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Exponential Mixing of the 2D Stochastic Navier-Stokes Dynamics*

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Abstract: We consider the Navier-Stokes equation on a two dimensional torus with a random force which is white noise in time, and excites only a finite number of modes. The number of excited modes depends on the viscosity ν, and grows like ν^{-3} when ν goes to zero. We prove that this Markov process has a unique invariant measure and is exponentially mixing in time.

1. Introduction

Homogenous isotropic turbulence is often mathematically modelled by the Navier Stokes equation subjected to an external stochastic driving force which is stationary in space and time and "large scale", which in particular means smooth in space. The status of the existence and uniqueness of solutions to the stochastic PDE parallels that of the deterministic one. In particular, in two dimensions, it holds under very general conditions.

However, for physical reasons, one is interested in the existence, uniqueness and properties of the stationary state of the resulting Markov process. While the existence of such a state follows with soft methods [10], uniqueness, i.e. ergodic and mixing properties of the process has been harder to establish. In a nonturbulent situation, i.e. with a sufficiently rough forcing this was established in [5] and for large viscosity in [8]. The first result for a smooth forcing was by Kuksin and Shirikyan [7] who considered a periodically kicked system with bounded kicks. In particular they could deal with the case where only a finite number of modes are excited by the noise (the number depends both on the viscosity and the size of the kicks). In [2], we proved uniqueness and exponential mixing for such a kicked system where the kicks have a Gaussian distribution, but we required that there be a nonzero noise for each mode. In this paper, we extend that analysis to the case where only finitely many modes are excited, and the forcing is white noise in time. An essential ingredient in our analysis is the Lyapunov-Schmidt type reduction

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introduced in [7], that allows to transform the original Markov process with infinite dimensional state space to a non-Markovian process with finite dimensional state space. We apply standard ideas of statistical mechanics (high temperature expansions) to this process to deduce mixing properties of the dynamics. While preparing this manuscript we received a preliminary draft [4] that claims similar results, using a somewhat more probabilistic approach. We thank these authors for communicating us their ideas, some of which helped us to simplify our arguments, especially in Sect. 8 below.

We consider the stochastic Navier-Stokes equation for the velocity field $u(t,x) \in \mathbf{R}^2$ defined on the torus $\mathbf{T} = (\mathbf{R}/2\pi\mathbf{Z})^2$:

$$du + ((u \cdot \nabla)u - \nu\nabla^2 u + \nabla p)dt = df, \tag{1}$$

where $f(t,x)$ is a Wiener process with covariance

$$Ef_\alpha(t,x)f_\beta(t',y) = \min\{t,t'\}C_{\alpha\beta}(x-y) \tag{2}$$

and $C_{\alpha\beta}$ is a smooth function satisfying $\sum_\alpha \partial_\alpha C_{\alpha\beta} = 0$. Equation (1) is supplemented with the incompressibility condition $\nabla \cdot u = 0 = \nabla \cdot f$, and we will also assume that the averages over the torus vanish: $\int_{\mathbf{T}} u(0,x) = 0 = \int_{\mathbf{T}} f(t,x)$, which imply that $\int_{\mathbf{T}} u(t,x) = 0$ for all times t.

It is convenient to change to dimensionless variables so that ν becomes equal to one. This is achieved by setting $u(t,x) = \nu u'(\nu t, x)$. Then u' satisfies (1), (2) with ν replaced by 1, and C by

$$C' = \nu^{-3}C.$$

From now on, we work with such variables and drop the primes. The dimensionless control parameter in the problem is the (rescaled) energy injection rate $\frac{1}{2}\mathrm{tr}\, C'(0)$, customarily written as $(\mathrm{Re})^3$,where Re is the Reynolds number:

$$\mathrm{Re} = \epsilon^{\frac{1}{3}}\nu^{-1},$$

and $\epsilon = \frac{1}{2}\mathrm{tr}\, C(0)$ is the energy injection rate in the original units (for explanations of the terminology see [6]).

In two dimensions, the incompressibility condition can be conveniently solved by expressing the velocity field in terms of the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$. First (1) implies the transport equation

$$d\omega + ((u \cdot \nabla)\omega - \nabla^2\omega)dt = db, \tag{3}$$

where $b = \partial_1 f_2 - \partial_2 f_1$ has the covariance

$$Eb(t,x)b(t',y) = \min\{t,t'\}(2\pi)^{-1}\gamma(x-y)$$

with $\gamma = -2\pi\nu^{-3}\Delta\mathrm{tr}C$.

Next, going to the Fourier transform, $\omega_k(t) = \frac{1}{2\pi}\int_{\mathbf{T}} e^{ik\cdot x}\omega(t,x)dx$, with $k \in \mathbf{Z}^2$; we may express u as $u_k = i\frac{(-k_2,k_1)}{k^2}\omega_k$, and write the vorticity equation as

$$d\omega(t) = F(\omega(t))dt + db(t), \tag{4}$$

where the drift is given by

$$F(\omega)_k = -k^2\omega_k + \frac{1}{2\pi}\sum_{l\in\mathbf{Z}^2\setminus\{0,k\}} \frac{k_1 l_2 - l_1 k_2}{|l|^2}\omega_{k-l}\omega_l \tag{5}$$

and $\{b_k\}$ are Brownian motions with $\bar{b_k} = b_{-k}$ and

$$Eb_k(t)b_l(t') = \min\{t, t'\}\delta_{k,-l}\,\gamma_k.$$

The dimensionless control parameter for the vorticity equation is

$$R = \sum_{k\in\mathbf{Z}^2} \gamma_k = 2\pi\gamma(0) \tag{6}$$

which is proportional to the ω injection rate, and also to the third power of the Reynolds number. We will be interested in the turbulent region $R \to \infty$; therefore, we will always assume below, when it is convenient, that R is sufficiently large.

For turbulence one is interested in the properties of the stationary state of the stochastic equation (4) in the case of *smooth* forcing (see [1] for some discussion of this issue) and, ideally, one would like to consider the case where one excites only a finite number of modes,

$$\gamma_k \neq 0, \quad k^2 \leq N,$$

with N of order of one. In this paper we assume that N scales as

$$N = \kappa R, \tag{7}$$

with κ an absolute constant fixed below. We take all the other $\gamma_k = 0$, although this condition can easily be relaxed. Let us denote the minimum of the covariance by

$$\rho = \min\{|\gamma_k| \mid |k|^2 \leq N\}.$$

Before stating our result, we need some definitions. Let P be the orthogonal projection in $H = L^2(\mathbf{T})$ to the subspace H_s of functions having zero Fourier components for $|k|^2 > N$. We will write

$$\omega = s + l$$

with $s = P\omega$, $l = (1 - P)\omega$ (respectively, the small k and large k parts of ω). Denote also by H_l the complementary subspace (containing the nonzero components of l). H is our probability space, equipped with $\mathcal{B}$, the Borel σ-algebra.

The stochastic equation (4) gives rise to a Markov process $\omega(t)$ and we denote by $P^t(\omega, E)$ the transition probability of this process.

Our main result is the

Theorem. *The stochastic Navier-Stokes equation* (4) *defines a Markov process with state space* $(H, \mathcal{B})$ *and for all* $R < \infty$, $\rho > 0$ *it has a unique invariant measure* μ *there. Moreover,* $\forall\omega \in H$, *for all Borel sets* $E \in H_s$ *and for all bounded Hölder continuous functions* F *on* H_l, *we have,*

$$\left|\int P^t(\omega, d\omega')1_E(s')F(l') - \int \mu(d\omega')1_E(s')F(l')\right| \leq C(\omega)||F||_\alpha e^{-mt}, \tag{8}$$

where $m = m(R, \rho, \alpha) > 0$, $||F||_\alpha$ *is the Hölder norm of exponent* α, *and* $C(\omega)$ *is a.s. finite.*

Remark 1. In a previous paper [1] we have shown that, with probability 1, the functions on the support of such a measure as constructed here are real analytic. In particular all correlation functions of the form

$$\int \mu(d\omega) \prod_i \nabla^{n_i} u(x_i)$$

exist.

Remark 2. The parameters in our problem are R and ρ. All constants that do not depend on them will be generically denoted by C or c. Besides, we write $C(X, Y, Z)$ for a "constant" depending only on X, Y, Z. These constants can vary from place to place, even in the same equation.

We close this section by giving the outline of the proof and explain its connection to ideas coming from Statistical Mechanics.

Let us start by observing that, if we neglect the nonlinear term in (4–5), we expect $\|\omega\|$ to be of order $R^{\frac{1}{2}}$, for typical realizations of the noise ($R^{\frac{1}{2}}$ is the typical size of the noise, and the $-k^2\omega_k$ term will dominate in Eq. (4) for larger values of $\|\omega\|$). It turns out that similar probabilistic estimates hold for the full Eq. (4) as shown in Sect. 3. Now, if $\|\omega\|$ is of size $R^{\frac{1}{2}}$, the $-k^2\omega_k$ term will dominate the nonlinear term (which is roughly of size $\|\omega\|^2$) in Eq. (4), for $|k| \geq \kappa R^{\frac{1}{2}}$, and one can expect that those modes (corresponding to l above) will behave somewhat like the solution of the heat equation and, in particular, that they will converge to a stationary state.

Thus, the first step is to express the l-modes in terms of the s-modes at previous times. This is done in Sect. 2 and produces a process for the s-modes that is no longer Markovian but has an infinite memory. In Statistical Mechanics, this would correspond to a system of unbounded spins (the s-modes) with infinite range interactions, with the added complications that, here, the measure is not given in a Gibbsian form, but only through a Girsanov formula, i.e. (23) below, and that time is continuous. Hence, we have to solve several problems: the possibility that ω be atypically large, the long range "interactions", and finally, showing that a version of the s-process with a suitable cutoff is ergodic and mixing.

The large ω problem is treated in Sect. 3, using probabilistic estimates developed in [1], which, in Statistical Mechanics, would be called stability estimates. The infinite memory problem is treated in Sects. 4 and 5, which are inspired by the idea of "high temperature expansion" in Statistical Mechanics, namely writing the Gibbs measure or, here, the Girsanov factor, as sum of products of factors having a finite range memory and which become smaller as that range increases. However, in the situation considered here, carrying out this expansion requires a careful and non standard partition of the phase space (explained in Sect. 4). The problem is that, even though for typical noise, hence for typical ω's, the l-modes depend exponentially weakly on their past (see Sect. 2), thus producing, typically, "interactions" that decay exponentially fast, they may depend sensitively on their past when the noise is large. In the language of Statistical Mechanics, atypically large noise produces long range correlations.

