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Dimension and Dynamical Entropy for Metrized C^*-Algebras

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Abstract: We introduce notions of dimension and dynamical entropy for unital C^* -algebras "metrized" by means of ${}_{\mathrm{c}}$Lip-norms, which are complex-scalar versions of the Lip-norms constitutive of Rieffel's compact quantum metric spaces. Our examples involve the UHF algebras M_{p^∞} and noncommutative tori. In particular we show that the entropy of a noncommutative toral automorphism with respect to the canonical ${}_{\mathrm{c}}$Lip-norm coincides with the topological entropy of its commutative analogue.

1. Introduction

The idea of a noncommutative metric space was introduced by Connes [5–7] who showed in a noncommutative-geometric context that a Dirac operator gives rise to a metric on the state space of the associated C^*-algebra. The question of when the topology thus obtained agrees with the weak* topology was pursued by Rieffel [25, 26], whose line of investigation led to the notion of a quantum metric space defined by specifying a Lip-norm on an order-unit space [27]. This definition includes Lipschitz seminorms on functions over compact metric spaces and more generally applies to unital C^*-algebras, the subspaces of self-adjoint elements of which form important examples of order-unit spaces. We would like to investigate here the structures which arise by essentially specializing and complexifying Lip-norms to obtain what we call "${}_{\mathrm{c}}$Lip-norms" on unital C^*-algebras. We thereby propose a notion of dimension for ${}_{\mathrm{c}}$Lip-normed unital C^*-algebras, along with two dynamical entropies (the second a measure-theoretic version of the first) which operate within the restricted domain of ${}_{\mathrm{c}}$Lip-norms satisfying the Leibniz rule (our version of noncommutative metrics).

Means for defining dimension appeared within Rieffel's work on quantum Gromov-Hausdorff distance in [27], where it is pointed out that Definition 13.4 therein gives rise to possible "quantum" versions of Kolmogorov ε-entropy. We take here a different approach which has its origin in Rieffel's prior study of Lip-norms in [25, 26], where the total boundedness of the set of elements of Lip-norm and order-unit norm no greater than 1 was shown to be a fundamental property. This total boundedness leads us to a definition of dimension using approximation by linear subspaces (Sect. 3). This also makes sense for order-unit spaces, but we will concentrate on C^*-algebras, the particular geometry of which will play a fundamental role in our examples, which involve the UHF algebras M_{p^∞} and noncommutative tori (Sect. 4). We will also show that for usual Lipschitz seminorms we recover the Kolmogorov dimension (Proposition 3.9).

We also use approximation by linear subspaces to define our two dynamical "product" entropies (Sect. 5). The definitions formally echo that of Voiculescu-Brown approximation entropy [33, 4], but here the algebraic structure enters in a very different way. One drawback of the Voiculescu-Brown entropy as a noncommutative invariant is the difficulty of obtaining nonzero lower bounds (however frequently the entropy is in fact positive) in systems in which the dynamical growth is not ultimately registered in algebraically or statistically commutative structures. We have in mind our main examples, the noncommutative toral automorphisms. For these we only have partial information about the Voiculescu-Brown entropy in the nonrational case (see [33, Sect. 5] and the discussion in the following paragraph), and even deciding when the entropy is positive is a problem (in the rational case, i.e., when the rotation angles with respect to pairs of canonical unitaries are all rational, we obtain the corresponding classical value, as follows from the upper bound established in [33, Sect. 5] along with the fact that the corresponding commutative toral automorphism sits as a subsystem, so that we can apply monotonicity). We show that for general noncommutative toral automorphisms the product entropy relative to the canonical "metric" coincides with the topological entropy of the corresponding toral homeomorphism (Sect. 7). In analogy with the relation between the discrete Abelian group entropy of a discrete Abelian group automorphism and the topological entropy of its dual [23], product entropy (which is an analytic version of discrete Abelian group entropy) may roughly be thought of as a "dual" counterpart of Voiculescu-Brown entropy, as illustrated by the key role played by unitaries in obtaining lower bounds for the former. When passing from the commutative to the noncommutative in an example like the torus, the "dual" unitary description persists (ensuring a metric rigidity that facilitates computations) while the underlying space and the transparency of the complete order structure vanish. The shift on M_{p^∞}, on the other hand, is equally amenable to analysis from the canonical unitary and complete order viewpoints due to the tensor product structure, and its value can be precisely calculated for both product entropy (Sect. 6) and Voiculescu-Brown entropy [33, Prop. 4.7] (see also [9, 33, 32, 13, 1] for computations with respect to other entropies which we will discuss below).

Since we are not dealing with discrete entities as in the discrete Abelian group entropy setting, product entropies will not be C^*-algebraic conjugacy invariants, but rather bi-Lipschitz C^*-algebraic conjugacy invariants (see Definition 2.8). In particular, if we consider ${}_{\mathrm{c}}$Lip-norms arising via the ergodic action of a compact group G equipped with a length function (see Example 2.13), the entropies will be "G-C^*-algebraic" invariants, that is, they will be invariant under C^*-algebraic conjugacies respecting the given group actions. To put this in context, we first point out that there have been two basic approaches to developing C^*-algebraic and von Neumann algebraic dynamical invariants which extend classical entropies. While the definitions of Voiculescu [33] and Brown [4] are based on local approximation, the measure-theoretic Connes-Narnhofer-Thirring (CNT) entropy [8] (a generalization of Connes-Størmer entropy [9]) and Sauvageot-Thouvenot entropy [29] take a physical observable viewpoint and are defined via the notions of an Abelian model and a stationary coupling with an Abelian system, respectively (see [31] for a survey). Because of the role played by Abelian systems in their respective definitions, the CNT and Sauvageot-Thouvenot entropies (which are known to coincide on nuclear C^*-algebras [29, Prop. 4.1]) function most usefully as invariants in the asymptotically Abelian situation. For instance, their common value for noncommutative 2-toral automorphisms with respect to the canonical tracial state is zero for a set of rotation parameters of full Lebesgue measure [20], while for rational parameters the corresponding classical value is obtained [16] and for the countable set of irrational rotation parameters for which the system is asymptotically Abelian (at least when restricted to a nontrivial invariant C^*-subalgebra generated by a pair of products of powers of the canonical unitaries) the value is positive when the associated matrix is hyperbolic (see [31, Chap. 9]) (in this case the Voiculescu-Brown entropy is thus also positive by [33, Prop. 4.6]). Other entropies which are not C^*-algebraic or von Neumann algebraic dynamical invariants have been introduced in [14, 32, 1]. The definitions of [14, 32] take a noncommutative open cover approach and hence are difficult to compute for examples like the noncommutative toral automorphisms (see the discussion in the last section of [14]). In [2] the Alicki-Fannes entropy [1] for general noncommutative 2-toral automorphisms was shown to coincide with the corresponding classical value if the dense algebra generated by the canonical unitaries is taken as the special set required by the definition. What is particular about the product entropies is that, from the perspective of noncommutative geometry as exemplified in noncommutative tori [24], they provide computable quantities which reflect the metric rigidity but require no additional structure to function (i.e., they are "metric" dynamical invariants).

The organization of the paper is as follows. In Sect. 2 we recall Rieffel's definition of a compact quantum metric space, and with this motivation then introduce ${}_{\mathrm{c}}$Lip-norms and the relevant maps for ${}_{\mathrm{c}}$Lip-normed unital C^*-algebras, after which we examine some examples. In Sect. 3 we introduce metric dimension and establish some properties, including its coincidence with Kolmogorov dimension for usual Lipschitz seminorms. Section 4 is subdivided into two subsections in which we compute the metric dimension for examples arising from compact group actions on the UHF algebras M_{p^∞} and noncommutative tori, respectively. The two subsections of Sect. 5 are devoted to introducing the two respective product entropies and recording some properties, and in Sects. 6 and 7 we carry out computations for the shift on M_{p^∞} and automorphisms of noncommutative tori, respectively.

In this paper we will be working exclusively with *unital* (i.e., "compact") C^*-algebras as generally indicated. For a unital C^*-algebra A we denote by 1 its unit, by $S(A)$ its state space, and by A_{sa} the real vector space of self-adjoint elements of A. Other general notation is introduced in Notation 2.2, 3.1, and 5.1.

2. ${}_{\mathrm{c}}$Lip-Norms on Unital C^*-Algebras

The context for our definitions of dimension and dynamical entropy will essentially be a specialization of Rieffel's notion of a compact quantum metric space to the complex-scalared domain of C^*-algebras. A compact quantum metric space is defined by specifying a Lip-norm on an order-unit space (see below), and this has a natural self-adjoint complex-scalared interpretation on a unital C^*-algebra in what will call a "${}_{\mathrm{c}}$Lip-norm" (Definition 2.3). In fact ${}_{\mathrm{c}}$Lip-norms will make sense in more general complex-scalared situations (e.g., operator systems), as will our definition of dimension (Definition 3.3), but we will stick to C^*-algebras as these will constitute our examples of interest and multiplication will ultimately enter the picture when we come to dynamical entropy, for which the Leibniz rule will play an important role.

We begin by recalling from [27] the definition of a compact quantum metric space. Recall that an *order-unit space* is a pair (A, e) consisting of a real partially ordered vector space A with a distinguished element e, called the *order unit*, such that, for each $a \in A$,

(1) there exists an $r \in \mathbb{R}$ with $a \leq re$, and
(2) if $a \leq re$ for all $r \in \mathbb{R}_{>0}$ then $a \leq 0$.

An order-unit space is a normed vector space under the norm

$$\|a\| = \inf\{r \in \mathbb{R} : -re \leq a \leq re\},$$

from which we can recover the order via the fact that $0 \leq a \leq e$ if and only if $\|a\| \leq 1$ and $\|e - a\| \leq 1$. A *state* on an order-unit space (A, e) is a norm-bounded linear functional on A whose dual norm and value on e are both 1 (which automatically implies positivity). The state space of A is denoted by $S(A)$. An important and motivating example of an order-unit space is provided by the space of self-adjoint elements of a unital C^*-algebra. In fact every order-unit space is isomorphic to some order-unit space of self-adjoint operators on a Hilbert space (see [27, Appendix 2]). Via Kadison's function representation we also see that order-unit spaces are precisely, up to isomorphism, the dense unital subspaces of spaces of affine functions over compact convex subsets of topological vector spaces (see [26, Sect. 1]).

Definition 2.1 ([27, Defs. 2.1 and 2.2]). *Let A be an order-unit space. A* **Lip-norm** *on A is a seminorm L on A such that*

(1) for all $a \in A$ we have $L(a) = 0$ if and only if $a \in \mathbb{R}e$, and
(2) the metric ρ_L defined on the state space $S(A)$ by

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in A \text{ and } L(a) \leq 1\}$$

induces the weak topology.*

A **compact quantum metric space** *is a pair (A, L) consisting of an order-unit space A with a Lip-norm L.*

As mentioned above, the subspace A_{sa} of self-adjoint elements in a unital C^*-algebra A forms an order-unit space, and so we can specialize Rieffel's definition in a more or less straightforward way to the C^*-algebraic context. We would like, however, our "Lip-norm" to be meaningfully defined on the C^*-algebra A as a vector space over the complex numbers. Such a "Lip-norm" should be invariant under taking adjoints, and thus, after introducing some notation, we make the following definition, which seems reasonable in view of Proposition 2.4.

Notation 2.2. *Let L be a seminorm on the unital C^*-algebra A which is permitted to take the value $+\infty$. We denote the sets $\{a \in A : L(a) < \infty\}$ and $\{a \in A : L(a) \leq r\}$ (for a given $r > 0$) by $\mathcal{L}$ and $\mathcal{L}_r$ (or in some cases for clarity by $\mathcal{L}^A$ and $\mathcal{L}_r^A$), respectively. For $r > 0$ we denote by A_r the norm ball $\{a \in A : \|a\| \leq r\}$. We write ρ_L to refer to the semi-metric defined on the state space $S(A)$ by*

$$\rho_L(\sigma, \omega) = \sup_{a \in \mathcal{L}_1} |\sigma(a) - \omega(a)|$$

for all $\sigma, \omega \in S(A)$. We write $\mathrm{diam}(S(A))$ *to mean the diameter of $S(A)$ with respect to the metric ρ_L. We say that L* **separates** $S(A)$ *if for every pair σ, ω of distinct states on A there is an $a \in \mathcal{L}$ such that $\sigma(a) \neq \omega(a)$, which is equivalent to ρ_L being a metric.*

Definition 2.3. *By a* $_{\mathrm{c}}$**Lip-norm** *on a unital C^*-algebra A we mean a seminorm L on A, possibly taking the value $+\infty$, such that*

(i) $L(a^*) = L(a)$ *for all* $a \in A$ *(adjoint invariance),*
(ii) for all $a \in A$ *we have* $L(a) = 0$ *if and only if* $a \in \mathbb{C}1$ *(ergodicity),*
(iii) L *separates* $S(A)$ *and the metric* ρ_L *induces the weak* topology on* $S(A)$.

Proposition 2.4. *Let L be a $_{\mathrm{c}}$Lip-norm on a unital C^*-algebra A. Then the restriction L' of L to the order-unit space $\mathcal{L} \cap A_{\mathrm{sa}}$ is a Lip-norm, and the restriction map from $S(A)$ to $S(\mathcal{L} \cap A_{\mathrm{sa}})$ is a weak* homeomorphism which is isometric relative to the respective metrics ρ_L and $\rho_{L'}$. Also, if L is any adjoint-invariant seminorm on A, possibly taking the value $+\infty$, such that the restriction L' to $\mathcal{L} \cap A_{\mathrm{sa}}$ is a Lip-norm which separates $S(A_{\mathrm{sa}}) \cong S(A)$, then L is a $_{\mathrm{c}}$Lip-norm, and the restriction from $S(A)$ to $S(\mathcal{L} \cap A_{\mathrm{sa}})$ is a weak* homeomorphism which is isometric relative to the respective metrics ρ_L and $\rho_{L'}$.*

Proof. The proposition is a consequence of the fact that if L is an adjoint-invariant seminorm then for any $\sigma, \omega \in S(A)$ the suprema of

$$|\sigma(a) - \omega(a)|$$

over the respective sets $\mathcal{L}_1$ and $\mathcal{L}_1 \cap A_{\mathrm{sa}}$ are the same, as shown in the discussion prior to Definition 2.1 in [27]. The second statement of the proposition also requires the fact that the ergodicity of L' (condition (1) of Definition 2.1) implies the ergodicity of L, which can be seen by noting that if $a \in A$ and $L(a) < \infty$ then setting $\mathrm{Re}(a) = (a + a^*)/2$ and $\mathrm{Im}(a) = (a - a^*)/2i$ (the real and imaginary parts of a) we have $L'(\mathrm{Re}(a)) = 0$ and $L'(\mathrm{Im}(a)) = 0$ by adjoint invariance, so that $\mathrm{Re}(a), \mathrm{Im}(a) \in \mathbb{R}1$ by condition (1) of Definition 2.1, and hence $a = \mathrm{Re}(a) + i\mathrm{Im}(a) \in \mathbb{C}1$. □

The following proposition follows immediately from Theorem 1.8 of [25] (note that the remark following Condition 1.5 therein shows that this condition holds in our case). Condition (4) in the proposition statement will provide the basis for our definitions of dimension and dynamical entropy.

