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Integrable Evolution Equations on the N-Dimensional Sphere

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Abstract: The problem of a classification of integrable evolution equations on the N-dimensional sphere is considered. We modify the main notions of the symmetry approach such as the formal symmetry and the canonical series of conserved densities to the case of such equations. Using these theoretical results, we solve several special classification problems. The main result is a complete classification of integrable isotropic evolution equations of third order on the sphere. An important class of anisotropic equations is also considered.

1. Introduction

In the paper [1] the following equation

$$U_t = \Big(U_{xx} + \frac{3}{2} < U_x,\ U_x > U\Big)_x + \frac{3}{2} < U,\ R(U) > U_x, \qquad < U,\ U >= 1 \quad (1)$$

was considered. Here $U = (U^1, \dots, U^{N+1})$ is an unknown vector, R is a constant symmetric matrix. Here and in the sequel $< \cdot, \cdot >$ stands for the standard scalar product in Euclidean space V. Without loss of generality it can be assumed that $R = diag(r_1, \dots, r_{N+1})$.

It was shown that this equation is integrable by the inverse scattering method for any N and R. If $N = 2$, then (1) is a higher symmetry of the Landau-Lifshitz equation. Besides, Eq. (1) defines an infinitesimal symmetry for the well-known Noemann system [2]

$$U_{xx} = -\Big(< U_x,\ U_x > + < U,\ R(U) > \Big) U + R(U), \qquad < U,\ U >= 1, \quad (2)$$

describing the dynamics of a particle on the sphere $\mathbb{S}^N$ under the influence of field with the quadratic potential $\mathcal{U} = \frac{1}{2} < U,\ R(U) >$. More precisely, if we eliminate the derivatives U_{xx} and U_{xxx} from (1) using Eq. (2), then the reduced system

$$U_t = \frac{1}{2}\Big(< U_x,\ U_x > + < U,\ R(U) > \Big) U_x - < U_x,\ R(U) > \ U + R(U_x)$$

is a symmetry for (2).

In this paper we are dealing with a problem of classification of integrable vector evolution equations similar to Eq. (1). Our main tool is the symmetry approach [20, 21, 23, 24] based on the observation that all integrable evolution equations with one spatial variable possess local higher symmetries (or, the same, higher commuting flows). We are developing a specific componentless version of this approach suitable for vector equations.

In Sect. 3 the main concepts of the symmetry approach are generalized to the case of equations of the form

$$U_t = f_n\, U_n + f_{n-1}\, U_{n-1} + \cdots + f_1\, U_1 + f_0\, U, \qquad U_i = \frac{\partial^i U}{\partial x^i}, \tag{3}$$

where $U(x,t)$ is an N-component vector, and f_i are scalar functions of variables

$$u_{[i,j]} =< U_i,\ U_j >, \qquad i \leqslant j, \tag{4}$$

$0 \leqslant i, j \leqslant n$.

It is clear that any Eq. (3) is invariant with respect to an arbitrary constant orthogonal transformation of the vector U. Equations of the form (3) are called **isotropic**.

The vector modified Korteweg-de Vries equation

$$U_t = U_{xxx} + < U,\ U > \ U_x, \tag{5}$$

gives us an example of an integrable isotropic equation. It is well known that this equation is integrable by the inverse scattering method for any N.

Different examples of integrable vector equations can be found in the papers [4, 5]. Some of them are closely related to such algebraic and geometrical objects as Jordan triple systems and symmetric spaces [3, 8, 9, 11].

In this paper we shall consider Eq. (3) that are integrable for arbitrary dimension N of the vector U. In addition, we assume that the coefficients f_i do not depend on N. By virtue of the arbitrariness of N, variables (4) will be regarded as **independent**. The functional independence of $u_{[i,j]},\ i \leqslant j$ is a crucial requirement in all our considerations. If N is fixed, we cannot suppose that. For instance, if $N = 3$, then the determinant of matrix A with entries $a_{ij} = u_{[i,j]},\ i, j = 1, 2, 3, 4$ identically equals to zero.

The signature of the scalar product is **inessential** for us. Furthermore, the assumptions that the space V is finite-dimensional and the field of constants is $\mathbb{R}$ are also unimportant. For instance, U could be a function of t, x, and y and the scalar product be

$$< U,\ V >= \int_{-\infty}^{\infty} U(t,x,y)\, V(t,x,y)\, dy.$$

Thus our formulas and statements are valid also for this particular sort of $1+2$-dimensional non-local equations.

A more restrictive class than Eq. (3) on $\mathbb{R}^N$ consists of equations

$$U_t = f_n\, U_n + f_{n-1}\, U_{n-1} + \cdots + f_1\, U_1 + f_0\, U, \qquad < U,\ U >= 1, \tag{6}$$

$U = (U^1, \ldots, U^{N+1})$, defined on the sphere $\mathbb{S}^N$. If $R = 0$, then (1) belongs to this class.

It is easy to see that the stereographic projection takes any Eq. (6) on $\mathbb{S}^N$ to some isotropic equation on $\mathbb{R}^N$. The converse statement is not true because, in general, the preimage of Eq. (3) on $\mathbb{R}^N$ under the stereographic projection is non-isotropic on $\mathbb{S}^N$.

In Sect. 2 we present a complete list of integrable isotropic equations

$$U_t = U_{xxx} + f_2\, U_{xx} + f_1\, U_x + f_0\, U, \tag{7}$$

on the sphere $\mathbb{S}^N$. A sketch of a proof of the corresponding classification theorem is contained in Sect. 4.

In order to prove that all equations from the list are really integrable, we find an auto-Bäcklund transformation, involving a "spectral" parameter, for each of the equations (see Sect. 5).

Equation (1) with non-trivial R is not isotropic. Nevertheless, equations of such type can be also treated in the framework of our componentless approach. To do this we assume that the coefficients f_i of Eq. (3), besides (4), depend on additional variables

$$v_{[i,j]} =< U_i,\ R(U_j) >, \qquad i \leqslant j, \tag{8}$$

where $0 \leqslant i, j \leqslant n$, and R is a constant symmetric matrix. We call such equations **anisotropic.** In the paper we consider anisotropic equations that are integrable for arbitrary symmetric matrix R. For this reason we regard the union of all scalar products (4) and (8) as a set of independent variables.

