diag}(e_i d\phi_0(r_i), 0, \dots, 0) g_i(x)^{-1} + O(1), \\ A(x) &\sim g_i(x) \text{diag}(e_i A_0(x - x_i), 0, \dots, 0) g_i(x)^{-1} + i g_i(x) d g_i(x)^{-1} + O(1),\end{aligned}$$

where $e_i = \pm 1$. The first Chern class of E evaluated on a small 2-sphere surrounding the i^{th} singularity is equal to e_i . Since the first Chern class of E evaluated on a large 2-torus $|z| = \text{const}$ is equal to $\ell_1 + \dots + \ell_m$, we have the relation

$$\sum_{j=1}^m \ell_j = \sum_{i=1}^n e_i.$$

We define the nonabelian charge of a monopole to be a vector $(k_1, k_2, \dots, k_{m-1})$ with components

$$k_p = n_- + \sum_{i=1}^p \ell_i.$$

Clearly, the integers ℓ_j are completely determined by the abelian charges e_i and the nonabelian charge (k_1, \dots, k_{m-1}) .

Given a periodic $U(m)$ monopole with singularities, one can decompose its fields into a trace-free and a trace part which separately satisfy the Bogomolny equation. The trace part is completely determined by the abelian charges e_i . The traceless part defines a $U(m)/U(1)$ periodic monopole with singularities. To understand the nature of the singularities, recall that a vector bundle with the structure group $U(m)/U(1) = SU(m)/\mathbb{Z}_m$ has a characteristic class with values in $H^2(X, \mathbb{Z}_m)$ which generalizes the second Stiefel-Whitney class. This class measures the obstruction for lifting the $U(m)/U(1)$ bundle to an $SU(m)$ bundle. In the physics literature it is known as the t'Hooft magnetic flux. One can show that the value of the t'Hooft magnetic flux on a small 2-sphere surrounding the i^{th} singularity is equal to $e_i \pmod{m}$. Thus the $U(m)/U(1)$ bundle corresponding to a $U(m)$ monopole with singularities cannot be lifted to an $SU(m)$ bundle. Note that for $m > 2$ the singularity with $e_i = +1$ is distinct from $e_i = -1$ even after passing to traceless fields.

Conversely, given a periodic $U(m)/U(1)$ monopole with singularities, one can unambiguously reconstruct a periodic $U(m)$ monopole with singularities. This happens because the trace part, being a periodic $U(1)$ monopole, with singularities, is completely determined by e_i and the asymptotics at infinity.

As in the case of $U(2)$ monopoles, there are constraints between various continuous parameters appearing in (22) and (24), namely

$$\sum_{j=1}^m \mu_j = - \sum_{i=1}^n e_i z_i, \quad (24)$$

$$\sum_{j=1}^m \alpha_j = \sum_{i=1}^n e_i \chi_i. \quad (25)$$

These constraints can be derived by computing A_{tr} and ϕ_{tr} and comparing with the known expressions for a $U(1)$ periodic monopole with singularities.

3. $N = 2$ Gauge Theories Compactified on a Circle

In this section we explain the relevance of periodic monopoles with singularities for understanding quantum properties of supersymmetric gauge theories. Chalmers and Hanany [4] were the first to realize that the metric on the moduli space of certain supersymmetric gauge theories in three dimensions is identical to the metric on the moduli space of BPS monopoles. This relation was used to great effect by many authors, notably by Hanany and Witten [14]. Later on, this relation was extended to four-dimensional $N = 2$ gauge theories compactified on a circle of arbitrary radius R [11, 20, 19], and it was shown that the quantum moduli space of many interesting theories of this kind coincides with the moduli space of self-duality equations or their reductions. Thus a difficult quantum-mechanical problem can often be converted to a much simpler problem of studying the moduli space of certain partial differential equations. In particular, in the decompactification limit $R \rightarrow \infty$ one can recover all the results of Seiberg, Witten, and others on the moduli space of four-dimensional $N = 2$ gauge theories.

The precise form of the PDE one has to study depends on the gauge theory in question. For example, certain finite $N = 2$ gauge theories (the so-called quiver theories) are solved in terms of instantons on $\mathbb{R}^2 \times T^2$ [20, 19]. $N = 2$ super-Yang-Mills with gauge group $SU(k)$ and no hypermultiplets are solved in terms of monopoles on $\mathbb{R}^2 \times \mathbb{S}^1$ [5]. We will see below that periodic monopoles with n singularities and nonabelian charge k are relevant for $N = 2SU(k)$ gauge theory with n massive hypermultiplets.

3.1. The geometry of the Coulomb branch. Consider an $N = 2$ $SU(k)$ gauge theory in a generic vacuum on the Coulomb branch, where the expectation value of the Higgs field in the vector multiplet breaks the gauge group down to its maximal torus. The low-energy effective theory is described by $k - 1$ abelian vector multiplets which contain $k - 1$ complex scalars ϕ , $k - 1$ photons A , and $2(k - 1)$ Majorana fermions. Thus the moduli space of the theory is a $k - 1$ -dimensional complex manifold. $N = 2$ supersymmetry requires the metric on the moduli space to be a special Kähler metric.

Now consider compactifying the theory on a circle of radius R . At length scales larger than R the theory is effectively three-dimensional. Its bosonic fields include $k - 1$ complex scalars, $k - 1$ periodic real scalars originating from Wilson lines of the four-dimensional photons along the compactified direction, and $k - 1$ periodic real scalars obtained by dualizing $k - 1$ three-dimensional photons. All in all, the moduli space of the effective three-dimensional theory is $4(k - 1)$ -dimensional. Its metric is required to be hyperkähler by supersymmetry.

Far from the origin of the Coulomb branch, the metric can be found by first flowing to the infrared in the four-dimensional theory, and then dimensionally reducing on a circle. This is possible because the low-energy effective theory in four-dimensions is free, and

thus no renormalization group flow occurs upon compactification. The resulting moduli space is fibered over the moduli space of the four-dimensional theory by $2(k - 1)$ -dimensional tori. The metric on the fibers is flat, and thus the metric on the total space far along the Coulomb branch has a $U(1)^{2k-2}$ isometry [28, 19].

As one moves towards the origin of the Coulomb branch, the form of the metric starts to deviate from this simple form. In particular, while the four-dimensional instantons respect the $U(1)^{2k-2}$ isometry, the Euclidean BPS monopoles wrapping the compactified direction do not. These effects are exponentially small far along the Coulomb branch, but are very important near the origin. They tend to smooth out the singularities of the naive metric obtain by dimensional reduction.

From the above discussion it is clear that the asymptotic behavior of the metric on the moduli space of the compactified theory is determined by the four-dimensional physics alone. If the four-dimensional theory is asymptotically free, or finite, the metric on the moduli space is locally flat at infinity. For the $SU(k)$ gauge theory with n hypermultiplets this happens if $2k \geq n$. The interpretation of this restriction in terms of periodic monopoles will be explained below.

3.2. String theory picture. The relation between periodic monopoles and $N = 2$ gauge theories emerges if one embeds these gauge theories into string theory in a particular way, which we now explain.

$N = 2$ $SU(k)$ gauge theories can be realized in IIA string theory by suspending k D4-branes between two parallel NS5-branes. We shall assume that the NS5-branes' world-volume is along the 0, 1, 2, 3, 4, 5 directions, and their positions in the 7, 8, 9 directions coincide. The NS5-brane with smaller (resp. larger) x^6 coordinate will be called the left (resp. right) NS5-brane. The k D4-branes are infinite in the 0, 1, 2, 3 directions and span a finite interval in the 6 direction. The two boundaries of the D4-brane worldvolume lie on the NS5-branes. The direction 3 will be assumed to be periodic with period $2\pi R$.

	0	1	2	③	4	5	6	7	8	9
NS5	x	x	x	x	x	x				
D4	x	x	x	x			x			

The world-volume theory on the D4-branes reduces in the infrared limit to the $\mathcal{N} = 2$ $SU(k)$ Yang-Mills theory on $\mathbb{R}^3 \times \mathbb{S}^1$, where x^0, x^1, x^2 are affine coordinates on \mathbb{R}^3 and x^3 parametrizes \mathbb{S}^1 .

In order to obtain a theory with n fundamental hypermultiplets, one should add D4-branes parallel to the original k D4-branes but located outside the interval in x^6 where the latter are located. These D4-branes end on either left or right NS5-branes and extend to either $x^6 = -\infty$ or $x^6 = +\infty$. The two kinds of D4-branes will be called left and right semi-infinite D4-branes, respectively, and their numbers denoted n_L and n_R .

The world-volume theory on the k suspended D4-branes is now an $N = 2$ $SU(k)$ gauge theory with $n_L + n_R$ hypermultiplets in the fundamental representation. Their masses are given by $n_L + n_R$ complex numbers which parametrize the positions of the semi-infinite D4-branes in the 45 plane. Since the direction 3 is periodic with period $2\pi R$, the gauge theory is compactified on a circle of radius R .

In three dimensions the mass of the hypermultiplet is parametrized by three real numbers rather than by one complex one. The same is true about the four-dimensional theory on a circle, except that one of the three real mass parameters takes values in $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ rather than in \mathbb{R} . Indeed, each hypermultiplet is associated with a global $U(1)$

symmetry. Gauging this $U(1)$ symmetry and letting the Wilson line of the corresponding photon along the compactified direction to be non-zero gives an effective mass to the three-dimensional hypermultiplet. In the above string theory picture the global $U(1)$ is identified with the $U(1)$ gauge group of the semi-infinite D4-brane, and the extra mass parameter is associated with the possibility of turning on a Wilson line along x^3 for the corresponding photon.

To interpret this brane configuration in terms of periodic monopoles, we perform T-duality along x^3 . The resulting configuration in Type IIB string theory consists of k D3-branes suspended between two NS5-branes, and $n_L + n_R$ semi-infinite D3-brane ending on the NS5-branes.

	0	1	2	③	4	5	6	7	8	9
NS5	x	x	x	x	x	x				
D3	x	x	x					x		

In the limit of $R \rightarrow 0$ the T-dual circle decompactifies and we end up with the Chalmers-Hanany-Witten brane configuration. As pointed out in [4], in the worldvolume theory on the two NS5-branes the suspended D3-branes appear as monopoles in the $SU(2)$ subgroup of $U(2)$, while the semiinfinite D3-branes appear as Dirac $U(1)$ monopoles minimally embedded in $U(2)$. These monopoles live in the part of the NS5-branes' world-volume orthogonal to the D3-branes, namely in the 3, 4, 5 directions. In the remaining directions (0, 1, 2) of the NS5-brane worldvolume the gauge field configuration on the NS5-branes is translationally-invariant.

For $R \neq 0$ the only difference is that the direction 3 is compact. This means that suspended D3-branes and semi-infinite D3-branes are nonabelian and Dirac monopoles on $\mathbb{R}^2 \times S^1$, respectively. It can be checked that a left semi-infinite D3-brane corresponds to a Dirac monopole with $e_i = +1$, while a right semi-infinite D3-brane corresponds to a Dirac monopole with $e_i = -1$. Thus we can identify $n_L = n_+$, $n_R = n_-$.

As in [4], one can argue that the metric on the moduli space of the suspended D3-branes does not receive quantum corrections and thus is identical to the classical metric on the moduli space of k periodic monopoles with $n = n_L + n_R$ singularities. On the other hand, this same metric must be the metric on the Coulomb branch of the $N = 2$ $SU(k)$ gauge theory with n fundamental hypermultiplets.