Proposition 2.5. *A seminorm L on a unital C^*-algebra A, possibly taking the value $+\infty$, is a $_{\mathrm{c}}$Lip-norm if and only if it separates $S(A)$ and satisfies*

(1) $L(a^*) = L(a)$ *for all* $a \in A$,
(2) for all $a \in A$ *we have* $L(a) = 0$ *if and only if* $a \in \mathbb{C}1$,
(3) $\sup\{|\sigma(a) - \omega(a)| : \sigma, \omega \in S(A) \text{ and } a \in \mathcal{L}_1\} < \infty$, *and*
(4) the set $\mathcal{L}_1 \cap A_1$ *is totally bounded in A for* $\|\cdot\|$.

When we come to dynamical entropy, $_{\mathrm{c}}$Lip-norms satisfying the Leibniz rule will be of central importance, and so we also make the following definition, which we may think of as describing one possible noncommutative analogue of a compact metric space (cf. Example 2.12).

Definition 2.6. *We say that a $_{\mathrm{c}}$Lip-norm L on a unital C^*-algebra A is a* **Leibniz** $_{\mathrm{c}}$**Lip-norm** *if it satisfies the Leibniz rule*

$$L(ab) \leq L(a)\|b\| + \|a\|L(b)$$

for all $a, b \in \mathcal{L}$.

Although we do not make lower semicontinuity a general assumption for ${}_{\mathrm{c}}$Lip-norms, it will typically hold in our examples, and has the advantage that we can recover the restriction of L to A_{sa} in a straightforward manner from ρ_L, as shown by the following proposition, which is a consequence of [26, Thm. 4.1] and Proposition 2.4.

Proposition 2.7. *Let L be a lower semicontinuous ${}_{\mathrm{c}}$Lip-norm on a unital C^*-algebra A. Then for all $a \in A_{\mathrm{sa}}$ we have*

$$L(a) = \sup \{|\sigma(a) - \omega(a)|/\rho_L(\sigma, \omega) : \sigma, \omega \in S(A) \text{ and } \sigma \neq \omega\}.$$

As for metric spaces, the essential maps in our ${}_{\mathrm{c}}$Lip-norm context are ones satisfying a Lipschitz condition, which puts a uniform bound on the amount of "stretching" as formalized in the following definition, for which we will adopt the conventional metric space terminology (see [37, Def. 1.2.1]).

Definition 2.8. *Let A and B be unital C^*-algebras with ${}_{\mathrm{c}}$Lip-norms L_A and L_B, respectively. A positive unital (linear) map $\phi : A \to B$ is said to be* **Lipschitz** *if there exists a $\lambda \geq 0$ such that*

$$L_B(\phi(a)) \leq \lambda L_A(a)$$

for all $a \in \mathcal{L}^A$. The least such λ is called the **Lipschitz number** *of ϕ. When ϕ is invertible and both ϕ and ϕ^{-1} are Lipschitz positive we say that ϕ is* **bi-Lipschitz**. *If*

$$L_B(\phi(a)) = L_A(a)$$

for all $a \in A$ then we say that ϕ is **isometric**. *The collection of bi-Lipschitz *-automorphisms of A will be denoted by* $\mathrm{Aut}_L(A)$.

The category of interest for dimension will be that of ${}_{\mathrm{c}}$Lip-normed unital C^*-algebras and Lipschitz positive unital maps, with the bi-Lipschitz positive unital maps forming the categorical isomorphisms. For entropy we will incorporate the algebraic structure in the definitions so that we will want our positive unital maps to be in fact *-homomorphisms. We remark that, as for usual metric spaces, the isometric maps are too rigid to be usefully considered as the categorical isomorphisms, and that our dimension and dynamical entropies will indeed be invariant under general bi-Lipschitz positive unital maps and bi-Lipschitz *-isomorphisms, respectively. We also remark that positive unital maps are C^*-norm contractive [28, Cor. 1], and hence any bi-Lipschitz positive unital map is C^*-norm isometric.

The following pair of propositions capture facts pertaining to Lipschitz maps. The first one is clear.

Proposition 2.9. *Let A, B, and C be unital C^*-algebras with respective ${}_{\mathrm{c}}$Lip-norms L_A, L_B, and L_C. If $\phi : A \to B$ and $\psi : B \to C$ are Lipschitz positive unital maps with Lipschitz numbers λ and ζ, respectively, then $\psi \circ \phi$ is Lipschitz with Lipschitz number bounded by the product $\lambda\zeta$.*

Lemma 2.10. *If L is a ${}_{\mathrm{c}}$Lip-norm on a unital C^*-algebra A and $a \in \mathcal{L} \cap A_{\mathrm{sa}}$ then denoting by $s(a)$ the infimum of the spectrum of a we have*

$$\|a - s(a)1\| \leq L(a)\mathrm{diam}(S(A)),$$

and hence for any $\sigma, \omega \in S(A)$ we have

$$\rho_L(\sigma, \omega) = \sup\{|\sigma(a) - \omega(a)| : a \in A_{\mathrm{sa}},\ L(a) \leq 1,\ \text{and } \|a\| \leq \mathrm{diam}(S(A))\}$$

Proof. Let a be an element of $\mathcal{L} \cap A_{\mathrm{sa}}$ and $s(a)$ the infimum of its spectrum. Then there are $\sigma, \omega \in S(A)$ such that $\sigma(a - s(a)1) = \|a - s(a)1\|$ and $\omega(a) = s(a)$. We then have

$$\begin{aligned}\|a - s(a)1\| &= |\sigma(a - s(a)1) - \omega(a - s(a)1)| \\ &= |\sigma(a) - \omega(a)| \\ &\leq L(a)\mathrm{diam}(S(A)).\end{aligned}$$

The second statement of the lemma follows by noting that, for any $\sigma, \omega \in S(A)$,

$$\rho_L(\sigma, \omega) = \sup\{|\sigma(a) - \omega(a)| : a \in A_{\mathrm{sa}} \text{ and } L(a) \leq 1\}$$

(see the first sentence in the proof of Proposition 2.5), while if $L(a) \leq 1$ then $\|a - s(a)1\| \leq \mathrm{diam}(S(A))$ from above,

$$L(a - s(a)1) = L(a) \leq 1$$

by the ergodicity of L, and

$$|\sigma(a - s(a)1) - \omega(a - s(a)1)| = |\sigma(a) - \omega(a)|.$$

□

Proposition 2.11. *If L is a lower semicontinuous Leibniz ${}_{\mathrm{c}}$Lip-norm on a unital C^*-algebra A and $u \in \mathcal{L}$ is a unitary then* Adu *is bi-Lipschitz, and the Lipschitz numbers of* Adu *and its inverse are bounded by* $2(1 + 2L(u)\mathrm{diam}(S(A)))$.

Proof. By the Leibniz rule and the adjoint-invariance of L, for any $a \in \mathcal{L}$ we have

$$L\left(uau^*\right) \leq L(u)\|a\| + L(a) + \|a\|L\left(u^*\right) = L(a) + 2\|a\|L(u).$$

For any $\sigma, \omega \in S(A)$ we therefore have, using Lemma 2.10 for the first equality,

$$\begin{aligned}&\rho_L(\sigma \circ \mathrm{Ad}u, \omega \circ \mathrm{Ad}u) \\ &\quad = \sup\{|\sigma\left(uau^*\right) - \omega\left(uau^*\right)| : a \in A_{\mathrm{sa}},\ L(a) \leq 1 \text{ and } \|a\| \leq \mathrm{diam}(S(A))\} \\ &\quad \leq \sup\{|\sigma(a) - \omega(a)| : a \in A_{\mathrm{sa}} \text{ and } L(a) \leq 1 + 2L(u)\mathrm{diam}(S(A))\} \\ &\quad \leq (1 + 2L(u)\mathrm{diam}(S(A))) \sup\{|\sigma(a) - \omega(a)| : a \in A_{\mathrm{sa}} \text{ and } L(a) \leq 1\} \\ &\quad = (1 + 2L(u)\mathrm{diam}(S(A)))\rho_L(\sigma, \omega).\end{aligned}$$

Since L is lower semicontinuous we can thus appeal to Proposition 2.7 to obtain, for any $a \in \mathcal{L} \cap A_{\mathrm{sa}}$,

$$\begin{aligned}L\left(uau^*\right) &= \sup_{\sigma,\omega\in S(A)} \frac{|(\sigma \circ \mathrm{Ad}u)(a) - (\omega \circ \mathrm{Ad}u)(a)|}{\rho_L(\sigma, \omega)} \\ &\leq \sup_{\sigma,\omega\in S(A)} \frac{|(\sigma \circ \mathrm{Ad}u)(a) - (\omega \circ \mathrm{Ad}u)(a)|}{\rho_L(\sigma \circ \mathrm{Ad}u, \omega \circ \mathrm{Ad}u)} \\ &\qquad \times \sup_{\sigma,\omega\in S(A)} \frac{\rho_L(\sigma \circ \mathrm{Ad}u, \omega \circ \mathrm{Ad}u)}{\rho_L(\sigma, \omega)} \\ &= L(a)(1 + 2L(u)\mathrm{diam}(S(A))).\end{aligned}$$

Thus, for any $a \in \mathcal{L}$, setting $\mathrm{Re}(a) = (a + a^*)/2$ and $\mathrm{Im}(a) = (a - a^*)/2i$ we have

$$\begin{aligned} L(uau^*) &\leq L(u\mathrm{Re}(a)u^*) + L(u\mathrm{Im}(a)u^*) \\ &\leq (L(\mathrm{Re}(a)) + L(\mathrm{Im}(a)))(1 + 2L(u)\mathrm{diam}(S(A))) \\ &\leq 2L(a)(1 + 2L(u)\mathrm{diam}(S(A))) \end{aligned}$$

using adjoint invariance. The same argument applies to $(\mathrm{Ad}u)^{-1} = \mathrm{Ad}u^*$, and so we obtain the result. □

We conclude this section with some examples of ${}_{\mathrm{c}}$Lip-norms.

Example 2.12 (commutative C^*-algebras). For a compact metric space (X, d) we define the Lipschitz seminorm L_d on $C(X)$ by

$$L_d(f) = \sup\{|f(x) - f(y)|/d(x, y) : x, y \in X \text{ and } x \neq y\},$$

from which we can recover d via the formula

$$d(x, y) = \sup\{|f(x) - f(y)| : f \in C(X) \text{ and } L_d(f) \leq 1\}.$$

The seminorm L_d is an example of a Leibniz ${}_{\mathrm{c}}$Lip-norm. For a reference on Lipschitz seminorms and the associated Lipschitz algebras see [37].

Example 2.13 (ergodic compact group actions). For us the most important examples of compact noncommutative metric spaces will be those which arise from ergodic actions of compact groups, as studied by Rieffel in [25]. Suppose γ is an ergodic action of a compact group G on a unital C^*-algebra A. Let e denote the identity element of G. We assume that G is equipped with a length function ℓ, that is, a continuous function $\ell : G \to \mathbb{R}_{\geq 0}$ such that, for all $g, h \in G$,

(1) $\ell(gh) \leq \ell(g) + \ell(h)$,
(2) $\ell\left(g^{-1}\right) = \ell(g)$, and
(3) $\ell(g) = 0$ if and only if $g = e$.

The length function ℓ and the group action γ combine to produce the seminorm L on A defined by

$$L(a) = \sup_{g \in G \setminus \{e\}} \frac{\left\|\gamma_g(a) - a\right\|}{\ell(g)},$$

which is evidently adjoint-invariant. It is easily verified that $L(a) = 0$ if and only if $a \in \mathbb{C}1$. Also, by [25, Thm. 2.3] the metric ρ_L induces the weak* topology on $S(A)$, and the Leibniz rule is easily checked, so that L is Leibniz ${}_{\mathrm{c}}$Lip-norm.

Example 2.14 (quotients). Let A and B be unital C^*-algebras and let $\phi : A \to B$ be a surjective unital positive linear map. For instance, ϕ may be a surjective unital *-homomorphism or a conditional expectation, as will be the case in our applications. Let L be a ${}_{\mathrm{c}}$Lip-norm on A. Then L induces a ${}_{\mathrm{c}}$Lip-norm L_B on B via the prescription

$$L_B(b) = \inf\{L(a) : a \in A \text{ and } \phi(a) = b\}$$

for all $b \in B$. This is the analogue of restricting a metric to a subspace. To see that L_B is indeed a ${}_{\mathrm{c}}$Lip-norm we observe that the restriction of ϕ to $\mathcal{L}^A \cap A_{\mathrm{sa}}$ yields a surjective morphism $\mathcal{L}^A \cap A_{\mathrm{sa}} \to \mathcal{L}^B \cap B_{\mathrm{sa}}$ of order-unit spaces (for surjectivity note that if $\phi(a) \in B_{\mathrm{sa}}$ then $\phi\left(\frac{1}{2}(a+a^*)\right) = \phi(a)$) so that we may appeal to [27, Prop. 3.1] to conclude that the restriction of L_B to $\mathcal{L} \cap B_{\mathrm{sa}}$ is a Lip-norm, so that L_B is a ${}_{\mathrm{c}}$Lip-norm by Proposition 2.4 (note that the restriction of L_B to $\mathcal{L} \cap B_{\mathrm{sa}}$ separates $S(B)$, since $\phi\left(\mathcal{L}^A\right) = \mathcal{L}^B$ and the restriction of L_A to $\mathcal{L} \cap A_{\mathrm{sa}}$ separates $S(A)$ by the first part of Proposition 2.4).

3. Dimension for ${}_{\mathrm{c}}$Lip-Normed Unital C^*-Algebras

Let A be a unital C^*-algebra with ${}_{\mathrm{c}}$Lip-norm L. Recall from Notation 2.2 our convention that $\mathcal{L}$ and $\mathcal{L}_r$ refer to the sets $\{a \in A : L(a) < \infty\}$ and $\{a \in A : L(a) \leq r\}$, respectively. The following notation will be extensively used for the remainder of the article.

Notation 3.1. *For a normed linear space $(X, \|\cdot\|)$ (which in our case will either be a C^*-algebra or a Hilbert space) we will denote by $\mathcal{F}(X)$ the collection of its finite-dimensional subspaces, and if Y and Z are subsets of X and $\delta > 0$ we will write $Y \subset_\delta Z$, and say that Z* **approximately contains** *Y* **to within** *δ, if for every $y \in Y$ there is an $x \in Z$ such that $\|y - x\| < \delta$. Using* $\dim X$ *to denote the vector space dimension of a subspace X, for any subset $Z \subset A$ and $\delta > 0$ we set*

$$D(Z, \delta) = \inf\{\dim X : X \in \mathcal{F}(A) \text{ and } Z \subset_\delta X\}$$

(or $D(Z, \delta) = \infty$ if the set on the right is empty) and if σ is a state on A then we set

$$D_\sigma(Z, \delta) = \inf\{\dim X : X \in \mathcal{F}(\mathcal{H}_\sigma) \text{ and } \pi_\sigma(Z)\xi_\sigma \subset_\delta X\}$$

(or $D_\sigma(Z, \delta) = \infty$ if the set on the right is empty), with $\pi_\sigma : A \to \mathcal{B}(\mathcal{H}_\sigma)$ referring to GNS representation associated to σ, with canonical cyclic vector ξ_σ.