This problem of sensitive dependence is coupled to the last problem, that of the convergence of the s-process with finite memory to a stationary state. We have to get lower bounds on transition probabilities and we can prove those (see Sect. 8) only when the s-modes remain for a sufficiently long time in a suitable region of the phase space; thus, if we did not control the sensitive dependence, we would not be able to carry out that last step. Finally, in Sect. 7, we prove the bounds on our "high temperature" expansion and, in Sect. 6, we use that expansion to prove the theorem. Note that, because we deal with a stochastic process, we never have to "exponentiate" our expansion, unlike what one would usually has to do in Statistical Mechanics (i.e., the analogue of the partition function here equals 1). The choice of κ in (7) is explained in Remark 2 of Sect. 4.

2. Finite Dimensional Reduction

We will use an idea of [7] to reduce the problem of the study of a Markov process with infinite dimensional state space to that of a non-Markovian process with finite dimensional state space.

For this purpose, write Eq. (4) for the small and large components of ω separately:

$$ds(t) = PF(s(t) + l(t))dt + db(t), \tag{9}$$

$$\frac{d}{dt}l(t) = (1 - P)F(s(t) + l(t)). \tag{10}$$

The idea of [7] is to solve the l equation for a given function s, thereby defining $l(t)$ as a function of the entire history of $s(t')$, $t' \leq t$. Then the s equation will have a drift with memory. Let us fix some notation. For a time interval I we denote the restriction of ω (or s, l respectively) to I by $\omega(I)$, and use the boldface notation $\mathbf{s}(I)$, to constrast it with $s(t)$, the value of s at a single time. $\|\cdot\|$ will denote the L^2 norm. In [1] it was proven that, for any $\tau < \infty$, there exists a set $\mathcal{B}_\tau$ of Brownian paths $b \in C([0, \tau], H_s)$ of full measure such that, for $b \in \mathcal{B}_\tau$, (4) has a unique solution with $\|\omega(t)\| < \infty$, $\|\nabla\omega(t)\| < \infty$ for all t (actually, $\omega(t)$ is real analytic). In particular, the projections s and l of this solution are in $C([0, \tau], H_{s(l)})$ respectively.

On the other hand, let us denote, given any $\mathbf{s} \in C([0, \tau], H_s)$, the solution - whose existence we will prove below – of (10), with initial condition $l(0)$ by $l(t, \mathbf{s}([0, t]), l(0))$. More generally, given initial data $l(t')$ at time $t' < \tau$ and $\mathbf{s}([t', \tau])$, the solution of (10) is denoted, for $\sigma \leq \tau$, by $l(\sigma, \mathbf{s}([t', \sigma]), l(t'))$ and the corresponding ω by $\omega(\sigma, \mathbf{s}([t', \sigma]), l(t'))$. The existence and key properties of those functions are given by:

Proposition 1. *Let $l(0) \in H_l$ and $s \in C([0, \tau], H_s)$. Then $l(\cdot, \mathbf{s}([0, t]), l(0)) \in C([0, \tau], H_l) \cap L^2([0, \tau], H_l^1)$, where $H_l^1 = H_l \cap H^1$, and H^1 is the first Sobolev space. In particular,*

$$\sup_{t\in[0,\tau]} \|l(t, \mathbf{s}([0, t]), l(0))\| \leq C(R, \sup_{t\in[0,\tau]} \|s(t)\|, \|l(0)\|), \tag{11}$$

where the notation $C(R, \sup_{t\in[0,\tau]} \|s(t)\|, \|l(0)\|)$ is defined in Remark 2, Sect. 1. Moreover, given two initial conditions l_1, l_2 and $t \leq \tau$,

$$\|l(t, \mathbf{s}([0, t]), l_1) - l(t, \mathbf{s}([0, t]), l_2)\| \leq \exp\left[-\kappa Rt + a\int_0^t \|\nabla\omega_1\|^2\right] \|l_1 - l_2\|, \tag{12}$$

where $a = (2\pi)^{-2} \sum |k|^{-4}$ and $\omega_1(t) = s(t) + l_1(t, \mathbf{s}([0, t]), l_1)$. The solution satisfies

$$l(t, \mathbf{s}([0, t]), l(0)) = l(t, \mathbf{s}([\tau, t]), l(\tau, \mathbf{s}([0, \tau]), l(0))). \tag{13}$$

Proof. The existence of l follows from standard a priori estimates which we recall for completeness. We have from (10) (see also (3)), for sufficiently smooth l,

$$\frac{1}{2}\frac{d}{dt}\|l\|^2 = -\|\nabla l\|^2 + (l, u \cdot \nabla s)$$

since, by incompressibility, $\nabla \cdot u = 0$, $(l, u \cdot \nabla l) = \frac{1}{2}\int \nabla \cdot (ul^2) = 0$. Use now the bound, for the functions d, v, b,

$$|(d, v \cdot \nabla b)| \leq \|d\|\|v\|_\infty\|\nabla b\| \leq \sqrt{a}\|d\|\|\Delta v\|\|\nabla b\|, \tag{14}$$

which follows from $\|v\|_\infty \leq (2\pi)^{-1}\sum_k \frac{|v(k)|k^2}{k^2}$ and Schwarz' inequality, and where $a = (2\pi)^{-2}\sum |k|^{-4}$. Using (14), $\alpha\beta \leq \frac{1}{2}(\alpha^2 + \beta^2)$ and $\|\Delta u\| = \|\nabla(s + l)\|$, we get:

$$\begin{aligned} |(l, u \cdot \nabla s)| &\leq \sqrt{a}\|l\|(\|\nabla s\| + \|\nabla l\|)\|\nabla s\| \\ &\leq \frac{\sqrt{a}}{2}(\|l\|^2 + \|\nabla s\|^4) + \frac{1}{2}\|\nabla l\|^2 + \frac{a}{2}\|l\|^2\|\nabla s\|^2. \end{aligned}$$

Hence,

$$\frac{d}{dt}\|l\|^2 \leq -\|\nabla l\|^2 + (\sqrt{a} + a\|\nabla s\|^2)\|l\|^2 + \sqrt{a}\|\nabla s\|^4. \tag{15}$$

The bound (11) on $\|l(t)\|$ follows then, by Gronwall's inequality, from (15) and the finiteness of $\sup_t \|\nabla s\|^2$ and of $\sup_t \|\nabla s\|^4$ (which follow from the finiteness of $\sup_t \|s\|^2$, since s has only finitely many nonzero Fourier coefficients). Finally, the boundedness of $\int_0^\tau \|\nabla l\|^2$ follows from (15) by integration.

For the second claim, let $\delta l(t) = l(t, s, l_1) - l(t, s, l_2) \equiv l_1(t) - l_2(t)$, and define $u^l = (1 - P)u$. We have:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\delta l\|^2 &= -\|\nabla\delta l\|^2 + (\delta l, \delta u^l \cdot \nabla\omega_1 + u_1 \cdot \nabla\delta l + \delta u^l \cdot \nabla\delta l) \\ &= -\|\nabla\delta l\|^2 + (\delta l, \delta u^l \cdot \nabla\omega_1) \end{aligned} \tag{16}$$

using, as above, $(\delta l, u_1 \cdot \nabla\delta l) = 0 = (\delta l, \delta u^l \cdot \nabla\delta l)$, and defining $\omega_1 = s + l_1$. Now, estimate, using (14) and $\|\Delta\delta u^l\| = \|\nabla\delta l\|$,

$$|(\delta l, \delta u^l \cdot \nabla\omega_1)| \leq \sqrt{a}\|\delta l\|\|\nabla\delta l\|\|\nabla\omega_1\| \leq \frac{1}{2}(\|\nabla\delta l\|^2 + a\|\delta l\|^2\|\nabla\omega_1\|^2). \tag{17}$$

So, by (7) and the fact that $l_k \neq 0$ only for $k^2 > N$,

$$\frac{d}{dt}\|\delta l\|^2 \leq -\kappa R\|\delta l\|^2 + a\|\delta l\|^2\|\nabla\omega_1\|^2, \tag{18}$$

which implies the claim (12) using Gronwall's inequality. The last claim (13) is obvious. □

Now, if $s = P\omega$ with ω as above being the solution of (4) with noise $b \in \mathcal{B}_\tau$ then the $l(s)$ constructed in the proposition equals $(1 - P)\omega$ and the stochastic process $s(t)$ satisfies the reduced equation

$$ds(t) = f(t)dt + db(t) \tag{19}$$

with

$$f(t) = PF(\omega(t)), \tag{20}$$

where $\omega(t)$ is the function on $C([0, t], H_s) \times H_l$ given by

$$\omega(t) = s(t) + l(t, \mathbf{s}([0, t]), l(0)). \tag{21}$$

Equation (19) has almost surely bounded paths and we have a Girsanov representation for the transition probability of the ω-process in terms of the s-variables

$$P^t(\omega(0), F) = \int \mu^t_{\omega(0)}(d\mathbf{s})F(\omega(t)) \tag{22}$$

with

$$\mu^t_{\omega(0)}(d\mathbf{s}) = e^{\int_0^t (f(\tau),\gamma^{-1}(ds(\tau)-\frac{1}{2}f(\tau)d\tau))}\nu^t_{s(0)}(d\mathbf{s}), \tag{23}$$

where $\nu^t_{s(0)}$ is the Wiener measure with covariance γ on paths $\mathbf{s} = \mathbf{s}([0, t])$ with starting point $s(0)$ and $(\cdot, \cdot)$ the ℓ^2 scalar product. We define the operator γ^{-1} in terms of its action on the Fourier coefficients:

$$(f, \gamma^{-1} f) = \sum_{|k|^2 \le N} |f_k|^2 \gamma_k^{-1}. \tag{24}$$

The Girsanov representation (22) is convenient since the problem of a stochastic PDE has been reduced to that of a stochastic process with finite dimensional state space. The drawback is that this process has infinite memory. In Sects. 4 and 5 we present a formalism, borrowed from statistical mechanics, that allows us to approximate it by a process with finite memory; the approximation will be controlled in Sect. 7, while the finite memory process will be studied in Sect. 8. This analysis is mostly done in the s-picture, but an important ingredient in it will be some a priori estimates on the transition probabilities of the original Markov process generated by (4) that we prove in the next section.

