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Eigenvalue Boundary Problems for the Dirac Operator

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Abstract: On a compact Riemannian spin manifold with mean-convex boundary, we analyse the ellipticity and the symmetry of four boundary conditions for the fundamental Dirac operator including the (global) APS condition and a Riemannian version of the (local) MIT bag condition. We show that Friedrich's inequality for the eigenvalues of the Dirac operator on closed spin manifolds holds for the corresponding four eigenvalue boundary problems. More precisely, we prove that, for both the APS and the MIT conditions, the equality cannot be achieved, and for the other two conditions, the equality characterizes respectively half-spheres and domains bounded by minimal hypersurfaces in manifolds carrying non-trivial real Killing spinors.

1. Introduction

The fundamental result of Lichnerowicz [Li] in the sixties regarding the spectrum of the Dirac operator D on closed spin manifolds, revealed subtle information on both the geometry and the topology of such manifolds (see for instance [BFGK, BHMM, Fr2, LM] and references therein). A basic lower bound for the eigenvalues λ of the Dirac operator is the Friedrich inequality [Fr1], which says that

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M R, \tag{F}$$

where R is the scalar curvature of the manifold and n its dimension. This inequality is sharp and the equality characterizes those geometries carrying non-trivial real Killing spinor fields (see also [Bä1]).

The Dirac operator has been also considered when the compact manifold has non-empty boundary in order to look for corresponding ellipticity and index theorems [APS,

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BW, GLP], to study its determinant [Bun, FGMSS] and to model physical situations in which the particle fields are confined in a bounded region of space [CJJTW, CJJT, J]. Each one of these situations requires a particular boundary condition to be imposed. However, from an analytical point of view, the ellipticity and other related properties of these boundary conditions are usually studied in an abstract and unified setting [BW, Hö, Se] based on the Calderón and Seeley theory of pseudo-differential operators.

In this paper, we study the spectrum of the fundamental Dirac operator on compact Riemannian spin manifolds with non-empty boundary, under four different boundary conditions: the global Atiyah-Patodi-Singer (APS) condition associated with the spectral resolution of the intrinsic Dirac operator on the boundary hypersurface; the local condition associated with a chirality (CHI) operator on the manifold (for example, if its dimension is even or if it is a space-like hypersurface in a Lorentzian manifold); the Riemannian version of the so-called (local) MIT bag condition; and finally a new global boundary condition obtained by a suitable modification of the APS condition (mAPS).

We show that these four conditions satisfy ellipticity criteria and the corresponding boundary problems are *well-posed* in the sense of Seeley [Se]. We prove (see Theorems 2, 3, 4 and 5) that the three APS, CHI and mAPS conditions make D a symmetric operator and so the corresponding spectra are real sequences tending to $+\infty$ and $-\infty$. Instead, under the MIT condition, the spectrum of the Dirac operator is an unbounded discrete set of complex numbers with positive imaginary part. Finally we prove that, under the four boundary conditions, one has the same lower bound (F) in terms of the minimum of the scalar curvature as in the closed case, provided that the mean curvature of the boundary hypersurface is non-negative. (In fact, in the case of the APS and CHI conditions this fact is proved in [HMZ1, HMZ2], by other means.)

The four conditions have different behavior with respect to the equality in (F). In fact, we show that such an equality is never achieved for the APS and MIT conditions, that it occurs for the CHI boundary condition if and only if the manifold is a half-sphere and that it is achieved for the mAPS condition if and only if the manifold admits a non-trivial real Killing spinor field and the boundary is a minimal hypersurface (this was a principal motivation to look for and introduce this new boundary condition). For example, all the domains enclosed in a sphere by embedded minimal hypersurfaces have the same first eigenvalue for the Dirac operator under the mAPS condition.

2. Riemannian Spin Manifolds and Their Boundaries

Consider an n-dimensional Riemannian spin manifold M with non-empty boundary ∂M and denote by $\langle\ ,\ \rangle$ its scalar product and by ∇ its corresponding Levi-Civita connection on the tangent bundle TM. We fix a spin structure (and so a corresponding orientation) on the manifold M and denote by $\mathbb{S}M$ the associated spinor bundle, which is a complex vector bundle of rank $2^{\left[\frac{n}{2}\right]}$. Then let

$$\gamma : \mathbb{C}\ell(M) \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{S}M)$$

be the Clifford multiplication, which provides a fibre preserving irreducible representation of the Clifford algebras constructed over the tangent spaces of M. When the dimension n is even, we have the standard chirality decomposition

$$\mathbb{S}M = \mathbb{S}M^+ \oplus \mathbb{S}M^-, \tag{1}$$

where the two direct summands are respectively the ± 1-eigenspaces of the endomorphism $\gamma(\omega_n)$, with $\omega_n = i^{\left[\frac{n+1}{2}\right]} e_1 \cdots e_n$, the complex volume form. It is well-known (see [LM]) that there are, on the complex spinor bundle $\mathbb{S}M$, a natural Hermitian metric $(\ ,\ )$ and a spinorial Levi-Civita connection, denoted also by ∇, which is compatible with both $(\ ,\ )$ and γ in the following sense:

$$X(\psi, \varphi) = (\nabla_X \psi, \varphi) + (\psi, \nabla_X \varphi), \tag{2}$$

$$\nabla_X (\gamma(Y)\psi) = \gamma(\nabla_X Y)\psi + \gamma(Y)\nabla_X \psi \tag{3}$$

for any tangent vector fields $X, Y \in \Gamma(TM)$ and any spinor fields $\psi, \varphi \in \Gamma(\mathbb{S}M)$ on M. Moreover, with respect to this Hermitian product on $\mathbb{S}M$, Clifford multiplication by vector fields is skew-Hermitian or equivalently

$$(\gamma(X)\psi, \gamma(X)\varphi) = |X|^2(\psi, \varphi). \tag{4}$$

Since the complex volume form ω_n is parallel with respect to the spinorial Levi-Civita connection, when $n = \dim M$ is even, the chirality decomposition (1) is preserved by ∇. From (4) one sees that it is an orthogonal decomposition.