Proposition 3.2. *$D(\mathcal{L}_1, \delta)$ is finite for every $\delta > 0$.*

Proof. Let $a \in \mathcal{L}$, and set $\mathrm{Re}(a) = (a + a^*)/2$ and $\mathrm{Im}(a) = (a - a^*)/2i$ (the real and imaginary parts of a). Let $s(\mathrm{Re}(a))$ and $s(\mathrm{Im}(a))$ be the infima of the spectra of $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$, respectively. Using Lemma 2.10 and the adjoint invariance of L we have

$$\begin{aligned}\|a - (s(\mathrm{Re}(a)) + is(\mathrm{Im}(a)))1\| &\leq \|\mathrm{Re}(a) - s(\mathrm{Re}(a))1\| + \|\mathrm{Im}(a) - s(\mathrm{Im}(a))1\| \\ &\leq L(\mathrm{Re}(a))\mathrm{diam}(S(A)) + L(\mathrm{Im}(a))\mathrm{diam}(S(A)) \\ &\leq 2L(a)\mathrm{diam}(S(A)).\end{aligned}$$

Set $r = 2\,\mathrm{diam}(S(A))$. Since $\mathcal{L}_1 \cap A_1$ is totally bounded by Proposition 2.5, so is $\mathcal{L}_r \cap A_r$ by a scaling argument. Let $\delta > 0$. Then there is an $X \subset \mathcal{F}(A)$ which approximately contains $\mathcal{L}_r \cap A_r$ to within δ, and if $a \in \mathcal{L}_1$ then from above we have

$$a - (s(\mathrm{Re}(a)) + is(\mathrm{Im}(a)))1 \in \mathcal{L}_r \cap A_r,$$

so that there exists an $x \in X$ with

$$\|a - (s(\mathrm{Re}(a)) + is(\mathrm{Im}(a)))1 - x\| < \delta.$$

But $(s(\mathrm{Re}(a)) + is(\mathrm{Im}(a)))1 - x \in \mathrm{span}(X \cup \{1\})$, and so we conclude that

$$\mathcal{L}_1 \subset_\delta \mathrm{span}(X \cup \{1\}).$$

Hence $D(\mathcal{L}_1, \delta)$ is finite. □

In view of Proposition 3.2 we make the following definition.

Definition 3.3. *We define the* **metric dimension** *of A with respect to L by*

$$\mathrm{Mdim}_L(A) = \limsup_{\delta\to 0^+} \frac{\log D(\mathcal{L}_1, \delta)}{\log \delta^{-1}}.$$

We may think of $D(\mathcal{L}_1, \delta)$ as the *δ-entropy* of A with respect to L in analogy with Kolmogorov ε-entropy [17], and indeed when L is a Lipschitz seminorm on a compact metric space (X, d) we will recover from $\mathrm{Mdim}_L(C(X))$ the Kolmogorov dimension (Proposition 3.9).

We emphasize that in using $D(\cdot, \cdot)$ in Definition 3.3 (and also in the definition of entropy in Sect. 5) we are not making any extra geometric assumptions in our finite-dimensional approximations by linear subspaces. For example, we are not requiring that these subspaces be images of positive or completely positive maps which are close to the identity on the set in question. In computing lower bounds we are thus left to rely on the Hilbert space geometry implicit in the C^*-algebraic structure, making repeated use of Lemma 3.8 below.

Proposition 3.4. *Let A and B be unital C^*-algebras with ${}_{\mathrm{c}}$Lip-norms L_A and L_B, respectively. Suppose $\phi : A \to B$ is a bi-Lipschitz positive unital map. Then*

$$\mathrm{Mdim}_{L_A}(A) = \mathrm{Mdim}_{L_B}(B).$$

Proof. Let $\lambda > 0$ be the Lipschitz number of ϕ. Then $\phi(\mathcal{L}_1^A) \subset \mathcal{L}_\lambda^B$, so that if $X \in \mathcal{F}(B)$ and $\mathcal{L}_\lambda^B \subset_\delta X$ then

$$\mathcal{L}_1^A \subset_\delta \phi^{-1}(X),$$

since ϕ is isometric for the C^*-norm (see the remark after Definition 2.8). As a consequence

$$D\left(\mathcal{L}_1^A, \delta\right) \leq D\left(\mathcal{L}_\lambda^B, \delta\right)$$

and so

$$\begin{aligned}
\mathrm{Mdim}_{L_A}(A) &= \limsup_{\delta\to 0^+} \frac{\log D(\mathcal{L}_1^A, \delta)}{\log \delta^{-1}} \\
&\leq \limsup_{\delta\to 0^+} \frac{\log D(\mathcal{L}_\lambda^B, \delta)}{\log \delta^{-1}} \\
&= \limsup_{\delta\to 0^+} \frac{\log D(\mathcal{L}_1^B, \lambda^{-1}\delta)}{\log \lambda^{-1}\delta^{-1}} \cdot \lim_{\delta\to 0^+} \frac{\log \lambda^{-1}\delta^{-1}}{\log \delta^{-1}} \\
&= \limsup_{\delta\to 0^+} \frac{\log D(\mathcal{L}_1^B, \lambda^{-1}\delta)}{\log \lambda^{-1}\delta^{-1}} \\
&= \mathrm{Mdim}_{L_B}(B).
\end{aligned}$$

The reverse inequality follows by a symmetric argument. $\square$

The following is immediate from Definition 3.3.

Proposition 3.5. *Let L and L' be ${}_{\mathrm{c}}$Lip-norms on a unital C^*-algebra such that $L \le L'$, that is, $L(a) \le L'(a)$ for all $a \in A$. Then*

$$\mathrm{Mdim}_L(A) \ge \mathrm{Mdim}_{L'}(A).$$

Proposition 3.6. *Let A and B be unital C^*-algebras, L_A a ${}_{\mathrm{c}}$Lip-norm on A, $\phi : A \to B$ a surjective positive unital map, and L_B the ${}_{\mathrm{c}}$Lip-norm on B induced from L_A by ϕ. Then*

$$\mathrm{Mdim}_{L_B}(B) \le \mathrm{Mdim}_{L_A}(A).$$

Proof. Since L_B is induced from L_A (Example 2.14) for any $b \in \mathcal{L}_1^B$ there is an $a \in A$ with $\phi(a) = b$ and $L(a) \le 2$. Thus if X is a linear subspace of A with $\mathcal{L}_2^A \subset_\delta X$ it follows that $\mathcal{L}_1^B \subset_\delta \phi(X)$. Hence

$$\begin{aligned}
\mathrm{Mdim}_{L_B}(B) &= \limsup_{\delta \to 0^+} \frac{\log D(\mathcal{L}_1^B, \delta)}{\log \delta^{-1}} \\
&\le \limsup_{\delta \to 0^+} \frac{\log D(\mathcal{L}_2^A, \delta)}{\log \delta^{-1}} \\
&= \limsup_{\delta \to 0^+} \frac{\log D(\mathcal{L}_1^A, 2^{-1}\delta)}{\log 2\delta^{-1}} \cdot \lim_{\delta \to 0^+} \frac{\log 2\delta^{-1}}{\log \delta^{-1}} \\
&= \mathrm{Mdim}_{L_A}(A).
\end{aligned}$$

□

Proposition 3.7. *Let A and B be unital C^*-algebras with ${}_{\mathrm{c}}$Lip-norms L_A and L_B, respectively. Let L be a ${}_{\mathrm{c}}$Lip-norm on $A \oplus B$ which induces L_A and L_B via the quotients onto A and B, respectively (see Example 2.14). Then*

$$\mathrm{Mdim}_L(A \oplus B) = \max\left(\mathrm{Mdim}_L(A), \mathrm{Mdim}_L(B)\right).$$

Proof. The inequality $\mathrm{Mdim}_L(A \oplus B) \ge \max(\mathrm{Mdim}_L(A), \mathrm{Mdim}_L(B))$ follows from Proposition 3.6. To establish the reverse inequality, let $\delta > 0$, and let $X \in \mathcal{F}(A)$ and $Y \in \mathcal{F}(B)$ be such that $\mathcal{L}_1^A \subset_\delta X$ and $\mathcal{L}_1^B \subset_\delta Y$. If $(a, b) \in \mathcal{L}_1^{A \oplus B}$ then $L(a)$ and $L(b)$ are no greater than 1, and hence there exist $x \in X$ and $y \in Y$ such that $\|x - a\| < \delta$ and $\|y - b\| < \delta$, so that

$$\|(x, y) - (a, b)\| < \delta.$$

Thus

$$\mathcal{L}_1^{A \oplus B} \subset_\delta \mathrm{span}\left(\{(x, 0) : x \in X\} \cup \{(0, y) : y \in Y\}\right),$$

and so we infer that

$$D\left(\mathcal{L}_1^{A \oplus B}, \delta\right) \le D\left(\mathcal{L}_1^A, \delta\right) + D\left(\mathcal{L}_1^B, \delta\right).$$

For each $\delta > 0$ the sum on the right in the above display is bounded by twice the maximum of its two summands, and so

$$\begin{aligned}\mathrm{Mdim}_L(A \oplus B) &= \limsup_{\delta \to 0^+} \frac{\log D\left(\mathcal{L}_1^{A\oplus B}, \delta\right)}{\log \delta^{-1}} \\ &\le \max\left(\limsup_{\delta \to 0^+} \frac{\log 2D\left(\mathcal{L}_1^{A}, \delta\right)}{\log \delta^{-1}}, \limsup_{\delta \to 0^+} \frac{\log 2D\left(\mathcal{L}_1^{B}, \delta\right)}{\log \delta^{-1}}\right) \\ &= \max(\mathrm{Mdim}_L(A), \mathrm{Mdim}_L(B)).\end{aligned}$$

□

As we show in Proposition 3.9 below, if L is a Lipschitz seminorm on a compact metric space (X, d) then $\mathrm{Mdim}_L(C(X))$ coincides with the Kolmogorov dimension [17, 18], whose definition we recall. Let (X, d) be a compact metric space. A set $E \subset X$ is said to be δ-*separated* if for any distinct $x, y \in E$ we have $d(x, y) > \delta$, while a set $F \in X$ is said to be δ-*spanning* if for any $x \in X$ there is a $y \in F$ such that $d(x, y) \le \delta$. We denote by $\mathrm{sep}(\delta, d)$ the largest cardinality of an δ-separated set and by $\mathrm{spn}(\delta, d)$ the smallest cardinality of a δ-spanning set. We furthermore denote by $N(\delta, d)$ the minimal cardinality of a cover of X by δ-balls. The *Kolmogorov dimension* of (X, d), which we will denote by $\mathrm{Kdim}_d(X)$, is the common value of the three expressions

$$\limsup_{\delta \to 0^+} \frac{\log \mathrm{sep}(\delta, d)}{\log \delta^{-1}}, \quad \limsup_{\delta \to 0^+} \frac{\log \mathrm{spn}(\delta, d)}{\log \delta^{-1}}, \quad \limsup_{\delta \to 0^+} \frac{\log N(\delta, d)}{\log \delta^{-1}}.$$

This also goes by other names in the literature, such as box dimension and limit capacity (see [22, Chap. 2]).

We will need the following lemma from [33], which will also be of use later on.

Lemma 3.8 ([33, Lemma 7.8])**.** *If B is an orthonormal set of vectors in a Hilbert space $\mathcal{H}$ and $\delta > 0$ then*

$$\inf\{\dim X : X \in \mathcal{F}(\mathcal{H}) \text{ and } X \subset_\delta B\} \ge (1 - \delta^2)\mathrm{card}(B).$$

Proposition 3.9. *Let (X, d) be a compact metric space, and let L be the associated Lipschitz seminorm on $C(X)$, that is,*

$$L(f) = \sup\{|f(x) - f(y)|/d(x, y) : x, y \in X \text{ and } x \neq y\}$$

for all $f \in C(X)$. Then

$$\mathrm{Mdim}_L(C(X)) = \mathrm{Kdim}_d(X).$$

Proof. Let $\delta > 0$ and let $\mathcal{U} = \{\mathcal{B}(x_1, \delta), \dots, \mathcal{B}(x_r, \delta)\}$ be a cover of X by δ-balls. Let $\Omega = \{f_1, \dots, f_r\}$ be a partition of unity subordinate to $\mathcal{U}$. If $f \in \mathcal{L}_1$ and x and y are points of X contained in the same member of $\mathcal{U}$, then

$$|f(x) - f(y)| < 2\delta.$$

Thus for any $x \in X$ we have

$$\begin{aligned}\left| f(x) - \sum_{1 \le i \le r} f(x_i) f_i(x) \right| &\le \sum_{1 \le i \le r} |f(x) - f(x_i)|\, f_i(x) \\ &\le \sum_{\{i : x \in \mathcal{B}(x_i, \delta)\}} |f(x) - f(x_i)|\, f_i(x) \\ &< 2\delta.\end{aligned}$$

Thus $\mathcal{L}_1 \subset_{2\delta} \operatorname{span}(\Omega)$, and since $\dim(\operatorname{span}(\Omega)) = \operatorname{card}(\mathcal{U})$ we conclude that

$$\operatorname{Mdim}_L(C(X)) = \limsup_{\delta \to 0^+} \frac{\log D(\mathcal{L}_1, 2\delta)}{\log \delta^{-1}} \leq \operatorname{Kdim}_d(X).$$

To establish the reverse inequality, let $\delta > 0$ and let $E = \{x_1, \ldots, x_r\}$ be a δ-separated set of maximal cardinality. The idea will be to consider the probability measure μ uniformly supported on E and to construct unitaries in $C(X)$ with sufficiently small Lipschitz seminorm which, when viewed as elements of $L^2(X, \mu)$, form an orthonormal basis, so that we can appeal to Lemma 3.8. For each $j = 1, \ldots, r$ define the function f_j by

$$f_j(x) = \max\left(0, 1 - \delta^{-1} d(x, x_j)\right)$$

for all $x \in X$, and observe that $L(f_j) = \delta^{-1}$. For each $k = 1, \ldots, r$ define the function g_k by

$$g_k = \sum_{j=1}^{n} \left[jkr^{-1}\right] f_j,$$

where $[\,\cdot\,]$ means take the fractional part. We then have $L(g_k) \leq \delta^{-1}$, as can be seen by alternatively expressing g_k as the join of the functions $\left[jkr^{-1}\right] f_j$ (note that the supports of the f_j's are pairwise disjoint) and applying the inequality $L(f \vee g) \leq \max(L(f), L(g))$ relating L to the lattice structure of real-valued functions on X. For each $k = 1, \ldots, r$ set

$$u_k = e^{2\pi i g_k}.$$

Repeated application of the Leibniz rule yields, for each $n \geq 1$,

$$\begin{aligned} L\left(\sum_{j=0}^{n} \frac{(2\pi i g_k)^j}{j!}\right) \leq \sum_{j=0}^{n} \frac{(2\pi)^j}{j!} L(g_k^j) &\leq \sum_{j=0}^{n} \frac{(2\pi)^j}{j!} j L(g_k) \\ &= 2\pi \left(\sum_{j=0}^{n-1} \frac{(2\pi)^j}{j!}\right) L(g_k) \\ &\leq 2\pi e^{2\pi} L(g_k), \end{aligned}$$

and thus, since the sequence $\left\{\sum_{j=0}^{n} \frac{(2\pi i g_k)^j}{j!}\right\}_{n \in \mathbb{N}}$ converges uniformly to u_k, we can appeal to the lower semicontinuity of L to obtain the estimate

$$L(u_k) \leq 2\pi e^{2\pi} L(g_k) \leq 2\pi e^{2\pi} \delta^{-1}.$$

Setting $U(\delta) = \{u_k : k = 1, \ldots, r\}$ and $C = 2\pi e^{2\pi}$, we thus have that the set $\{C^{-1}u : u \in U(\delta)\}$, which we will simply denote by $C^{-1}U(\delta)$, lies in $\mathcal{L}_1$ if $\delta \leq C$.