3. A Priori Estimates on the Transition Probabilities

The memory in the process (19) is coming from the dependence of the solution of (10) on its initial conditions. By Proposition 1, the dependence is weak if $\int_0^t \|\nabla\omega\|^2$ is less than cR for a suitable c. We localize the time intervals where this condition holds by inserting a suitable partition of unity in the expression (22). We shall show (in Sect. 8 below) that, during such time intervals, the s process behaves qualitatively like an ergodic Markov process. In this section we show that the complementary time intervals occur with small probability.

Let us first explain the partition of unity. We define, for each unit interval $[n-1, n] \equiv \mathbf{n}$, a quantity measuring the size of ω on that interval by:

$$D_n = \frac{1}{2} \sup_{t \in \mathbf{n}} \|\omega(t)\|^2 + \int_{\mathbf{n}} \|\nabla \omega(t)\|^2 dt. \tag{25}$$

Let $\{\phi_k\}_{k \in \mathbf{N}}$ be a smooth partition of unity for $\mathbf{R}^+$, with the support of ϕ_k contained in $[2^k R, 2^{k+2} R]$ for $k > 0$, and in $[0, 4R]$ for $k = 0$. Set, for $\mathbf{k} \in \mathbf{N}^t$,

$$\chi_{\mathbf{k}}(\omega) = \prod_n \phi_{k_n}(D_n(\omega)). \tag{26}$$

We insert $1 = \sum_{\mathbf{k}} \chi_{\mathbf{k}}$ in (23), to get

$$\mu^t_{\omega(0)}(d\mathbf{s}) = \sum_{\mathbf{k}} \chi_{\mathbf{k}} \mu^t_{\omega(0)}(d\mathbf{s}). \tag{27}$$

The following proposition bounds the probability of the unlikely event that we are interested in:

Proposition 2. *There exist constants $c > 0$, $c' < \infty$, $\beta_0 < \infty$, such that for all t, t', $1 \le t < t'$ and all $\beta \ge \beta_0$,*

$$P\left(\sum_{n=t}^{t'-1} D_n(\omega) \ge \beta R |t' - t| \middle| \omega(0) \right) \le \exp\left(\tfrac{1}{R} c' e^{-t} \|\omega(0)\|^2 \right) \exp(-c\beta |t' - t|). \tag{28}$$

In order to prove Proposition 2, we need some lemmas. We will start with a probabilistic analogue of the so-called enstrophy balance:

Lemma 3.1. *For all $\omega(0) \in L^2$, and all $t \ge 0$,*

$$E\left[e^{\frac{1}{4R} \|\omega(t)\|^2} \;\middle|\; \omega(0) \right] \le 3 e^{\frac{1}{4R} e^{-t} \|\omega(0)\|^2}, \tag{29}$$

and

$$P\left(\|\omega(t)\|^2 \ge D | \omega(0) \right) \le 3 e^{-\frac{D}{4R}} e^{\frac{1}{4R} e^{-t} \|\omega(0)\|^2}. \tag{30}$$

Remark. This lemma shows that the distribution of $\|\omega(t)\|^2$ satisfies an exponential bound on scale R with a prefactor whose dependence on the initial condition decays exponentially in time. Thus, if $\|\omega(0)\|^2$ is of order D, $\|\omega(t)\|^2$ will be, with large probability, of order R after a time of order $\log D$.

Proof. Let $x(\tau) = \lambda(\tau) \|\omega(\tau)\|^2 = \lambda(\tau) \sum_k |\omega_k|^2$ for $0 \le \tau \le t$. Then by Ito's formula (remember that, by (6), $\sum_k \gamma_k = R$ and thus $\gamma_k \le R$, $\forall k$):

$$\begin{aligned} \frac{d}{d\tau} E[e^x] &= E\left[\left(\dot{\lambda} \lambda^{-1} x - 2\lambda \sum_k k^2 |\omega_k|^2 + \lambda \sum_k \gamma_k + 2\lambda^2 \sum_k \gamma_k |\omega_k|^2 \right) e^x \right] \\ &\le E\left[((\dot{\lambda} \lambda^{-1} - 2 + 2\lambda R) x + \lambda R) e^x \right], \end{aligned} \tag{31}$$

where E denotes the conditional expectation, given $\omega(0)$, and where we used the Navier-Stokes equation (3), $|k| \ge 1$ for $\omega_k \ne 0$, and the fact that the nonlinear term does not contribute (using integration by parts and $\nabla \cdot u = 0$). Take now $\lambda(\tau) = \frac{1}{4R} e^{(\tau - t)}$ so that $\lambda \leq \frac{1}{4R}$, $\dot{\lambda}\lambda^{-1} = 1$, $\dot{\lambda}\lambda^{-1} - 2 + 2\lambda R \leq -\frac{1}{2}$ and $\lambda R \leq \frac{1}{4}$. So,

$$\frac{d}{d\tau} E[e^x] \leq E\left[\left(\tfrac{1}{4} - \tfrac{1}{2}x\right) e^x\right] \leq \tfrac{1}{2} - \tfrac{1}{4} E[e^x],$$

where the last inequality follows by using $(1 - 2x)e^x \leq 2 - e^x$. Thus, Gronwall's inequality implies that:

$$E\left[e^{x(\tau)}\right] \leq e^{-\frac{\tau}{4}} e^{x(0)} + 2 \leq 3e^{x(0)},$$

i.e., using the definition of $\lambda(\tau)$,

$$E\left[\exp\left(\frac{e^{\tau - t}}{4R} \|\omega(\tau)\|^2\right)\right] \leq 3 \exp\left(\frac{e^{-t}\|\omega(0)\|^2}{4R}\right),$$

This proves (29) by putting $\tau = t$; (30) follows from (29) by Chebychev's inequality. $\square$

Since the D_n in (28) is the supremum over unit time intervals of

$$D_t(\omega) = \frac{1}{2}\|\omega(t)\|^2 + \int_{n-1}^{t} \|\nabla\omega\|^2 d\tau \quad n - 1 \leq t \leq n, \tag{32}$$

which does not involve only $\|\omega(t)\|^2$, we need to control also the evolution of $D_t(\omega)$ over a unit time interval, taken, for now, to be [0, 1]. From the Navier-Stokes equation (3) and Ito's formula, we obtain

$$D_t(\omega) = D_0(\omega) + Rt + \int_0^t (\omega, db) \tag{33}$$

(since the nonlinear term does not contribute, as in (31)). Our basic estimate is:

Lemma 3.2. *There exist $C < \infty$, $c > 0$ such that, $\forall A \geq 3D_0(\omega)$*

$$P\left(\sup_{t\in[0,1]} D_t(\omega) \geq A|\omega(0)\right) \leq Ce^{-\frac{cA}{R}}. \tag{34}$$

Remark. While the previous lemma showed that $\|\omega(t)\|^2$ tends to decrease as long as it is larger than $\mathcal{O}(R)$, this lemma shows that, in a unit interval, $D_t(\omega)$ does not increase too much relative to $D_0(\omega) = \frac{1}{2}\|\omega(0)\|^2$. Thus, by combining these two lemmas, we see that $D_n(\omega) = \sup_{t\in[n-1,n]} D_t(\omega)$ is, with large probability, less than $\|\omega(0)\|^2$, when the latter is larger than $\mathcal{O}(R)$, at least for $n \geq n_0$ not too small. This is the content of Lemma 3.3 below. Thus, it is unlikely that $D_n(\omega)$ remains much larger than R over some interval of (integer) times, and this fact will be the basis of the proof of Proposition 2.

Proof. From (33), we get that

$$P\left(\sup_{t\in[0,1]} D_t(\omega) \geq A\middle|\omega(0)\right) \leq P\left(\sup_{t\in[0,1]} \left|\int_0^t (\omega, db)\right| \geq (A - D_0 - R)\middle|\omega(0)\right). \tag{35}$$

The process $t \to \int_0^t (\omega, db)$ is a continuous martingale so, by Doob's inequality (see e.g. [9], p.24), the submartingale $x_t \equiv |\int_0^t (\omega, db)|$ satisfies the bounds

$$E((\sup_t x_t)^p) \le \left(\frac{p}{p-1}\right)^p E\left(x_1^p\right) \quad \forall p \ge 2, \tag{36}$$

where E denotes the conditional expectation, given $\omega(0)$. These imply

$$E(e^{\varepsilon \sup x_t}) \le 5E(e^{\varepsilon x_1}), \tag{37}$$

where ε will be chosen small below (to derive (37), expand both exponentials, use (36) and $\left(\frac{p}{p-1}\right)^p \le 4$ for $p \ge 2$; for $p = 1$, use $Ea \le \frac{1}{2}(\alpha + \alpha^{-1} Ea^2)$ for $a \ge 0$ and take $\alpha = 2$). Since

$$E(e^{\varepsilon x_1}) \le \frac{1}{2}\left(E\left(e^{\varepsilon \int_0^1 (\omega, db)}\right) + E\left(e^{-\varepsilon \int_0^1 (\omega, db)}\right)\right), \tag{38}$$

using Novikov's bound, we get

$$\begin{aligned} E\left(e^{\pm\varepsilon \int_0^1 (\omega, db)}\right) &\le \left(E\left(e^{2\varepsilon^2 \int_0^1 d\tau (\omega(\tau), \gamma\omega(\tau))}\right)\right)^{1/2} \\ &\le \left(\int_0^1 d\tau E\left(e^{2\varepsilon^2 (\omega(\tau), \gamma\omega(\tau))}\right)\right)^{1/2} \\ &\le \left(\int_0^1 d\tau E\left(e^{2\varepsilon^2 R \|\omega(\tau)\|^2}\right)\right)^{1/2}, \end{aligned} \tag{39}$$

where the last two inequalities follow from Jensen's inequality, applied to $e^{2\varepsilon^2 \int_0^1 d\tau (\omega(\tau), \gamma\omega(\tau))}$, and from $\gamma_k \le R$ (see (6)).