In this setting, the (fundamental) Dirac operator D on the manifold M is the first order elliptic differential operator acting on spinor fields given locally by

$$D = \sum_{i=1}^{n} \gamma(e_i)\nabla_{e_i},$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame in TM. When $n = \dim M$ is even, D interchanges the chirality subbundles $\mathbb{S}M^{\pm}$.

The boundary hypersurface ∂M is also an oriented Riemannian manifold with the induced orientation and metric. If $\nabla^{\partial M}$ stands for the Levi-Civita connection of the induced metric we have the Gauss and Weingarten equations

$$\nabla_X Y = \nabla_X^{\partial M} Y + \langle AX, Y\rangle N, \qquad \nabla_X N = -AX,$$

for any vector fields X, Y tangent to ∂M, where A is the shape operator or Weingarten endomorphism of the hypersurface ∂M corresponding to the unit normal field N compatible with the given orientation. As the normal bundle of the boundary hypersurface is trivial, the Riemannian manifold ∂M is also a spin manifold and so we will have the corresponding spinor bundle $\mathbb{S}\partial M$, the Clifford multiplication $\gamma^{\partial M}$, the spinorial Levi-Civita connection $\nabla^{\partial M}$ and the intrinsic Dirac operator $D^{\partial M}$. It is not difficult to show (see [Bä2, BFGK, HMZ1, HMZ3, Bur, Tr, Mo]) that the restricted Hermitian bundle

$$\mathbf{S} := \mathbb{S}M_{|\partial M}$$

can be identified with the intrinsic Hermitian spinor bundle $\mathbb{S}\partial M$, provided that $n = \dim M$ is odd. Instead, if $n = \dim M$ is even, the restricted bundle $\mathbf{S}$ could be identified with the sum $\mathbb{S}\partial M \oplus \mathbb{S}\partial M$. With such identifications, for any spinor field $\psi \in \Gamma(\mathbf{S})$ on the boundary hypersurface ∂M and any vector field $X \in \Gamma(T\partial M)$, define on the restricted bundle $\mathbf{S}$, the Clifford multiplication $\gamma^{\mathbf{S}}$ and the connection $\nabla^{\mathbf{S}}$ by

$$\gamma^{\mathbf{S}}(X)\psi = \gamma(X)\gamma(N)\psi, \tag{5}$$

$$\nabla_X^{\mathbf{S}}\psi = \nabla_X\psi - \tfrac{1}{2}\gamma^{\mathbf{S}}(AX)\psi = \nabla_X\psi - \tfrac{1}{2}\gamma(AX)\gamma(N)\psi\,. \tag{6}$$

Then it is easy to see that $\gamma^{\mathbf{S}}$ and $\nabla^{\mathbf{S}}$ correspond respectively to $\gamma^{\partial M}$ and $\nabla^{\partial M}$, for n odd, and to $\gamma^{\partial M} \oplus -\gamma^{\partial M}$ and $\nabla^{\partial M} \oplus \nabla^{\partial M}$, for n even. Then, $\gamma^{\mathbf{S}}$ and $\nabla^{\mathbf{S}}$ satisfy the same compatibilty relations (2), (3) and (4) and together with the following additional identity:

$$\nabla^{\mathbf{S}}_X (\gamma(N)\psi) = \gamma(N)\nabla^{\mathbf{S}}_X \psi.$$

As a consequence, the hypersurface Dirac operator $\mathbf{D}$ acts on smooth sections $\psi \in \Gamma(\mathbf{S})$ as

$$\mathbf{D}\psi := \sum_{j=1}^{n-1} \gamma^{\mathbf{S}}(u_j)\nabla^{\mathbf{S}}_{u_j}\psi = \frac{n-1}{2}H\psi - \gamma(N)\sum_{j=1}^{n-1}\gamma(u_j)\nabla_{u_j}\psi,$$

where $\{u_1, \ldots, u_{n-1}\}$ is a local orthonormal frame tangent to the boundary ∂M and $H = (1/(n-1))\operatorname{trace} A$ is its mean curvature function, coincides with the intrinsic Dirac operator $D^{\partial M}$ on the boundary, for n odd, and with the pair $D^{\partial M} \oplus -D^{\partial M}$, for n even. In the particular case where the field $\psi \in \Gamma(\mathbf{S})$ is the restriction of a spinor field $\psi \in \Gamma(\mathbb{S}M)$ on M, this means that

$$\mathbf{D}\psi = \frac{n-1}{2}H\psi - \gamma(N)D\psi - \nabla_N\psi. \tag{7}$$

Note that we always have the anticommutativity property

$$\mathbf{D}\gamma(N) = -\gamma(N)\mathbf{D} \tag{8}$$

and so, when ∂M is compact, the spectrum of $\mathbf{D}$ is symmetric with respect to zero and coincides with the spectrum of $D^{\partial M}$, for n odd, and with $\operatorname{Spec}(D^{\partial M}) \cup -\operatorname{Spec}(D^{\partial M})$, for n even.

3. A Spinorial Reilly Inequality

Our main goal in this paper is to estimate the eigenvalues of the Dirac operator D on the compact Riemannian spin manifold M under suitable boundary conditions. By examining their limiting cases, one can study the geometry of certain hypersurfaces. When the manifold M is closed (compact without boundary), D a self-adjoint elliptic operator of order one and so its spectrum is a discrete unbounded sequence of real numbers. When the boundary ∂M is non-empty, we shall see in the next section that there are boundary conditions for which one may have a discrete and not necessarily real spectrum for the Dirac operator with finite dimensional eigenspaces and smooth eigenspinors. The defect of symmetry of D on the manifold with boundary M appears by integrating by parts to obtain

$$\int_M (D\psi, \varphi) - \int_M (\psi, D\varphi) = -\int_{\partial M} (\gamma(N)\psi, \varphi), \tag{9}$$

where $\psi, \varphi \in \Gamma(\mathbb{S}M)$ and N is the inner unit normal field along the boundary. When the considered boundary condition forces the boundary integral on the r.h.s of (9) to vanish, the spectrum is necessarily real.