Section 7 contains new nontrivial examples of integrable anisotropic evolution equations of third order on the N-dimensional sphere.

2. Classification Results for the Isotropic Case

In this section we formulate some classification statements concerning integrable evolution equations of third order on the N-dimensional sphere. This classification problem is much simpler than the similar problem on $\mathbb{R}^N$. Indeed, the set of independent variables (4) on $\mathbb{S}^N$ is reduced because of the constraint $u_{[0,0]} = 1$. Differentiating this identity, we can express all variables of the form $u_{[0,k]},\ k \geqslant 1$ in terms of the remaining independent scalar products

$$u_{[i,j]} =< U_i,\ U_j >, \qquad 1 \leqslant i \leqslant j. \tag{9}$$

For example, $u_{[0,1]} = 0,\ u_{[0,2]} = -u_{[1,1]}$, and so on. Therefore the coefficients of Eq. (7) on $\mathbb{S}^N$ *a priori* depend on only three independent variables $u_{[1,1]}$, $u_{[1,2]}$, and $u_{[2,2]}$ whereas in the case of $\mathbb{R}^N$ they are functions of six variables $u_{[0,0]}$, $u_{[0,1]}$, $u_{[1,1]}$, $u_{[0,2]}$, $u_{[1,2]}$, and $u_{[2,2]}$.

Theorem 1. *Suppose that equation*

$$U_t = U_{xxx} + f_2\, U_{xx} + f_1\, U_x + f_0\, U, \qquad f_i = f_i(u_{[1,1]}, u_{[1,2]}, u_{[2,2]}) \tag{10}$$

on the sphere $< U,\ U >= 1$ *possesses an infinite series of commuting flows of the form*

$$U_{\tau_k} = g_k\, U_k + g_{k-1}\, U_{k-1} + \cdots + g_1\, U_x + g_0\, U, \qquad k \to \infty,$$

whose coefficients g_i *depend on variables (9); then this equation belongs to the following list:*

$$U_t = U_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,U_{xx} + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2\,(1 + a\,u_{[1,1]})} + c\right) U_x, \tag{11}$$

$$U_t = U_{xxx} + \frac{3}{2}\left(\frac{a^2\,u_{[1,2]}^2}{1 + a\,u_{[1,1]}} - a\,(u_{[2,2]} - u_{[1,1]}^2) + u_{[1,1]} + c\right) U_x + 3\,u_{[1,2]}\,U, \tag{12}$$

$$U_t = U_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,U_{xx} + \left(\frac{3}{2}\frac{u_{[2,2]}}{u_{[1,1]}} + c\right) U_x, \tag{13}$$

$$\begin{aligned} U_t \;&=\; U_{xxx} - 3\,\frac{(q+1)\;u_{[1,2]}}{2\,q\,u_{[1,1]}}\,U_{xx} + 3\,\frac{(q-1)\;u_{[1,2]}}{2\,q}\,U \\ &+ \frac{3}{2}\left(\frac{(q+1)\,u_{[2,2]}}{u_{[1,1]}} - \frac{(q+1)\,a\,{u_{[1,2]}}^2}{q^2 u_{[1,1]}} + u_{[1,1]}\,(1-q) + c\right) U_x, \end{aligned} \tag{14}$$

where a and c are arbitrary constants, $q = \varepsilon\sqrt{1 + a\,u_{[1,1]}}$, $\quad \varepsilon^2 = 1$.

Theorem 2. *If Eq. (10) on $\mathbb{S}^N$ possesses an infinite series of conservation laws $(\rho_k)_t = (\sigma_k)_x,\quad k \to \infty$, where ρ_k and σ_k are functions of variables (9), then this equation belongs to the same list (11)–(14).*

Remark 1. The constant c can be removed by the Galilean transformation and below we will omit this constant as trivial. The constant a can be reduced to $a = 0$ or to $a = \pm 1$ by an appropriate scaling of x and t. Thus the list contains rather many non-equivalent equations over $\mathbb{R}$. In particular, Eq. (1) with $R = 0$ coincides with (12), where $a = 0$. Equation (14) with $a = 0$ and $\varepsilon = -1$ reads as

$$U_t = U_{xxx} + (3\,u_{[1,1]} + c)\,U_x + 3\,u_{[1,2]}\,U. \tag{15}$$

If $a = 0$ and $\varepsilon = 1$ then Eq. (14) becomes

$$U_t = U_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,U_{xx} + \left(3\,\frac{u_{[2,2]}}{u_{[1,1]}} + c\right) U_x. \tag{16}$$

Remark 2. In the process of the proof of Theorems 1 and 2 (see Sect. 4) it has been checked that all Eqs. (11)–(14) have non-trivial local conservation laws of orders 1,2,3 and 4. Moreover, we have verified that each of these equations possesses a higher symmetry of fifth order. For example, the fifth order symmetry of Eq. (15) has the following form:

$$\begin{aligned} U_\tau &= U_5 + 5\,u_{[1,1]}\,U_3 + 15\,u_{[1,2]}\,U_2 + 5\,\left(3\,{u_{[1,1]}}^2 + 2\,u_{[2,2]} + 3\,u_{[1,3]}\right)\,U_1 \\ &\quad + \;5\big(6\,u_{[1,2]}u_{[1,1]} + 2\,u_{[2,3]} + u_{[1,4]}\big)\;U. \end{aligned}$$

We are sure that all our equations have infinite series of symmetries and conserved densities, but of course it should be rigorously proved. From our viewpoint the existence of higher symmetries and/or conservation laws is a very efficient way to list all integrable cases. But there is a little help from symmetries and conservation laws for integrating of a given equation. That is why we find in Sect. 5 auto-Bäcklund transformations for all equations of the list.

Remark 3. The coincidence of the lists from Theorems 1 and 2 shows that the so-called Burgers type equations of the form (10) do not exist on $\mathbb{S}^N$. Recall that the Burgers type equations (C-integrable equations in the terminology by F. Calogero [22]) possess higher symmetries but have no higher conservation laws.