Note that the monopole charge k is related to the rank of the gauge group of the $N = 2$ gauge theory. If we require the $N = 2$ gauge theory to be asymptotically free or finite, k must obey $2k \geq n$. In Sect. 2 we derived the same restriction on the monopole charge by analyzing the asymptotic behavior of the periodic monopole. One may wonder which assumption about monopoles corresponds to the requirement that the gauge theory be asymptotically free or finite. The answer is quite simple: we assumed that the periodic monopole configuration breaks the gauge group $U(2)$ down to $U(1) \times U(1)$ for large $|z|$. This is reflected in the fact that the difference of the eigenvalues of the Higgs field either goes to infinity or approaches a finite non-zero limit for $|z| \rightarrow \infty$. That is, the assumption of asymptotic freedom or finiteness is equivalent to the assumption of maximal symmetry breaking at infinity.

The string theory picture can also be used to anticipate the result of the Nahm transform applied to periodic monopoles. To this end we perform an S-duality and then a T-duality along the direction 3. The resulting brane configuration consists only of D4-branes.

	0	1	2	③	4	5	6	7	8	9
D4	x	x	x		x	x				
D4	x	x	x	x				x		

Such a configuration of D4-branes is described by the Hitchin equations on a cylinder parametrized by x^3 and x^6 [20, 19]. The gauge group is $SU(K)$, where the number K depends on n_{\pm} and k . We will see below that $K = \max(n_+, n_-, k)$.

3.3. Periodic $U(m)$ monopoles and $N = 2$ Gauge theories. The relation between periodic $U(2)$ monopoles with and without singularities and quantum $N = 2$ gauge theories can be extended to $U(m)$ monopoles. Consider the following gauge theory: the gauge group is $SU(k_1) \times SU(k_2) \times \cdots \times SU(k_{m-1})$, the matter consists of $m-2$ hypermultiplets in the representations

$$(k_1, \bar{k}_2, 1, \dots, 1), (1, k_2, \bar{k}_3, 1, \dots, 1), \dots, (1, \dots, 1, k_{m-2}, \bar{k}_{m-1}),$$

of the gauge group, n_- hypermultiplets in the representation $(\bar{k}_1, 1, \dots, 1)$, and n_+ hypermultiplets in the representation $(1, \dots, 1, k_{m-1})$. For $m = 2$ this gauge theory reduces to the one studied in the previous section. Its string theory realization consists of m parallel NS5-branes along 0, 1, 2, 3, 4, 5 separated in the x^6 direction, $m-1$ stacks of D4-branes suspended between the successive NS5-branes such that the j^{th} stack contains k_j D4-branes, n_- semi-infinite D4-branes ending on the right-most NS5-brane, and n_+ semi-infinite D4-branes ending on the left-most NS5-brane [31].

Performing T-duality along the x^3 direction converts all the D4-branes into D3-branes. The resulting brane configuration in Type IIB string theory is identical to the one considered by Hanany and Witten [14], except that the direction 3 is periodic. Since we have m NS5-branes, their worldvolume theory has gauge group $U(m)$. The above brane configuration is represented in this worldvolume theory by a BPS monopole with a nonabelian charge (k_1, \dots, k_{m-1}) with $n_+ + n_-$ Dirac-type singularities of which n_- have $e_i = -1$ and n_+ have $e_i = +1$ [14, 6, 8]. We conclude that the moduli space of a $U(m)$ periodic monopole with n_+ singularities with $e_i = +1$, n_- singularities with $e_i = -1$ and a nonabelian charge (k_1, \dots, k_{m-1}) is identical to the quantum Coulomb branch of the $N = 2$ gauge theory described above compactified on a circle.

Performing S-duality and then T-dualizing x^3 again yields a configuration consisting solely of D4-branes. Such configurations have been studied in [20, 19] in the case when the direction 6 is periodic and $k_1 = k_2 = \cdots = k_{m-1}$. The results of [20, 19] suggest that brane configurations of this type are described by the $SU(K)$ Hitchin equations on a cylinder, where K is some integer number. As explained below, the Hitchin equations are related to periodic monopoles by means of the Nahm transform. The integer K will be determined to be $\max(n_+, n_-, k_1, \dots, k_{m-1})$.

4. Nahm Transform

This section describes the Nahm transform for periodic $U(2)$ monopoles with singularities, as well as certain algebro-geometric data associated to monopoles.

4.1. Direct Nahm transform. Given a periodic monopole (A, ϕ) with singularities we define a family of Dirac-type operators parametrized by a point $(r, t) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ as follows. Let L be a line bundle over $\mathbb{R}^2 \times \mathbb{S}^1$ with a flat unitary connection $a = -td\chi$ whose only non-zero component is along \mathbb{S}^1 . The variable t takes values in the dual circle \mathbb{R}/\mathbb{Z} which we denote $\hat{\mathbb{S}}^1$. Let σ_i be Pauli matrices and let r be a real number. Let S be the trivial rank-two vector bundle on $\mathbb{R}^2 \times \mathbb{S}^1$. The Pauli matrices can be regarded

as morphisms $S \rightarrow S$. We define a first-order differential operator $D : S \otimes E \rightarrow S \otimes E$ by

$$D = \sigma \cdot d_{A+a} - (\phi - r). \quad (26)$$

With some abuse of terminology we shall refer to it as “the Dirac operator”, and to the bundle S as “the spin bundle”.

We shall use a multi-valued complex coordinate $s = r + it$ on $\mathbb{R} \times \hat{S}^1$. It is important to know for which values of s the operator D is Fredholm. If both ℓ_1 and ℓ_2 are non-zero, then it is easy to see that D is Fredholm for all $s \in \mathbb{R} \times \hat{S}^1$. If either ℓ_1 or ℓ_2 are zero, then one or both of the eigenvalues of the Higgs field stay finite for $|z| \rightarrow \infty$. Depending on whether $\ell_1 = 0$ or $\ell_2 = 0$, this eigenvalue is equal to v_1 or v_2 . In this case D can fail to be Fredholm only if $r = \text{Res}$ is equal to one of the finite eigenvalues, and at the same time $t = \text{Im}s$ is equal to b_1 or b_2 . This can be stated more concisely by introducing a non-unitary connection $A - i\phi d\chi$, and letting $V(z)$ be its holonomy along the χ direction at a point $z \in \mathbb{C}$. If both ℓ_1 and ℓ_2 are non-zero, the eigenvalues of $V(z)$ do not approach a finite limit as $z \rightarrow \infty$; if $\ell_1 = 0, \ell_2 \neq 0$, then one of the eigenvalues of $V(z)$ approaches a finite limit $w_1 = e^{v_1 + ib_1}$; if $\ell_1 \neq 0, \ell_2 = 0$, then one of the eigenvalues of $V(z)$ approaches a finite limit $w_2 = e^{v_2 + ib_2}$; if $\ell_1 = \ell_2 = 0$, then the eigenvalues of $V(z)$ approach w_1 and w_2 . Let \mathcal{K} be the set of points on $\mathbb{R} \times \hat{S}^1$ such that $e^{2\pi s}$ is equal to one of the limiting eigenvalues of $V(z)$. \mathcal{K} consists of at most two distinct points. The operator D is Fredholm for $s \in (\mathbb{R} \times \hat{S}^1) \setminus \mathcal{K}$ because $D^\dagger D$ has a mass gap.

The Weitzenböck formula [5] implies that the operator $D^\dagger D$ is positive-definite on the space of functions of rapid decrease. It is also easy to see that all elements of the L^2 kernel of D must be decreasing rapidly at infinity for $s \notin \mathcal{K}$, and therefore the L^2 kernel of D is empty for $s \notin \mathcal{K}$. It follows that the L^2 kernel of D^\dagger is a vector bundle on $(\mathbb{R} \times \hat{S}^1) \setminus \mathcal{K}$ of rank $-\text{Ind}D$. We denote this bundle \hat{E} . We will show below that for a periodic monopole with singularities $\text{Ind}D = -\max(n_+, n_-, k)$.

We now endow \hat{E} with a unitary connection \hat{A} and a section of $\text{End}\hat{E}$. The space of all L^2 sections of $S \otimes E$ forms a trivial unitary bundle over $\mathbb{R} \times \hat{S}^1$ with a trivial connection. The bundle $\text{Ker}D^\dagger$ is a subbundle in it. Let P denote the orthogonal projector to $\text{Ker}D^\dagger$. The induced connection on $\text{Ker}D^\dagger$ is given by

$$\hat{A} = iPds \frac{\partial}{\partial s} + iPd\bar{s} \frac{\partial}{\partial \bar{s}}. \quad (27)$$

The Higgs field $\hat{\phi} \in \Gamma(\text{End}\hat{E})$ is defined by

$$\hat{\phi}(s) = Pz. \quad (28)$$

A computation similar to Nahm’s original computation shows [5] that \hat{A} and $\hat{\phi}$ satisfy the Hitchin equations:

$$\bar{\partial}_{\hat{A}} \hat{\phi} = 0 \quad (29)$$

$$\hat{F}_{s\bar{s}} + \frac{i}{4} [\hat{\phi}, \hat{\phi}^\dagger] = 0. \quad (30)$$

Thus to any periodic $U(2)$ monopole with singularities we can associate a solution of the Hitchin equations on a cylinder, with zero, one, or two points deleted, depending on the values of ℓ_1, ℓ_2 .

4.2. Reformulation of the Nahm transform. Here we present a useful reformulation of the Nahm transform in which one can easily recognize its cohomological origin. The cohomological formulation of the Nahm transform is made explicit in [5]. If we want to solve the Dirac equation

$$D^\dagger \theta = D^\dagger \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = 0, \quad (31)$$

we can do it in two steps. First, let us find a solution of the equation

$$2(\partial_{\bar{z}} - iA_{\bar{z}})\check{\theta}_1 - (\partial_\chi - \phi - iA_\chi + s)\check{\theta}_2 = 0, \quad (32)$$

where $\check{\theta}_1$ and $\check{\theta}_2$ are L^2 sections of E . If we define a first-order differential operator

$$\bar{D}_1^s = (2(\partial_{\bar{z}} - iA_{\bar{z}}), -(\partial_\chi - \phi - iA_\chi + s))$$

which acts from $E \oplus E$ to E , and let $\check{\theta} = \check{\theta}_1 \oplus \check{\theta}_2 \in \Gamma(E \oplus E)$, then the above equation can be rewritten as

$$\bar{D}_1^s \check{\theta} = 0.$$

Now let us define an operator \bar{D}_0^s which acts from E to $E \oplus E$:

$$\bar{D}_0^s = \begin{pmatrix} \partial_\chi - \phi - iA_\chi + s \\ 2(\partial_{\bar{z}} - iA_{\bar{z}}) \end{pmatrix}. \quad (33)$$

Bogomolny equations imply that $\bar{D}_1^s \bar{D}_0^s = 0$. Thus if $\check{\theta}$ solves Eq. (32), then $\check{\theta} + \bar{D}_0^s \rho$ also solves Eq. (32) for any $\rho \in \Gamma(E)$.

In the second step, we look for $\rho \in \Gamma(E)$ such that $\check{\theta} + \bar{D}_0^s \rho$ is square-integrable and solves the Dirac equation. Note that D^\dagger can be written in the form

$$D^\dagger = \begin{pmatrix} (\bar{D}_0^s)^\dagger \\ -\bar{D}_1^s \end{pmatrix}. \quad (34)$$

For

$$\theta = \check{\theta} + \bar{D}_0^s \rho \quad (35)$$

to solve $D^\dagger \theta = 0$, the section ρ must satisfy

$$(\bar{D}_0^s)^\dagger \bar{D}_0^s \rho = -(\bar{D}_1^s)^\dagger \check{\theta}. \quad (36)$$

Now observe that the operator $(\bar{D}_0^s)^\dagger \bar{D}_0^s$ is positive-definite for $s \notin \mathcal{K}$, and therefore the above equation has a unique square-integrable solution. Thus there is a one-to-one correspondence between solutions of the Dirac equation $D^\dagger \theta = 0$, and solutions of the equation $\bar{D}_1^s \check{\theta} = 0$ modulo sections of $E \oplus E$ of the form $\bar{D}_0^s \rho$.