A basic tool to relate the eigenvalues of the Dirac operator and the geometry of the manifold M and that of its boundary ∂M will be, as in the closed case (see [Fr1]), the integral version of the Schrödinger-Lichnerowicz formula

$$D^2 = \nabla^*\nabla + \frac{1}{4}R,$$

where R is the scalar curvature of M. In fact, given a spinor field ψ on M, taking into account the formula above, if we compute the divergence of the one-form α defined by

$$\alpha(X) = (\gamma(X)D\psi + \nabla_X\psi, \psi), \qquad \forall X \in TM$$

and integrate, one gets

$$-\int_{\partial M}(\gamma(N)D\psi + \nabla_N\psi, \psi) = \int_M \left(|\nabla\psi|^2 - |D\psi|^2 + \frac{1}{4}R|\psi|^2\right),$$

which by (7), could be written as

$$\int_{\partial M}\left((\mathbf{D}\psi, \psi) - \frac{n-1}{2}H|\psi|^2\right) = \int_M \left(|\nabla\psi|^2 - |D\psi|^2 + \frac{1}{4}R|\psi|^2\right).$$

Finally, we will use the spinorial Schwarz inequality

$$|D\psi|^2 \leq n|\nabla\psi|^2, \qquad \forall\psi \in \Gamma(\mathbb{S}M),$$

where the equality is achieved only by the so-called twistor spinors, that is, those satisfying the following over-determined first order equation

$$\nabla_X\psi = -\frac{1}{n}\gamma(X)D\psi, \qquad \forall X \in TM.$$

Then we get the following integral inequality, called *Reilly inequality* [HMZ1, HMZ3] because of its similarity with the corresponding one obtained in [Re] for the Laplace operator,

$$\int_{\partial M}\left((\mathbf{D}\psi, \psi) - \frac{n-1}{2}H|\psi|^2\right) \geq \int_M \left(\frac{1}{4}R|\psi|^2 - \frac{n-1}{n}|D\psi|^2\right), \tag{10}$$

with equality only for twistor spinors on M.

4. Ellipticity of the Boundary Conditions

Now we introduce suitable boundary conditions for the fundamental Dirac operator D. On a compact Riemannian spin manifold M with boundary, the Dirac operator D : $\Gamma(\mathbb{S}M) \to \Gamma(\mathbb{S}M)$ has an infinite dimensional kernel and a closed image with finite codimension. We look for conditions B to be imposed on the restrictions to the boundary ∂M of the spinor fields on M so that this kernel becomes finite dimensional and then the boundary problem

$$\begin{cases} D\psi &= \Phi \quad \text{on } M \\ B\psi_{|\partial M} &= \chi \quad \text{along } \partial M, \end{cases} \tag{BP}$$

for $\Phi \in \Gamma(\mathbb{S}M)$ and $\chi \in \Gamma(\mathbf{S})$, is of Fredholm type. In this case, we will have smooth solutions for any data Φ and χ belonging to a certain subspace with finite codimension and these solutions will be unique up to a finite dimensional kernel.

To our knowledge, the study of boundary conditions suitable for an elliptic operator $\mathcal{D}$ (of any order, although for simplicity, we only consider first order operators) acting on smooth sections of a Hermitian vector bundle $F \to M$ has been first done in the fifties by Lopatinsky and Shapiro ([Hö, Lo]), but the main tool was discovered by Calderón in the sixties: the so-called Calderón projector

$$\mathcal{P}_+(\mathcal{D}) : H^{\frac{1}{2}}(F_{|\partial M}) \longrightarrow \{\psi_{|\partial M} \mid \psi \in H^1(F), \mathcal{D}\psi = 0\}.$$

This is a pseudo-differential operator of order zero (see [BW, Se]) with principal symbol $\mathfrak{p}_+(\mathcal{D}) : T\partial M \to \mathrm{End}_{\mathbb{C}}(F)$ depending only on the principal symbol $\sigma_{\mathcal{D}}$ of the operator $\mathcal{D}$ and can be calculated as follows:

$$\mathfrak{p}_+(\mathcal{D})(u) = -\frac{1}{2\pi i}\int_\Gamma \left[(\sigma_{\mathcal{D}}(N))^{-1}\sigma_{\mathcal{D}}(u) - \zeta I\right]^{-1} d\zeta, \tag{11}$$

for any $p \in \partial M$ and $u \in T_p\partial M$, where N is the inner unit normal along the boundary ∂M and Γ is a positively oriented cycle in the complex plane enclosing the poles of the integrand with negative imaginary part. Although the Calderón projector is not unique for a given elliptic operator $\mathcal{D}$, its principal symbol is uniquely determined by $\sigma_{\mathcal{D}}$. One of the important features of the Calderón projector is that its principal symbol detects the *ellipticity of a boundary condition*, or in other words, if the corresponding boundary problem (BP) is a *well-posed problem* (according to Seeley in [Se]). In fact (cfr. [Se] or [BW, Chap. 18]),

A pseudo-differential operator

$$B : L^2(F_{|\partial M}) \longrightarrow L^2(V),$$

where $V \to \partial M$ is a complex vector bundle over the boundary, is called a **(global) elliptic boundary condition** *when its principal symbol $b : T\partial M \to \mathrm{Hom}_{\mathbb{C}}(F_{|\partial M}, V)$ satisfies that, for any non-trivial $u \in T_p\partial M$, $p \in \partial M$, the restriction*

$$b(u)_{|\mathrm{image}\,\mathfrak{p}_+(\mathcal{D})(u)} : \mathrm{image}\,\mathfrak{p}_+(\mathcal{D})(u) \subset F_p \longrightarrow V_p$$

is an isomorphism onto image $b(u) \subset V_p$*. Moreover, if* $\mathrm{rank}\, V = \dim \mathrm{image}\,\mathfrak{p}_+(\mathcal{D})(u)$*, we say that B is a* **local elliptic boundary condition**.