Remark 4. Equation (13) on $\mathbb{R}^N$ has been found the papers [4, 6, 11]. This equation is related to vector triple Jordan systems. It is a vector generalization of the well-known Swartz-KdV equation

$$v_t = v_{xxx} - \frac{3}{2}\frac{v_{xx}^2}{v_x}.$$

Remark 5. In the case of one-dimensional sphere Eqs. (11), (12) with $a = 0$ can be reduced to the potential mKdV equation

$$v_t = v_{xxx} + v_x^3$$

by the stereographic projection and some point-wise transformations. Equations (11) and (12) with $a = -1$ are reduced to

$$v_t = v_{xxx} - \frac{1}{8}Q''\, v_x + \frac{3}{32}\frac{(Q - 4v_x^2)_x^2}{v_x\,(Q - 4v_x^2)}, \tag{17}$$

where $Q(v) = (v^2 + 1)^2$. Equation (17) with $Q(v)$ being an arbitrary polynomial of fourth degree is known as the Calogero-Degasperis equation (see [13]). Our particular case corresponds to a trigonometric degeneration of the elliptic curve implicitly involved in (17).

Equation (14) is reduced to the following integrable equation:

$$v_t = v_{xxx} - \frac{6\,a\,v_x\,v_{xx}^2}{1 + 4\,a{v_x}^2} + 8v_x^3.$$

Remark 6. It would be interesting to find a geometrical interpretation of Eqs. (11)–(14) along the lines of [10]. Here we only note that Eqs. (11) and (13) admit the following constraint: $< U_x,\, U_x >= 1$. This means that the t-deformation of an initial curve $U(x)$ on the sphere by virtue of these equations preserves the length.

3. Canonical Densities

In the papers [12, 20] the concept of formal symmetry for one-component evolution equations of the form

$$u_t = F(u,\, u_1,\, u_2, \ldots, u_n), \qquad u_i = \frac{\partial^i u}{\partial x^i} \tag{18}$$

has been introduced. By definition, the formal symmetry (or the formal recursion operator) is a series of the form

$$L = a_1\, D_x + a_0\, + a_{-1}\, D_x^{-1} + a_{-2}\, D_x^{-2} + \cdots, \qquad a_i = a_i(u,\, u_1, \ldots, u_{n_i}), \tag{19}$$

satisfying the following operator relation:

$$L_t = [F_*,\, L], \qquad F_* = \sum_0^n \frac{\partial^i F}{\partial u_i}\, D_x^i. \tag{20}$$

Here D_x is the total derivative operator with respect to x:

$$D_x = \sum_0^\infty u_{i+1}\,\frac{\partial^i}{\partial u_i},$$

F_* is the Frechét derivative of the right-hand side of Eq. (18). It was shown in [12, 19] that the formal symmetry exists for any Eq. (18) possessing an infinite series of local higher symmetries or conservation laws.

The residues $\rho_i = \operatorname{res} L^i$ are local conserved densities for Eq. (18). They are called the **canonical densities**. In [19] it was proved that if Eq. (18) has an infinite series of conservation laws, then the canonical densities ρ_i are trivial for all even i.

There is an alternative way [14, 15] to compute the canonical densities. It is based on identities for the logarithmic derivative of a formal eigenfunction for the operator $\frac{\partial}{\partial t} - F_*$. This algorithm deals with commutative Laurent series in contrast with non-commutative series similar to (19).

In this section we define an infinite sequence of necessary integrability conditions for Eq. (3). These conditions

$$D_t\,\rho_i = D_x\,\theta_i, \qquad i = 0, 1, 2, \ldots \tag{21}$$

have the form of conservation laws, where $\rho_i,\ \theta_i$ are some functions of variables (4), which can be recursively found in terms of the coefficients f_i of Eq. (3).

These conditions are very close to the canonical conservation laws from the papers [12, 20, 21, 7, 14, 15, 23] but do not coincide with them. Our conditions are more convenient for classification problems related to Eq. (3) since they are much simpler than the standard canonical densities for multi-component systems.

Theorem 3. **(i)** *If Eq. (3) possesses an infinite series of commuting flows of the form*

$$U_\tau = g_m\,U_m + g_{m-1}\,U_{m-1} + \cdots + g_1\,U_1 + g_0\,U, \tag{22}$$

then there exists a formal series

$$L = a_1\,D_x + a_0\, + a_{-1}\,D_x^{-1} + a_{-2}\,D_x^{-2} + \cdots\,, \tag{23}$$

satisfying the operator relation

$$L_t = [A,\ L], \qquad A = \sum_0^n f_i\,D_x^i. \tag{24}$$

Here g_i, a_i are some functions of variables (4), f_i are the coefficients of Eq. (3).

(ii) *The following functions*

$$\rho_{-1} = \frac{1}{a_1}, \qquad \rho_0 = \frac{a_0}{a_1}, \qquad \rho_i = \operatorname{res} L^i, \qquad i \in \mathbb{N} \tag{25}$$

are conserved densities for Eq. (3).

(iii) *If Eq. (3) possesses an infinite series of conserved densities depending on variables (4), then there exists a series L satisfying (24), and a series S of the form*

$$S = s_1\,D_x + s_0\, + s_{-1}\,D_x^{-1} + s_{-2}\,D_x^{-2} + \cdots \tag{26}$$

such that

$$S_t + A^T S + S A = 0, \qquad S^T = -S, \tag{27}$$

where the superscript T stands for formal conjugation.

(iiii) *Under the conditions of item (iii) all densities (25) corresponding to even i are trivial i.e., $\rho_{2k} = D_x(\sigma_k)$ for some functions σ_k of variables (4).*

Comments. In [21, 7] the notion of the formal symmetry was generalized to the case of systems of evolution equations. In these papers the formal symmetry is a series with **matrix** coefficients satisfying (20). In our paper both the operators A and L are scalar objects and A does not coincide with the Frechét derivative F_* of the right-hand side of the system.

In [7–9] Sergey Svinolupov has described integrable cases for several classes of N-component polynomial systems using the existence of the formal symmetry as a necessary condition of integrability. In these papers he imposed very serious restrictions on the structure of the right-hand side and only a collection of unknown constants was to be determined. Any attempts to solve more general classification problems for N-component systems with the help of the standard component-wise approach lead to computational difficulties which cannot be overcome.

The use of the scalar series L defined by formula (24) instead the formal symmetry makes possible a complete classification of isotropic integrable systems of the form $U_t = U_3 + f_2 U_2 + f_1 U_1 + f_0 U$ on $\mathbb{R}^N$ without any assumptions about the structure of the coefficients f_i. We are planning to publish a separate paper devoted to this problem.