The benefit of the new description is that it simplifies the definition of $\hat{\phi}$ and the holomorphic structure on \hat{E} . Since the operators \bar{D}_1^s and \bar{D}_0^s commute with ∂_s , as well as with multiplication by a holomorphic function of z , the new definition of $\hat{\phi}$ is simply

$$\hat{\phi} = z,$$

and the new definition of the $\bar{\partial}$ operator on \hat{E} is simply

$$\bar{\partial}_{\hat{A}} = \frac{\partial}{\partial \bar{s}}.$$

4.3. Monopole spectral data. An important role in the subsequent analysis is played by the *spectral data*, which are algebro-geometric data associated to every periodic monopole with singularities. To define the spectral data, consider the already familiar connection $B = A - i\phi d\chi$ and its holonomy $V(z)$ around the circle \mathbb{S}^1 parametrized by χ at a point $z \in \mathbb{C} \cong \mathbb{R}^2$. $V(z)$ is a section of a bundle obtained by restricting E to the plane $\chi = 0$. This bundle has a natural holomorphic structure given by $\bar{\partial}_A$, and the Bogomolny equation ensures that $V(z)$ is a holomorphic section [5]. Thus the coefficients of the characteristic polynomial of $V(z)$ are holomorphic functions of z , and the equation

$$\det\left(e^{2\pi s} - V(z)\right) = 0 \quad (37)$$

defines a holomorphic curve in $\mathbb{C} \times \mathbb{C}^*$, where we identified \mathbb{C}^* with the cylinder $\mathbb{R} \times \hat{\mathbb{S}}^1$ parametrized by s via the exponential map. This curve \mathbf{S} will be called the monopole spectral curve. Since each point of the curve \mathbf{S} corresponds to an eigenvalue of $V(z)$, there is a well-defined sheaf M on \mathbf{S} consisting of the eigenvectors of $V(z)$. For a general monopole the spectral curve is nonsingular, and M is the sheaf of local sections of a line bundle [5]. The line bundle M has a natural holomorphic structure defined as follows. For s and z related by Eq. (37), a section of M is represented by a section θ of the bundle E satisfying

$$\left(\frac{\partial}{\partial\chi} - iA_\chi - \phi + s\right)\theta = 0. \quad (38)$$

It is a holomorphic section of M if and only if

$$\left(\frac{\partial}{\partial\bar{z}} - iA_{\bar{z}}\right)\theta = 0. \quad (39)$$

These two equations are consistent because of Bogomolny equations. We will call M the spectral line bundle, and the pair (S, M) the monopole spectral data.

4.4. Hitchin spectral data. To every solution of the Hitchin equations on $\mathbb{C}^* \cong \mathbb{R} \times \hat{\mathbb{S}}^1$ one can also associate a holomorphic curve \mathbf{C} and a sheaf N on it. The equation of the curve is the characteristic equation of $\hat{\phi}(s)$:

$$\det\left(z - \hat{\phi}(s)\right) = 0. \quad (40)$$

It defines a holomorphic curve in $\mathbb{C} \times \mathbb{C}^*$ because $\hat{\phi}$ is a holomorphic section of \hat{E} by virtue of the Hitchin equations. The sheaf N is the sheaf of eigenvectors of $\hat{\phi}(s)$. If the curve \mathbf{C} is nonsingular, then N is a line bundle [5]. The holomorphic structure on N is defined as follows: a section ψ of N is holomorphic if and only if

$$\left(\frac{\partial}{\partial\bar{s}} - i\hat{A}_{\bar{s}}\right)\psi = 0. \quad (41)$$

Since $\partial_{\bar{s}} - i\hat{A}_{\bar{s}}$ commutes with $\hat{\phi}$, this definition is consistent. The pair (\mathbf{C}, N) is called the Hitchin spectral data.

4.5. Coincidence of the spectral data. To any periodic monopole we can thus associate two kinds of spectral data: the monopole spectral data, and the Hitchin spectral data of its Nahm transform. A fact of paramount importance is that these two kinds of spectral data coincide. In [5] we proved this for periodic monopoles without singularities, but the argument applies to the present case just as well. Below we sketch the construction of the isomorphism between the two kinds of spectral data.

Suppose that a point $(\zeta, e^{2\pi\sigma})$ belongs to the spectral curve $\mathbf{C} \subset \mathbb{C} \times \mathbb{C}^*$. If Θ represents a holomorphic section of N , it satisfies

$$\hat{\phi}(\sigma)\Theta = \zeta\Theta, \quad (42)$$

and

$$(\partial_{\bar{\sigma}} - i\hat{A}_{\bar{\sigma}})\Theta = 0. \quad (43)$$

From the point of view of the Nahm transform, Θ is a zero mode of D^\dagger twisted by σ . As explained above, we can also think of Θ as a section $\check{\theta} \in \Gamma(E \oplus E)$ satisfying $\check{D}_1^\sigma \check{\theta} = 0$, modulo the equivalence relation $\check{\theta} \sim \check{\theta} + \check{D}_0^\sigma \rho$. From this point of view, Eq. (42) is equivalent to $(z - \zeta)\check{\theta} = \check{D}_0^\sigma \psi$. The latter equation implies that $\psi|_{z=\zeta}$ represents a section of the sheaf M as defined in Eq. (38). The statement that the holomorphic line bundles M and N are isomorphic means that the condition

$$\left(\frac{\partial}{\partial \bar{\zeta}} - iA_{\bar{\zeta}} \right) \psi|_{z=\zeta} = 0 \quad (44)$$

on ψ is equivalent to imposing the condition (43) on Θ .

4.6. Index computation. Let us now justify the claim that $\text{Ind} D = -\max(n_+, n_-, k)$. The operator D is a Callias-type, or Dirac-Schroedinger, operator. There are a number of index theorems for these kind of operators which express $\text{Ind} D$ in terms of Chern classes of the eigenbundles of the Higgs field on the boundary. Unfortunately, none of these theorems apply to the present situation, as they usually do not allow for singularities of the fields.

Instead of using this direct approach, we will give two indirect arguments which show that $\text{Ind} D = -\max(n_+, n_-, k)$. The first argument uses the fact that the monopole spectral curve coincides with the Hitchin spectral curve. By definition of the Nahm transform, $-\text{Ind} D = \text{rank} \hat{E}$, which in turn is equal to the number of times the Hitchin spectral curve covers the w -plane. On the other hand, it is shown in the next section that the monopole spectral curve has the form

$$A(z)w^2 - B(z)w + C(z) = 1,$$

where $w = e^{2\pi s}$, and $A(z)$, $B(z)$, and $C(z)$ are polynomials of degree n_- , k , and n_+ , respectively. This curve covers the w -plane $\max(n_+, n_-, k)$ times, which proves our statement.

The second argument uses a reformulation of the Nahm transform presented in Subsect. 4.2. It provides some information on the spatial structure of the zero modes of D^\dagger . Consider a complex of sheaves of vector spaces

$$K^\sigma : 0 \rightarrow E \xrightarrow{\check{D}_0^\sigma} E \oplus E \xrightarrow{\check{D}_1^\sigma} E \rightarrow 0,$$

where the sections of E are assumed to be rapidly decaying. The value of σ is arbitrary, except that we require $\sigma \notin \mathcal{K}$. As shown in Sect. 4.2, $H^1(K^\sigma)$ is naturally isomorphic to $\text{Ker} D^\dagger$ twisted by $s = \sigma$. It remains to compute the dimension of $H^1(K^\sigma)$. To this end consider another complex of sheaves:

$$0 \rightarrow E \xrightarrow{(z-\zeta)} E \xrightarrow{rest.} E|_{z=\zeta} \rightarrow 0,$$

where $rest.$ is the restriction map, and $E|_{z=\zeta}$ is concentrated on the circle $z = \zeta$. This complex is not exact; nevertheless it leads to a long exact sequence in $\tilde{\mathcal{D}}^\sigma$ -cohomology [5]:

$$0 \rightarrow H_{\tilde{\mathcal{D}}^\sigma}^0(\mathbb{S}^1, E|_{z=\zeta}) \rightarrow H^1(K^\sigma) \xrightarrow{z-\zeta} H^1(K^\sigma) \xrightarrow{rest.} H_{\tilde{\mathcal{D}}^\sigma}^1(\mathbb{S}^1, E|_{z=\zeta}) \rightarrow 0.$$

This exact sequence implies that the dimension of $H^1(K^\sigma)$ is equal to the number of points at which the spectral curve intersects the line $s = \sigma$. Using the explicit form of the monopole spectral curve, one can easily see that this number is equal to $\max(n_+, n_-, k)$.

5. Boundary Conditions for Hitchin Data

5.1. General remarks. The boundary conditions on the Hitchin data will be determined mainly by studying the monopole spectral curve. It has the general form

$$w^2 - b(z)w + c(z) = 0.$$

The functions $b(z) = \text{Tr} V(z, 2\pi)$ and $c(z) = \det V(z, 2\pi)$ are known to be holomorphic on $\mathbb{C} \setminus \{z_1, \dots, z_n\}$. We now show that they are rational functions.

Using the known asymptotic behavior of ϕ and A near the singularities, we compute that $c(z)$ has a simple zero at $z = z_i$ if $e_i = +1$ and a simple pole if $e_i = -1$. As for $b(z)$, it has a simple pole at $z = z_i$ if $e_i = -1$ and is regular if $e_i = 1$. Thus $b(z)$ and $c(z)$ are meromorphic on \mathbb{C} .

Using the known asymptotic behavior of ϕ and A at infinity, we obtain the following asymptotic formulas for $b(z)$ and $c(z)$ for $z \rightarrow \infty$:

$$b(z) = z^{\ell_1} e^{\mathfrak{v}_1} \left(1 + \frac{\mu_1}{z} + O(1/z^2) \right) + z^{\ell_2} e^{\mathfrak{v}_2} \left(1 + \frac{\mu_2}{z} + O(1/z^2) \right), \quad (45)$$

$$c(z) = z^{\ell_1 + \ell_2} e^{\mathfrak{v}_1 + \mathfrak{v}_2} \left(1 + \frac{\mu_1 + \mu_2}{z} + O(1/z^2) \right). \quad (46)$$

Here $\mathfrak{v}_1 = v_1 + ib_1$, $\mathfrak{v}_2 = v_2 + ib_2$. Hence $b(z)$ and $c(z)$ are meromorphic on \mathbb{P}^1 , i.e. rational functions on \mathbb{C} . Moreover, the above information about the poles of $b(z)$ and $c(z)$ implies

$$b(z) = \frac{B(z)}{A(z)}, \quad c(z) = \frac{C(z)}{A(z)},$$

where

$$A(z) = \prod_{e_i=-1} (z - z_i), \quad C(z) = e^{\mathfrak{v}_1 + \mathfrak{v}_2} \prod_{e_i=1} (z - z_i),$$

and $B(z)$ is a polynomial of degree k . Thus the monopole spectral curve can be rewritten in the following form:

$$A(z)w^2 - B(z)w + C(z) = 1. \quad (47)$$

It is understood here that the points (w, z) on the curve satisfying $A(z) = 0$ must be deleted. It is important to note that the known asymptotics of $b(z)$ and $c(z)$ determine the leading and the next-to-leading coefficients of $B(z)$ in terms of the monopole parameters. For example, the leading coefficient of $B(z)$ is given by

$$\begin{cases} e^{v_1}, & \ell_1 > \ell_2 \\ e^{v_1} + e^{v_2}, & \ell_1 = \ell_2. \end{cases}$$

The precise expression for the next-to-leading coefficient will not be needed here. The remaining $k - 2$ coefficients of $B(z)$ are the moduli of the monopole.