When B is a local operator this definition yields the so-called Lopatinsky-Shapiro conditions for ellipticity (see for example [Hö]). When these definitions and the subsequent theorems are applied to the case where the vector bundle F is the spinor bundle $\mathbb{S}M$ and the elliptic operator $\mathcal{D}$ is the Dirac operator D on the spin Riemannian manifold M, we obtain the following well-known facts in the setting of the general theory of boundary problems for elliptic operators (see for example [BrL, BW, GLP, Hö, Se]):

Proposition 1. *Let M be an n-dimensional compact Riemannian spin manifold with non-empty boundary ∂M. Consider the restriction* **S** *to the boundary ∂M of the spinor bundle $\mathbb{S}M$ of M. A pseudo-differential operator*

$$B : L^2(\mathbf{S}) \longrightarrow L^2(V),$$

where $V \to \partial M$ is a Hermitian vector bundle, is an elliptic boundary condition for the fundamental Dirac operator D of M if and only if its principal symbol $b : T\partial M \to \mathrm{Hom}_{\mathbb{C}}(\mathbf{S}, V)$ satisfies the following two conditions:

$$\ker b(u) \cap \{\eta \in \mathbb{S}M_p \mid i\gamma(N)\gamma(u)\eta = -|u|\eta\} = \{0\},$$
$$\dim \mathrm{image}\, b(u) = \tfrac{1}{2}\dim \mathbb{S}M_p = 2^{[\frac{n}{2}]-1}.$$

Moreover, if V is a bundle with rank $\frac{1}{2}\dim \mathbb{S}M_p = 2^{[\frac{n}{2}]-1}$, we have a local elliptic boundary condition. When these ellipticity conditions are satisfied, the problem (BP) *is of Fredholm type and the corresponding eigenvalue boundary problem*

$$\begin{cases} D\psi = \lambda\psi & \text{on } M \\ B\psi_{|\partial M} = 0 & \text{along } \partial M, \end{cases} \tag{EBP}$$

has a discrete spectrum with finite dimensional eigenspaces consisting of smooth spinor fields, unless it is the whole complex plane.

Proof. Since the principal symbol σ_D of the Dirac operator D on M is given by

$$\sigma_D(v) = i\gamma(v), \qquad \forall v \in TM,$$

then by (11), the principal symbol of the Calderón projector of the Dirac operator is given by

$$\mathfrak{p}_+(D)(u) = -\frac{1}{2|u|}(i\gamma(N)\gamma(u) - |u|I) = \frac{1}{2|u|}\left(i\gamma^{\mathbf{S}}(u) + |u|I\right),$$

for each non-trivial $u \in T\partial M$ and where $\gamma^{\mathbf{S}}$ is identified in (5) as the intrinsic Clifford product on the boundary. As the endomorphism $i\gamma(N)\gamma(u) = -i\gamma^{\mathbf{S}}(u)$ is self-adjoint and its square is $|u|^2$ times the identity map, then it has exactly two eigenvalues, say $|u|$ and $-|u|$, whose eigenspaces are of the same dimension $\frac{1}{2}\dim \mathbb{S}M_p = 2^{[\frac{n}{2}]-1}$, since they are interchanged by $\gamma(N)$. Hence the symbol $\mathfrak{p}_+(D)(u)$ is, up to a constant, the orthogonal projection onto the eigenspace corresponding to the eigenvalue $-|u|$ and so

$$\operatorname{image}\mathfrak{p}_+(D)(u) = \{\eta \in \mathbb{S}M_p \mid i\gamma(N)\gamma(u)\eta = -|u|\eta\},$$
$$\dim \operatorname{image}\mathfrak{p}_+(D)(u) = \tfrac{1}{2}\dim \mathbb{S}M_p = 2^{[\frac{n}{2}]-1}.$$

From these equalities and from the definition of ellipticity for the boundary condition represented by the pseudo-differential operator B, we have that the first equation in the statement of this proposition is equivalent to the injectivity of the map $b(u)_{|\operatorname{image}\mathfrak{p}_+(D)(u)}$. The second one implies that $\dim \operatorname{image} b(u) = \dim \operatorname{image} \mathfrak{p}_+(D)(u)$ and so, together with the injectivity above, this means that $b(u)_{|\operatorname{image}\mathfrak{p}_+(D)(u)}$ is surjective. So we have proved that the two claimed conditions are equivalent to the ellipticity of the boundary condition B for the Dirac operator D on M. Now, from this ellipticity, one may deduce that the problems (BP) and (EBP) are of Fredholm type and the remaining assertions on eigenvalues and eigenspaces follow in a standard way (see [BW, Hö]). □

5. Four Boundary Conditions

In this last section, on a compact Riemannian spin manifold M with boundary, we will study the ellipticity of four boundary conditions for the Dirac operator D, where two of them are of global nature and the others are of local type. We will prove that, under each of these conditions, the square of any eigenvalue of D is bounded from below in terms of the minimum of the scalar curvature R of M. In fact, we show that, under the four boundary conditions, Friedrich's inequality (F) is still true. In the case of closed manifolds, the equality is achieved only when the manifold carries some non-trivial (real) Killing spinor fields. The important point is that these four conditions behave differently with respect to the equality case: two of them are never achieved and the others characterize half-spheres and domains enclosed by embedded minimal hypersurfaces in manifolds with non-trivial Killing spinors.

5.1. The Atiyah-Patodi-Singer (APS) condition. Atiyah, Patodi and Singer introduced in [APS] this well-known boundary condition in order to establish index theorems for compact manifolds with boundary. Later, this condition has been used to study the positive mass and the Penrose inequalities (see [He2, Wi]). Such a condition does not allow to model confined particle fields since, from the physical point of view, its global nature is interpreted as a causality violation. Although it is a well-kown fact that the APS condition is an elliptic boundary condition, we are going to sketch the proof in the setting of Proposition 1, for two reasons: first for completeness and second for pointing out that the APS condition for a chiral Dirac operator covers both cases of odd and even dimension, although the latter case is not referred to the spectral resolution of the intrinsic Dirac operator $D^{\partial M}$ but to the system $D^{\partial M} \oplus -D^{\partial M}$.