Reduced proof of Theorem 3. In many respects the proof is analogous to one used in [20] for the scalar case.

(i) Let us rewrite Eq. (3) and its higher symmetry (22) in the form

$$U_t = A(U), \quad U_\tau = B(U), \qquad B = \sum_0^m g_i\, D_x^i. \tag{28}$$

The compatibility of Eqs. (28) implies the following operator identity:

$$B_t - [A,\ B] = A_\tau.$$

For m large enough we can ignore the right-hand side of this relation. In other words, the operator B approximately satisfies (24). But then the first order series $L_m = B^{1/m}$ also approximately satisfies (24). A rigorous assembling of approximate solutions L_m into one exact solution L can be done in the same way as in [20].

(ii) The statement follows from the known Adler's formula (see [20]).

(iii) Let us represent the variational derivative

$$\frac{\delta\rho}{\delta U} = \sum_{i\leqslant j}(-D_x)^i\Big(\frac{\partial\rho}{\partial u_{[i,j]}}\, U_j\Big) + (-D_x)^j\Big(\frac{\partial\rho}{\partial u_{[i,j]}}\, U_i\Big) \tag{29}$$

of arbitrary conserved density ρ of order m in the form

$$\frac{\delta\rho}{\delta U} = S_m(U),$$

where S_m is a scalar differential operator of order $2m$, whose coefficients depend on variables (4). As it is well known [28], this variational derivative satisfies the following equation:

$$\left(\frac{\delta\rho}{\delta U}\right)_t = -\Big(A(U)\Big)_*^T \frac{\delta\rho}{\delta U},$$

where the subscript $*$ means the Frechét derivative and the superscript T stands for formal conjugation. It follows from this equation that the operator S_m approximately satisfies (27). To conclude the proof of Theorem 3 it suffices to repeat the corresponding reasoning from [20].

Remark 7. In the same way, this theorem can be proved for the isotropic equations on the sphere and for the anisotropic equations.

4. Systems of Third Order. Isotropic Case

Finding from (24) coefficients a_1, a_0, a_{-1} of the series L, it is easy to verify that for Eq. (10) the densities ρ_0 and ρ_1 are expressed in terms of the coefficients f_i by the following formulae:

$$\rho_0 = -\frac{1}{3}\, f_2, \tag{30}$$

$$\rho_1 = \frac{1}{9}\, f_2^2 - \frac{1}{3}\, f_1 + \frac{1}{3}\, D_x\, f_2. \tag{31}$$

The corresponding functions θ_i can be found from (21). The fact that the left-hand sides of (21) are total x-derivatives imposes rigid restrictions (see below) to the coefficients f_i of (7).

Expressions for ρ_i, $i > 1$ involve the coefficients f_k and the functions θ_j with $j \leqslant i-2$. Using a technique developed in the papers [14, 15], one can obtain the following recursion formula:

$$\begin{aligned}\rho_{n+2} = {} & \frac{1}{3}\Big[\theta_n - f_0\,\delta_{n,0} - 2\, f_2\,\rho_{n+1} - f_2\, D_x\rho_n - f_1\,\rho_n\Big] \\ & -\frac{1}{3}\Big[f_2 \sum_{s=0}^{n} \rho_s\,\rho_{n-s} + \sum_{0\leqslant s+k\leqslant n} \rho_s\,\rho_k\,\rho_{n-s-k} + 3\sum_{s=0}^{n+1}\rho_s\,\rho_{n-s+1}\Big] \\ & - D_x\Big[\rho_{n+1} + \frac{1}{2}\sum_{s=0}^{n}\rho_s\,\rho_{n-s} + \frac{1}{3} D_x\rho_n\Big], \quad n \geqslant 0,\end{aligned} \tag{32}$$

where $\delta_{i,j}$ is the Kronecker delta and $\rho_0,\ \rho_1$ are defined by (30), (31).

According to this formula,

$$\rho_2 = -\frac{1}{3}\, f_0 + \frac{1}{3}\,\theta_0 - \frac{2}{81}\, f_2^3 + \frac{1}{9}\, f_1\, f_2 - D_x\left(\frac{1}{9}\, f_2^2 + \frac{2}{9}\, D_x\, f_2 - \frac{1}{3}\, f_1\right)$$

and so on.

In order to show how to manipulate with conditions (21) we consider Eqs. (10) on $\mathbb{S}^N$ and present a proof of Theorem 2. To perform the corresponding computations some special Maple routines were written.

The equation of the sphere $<U,\ U>= 1$ gives rise to the relation $<U,\ U_t>= 0$. It follows from this that any Eq. (10) on $\mathbb{S}^N$ has the following form:

$$U_t = U_3 + f_2\, U_2 + f_1\, U_1 + (f_2\, u_{[1,1]} + 3\, u_{[1,2]})\, U_0.$$

Thus for Eqs. (10) on $\mathbb{S}^N$, we have to replace f_0 by $f_2\, u_{[1,1]} + 3\, u_{[1,2]}$ in conditions (32).

Lemma 1. *Suppose Eq. (10) on $\mathbb{S}^N$ possesses an infinite series of conserved densities depending on variables (4); then the equation has the form*

$$\begin{aligned} U_t \; &= U_3 + A\, u_{[1,2]}\, U_2 + (B\, u_{[2,2]} + C\, u_{[1,2]}^2 + D\, u_{[1,2]} + E)\, U_1 \\ &\quad + (u_{[1,1]}\, A + 3)\, u_{[1,2]}\, U_0, \end{aligned} \tag{33}$$

where A, B, C, D, E are some functions of variable $u_{[1,1]}$.

Proof. It follows from (30) and the item (iiii) of Theorem 3 that $f_2 = D_x(\sigma_0)$ for some function σ_0. Since f_2 does not depend on the third order derivative, we see that σ_0 may depend on the variable $u_{[1,1]}$ only. Thus $f_2 = 2\sigma_0'\, u_{[1,2]}$ and the function A from (33) coincides with $2\sigma_0'$.