In [31] it was shown that the curve (47) is the Seiberg-Witten curve for the $N = 2$, $d = 4$ gauge theory with gauge group $SU(k)$ and n fundamental hypermultiplets. The masses of the hypermultiplets are the zeros of $A(z)$ and $C(z)$, which from the monopole point of view are just the positions of the Dirac-type singularities. The reason for this ‘‘coincidence’’ was explained in Sect. 3.

We proved in Sect. 4 that the Hitchin spectral curve is identical to the monopole spectral curve. We now use the form of the spectral curve to determine the asymptotic behavior of the Hitchin data. The results depend on the relative magnitude of the numbers k, n_+, n_- . There are seven possible cases to consider. But note that the substitution $\phi \rightarrow -\phi$, $\chi \rightarrow -\chi$ leaves the Bogomolny equation invariant and in terms of the spectral curve maps w to $1/w$ leaving z invariant. Thus this map interchanges n_+ and n_- , and without loss of generality we may assume that $n_- \leq k$. This leaves us with four cases to consider.

5.2. The case $n_- < k = n_+$. The rank of \hat{E} is $n_+ = k$. The set $\mathcal{K} \subset \mathbb{R} \times \hat{\mathbb{S}}^1$ consists of a single point $w = w_2 = e^{v_2}$. Thus we need to understand the behavior of \hat{A} and $\hat{\phi}$ for $|r| \rightarrow \infty$, as well as near $w = w_2$.

We begin with the region of large $|r|$. The spectral curve equation implies that the eigenvalues of $\hat{\phi}$ for $r \rightarrow +\infty$ ($w \rightarrow \infty$) asymptote to

$$a_1, a_2, \dots, a_{n_-}, e^{(2\pi s - v_1)/(k - n_-)}, \omega_{k - n_-} e^{(2\pi s - v_1)/(k - n_-)}, \dots, \omega_{k - n_-}^{k - n_- - 1} e^{(2\pi s - v_1)/(k - n_-)}, \quad (48)$$

where $\omega_p = e^{2\pi i/p}$, and a_1, \dots, a_{n_-} are the roots of $A(z)$. For $r \rightarrow -\infty$ ($w \rightarrow 0$) the n_+ eigenvalues of $\hat{\phi}(s)$ are all distinct and approach the n_+ roots of $C(z)$.

To determine the behavior of \hat{A} , note first that the curvature of \hat{A} goes to zero for large $|r|$ [5]. Indeed, the computations in [5] imply that

$$F_{\hat{A}} = iP\sigma_3(D^\dagger D)^{-1}P, \quad (49)$$

where P is the projector onto the kernel of D^\dagger . Now, it is easy to see that the L^2 norm of the Green’s function of $D^\dagger D$ is bounded from above by $const/|r|^{3/2}$, and so is the norm

of $F_{\hat{A}}$. Since $F_{\hat{A}}$ goes to zero at least as fast as $1/|r|^{3/2}$, \hat{A} has well-defined limiting holonomies around $\hat{\mathbb{S}}^1$ for $r \rightarrow \pm\infty$.

Since the Hitchin equations relate $F_{\hat{A}}$ and $[\hat{\phi}, \hat{\phi}^\dagger]$, we see that $[\hat{\phi}, \hat{\phi}^\dagger]$ goes to zero for large $|r|$, and therefore in this limit the eigenvectors of $\hat{\phi}$ become orthogonal. Furthermore, since for $r \rightarrow -\infty$ the eigenvalues of $\hat{\phi}$ approach constants, in this limit the holonomy of \hat{A} becomes diagonal in the basis of the eigenvectors of $\hat{\phi}$.

It is possible to find the eigenvalues of the limiting holonomy by analyzing the Nahm transform in more detail. Instead we will use a shortcut. As explained in the next section, the Nahm transform admits an inverse. The inverse Nahm transform involves finding the kernel of a family of Dirac-type operator \hat{D}^x parametrized by a point $x = (z, \chi) \in \mathbb{R}^2 \times \mathbb{S}^1$. The fields $A(x), \phi(x)$ are expressed through the overlaps of the zero-modes of \hat{D}^x . It will be seen that \hat{D}^x can fail to be Fredholm only when $\exp(i\chi)$ coincides with one of the eigenvalues of the limiting holonomies of \hat{A} , and z coincides with the corresponding limiting eigenvalue of $\hat{\phi}$. For all other values of $x = (z, \chi)$, the operator \hat{D}^x is Fredholm, and the fields $A(x), \phi(x)$ are nonsingular. We already know that the limiting eigenvalues of $\hat{\phi}$ for $r \rightarrow -\infty$ are precisely the z -coordinates of the singularities with $e_i = +1$. Hence the limiting eigenvalues of \hat{A} must be the χ -coordinates of these singularities. Thus, if we denote these χ -coordinates by $\chi_{n+1}, \dots, \chi_n$, the holonomy of \hat{A} in the basis of the eigenvectors of $\hat{\phi}$ and in the limit $r \rightarrow -\infty$ is

$$\text{diag}(e^{i\chi_{n+1}}, e^{i\chi_{n+2}}, \dots, e^{i\chi_n}). \quad (50)$$

Now let us find the limiting holonomy of \hat{A} for $r \rightarrow +\infty$. Note that the Hitchin equations together with the limiting behavior of the eigenvalues of $\hat{\phi}$ imply that in the basis of the eigenvectors of $\hat{\phi}$ the limiting holonomy is

$$\begin{pmatrix} e^{i\beta_1} & & & & & \\ & \dots & & & & \\ & & e^{i\beta_{n_-}} & & & \\ & & & & & \\ & & & & & V_{k-n_-} e^{i\tilde{\alpha}_1/(k-n_-)} \end{pmatrix}, \quad (51)$$

where $\beta_1, \dots, \beta_{n_-}, \tilde{\alpha}_1 \in \mathbb{R}/(2\pi\mathbb{Z})$, and V_p is $p \times p$ ‘‘shift’’ matrix given by

$$V_p = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \quad (52)$$

The ‘‘shift’’ matrix appears because $k - n_-$ of the limiting eigenvalues of $\hat{\phi}$ are cyclically permuted as one goes around $\hat{\mathbb{S}}^1$. Note also that $\tilde{\alpha}_1$ takes values in $\mathbb{R}/(2\pi\mathbb{Z})$ because a shift $\tilde{\alpha}_1 \rightarrow \tilde{\alpha}_1 + 2\pi m$, $m \in \mathbb{Z}$, can be undone by a gauge transformation.

It remains to determine $\beta_1, \dots, \beta_{n_-}$ and $\tilde{\alpha}_1$. The first n_+ eigenvalues of the limiting holonomy are associated with the eigenvalues of $\hat{\phi}$ which become constant in the limit $r \rightarrow +\infty$. Using the same shortcut as above, we find that β_i is equal to χ_i , the χ -coordinate of the i^{th} singularity with $e_i = -1$. We will discuss how to express $\tilde{\alpha}_1$ in terms of the parameters of the monopole in the end of this subsection.

Now let us determine the behavior of \hat{A} and $\hat{\phi}$ near the point $w = w_2$. From the spectral curve equation it is easy to see that the function $z(w)$ considered as a meromorphic function on the curve has a simple pole at $w = w_2 = \exp(v_2)$ with residue $e^{v_2} \mu_2$. This implies that the Higgs field $\hat{\phi}$ behaves as

$$\hat{\phi}(s) \sim \frac{R_1}{s - s_2} + O(1),$$

where $s_2 = \frac{1}{2\pi} \log w_2$, and R is a rank-one matrix whose non-zero eigenvalue is $e^{v_2} \mu_2$.

Now let us show that $[R, R^\dagger] = 0$. To this end it is sufficient to demonstrate that $[\hat{\phi}, \hat{\phi}^\dagger]$ grows at most as $1/|s - s_2|$ in the limit $s \rightarrow s_2$. To estimate the norm of this commutator, note that by virtue of the Hitchin equations and (49) we have

$$[\hat{\phi}, \hat{\phi}^\dagger] = -4P\sigma_3(D^\dagger D)^{-1}P.$$

Now the required estimate on the commutator follows from a simple estimate of the L^2 norm of $(D^\dagger D)^{-1}$.

Having determined the behavior of $\hat{\phi}$ near $s = s_2$, we now turn to \hat{A} . Our strategy will be the following. Specifying a unitary connection \hat{A} is equivalent to specifying a holomorphic structure on \hat{E} , as well as a Hermitian metric on \hat{E} . In particular, the behavior of \hat{A} near $s = s_2$ is determined by the rate of growth of holomorphic sections of \hat{E} near $s = s_2$. As a basis of holomorphic sections we will use the eigenvectors of $\hat{\phi}$. They can be reinterpreted as holomorphic sections of the spectral line bundle. Their norm can be estimated using the ‘‘cohomological’’ reformulation of the Nahm transform described in Sect. 4.2. The same strategy was used in [3] to study the Nahm transform of instantons on $\mathbb{R}^2 \times T^2$, and we will make use of some of the results of that paper.

An important role in this argument is played by the holomorphic isomorphism of the spectral line bundles M and N whose construction we now recall. Let $(\zeta, e^{2\pi\sigma})$ be the coordinates of a point on the spectral curve \mathcal{C} . In the neighbourhood of $\sigma = s_2$ ζ has a simple pole as a function of σ , $\zeta \sim 1/(\sigma - s_2)$. If Θ is a vector in the fiber of the spectral line bundle over the point $(\zeta, e^{2\pi\sigma})$, then

$$\hat{\phi}(\sigma)\Theta = \zeta\Theta.$$

The fact that ζ diverges for $\sigma \rightarrow s_2$ means that Θ is an eigenvector of ϕ , unique up to a multiple, whose eigenvalue diverges in this limit. Let θ be the corresponding class in $\bar{\mathcal{D}}$ -cohomology, as described in Sect. 4.5. This implies

$$(z - \zeta)\theta = \bar{D}_0^\sigma \psi_\zeta(z, \chi), \quad (53)$$

for some $\psi_\zeta(z, \chi) \in \Gamma(E)$. Then $\psi_\zeta(\zeta, \chi)$ is an element in the fiber $M \rightarrow \mathbf{S}$, and the isomorphism between M and N identifies it with Θ . Now let us start varying σ so that $\Theta(\sigma)$ is a holomorphic section of the spectral line bundle. This is equivalent to saying that the section $\psi_\zeta(\zeta, \chi)$ is holomorphic with respect to $\bar{\partial}_A$, i.e.

$$(\partial_{\bar{\zeta}} - iA_{\bar{\zeta}})\psi_\zeta(\zeta, \chi) = 0. \quad (54)$$

From the asymptotics of A we infer that when ζ tends to infinity the norm of such a section behaves as $|\zeta|^{-n - \alpha_2/(2\pi)}$ with an integer n .