Next, let μ be the probability measure uniformly supported on E and let $\pi_\mu : C(X) \to \mathcal{B}(L^2(X, \mu))$ be the associated GNS representation, with canonical cyclic vector ξ_μ. Then, for each $k = 1, \ldots, r$, $\pi_\mu(u_k)\xi_\mu$ is the unit vector

$$\left(1, e^{2\pi i k r^{-1}}, \left(e^{2\pi i k r^{-1}}\right)^2, \ldots, \left(e^{2\pi i k r^{-1}}\right)^{r-1}\right)$$

under the obvious identification of $L^2(X,\mu)$ with $\mathbb{C}^r$ which respects the order of the indexing of the points $x_1,\dots,x_r$. Hence we see that the set $\{\pi_\mu(u)\xi_\mu : u \in U(\delta)\}$ forms an orthonormal basis for $L^2(X,\mu)$, and so by Lemma 3.8 we have

$$D_\mu(U(\delta),2^{-1}) \geq (1-2^{-2})\,\mathrm{card}(U(\delta)) = \tfrac{3}{4}\mathrm{card}(E) = \tfrac{3}{4}\mathrm{sep}(\delta,d),$$

(for the meaning of $D_\mu(\cdot,\cdot)$ see Notation 3.1).

Carrying out the above construction for each $\delta > 0$, we then have

$$\begin{aligned}
\mathrm{Mdim}_L(C(X)) &= \limsup_{\delta\to 0^+} \frac{\log D(\mathcal{L}_1, 2^{-1}C^{-1}\delta)}{\log 2C\delta^{-1}} \\
&= \limsup_{\delta\to 0^+} \frac{\log D(\mathcal{L}_1, 2^{-1}C^{-1}\delta)}{\log \delta^{-1}} \\
&\geq \limsup_{\delta\to 0^+} \frac{\log D(C^{-1}\delta U(\delta), 2^{-1}C^{-1}\delta)}{\log \delta^{-1}} \\
&= \limsup_{\delta\to 0^+} \frac{\log D(U(\delta), 2^{-1})}{\log \delta^{-1}} \\
&\geq \limsup_{\delta\to 0^+} \frac{\log D_\mu(U(\delta), 2^{-1})}{\log \delta^{-1}} \\
&\geq \limsup_{\delta\to 0^+} \frac{\log \frac{3}{4}\,\mathrm{sep}(\delta,d)}{\log \delta^{-1}} \\
&= \mathrm{Kdim}_d(X).
\end{aligned}$$

□

4. Group Actions and Dimension

Here we compute the dimension for some examples in which the ${}_{\mathrm{c}}$Lip-norm is defined by means of an ergodic compact group action.

4.1. The UHF algebra M_{p^∞}. We consider here the infinite tensor product $M_p^{\otimes\mathbb{Z}}$ (usually denoted M_{p^∞}) of $p\times p$ matrix algebras M_p over $\mathbb{C}$ with the infinite product of Weyl actions. As shown in [21] there is a unique ergodic action of $G=\mathbb{Z}_p\times\mathbb{Z}_p$ on a simple C^*-algebra up to conjugacy, namely the Weyl action on M_p, defined as follows. Let ρ be the p^{th} root of unity $e^{2\pi i/p}$, and consider the unitary $u=\mathrm{diag}(1,\rho,\rho^2,\dots,\rho^{p-1})$ along with the unitary v which has 1's on the superdiagonal and in the bottom left-hand entry and 0's elsewhere. Then we have

$$vu = \rho uv,$$

and u and v generate M_p C^*-algebraically. The Weyl action $\gamma : G\to \mathrm{Aut}(M_p)$ is given by the following specification on the generators u and v:

$$\begin{aligned}
\gamma_{(r,s)}(u) &= \rho^r u, \\
\gamma_{(r,s)}(v) &= \rho^s v.
\end{aligned}$$

We may then consider the infinite product action $\gamma^{\otimes\mathbb{Z}}$ of the product group $G^{\mathbb{Z}}$ on $M_p^{\otimes\mathbb{Z}}$.

Consider the metric on G obtained by viewing G as a subgroup of $\mathbb{R}^2/\mathbb{Z}^2$ with the metric induced from the Euclidean metric on $\mathbb{R}^2$, and let ℓ be the length function on G defined by taking the distance to 0. Given $0 < \lambda < 1$ we define the length function ℓ_λ on $G^{\mathbb{Z}}$ by

$$\ell_\lambda\left((g_j, h_j)_{j\in\mathbb{Z}}\right) = \sum_{j\in\mathbb{Z}} \lambda^{|j|}\ell\left((g_j, h_j)\right).$$

We could also define length functions on $G^{\mathbb{Z}}$ by using suitable choices of weightings of ℓ on the factors other than the above geometric ones (and in many cases compute the metric dimension as in Proposition 4.1 below), but for simplicity we will restrict our attention to length functions of the form ℓ_λ. Let L be the ${}_{\mathrm{c}}$Lip-norm on $M_p^{\otimes\mathbb{Z}}$ arising from the action $\gamma^{\otimes\mathbb{Z}}$ and the length function ℓ_λ.

Proposition 4.1. *We have*

$$\mathrm{Mdim}_L\left(M_p^{\otimes\mathbb{Z}}\right) = \frac{4\log p}{\log\lambda^{-1}}.$$

Proof. For each n consider the conditional expectation E_n of $M_p^{\otimes\mathbb{Z}}$ onto the subalgebra $M_p^{\otimes[-n,n]}$ given by

$$E_n(a) = \int_{G^{\mathbb{Z}\setminus[-n,n]}} \gamma_g^{\otimes\mathbb{Z}}(a)\,dg,$$

where dg is normalized Haar measure on $G^{\mathbb{Z}}$ and $G^{\mathbb{Z}\setminus[-n,n]}$ is the subgroup of $G^{\mathbb{Z}}$ of elements which are the identity at the coordinates in the interval $[-n, n]$. Then for each $a \in \mathcal{L}$ we have

$$\begin{aligned}
\|E_n(a) - a\| &= \left\| \int_{G^{\mathbb{Z}\setminus[-n,n]}} \left(\gamma_g^{\otimes\mathbb{Z}}(a) - a\right) dg \right\| \\
&\le \int_{G^{\mathbb{Z}\setminus[-n,n]}} \left\|\gamma_g^{\otimes\mathbb{Z}}(a) - a\right\| dg \\
&\le \int_{G^{\mathbb{Z}\setminus[-n,n]}} L(a)\ell_\lambda(g)\,dg \\
&\le L(a)\frac{2\lambda^{n+1}}{1-\lambda}.
\end{aligned}$$

Let $\delta > 0$. If δ is sufficiently small there is an $n \in \mathbb{N}$ such that

$$2\lambda^{n+1}(1-\lambda)^{-1} \le \delta \le 2\lambda^n(1-\lambda)^{-1}.$$

Then, in view of the above estimate on $\|E_n(a) - a\|$ when $a \in \mathcal{L}_1$, we have that $\mathcal{L}_1$ is approximately contained in $M_p^{\otimes[-n,n]}$ to within δ. Since $M_p^{\otimes[-n,n]}$ has linear dimension $p^{2(2n+1)}$ it follows that

$$\begin{aligned}
\frac{\log D(\mathcal{L}_1, \delta)}{\log\delta^{-1}} &\le \frac{\log D(\mathcal{L}_1, 2\lambda^{n+1}(1-\lambda)^{-1})}{\log(2(1-\lambda)\lambda^{-n})} \\
&\le \frac{(4n+2)\log p}{\log(2(1-\lambda)\lambda^{-n})}
\end{aligned}$$

and so

$$\begin{aligned}\mathrm{Mdim}_L\left(M_p^{\otimes\mathbb{Z}}\right) &= \limsup_{n\to\infty}\frac{\log D(\mathcal{L}_1,\delta)}{\log\delta^{-1}}\\ &\le \lim_{n\to\infty}\frac{(4n+2)\log p}{\log(2(1-\lambda)\lambda^{-n})}\\ &= \frac{4\log p}{\log\lambda^{-1}}.\end{aligned}$$

To prove the reverse inequality, consider for each $n\in\mathbb{N}$ the subset

$$U_n=\Big\{u^{i_{-n}}v^{j_{-n}}\otimes u^{i_{-n+1}}v^{j_{-n+1}}\otimes\cdots\otimes u^{i_n}v^{j_n}:\\ 0\le i_k,\, j_k\le p-1 \text{ for } k=-n,\dots,n\Big\}$$

of $M_p^{\otimes[-n,n]}$ (i.e., all elementary tensors in $M_p^{\otimes[-n,n]}$ whose components are Weyl unitaries in the respective copies of M_p). It is easily checked that the ${}_{\mathrm{c}}$Lip-norm of any element in U_n is bounded by $2\left(1+2\sum_{k=1}^n\lambda^k\right)\le(4n+2)\lambda^n$. Now the product of any two distinct products of powers of Weyl generators in M_p is zero under evaluation at the unique tracial state τ on $M_p^{\otimes\mathbb{Z}}$, as can be seen from the commutation relation between u and v. Thus, since τ is a tensor product of traces in its restriction to $M_p^{\otimes[-n,n]}$, the product of any two distinct elements of Ω_n is zero under evaluation by τ. This implies that $\pi_\tau(U_n)\xi_\tau$ is an orthonormal set in the GNS representation Hilbert space associated to τ with canonical cyclic vector ξ_τ, and so by Lemma 3.8 we have

$$D_\tau\left(U_n,2^{-1}\right)\ge\left(1-2^{-1}\right)\mathrm{card}\,(\pi_\tau(U_n)\xi_\tau)=\tfrac{3}{4}p^{2(2n+1)}.$$

Thus setting

$$W_n=\left\{(4n+2)^{-1}\lambda^n w: w\in U_n\right\}$$

(which is contained in $\mathcal{L}_1$) we have

$$\begin{aligned}D(W_n,(4n+2)^{-1}\lambda^{-n}2^{-1}) &\ge D_\tau(W_n,(4n+2)^{-1}\lambda^{-n}2^{-1})\\ &\ge D_\tau(U_n,2^{-1})\\ &\ge \tfrac{3}{4}p^{2(2n+1)}\end{aligned}$$

and so

$$\begin{aligned}\mathrm{Mdim}_L\left(M_p^{\otimes\mathbb{Z}}\right) &\ge \limsup_{n\to\infty}\frac{\log(D(W_n,(4n+2)^{-1}\lambda^{-n}2^{-1})}{\log((4n+2)^{-1}\lambda^{-n}2^{-1})}\\ &\ge \limsup_{n\to\infty}\frac{\log\frac{3}{4}+(4n+2)\log p}{\log((4n+2)^{-1}2^{-1})+n\log\lambda^{-1}}\\ &= \frac{4\log p}{\log\lambda^{-1}},\end{aligned}$$

completing the proof. □

Because we have used the canonical unitary description of $M_p^{\otimes\mathbb{Z}}$ in an essential way, we cannot expect to be able to carry out a computation for much more general types of tensor products by extending the arguments of this subsection, although such a computation would be possible, for example, for tensor products of noncommutative tori, in which case we could incorporate the methods of the next subsection.

4.2. Noncommutative tori. Let $\rho : \mathbb{Z}^p \times \mathbb{Z}^p \to \mathbb{T}$ be an antisymmetric bicharacter and for $1 \le i, j \le k$ set

$$\rho_{ij} = \rho\left(e_i, e_j\right),$$

where $\{e_1, \dots, e_p\}$ is the standard basis for $\mathbb{Z}^p$. The universal C^*-algebra A_ρ generated by unitaries $u_1, \dots, u_p$ satisfying

$$u_j u_i = \rho_{ij} u_i u_j$$

is referred to as a *noncommutative p-torus*. Slawny showed in [30] that A_ρ is simple if and only if ρ is nondegenerate (meaning that $\rho(g, h) = 1$ for all $h \in \mathbb{Z}^p$ implies that $g = 0$), and these two conditions are furthermore equivalent to the existence of a unique tracial state on A_ρ (see [11]).

Let A_ρ be a noncommutative p-torus with generators $u_1, \dots, u_p$. There is an ergodic action $\gamma : \mathbb{T}^p \cong (\mathbb{R}/\mathbb{Z})^p \to \mathrm{Aut}(A_\rho)$ determined by

$$\gamma_{(t_1,\dots,t_p)}(u_j) = e^{2\pi i t_j} u_j$$

(see [21]). We will consider the ${}_{\mathrm{c}}$Lip-norm L arising from the action γ as in Example 2.13, with the length function given by taking the distance to 0 with respect to the metric induced from the Euclidean metric on $\mathbb{R}^p$ scaled by 2π (scaling will not affect the value of $\mathrm{Mdim}_L(A_\rho)$ but our choice of length function ensures for convenience that $L(u_j) = 1$ for each $j = 1, \dots, p$). We denote by τ the tracial state defined by

$$\tau(a) = \int_{\mathbb{T}^p} \gamma_{(t_1,\dots,t_p)}(a)\, d\left(t_1, \dots, t_p\right)$$

for all $a \in A_\rho$, where $d(t_1, \dots, t_p)$ is normalized Haar measure on $\mathbb{T}^p \cong (\mathbb{R}/\mathbb{Z})^p$.

For $(n_1, \dots, n_p) \in \mathbb{N}^p$ let $R(n_1, \dots, n_p)$ denote the set of points $(k_1, \dots, k_p)$ in $\mathbb{Z}^p$ such that $|k_i| \le n_i$ for $i = 1, \dots, p$. For each $a \in A_\rho$, we define for each $(n_1, \dots, n_p) \in \mathbb{N}^p$ the partial Fourier sum

$$s_{(n_1,\dots,n_p)}(a) = \sum_{(k_1,\dots,k_p)\in R(n_1,\dots,n_p)} \tau\left(a u_p^{-k_p} \cdots u_1^{-k_1}\right) u_1^{k_1} \cdots u_p^{k_p}$$

and for each $n \in \mathbb{N}$ the Cesàro mean

$$\sigma_n(a) = \left(\sum_{(n_1,\dots,n_p)\in R(n,n,\dots,n)} s_{(n_1,\dots,n_p)}(a)\right) \Bigg/ (n+1)^p.$$

Weaver showed in [35, Thm. 22] for the case $p = 2$ that $\sigma_n(a) \to a$ in norm for all $a \in \mathcal{L}$. To compute $\mathrm{Mdim}_L(A_\rho)$ we will need a handle on the rate of this convergence, and so we have in Lemma 4.3 below an extension to the noncommutative case of a standard result in classical Fourier analysis (see for example [15]). To make the required estimate we will use the expression for $\sigma_n(a) - a$ given by the following lemma, which can be proved in the same way as its specialization to the case $p = 2$, which appears in a more general form in [36] as Lemma 3.1 and is established in the course of the proof of [35, Thm. 22].