So, altogether, we have, by Chebychev's inequality and (37–39):

$$\begin{aligned} &P\left(\sup_{t\in[0,1]} \left|\int_0^t (\omega, db)\right| \ge (A - D_0 - R) \middle| \omega(0)\right) \\ &\le 5e^{-\varepsilon(A - D_0 - R)} \left(\int_0^1 d\tau E\left(e^{2\varepsilon^2 R \|\omega(\tau)\|^2}\right)\right)^{1/2}. \end{aligned} \tag{40}$$

Now, combine this with (35) and (29) in Lemma 3.1 above, choosing $2\varepsilon^2 R = \frac{1}{4R}$, i.e. $\varepsilon = \frac{1}{\sqrt{8}R}$, to get

$$P\left(\sup_{t\in[0,1]} D_t(\omega) \ge A|\omega(0)\right) \le 15e^{-\varepsilon(A - D_0 - R)} e^{D_0/4R} \tag{41}$$

which yields (34) for $A \ge 3D_0(\omega)$ and $C = 15e^{\frac{1}{\sqrt{8}}}$ and $c = \frac{1}{3}\left(\frac{2}{\sqrt{8}} - \frac{1}{4}\right)$. □

Let A_k be, for $k > 0$, the interval $[2^k R, 2^{k+1} R]$ and let $A_0 = [0, 2R]$. Given an integer n_0 define, for $k, k' \geq 0$,

$$P(k|k') = \sup_{\omega'(0)} P(D_{n_0}(\omega)) \in A_k|\omega'(0)) \equiv \sup_{\omega'(0)} P(k|\omega'(0)), \tag{42}$$

where the supremum is taken over $\omega'(0)$ such that $\|\omega'(0)\|^2 \leq 2^{k'+1}R$ (the intervals labelled by k will play a role similar to the k's introduced in (26), but, since we do not need a smooth partition of unity here, we use a more conventional partition). Observe that we have $\forall k, k' \geq 0$,

$$P(k|k') \leq 1. \tag{43}$$

The main ingredient in the proof of Proposition 2 is

Lemma 3.3. *There exist constants $c > 0$, $C < \infty$ such that*

$$P(k|k') \leq C \exp\big(-c2^k\big) \exp(e^{-(n_0-1)}2^{k'-1}). \tag{44}$$

Proof. We split

$$\begin{aligned} P(k|\omega'(0)) = E\Big(1_{A_k}(D_{n_0}(\omega))1\left(\|\omega(n_0-1)\|^2 > \tfrac{2}{3}2^k R\right)|\omega'(0)\Big) \\ +E\Big(1_{A_k}(D_{n_0}(\omega))1\left(\|\omega(n_0-1)\|^2 \leq \tfrac{2}{3}2^k R\right)|\omega'(0)\Big), \end{aligned}$$

where 1_{A_k} is the indicator function of the interval A_k, and $1(X)$ is the indicator function of the event X. Hence, we may bound

$$\begin{aligned} P(k|k') \leq \sup P\Big(\|\omega(n_0-1)\|^2 > \tfrac{2}{3}2^k R|\omega'(0)\Big) \\ + \sup E\Big(1_{A_k}(D_{n_0}(\omega))|\omega(n_0-1)\Big), \end{aligned} \tag{45}$$

where the supremum in the first term is taken over $\omega'(0)$ such that $\|\omega'(0)\|^2 \leq 2^{k'+1}R$ and, in the second term, over $\omega(n_0-1)$ such that $\|\omega(n_0-1)\|^2 \leq \frac{2}{3}2^k R$.

Using Lemma 3.1, we bound the first term of (45) :

$$P\Big(\|\omega(n_0-1)\|^2 > \tfrac{2}{3}2^k R\big|\omega'(0)\Big) \leq 3\exp\left(-\tfrac{2^k}{6}\right)\exp\left(e^{-(n_0-1)}2^{k'-1}\right). \tag{46}$$

And, using Lemma 3.2, and the fact that the support of 1_{A_k} is in $[2^k R, 2^{k+1} R]$ for $k > 0$, we bound the second term of (45), for $k > 0$, by

$$\begin{aligned} E\left(1_{A_k}\left(D_{n_0}(\omega)\right)\big|\omega(n_0-1)\right) &\leq P\left(\sup_{t\in[n_0-1,n_0]} D_t(\omega) \geq 2^k R\big|\omega(n_0-1)\right) \\ &\leq C\exp(-c2^k), \end{aligned} \tag{47}$$

since $\omega(n_0-1)$ is such that $2^k R \geq \frac{3}{2}\|\omega(n_0-1)\|^2 = 3D_0(\omega)$. For $k = 0$, (47) obviously holds also. This proves (44). □

Proof of Proposition 2. By Lemma 3.3, we may find n_0 so that $\exists\, c > 0$, $C < \infty$ such that

$$P(k|k') \leq C \exp(-c2^k) \quad \text{for } k \geq k'. \tag{48}$$

Let us fix such n_0. Let $\mathcal{D}$ be the sum of D_n in (28) and $\mathcal{D}_\tau$ the same sum with n restricted to the lattice $n_0\mathbf{Z} + \tau$. We can write:

$$P\left(\mathcal{D} \geq \beta R|t' - t|\Big|\omega(0)\right) \leq \sum_{\tau=0}^{n_0-1} P\left(\mathcal{D}_\tau \geq \frac{\beta R|t' - t|}{n_0}\Big|\omega(0)\right).$$

So, since $|t' - t| \geq 1$, by changing the values of c, and β_0 in (28), it suffices to prove (28) for $\mathcal{D}$ replaced by $\mathcal{D}_\tau$, $\tau = 0, \ldots, n_0 - 1$; and, since all the terms are similar, we shall consider only $\tau = 0$. Finally, by redefining t, t', it is enough to bound by the RHS of (28) the probability of the event

$$\sum_{n=t}^{t'-1} D_{nn_0}(\omega) \geq \beta R|t' - t|.$$

Using the Markov property, the definition (42) of $P(k|k')$, and the fact that $D_{nn_0} \in A_k$ means that $D_{nn_0} \leq 2^{k_{nn_0}+1}R$, we see that it suffices (changing again c and β_0) to prove the estimate (28) for the expression

$$\sum_{\{k_{nn_0}\}} 1\left(\sum_{n=t}^{t'-1} 2^{k_{nn_0}} \geq \beta|t - t'|\right) \prod_{n=t}^{t'-2} P(k_{(n+1)n_0}|k_{nn_0}) P(k_{tn_0}|\omega(0)). \tag{49}$$

We bound (49), using Chebychev's inequality, by

$$\begin{aligned}(49) \leq \exp(-\varepsilon\beta|t' - t|) &\sum_{\{k_{nn_0}\}} \exp\left(\varepsilon \sum_{n=t}^{t'-1} 2^{k_{nn_0}}\right) \\ &\times \prod_{n=t}^{t'-2} P\left(k_{(n+1)n_0}|k_{nn_0}\right) P\left(k_{tn_0}|\omega(0)\right),\end{aligned} \tag{50}$$

where ε will be chosen small below.

Consider now $\sum_k \exp(\varepsilon 2^k) P(k|k')$. Splitting this sum into $\sum_{0 \leq k \leq k'-1}$ and $\sum_{k \geq k'}$ and using (43) for the first sum and (48) for the second, we get:

$$\sum_k \exp(\varepsilon 2^k) P(k|k') \leq k' \exp(\varepsilon 2^{k'-1}) + e^a, \tag{51}$$

where $e^a \equiv C \sum_{k=0}^{\infty} \exp((\varepsilon - c)2^k)$ is bounded as long as (say) $\varepsilon \leq c/2$. Moreover, we can bound $k' \exp(\varepsilon 2^{k'-1}) + e^a \leq e^{c_1} \exp\left(\varepsilon 2^{-\frac{1}{2}} 2^{k'}\right)$. Altogether, we have:

$$\sum_k \exp(\varepsilon 2^k) P(k|k') \leq e^{c_1} \exp\left(\varepsilon 2^{-\frac{1}{2}} 2^{k'}\right). \tag{52}$$

Let us apply this first to the sum over $k_{(t'-1)n_0}$, then $k_{(t'-2)n_0}$ and so on. The result of (52) is that, apart from the prefactor e^{c_1}, we obtain, when we sum over $k_{(t'-2)n_0}$, the same summand as in the first sum, but with ε replaced by $\varepsilon + \varepsilon 2^{-\frac{1}{2}}$. And, after m steps we have ε replaced by $\varepsilon \sum_{l=0}^{m} 2^{-\frac{1}{2}l}$. Thus, we can use this inductively on $P\left(k_{(n+1)n_0}|k_{nn_0}\right)$ for all n, with $t \leq n \leq t'-2$, as long as $\varepsilon \sum_{l=0}^{\infty} 2^{-\frac{1}{2}l} = \varepsilon \left(\frac{1}{1-2^{-\frac{1}{2}}}\right) \leq c/2$, which holds for ε small enough. Thus, we obtain, $\forall t' > t$, a bound for the sum in (50)

$$e^{c_1|t'-1-t|} \sum_{k_{tn_0}} \exp(c_2 \varepsilon 2^{k_{tn_0}}) P(k_{tn_0}|\omega(0)) \tag{53}$$

with $c_2 = \frac{1}{1-2^{-\frac{1}{2}}}$. Observe that, using (42) and (44), with n_0 replaced by t and k' being the smallest k such that $\|\omega(0)\|^2 \leq 2^{k+1}R$, we may bound

$$P(k_{tn_0}|\omega(0)) \leq C \exp(-c 2^{k_{tn_0}}) \exp(e e^{-t} 2^{k'-1}).$$

Then the sum over k_{tn_0} in (53) can be bounded, since $\sum_{k_{tn_0}} \exp((c_2\varepsilon - c)2^{k_{tn_0}}) \leq C$ for ε small, and we get:

$$\sum_{k_{tn_0}} \exp(c_2 \varepsilon 2^{k_{tn_0}}) P(k_{tn_0}|\omega(0)) \leq C \exp(e e^{-t} 2^{k'-1}).$$