Since M and N are holomorphically equivalent, the corresponding Θ is going to satisfy $(\partial_{\bar{\sigma}} - i\hat{A}_{\bar{\sigma}})\Theta(\sigma) = 0$. For any $\mu(z, \chi) \in \Gamma(E)$ we can replace $\psi_\zeta(z)$ with

$\psi_\zeta(z) + (z - \zeta)\mu(z, \chi)$ without spoiling the holomorphicity condition (54). If we define a representative $\check{\theta}$ of the class Θ by $\check{\theta}_\zeta(z) = \bar{D}_0\psi_\zeta(z)/(z - \zeta)$ such a change of ψ_ζ will change $\check{\theta}_\zeta$ by $\bar{D}_0\mu$. As explained in Sect. 4.2, from $\check{\theta}_\zeta$ we can construct an L^2 solution of the Dirac equation $D^\dagger\theta_\sigma = 0$:

$$\theta_\sigma = \check{\theta}_\zeta - \bar{D}_0^\sigma \left((\bar{D}_0^\sigma)^\dagger \bar{D}_0^\sigma \right)^{-1} (\bar{D}_0^\sigma)^\dagger \check{\theta}_\zeta. \quad (55)$$

The norm of Θ_σ is defined to be the L^2 norm of θ_σ . Now we use Lemma 7.5 of [3] to estimate this norm:

$$|\bar{D}_0^\sigma \left((\bar{D}_0^\sigma)^\dagger \bar{D}_0^\sigma \right)^{-1} (\bar{D}_0^\sigma)^\dagger \check{\theta}_\zeta| \leq \frac{\text{const}}{|\sigma - s_2|} |(\bar{D}_0^\sigma)^\dagger \check{\theta}_\zeta|. \quad (56)$$

Thus the norm of θ_σ is bounded from above by a multiple of $|\sigma - s_2|^{-1+n+\alpha_2/(2\pi)}$.

In a similar manner one can estimate the norms of the holomorphic sections which correspond to the eigenvalues of $\hat{\phi}$ which stay finite in the limit $s \rightarrow s_2$. We find that their norms remain bounded.

Now we are ready to determine the behavior of \hat{A} near the singularity. Since the norms of all holomorphic sections grow not faster than powers of $|\sigma - s_2|$, \hat{A} must have a simple pole at this point:

$$\hat{A}_s(\sigma) = \frac{Q}{\sigma - s_2} + O(1).$$

The residue Q must satisfy $[Q, Q^\dagger] = 0$, for the same reasons as the residue of $\hat{\phi}$. Furthermore, it must satisfy $[Q^\dagger, R] = 0$ for the equation $\bar{\partial}_{\hat{A}}\hat{\phi} = 0$ to be satisfied. Thus it is possible to choose a basis in the fiber of \hat{E} over $s = s_2$ such that both Q and R are diagonal. The above estimates on the norm of holomorphic eigenvectors of $\hat{\phi}$ then imply that the eigenvalues of Q restricted to $\text{Ker } R$ are zero, and therefore Q is a rank-one matrix. It is easy to see that by a gauge transformation one can make the single non-zero eigenvalue of Q to be purely imaginary. We will denote this purely imaginary number by $i\tilde{\alpha}_2/(4\pi)$. Further gauge transformations can shift $\tilde{\alpha}_2$ by multiples of 2π , so we should regard $\tilde{\alpha}_2$ as taking values in $\mathbb{R}/(2\pi\mathbb{Z})$.

It remains to understand how the parameters $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ depend on the monopole parameters. Since $\text{Tr } F_{\hat{A}} = 0$, we can obtain a relation between various limiting holonomies of \hat{A} by integrating this equation over \hat{X} and using the Stokes' theorem:

$$\sum_{i=1}^n e_i \chi_i = \tilde{\alpha}_1 + \tilde{\alpha}_2. \quad (57)$$

This equation leaves the individual values of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ undetermined. We conjecture that $\tilde{\alpha}_1 = \alpha_1$ and $\tilde{\alpha}_2 = \alpha_2$. One piece of evidence in favor of this is that the relation (57) is automatically satisfied due to (21). More compelling evidence is provided by our estimate of the rate of growth of holomorphic sections of \hat{E} near the point $s = s_2$. We found that the sections in the image of Q are bounded from above by a multiple of $|s - s_2|^{-1+n+\alpha_1/(2\pi)}$, where n is an integer. If this bound were saturated, the equality $\alpha_1 = \tilde{\alpha}_1$ would follow. Then $\tilde{\alpha}_2 = \alpha_2$ would also follow by combining (57) and (21).

5.3. *The case $n_- < k > n_+$.* The rank of \hat{E} is k . The set \mathcal{K} is empty, i.e. the Hitchin data are non-singular for all s . The fact that ϕ is non-singular can be seen from the spectral curve equation

$$B(z) - wA(z) - \frac{C(z)}{w} = 0. \quad (58)$$

If $k > n_{\pm}$, then the roots of this polynomial equation in z have no singularities as functions of $w \in \mathbb{C}^*$.

The spectral curve equation implies that for $r \rightarrow +\infty$ the asymptotics of the eigenvalues of $\hat{\phi}$ are given by Eq. (48). The asymptotic behavior of the eigenvalues of $\hat{\phi}$ for $r \rightarrow -\infty$ is analogous:

$$c_1, c_2, \dots, c_{n_+}, e^{(2\pi s - v_1)/(n_+ - k)}, \\ \omega_{n_+ - k} e^{(2\pi s - v_2)/(n_+ - k)}, \dots, \omega_{n_+ - k}^{k - n_+ - 1} e^{(2\pi s - v_2)/(n_+ - k)}, \quad (59)$$

where c_1, \dots, c_{n_+} are the roots of $C(z)$. Note that all eigenvalues of $\hat{\phi}$ either approach a constant value, or grow exponentially with $|r|$.

Since the curvature of \hat{A} decays as $|r|^{-3/2}$ for large $|r|$, \hat{A} has well-defined limiting holonomies around $\hat{\mathbb{S}}^1$. The same reasoning as in the previous subsection implies that in the basis of the eigenvectors of $\hat{\phi}$ the limiting holonomy for $r \rightarrow +\infty$ must have the form

$$\begin{pmatrix} e^{i\chi_1} & & & \\ & \dots & & \\ & & e^{i\chi_{n_-}} & \\ & & & V_{k-n_-} e^{i\tilde{\alpha}_1/(k-n_-)} \end{pmatrix}, \quad (60)$$

where $\chi_i, i = 1, \dots, n_-$, are the χ -coordinates of the singularities with $e_i = -1$, and $\tilde{\alpha}_1 \in \mathbb{R}/(2\pi\mathbb{Z})$. For $r \rightarrow -\infty$ the limiting holonomy in the basis of the eigenvectors of ϕ is given by

$$\begin{pmatrix} e^{i\chi_{n_-+1}} & & & \\ & \dots & & \\ & & e^{i\chi_{n_-+n_+}} & \\ & & & V_{k-n_+} e^{-i\tilde{\alpha}_2/(k-n_+)} \end{pmatrix}, \quad (61)$$

where $\chi_{n_-+1}, \dots, \chi_{n_-+n_+}$ are the χ -coordinates of the singularities with $e_i = +1$, and $\tilde{\alpha}_2 \in \mathbb{R}/(2\pi\mathbb{Z})$.

It remains to express the parameters $\tilde{\alpha}_1, \tilde{\alpha}_2$ through the parameters of the monopole. The Stokes' theorem again implies (57), but leaves the individual values of $\tilde{\alpha}_1, \tilde{\alpha}_2$ undetermined. We conjecture that in fact $\tilde{\alpha}_1 = \alpha_1$ and $\tilde{\alpha}_2 = \alpha_2$. The main evidence in favor of this conjecture is the upper bound on the rate of growth of holomorphic sections for $r \rightarrow \pm\infty$. For example, consider the limit $r \rightarrow +\infty$. The bundle \hat{E} has a subbundle spanned by the eigenvectors of $\hat{\phi}$ with diverging eigenvalues. The rank of this subbundle is $k - n_-$. Let L be its top exterior power. L is a line bundle which inherits from \hat{E} a holomorphic structure, as well as a Hermitian inner product. It is easy to see that holomorphic sections of this bundle grow as $e^{r(2\pi n - \tilde{\alpha}_1)}$, where $n \in \mathbb{Z}$, for $r \rightarrow +\infty$. On the other hand, one can estimate the rate of growth using the coincidence of the spectral data, as in Subsect. 5.2, and get that the holomorphic sections of L are bounded by $e^{r(2\pi n - \alpha_1)}$, $n \in \mathbb{Z}$. If the bound is saturated, then $\tilde{\alpha}_1 = \alpha_1$, and consequently $\tilde{\alpha}_2 = \alpha_2$.

where Q_i is a rank-one matrix satisfying $[Q_i^\dagger, R_i] = 0$. The only non-zero eigenvalue of Q_i can be made purely imaginary by a gauge transformation and will be denoted $i\tilde{\alpha}_i/(4\pi)$. Stokes' theorem implies the relation (57), as before. The estimate of the rate of growth of the holomorphic sections of \hat{E} near the points $s = s_i$ suggests that in fact $\tilde{\alpha}_i = \alpha_i$, $i = 1, 2$.

In the case $n_- = k = n_+$ the Nahm transform for periodic monopoles resembles very much the Nahm transform for doubly-periodic instantons studied in [18]. Recall that doubly-periodic instantons are solutions of the $U(2)$ self-duality equation on $\mathbb{R}^2 \times T^2$ with finite action and vanishing first Chern class. There the Nahm transform is described by solutions of the Hitchin equations on a torus with two punctures [20, 19, 18]. The behavior of \hat{A} and $\hat{\phi}$ at the punctures is the same as above. The rank of the Hitchin bundle is given by the second Chern class of the instanton bundle on $\mathbb{R}^2 \times T^2$.

The Nahm data for periodic monopoles can be regarded as a limiting case of the Nahm data for doubly-periodic instantons. The cylinder $\mathbb{R} \times \hat{\mathbb{S}}^1$ can be regarded as a degeneration of the torus of [18]. For example, if the torus is realized as a quotient of \mathbb{C} by the lattice generated by 1 and τ , one can consider the limit $\text{Im}\tau \rightarrow +\infty$. The positions of the punctures should be held fixed in this limit. The nonabelian monopole charge k corresponds to the instanton number.

One can see the reason for this by looking at the monopole side of the story. The case $k = n_+ = n_-$ is special in that the eigenvalues of the monopole Higgs field ϕ approach constants at infinity. Now recall that the Bogomolny equation is a reduction of the self-duality equation to three dimensions. Thus a periodic monopole can be regarded as an instanton on $Y = \mathbb{R}^2 \times \mathbb{S}_\chi^1 \times \mathbb{S}_\theta^1$ invariant with respect to the translations of the circle \mathbb{S}_θ^1 . The relation between the self-dual connection \tilde{A} on Y and the monopole fields on $X = \mathbb{R}^2 \times \mathbb{S}_\chi^1$ is given by

$$\tilde{A} = \psi^*(A) + \psi^*(\phi)d\theta,$$

where $\psi : Y \rightarrow X$ is the natural projection. The connection \tilde{A} is self-dual for any k and n_\pm , but the case $k = n_+ = n_-$ is special, because only in this case the large- $|z|$ behavior of \tilde{A} is that of a doubly-periodic instanton as defined in [18]. Of course, unlike in [18], our \tilde{A} has singularities for $z = z_i$, $\chi = \chi_i$.