Recall the classical Fejér kernel K_n defined by

$$K_n(t) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{2\pi i k t} = \frac{1}{n+1}\left(\frac{\sin((n+1)t/2)}{\sin(t/2)}\right)^2.$$

Lemma 4.2. *If $a \in A_\rho$ then for all $n \in \mathbb{N}$ we have*

$$\sigma_n(a) = \int_{\mathbb{T}^p} \gamma_{(t_1,\dots,t_p)}(a) K_n(t_1) \cdots K_n(t_p)\, d\left(t_1, \dots, t_p\right)$$

and

$$a - \sigma_n(a) = \sum_{k=1}^{p} \int_{\mathbb{T}^{k-1}} \gamma_{(t_1,\dots,t_{k-1},0,\dots,0)} \left(\int_{\mathbb{T}} (a - \gamma_{r_k(t_k)}(a)) K_n(t_k) dt_k \right) \times K_n(t_1) \cdots K_n(t_{k-1})\, d(t_1, \dots, t_{k-1}),$$

with the integrals taken in the Riemann sense and $r_k(t)$ denoting the p-tuple which is t at the k^{th} coordinate and 0 elsewhere.

Notice that the right-hand expression in the second display of the statement of Lemma 4.2 is a telescoping sum, so that the second display is an immediate consequence of the first display in view of the fact that the integral of the Fejér kernel over $\mathbb{T}$ is 1. Note also that the first display shows that $\|\sigma_n(a)\| \leq \|a\|$ for all $n \in \mathbb{N}$ and $a \in A_\rho$, a fact which will be of use in the proof of Proposition 7.4.

Lemma 4.3. *If $a \in \mathcal{L}^{A_\rho}$ then there is a $C > 0$ such that*

$$\|a - \sigma_n(a)\| < L(a) C \frac{\log n}{n}$$

for all $n \in \mathbb{N}$.

Proof. It suffices to show that each of the summands on the right-hand side of the second display of Lemma 4.2 is bounded by $Mn^{-1}\log n$ for some $M > 0$ and all $n \in \mathbb{N}$. We thus observe that if $1 \leq k \leq p$ then, with $r_k(t)$ denoting the p-tuple which is t at the k^{th} coordinate and 0 elsewhere,

$$\begin{aligned}
&\left\| \int_{\mathbb{T}^{k-1}} \gamma_{(t_1,\dots,t_{k-1},0,\dots,0)} \left(\int_{\mathbb{T}} (a - \gamma_{r_k(t_k)}(a)) K_n(t_k) dt_k \right) \right. \\
&\qquad \left. \times\, K_n(t_1) \cdots K_n(t_{k-1})\, d(t_1, \dots, t_{k-1}) \right\| \\
&\leq \int_{\mathbb{T}^{k-1}} \left\| \int_{\mathbb{T}} (a - \gamma_{r_k(t_k)}(a)) K_n(t_k) dt_k \right\| K_n(t_1) \cdots K_n(t_{k-1})\, d(t_1, \dots, t_{k-1}) \\
&\leq \int_{\mathbb{T}} \|a - \gamma_{r_k(t_k)}(a)\| K_n(t_k)\, dt_k \\
&\leq L(a) \int_{\mathbb{T}} |t| K_n(t) dt.
\end{aligned}$$

Estimating the integral $\int_{\mathbb{T}} |t| K_n(t)\, dt$ is a standard exercise from classical Fourier analysis (see [15, Exercise 3.1]): using the fact that $|\sin(\pi t)| > 2|t|$ and hence

$$K_n(t) \leq \min\left(n + 1, \frac{1}{4(n+1)t^2}\right)$$

for $0 < |t| < \frac{1}{2}$, we readily obtain, for the integral of $|t| K_n(t)$ over each of the intervals $[-\frac{1}{2}, -\frac{1}{2(n+1)}]$, $[-\frac{1}{2(n+1)}, \frac{1}{2(n+1)}]$, and $[\frac{1}{2(n+1)}, \frac{1}{2}]$, an upper bound of $n^{-1}\log n$ times some constant independent of n, yielding the result. $\square$

Proposition 4.4. *We have*

$$\mathrm{Mdim}_L(A_\rho) = p.$$

Proof. Let $\delta > 0$, and assume δ is sufficiently small so that there is an $n \in \mathbb{N}$ such that

$$C(n+1)^{-1}\log(n+1) \leq \delta \leq Cn^{-1}\log n.$$

Lemma 4.3 then yields

$$\begin{aligned}\frac{\log D(\mathcal{L}_1, \delta)}{\log \delta^{-1}} &\leq \frac{\log D(\mathcal{L}_1, Cn^{-1}\log n)}{\log(C(n+1)^{-1}\log(n+1))^{-1}}\\ &\leq \frac{p\log(2n+1)}{\log(Cn^{-1}\log n)^{-1}}\end{aligned}$$

so that

$$\mathrm{Mdim}_L(A_\rho) \leq \limsup_{n\to\infty} \frac{p\log(2n+1)}{\log(Cn^{-1}\log n)^{-1}} = p.$$

To prove the reverse inequality, for each $n \in \mathbb{N}$ consider the set

$$U_n = \left\{u_1^{k_1}u_2^{k_2}\cdots u_p^{k_p} : |k_i| \leq n \text{ for } i = 1, \ldots, p\right\}$$

of unitaries in A_ρ. By repeated application of the Leibniz inequality and using the fact that $L(u_i) = 1$ for each $i = 1, \ldots, p$ we have the following estimate for the ${}_{\mathrm{c}}$Lip-norm of an arbitrary element of U_n:

$$L\left(u_1^{k_1}u_2^{k_2}\cdots u_p^{k_p}\right) \leq k_1L(u_1) + k_2L(u_2) + \cdots + k_pL(u_p) \leq pn.$$

Thus the set $W_n = \{(pn)^{-1}u : u \in U_n\}$ is contained in $\mathcal{L}_1$. Now products of distinct elements of the (self-adjoint) set U_n evaluate to zero under the tracial state τ, so that, in the GNS representation Hilbert space associated to τ with canonical cyclic vector ξ_τ, $\pi_\tau(U_n)\xi_\tau$ forms an orthonormal set of vectors. Thus, given $\delta > 0$ we can apply Proposition 3.8 to obtain, for each $n \geq 1$,

$$D\left(W_n, (pn)^{-1}\delta\right) \geq D(U_n, \delta) \geq D_\tau(U_n, \delta) \geq (1-\delta^2)(2n+1)^p$$

so that, assuming $\delta < 1$,

$$\begin{aligned}\mathrm{Mdim}_L(A_\rho) &\geq \limsup_{n\to\infty} \frac{\log D(W_n, (pn)^{-1}\delta)}{\log(pn\delta^{-1})}\\ &\geq \limsup_{n\to\infty} \frac{\log(1-\delta^2) + p\log(2n+1)}{\log(pn\delta^{-1})}\\ &= p,\end{aligned}$$

as desired. □

5. Product Entropies

We now study dynamics within the framework of unital C^*-algebras with Leibniz ${}_{\mathrm{c}}$Lip-norms, concentrating on iterative growth as captured in the "product" entropy of Subsect. 5.1 and its measure-theoretic version in Subsect. 5.2. That the Leibniz rule is important here can be seen by examining the proofs of Propositions 5.4 and 5.6 (although the latter only requires that $\mathcal{L}$ be closed under multiplication).

5.1. Product entropy. We begin by introducing some notation.

Notation 5.1. *For any set X we will denote by $Pf(X)$ the collection of finite subsets of X. If $X_1, X_2, \dots, X_n$ are subsets of the C^*-algebra A we will use the notation $X_1 \cdot X_2 \cdot \cdots \cdot X_n$ or $\prod_{j=1}^n X_j$ to refer to the set*

$$\{a_1 a_2 \cdots a_n : a_i \in X_i \textit{ for each } i = 1, \dots, n\}.$$

Recall from Notation 2.2 that, for a C^*-algebra A and $r > 0$, A_r refers to the set $\{a \in A : \|a\| \leq r\}$. For the meaning of $D(\cdot\,,\cdot)$ see Notation 3.1.

Definition 5.2. *Let A be a unital C^*-algebra with Leibniz ${}_{\mathrm{c}}$Lip-norm L, and let $\alpha \in \mathrm{Aut}_L(A)$. For $\Omega \in Pf(\mathcal{L} \cap A_1)$ and $\delta > 0$ we define*

$$\begin{aligned}
\mathrm{Entp}_L(\alpha, \Omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log D\left(\Omega \cdot \alpha(\Omega) \cdot \alpha^2(\Omega) \cdot \cdots \cdot \alpha^{n-1}(\Omega), \delta\right),\\
\mathrm{Entp}_L(\alpha, \Omega) &= \sup_{\delta>0} \mathrm{Entp}_L(\alpha, \Omega, \delta),\\
\mathrm{Entp}_L(\alpha) &= \sup_{\Omega \in Pf(\mathcal{L}\cap A_1)} \mathrm{Entp}_L(\alpha, \Omega).
\end{aligned}$$

We will call $\mathrm{Entp}_L(\alpha)$ *the* **product entropy** *of* α.

We record in the following proposition the evident fact that $\mathrm{Entp}_L(A)$ is invariant under bi-Lipschitz *-isomorphisms.

Proposition 5.3. *Let A and B be unital C^*-algebras with Leibniz ${}_{\mathrm{c}}$Lip-norms L_A and L_B, repectively. Let $\alpha \in \mathrm{Aut}_{L_A}(A)$ and $\beta \in \mathrm{Aut}_{L_B}(B)$. Suppose $\Gamma : A \to B$ is a bi-Lipschitz *-isomorphism which intertwines α with β (i.e., $\Gamma \circ \alpha = \beta \circ \Gamma$). Then*

$$\mathrm{Entp}_L(\alpha) = \mathrm{Entp}_L(\beta).$$

The entropy $\mathrm{Entp}(\alpha)$ is related to the metric dimension of A by the following inequality, which formally parallels a familiar fact about topological entropy (see [10, Prop. 14.20]). We remark that we don't know whether the Lipschitz number of a bi-Lipschitz automorphism α can be strictly less than 1, although it is evident that in general at least one of α and α^{-1} must have Lipschitz number at least 1.

Proposition 5.4. *If $\alpha \in \mathrm{Aut}_L(A)$ and $\mathrm{Mdim}_L(A)$ is finite then*

$$\mathrm{Entp}_L(\alpha) \leq \mathrm{Mdim}_L(A) \cdot \log \max(\lambda, 1),$$

where λ is the Lipschitz number of α.

Proof. Let $\Omega \in Pf(\mathcal{L} \cap A_1, \delta)$ and $\delta > 0$. Set $M = \max_{a\in\Omega} L(a)$. Then by repeated application of the Leibniz inequality we see that elements of the set

$$\Omega_n = \Omega \cdot \alpha(\Omega) \cdot \alpha^2(\Omega) \cdot \cdots \cdot \alpha^{n-1}(\Omega)$$

have ${}_{\mathrm{c}}$Lip-norm at most $M(1+\lambda+\lambda^2+\cdots+\lambda^{n-1})$, which is bounded above by $Mn\lambda^n$. Hence $\mathcal{L}_1$ contains the set $\{(Mn\lambda^n)^{-1}a : a \in \Omega_n\}$, which we will denote simply by $(Mn\lambda^n)^{-1}\Omega_n$. It follows that

$$\begin{aligned}
\mathrm{Entp}_L(\alpha, \Omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log D(\Omega_n, \delta) \\
&= \limsup_{n\to\infty} \frac{1}{n} \log D((Mn\lambda^n)^{-1}\Omega_n, (Mn\lambda^n)^{-1}\delta) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \log D(\mathcal{L}_1, (Mn\lambda^n)^{-1}\delta).
\end{aligned}$$

If $\lambda < 1$ then this last limit supremum is clearly zero. If on the other hand $\lambda \geq 1$ then

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log D(\mathcal{L}_1, (Mn\lambda^n)^{-1}\delta) \\
&\quad\leq \limsup_{n\to\infty} \frac{1}{n} \frac{\log D(\mathcal{L}_1, (Mn\lambda^n)^{-1}\delta)}{\log(Mn\lambda^n\delta^{-1})} \log(Mn\lambda^n\delta^{-1}) \\
&\quad= \limsup_{n\to\infty} \frac{\log D(\mathcal{L}_1, (Mn\lambda^n)^{-1}\delta)}{\log(Mn\lambda^n\delta^{-1})} \cdot \lim_{n\to\infty} \frac{1}{n} \log(Mn\lambda^n\delta^{-1}) \\
&\quad= \mathrm{Mdim}_L(A) \cdot \log\lambda.
\end{aligned}$$

We thus obtain the result by taking the supremum over all Ω and δ. □

Corollary 5.5. *If* $\mathrm{Mdim}_L(A)$ *is finite and* $\alpha \in \mathrm{Aut}_L(A)$ *is Lipschitz isometric then* $\mathrm{Entp}_L(\alpha) = 0$. *In particular* $\mathrm{Entp}_L(\mathrm{id}_A) = 0$.

Corollary 5.5 shows that the appropriate domain for our notion of entropy as a measure of dynamical growth is the class of ${}_{\mathrm{c}}$Lip-normed unital C^*-algebras A for which $\mathrm{Mdim}_L(A)$ is finite, in analogy to the situation of topological approximation entropies [4, 33] which function under conditions of "finiteness" like nuclearity or exactness.

Proposition 5.6. *If* $\alpha \in \mathrm{Aut}_L(A)$ *and* $k \in \mathbb{Z}$ *then* $\mathrm{Entp}_L(\alpha^k) = |k| \,\mathrm{Entp}_L(\alpha)$.