Moreover, we have, by definition of k', $2^{k'} \leq c \frac{\|\omega(0)\|^2}{R}$. Thus, we obtain the bound (28) for (49), for β_0 large enough (e.g. take $\frac{1}{2}\varepsilon\beta \geq \frac{1}{2}\varepsilon\beta_0 \geq c_1 + \log C$, use $|t'-t| \geq 1$, and, in (28), take $c = \frac{\varepsilon}{2}$), by combining these inequalities with (50) and (53). □

4. Partition of the Path Space

Consider the expression (27) for the measure μ. Given $\mathbf{k}$, we will now decompose the time axis into regions where Eq. (10) may have sensitive dependence on initial conditions and the complement of those regions. Motivated by Proposition 2, let us consider, for time intervals L, the expressions

$$\gamma_L = \sum_{\mathbf{n} \subset L} 2^{k_n}. \tag{54}$$

Let T be a number to be fixed later (in Sects. 6–8), depending on ρ, the minimum of the noise covariance. Define

$$\beta(L) = \begin{cases} \beta|L| & \text{if } |L| > \frac{1}{2}T \\ \frac{1}{2}\beta T & \text{if } |L| \leq \frac{1}{2}T \end{cases}. \tag{55}$$

β is a constant to be fixed later (see Remark 2 below). Call the time intervals with end points on the lattice $T\mathbf{Z}$ T-intervals, and, for an interval $L = [m, n]$, let $\bar{L}$ be the smallest T-interval containing $[m, n]$. Consider the set $\mathcal{L}$ of intervals L such that, either

$$\gamma_L > \beta(L), \tag{56}$$

or $L = [(n-1)T, nT]$, so that

$$2^{k_{nT}} > \beta' T, \tag{57}$$

where $\beta' < \beta$ is a constant also to be fixed later (see Remark 2 below). Let $\bar{\mathcal{L}}$ be the union of all $\bar{L}$ with $L \in \mathcal{L}$. We call the connected components of $\bar{\mathcal{L}}$ *large* intervals and the T-intervals of length T in its complement *small* intervals. Note that intervals of length T can be either small or large (those of length at least $2T$ are always large). Hence, we introduce *labels* small/large on those intervals. By construction, two large intervals are always separated by at least one small one.

Remark 1. "Large" and "small" refer to $\omega(J)$ being large or small, not to the size of the interval. We use this slightly misleading terminology for the sake of brevity. γ_L are the natural random variables entering in the sensitive dependence estimate (12) and whose probability distribution was studied in Proposition 2. Since the estimate (28) involves the initial condition at the beginning of the time interval we consider and, since this initial condition is the size of ω at the end of a time interval where (56) is violated, we need to be sure it does not dominate the bound (28). For that reason, we include in our set of unlikely events also the ones defined by (57).

Remark 2. The three constants in our construction, κ, β, β' entering (7), (56) and (57) are fixed as follows: $\beta' \geq \beta_0'$, $\beta \geq \beta(\beta')$ and $\kappa \geq \kappa(\beta)$.

Remark 3. The virtues of this partition of phase space can be seen in Lemma 4.1 and 7.4 below. The bound (59) and Proposition 2 will imply that large intervals are unprobable. On the other hand, (58) and (139) will allow us to show that the argument of the exponential in (12) is less than $-cR|J|$, when the interval $[0, t]$ is replaced by an interval J strictly including one of the intervals constructed here. This property will be essential in order to obtain bounds on the terms of the expansion constructed in the next section.

Taken together, the small and large intervals form a partition $\pi(\mathbf{k}) = J_1, \ldots, J_N$ of the total time interval $[0, t]$. We arrange them in temporal order and write $J_i = [\tau_{i-1}, \tau_i]$ with $\tau_0 = 0$, $\tau_N = t$.

Our construction has the following properties:

Lemma 4.1. *Let $J = [\tau', \tau]$ be a T-interval $J \in \pi(\mathbf{k})$.*
(a) *If J is small, then*

$$\sum_{\mathbf{n} \subset J} 2^{k_n} \leq \beta T \text{ and } 2^{k_\tau} \leq \beta' T, \tag{58}$$

(b) *If J is large, then J may be written as a union $J' \cup J''$ so that*

$$\gamma_{J'} > \tfrac{1}{4}\beta|J'| \tag{59}$$

and J'' is a union of intervals $[(n-1)T, nT]$ satisfying (57).

Remark 4. At both ends of any interval, either large or small, we have $2^{k_n} \leq \beta T$ (otherwise the interval would be large, not small, or would not end there). Note that we have β here, not the smaller β' of (58). So, if ω is such that $D_n(\omega)$ is in the support of ϕ_{k_n}, we have:

$$\|\omega(\tau)\|^2 \leq 8\beta RT, \tag{60}$$

where τ is the endpoint of the interval.

Proof of Lemma 4.1. (a) A small interval cannot be an L for which (56) holds nor an interval $[(n-1)T, nT]$ satisfying (57); hence, (58) holds.

For (b), let J' be the union of the $\bar{L}$ in J with L such that (56) holds. We may cover J' by a subset $\bar{L}_i$, $i = 1, \dots p$, of these intervals, in such a way that $\bar{L}_i \cap \bar{L}_j = \emptyset$ for $|i-j| > 1$. From (56, 55), we deduce that $\gamma_{\bar{L}} > \frac{1}{2}\beta|\bar{L}|$ and then,

$$\gamma_{J'} \geq \frac{1}{2}\sum \gamma_{\bar{L}_j} \geq \frac{1}{4}\beta|J'|. \quad \square$$

In order to obtain the analogue of what in Statistical Mechanics is called the high temperature expansion, we need to write the sum in (27) as a sum of products of independent factors. As a first step in that direction, we would like to express the sum in (27) as a sum of partitions $\pi = (J_1, \dots, J_n)$ of $[0, t]$ into T-intervals and sums over $\mathbf{k}_i \in \mathbf{N}^{J_i}$. However, a moment's thought reveals that the sum over $\mathbf{k}$ creates correlations between the different $\mathbf{k}_i$. E.g. J_i being small is a very nonlocal condition in terms of $\mathbf{k}$: nowhere in the whole interval $[0, t]$ can there be a k_n large enough to create a $L \in \mathcal{L}$ that intersects J_i. Given an arbitrary T-interval J and $\mathbf{k} \in \mathbf{N}^J$, we may define, in the same way as we did above for $[0, t]$, the partition $\pi(\mathbf{k})$ of J into small and large intervals. In particular, $\pi(\mathbf{k}) = \{J\}$ means, if $|J| = T$, that $\mathbf{k}$ is such that J is small or large depending on the label on J and, if $|J| > T$, that $\mathbf{k}$ is such that J is large. Then, we have:

Lemma 4.2. *Let $\pi = \{J_1, \dots, J_N\}$ be a partition of $[0, t]$ into T-intervals and let $\mathbf{k}_i \in \mathbf{N}^{J_i}$ be given such that $\pi(\mathbf{k}_i) = \{J_i\}$. Let $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_N)$. Then $\pi(\mathbf{k}) = \cup_i \pi(\mathbf{k}_i)$ if and only if the $\mathbf{k}_i$ satisfy the constraints*

$$\forall L \subset J_i \cup J_{i+1} \text{ so that } L \cap J_i \neq \emptyset \neq L \cap J_{i+1} \ : \ \ \gamma_L \leq \beta(L) \tag{61}$$

for all $i = 1, \dots, N-1$.

Proof. Assume first that $\pi(\mathbf{k}) = \cup_i \pi(\mathbf{k}_i)$. Hence $\pi(\mathbf{k}) = \pi$ and by the definition of $\pi(\mathbf{k})$, every L such that $\gamma_L > \beta(L)$ is contained in some J_i. Thus, (61) holds.

For the converse, observe first that, by the definition of the partitions $\pi(\mathbf{k})$ and $\cup_i \pi(\mathbf{k}_i)$, their sets of small and large J's are entirely determined by the set of connected components of $\mathcal{L}$ given by $\mathbf{k}$ on $[0, t]$ for $\pi(\mathbf{k})$ and the set of connected components of $\mathcal{L}_i$ given by $\mathbf{k}_i$ on each J_i for $\cup_i \pi(\mathbf{k}_i)$. Thus it is enough to show that their connected components coincide. The intervals satisfying (57) obviously coincide. By definition of γ_L and of the large intervals, each connected component of $\mathcal{L}_i$ must be contained in a connected component of $\mathcal{L}$, since $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_N)$. Now, using (61), we show the converse, which will establish the claim. Let L be a connected component of $\mathcal{L}$. If $L \subset J_i$, then L is a connected component of $\mathcal{L}_i$. Thus, if the claim is not true, there must exist a connected component L of $\mathcal{L}$, such that L is not included in any J_i and such that $\gamma_L > \beta(L)$. By (61), L cannot be included in two adjacent J's either. Thus, there must be a connected L with $\gamma_L > \beta(L) = \beta|L|$ and J_i such that $L \cap J_{i-1} \neq \emptyset$, $L \cap J_{i+1} \neq \emptyset$. Then $L = L_1 \cup L_2$ with L_1, L_2 having the midpoint of J_i as a common boundary point. Hence, by (61), $\gamma_{L_i} \leq \beta(L_i) = \beta|L_i|$ since $|L_i| > \frac{1}{2}T$. Thus $\gamma_L = \gamma_{L_1} + \gamma_{L_2} \leq \beta|L| = \beta(L)$, which is a contradiction. $\square$

Consider now the sum (27). Let $\pi(\mathbf{k}) = \{J_1, \dots, J_N\}$. Define the Girsanov factor

$$g_{J_i}(\omega) = e^{\int_{J_i}(f(t), \gamma^{-1}(ds(t) - \frac{1}{2}f(t)dt))}, \tag{62}$$

where we recall that $f(t)$ and ω, given by (20) and (21), and thus g_{J_i}, depend on the whole past i.e. on $\mathbf{s}([0, \tau_i])$ and $l(0)$. Let $\mathbf{k}_i \in \mathbf{N}^{J_i}$ be the restriction of $\mathbf{k}$ to J_i, let us denote by $\chi_{\mathbf{k}_i}$ the corresponding product (26), and let

$$\mu_{\mathbf{k}_i}(d\mathbf{s}(J_i)) = \chi_{\mathbf{k}_i} g_{J_i} \nu^{|J_i|}_{s(\tau_{i-1})}(d\mathbf{s}(J_i)). \tag{63}$$