Precisely, this condition can be described as follows. Choose the Hermitian bundle V (of Proposition 1) over the boundary hypersurface ∂M as the restricted spinor bundle $\mathbf{S}$ defined in Sect. 2, and define $B_{\text{APS}} : L^2(\mathbf{S}) \to L^2(\mathbf{S})$ as the orthogonal projection onto the subspace spanned by the eigenspinors corresponding to the non-negative eigenvalues of the self-adjoint intrinsic operator $\mathbf{D}$. Atiyah, Patodi and Singer showed in [APS] (see also [BW, Prop. 14.2]) that B_{APS} is a zero order pseudo-differential operator whose principal symbol b_{APS} satisfies the following fact: for each $p \in \partial M$ and $u \in T_p\partial M - \{0\}$, the map $b_{\text{APS}}(u)$ is the orthogonal projection onto the eigenspace of $\sigma_{\mathbf{D}}(u) = i\gamma^{\mathbf{S}}(u)$ corresponding to the positive eigenvalue $|u|$. That is

$$b_{\text{APS}}(u) = \frac{1}{2}\left(i\gamma^{\mathbf{S}}(u) + |u|I\right) = \frac{1}{2}\left(-i\gamma(N)\gamma(u) + |u|I\right), \tag{12}$$

and so the principal symbol b_{APS} of the APS operator coincides, up to a constant, with the principal symbol $\mathfrak{p}_+(D)$ of the Calderón projector of D. From this, it is immediate to see that the two ellipticity conditions in Proposition 1 are satisfied.

The following result (see [HMZ1, HMZ2] and also [FS, Theorem 10] for a weaker version) provides a lower bound for the eigenvalues of the Dirac operator with the APS boundary condition. We give a short proof of this estimate to illustrate the difference with the other boundary conditions.

Theorem 2. *Let M be a compact Riemannian spin manifold whose non-empty boundary ∂M has non-negative mean curvature (w.r.t. the inner normal). Under the APS boundary condition, the spectrum of the Dirac operator D of M is a sequence of unbounded real numbers $\{\lambda_k^{\text{APS}} \mid k \in \mathbb{Z}\}$ which satisfy the following strict inequality:*

$$\left(\lambda_k^{\text{APS}}\right)^2 > \frac{n}{4(n-1)} \inf_M R, \qquad k \in \mathbb{Z}.$$

Proof. We know that the APS condition referred to the spectral resolution of $\mathbf{D}$ is an elliptic boundary condition. Moreover, from the supercommutativity relation (8), we have

$$B_{\text{APS}}\,\gamma(N) + \gamma(N) B_{\text{APS}} = \gamma(N)(I + \pi_0), \tag{13}$$

$$B_{\text{APS}}\,\mathbf{D} = \mathbf{D} B_{\text{APS}}, \tag{14}$$

where π_0 is the L^2-orthogonal projection on the space of harmonic spinors of $\mathbf{D}$. This implies that, if $\psi, \varphi \in \Gamma(\mathbb{S})$ satisfy $B_{\text{APS}}\,\psi = B_{\text{APS}}\,\varphi = 0$ (and subsequently $\pi_0\psi = \pi_0\varphi = 0$), then

$$B_{\text{APS}}\,\gamma(N)\psi = \gamma(N)\psi, \qquad \text{and} \qquad B_{\text{APS}}\,\gamma(N)\varphi = \gamma(N)\varphi.$$

As a consequence we deduce that, for $\psi, \varphi \in \Gamma(\mathbb{S}M)$,

$$\int_{\partial M} (\gamma(N)\psi, \varphi) = \int_{\partial M} (B_{\mathrm{APS}}\, \gamma(N)\psi, \varphi) = \int_{\partial M} (\gamma(N)\psi, B_{\mathrm{APS}}\, \varphi) = 0$$

and under the APS boundary condition, by (9) the Dirac operator D on the bulk manifold M is a symmetric operator. Then the corresponding spectrum is real and so an unbounded discrete sequence (see [Hö, GLP] for instance).

Now consider an eigenspinor $\psi \in \Gamma(\mathbb{S}M)$ associated with an arbitrary eigenvalue λ_k^{APS}, $k \in \mathbb{Z}$, in the Reilly inequality (10). Then

$$\int_M \left(\frac{1}{4}R - \frac{n-1}{n}\left(\lambda_k^{\mathrm{APS}}\right)^2\right)|\psi|^2 \le \int_{\partial M} (\mathbf{D}\psi, \psi),$$

since $H \ge 0$. But, using $B_{\mathrm{APS}}\, \psi = 0$ and the commutativity (13), one gets

$$\int_{\partial M} (\mathbf{D}\psi, \psi) \le 0$$

and the equality holds only when the restriction $\psi_{|\partial M}$ vanishes. Hence

$$\left(\lambda_k^{\mathrm{APS}}\right)^2 \ge \frac{n}{4(n-1)} \inf_M R$$

and the equality is achieved if and only if the eigenspinor ψ is simultaneously a twistor spinor (and so a real Killing spinor) and its restriction $\psi_{|\partial M}$ is zero. But, since a real Killing spinor is of constant length, then $\psi = 0$, which is impossible. Hence, the inequality above is strict. □

5.2. The condition associated with a chirality (CHI) operator. This type of (local) boundary condition has already been considered in the context of comparison results [Bun], to estimate the mass of asymptotically flat manifolds including black holes [GHHP, He1] and also in order to study eigenvalue estimates [FS, HMZ2]. By contrast to the APS condition, which exists on any spin manifold with boundary, the second boundary condition which we shall consider is subjected to the existence on the manifold M of a linear map $G : \gamma(\mathbb{S}M) \to \gamma(\mathbb{S}M)$ satisfying

$$G^2 = I, \qquad (G\psi, G\varphi) = (\psi, \varphi), \tag{15}$$

$$\nabla_X (G\psi) = G\nabla_X\psi, \qquad \gamma(X)G\psi = -G\gamma(X)\psi \tag{16}$$

for each vector field X and spinor fields ψ, φ on M. This map G is often called a *chirality operator* because, when the dimension n of M is even, the standard candidate is $G = \gamma(\omega_n)$, the Clifford multiplication by the complex volume form ω_n which yields to the chirality decomposition (1) of the spinor bundle. In this case, G would be nothing but the usual conjugation changing chirality of spinors. But there is another important situation where such an operator appears, this is when the manifold M is a spacelike hypersurface of a Lorentzian manifold $\widetilde{M}$ of dimension $n+1$ and both the Riemannian and spinorial structures on M are the induced ones from $\widetilde{M}$. In this case one can choose $G = \gamma(T)$, the Clifford multiplication by a unit time-like normal field T on M (see for example [He1]).