Proof. Suppose first that $k \geq 0$. Let $\Omega \in Pf(\mathcal{L} \cap A_1)$ and $\delta > 0$, and suppose $1 \in \Omega$. Then

$$\prod_{j=0}^{n-1} \alpha^{jk}(\Omega) \subset \prod_{j=0}^{(n-1)k} \alpha^j(\Omega)$$

so that

$$\begin{aligned}\mathrm{Entp}_L(\alpha^k,\Omega,\delta) &= \limsup_{n\to\infty}\frac{1}{n}\log D\left(\prod_{j=0}^{n-1}\alpha^{jk}(\Omega),\delta\right)\\ &\leq k\limsup_{n\to\infty}\frac{1}{kn}\log D\left(\prod_{j=0}^{(n-1)k}\alpha^{j}(\Omega),\delta\right)\\ &= k\,\mathrm{Entp}_L(\alpha,\Omega,\delta).\end{aligned}$$

On the other hand setting $\Omega_k = \prod_{j=0}^{k-1}\alpha^j(\Omega)$ (which is contained in $Pf(\mathcal{L}\cap A_1)$ in view of the Leibniz rule) we have

$$\prod_{j=0}^{\lfloor\frac{n}{k}\rfloor}\alpha^j(\Omega_k)\subset\prod_{j=0}^{n-1}\alpha^j(\Omega)$$

so that

$$\begin{aligned}\mathrm{Entp}_L(\alpha^k,\Omega_k,\delta) &= \limsup_{n\to\infty}\frac{k}{n}\log D\left(\prod_{j=0}^{\lfloor\frac{n}{k}\rfloor}\alpha^{j}(\Omega_k),\delta\right)\\ &\leq k\limsup_{n\to\infty}\frac{1}{n}\log D\left(\prod_{j=0}^{n-1}\alpha^{j}(\Omega),\delta\right)\\ &= k\,\mathrm{Entp}_L(\alpha,\Omega,\delta),\end{aligned}$$

and hence

$$\mathrm{Entp}_L\left(\alpha^k,\Omega_k,\delta\right)\leq k\,\mathrm{Entp}_L\left(\alpha,\Omega,\delta\right).$$

Taking the supremum over all $\Omega\in Pf(\mathcal{L}\cap A_1)$ and $\delta>0$ yields $\mathrm{Entp}_L(\alpha^k) = k\,\mathrm{Entp}_L(\alpha)$.

To prove the assertion for $k<0$ it suffices, in view of the first part, to show that $\mathrm{Entp}_L(\alpha^{-1}) = \mathrm{Entp}_L(\alpha)$. Since

$$\alpha^{-n+1}\left(\prod_{j=0}^{n-1}\alpha^j(\Omega)\right)=\prod_{j=0}^{n-1}\alpha^{-j}(\Omega)$$

we have

$$D\left(\prod_{j=0}^{n-1}\alpha^j(\Omega),\delta\right)=D\left(\prod_{j=0}^{n-1}\alpha^{-j}(\Omega),\delta\right)$$

and hence

$$\mathrm{Entp}_L(\alpha,\Omega,\delta)=\mathrm{Entp}_L\left(\alpha^{-1},\Omega,\delta\right),$$

from which we reach the conclusion by taking the supremum over all $\Omega\in Pf(\mathcal{L}\cap A_1)$ and $\delta>0$. $\square$

The following proposition is clear from Definition 5.2.

Proposition 5.7. *Let A be a unital C^*-algebra with ${}_{\rm c}$Lip-norm L_A and $B \subset A$ a unital C^*-subalgebra with ${}_{\rm c}$Lip-norm L_B such that L_B is the restriction of L_A to B. Suppose that there is a C^*-norm contractive idempotent linear map of A onto B. If $\alpha \in \mathrm{Aut}_L(A)$ leaves B invariant then*

$$\mathrm{Entp}_{L_B}(\alpha|_B) \leq \mathrm{Entp}_{L_A}(\alpha).$$

Proposition 5.8. *Let A and B be unital C^*-algebras, L_A a Leibniz ${}_{\rm c}$Lip-norm on A, $\phi : A \to B$ a surjective unital *-homomorphism, and L_B the Leibniz ${}_{\rm c}$Lip-norm induced on B via ϕ. Suppose there exists a positive C^*-norm contractive (not necessarily unital) Lipschitz map $\psi : B \to A$ such that $\phi \circ \psi = \mathrm{id}_B$. Let $\alpha \in \mathrm{Aut}_{L_A}(A)$ and $\beta \in \mathrm{Aut}_{L_B}(B)$ and suppose $\phi \circ \alpha = \beta \circ \phi$. Then*

$$\mathrm{Entp}_{L_B}(\beta) \leq \mathrm{Entp}_{L_A}(\alpha).$$

Proof. Let $\Omega \in Pf(\mathcal{L}^B \cap B_1)$ and $\delta > 0$. Since ψ is norm-decreasing we have $\psi(\Omega) \in Pf(\mathcal{L}^A \cap A_1)$. Now if $X \in \mathcal{F}(A)$ is such that

$$\psi(\Omega) \cdot \alpha(\psi(\Omega)) \cdot \dots \cdot \alpha^{n-1}(\psi(\Omega)) \subset_\delta X,$$

then

$$\begin{aligned}
\Omega \cdot \beta(\Omega) \cdot \dots \cdot \beta^{n-1}(\Omega) &= (\phi \circ \psi)(\Omega) \cdot \beta((\phi \circ \psi)(\Omega)) \cdot \dots \cdot \beta^{n-1}((\phi \circ \psi)(\Omega)) \\
&= \phi(\psi(\Omega)) \cdot \phi(\alpha(\psi(\Omega))) \cdot \dots \cdot \phi(\alpha^{n-1}(\psi(\Omega))) \\
&= \phi(\psi(\Omega) \cdot \alpha(\psi(\Omega)) \cdot \dots \cdot \alpha^{n-1}(\psi(\Omega))) \\
&\subset_\delta \phi(X)
\end{aligned}$$

and so

$$D(\Omega \cdot \beta(\Omega) \cdot \dots \cdot \beta^{n-1}(\Omega), \delta) \leq D(\psi(\Omega) \cdot \alpha(\psi(\Omega)) \cdot \dots \cdot \alpha^{n-1}(\psi(\Omega)), \delta),$$

from which the proposition follows. □

5.2. Product entropy with respect to an invariant state. We define now a version of $\mathrm{Mdim}_L(A)$ relative to a dynamically invariant state σ. As in Subsect. 5.1 we are assuming that L is a Leibniz ${}_{\rm c}$Lip-norm. For the meaning of $D_\sigma(\cdot\,,\cdot)$ see Notation 3.1.

Definition 5.9. *Let $\alpha \in \mathrm{Aut}_L(A)$ and let σ be a state of A which is α-invariant, i.e., $\sigma \circ \alpha = \sigma$. For $\Omega \in Pf(\mathcal{L} \cap A_1)$ and $\delta > 0$ we define*

$$\begin{aligned}
\mathrm{Entp}_{L,\sigma}(\alpha, \Omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log D_\sigma\left(\Omega \cdot \alpha(\Omega) \cdot \alpha^2(\Omega) \cdot \dots \cdot \alpha^{n-1}(\Omega), \delta\right), \\
\mathrm{Entp}_{L,\sigma}(\alpha, \Omega) &= \sup_{\delta>0} \mathrm{Entp}_{L,\sigma}(\alpha, \Omega, \delta), \\
\mathrm{Entp}_{L,\sigma}(\alpha) &= \sup_{\Omega \in Pf(\mathcal{L}\cap A_1)} \mathrm{Entp}_{L,\sigma}(\alpha, \Omega).
\end{aligned}$$

We will call $\mathrm{Entp}_{L,\sigma}(\alpha)$ *the* **product entropy** *of α with respect to σ.*

The following two propositions follow immediately from the definition.

Proposition 5.10. *Let A and B be unital C^*-algebras with respective Leibniz ${}_{\mathrm{c}}$Lip-norms L_A and L_B. Let $\alpha \in \mathrm{Aut}_{L^A}(A)$ and $\beta \in \mathrm{Aut}_{L^B}(B)$, and let σ and ω be α- and β-invariant states on A and B, respectively. Suppose $\Gamma : A \to B$ is a bi-Lipschitz *-isomorphism such that $\Gamma \circ \alpha = \beta \circ \Gamma$ and $\omega \circ \Gamma = \sigma$. Then*

$$\mathrm{Entp}_{L,\sigma}(\alpha) = \mathrm{Entp}_{L,\omega}(\beta).$$

Proposition 5.11. *Let A be a unital C^*-algebra with Leibniz ${}_{\mathrm{c}}$Lip-norm L_A and $B \subset A$ a unital C^*-subalgebra with Leibniz ${}_{\mathrm{c}}$Lip-norm L_B such that L_B is the restriction of L_A to B. Let σ be a state on A with $\sigma \circ \alpha = \sigma$, and suppose that there is a idempotent linear map of A onto B which is contractive for the Hilbert space norm under the GNS construction associated to σ. If $\alpha \in \mathrm{Aut}_L(A)$ leaves B invariant then*

$$\mathrm{Entp}_{L_B,\sigma}(\alpha|_B) \leq \mathrm{Entp}_{L_A,\sigma}(\alpha).$$

The next proposition can be established in the same way as its counterpart Proposition 5.6 in Subsect. 5.1.

Proposition 5.12. *If $\alpha \in \mathrm{Aut}_L(A)$, σ is an α-invariant state on A, and $k \in \mathbb{Z}$, then* $\mathrm{Entp}_{L,\sigma}(\alpha^k) = |k|\,\mathrm{Entp}_{L,\sigma}(\alpha)$.

Proposition 5.13. *If $\alpha \in \mathrm{Aut}_L(A)$ and σ is an α-invariant state on A then*

$$\mathrm{Entp}_{L,\sigma}(\alpha) \leq \mathrm{Entp}_L(\alpha).$$

Proof. It suffices to show that, for a given $\Omega \in Pf(\mathcal{L} \cap A_1)$ and $\delta > 0$,

$$D_\sigma(\Omega, \delta) \leq D(\Omega, \delta),$$

and for this inequality we need only observe that if X is a finite-dimensional subspace of A such that $\Omega \subset_\delta X$, then whenever $a \in \Omega$ and $x \in X$ satisfy $\|a - x\| < \delta$ we have

$$\|\pi(a)\xi_\sigma - \pi(x)\xi_\sigma\|_\sigma = \|\pi(a - x)\xi_\sigma\|_\sigma \leq \|\pi(a - x)\| \leq \|a - x\| < \delta,$$

so that $\pi(X)\xi_\sigma$ is a subspace of $\mathcal{H}_\sigma$ with $\pi(\Omega)\xi_\sigma \subset_\delta \pi(X)\xi_\sigma$ and $\dim \pi(\Omega)\xi_\sigma \leq \dim X$. $\square$

Corollary 5.14. *If $\mathrm{Mdim}_L(A)$ is finite and $\alpha \in \mathrm{Aut}_L(A)$ is Lipschitz isometric then* $\mathrm{Entp}_{L,\sigma}(\alpha) = 0$. *In particular* $\mathrm{Entp}_{L,\sigma}(\mathrm{id}_A) = 0$.

Proof. This follows by combining Proposition 5.13 with Corollary 5.5. $\square$

6. Tensor Product Shifts

The fundamental prototypical system for topological entropy is the shift on the infinite product $\{1, \ldots, p\}^{\mathbb{Z}}$, with entropy $\log p$. Here we consider the noncommutative analogue of this map, the (right) shift on the infinite tensor product $M_p^{\otimes\mathbb{Z}}$ of $p \times p$ matrix algebras M_p over $\mathbb{C}$, here with the Leibniz ${}_{\mathrm{c}}$Lip-norm L furnished by the infinite product $\gamma^{\otimes\mathbb{Z}} : G^{\mathbb{Z}} \to \mathrm{Aut}\left(M_p^{\otimes\mathbb{Z}}\right)$ of Weyl actions and length function ℓ_λ (for a given $0 < \lambda < 1$) as described in Subsect. 4.1.

Before computing the entropy of the shift we will show that it is a bi-Lipschitz *-automorphism.

Proposition 6.1. *The shift α on $M_p^{\otimes\mathbb{Z}}$ is a bi-Lipschitz *-automorphism, and α and its inverse have Lipschitz numbers bounded by λ.*

Proof. Let $T : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be the right shift homeomorphism. Then it is readily seen that if a is an elementary tensor in $M_p^{[m,n]} \subset M_p^{\otimes\mathbb{Z}}$ for some $m, n \in \mathbb{Z}$ then $\gamma_g^{\otimes\mathbb{Z}}(\alpha(a)) = \alpha\left(\gamma_{Tg}^{\otimes\mathbb{Z}}(a)\right)$ for all $g \in G^{\mathbb{Z}}$, and since such a generate $M_p^{\otimes\mathbb{Z}}$ we have $\gamma_g^{\otimes\mathbb{Z}} \circ \alpha = \alpha \circ \gamma_{Tg}^{\otimes\mathbb{Z}}$ for all $g \in G^{\mathbb{Z}}$. Thus, for any $a \in M_p^{\otimes\mathbb{Z}}$,

$$\begin{aligned}
L(\alpha(a)) &= \sup_{g \in G^{\mathbb{Z}} \setminus \{e\}} \frac{\left\| \gamma_g^{\otimes\mathbb{Z}}(\alpha(a)) - \alpha(a) \right\|}{\ell_\lambda(g)} \\
&= \sup_{g \in G^{\mathbb{Z}} \setminus \{e\}} \frac{\left\| \alpha\left(\gamma_{Tg}^{\otimes\mathbb{Z}}(a)\right) - \alpha(a) \right\|}{\ell_\lambda(g)} \\
&\le \sup_{g \in G^{\mathbb{Z}} \setminus \{e\}} \frac{\left\| \gamma_{Tg}^{\otimes\mathbb{Z}}(a) - a \right\|}{\ell_\lambda(Tg)} \cdot \sup_{g \in G^{\mathbb{Z}} \setminus \{e\}} \frac{\ell_\lambda(Tg)}{\ell_\lambda(g)} \\
&\le L(a) \cdot L(T),
\end{aligned}$$

where $L(T)$ is the Lipschitz number of the homeomorphism T with respect to the metric defining ℓ_λ (see Subsect. 4.1), and it is straightforward to verify that $L(T) = \lambda$. We can argue similarly for α^{-1} to reach the desired conclusion. □

Proposition 6.2. *Let α be the shift on $M_p^{\otimes\mathbb{Z}}$ and $\tau = tr_p^{\otimes\mathbb{Z}}$ the unique (and hence α-invariant) tracial state on $M_p^{\otimes\mathbb{Z}}$. Then*

$$\mathrm{Entp}_{L,\tau}(\alpha) \ge 2 \log p.$$

Proof. Let $u, v \in M_p^{\otimes\mathbb{Z}}$ be the Weyl generators for the zeroth copy of M_p (identified as a subalgebra of $M_p^{\otimes\mathbb{Z}}$) and let Ω be the finite subset $\{u^i v^j : 0 \le i, j \le k-1\}$ of $\mathcal{L} \cap \left(M_p^{\otimes\mathbb{Z}}\right)_1$. Then the set $\Omega_n = \Omega \cdot \alpha(\Omega) \cdot \alpha^2(\Omega) \cdot \cdots \cdot \alpha^{n-1}(\Omega)$ is precisely the subset

$$\left\{ u^{i_0} v^{j_0} \otimes u^{i_1} v^{j_1} \otimes \cdots \otimes u^{i_{n-1}} v^{j_{n-1}} : 0 \le i_k, j_k \le p-1 \text{ for } k = 0, \ldots, n-1 \right\}$$

of $M_p^{\otimes[0,n]}$ as considered sitting in $M_p^{\otimes\mathbb{Z}}$. Thus $\pi_\tau(\Omega_n)\xi_\tau$ is an orthonormal set in the GNS representation Hilbert space associated to τ with canonical cyclic vector ξ_τ (see the second half of the proof of Proposition 4.1), and so by Lemma 3.8 for any $\delta > 0$ we have

$$D_\tau(\Omega_n, \delta) \ge (1 - \delta^2)\mathrm{card}(\pi_\tau(\Omega_n)\xi_\tau) = (1 - \delta^2) p^{2n}.$$

Thus if $\delta < 1$ we obtain

$$\mathrm{Entp}_{L,\sigma}(\alpha, \Omega, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log D_\sigma(\Omega_n, \delta) \ge 2 \log p,$$

which yields the proposition. □

Note that by Propositions 5.4, 4.1, and 6.1 the shift α satisfies

$$\mathrm{Entp}_L(\alpha) \leq 4 \log p.$$

The following proposition yields the sharp upper bound of $2 \log p$.