To specify the form of the coefficient f_1, let us consider the condition $D_t(\rho_1) = D_x(\theta_1)$, where ρ_1 is given by (31). Using the main property of the Euler operator, we can eliminate the unknown function θ_1. It is well known [28] that $\frac{\delta}{\delta U}\, D_x(\theta) = 0$ for any function θ. Therefore we have

$$\frac{\delta}{\delta U}\Big((\rho_1)_t\Big) = 0. \tag{34}$$

It is easily verified that

$$\frac{\delta}{\delta U}\Big((\rho_1)_t\Big) = 2\Big(D_x\frac{\partial f_1}{\partial u_{[2,2]}} - \frac{2}{3}\frac{\partial f_1}{\partial u_{[2,2]}} A u_{[1,2]}\Big)\, U_6 + \cdots, \tag{35}$$

where the dots denote the terms which do not contain U_6. Equating to zero the coefficient of U_6 in (35) we get

$$\begin{aligned} &2\frac{\partial^2 f_1}{\partial u_{[2,2]}^2}\, u_{[2,3]} + \frac{\partial^2 f_1}{\partial u_{[2,2]}\partial u_{[1,2]}}\, (u_{[1,3]} + u_{[2,2]}) \\ &+ 2\frac{\partial^2 f_1}{\partial u_{[2,2]}\partial u_{[1,1]}}\, u_{[1,2]} - \frac{2}{3}\frac{\partial f_1}{\partial u_{[2,2]}} A u_{[1,2]} = 0. \end{aligned} \tag{36}$$

This obviously implies

$$\frac{\partial^2 f_1}{\partial u_{[2,2]}^2} = \frac{\partial^2 f_1}{\partial u_{[2,2]}\partial u_{[1,2]}} = 0.$$

Taking these relations into account, let us equate the coefficients at $u_{[3,3]}u_{[1,4]}\, U_1$ and $u_{[2,4]}u_{[1,4]}\, U_1$ in (35) to zero. It is not hard to check that as the result we obtain

$$\frac{\partial^3 f_1}{\partial u_{[1,2]}^3}\Big(u_{[1,1]}\frac{\partial f_1}{\partial u_{[2,2]}} - 15\Big) = 0$$

and

$$\frac{\partial^3 f_1}{\partial u_{[1,2]}^3}\Big(2u_{[1,1]}\frac{\partial f_1}{\partial u_{[2,2]}} - 39\Big) = 0.$$

It follows from these two identities that $\dfrac{\partial^3 f_1}{\partial u_{[1,2]}^3} = 0$. This concludes the proof of Lemma 1.

Remark 8. The statement of Lemma 1 is a generalization of the fact (see [23]) that for integrable scalar equations of the form

$$u_t = u_3 + F(u, u_1, u_2)$$

the function F is a second degree polynomial in u_2.

It follows from (36) that

$$3B' = A\,B. \tag{37}$$

One more important relation can be derived from

$$\frac{\delta}{\delta U}\,(\rho_2) = 0.$$

Vanishing of the coefficient of the highest derivative U_4 in this condition implies

$$A^2 + 2B\,A' u_{[1,1]} + 2A\,B - 3A' = 0. \tag{38}$$

At last, the conditions obtained from the coefficients at $u_{[1,6]} u_{[1,2]}\,U_1$ and $u_{[1,5]}\,U_3$ in (34) implies that

$$B''\,B^2 - 2B'^3\,u_{[1,1]} - 4B'^2\,B = 0. \tag{39}$$

It is easy to find from (37)–(39) that

Case 1) $\quad B = \mu,$

or Case 2) $\quad B = \dfrac{\lambda}{u_{[1,1]}}, \quad \lambda \neq 0,$

or Case 3) $\quad B^2\,u_{[1,1]} - 3B = \nu, \quad \nu \neq 0,$

where μ, λ, ν are constants.

Consider, for example, case 2. Relation (37) leads to $A = -\frac{3}{u_{[1,1]}}$. Equating to zero the coefficient at $u_{[1,5]}\,U_3$ in (34), we obtain

$$(2\lambda - 3)(C\,u_{[1,1]}^2 + \lambda - 3) = 0. \tag{40}$$

The coefficient at $u_{1,6]} u_{[1,2]}\,U_1$ gives us

$$3C'\,u_{[1,1]} - 2u_{[1,1]}^2 C^2 + 2(6 - \lambda)C = 0. \tag{41}$$

Moreover, one can derive from (34) that $D = 0$ and E is a constant. Thus if $\lambda \neq \frac{3}{2}$, we have the following equation:

$$U_t = U_3 - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,U_2 + \left(\lambda\,\frac{u_{[2,2]}}{u_{[1,1]}} + (3 - \lambda)\,\frac{u_{[1,2]}^2}{u_{[1,1]}^2} + c\right) U_1. \tag{42}$$

For any constant λ, this equation satisfies conditions (21) with $i = 0, 1, 2, 3, 4$. It can be easily checked that the conditions with $i = 1$ and $i = 3$ yield non-trivial conserved densities

$$\rho_1 = \frac{u_{[1,1]}\,u_{[2,2]} - u_{[1,2]}^2}{u_{[1,1]}^2}$$

and

$$\rho_3 = \frac{u_{[3,3]}}{u_{[1,1]}} - \frac{(u_{[1,3]} - u_{[2,2]})^2}{u_{[1,1]}^2} - \frac{u_{[2,2]}^2}{u_{[1,1]}^2} + \frac{(9-\lambda)}{2}\frac{(u_{[1,1]}\,u_{[2,2]} - u_{[1,2]}^2)^2}{u_{[1,1]}^4}$$

of second and third orders. It follows from (21) with $i = 5$ that $\lambda = 3$ and we obtain Eq. (16). Note that Eq. (42) with arbitrary λ provides us an example of a non-integrable equation having a local higher conservation law of third order.

In the case $\lambda = \frac{3}{2}$, the general solution

$$C = \frac{3}{2u_{[1,1]}^2\,(1 + a\,u_{[1,1]})}$$

of the Bernoulli equation (41) immediately gives rise to Eq. (11). The particular solution $C = 0$ corresponds to Eq. (13).

Case 1 can be subdivided into two subcases: $\mu = 0$ and $\mu \neq 0$. For both subcases explicit expressions for the coefficients of (33) can be easily obtained. But to specify the values of some constants in these formulae it is needed to use conditions (21) up to $i = 6$. As the result, we obtain Eqs. (15) and (12).