Anyway, given such a chirality operator G on M, the fibre preserving endomorphism $\gamma(N)G : \Gamma(\mathbf{S}) \to \Gamma(\mathbf{S})$, acting on sections of the restricted spinor bundle, is self-adjoint with respect to the pointwise Hermitian product, whose square is the identity. Hence it has two eigenvalues $+1$ and -1 whose corresponding eigenspaces are interchanged by, for example, the isomorphism $\gamma(N)$. Hence the eigensubbundle V over ∂M corresponding to the eigenvalue -1 verifies

$$\text{rank}\, V = \frac{1}{2}\text{rank}\, \mathbf{S} = 2^{[\frac{n}{2}]-1}.$$

Define now the boundary condition $B_{\text{CHI}} : L^2(\mathbf{S}) \to L^2(V)$ as the linear operator

$$B_{\text{CHI}} = \frac{1}{2}\left(I - \gamma(N)G\right),$$

that is, the orthogonal projection onto the eigensubbundle V. This is a differential operator of order zero and so its principal symbol $b_{\text{CHI}}(u)$, on each vector $u \in T\partial M$, coincides with the operator itself, that is,

$$b_{\text{CHI}}(u) = \frac{1}{2}(I - \gamma(N)G), \qquad \forall u \in T\partial M$$

and, in particular,

$$\dim \text{image}\, b_{\text{CHI}}(u) = \text{rank}\, V = 2^{[\frac{n}{2}]-1}.$$

Now it is easy to check that the two conditions in Proposition 1 are satisfied and so B_{CHI} is a local elliptic boundary condition. We have

Theorem 3. *Let M be a compact Riemannian spin manifold whose non-empty boundary ∂M has non-negative mean curvature (w.r.t. the inner normal). Under the CHI boundary condition, the spectrum of the Dirac operator D of M associated with a chirality operator on M, is a non-decreasing sequence of real numbers*

$$\left\{\lambda_k^{\text{CHI}} \mid k \in \mathbb{Z}\right\}$$

with $\lim_{k\to\pm\infty} \lambda_k^{\text{CHI}} = \pm\infty$ *and satisfying the following inequality:*

$$\left(\lambda_k^{\text{CHI}}\right)^2 \geq \frac{n}{4(n-1)} \inf_M R, \qquad k \in \mathbb{Z}.$$

Moreover the equality holds (for $\lambda^{\text{CHI}} = \lambda_{\pm 1}^{\text{CHI}}$) if and only if M is isometric to the half-sphere with radius $n/2|\lambda^{\text{CHI}}|$.

Proof. The spectrum being real is a consequence of the fact that the Dirac operator D is symmetric when it acts on spinor fields $\psi \in \Gamma(\mathbb{S}M)$ such that $B_{\text{CHI}}\, \psi_{|\partial M} = 0$, that is, $\gamma(N)G\psi = \psi$. In fact, if $\varphi \in \Gamma(\mathbb{S}M)$ is another field satisfying the same boundary condition, we have from (15) and (16),

$$(\gamma(N)\psi, \varphi) = (G\gamma(N)\psi, G\varphi) = (\psi, \gamma(N)\varphi) = -(\gamma(N)\psi, \varphi).$$

Integrating over ∂M this pointwise equality and using (9) one has the symmetry property.

Consider a smooth spinor field on M such that

$$D\psi = \lambda^{\text{CHI}}\, \psi, \qquad \text{and} \qquad B_{\text{CHI}}\, \psi_{|\partial M} = 0$$

and plug it into the Reilly inequality (10). As in the proof of Theorem 2, we will have the desired inequality if we are able to show that the *mass term* on the boundary

$$\int_{\partial M} (\mathbf{D}\psi, \psi)$$

is non-positive. But, in this case, this term is exactly zero. In fact we only have to realize that (16), (5) and (6) imply

$$\mathbf{D}G = G\mathbf{D}.$$

Then, we have the following pointwise equation:

$$(\mathbf{D}\psi, \psi) = (\gamma(N)G\mathbf{D}\psi, \gamma(N)G\psi) = -(\mathbf{D}\psi, \psi)$$

because of (8) and $\gamma(N)G\psi = \psi$. As a consequence we have the claimed inequality.

Suppose now that equality is achieved. As in the proof of Theorem 2, we have equality in (10) and so the eigenspinor ψ is a twistor spinor and hence a non-trivial real Killing spinor. In fact

$$\nabla_v \psi = -\frac{\lambda^{\mathrm{CHI}}}{n}\gamma(v)\psi, \qquad \forall v \in TM. \tag{17}$$

This implies that the length $|\psi|^2$ is a non-zero constant and that M is an Einstein manifold with scalar curvature $R = 4n(n-1)\left(\lambda^{\mathrm{CHI}}\right)^2$ (see for example [BFGK]). Since the assumption

$$\int_{\partial M} H|\psi|^2 \geq 0,$$

has been used to get the inequality, we deduce that $H = 0$, that is, the boundary is a minimal hypersurface. Consider now the smooth function $f = (G\psi, \psi)$ which takes real values, since from (15), G is pointwise self-adjoint. Moreover, if we take $\varphi = G\psi$ in equality (9), considering that D and G anticommute (because of (16) and the boundary condition $\gamma(N)G\psi = \psi$), we have

$$2\lambda^{\mathrm{CHI}} \int_M f = \int_{\partial M} |\psi|^2.$$

This yields the following two important facts:

$$\lambda^{\mathrm{CHI}} \neq 0 \qquad \text{and} \qquad f \not\equiv 0.$$

On the other hand, since ψ is a Killing spinor and from (15), one can easily compute that the Hessian of the function f is given by

$$\nabla^2 f = -\left(\frac{2\lambda^{\mathrm{CHI}}}{n}\right)^2 f\langle\ ,\ \rangle.$$