The origin of these singularities can be understood by analyzing how the limiting procedure described above affects the instantons. A doubly-periodic $U(2)$ instanton with charge 1 can be regarded as made of two ‘‘monopole’’ constituents. Each constituent has a fixed size, so its moduli describe its position on $\mathbb{R}^2 \times T^2$. Thus a charge 1 instanton has 8-dimensional moduli space. (This interpretation of an instanton as a combination of two monopoles also arises for calorons, i.e. instantons on $\mathbb{R}^3 \times \mathbb{S}^1$, see [22] for details.) The sizes of the two constituents are determined by the asymptotic behavior of the components of A along the T^2 and need not be the same. In particular, one can take a limit in which the size of one of the constituents goes to zero, while the size of the other one stays finite. A point-like constituent monopole is nothing but a Dirac-type singularity on $\mathbb{R}^2 \times T^2$ of the kind considered in this paper. Thus for $k = n_- = n_+$ periodic monopoles with singularities can be obtained as a limit of doubly-periodic $U(2)$ instantons with instanton charge k .

The degeneration of a doubly-periodic instanton into a periodic monopole with singularities can be easily seen by slightly modifying the brane configuration discussed in Sect. 3. A doubly-periodic $U(2)$ instanton of charge k is described by a brane configuration with k D3-branes and 2 NS5-branes, but with the x^6 direction compactified on a

circle. As described in Sect. 3, a D3-brane can end on the NS5-brane, and therefore each D3-brane breaks into two segments suspended between the NS5-branes and capable of moving independently. These suspended segments represent the constituent monopoles. The size of a constituent is inversely proportional to the length of the segment. In order to obtain a charge k periodic monopole with $2k$ singularities one has to take the limit in which the NS5-branes are very close together, so that half of the segments are much longer than the other half.

6. Inverse Nahm Transform

In this section we construct and study the inverse Nahm transform which associates a periodic monopole with singularities to a solution of the Hitchin equations on $(\mathbb{R} \times \hat{\mathbb{S}}^1) \setminus \mathcal{K}$ with the spectral curve and asymptotic behavior described in Sect. 5.

6.1. Construction of the monopole fields. Let $\hat{A}, \hat{\phi}$ be a solution of the Hitchin equations with $\text{rank}(E) = K$. We will use these data to define a family of Dirac-type operators parametrized by a point $(z, \chi) \in \mathbb{C} \times \mathbb{S}^1$. Let $\hat{a} = -\chi dt$ be a unitary connection on a trivial line bundle on $\mathbb{R} \times \hat{\mathbb{S}}^1$. We define a Dirac type-operator from $\hat{E} \oplus \hat{E}$ to $\hat{E} \oplus \hat{E}$ by

$$\hat{D} = \begin{pmatrix} -\hat{\phi} + z & 2\partial_{\hat{A}+\hat{a}} \\ 2\bar{\partial}_{\hat{A}+\hat{a}} & -\hat{\phi}^\dagger + \bar{z} \end{pmatrix}. \quad (65)$$

Now let us assume that \hat{A} and $\hat{\phi}$ satisfy the kind of boundary conditions described in Sect. 5. Standard arguments show that \hat{D} can fail to be Fredholm only if z is equal to one of the asymptotic eigenvalues of $\hat{\phi}$ and $e^{i\chi}$ is equal to the corresponding eigenvalue of the limiting holonomy of \hat{A} . (We recall that the boundary conditions of Sect. 5 imply that the holonomy of \hat{A} in the limit $|r| \rightarrow \infty$ preserves all the eigenvectors of $\hat{\phi}$ corresponding to finite limiting eigenvalues and permutes the rest of the eigenvectors.) Let us denote the set of such points of $\mathbb{C} \times \mathbb{S}^1$ by \mathcal{M} . Obviously, \mathcal{M} is a finite set whose cardinality does not exceed $2K$.

The Weitzenböck formula [5] implies that the L^2 kernel of \hat{D} is trivial, therefore $\text{Ker } \hat{D}^\dagger$ is a vector bundle on $\mathbb{R}^2 \times \mathbb{S}^1$ of rank $-\text{Ind } \hat{D}$. It is a subbundle of a trivial infinite-dimensional bundle on $\mathbb{R}^2 \times \mathbb{S}^1$ whose fiber consists of all L^2 sections of $\hat{E} \oplus \hat{E}$. The latter bundle has a natural Hermitian inner product. Let \hat{P} be the corresponding projector to $\text{Ker } \hat{D}^\dagger$. We define a connection A on $E = \text{Ker } \hat{D}^\dagger$ by

$$d_A = \hat{P} \left(dz \frac{\partial}{\partial z} + d\bar{z} \frac{\partial}{\partial \bar{z}} + d\chi \frac{\partial}{\partial \chi} \right) \hat{P}. \quad (66)$$

We define a Higgs field $\phi \in \Gamma(\text{End}(E))$ by

$$\phi = \hat{P} r \hat{P}. \quad (67)$$

Obviously, $\phi^\dagger = \phi$. These formulas are well-defined because the elements of the L^2 kernel of \hat{D}^\dagger decay at least exponentially for $(z, \chi) \notin \mathcal{M}$, as can be easily verified.

We claim that A, ϕ satisfy the Bogomolny equation. The computation demonstrating this is exactly the same as in [5]. The proof that the composition of the direct and inverse Nahm transform is the identity is also the same as in [5].

We now want to show that the solution of the Bogomolny equation obtained in this way is a periodic $U(2)$ monopole with singularities in the sense of Sect. 2. This will imply that there is a one-to-one correspondence between periodic $U(2)$ monopoles with singularities and the Hitchin data of the kind described in Sect. 5. As a first step, we need to show that $-\text{Ind} \hat{D} = 2$, and therefore the monopole bundle obtained by the inverse Nahm transform has the rank two. The argument for this is exactly the same as in [5]. One proves that $\dim \text{Ker} \hat{D}^\dagger$ for $z = z_0, \chi = \chi_0$ is equal to the number of points at which the Hitchin spectral curve intersects the line $z = z_0$ (provided $(z_0, \chi_0) \notin \mathcal{M}$). On the other hand, one of our assumptions about the Hitchin system was that the Hitchin spectral curve has the form

$$A(z)w^2 - B(z)w + C(z) = 0,$$

where $w \neq 0$, and the roots of $A(z)$ and $C(z)$ are the asymptotic eigenvalues of $\hat{\phi}$. It follows that $\dim \text{Ker} \hat{D}^\dagger = 2$ for all $(z, \chi) \notin \mathcal{M}$.

6.2. Asymptotic behavior of the monopole fields. It remains to show that the monopole has the right behavior near the points of \mathcal{M} , as well as for $|z| \rightarrow \infty$. Suppose that the boundary conditions for the Hitchin data are such that n_- of the eigenvalues of $\hat{\phi}$ approach constants as $r \rightarrow +\infty$ and n_+ of them approach constants as $r \rightarrow -\infty$. Taking into account the $w \rightarrow 1/w$ symmetry we may assume that $n_- \leq n_+$ without loss of generality. Recall now that we assumed that the spectral curve defined by the equation

$$\det(z - \hat{\phi}(s)) = 0 \tag{68}$$

has the form

$$A(z)w^2 - B(z)w + C(z) = 0, \tag{69}$$

where $w = e^{2\pi s}$, and $A(z)$ is a polynomial of degree n_- , $C(z)$ is a polynomial of degree n_+ . If we denote by k the degree of $B(z)$, then it is easy to see that the rank of the Hitchin system K is $\max(n_+, n_-, k)$. Moreover, analyzing the four possible boundary conditions for $\hat{\phi}$, one can easily see that $2k \geq n_+ + n_-$, and if one normalizes the leading coefficient of $A(z)$ to be one, then the leading coefficient of $C(z)$ is

$$e^{v_1 + v_2},$$

and the leading coefficient of $B(z)$ is

$$\begin{cases} e^{v_1}, & \ell_1 > \ell_2, \\ e^{v_1} + e^{v_2}, & \ell_1 = \ell_2, \end{cases}$$

where $\ell_1 = k - n_-$, $\ell_2 = n_+ - k$.

We now use these properties of the spectral curve to find the behavior of ϕ and A for large $|z|$ as well as near the points of \mathcal{M} .

To analyze the behavior for $|z| \rightarrow \infty$, note the following formula for the components of curvature of A [5]:

$$F_{\bar{z}\chi} = 2i \hat{P} \sigma_- (\hat{D}^\dagger \hat{D})^{-1} \hat{P}, \quad F_{z\bar{z}} = i \hat{P} \sigma_3 (\hat{D}^\dagger \hat{D})^{-1} \hat{P}. \tag{70}$$

It is easy to see that the L^2 norm of $(\hat{D}^\dagger \hat{D})^{-1}$ is bounded from above by a multiple of $1/|z|$. Hence the components of curvature decay at least as fast as that, and then the Bogomolny equation implies that the covariant differential of ϕ goes to zero for $|z| \rightarrow \infty$. It follows that for large $|z|$ the eigenvalues of ϕ are independent of χ , and furthermore that the eigenvalues of $V(z)$ factorize into the eigenvalues of the holonomy of A and the eigenvalues of $e^{2\pi\phi(z,\chi)}$. Thus the behavior of the eigenvalues of ϕ for large $|z|$ can be read off the asymptotic behavior of the eigenvalues of $V(z)$, which are encoded in the spectral curve. One can easily see that this yields (6) with $\ell_1 = k - n_-$ and $\ell_2 = n_+ - k$.

The behavior of A for large $|z|$ can be inferred from Theorem 10.5 of [17] about the behavior of solutions of the Bogomolny equation. (There the theorem is stated for the Bogomolny equation on \mathbb{R}^3 , but the proof goes through for $\mathbb{R}^2 \times \mathbb{S}^1$ as well). This theorem asserts that if the difference between the eigenvalues of the Higgs field is bounded from below for $|z| \rightarrow \infty$, then $d_A \phi$ is proportional to ϕ with exponential accuracy. Thus, up to corrections of order $\exp(-\delta|z|)$, $\delta > 0$, the connection A preserves the splitting of E into the eigenbundles of ϕ . Hence if we use the eigenvectors of ϕ as the orthonormal frame of E , the $U(2)$ Bogomolny equation splits into a pair of the decoupled $U(1)$ Bogomolny equations. Solving the rank-one Bogomolny equation for A we find that the asymptotic behavior of A is given by (9) with some α_1, α_2 . One can also check that (7) is satisfied.

We now turn to the behavior of ϕ and A near the points of \mathcal{M} . Recall that their z -coordinates are given by the roots of $A(z)$ and $C(z)$, while their χ coordinates are given by the corresponding limiting eigenvalues of the holonomy of \hat{A} . From the spectral curve equation we see that near such a point $z = z_i$ one of the eigenvalues of $V(z)$ either diverges as $(z - z_i)^{-1}$, or goes to zero as $z - z_i$. Obviously, this can happen only if the Higgs field ϕ becomes singular at $z = z_i$, $\chi = \chi_i$.