We can then write

$$\chi_{\mathbf{k}} \mu^t_{\omega(0)}(d\mathbf{s}) = \prod_{i=1}^{N} \mu_{\mathbf{k}_i}(d\mathbf{s}(J_i)). \tag{64}$$

Let π be a partition of $[0, t]$ into T-intervals with labels "small" or "large" on the ones of length T. Let us define, for such a labelled T-interval J, $1_J(\mathbf{k})$ to be the indicator function for the set of $\mathbf{k} \in \mathbf{N}^J$ such that $\pi(\mathbf{k}) = \{J\}$ (i.e. if $|J| = T$ 1_J is supported on $\mathbf{k}$ so that J is small or large depending on the label and if $|J| > T$ on $\mathbf{k}$ so that J is large). For two adjacent T-intervals J, J' let $1_{JJ'}(\mathbf{k}, \mathbf{k}')$ be the indicator function for the set of $(\mathbf{k}, \mathbf{k}') \in \mathbf{N}^J \times \mathbf{N}^{J'}$, such that $\gamma_L \leq \beta(L)$ for all $L \subset J \cup J'$ which intersect both J and J'. Using Lemma 4.2, we may then write Eq. (27) as

$$\mu^t_{\omega(0)}(d\mathbf{s}) = \sum_{\pi} \sum_{\mathbf{k}_1 \ldots \mathbf{k}_N} \prod_{i=1}^{N} 1_{J_i}(\mathbf{k}_i) \mu_{\mathbf{k}_i}(d\mathbf{s}(J_i)) \prod_{i=1}^{N-1} 1_{J_i J_{i+1}}(\mathbf{k}_i, \mathbf{k}_{i+1}). \tag{65}$$

Note that this expression has a Markovian structure in the sets J_i, but each $\mu_{\mathbf{k}_i}$ depends on the whole past history. In the next section, we shall decouple this dependence.

5. Decoupling

By decoupling we mean that we shall write $\mu^t_{\omega(0)}$ as a product of measures whose dependence on the past extends only over two adjacent intervals, and corrections. To achieve that, consider $\mu_{\mathbf{k}_i}$, for $i > 2$; remember that $[0, t]$ is partitioned into intervals $J_i = [\tau_{i-1}, \tau_i]$ with $\tau_0 = 0$, $\tau_N = t$. Fix $j < i$, and introduce the drift with memory on $[\tau_{j-1}, t]$:

$$f_j(t) = PF(\omega_j(t)), \tag{66}$$

where

$$\omega_j(t) = (s(t), l(t, \mathbf{s}([\tau_{j-1}, t]), 0))$$

is the solution of (9, 10), with initial condition $l(\tau_{j-1}) = 0$. We denote by g_{ij} the Girsanov factor $g_{J_i}(\omega_j)$ (given by (62), with $f(t)$ replaced by $f_j(t)$). Note that it depends only on the history $\mathbf{s}([\tau_{j-1}, \tau_i])$.

Since the characteristic function $\chi_{\mathbf{k}_i}$ also depends on the past through the ω dependence of (26), we need to decouple this too. We let

$$\chi_{\mathbf{k}_i j} = \prod_{\mathbf{n} \subset J_i} \phi_{k_n}(D_n(\omega_j)). \tag{67}$$

We can now define the decoupled measure for $j = 2, \ldots, i-1$:

$$\mu_{\mathbf{k}_i j}(d\mathbf{s}(J_i) | \mathbf{s}([\tau_{j-1}, \tau_{i-1}])) = \chi_{\mathbf{k}_i j} g_{ij} \nu^{|J_i|}_{s(\tau_{i-1})}(d\mathbf{s}(J_i)); \tag{68}$$

this measure is defined on the paths on the time interval J_i and depends on the past up to and including the interval J_j. To connect to (64), we write, for $i \geq 3$, a telescopic sum

$$\mu_{\mathbf{k}_i} = \mu_{\mathbf{k}_i i-1} + \sum_{j=1}^{i-2} (\mu_{\mathbf{k}_i j} - \mu_{\mathbf{k}_i j+1}) \equiv \sum_{j=1}^{i-1} \mu_{\mathbf{k}_i, j} \ , \tag{69}$$

where by definition $\mu_{\mathbf{k}_i 1} = \mu_{\mathbf{k}_i}$; note that this term is the only one depending on $l(0)$. For $i = 1, 2$ we will set by convention $j_i = i - 1$, $\mathbf{s}([\tau_{-1}, \tau_0]) = \omega(0)$, and define $\mu_{\mathbf{k}_i j_i} = \mu_{\mathbf{k}_i}$. Inserting (69) into (65), we get

$$\mu^t_{\omega(0)}(d\mathbf{s}) = \sum_\pi \sum_{\mathbf{k}_1 \ldots \mathbf{k}_N} \sum_{\mathbf{j}} \prod_{i=1}^N 1_{J_i}(\mathbf{k}_i) \mu_{\mathbf{k}_i, j_i}(d\mathbf{s}(J_i) | \mathbf{s}([\tau_{j_i - 1}, \tau_{i-1}]))$$
$$\times \prod_{i=1}^{N-1} 1_{J_i J_{i+1}}(\mathbf{k}_i, \mathbf{k}_{i+1}). \tag{70}$$

One should realize that the leading term in the sum (70) is the one with all $j_i = i - 1$ and $\mathbf{k}$ such that the partition $\pi(\mathbf{k})$ consists of only small intervals. Indeed, $\mu_{\mathbf{k}_i, j_i}$ with $j_i \neq i - 1$ describes the change of $\mu_{\mathbf{k}_i}$ under variation in the distant past. This will be shown to be small as a consequence of Proposition 1 and Lemma 4.1. On the other hand, the occurrence of large intervals will be shown to have a small probability, using Proposition 2.

Therefore, we will group all these small terms as follows. Consider the set

$$L' = \bigcup_{j_i < i-1} [\tau_{j_i - 1}, \tau_i] \bigcup_{J_i \text{ large}} (J_i \cup J_{i+1}), \tag{71}$$

where we have grouped the terms mentioned above, and also included the small intervals following the large ones for later convenience. Since our initial condition $\omega(0)$ is arbitrary, it is convenient to include also the intervals J_1, J_2, and to let

$$L = J_1 \cup J_2 \cup L'. \tag{72}$$

Let $K_1, \ldots, K_N$ be the partition of $[0, t]$ into T-intervals, in chronological order, where the K_l's, which are unions of intervals J_i, are given by the connected components of L and by the small intervals $J_i \subset L^c$. In the first case, $|K_l| \geq 2T$, since we always attach to a large interval J_i the interval J_{i+1}, see (72, 71); in the second case, $|K_l| = T$.

Fix now $K = [\tau_0, \tau]$, a T-interval and let $J_0 = [\tau_0 - T, \tau_0]$ if $0 \notin K$. Let $\mathbf{k}_+, \mathbf{k}_0 \in \mathbf{N}^T$ and $\mathbf{s}(J_0) \in C(J_0, H_s)$. We define

$$\mu_K(d\mathbf{s}(K), \mathbf{k}_+ | \mathbf{s}(J_0), \mathbf{k}_0) = \sum_\pi \sum_{\mathbf{k}_1 \ldots \mathbf{k}_{N-1}} \sum_{\mathbf{j}} \prod_{i=1}^N 1_{J_i}(\mathbf{k}_i) \mu_{\mathbf{k}_i, j_i}(d\mathbf{s}(J_i) | \mathbf{s}([\tau_{j_i - 1}, \tau_{i-1}]))$$
$$\times \prod_{i=0}^{N-1} 1_{J_i J_{i+1}}(\mathbf{k}_i, \mathbf{k}_{i+1}), \tag{73}$$

where $\mathbf{k}_N = \mathbf{k}_+$, for $i = 1$, $\mathbf{s}([\tau_{j_i - 1}, \tau_{i-1}]$ is replaced by $\mathbf{s}(J_0)$, and the sum is over π and $\mathbf{j}$ so that K equals L' of (71) if $0 \notin K$, or L of (72) if $0 \in K$. In the latter case, we replace $\mathbf{s}(J_0)$, $\mathbf{k}_0$ by $\omega(0)$ and the last product starts at $i = 1$. Note that, because of the presence of J_{i+1} in (71), the last interval in K is small. With this definition, we can then rewrite (70) as:

$$\mu^t_{\omega(0)}(d\mathbf{s}) = \sum_{\pi} \sum_{\mathbf{k}_1 \ldots \mathbf{k}_M} \prod_{i=1}^{M} \mu_{K_i}(d\mathbf{s}(K_i), \mathbf{k}_i | \mathbf{s}(J_{i-1}), \mathbf{k}_{i-1}), \tag{74}$$

where the sum is over partitions $\pi = (K_1, \ldots, K_M)$ of $[0, t]$ into T-intervals $K_i = [\tau_{i-1}, \tau_i]$ so that $|K_1| \geq 2T$ (because we included J_1, J_2 into K_1, see (72)), and for $i = 1$, $\mathbf{s}(J_{i-1}), \mathbf{k}_{i-1}$ is replaced by ω_0. Note that all the K_l's so that $|K_l| = T$ are small intervals, and, in that case, K_l coincides with an interval $J_i = [\tau_i - T, \tau_i]$.