In Case 3 we have

$$B = \frac{3}{2}\frac{q+1}{u_{[1,1]}}, \qquad A = -\frac{3}{2}\frac{q+1}{q\,u_{[1,1]}}, \qquad q = \varepsilon\sqrt{1 + a\,u_{[1,1]}}, \qquad \varepsilon^2 = 1,$$

where $a = 4\nu/9$. The remaining coefficients are easily derived from condition (21) with $i = 3$. As the result, we obtain Eq. (14).

5. Bäcklund Transformations for Equations of the List

We have to prove somehow the integrability of all equations from the list (11)–(14) obtained with the help of the necessary integrability conditions (21). The usual way for that is to find Lax representations or Miura transformations between "new" equations from the list and equations known to be integrable. But we choose another possibility.

In this section we present first order auto-Bäcklund transformations for all equations from the list. Such a transformation involving an arbitrary parameter allows us to build up both multi-solitonic and finite-gap solutions even if the Lax representation is not known (see [18]). That's why the existence of an auto-Bäcklund transformation with additional "spectral" parameter λ is a convincing evidence of integrability.

For the scalar evolution equations, the auto-Bäcklund transformation of first order is a relation between two solutions u and v of the same equation and their derivatives u_x and v_x. Writing this constraint as $u_x = \phi(u, v, v_x)$, we can express all derivatives of u in terms of $u, v, v_x, \ldots, v_i, \ldots$. The last variables are regarded as independent.

In the vector case, the independent variables are vectors

$$U,\ V,\ V_1,\ V_2,\ \ldots\ V_i \ldots, \tag{43}$$

and all their scalar products

$$v_{[i,j]} \stackrel{def}{=} < V_i,\ V_j >, \qquad w_i \stackrel{def}{=} < U,\ V_i >, \qquad i, j \geqslant 0. \tag{44}$$

In this paper we consider special vector auto-Bäcklund transformations of the form

$$U_1 = h\, V_1 + f\, U + g\, V, \tag{45}$$

where f, g and h are functions of variables (44) depending on first derivatives at most. Since V lies on the sphere $< V,\ V >= 1$, we assume without loss of generality that the arguments of f, g and h are $w_0 =< U,\ V >,\ \ w_1 =< U,\ V_1 >,\ \ v_{[1,1]} =< V_1,\ V_1 >$.

Since $< U,\ U >= 1$, and $< U,\ U_1 >= 0$, it follows from (45) that

$$h = -\frac{f + w_0\, g}{w_1}. \tag{46}$$

To find an auto-Bäcklund transformation for Eq. (10), we differentiate (45) with respect to t by virtue of (10) and express all vector and scalar variables in terms of independent variables (43) and (44). By definition of Bäcklund transformation, the expression thus obtained must be identically zero. Splitting this expression with respect to the independent variables different from w_0, w_1, $v_{[1,1]}$, we derive an overdetermined system of non-linear PDEs for the functions f and g. If the system has a solution depending on an essential parameter λ, this solution gives us the auto-Bäcklund transformation we are looking for.

We present below the result of our computations.

In the case of Eq. (12) (where we put $c = 0$) the auto-Bäcklund transformation reads as follows:

$$U_1 = -V_1 + \left(w_1 \frac{1 - \lambda\, a\, w_0}{1 + w_0} - w_0\, G\right) U + \left(\frac{w_1\, (1 + \lambda\, a)}{1 + w_0} + G\right) V, \tag{47}$$

where

$$G = \sqrt{\frac{\lambda\, (2 + \lambda\, a - \lambda\, a\, w_0)(1 + a\, v_{[1,1]})}{(1 + w_0)}}.$$

In particular, if $a = 0$, then we have

$$U_1 = -V_1 + \frac{w_1}{1 + w_0}\, \big(U + V\big) + \frac{\lambda}{\sqrt{1 + w_0}}\, \big(V - w_0\, U\big). \tag{48}$$

The vector Schwartz-KdV Eq. (13) admits the auto-Bäcklund transformation of the form

$$U_1 = G\, \Big(w_1 (U + V) - (1 + w_0)\, V_1\Big), \tag{49}$$

where

$$G = \frac{1}{\lambda} \left(1 + \sqrt{\frac{\lambda + 1 + w_0}{1 + w_0}}\right)^2. \tag{50}$$

The auto-Bäcklund transformation for Eq. (11) is defined by (49), where

$$G = \frac{1}{1 + w_0} + \frac{2}{\lambda} + \frac{2}{\lambda} \sqrt{\frac{(\lambda + 1 + w_0)(1 + a v_{[1,1]})}{a v_{[1,1]} (1 + w_0)}}. \tag{51}$$

Note that Eq. (13) is a limit of (11) as $a \to \infty$. The corresponding limit value of (51) gives us (50).

Formula (51) is not valid if $a = 0$. To derive from (51) an auto-Bäcklund transformation for Eq. (11) with $a = 0$, one must put $\lambda = \lambda'/a$ at first. After that the limit of (51) as $a \to 0$ gives rise to

$$G = \frac{\sqrt{v_{[1,1]}} + \lambda\sqrt{1+w_0}}{(1+w_0)\sqrt{v_{[1,1]}}}.$$

Finally, the auto-Bäcklund transformation for Eq. (14) is defined by the following expression:

$$U_1 = F\,(V_1 - w_1\,U) + G\,(V - w_0\,U), \tag{52}$$

where

$$F = \frac{\lambda a\, w_1}{p-1} + R, \qquad R = \sqrt{1 + \lambda^2 a(1 - w_0^2)}, \qquad p = \varepsilon\sqrt{1 + a\, v_{[1,1]}},$$
$$G = \frac{w_0\, w_1\, F}{1 - w_0^2} + w_1\left(2\lambda^2 a + \frac{1}{1 - w_0^2}\right) + \lambda(p-1)R + \frac{\lambda a\, w_1^2\, R}{(p-1)(1-w_0^2)},$$

ε and a are the constants from (14) and λ is the Bäcklund parameter.