To determine the nature of the singularity, we first estimate how fast F_A and ϕ can grow near the singular point. It is easy to show that the norm of $(\hat{D}^\dagger \hat{D})^{-1}$ is bounded by a multiple of $1/r_i^2$, where r_i is the distance to the singular point. By (70), the same is true about the norm of F_A . Further, one can show that for sufficiently small r_i all the elements of $\text{Ker } \hat{D}^\dagger$ are bounded by a multiple of $\exp(-|r|r_i)$. (This is due to the exponential decay of the Green's function $(\hat{D}^\dagger \hat{D})^{-1}$.) Then from the definition of ϕ one can easily see that

$$\|\phi\| \leq \frac{\text{const}}{r_i}.$$

In the appendix we prove that any solution of the $U(2)$ Bogomolny equation with such a singularity has the following form:

$$\phi(x) \sim h(x) \begin{pmatrix} m_1 \phi_0(r_i) & 0 \\ 0 & m_2 \phi_0(r) \end{pmatrix} h(x)^{-1} + O(1), \quad (71)$$

$$d_A \phi(x) \sim h(x) \begin{pmatrix} m_1 d\phi_0(r_i) & 0 \\ 0 & m_2 d\phi_0(r) \end{pmatrix} h(x)^{-1} + O(1), \quad (72)$$

$$A(x) \sim h(x) \begin{pmatrix} m_1 A_0(x - x_i) & 0 \\ 0 & m_2 A_0(x - x_i) \end{pmatrix} h(x)^{-1} \\ + ih(x) dh(x)^{-1} + O(1). \quad (73)$$

Here $h(x)$ is a $U(2)$ -valued function defined in the neighborhood of the singular point, m_1, m_2 are integers, and $\phi_0(r)$ and $A_0(x)$ have been defined in Sect. 2. The idea of the proof is to lift the monopole to an instanton on the Taub-NUT space with a point deleted,

use the Uhlenbeck compactification theorem to show that the instanton can be continued to the deleted point, and then project the instanton back to three dimensions.

Using these formulas for A and ϕ it is straightforward to compute $V(z)$ near a singular point and compare with what the spectral curve predicts. The results match if and only if one of the m_i is zero, and the other one is 1 or -1 , depending on whether z_i is a root of $A(z)$ or $C(z)$. This completes the demonstration that the inverse Nahm transform produces a periodic monopole with singularities out of any solution of the Hitchin equations with the boundary conditions as in Sect. 5.

7. Nahm Transform for Periodic $U(m)$ Monopoles

Periodic $U(m)$ monopoles with singularities defined in Sect. 2.3 can be analyzed in the same manner as $U(2)$ monopoles. Let us describe the result of the Nahm transform applied to a periodic $U(m)$ monopole with n_+ (resp. n_-) singularities of positive (resp. negative) Chern class and nonabelian charges $(k_1, k_2, \dots, k_{m-1})$.

The Nahm transform yields a solution of the Hitchin equations on a cylinder with several points deleted. The rank of the Hitchin bundle is equal to $\max(n_+, n_-, k_1, \dots, k_{m-1})$. The deleted points are determined as follows. Recall that the integers ℓ_1, \dots, ℓ_m which determine the behavior of the eigenvalues of ϕ at infinity are given by

$$\begin{aligned} l_1 &= k_1 - n_+, \\ l_2 &= k_2 - k_1, \\ &\dots \\ l_{m-1} &= k_{m-1} - k_{m-2}, \\ l_m &= n_- - k_{m-1}. \end{aligned} \tag{74}$$

These numbers are ordered: $\ell_1 \geq \ell_2 \geq \dots \geq \ell_m$. If $\ell_i = 0$, then the i^{th} eigenvalue of ϕ approaches a finite value v_i for $|z| \rightarrow \infty$, and the corresponding eigenvalue of the holonomy of A along S^1 approaches a constant value e^{b_i} . Set $v_i = v_i + ib_i$. The deleted points are in one-to-one correspondence with i such that $\ell_i = 0$, and they are located at $w = w_i = e^{v_i}$.

The Higgs field $\hat{\phi}$ has a simple pole at each of the deleted points:

$$\hat{\phi} \sim \frac{R_i}{s - s_i}.$$

Here $s_i = v_i/(2\pi)$ and R_i is a rank-one matrix satisfying $[R_i, R_i^\dagger] = 0$ whose only non-zero eigenvalue is equal to $\mu_i w_i$ (see (22) for the definition of μ_i). The connection \hat{A} also has a simple pole at the deleted point:

$$\hat{A} \sim \frac{Q_i}{s - s_i},$$

where Q_i is a rank-one matrix satisfying $[Q_i^\dagger, R_i] = 0$.

Before we describe the behavior of the Hitchin data for $r \rightarrow \pm\infty$, let us note two alternative definitions of the rank K in terms of ℓ_i . Let j_+ be the number of strictly positive ℓ_i . It is easy to see that

$$K = n_- + \sum_{i=1}^{j_+} \ell_i. \tag{75}$$

Similarly, if j_- is the number of strictly negative ℓ_i , then we have an identity

$$K = n_+ - \sum_{i=m}^{m-j_-+1} \ell_i. \quad (76)$$

The curvature of \hat{A} goes to zero as $\text{const}/|r|^{3/2}$ for $|r| \rightarrow \infty$, therefore \hat{A} has well-defined limiting holonomies for $r \rightarrow \pm\infty$.

For $r \rightarrow +\infty$, n_- of the eigenvalues of $\hat{\phi}$ approach constant values

$$z_1, z_2, \dots, z_{n_-},$$

where z_i is the z -coordinate of the i^{th} singularity with $e_i = -1$. The corresponding eigenvectors are also the eigenvectors of the limiting holonomy of \hat{A} with eigenvalues

$$e^{i\chi_1}, e^{i\chi_2}, \dots, e^{i\chi_{n_-}},$$

where χ_i is the χ -coordinate of the i^{th} singularity with $e_i = -1$.

For $r \rightarrow +\infty$, ℓ_1 of the eigenvalues of $\hat{\phi}$ asymptote to

$$\omega_{\ell_1}^j \exp\left(\frac{2\pi s - \mathfrak{v}_1}{\ell_1}\right), \quad j = 1, \dots, \ell_1,$$

where ω_p denotes $\exp(2\pi i/p)$. The limiting holonomy of \hat{A} preserves the subspace spanned by the corresponding eigenvectors and its restriction to this subspace is equal to

$$V_{\ell_1} \exp(2\pi i\alpha_1/\ell_1),$$

where V_{ℓ_1} is the $\ell_1 \times \ell_1$ ‘‘shift matrix’’ defined in (52). Further, for $r \rightarrow +\infty$, ℓ_2 of the eigenvalues of $\hat{\phi}$ approach

$$\omega_{\ell_2}^j \exp\left(\frac{2\pi s - \mathfrak{v}_2}{\ell_2}\right), \quad j = 1, \dots, \ell_2,$$

and so on, until we reach ℓ_{j_+} eigenvalues of $\hat{\phi}$ which approach

$$\omega_{\ell_{j_+}}^j \exp\left(\frac{2\pi s - \mathfrak{v}_{j_+}}{\ell_{j_+}}\right), \quad j = 1, \dots, \ell_{j_+}.$$

By (75), we have described the behavior of all K eigenvalues of $\hat{\phi}$ and the limiting holonomy of \hat{A} . Note that since all the numbers $\ell_1, \dots, \ell_{j_+}$ are positive, the eigenvalues of $\hat{\phi}$ which do not approach finite values grow exponentially as $r \rightarrow +\infty$.

For $r \rightarrow -\infty$ the situation is similar. n_+ of the eigenvalues of $\hat{\phi}$ approach constant values

$$z_{n_-+1}, z_{n_-+2}, \dots, z_{n_-+n_+},$$

where z_{n_-+i} is the z -coordinate of the i^{th} singularity with $e_i = +1$. The corresponding eigenvectors are also the eigenvectors of the limiting holonomy of \hat{A} with eigenvalues

$$e^{i\chi_{n_-+1}}, e^{i\chi_{n_-+2}}, \dots, e^{i\chi_{n_-+n_+}},$$

where χ_{n-+i} is the χ -coordinate of the i^{th} singularity with $e_i = +1$. $|\ell_m|$ of the eigenvalues of $\hat{\phi}$ asymptote to

$$\omega_{\ell_m}^j \exp\left(\frac{2\pi s - \mathbf{v}_m}{\ell_m}\right), \quad j = 1, \dots, |\ell_m|.$$

The limiting holonomy of \hat{A} preserves the subspace spanned by the corresponding eigenvectors and its restriction to this subspace is equal to

$$V_{\ell_m} \exp(2\pi i \alpha_m / \ell_m).$$

$|\ell_{m-1}|$ eigenvalues of $\hat{\phi}$ asymptote to

$$\omega_{\ell_{m-1}}^j \exp\left(\frac{2\pi s - \mathbf{v}_{m-1}}{\ell_{m-1}}\right), \quad j = 1, \dots, |\ell_{m-1}|,$$

and so on, until we reach the $|\ell_{m-j-+1}|$ eigenvalues of $\hat{\phi}$ which asymptote to

$$\omega_{\ell_{m-j-+1}}^j \exp\left(\frac{2\pi s - \mathbf{v}_{m-j-+1}}{\ell_{m-j-+1}}\right), \quad j = 1, \dots, |\ell_{m-j-+1}|.$$

By (76), we have described the behavior of all K eigenvalues of $\hat{\phi}$ and the limiting holonomy of \hat{A} . Note that since all the numbers $\ell_m, \dots, \ell_{m-j-+1}$ are negative, the eigenvalues of $\hat{\phi}$ which do not approach finite values grow exponentially as $r \rightarrow -\infty$.

The spectral curve corresponding to such a periodic monopole can be determined either by computing the characteristic polynomial of $\hat{\phi}(w)$, or directly from the definition of the monopole. The latter way is simpler and yields the following equation in $\mathbb{C} \times \mathbb{C}^*$:

$$A(z)w^m + B_1(z)w^{m-1} + B_2(z)w^{m-2} + \dots + B_{m-1}(z)w + C(z) = 0.$$

Here

$$A(z) = \prod_{i=1}^{n-} (z - z_i), \quad C(z) = e^{\mathbf{v}_1 + \dots + \mathbf{v}_m} \prod_{i=n-+1}^n (z - z_i),$$

and B_i is a polynomial of degree k_i whose leading and next-to-leading coefficients are determined by the asymptotics of the monopole fields. Note that this is precisely the Seiberg-Witten curve for the $\mathcal{N} = 2$ $d = 4$ gauge theory corresponding to our periodic $U(m)$ monopole [31].

8. Concluding Remarks

In this paper we have studied the periodic monopole with singularities. From the physical point of view, they are of interest for several reasons. First, as we have shown above, their moduli spaces can be used to “solve” $\mathcal{N} = 2$ $d = 4$ gauge theories compactified on a circle. Although computing the metric on the moduli space of periodic monopoles is hard, it is still an infinitely easier problem than summing up an infinite number of instantons in a quantum field theory, including monopole loops wrapping the compactified direction. At the moment it is not clear if any of these metrics can be computed in a closed form, but it seems reasonably straightforward to compute the asymptotic expansion far along the Coulomb branch. This problem will be addressed in [9].

Second, we have seen that the Nahm transform for periodic monopoles with singularities is described by the Hitchin equations on a cylinder, and that the Hitchin spectral curve is the Seiberg-Witten curve of the corresponding gauge theory. Recall now that the space of solutions of the Hitchin equations is an algebraically completely integrable system, $\hat{\phi}$ being the Lax operator, and s being the spectral parameter [16, 24]. Thus our approach allows to associate to any $N = 2d = 4$ gauge theory which admits a brane realization an integrable system, so that the Seiberg-Witten curve is given by the characteristic polynomial of the Lax operator. The relation between $N = 2d = 4$ gauge theories and completely integrable systems was noted previously [12, 25, 10], but its origin remained somewhat mysterious. For finite $N = 2$ gauge theories this question was clarified in [20, 19] (see also [13]). In our work on nonsingular periodic monopoles [5], we described the Hitchin system corresponding to an $N = 2$ super-Yang-Mills theory without matter, and in this paper we generalized this to a much larger class of theories with product gauge groups and hypermultiplets.