The expansion in (74) has a Markovian structure in the pairs $\sigma = (\mathbf{s}(K), \mathbf{k}) \equiv (\mathbf{s}, \mathbf{k})$, and it is convenient to set

$$\mu_K(d\sigma | \sigma') = \mu_K(d\mathbf{s}, \mathbf{k} | \mathbf{s}', \mathbf{k}'). \tag{75}$$

We write for the convolution of such kernels:

$$\mu_K \mu_{K''}(d\sigma | \sigma'') = \int \mu_K(d\sigma | \sigma') \mu_{K''}(d\sigma' | \sigma''), \tag{76}$$

where the integral means both the integral over $d\mathbf{s}'$ and the sum over $\mathbf{k}'$. For $|K| = T$, i.e. for a small interval, we drop the index K altogether in our notation and write μ^n for the n-fold convolution. With these preparations, let us then consider the expression (22) when the function F depends only on s:

$$\begin{aligned} P^t(\omega(0), F) &= \int \mu^t_{\omega(0)}(d\mathbf{s}) F(s(t)) \\ &= \sum_{\mathbf{K}, \mathbf{n}} \int \mu^{n_M} \mu_{K_M} \ldots \mu^{n_1} \mu_{K_1}(d\sigma | \omega(0)) F(s(t)), \end{aligned} \tag{77}$$

where $\mathbf{K} = (K_1, \ldots, K_M)$ are disjoint T-intervals of length at least $2T$, $\sum |K_i| + T \sum n_i = t$, $n_M \geq 0$, $n_i > 0$ for other i's and $M \geq 1$.

There are two kinds of transition kernels in (77), the unlikely ones μ_K and the likely ones μ^n. The latter will be responsible for the convergence to stationarity and we will discuss them next. Let $\sigma = (\mathbf{s}, \mathbf{k})$ with $\mathbf{s} = \mathbf{s}(J)$ and $J = [\tau, \tau + T]$, $J_0 = [\tau - T, \tau]$. Define

$$P(d\mathbf{s} | \mathbf{s}') = g_J(\omega) \nu^T_{s(\tau)}(d\mathbf{s}), \tag{78}$$

where $\omega(t) = (s(t), l(t, \mathbf{s} \vee \mathbf{s}'([\tau - T, t]), 0)$, with $\mathbf{s} \vee \mathbf{s}'$ being the configuration on $[\tau - T, \tau + T]$ coinciding with $\mathbf{s}'$ on $[\tau - T, \tau]$, and with $\mathbf{s}$ on $[\tau, \tau + T]$; we put $P(d\mathbf{s}|\mathbf{s}') = 0$ if $s(\tau) \neq s'(\tau)$; $g_J(\omega)$ is the Girsanov factor (62) (which here, of course, because of the definition of ω, depends only on $\mathbf{s} \vee \mathbf{s}'([\tau - T, t])$). Let also

$$\chi_{\mathbf{k}}(\mathbf{s}, \mathbf{s}') = \chi_{\mathbf{k}}(\omega) 1_J(\mathbf{k}), \tag{79}$$

where $1_J(\mathbf{k})$ is supported on $\mathbf{k}$ so that J is a small interval. Then, (75) in the special case $|K| = T$ gives:

$$\mu(d\sigma | \sigma') = \chi_{\mathbf{k}}(\mathbf{s}, \mathbf{s}') P(d\mathbf{s} | \mathbf{s}') 1_{J_0 J}(\mathbf{k}', \mathbf{k}). \tag{80}$$

Let $\bar{\mu}$ be given by (80) without the $1_{J_0 J}(\mathbf{k}', \mathbf{k})$ factor:

$$\bar{\mu}(d\sigma|\sigma') = \chi_{\mathbf{k}}(\mathbf{s}, \mathbf{s}')P(d\mathbf{s}|\mathbf{s}') \tag{81}$$

and write

$$\mu = \bar{\mu} + \Delta. \tag{82}$$

$\Delta(d\sigma|\sigma')$ is a measure of small total mass, since it is supported on σ's such that large intervals L intersect two adjacent small ones. So, let us expand:

$$\mu^n = (\bar{\mu} + \Delta)^n = \sum \bar{\mu}^{n_1}\Delta^{n_2}\dots\bar{\mu}^{n_{k-1}}\Delta^{n_k}. \tag{83}$$

We will state now the basic bounds for the transition kernels that allow us to control the expansions (80) and (83). Remember that the initial states $\mathbf{s}'$ in our kernels are on small intervals $J_0 = [\tau - T, \tau]$ (except for the μ_K with $0 \in K$ which has $\omega(0)$ as initial state). This means that $\omega'(t) = \omega(t, \mathbf{s}'([\tau - T, t]), l'(\tau - T))$ is constrained to be on the support of the $\chi_{\mathbf{k}}$ with $\mathbf{k}$ such that J_0 is small. This implies that all the transition kernels have initial states $\mathbf{s}' \in C_s \subset C(J_0, H_s)$ given by (see (26) and the support of ϕ_k)

$$C_s = \left\{ \mathbf{s}' \mid \sum_{\mathbf{n} \subset J_0} D_n(\omega') \leq 4\beta RT,\ D_\tau(\omega') \leq 4\beta' RT \right\}. \tag{84}$$

The first proposition controls the unlikely events of having either Δ^n, $n \geq 1$, or μ_K with $|K| \geq 2T$ (or both):

Proposition 3. *There exists $c > 0$, $c' < \infty$, $T_0 = T_0(\rho, R) < \infty$ such that, $\forall T \geq T_0$, and for $|K| \geq 2T$, or $m \geq 2$, or $m = 1$ and $|K| \geq T$,*

$$\sup_{\mathbf{k}'} \sup_{\mathbf{s}' \in C_s} \int |\Delta^m \mu_K(d\sigma|\sigma')| \leq e^{-c(|K|+Tm)} C_K(\omega(0)), \tag{85}$$

where the sup is over $\mathbf{k}'$ so that J_0 is small, if $0 \notin K$. $C_K(\omega(0)) = 1$ if $0 \notin K$ and

$$C_K(\omega(0)) = e^{c'\beta' T} e^{\frac{\|\omega(0)\|^2}{8R}} \tag{86}$$

if $0 \in K$.

For the likely events we look more closely at $\bar{\mu}^n$:

$$\bar{\mu}^n(d\sigma|\sigma') = \int \bar{\mu}(d\sigma|\sigma'')\lambda^{n-1}(d\mathbf{s}''|\mathbf{s}') \tag{87}$$

with λ given by, see (81),

$$\lambda(d\mathbf{s}|\mathbf{s}') = \sum_{\mathbf{k}} \bar{\mu}(d\sigma|\sigma') = \sum_{\mathbf{k}} \chi_{\mathbf{k}}(\mathbf{s}, \mathbf{s}')P(d\mathbf{s}|\mathbf{s}'). \tag{88}$$

The content of the following proposition is that λ^n relaxes to equilibrium:

Proposition 4. *There exist $\delta = \delta(\rho, R) > 0$, $p = p(\rho, R) < \infty$, such that, $\forall T \geq T_0$,*

$$\sup_{\mathbf{s}' \in C_s} \int |\lambda^p(d\mathbf{s}|\mathbf{s}') - \lambda^p(d\mathbf{s}|0)| \leq 1 - \delta. \tag{89}$$

6. Proof of the Theorem

The proof of the theorem is rather straightforward, given the estimates, stated in Propositions 3 and 4, on the measures in (77). Note that the length T of the intervals entering in the expansion (77) is a parameter that has not yet been fixed. For simplicity, we shall consider only times t in (8) that are multiples of T; the general case is easy to obtain.

We divide the proof into two parts: in the first one, $F = 1$ in (8) and, in the second, F is a general Hölder continuous function.

In the case $F = 1$, we integrate a function, 1_E, depending only on s' and we may use (77). Let

$$\mu_0(d\mathbf{s}) \equiv \lambda^p(d\mathbf{s}|0)$$

and rewrite (89) as

$$\sup_{\mathbf{s}'\in C_s} \int |\lambda^p(d\mathbf{s}|\mathbf{s}') - \mu_0(d\mathbf{s})| \leq 1 - \delta. \tag{90}$$

In (77), first, expand each μ^{n_i} factor, for $i = 1, \dots M$ using (83):

$$\mu^{n_i} = (\bar{\mu} + \Delta)^{n_i} = \sum \bar{\mu}^{n_{i1}} \Delta^{m_{i1}} \dots \bar{\mu}^{n_{ik_i}} \Delta^{m_{ik_i}}. \tag{91}$$

Then we write, using (87),

$$\bar{\mu}^{n_{ij}} = \bar{\mu}(\lambda^p - \mu_0 + \mu_0)^{\left[\frac{n_{ij}-1}{p}\right]} \lambda^{q_{ij}}, \tag{92}$$

where λ is defined by (88) and $n_{ij} - 1 = \left[\frac{n_{ij}-1}{p}\right] p + q_{ij}$, i.e. $q_{ij} < p$. Finally, expand each of the resulting factors

$$(\lambda^p - \mu_0 + \mu_0)^{\left[\frac{n_{ij}-1}{p}\right]} = \sum M_{a_{ij}}(\lambda^p - \mu_0, \mu_0), \tag{93}$$

where $M_{a_{ij}}$ is a monomial, of degree a_{ij} in the first variable. This way we end up with an expansion of $P^t(\omega(0), 1_E)$ in terms of products of μ_K with $|K| \geq 2T$, Δ^m, $\lambda^p - \mu_0$, $\Delta\bar{\mu}$, $\bar{\mu}$, λ^q, with $q < p$, and of μ_0.

Consider now two initial conditions $\omega(0) = \omega_0, \omega_0'$ and let $\|\omega_0'\| \leq \|\omega_0\|$. Let $t_0 = \frac{C}{\beta R}\|\omega_0\|^2 + T$, and perform the expansion (91–93) for the factors μ^{n_i} that occur after t_0 in (77). Let, for $n \geq t_0$, $P_n^t(\omega_0, 1_E)$ consist of all the terms in the resulting sum that have $\mu_0(d\mathbf{s})$ with $\mathbf{s} = \mathbf{s}([(n-1)T, nT])$ as one of the factors in the product. Note that, if n is larger than t_0, such terms always exist. Indeed, $D_1 \geq \frac{1}{2}\|\omega_0\|^2$ forces $2^{k_1} \geq \frac{1}{8R}\|\omega_0\|^2$ and thus implies that the origin is contained in a large interval of length $\frac{C}{\beta}2^{k_1}$; but longer intervals are not forced by the initial condition, and so, μ^{n_i} factors are not forbidden in (77), after t_0. The same will be true for ω_0', since $\|\omega_0'\| \leq \|\omega_0\|$. Since, $\mu_0(d\mathbf{s})$ is independent of the past, the sum in $P_n^t(\omega_0, 1_E)$ factorizes and, for the times before $(n-p)T$, we recover the full $P^{(n-p)T}$. We have then

$$P_n^t(\omega_0, 1_E) = \int P^{(n-p)T}(\omega_0, d\mathbf{s}')\mu_0(d\mathbf{s}) f(n, \mathbf{s}, E) = \int \mu_0(d\mathbf{s}) f(n, \mathbf{s}, E),$$

since $P^{(n-p)T}(\omega_0, H_s) = 1$. Thus,

$$P_n^t(\omega_0, 1_E) = P_n^t(\omega_0', 1_E),$$

and we conclude that

$$P^t(\omega_0, 1_E) - P^t(\omega_0', 1_E) = R^t(\omega_0, 1_E) - R^t(\omega_0', 1_E),$$

where $R^t(\omega_0, 1_E)$ is given by the same sum as $P^t(\omega_0, 1_E)$ except for the terms that have a factor $\mu_0(d\mathbf{s})$ with $\mathbf{s} = \mathbf{s}([(n-1)T, nT])$ for $n \geq t_0$.