In other words, the function f is a non-trivial solution on M of the Obata equation. But using again the boundary condition satisfied by the eigenspinor ψ we see that

$$f_{|\partial M} = (G\psi_{|\partial M}, \psi_{|\partial M}) = (\gamma(N)G\psi_{|\partial M}, \gamma(N)\psi_{|\partial M}) = (\psi_{|\partial M}, \gamma(N)\psi_{|\partial M}),$$

and hence $f_{|\partial M}$ is identically zero since $(\psi_{|\partial M}, \gamma(N)\psi_{|\partial M})$ is a purely imaginary function. Now we apply the boundary version of the Obata theorem found by Reilly in [Re] in order to conclude that M is isometric to the required half-sphere. □

5.3. The Riemannian version of the MIT bag condition. In the seventies, some physicists at the Massachusetts Institute of Technology proposed a model for elementary particles (see [CJJTW, CJJT, J]) which has been called later the MIT bag model. It works with fields confined in a finite region of the space which, in the massless spin-$\frac{1}{2}$ case, are modeled by spinor fields satisfying the (Lorentzian) Dirac equation defined in the region of the space-time swept out by a bounded *bag* of a given rest-space. These solutions of the Dirac equation should obey a local boundary condition, which we shall examine now in the Riemannian setup. It is interesting to point out that such a Riemannian version of the MIT boundary condition has been used in another context (see [FGMSS, HMZ3]), because of its invariance under conformal changes of the metric of the manifold.

Consider the pointwise endomorphism $i\gamma(N) : \Gamma(\mathbf{S}) \to \Gamma(\mathbf{S})$ acting on sections of the spinor bundle of the compact Riemannian spin manifold M restricted to the boundary hypersurface ∂M, where N is the inner unit normal field along ∂M. The square of this endomorphism is the identity, its eigenvalues are ± 1 with the same multiplicity. In a similar way to the CHI condition, we denote by $V \to \partial M$ the eigensubbundle of $\mathbf{S}$ corresponding to the eigenvalue -1 and so we have again

$$\operatorname{rank} V = \frac{1}{2}\operatorname{rank}\mathbf{S} = 2^{[\frac{n}{2}]-1}.$$

The new boundary condition $B_{\mathrm{MIT}} : L^2(\mathbf{S}) \to L^2(V)$ is also the corresponding orthogonal projection

$$B_{\mathrm{MIT}} = \frac{1}{2}(I - i\gamma(N))$$

onto V. Hence, for each vector u tangent to the boundary, we have the following principal symbol b_{MIT} with the following properties:

$$b_{\mathrm{MIT}}(u) = \frac{1}{2}(I - i\gamma(N)), \qquad \dim \operatorname{image} b_{\mathrm{MIT}} = \operatorname{rank} V = 2^{[\frac{n}{2}]-1}.$$

As above, from this it is immediate to check the ellipticity conditions given by Proposition 1. Now we state the corresponding result estimating the eigenvalues of the corresponding eigenvalue boundary problem.

Theorem 4. *Let M be a compact Riemannian spin manifold whose non-empty boundary ∂M has non-negative mean curvature (w.r.t. the inner normal). Under the MIT bag boundary condition, the spectrum of the Dirac operator D of M is an unbounded discrete set of complex numbers λ^{MIT} with positive imaginary part which satisfy the following inequality*

$$\left|\lambda^{\mathrm{MIT}}\right|^2 > \frac{n}{4(n-1)} \inf_M R.$$

Proof. Let λ^{MIT} be an eigenvalue of the considered problem. That is

$$D\psi = \lambda^{\mathrm{MIT}}\psi, \qquad B_{\mathrm{MIT}}\psi = 0, \qquad \text{i.e.,} \quad i\gamma(N)\psi = \psi$$

for a non-trivial spinor field ψ on M. Then, taking such a spinor in (9) and choosing $\varphi = i\psi$ we obtain

$$2\Im(\lambda^{\mathrm{MIT}}) \int_M |\psi|^2 = \int_{\partial M} |\psi|^2$$

and so the eigenvalue has non-negative imaginary part. If $\Im(\lambda^{\mathrm{MIT}}) = 0$, then the restriction $\psi_{|\partial M}$ would vanish and the unique continuation principle (see for intance [BW, Chap. 8]) would imply that ψ is identically zero on the whole of M. Hence all the eigenvalues for the MIT boundary condition belong to the upper complex half-plane and so [Hö] that spectrum has to be discrete.

In order to obtain the estimate of the length of the λ^{MIT} we will proceed as in the two previous cases by taking the associated eigenspinor ψ in the Reilly inequality (10) and recalling that we are assuming that ∂M is mean-convex, i.e., $H \geq 0$. Then

$$\int_M \left(\frac{1}{4}R - \frac{n-1}{n}\left|\lambda^{\mathrm{MIT}}\right|^2\right)|\psi|^2 \leq \int_{\partial M} (\mathbf{D}\psi, \psi).$$

Here the mass integrand is identically zero, since by (8), one has

$$(\mathbf{D}\psi, \psi) = (i\gamma(N)\mathbf{D}\psi, i\gamma(N)\psi) = -(\mathbf{D}\psi, \psi).$$

This proves the inequality. We still need to prove that equality could not be achieved. Assume the contrary, that is

$$\left|\lambda^{\mathrm{MIT}}\right|^2 = \frac{n}{4(n-1)} \inf_M R,$$

then we also have equality in (10) and ψ is a twistor eigenspinor. In other words, ψ is a Killing spinor field with associated constant $-\lambda^{\mathrm{MIT}}/n$. But it is well-known [BFGK] that non-trivial imaginary Killing spinors only live on Einstein manifolds with negative scalar curvature. This contradicts the equality above. □

5.4. A new boundary condition for the Dirac operator. We have just studied, in a unified frame, the spectra of the fundamental Dirac operator on a compact Riemannian manifold with boundary, under three more or less known (global and local) boundary conditions. Finally, we introduce a new global condition which, in our opinion is of special interest, since the corresponding eigenvalues satisfy again (F) and where the limiting case includes relevant geometries.