Proposition 6.3. *With α the shift we have*

$$\mathrm{Entp}_L(\alpha) \leq 2 \log p.$$

Proof. Let $\Omega \in Pf\left(\mathcal{L} \cap \left(M_p^{\otimes\mathbb{Z}}\right)_1\right)$ and $\delta > 0$. Set $C = \max_{a\in\Omega} L_\lambda(a)$. For each n consider the conditional expectation $E_n : M_p^{\otimes\mathbb{Z}} \to M_p^{\otimes[-n,n]}$ given by

$$E_n(a) = \int_{G^{\mathbb{Z}\setminus[-n,n]}} \gamma_g^{\otimes\mathbb{Z}}(a)\, dg,$$

where dg is normalized Haar measure on $G^{\mathbb{Z}}$. We then have

$$\begin{aligned}
\|E_n(a) - a\| &= \left\| \int_{G^{\mathbb{Z}\setminus[-n,n]}} \left(\gamma_g^{\otimes\mathbb{Z}}(a) - a\right) dg \right\| \\
&\leq \int_{G^{\mathbb{Z}\setminus[-n,n]}} \left\|\gamma_g^{\otimes\mathbb{Z}}(a) - a\right\| dg \\
&\leq \int_{G^{\mathbb{Z}\setminus[-n,n]}} C\ell_\lambda(g)\, dg \\
&\leq \frac{2C\lambda^{n+1}}{1-\lambda}.
\end{aligned}$$

If $a_1, \dots, a_n \in \Omega$ then, estimating the norm of differences of products in the usual way and using the fact that the conditional expectations are norm-decreasing, we have

$$\begin{aligned}
&\left\| E_{\lceil\sqrt{n}\rceil}(a_1)\alpha(E_{\lceil\sqrt{n}\rceil}(a_2)) \cdots \alpha^{n-1}(E_{\lceil\sqrt{n}\rceil}(a_n)) - a_1\alpha(a_2)\cdots\alpha^{n-1}(a_n) \right\| \\
&\leq \sum_{k=1}^{n} \left\| \alpha^{k-1}(E_{\lceil\sqrt{n}\rceil}(a_k)) - \alpha^{k-1}(a_k) \right\| \\
&= \sum_{k=1}^{n} \left\| E_{\lceil\sqrt{n}\rceil}(a_k) - a_k \right\| \\
&\leq \frac{2Cn\lambda^{\lceil\sqrt{n}\rceil+1}}{1-\lambda},
\end{aligned}$$

which is smaller than δ for all n greater than some $n_0 \in \mathbb{N}$ (here $\lceil\cdot\rceil$ denotes the ceiling function).

Next we observe that the product

$$E_{\lceil\sqrt{n}\rceil}(a_1)\alpha\left(E_{\lceil\sqrt{n}\rceil}(a_2)\right)\cdots\alpha^{n-1}\left(E_{\lceil\sqrt{n}\rceil}(a_n)\right)$$

is contained in the subalgebra $M_p^{\otimes[-\lceil\sqrt{n}\rceil,\lceil\sqrt{n}\rceil+n]}$ of $M_p^{\otimes\mathbb{Z}}$, and this subalgebra has linear dimension $p^{2(2\lceil\sqrt{n}\rceil+n)}$. In view of the first paragraph, for all $n \geq n_0$, the set Ω_n is approximately contained in $M_p^{\otimes[-\lceil\sqrt{n}\rceil,\lceil\sqrt{n}\rceil+n]}$ to within δ, and so we have

$$\mathrm{Entp}_L(\alpha, \Omega, 2\delta) \leq \limsup_{n\to\infty} \frac{1}{n} \log p^{2(2\lceil\sqrt{n}\rceil+n)} = 2\log p.$$

The proposition now follows by taking the supremum over all Ω and δ. $\square$

As a consequence of Propositions 6.2, 6.3, and 5.13 we obtain the following.

Proposition 6.4. *With α the shift and τ the unique tracial state on $M_p^{\otimes\mathbb{Z}}$ we have*

$$\mathrm{Entp}_L(\alpha) = \mathrm{Entp}_{L,\tau}(\alpha) = 2\log p.$$

7. Noncommutative Toral Automorphisms

Let A_ρ be a noncommutative p-torus with generators $u_1, \ldots, u_p$, canonical ergodic action $\gamma : \mathbb{T}^p \to \mathrm{Aut}(A_\rho)$, and associated Leibniz ${}_{\mathrm{c}}$Lip-norm L and γ-invariant tracial state τ, as defined in Subsect. 4.2. We let $\pi_\tau : A_\rho \to \mathcal{B}(\mathcal{H}_\tau)$ be the GNS representation associated to τ, with canonical cyclic vector ξ_τ. Let $T = (s_{ij})$ be a $p \times p$ integral matrix with $\det T = \pm 1$, and suppose that T defines an automorphism α_T of A_ρ via the specifications

$$\alpha_T(u_j) = u_1^{s_{1j}} \cdots u_p^{s_{pj}}$$

on the generators (this will always be the case if $\det T = 1$ owing to the universal property of noncommutative tori). These noncommutative versions of toral automorphisms were introduced in the case $p = 2$ in [34] and [3]. Since τ is zero on products of powers of generators which are not equal to the unit, we see that it is invariant under the automorphism α_T and the action γ. Fix a $t = (t_1, \ldots, t_p) \in \mathbb{T}^p \cong (\mathbb{R}/\mathbb{Z})^p$ and consider the automorphism γ_t coming from the action γ. We will compute the entropies $\mathrm{Entp}_L(\alpha_T \circ \gamma_t)$ and $\mathrm{Entp}_{L,\tau}(\alpha_T \circ \gamma_t)$ and furthermore show that their common value bounds above the entropies $\mathrm{Entp}_L(\mathrm{Ad}u \circ \alpha_T \circ \gamma_t)$ and $\mathrm{Entp}_{L,\tau}(\mathrm{Ad}u \circ \alpha_T \circ \gamma_t)$ for any unitary $u \in \mathcal{L}$. We remark that in the case $p = 2$, when A_ρ is a rotation C^*-algebra A_θ, Elliott showed in [12] that if the angle θ satisfies a generic Diophantine property then all automorphisms preserving the dense *-subalgebra of smooth elements (i.e., all "diffeomorphisms") are of the form $\mathrm{Ad}u \circ \alpha_T \circ \gamma_t$, where u is a smooth unitary (and hence of finite ${}_{\mathrm{c}}$Lip-norm).

Proposition 7.1. *The *-automorphism $\alpha = \mathrm{Ad}u \circ \alpha_T \circ \gamma_t$ is bi-Lipschitz, and α and its inverse have Lipschitz numbers bounded by*

$$2\|T\|(1 + 2L(u)\mathrm{diam}(S(A)))$$

and

$$2\left\|T^{-1}\right\|(1 + 2L(u)\mathrm{diam}(S(A))),$$

respectively, where $\|T\|$ and $\|T^{-1}\|$ are the respective norms of T and T^{-1} as operators on the real inner product space $\mathbb{R}^p$.

Proof. If we consider T as acting on $\mathbb{T}^p$ then $\gamma_g \circ \alpha = \alpha \circ \gamma_{Tg}$ for all $g \in \mathbb{T}^p$, as can be seen by checking this equation on the generators $u_1, \dots, u_p$. As in the proof of Proposition 6.1 we thus have, for any $a \in \mathcal{L}$, the bound

$$L(\alpha(a)) \leq L(a) \cdot L(T),$$

where $L(T)$ is the Lipschitz number of the homeomorphism T. If we consider T as an operator on $\mathbb{R}^p$, then its Lipschitz number is $\|T\|$ by definition of the operator norm, and so by linearity the Lipschitz number $L(T)$ of T on the quotient $\mathbb{T}^p \cong \mathbb{R}^p/\mathbb{Z}^p$ must again be $\|T\|$. Next note that γ_t is isometric, for if $a \in \mathcal{L}$ then

$$L(\gamma_t(a)) = \sup_{s \in \mathbb{T}^p \setminus \{0\}} \frac{\|\gamma_{s+t}(a) - \gamma_t(a)\|}{\ell(s)} = \sup_{s \in \mathbb{T}^p \setminus \{0\}} \frac{\|\gamma_s(a) - a\|}{\ell(s)} = L(a).$$

Also, since L is readily checked to be lower semicontinuous, by Proposition 2.11 the Lipschitz number of $\mathrm{Ad}u$ is bounded by $2(1 + 2L(u)\mathrm{diam}(S(A)))$. Thus by Proposition 2.9 we get the desired bound on the Lipschitz number of $\mathrm{Ad}u \circ \alpha_T \circ \gamma_t$. A similar argument can be applied to $(\mathrm{Ad}u \circ \alpha_T \circ \gamma_t)^{-1} = \gamma_{-t} \circ \alpha_{T^{-1}} \circ \mathrm{Ad}u^*$. □

Proposition 7.2. *We have*

$$\mathrm{Entp}_{L,\tau}(\alpha_T \circ \gamma_t) \geq \sum_{|\lambda_i| \geq 1} \log |\lambda_i|,$$

where $\lambda_1, \dots, \lambda_p$ are the eigenvalues of T counted with spectral multiplicity.

Proof. Let K be a finite subset of $\mathbb{Z}^p$ and set

$$U_K = \left\{ u_1^{k_1} u_2^{k_2} \cdots u_p^{k_p} : (k_1, \dots, k_p) \in K \right\}.$$

The elements of U_K, being products of powers of generators, all have finite ${}_{\mathrm{c}}$Lip-norm. Observe that $\alpha_T \circ \gamma_t$ takes a product of the form $\eta u_1^{k_1} u_2^{k_2} \cdots u_p^{k_p}$, with η a complex number of modulus one, to a product of the same form, with the exponents on the u_i's respecting the action of the group automorphism ζ_T of $\mathbb{Z}^p$ defined via the action of T. Thus if K is a finite subset of $\mathbb{Z}^p$ then the set

$$U_K \cdot (\alpha_T \circ \gamma_t)(U_K) \cdot \dots \cdot (\alpha_T \circ \gamma_t)^{n-1}(U_K)$$

contains a subset $U_{K,n}$ of the form

$$\left\{ \eta_{(k_1,\dots,k_p)} u_1^{k_1} u_2^{k_2} \cdots u_p^{k_p} : (k_1, \dots, k_p) \in K + \zeta_T K + \cdots + \zeta_T^{n-1} K \right\},$$

where each $\eta_{(k_1,\dots,k_p)}$ is a complex number of modulus one. Note that $\pi_\tau(U_{K,n})\xi_\tau$ is an orthonormal set of vectors in the GNS representation Hilbert space associated to τ with canonical cyclic vector ξ_τ, since the product of any two distinct vectors in this set is a scalar multiple of a product of the form $u_1^{k_1} u_2^{k_2} \cdots u_p^{k_p}$ with the k_i's not all zero, in which case evaluation under τ yields zero. It thus follows from Lemma 3.8 that if $\delta > 0$ then

$$\begin{aligned} D_\tau(U_{K,n}, \delta) &\geq (1 - \delta^2)\mathrm{card}(\pi(U_{K,n})\xi_\tau) \\ &= (1 - \delta^2)\mathrm{card}(K + \zeta_T K + \cdots + \zeta_T^{n-1} K), \end{aligned}$$

so that whenever $\delta < 1$ we get

$$\begin{aligned}\mathrm{Entp}_{L,\sigma}(\alpha_t \circ \gamma_t, U_K, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log D_\tau(U_K \cdot \alpha(U_K) \cdot \cdots \cdot \alpha^{n-1}(U_K), \delta),\\ &\geq \limsup_{n\to\infty} \frac{1}{n} \log D_\tau(U_{K,n}, \delta)\\ &\geq \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{card}(K + \zeta_T K + \cdots + \zeta_T^{n-1} K).\end{aligned}$$

We thus reach the desired conclusion by recalling from the computation of the discrete Abelian group entropy of ζ_T [23] that

$$\lim_K \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{card}(K + \zeta_T K + \cdots + \zeta_T^{n-1} K) = \sum_{|\lambda_i|\geq 1} \log|\lambda_i|,$$

where the limit is taken with respect to the net of finite subsets K of $\mathbb{Z}^p$. □

To compute upper bounds we need a couple of lemmas.

Lemma 7.3. *Let ζ_T be the group automorphism of $\mathbb{Z}^p$ defined via the action of an $p \times p$ integral matrix T with $\det(T) = \pm 1$. Let $\lambda_1, \ldots, \lambda_p$ be the eigenvalues of T counted with spectral multiplicity. For each $m \in \mathbb{N}$ let K_m be the cube*

$$\left\{(k_1, \ldots, k_p) \in \mathbb{Z}^p : |k_i| \leq m \text{ for each } i = 1, \ldots, p\right\}$$

and define recursively for $n \geq 0$ the sets $L_{m,n} \in \mathbb{Z}^p$ by $L_{m,0} = K_m$ and

$$L_{m,n+1} = \zeta_T(L_{m,n}) + K_m.$$

Then for every $\delta > 0$ there is a $Q > 0$ such that, for all $m, n \in \mathbb{N}$,

$$\operatorname{card}\left(L_{m,0} + L_{m,1} + \cdots + L_{m,n-1}\right) \leq Q(mn^2)^p(1+\delta)^n \prod_{|\lambda_i|\geq 1} |\lambda_i|^n.$$

Proof. For any subset K of $\mathbb{Z}^p$ we will denote its convex hull as a subset of $\mathbb{R}^p$ by $\tilde{K}$. With ζ_T also referring to the linear map on $\mathbb{R}^p$ defined by T, we consider the convex set $\tilde{L}_{m,0} + \tilde{L}_{m,1} + \cdots + \tilde{L}_{m,n-1}$. By amplifying this set by a linear factor of 2^p we can ensure that it contains every cube of unit side length centred at some point in $L_{m,0} + L_{m,1} + \cdots + L_{m,n-1}$, so that

$$\operatorname{card}\left(L_{m,0} + L_{m,1} + \cdots + L_{m,n-1}\right) \leq 2^p \operatorname{vol}\left(\tilde{L}_{m,0} + \tilde{L}_{m,1} + \cdots + \tilde{L}_{m,n-1}\right).$$

To estimate this volume on the right we assemble a basis $\mathcal{B}$ of $\mathbb{R}^p$ by picking a basis for the spectral subspace associated to each real eigenvalue and each pair of conjugate complex eigenvalues. Working from this point on with respect to the basis $\mathcal{B}$, we note that the sets $\tilde{K}_m$ are now parallelipipeds, and they can be contained in cubes B_m centred at 0 of side length rm for some $r > 0$ independent of m by the linearity of our basis change. If we define the sets $M_{m,n}$ recursively by $M_{m,0} = B_m$ and

$$M_{m,n+1} = \zeta_T(M_{m,n}) + B_m,$$

then the set $M_{m,0} + M_{m,1} + \cdots + M_{m,n-1}$ is a p-dimensional rectangular box which is centred at the origin with each face perpendicular to some coordinate axis, and this box contains $\tilde{L}_{m,0} + \tilde{L}_{m,1} + \cdots + \tilde{L}_{m,n-1}$, so that it suffices to show that

$$\operatorname{vol}\left(M_{m,0} + M_{m,1} + \cdots + M_{m,n-1}\right)$$

is bounded by the last expression in the lemma statement for some $C > 0$.