Auto-Bäcklund transformations for Eq. (15) and (16) can be obtained from these formulas by setting $a = 0$ and $\varepsilon = -1$ or $\varepsilon = 1$ correspondingly. For Eq. (15) the Bäcklund transformation takes the following form:

$$U_1 = V_1 + \frac{w_1}{1 - w_0}\big(V - U\big) - 2\lambda\,\big(V - w_0\,U\big),$$

and for Eq. (16) the Bäcklund transformation is given by

$$U_1 = \frac{w_1\,(v_{[1,1]} + 2\lambda\, w_1)}{v_{[1,1]}\,(1 - w_0)}\,(V - U) + \left(1 + 2\lambda\,\frac{w_1}{v_{[1,1]}}\right)\,V_1.$$

6. Soliton Solutions

Using the auto-Bäcklund transformations from the previous section, one can find particular solutions of Eqs. (11)–(14). Here we construct soliton type solutions for (12) with $c = 0$.

Let us take for V a constant solution C^1 of (12), where $< C^1, C^1 >= 1$. Taking into account that $v_{[i,j]} = 0,\ w_j = 0$ for $j > 0$, we obtain from (47) that

$$U_x = g(C^1 - w_0\,U), \tag{53}$$

where

$$g = \sqrt{\lambda\,\frac{2 + \lambda\, a - \lambda\, a\, w_0}{1 + w_0}}.$$

It follows from (53) that

$$w_{0,x} = g(1 - w_0^2). \tag{54}$$

Using (53), (54), it can easily be checked that Eq. (12) reduces to $U_t = \lambda g(C^1 - w_0\, U)$. It means that the solution U depends on $x + \lambda t$.

It is clear that $w_0^2 \leqslant 1$. Equation (53) has a trivial solution $w_0^2 = 1,\ U = w_0 C^1$. The expression

$$\frac{U - C^1 w_0}{\sqrt{1 - w_0^2}}$$

is a first integral of (53). Hence, non-trivial solutions of (53) locally can be represented as

$$U = C^1 w_0 + C^2 \sqrt{1 - w_0^2},$$

where the constant vector C^2 satisfies the conditions

$$< C^2, C^2 >= 1, \qquad < C^1,\ C^2 >= 0.$$

The simplest way to solve Eq. (54) is to note that this equation is equivalent to the following equation for the function g:

$$g_x = \lambda - g^2.$$

Analytical properties of the solution $U(x + \lambda t)$ essentially depend on the sign of λ. If $\lambda = -k^2 < 0$, then the solution is periodic:

$$U = C^1 \left(\frac{2(ak^2 - 1)}{ak^2 + \tan^2 \psi} - 1 \right) + 2C^2 \, \frac{\sqrt{ak^2 - 1}}{\cos \psi \, (ak^2 + \tan^2 \psi)}, \tag{55}$$

where $a > k^{-2},\ g = -k \tan \psi,\ \psi = k(x - k^2 t)$.

If $\lambda = k^2 > 0$, there exists a particular case $g^2 = k^2$, where $\lambda a = -1$, $w_0 = \tanh \varphi,\ \varphi = k(x + k^2 t)$ and

$$U = C^1 \tanh \varphi + C^2 \cosh^{-1} \varphi. \tag{56}$$

In the generic case we have two kinds of solutions:

(1) if $g^2 < k^2$, then $a < -k^{-2},\ g = k \tanh \varphi,\ \varphi = k(x + k^2 t)$ and

$$U = C^1 \left(\frac{2(ak^2 + 1)}{ak^2 + \tanh^2 \varphi} - 1 \right) + 2C^2 \, \frac{\sqrt{|a|k^2 - 1}}{\cosh \varphi \, (ak^2 + \tanh^2 \varphi)}. \tag{57}$$

(2) if $g^2 > k^2$, then $a > -k^{-2},\ g = k \coth \varphi,\ \varphi = k(x + k^2 t)$ and

$$U = C^1 \left(\frac{2(ak^2 + 1)}{ak^2 + \coth^2 \varphi} - 1 \right) + 2C^2 \, \frac{\sqrt{ak^2 + 1}}{\sinh \varphi \, (ak^2 + \coth^2 \varphi)}. \tag{58}$$

We see that the form of solitons and periodic waves described by Eq. (12) essentially depend on their propagation velocities. Really, if $a < 0$ then the rapid solitons have the form (57) and the slow solitons are of the form (58). If $a > 0$, then both solitons (58) and periodic waves (55) exist but the latter cannot propagate with small velocities.

All solutions (55)–(58) have only two independent components $U^1 =< U, C^1 >$ and $U^2 =< U, C^2 >$. We present below plots of the initial profiles of U_1 and U_2 for some solutions:

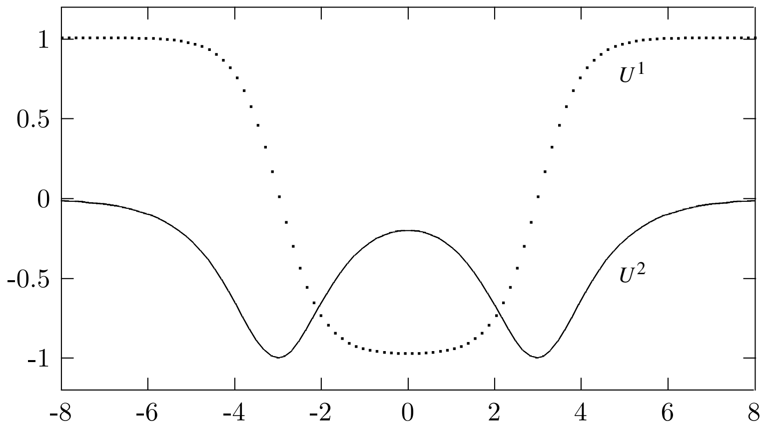

Fig. 1. Soliton solution (57) for $a = -1,\ k^2 = 100/99$

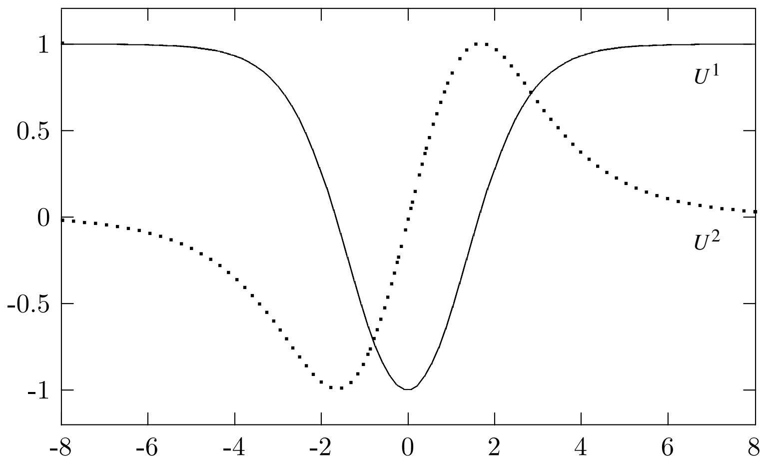

Fig. 2. Soliton solution (58) for $a = -1,\ k = 2/3$

7. Equations on the Sphere. Anisotropic Case

In this section we present a list of anisotropic integrable equations similar to Eq. (1). Equation (1) and its symmetries contain both the scalar products $u_{ij} =< U_i,\ U_j >$ and $v_{ij} =< U_i,\ R(U_j) >$. Since R is an arbitrary symmetric operator, we regard $< U,\ V >$ and $< U,\ R(V) >$ as two independent scalar products on the same vector space. The theory of canonical densities developed in Sect. 3 can be easily generalized to the case of Eqs. (3), whose coefficients f_i depend on variables (4) and (8).

The following statement is an extension of Theorems 1 and 2 to the anisotropic case.

Theorem 4. *Suppose Eq. (10) with*

$$f_i = f_i(u_{[1,1]},\ u_{[1,2]},\ u_{[2,2]},\ v_{[0,0]},\ v_{[0,1]},\ v_{[1,1]})$$

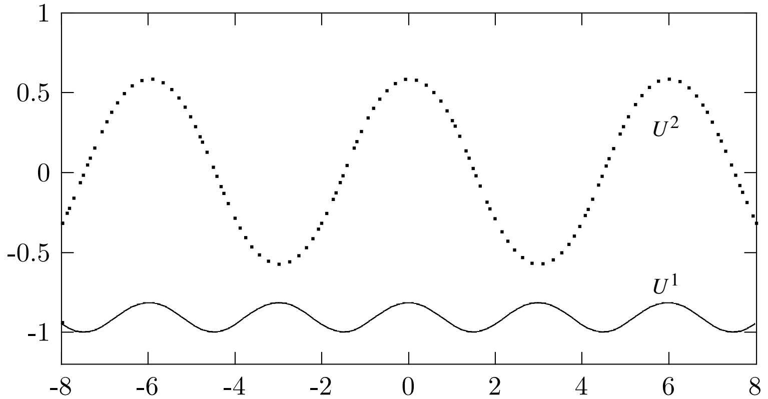

Fig. 3. Periodic solution (55) for $a = 1,\ k = 1.05$

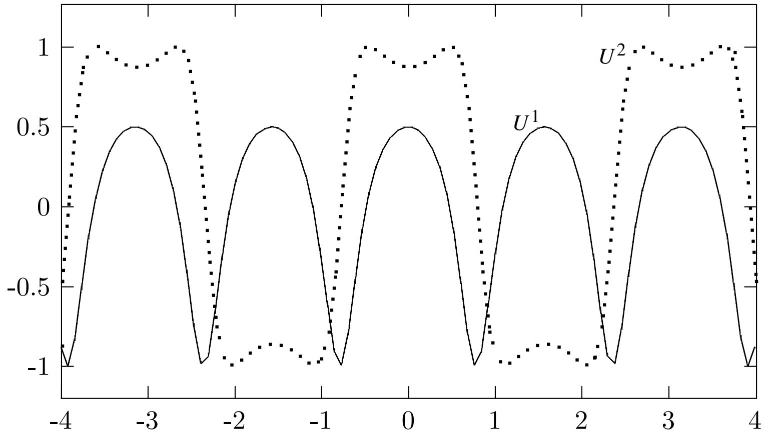

Fig. 4. Periodic solution (55) for $a = 1,\ k = 2$

on the sphere $\mathbb{S}^N$ has an infinite series of commuting flows or conserved densities; then this equation is one of (11)–(14) or belongs to the following list:

$$U_t = U_3 + \left(\frac{3}{2}\, u_{[1,1]} + c\, v_{[0,0]}\right) U_1 + 3\, u_{[1,2]}\, U_0, \tag{59}$$

$$U_t = U_3 - 3\, \frac{u_{[1,2]}}{u_{[1,1]}}\, U_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2} + \frac{c\, v_{[1,1]}}{u_{[1,1]}}\right) U_1, \tag{60}$$

$$U_t = U_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\, U_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2} - \frac{(v_{[0,1]} + u_{[1,2]})^2}{(u_{[1,1]} + v_{[0,0]} + a)\, u_{[1,1]}} + \frac{v_{[1,1]}}{u_{[1,1]}}\right) U_1. \tag{61}$$

Equation (59) coincides with (1). Each of the two remaining equations satisfies conditions (21) with $i \leqslant 9$ and possesses local conserved densities of orders 1, 2, 3 and 4. We also verified that these equations have fifth order symmetries. Though Eqs. (60) and (61) are definitely integrable, in order to prove this it is necessary to find Lax representations or auto-Bäcklund transformations for them. It will be done in a separate paper.

In the case $N = 1$, after the trigonometric parameterization of the circle

$$u^1 = \frac{\tan^2(s) - 1}{\tan^2(s) + 1}, \qquad u^2 = \frac{2\tan(s)}{\tan^2(s) + 1},$$

both Eqs. (59) and (60) become

$$s_t = s_{xxx} + 2\, s_x^3 + \frac{3}{4}\Big(c_1 + c_2 \cos(4s)\Big)\, s_x.$$

The latter equation is well known in the theory of integrable PDEs [13, 25].

The rational parameterization

$$u^1 = \frac{v^2 - 1}{v^2 + 1}, \qquad u^2 = \frac{2\,v}{v^2 + 1}$$

of the circle brings Eq. (61) with $N = 1$ to the form (17), where $Q = \alpha\, v^4 + \beta\, v^2 + \alpha$ with arbitrary parameters α and β. Thus (61) is an integrable vector generalization of the generic Calogero-Degasperis equation (see Remark 5).

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