We again choose the bundle $V \to \partial M$ to be the restricted spinor bundle $\mathbf{S}$ and introduce the following operator $B_{\mathrm{mAPS}} : L^2(\mathbf{S}) \to L^2(\mathbf{S})$ given by

$$B_{\mathrm{mAPS}} = B_{\mathrm{APS}}(I + \gamma(N)),$$

which is the composition of the zero order differential operator $I + \gamma(N)$ and the APS pseudo-differential operator. This composition is also a pseudo-differential operator of zero order (see for example [LM]) and, from (12), its principal symbol b_{mAPS} satisfies for all $u \in T\partial M$, the relation

$$\begin{aligned} b_{\mathrm{mAPS}}(u) &= b_{\mathrm{APS}}(u)(I + \gamma(N)) \\ &= \frac{1}{2}(-i\gamma(N)\gamma(u) + |u|I)(I + \gamma(N)). \end{aligned}$$

The first ellipticity condition in Proposition 1 arises now immediately. For the second one, take into account that $I + \gamma(N)$ is an isomorphism and so

$$\dim \operatorname{image} b_{\mathrm{mAPS}}(u) = \dim \operatorname{image} b_{\mathrm{APS}}(u) = 2^{[\frac{n}{2}]-1}.$$

Once we checked the ellipticity of the proposed boundary condition, we have now:

Theorem 5. *Let M be a compact Riemannian spin manifold whose non-empty boundary ∂M has non-negative mean curvature (w.r.t. the inner normal). Under the modified APS boundary condition,*

$$B_{\mathrm{mAPS}} = B_{\mathrm{APS}}\,(I + \gamma(N)) = 0,$$

the spectrum of the Dirac operator D of M is a non-decreasing sequence of real numbers $\{\lambda_k \mid k \in \mathbb{Z}\}$ tending to $\pm\infty$ which satisfy the following inequality:

$$\lambda_k^2 \geq \frac{n}{4(n-1)} \inf_M R, \qquad k \in \mathbb{Z}.$$

Moreover, the equality holds if and only if M carries a non-trivial real Killing spinor field with negative Killing constant and the boundary ∂M is minimal.

Proof. We first prove that, under the boundary condition B_{mAPS}, D is symmetric. In fact, let $\psi, \varphi \in \Gamma(\mathbb{S}M)$ be such that

$$B_{\mathrm{mAPS}}\,\psi = B_{\mathrm{APS}}\,(\psi + \gamma(N)\psi) = 0, \qquad B_{\mathrm{mAPS}}\,\varphi = B_{\mathrm{APS}}\,(\varphi + \gamma(N)\varphi) = 0.$$

Using (14), we deduce that

$$B_{\mathrm{APS}}\,(\gamma(N)\psi - \psi) = \gamma(N)\psi - \psi.$$

Now as B_{APS} is an orthogonal L^2-projection, we have

$$\int_{\partial M} (\gamma(N)\psi, \varphi) = \int_{\partial M} (\psi + B_{\mathrm{APS}}\,\chi, \varphi) = \int_{\partial M} (\psi, \varphi) + \int_{\partial M} (\chi, B_{\mathrm{APS}}\,\varphi),$$

where we have put $\chi = \gamma(N)\psi - \psi$. But $B_{\mathrm{APS}}\,\varphi = -B_{\mathrm{APS}}\,\gamma(N)\varphi$ and hence

$$\int_{\partial M} (\gamma(N)\psi, \varphi) = \int_{\partial M} (\psi, \varphi) - \int_{\partial M} (B_{\mathrm{APS}}\,\chi, \gamma(N)\varphi).$$

Finally, we use that $B_{\mathrm{APS}}\,\chi = \chi$ to get

$$\int_{\partial M} (\gamma(N)\psi, \varphi) = \int_{\partial M} (\psi, \gamma(N)\varphi).$$

As a consequence we have from (9) the required symmetry. So the considered spectrum is real.

We take again an eigenspinor ψ corresponding to an eigenvalue λ_k, $k \in \mathbb{Z}$, satisfying the boundary condition

$$B_{\mathrm{mAPS}}\,\psi = B_{\mathrm{APS}}\,(\psi + \gamma(N)\psi) = 0,$$

which we plug in inequality (10). Under the assumption $H \geq 0$, the claimed inequality follows, if we show that the boundary mass term

$$\int_{\partial M} (\mathbf{D}\psi, \psi)$$

vanishes. In fact, the supercommutativity relation (8) implies that

$$(\mathbf{D}\psi, \psi) = \frac{1}{2}(\mathbf{D}(\psi + \gamma(N)\psi), \psi - \gamma(N)\psi).$$

But, since $B_{\text{APS}}\,(\psi+\gamma(N)\psi)=0$ and $B_{\text{APS}}\,(\psi-\gamma(N)\psi)=\psi-\gamma(N)\psi$, a suitable use of (13) gives

$$\int_{\partial M}(\mathbf{D}(\psi+\gamma(N)\psi),\psi-\gamma(N)\psi)=0.$$

If the equality occurs we deduce, as in the three preceding cases, that ψ is a non-trivial real Killing spinor on M. Then its length is a non-trivial constant and so H must be zero as claimed. Moreover, from (7) and (8) we get

$$\mathbf{D}(\psi+\gamma(N)\psi)=-\frac{n-1}{n}\lambda(\psi+\gamma(N)\psi).$$

Since $B_{\text{mAPS}}\,\psi=0$ (and so $\pi_0\psi=0$) we deduce that $\lambda>0$.

Conversely, assume that M is a compact Riemannian spin manifold with minimal boundary ∂M carrying a non-trivial Killing spinor ψ with a real Killing constant $-\lambda/n<0$. It is clear that $D\psi=\lambda\psi$. Moreover, from (7) and the fact that $\nabla_N\psi=-(\lambda/n)\gamma(N)\psi$ we have that the restriction of ψ to the boundary satisfies

$$\mathbf{D}\psi=-\frac{n-1}{n}\gamma(N)\psi.$$

From this and (8) we have that

$$\mathbf{D}(\psi+\gamma(N)\psi)=-\frac{n-1}{n}\lambda(\psi+\gamma(N)\psi).$$

Since we assumed $\lambda>0$, the spinor field $\psi+\gamma(N)\psi$ is an eigenspinor of $\mathbf{D}$ associated with a negative eigenvalue. Then its APS projection has to vanish and so $B_{\text{mAPS}}\,\psi=0$. □

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