Let v be a vector in $\mathcal{B}$ associated to a real eigenvalue λ or a complex conjugate pair $\{\lambda, \bar{\lambda}\}$. We can then find a $Q > 0$ such that for all $n \in \mathbb{N}$ the length of the vector $T^n(v)$ is bounded by

$$Q(1+\delta)^n |\lambda|^n,$$

where the factor $(1+\delta)^n$ is required to handle additional polynomial growth in the presence of a possible non-trivial generalized eigenspace. In view of the recursion defining $M_{m,n}$ we then see that any scalar multiple of v which lies in $M_{m,n}$ must be bounded in length by

$$Qrm(1+\delta)^{n-1}|\lambda|^{n-1} + Qrm(1+\delta)^{n-2}|\lambda|^{n-2} + \cdots + Qrm,$$

which in turn is bounded by

$$Qrmn(1+\delta)^n \max(|\lambda|^n, 1).$$

It follows that any scalar multiple of v contained in $M_{m,0} + M_{m,1} + \cdots + M_{m,n-1}$ is bounded in length by

$$Qrm \sum_{j=0}^{n-1} j(1+\delta)^j \max(|\lambda|^j, 1),$$

and this expression is less than

$$Qrmn^2(1+\delta)^n \max(|\lambda|^n, 1).$$

Since the set $M_{m,0} + M_{m,1} + \cdots + M_{m,n-1}$ is a rectangular box squarely positioned with respect to the basis $\mathcal{B}$ and centred at the origin (as described above), we combine these length estimates to conclude that

$$\operatorname{vol}(M_{m,0} + M_{m,1} + \cdots + M_{m,n-1}) \leq (Qrmn^2)^p (1+\delta)^n \prod_{|\lambda_i| \geq 1} |\lambda_i|^n,$$

which yields the result. $\square$

Proposition 7.4. *Suppose $u \in A_\rho$ is a unitary with $L(u) < \infty$. Then*

$$\operatorname{Entp}_L(\operatorname{Ad} u \circ \alpha_T \circ \gamma_t) \leq \sum_{|\lambda_i| \geq 1} \log |\lambda_i|,$$

where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of T counted with spectral multiplicity.

Proof. Set $\alpha = \mathrm{Ad}u \circ \alpha_T \circ \gamma_t$ for notational brevity. Let $\Omega \in Pf(\mathcal{L} \cup (A_\rho)_1)$ and $\delta > 0$. By Lemma 4.3 we can find an $C > 0$ such that

$$\|a - \sigma_n(a)\| \leq C\frac{\log n}{n}$$

for all $n \in \mathbb{N}$ and $a \in \Omega \cup \{u\}$, where $\sigma_n(a)$ is the n^{th} Cesàro mean, as defined in the paragraph preceding the statement of Lemma 4.2. Since $\sigma_n(u^*) = \sigma_n(u)^*$, we also then have

$$\left\|u^* - \sigma_n(u^*)\right\| \leq C\frac{\log n}{n}$$

for all $n \in \mathbb{N}$. Furthermore

$$\left\|\alpha^j(a) - \alpha^j(\sigma_n(a))\right\| \leq C\frac{\log n}{n}$$

for all $j, n \in \mathbb{N}$. By applying the triangle inequality n times in the usual way to estimate differences of products and using the fact that the operation of taking a Cesàro is norm-decreasing (as can be seen from the first display in the statement of Lemma 4.2), we then have, for any $a_1, \dots, a_n \in \Omega$,

$$\left\|a_1\alpha(a_2)\cdots\alpha^{n-1}(a_n) - \sigma_{n^2}(a_1)\alpha(\sigma_{n^2}(a_2))\cdots\alpha^{n-1}(\sigma_{n^2}(a_n))\right\| \leq C\frac{\log n^2}{n},$$

and this last quantity is less than δ for all n greater than or equal to some $n_0 \in \mathbb{N}$.

With the notation of the statement of Lemma 7.3 we next note that for any $a \in A$ and $n \in \mathbb{N}$ we have by definition

$$\sigma_{n^2}(a) \in \operatorname{span}\left\{u_1^{k_1}u_2^{k_2}\cdots u_p^{k_p} : (k_1, \dots, k_p) \in K_{n^2}\right\},$$

while if

$$a \in \operatorname{span}\left\{u_1^{k_1}u_2^{k_2}\cdots u_p^{k_p} : (k_1, \dots, k_p) \in K\right\}$$

for some finite $K \subset \mathbb{Z}^p$ then

$$(\mathrm{Ad}u)(\sigma_{n^2}(a)) \in \operatorname{span}\left\{u_1^{k_1}u_2^{k_2}\cdots u_p^{k_p} : (k_1, \dots, k_p) \in K + K_{2n^2}\right\}$$

for all $n \in \mathbb{N}$ (the factor of 2 in the subscript of K_{2n^2} is required to handle multiplication of a by both u and u^*). Thus, since γ_t commutes with the operation of taking a Cesàro sum of a given order, the set of all products $\sigma_{n^2}(a_1)\alpha(\sigma_{n^2}(a_2))\cdots\alpha^{n-1}(\sigma_{n^2}(a_n))$ with $a_i \in \Omega$ for $i = 1, \dots, n$ is contained in the subspace

$$X_n = \operatorname{span}\left\{u_1^{k_1}u_2^{k_2}\cdots u_p^{k_p} : (k_1, \dots, k_p) \in L_{2n^2,0} + L_{2n^2,1} + \cdots + L_{2n^2,n-1}\right\},$$

again using the notation in the statement of Lemma 7.3 (taking $m = 2n^2$ here). In view of the first paragraph X_n approximately contains $\Omega \cdot \alpha(\Omega) \cdot \cdots \cdot \alpha^{n-1}(\Omega)$ to within 2δ for all $n \geq n_0$, and by Lemma 7.3 there exists a $Q > 0$ such that

$$\dim(X_n) \leq (2Qn^3)^p(1+\delta)^n \prod_{|\lambda_i| \geq 1} |\lambda_i|^n$$

for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned}\mathrm{Entp}_L(\alpha, \Omega, 2\delta) &\leq \limsup_{n\to\infty} \frac{1}{n} \log \left((2Qn^3)^p (1+\delta)^n \prod_{|\lambda_i|\geq 1} |\lambda_i|^n \right) \\ &= \log(1+\delta) + \sum_{|\lambda_i|\geq 1} \log |\lambda_i|.\end{aligned}$$

Taking the supremum over all $\delta > 0$ then yields

$$\mathrm{Entp}_L(\alpha, \Omega) \leq \sum_{|\lambda_i|\geq 1} \log |\lambda_i|,$$

from which the proposition follows. □

Theorem 7.5. *We have*

$$\mathrm{Entp}_L(\alpha_T \circ \gamma_t) = \mathrm{Entp}_{L,\tau}(\alpha_T \circ \gamma_t) = \sum_{|\lambda_i|\geq 1} \log |\lambda_i|,$$

where $\lambda_1, \dots, \lambda_p$ *are the eigenvalues of* T *counted with spectral multiplicity. In particular,*

$$\mathrm{Entp}_L(\alpha_T) = \mathrm{Entp}_{L,\tau}(\alpha_T) = \sum_{|\lambda_i|\geq 1} \log |\lambda_i|.$$

Proof. This follows by combining Propositions 7.2, 7.4, and 5.13. □

We also have the following, which is a consequence of Propositions 5.13 and 7.4.

Proposition 7.6. *If* $u \in A$ *is a unitary with* $L(u) < \infty$*, then*

$$\mathrm{Entp}_L(\mathrm{Ad}u) = \mathrm{Entp}_{L,\tau}(\mathrm{Ad}u) = 0.$$

It is readily seen that if u is a unitary of the form $\eta u_1^{k_1} u_2^{k_2} \cdots u_p^{k_p}$ for some integers $k_1, \dots, k_p$ and complex number η of unit modulus, then the automorphism $\mathrm{Ad}u \circ \alpha_T \circ \gamma_t$ can be alternatively expressed as $\alpha_T \circ \gamma_{t'}$ for some $t' \in \mathbb{T}^p$, in which case Theorem 7.5 applies. We leave open the problem of computing the product entropies of $\mathrm{Ad}u \circ \alpha_T \circ \gamma_t$ when $u \in \mathcal{L}$ is a unitary not of this form and the eigenvalues of T do not all lie on the unit circle. We expect however that the entropies are positive when α_T is asymptotically Abelian (see [19] for a description of when this occurs in the case $p = 2$) and the partial Fourier sums or Cesàro means of u converge sufficiently fast to u, for we could then aim to apply the argument of the proof of Proposition 7.2 up to a degree of approximation.

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References

1. Alicki, R., Fannes, M.: Defining quantum dynamical entropy. Lett. Math. Phys. **32**, 75–82 (1994)
2. Andries, J., Fannes, M., Tuyls, P., Alicki, R.: The dynamical entropy of the quantum Arnold cat map. Lett. Math. Phys. **35**, 375–383 (1995)
3. Brenken, B.: Representations and automorphisms of the irrational rotation algebra. Pacific J. Math. **111**, 257–282 (1984)
4. Brown, N.P.: Topological entropy in exact C^*-algebras. Math. Ann. **314**, 347–367 (1999)
5. Connes, A.: Compact metric spaces, Fredholm modules and hyperfiniteness. Ergod. Th. Dynam. Sys. **9**, 207–220 (1989)
6. Connes, A.: *Noncommutative Geometry*. San Diego: Academic Press, 1994
7. Connes, A.: Gravity coupled with matter and the foundation of non-commutative geometry. Commun. Math. Phys. **182**, 155–176 (1996)
8. Connes, A., Narnhofer, H., Thirring, W.: Dynamical entropy of C^*-algebras and von Neumann algebras. Commun. Math. Phys. **112**, 691–719 (1987)
9. Connes, A., Størmer, E.: Entropy of automorphisms of II_1-von Neumann algebras. Acta. Math. **134**, 289–306 (1975)
10. Denker, M., Grillenberger, C., Sigmund, K.: Ergodic theory on compact spaces. Lecture Notes in Math, Vol. 527. Berlin: Springer-Verlag, 1976
11. Elliott, G.A.: On the K-theory of the C^*-algebra generated by a projective representation of a torsion-free discrete abelian group. In: *Operator Algebras and Group Representations*, Vol. I. Boston: Pitman, 1984, pp. 159–164
12. Elliott, G.A.: The diffeomorphism group of the irrational rotation C^*-algebra. C. R. Math. Rep. Acad. Sci. Canada **8**, 329–334 (1986)
13. Hudetz, T.: Quantum topological entropy: First steps of a "pedestrian" approach. In: *Quantum Probability & Related Topics*, River Edge, NJ: World Scientific, 1993, pp. 237–261
14. Hudetz, T.: Topological entropy for appropriately approximated C^*-algebras. J. Math. Phys. **35**, 4303–4333 (1994)
15. Katznelson, Y.: *An Introduction to Harmonic Analysis*, Second Edition. New York: Dover Publications, 1976
16. Klimek, S., Leśniewski, A.: Quantized chaotic dynamics and non-commutative KS entropy. Ann. Physics **248**, 173–198 (1996)
17. Kolmogorov, A.N., Tihomirov, V.M.: ε-entropy and ε-capacity of sets in functional analysis. Amer. Math. Soc. Trans. (2) **17**, 277–364 (1961)
18. Makarov, B.M., Goluzina, M.G., Lodkin, A.A., Podkorytov, A.N.: Selected problems in real analysis. Translations of Mathematical Monographs, Vol. 107. Providence: AMS, 1992
19. Narnhofer, H.: Ergodic properties of automorphisms on the rotation algebra. Rep. Math. Phys. **39**, 387–406 (1997)
20. Narnhofer, H., Thirring, W.: C^*-dynamical systems that are asymptotically highly anticommutative. Lett. Math. Phys. **35**, 145–154 (1995)
21. Olesen, D., Pedersen, G.K., Takesaki, M.: Ergodic actions of compact Abelian groups. J. Operator Theory **3**, 237–269 (1980)
22. Pesin, Ya. B.: *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*. Chicago: The University of Chicago Press, 1997
23. Peters, J.: Entropy on discrete abelian groups. Adv. Math. **33**, 1–13 (1979)
24. Rieffel, M.A.: Noncommutative tori – A case study of noncommutative differentiable manifolds. Contemporary Math. **105**, 191–211 (1990)
25. Rieffel, M.A.: Metrics on states from actions of compact groups. Doc. Math. **3**, 215–229 (1998)
26. Rieffel, M.A.: Metrics on state spaces. Doc. Math. **4**, 559–600 (1999)
27. Rieffel, M.A.: Gromov-Hausdorff distance for quantum metric spaces. arXiv:math.OA/0011063 v2 (2001)
28. Russo, B., Dye, H.A.: A note on unitary operators in C^*-algebras. Duke Math. J. **33**, 413–416 (1966)
29. Sauvageot, J.-L., Thouvenot, P.: Une nouvelle définition de l'entropie dynamique des systèmes non-commutatifs. Commun. Math. Phys. **145**, 411–423 (1992)
30. Slawny, J.: On factor representations and the C^*-algebra of canonical commutation relations. Commun. Math. Phys. **24**, 151–170 (1972)
31. Størmer, E.: A survey of noncommutative dynamical entropy. In: *Classification of Nuclear C^*-algebras. Entropy in Operator Algebras*. Berlin: Springer, 2002, pp. 147–198
32. Thomsen, K.: Topological entropy for endomorphisms of local C^*-algebras. Commun. Math. Phys. **164**, 181–193 (1994)
33. Voiculescu, D.V.: Dynamical approximation entropies and topological entropy in operator algebras. Commun. Math. Phys. **170**, 249–281 (1995)

34. Watatani, Y.: Toral automorphisms on irrational rotation algebras. Math. Japon. **26**, 479–484 (1981)
35. Weaver, N.: Lipschitz algebras and derivations of von Neumann algebras. J. Funct. Anal. **139**, 261–300 (1996)
36. Weaver, N.: α-Lipschitz algebras on the noncommutative torus. J. Operator Theory **39**, 123–138 (1998)
37. Weaver, N.: *Lipschitz Algebras*. River Edge, NJ: World Scientific, 1999

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