Third, in Witten's approach to $N = 2$ gauge theories [31], the Seiberg-Witten curve describes the world-volume of the M-theory fivebrane. Then BPS monopoles wrapping the compactified direction are represented by Euclidean M2-branes whose boundaries lie on the M5-brane. Computing the contribution of such configurations to the metric on the moduli space seems rather difficult. On the other hand, our methods in principle allow to compute the contribution of an arbitrary number of wrapped BPS monopoles. Conceivably, this could shed some light on the dynamics of open M2-branes.

From the mathematical point of view, periodic monopoles with singularities are of interest because their moduli spaces provide new examples of hyperkähler manifolds without any continuous isometries. The interpretation of the moduli spaces in terms of $N = 2$ gauge theories makes it clear that they are complete (because the Higgs branch is absent) and locally flat at infinity (because the gauge theories are asymptotically free or finite). In the case when the rank of the gauge group is one, the moduli space is an elliptic fibration over \mathbb{C} . No examples of complete hyperkähler metric on elliptic fibrations which are locally flat at infinity were known prior to this work (an incomplete example is provided by the so-called Ooguri-Vafa metric [26]), and it is an interesting question which fibrations admit such metrics. We will suggest a possible answer to this question in [9].

Acknowledgements. It is our pleasure to thank Nigel Hitchin, Marcos Jardim, and Tony Pantev for discussions. S.Ch. is grateful to the Institute for Advanced Study, Princeton, for hospitality during the final stage of this work. S.Ch. was supported in part by NSF grant PHY9819686. A.K. was supported in part by DOE grant DE-FG02-90ER40542.

A. Singularities of the Monopole Fields

Let $U \subset \mathbb{R}^3$ be a punctured neighborhood of the origin, and let E be a rank two unitary vector bundle on U . Let A be a unitary connection on E and ϕ be a Hermitian section of $\text{End}(E)$ such that the Bogomolny equation $F_A = *d_A\phi$ is satisfied and

$$\|\phi\| \leq \frac{C_1}{|x|}, \quad (77)$$

$$\|F_A\| \leq \frac{C_2}{|x|^2}, \quad (78)$$

where $C_1, C_2 > 0$. We are going to show that there exist integers m_1, m_2 and a $U(2)$ -valued function $h(x)$ defined on U such that

$$\phi(x) \sim h(x) \begin{pmatrix} m_1 \phi_0(r) & 0 \\ 0 & m_2 \phi_0(r) \end{pmatrix} h(x)^{-1} + O(1), \quad (79)$$

$$A(x) \sim h(x) \begin{pmatrix} m_1 A_0(x) & 0 \\ 0 & m_2 A_0(x) \end{pmatrix} h(x)^{-1} + i h(x) dh(x)^{-1} + O(1). \quad (80)$$

Here $r = |x|$, $\phi_0(r) = -1/(2r)$, and $A_0(x)$ is defined in Sect. 2.

Consider a four-dimensional noncompact manifold X with coordinates (x, t) , $x \in U$, $\theta \in \mathbb{R}/(4\pi\mathbb{Z})$ and the metric

$$ds^2 = V(x) dx^i dx^j \delta_{ij} + V(x)^{-1} (d\theta + \omega_i(x) dx^i)^2, \quad (81)$$

where

$$V = 1 + \frac{1}{|x|^2},$$

and $\omega = \omega_i(x) dx^i$ is a unitary connection on a line bundle on U satisfying the $U(1)$ Bogomolny equation:

$$d\omega = *dV.$$

It is easy to see that $\omega(x) = -A_0(x)$, and therefore the degree of the line bundle is -1 .

The metric (81) is of the Taub-NUT (or Gibbons-Hawking) type, and it is well known that it is hyperkähler and admits a tri-holomorphic $U(1)$ action generated by the vector field $\frac{\partial}{\partial \theta}$. The manifold X endowed with this metric admits a nonsingular partial completion obtained by adding a single point p over $x = 0$. This point is invariant with respect to the $U(1)$ action mentioned above. We denote this partial completion by \tilde{X} .

Let ψ be the projection $X \rightarrow U$. Let \tilde{A} be a connection on $\psi^*(E)$ defined by

$$\tilde{A} = \psi^*(A) + \psi^*(V^{-1}\phi)(d\theta + \omega). \quad (82)$$

It is easy to check that it is a self-dual $U(1)$ -invariant connection on $\psi^*(E)$. The self-duality holds because A and ϕ satisfy the Bogomolny equation. (This observation is due to P. Kronheimer [23].) We claim that both $\psi^*(E)$ and \tilde{A} can be continued in a unique manner to the point p while preserving $U(1)$ -invariance and self-duality. Indeed, the action density of \tilde{A} is given by

$$\begin{aligned} \text{Tr } F_{\tilde{A}} \wedge *F_{\tilde{A}} &= 2\psi^* \left(V^{-2} \text{Tr } F_A \wedge *F_A + dV^{-1} \wedge *dV^{-1} \text{Tr } \phi^2 \right) d\theta \\ &\quad - d \left[\psi^* \left(*dV^{-1} \text{Tr } \phi^2 \right) d\theta \right]. \end{aligned} \quad (83)$$

Using (77), one can easily see that the integral of the action density over X converges. By Uhlenbeck's compactification theorem [30], there is a unique smooth continuation of $\psi^*(E)$ and \tilde{A} to p .

Obviously, the resulting connection is invariant with respect to $\frac{\partial}{\partial \theta}$ and self-dual. It is easy to see that in a θ -invariant gauge any such connection has the following form in the neighborhood of p :

$$\tilde{A} = h(x) \begin{pmatrix} \left(\frac{m_1}{2} + O(x)\right) d\theta & 0 \\ 0 & \left(\frac{m_2}{2} + O(x)\right) d\theta \end{pmatrix} h(x)^{-1} + i h(x) dh(x)^{-1} + a_i(x) dx^i,$$

where m_1, m_2 are integers, $a_i(x)$ are smooth $u(2)$ -valued functions, and $h(x)$ is a smooth $U(2)$ -valued function in the punctured neighborhood of $x = 0$. The geometric meaning of m_1 and m_2 is the following: they are the weights of the $U(1)$ action on the fiber of $\psi^*(E)$ at p . Solving (82) for A and ϕ and recalling that $\omega(x) = -A_0(x)$, we obtain (79–80).

References

1. Atiyah, M., Hitchin, N.: *The Geometry and Dynamics of Magnetic Monopoles*. Princeton, USA: University Press, 1988
2. Banks, T., Douglas, M.R., Seiberg, N.: Probing F theory with branes. *Phys. Lett.* **B387**, 278 (1996) [hep-th/9605199]
3. Biquard, O., Jardim, M.: Asymptotic behaviour and the moduli space of doubly-periodic instantons. Preprint math.dg/0005154
4. Chalmers, G., Hanany, A.: Three dimensional gauge theories and monopoles. *Nucl. Phys.* **B489**, 223 (1997) [hep-th/9608105]
5. Cherkis, S.A., Kapustin, A.: Nahm transform for periodic monopoles and $N = 2$ super Yang-Mills theory. *Commun. Math. Phys.* **218**, 333 (2001) [hep-th/0006050]
6. Cherkis, S.A., Kapustin, A.: Singular monopoles and gravitational instantons. *Commun. Math. Phys.* **203**, 713 (1999) [hep-th/9803160]
7. Cherkis, S.A., Kapustin, A.: $D(k)$ gravitational instantons and Nahm equations. *Adv. Theor. Math. Phys.* **2**, 1287 (1999) [hep-th/9803112]
8. Cherkis, S.A., Kapustin, A.: Singular monopoles and supersymmetric gauge theories in three dimensions. *Nucl. Phys.* **B525**, 215 (1998) [hep-th/9711145]
9. Cherkis, S.A., Kapustin, A.: Hyperkähler metrics from periodic monopoles. *Phys. Rev. D* **65**, 084015 (2002) [hep-th/0109141]
10. Donagi, R., Witten, E.: Supersymmetric Yang-Mills theory and integrable systems. *Nucl. Phys.* **B460**, 299 (1996) [hep-th/9510101]
11. Ganor, O., Sethi, S.: New perspectives on Yang-Mills theories with sixteen supersymmetries. *JHEP* **9801**, 007 (1998) [hep-th/9712071]
12. Gorsky, A. et al.: Integrability and Seiberg-Witten exact solution. *Phys. Lett.* **B355**, 466 (1995) [hep-th/9505035]
13. Gukov, S.: Seiberg-Witten solution from matrix theory. Preprint hep-th/9709138
14. Hanany, A., Witten, E.: Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics. *Nucl. Phys.* **B492**, 152 (1997) [hep-th/9611230]
15. Hitchin, N.J.: Twistor construction of Einstein metrics. In: *Global Riemannian Geometry (Durham, 1983)*, T.J. Willmore, N.J. Hitchin (eds), Chichester: Horwood, 1984, p. 115
16. Hitchin, N.: Stable bundles and integrable systems. *Duke Math. J.* **54**, 91 (1987)
17. Jaffe, A., Taubes, C.: *Vortices and Monopoles. Structure of Static Gauge Theories*. Boston: Birkhäuser, 1980
18. Jardim, M.: Construction of doubly-periodic instantons. *Commun. Math. Phys.* **216**, 1 (2001) [math.dg/9909069]; Nahm transform of doubly-periodic instantons. Preprint math.dg/9910120; Spectral curves and the Nahm transform of doubly-periodic instantons. Preprint math.ag/9909146
19. Kapustin, A.: Solution of $N = 2$ gauge theories via compactification to three dimensions. *Nucl. Phys.* **B534**, 531 (1998) [hep-th/9804069]
20. Kapustin, A., Sethi, S.: The Higgs branch of impurity theories. *Adv. Theor. Math. Phys.* **2**, 571 (1998) [hep-th/9804027]
21. Klemm, A. et al.: Self-dual strings and $N = 2$ supersymmetric field theory. *Nucl. Phys.* **B477**, 746 (1996) [hep-th/9604034]
22. Kraan, T.C., van Baal, P.: Monopole constituents inside $SU(N)$ calorons. *Phys. Lett.* **B435**, 389 (1998) [hep-th/9806034]
23. Kronheimer, P.B.: Monopoles and Taub-NUT metrics. M.Sc. Thesis, Oxford, 1985
24. Markman, E.: Spectral curves and integrable systems. *Comp. Math.* **93**, 255 (1994)
25. Martinec, E., Warner, N.: Integrable systems and supersymmetric gauge theory. *Nucl. Phys.* **B459**, 97 (1996) [hep-th/9509161]
26. Ooguri, H., Vafa, C.: Summing up D-instantons. *Phys. Rev. Lett.* **77**, 3296 (1996) [hep-th/9608079]
27. Seiberg, N., Witten, E.: Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory. *Nucl. Phys.* **B426**, 19 (1994), Erratum *ibid.* **B430**, 485 (1994) [hep-th/9407087]; Monopoles, duality, and chiral symmetry breaking in $N = 2$ supersymmetric QCD. *Nucl. Phys.* **B431**, 484 (1994) [hep-th/9408099]

28. Seiberg, N., Witten, E.: Gauge dynamics and compactification to three dimensions. In: *The Mathematical Beauty of Physics (Saclay, 1996)*. River Edge, NJ: World Sci. Publishing, 1997, p. 333 [hep-th/9607163]
29. Simpson, C.T.: Harmonic bundles on noncompact curves. *J. Am. Math. Soc.* **3**, 713 (1990)
30. Uhlenbeck, K.K.: Removable singularities in Yang-Mills fields. *Commun. Math. Phys.* **83**, 11 (1982)
31. Witten, E.: Solutions of four-dimensional field theories via M-theory. *Nucl. Phys.* **B500**, 3 (1997) [hep-th/9703166]

Communicated by A. Connes