We will estimate $|R^t(\omega_0, 1_E)|$. Note that it contains only, after time t_0, the factors μ_K with $|K| \geq 2T$, Δ^m, $\lambda^p - \mu_0$, $\Delta\bar{\mu}$, $\bar{\mu}$, λ^q, with $q < p$, i.e. no μ_0 factors. Let us count the powers of the various factors in this expansion, using the definitions in Eqs. (91), (92) and (93). The number N_Δ of Δ-factors is

$$N_\Delta = \sum_{ij} m_{ij}.$$

To count the number of $\lambda^p - \mu_0$ factors, note that only the term with $a_{ij} = \left[\frac{n_{ij}-1}{p}\right]$ in (93) enters (all the others having at least one μ_0);

$$\sum_{ij} a_{ij} = \sum_{ij} \left[\frac{n_{ij}-1}{p}\right] \geq \frac{1}{p}\sum_{ij}(n_{ij} - 2) = \frac{1}{p}\sum_i n_i - \frac{1}{p}\sum_{ij}(m_{ij}+2), \quad (94)$$

where, in the last step, we used (91). Since $\sum_i 1 = M'$, where M' is the number of K_i factors in (77) that do not occur before t_0, we get $\sum_{ij}(m_{ij}+2) \leq 3N_\Delta + 2M'$, where the first term bounds the sum over $m_{ij} \neq 0$, and the second the sum over $m_{ij} = 0$. Thus,

$$(94) \geq \frac{1}{p}\left(\sum_i n_i - 3N_\Delta - 2M'\right) \geq \frac{1}{p}\left(\frac{t-t_0}{T} - \frac{1}{T}\sum_i(|K_i|+2) - 3N_\Delta\right), \quad (95)$$

where in the last step we used $T\sum n_i + \sum |K_i| \geq t - t_0$ (remembering that we use the expansion in (91–93) only after time t_0).

In order to bound $|R^t(\omega_0, 1_E)|$, which is a sum of terms, we shall first bound all the factors in each term. For μ_K, for $|K| \geq 2T$, and Δ^m, $m \geq 2$ we use (85) and, writing $\Delta\bar{\mu} = \Delta\mu - \Delta^2$, we obtain a bound like (85) (with another c), for $\Delta\bar{\mu}$ instead of $\Delta\mu$; for $\lambda^p - \mu_0$, we use (89). The other terms have simple bounds: since $\bar{\mu}(d\sigma|\sigma')$, defined in (81), is positive, we have

$$\sup_{\sigma'} \int |\bar{\mu}(d\sigma|\sigma')| = \sup_{\sigma'} \int \bar{\mu}(d\sigma|\sigma') \leq 1, \quad (96)$$

and, similarly, by (88),

$$\sup_{\mathbf{s}'} \int |\lambda^q(d\mathbf{s}|\mathbf{s}')| \leq 1.$$

We also have, for Δ^m with $m = 1$,

$$\sup_{\sigma'} \int |\Delta(d\sigma|\sigma')| \leq \sup_{\sigma'} \int |\bar{\mu}(d\sigma|\sigma')| + \sup_{\sigma'} \int |\mu(d\sigma|\sigma')| \leq 2.$$

Observe that the last three factors occur always next to other factors: $\bar{\mu}$ or Δ at the beginning or the end of the products in (91) (actually, there is, in the full expansion, at most one factor Δ not multiplied by $\bar{\mu}$ or by μ_K) or λ^q at the end of the product in (92). So, the summation in R^t runs only over the sets K_i in (77) and over the occurrences of

Δ in (91) (since only the term without a μ_0 factor in (93) enters in R^t). Combining this observation, all the above inequalities and (95), we can bound R^t by a sum of

$$C(\omega(0))(1-\delta)^{\frac{1}{p}\left(\frac{t-t_0}{T}\right)}e^{-c(\sum|K_i|+TN_\Delta)},$$

over the subsets K' consisting of the union of the T-intervals K_i and of the T-intervals where Δ occurs; here, $C(\omega(0)) = C_K(\omega(0))$, given by (86). Since t_0 depends on ω_0, we may absorb the factor $(1-\delta)^{\frac{1}{p}\left(\frac{-t_0}{T}\right)}$ into $C(\omega(0))$, and we get:

$$|R^t(\omega_0, 1_E)| \le C(\omega(0))(1-\delta)^{\frac{1}{p}(\frac{t}{T})}\sum_{K'} e^{-c|K'|} \le C(\omega(0))(1-\delta')^{\frac{t}{T}}e^{e^{-cT}\frac{t}{T}}, \quad (97)$$

since the sums over subsets of $[0, t]$ made of T-intervals can be identified with sums over subsets of $[0, t/T]$ (remember that t is a multiple of T). δ' is defined by $1-\delta' = (1-\delta)^{\frac{1}{p}}$ and, like δ and p, is independent of T. Therefore, choosing T large enough, (97) can be bounded by $C(\omega(0))e^{-mt}$ for some $m > 0$ depending on R and T, i.e. on R and ρ (T will be chosen as a function of ρ in the next section). Using a similar bound for $R^t(\omega_0', 1_E)$, we obtain that

$$|P^t(\omega(0), 1_E) - P^t(\omega'(0), 1_E)| \le C(\omega(0))e^{-mt}. \quad (98)$$

From this, the existence of the limit $\lim_{t\to\infty} P^t(\omega(0), 1_E)$ follows: indeed, write, for $t > t'$,

$$P^t(\omega(0), 1_E) - P^{t'}(\omega(0), 1_E) = \int P^{t-t'}(\omega(0), d\omega)(P^{t'}(\omega, 1_E) - P^{t'}(\omega(0), 1_E)) \quad (99)$$

and use (98)

$$|P^{t'}(\omega, 1_E) - P^{t'}(\omega(0), 1_E)| \le (C(\omega) + C(\omega(0)))e^{-mt'}.$$

Then we have, by (30) and (86),

$$\int P^{t-t'}(\omega(0), d\omega)C(\omega) \le 3e^{c'\beta' T}\sum_{n\ge|\|\omega_0\|^2} e^{\frac{n}{8R}}e^{-\frac{n}{4R}} + C(\omega(0)) = C'(\omega(0)). \quad (100)$$

Hence, $\lim_{t\to\infty} P^t(\omega(0), 1_E)$ exists, and (8) with $F(l') = 1$ also follows.

Now, consider (8) for a general $F = F(l')$. Write $F = F - F_0 + F_0$, where, by definition, $F_0(l) = F(l(t, s([\frac{t}{2}, t]), 0))$. Then,

$$P^t(\omega(0), F) = \int \mu^t_{\omega(0)}(d\mathbf{s})(F - F_0) + \int \mu^t_{\omega(0)}(d\mathbf{s})F_0. \quad (101)$$

Let us start with the first term. We write it as

$$\int \mu^t_{\omega(0)}(d\mathbf{s})(F - F_0) = \int \mu^t_{\omega(0)}(d\mathbf{s})(F - F_0)1_\omega + \int \mu^t_{\omega(0)}(d\mathbf{s})(F - F_0)(1 - 1_\omega), \quad (102)$$

where 1_ω is the indicator function of the event $\|\omega(\frac{t}{2})\|^2 > Rt$. By the probabilistic estimate (30) the first term may be bounded by

$$2\|F\|_\infty P(\|\omega(\tfrac{t}{2})\|^2 > Rt|\omega(0)) \le C(\omega(0))\|F\|_\alpha e^{-ct}, \tag{103}$$

where $\|F\|_\alpha$ is the Hölder norm of F.

For the second term, write it as a sum

$$\int \mu^t_{\omega(0)}(d\mathbf{s})(F - F_0)(1 - 1_\omega)1_D + \int \mu^t_{\omega(0)}(d\mathbf{s})(F - F_0)(1 - 1_\omega)(1 - 1_D), \tag{104}$$

where 1_D is the indicator function of the event $a\sum_{n=\frac{t}{2}+1}^{t} D_n(\omega) > \frac{\kappa}{2}Rt$. Using again the probabilistic estimates, we have, by (28) (with 0 replaced by $\frac{t}{2}$) and the constraint $1 - 1_\omega$, i.e. $\|\omega(\frac{t}{2})\|^2 \le Rt$, that, for κ large:

$$\left|\int \mu^t_{\omega(0)}(d\mathbf{s})(F - F_0)(1 - 1_\omega)1_D\right| \le C\|F\|_\alpha e^{-ct}. \tag{105}$$

For the second term in (104), we use the fact that F is Hölder continuous:

$$|F - F_0| \le \|F\|_\alpha \|l(t, \mathbf{s}([\tfrac{t}{2}, t], 0) - l(t, \mathbf{s}([0, t], l_0)\|^\alpha,$$

and

$$\|l(t, \mathbf{s}([\tfrac{t}{2}, t], 0) - l(t, \mathbf{s}([0, t], l_0)\| = \|l(t, \mathbf{s}([\tfrac{t}{2}, t], 0) - l(t, \mathbf{s}([\tfrac{t}{2}, t], l(\tfrac{t}{2}))\| \le e^{-cRt},$$

which follows from (12), with $[0, t]$ replaced by $[\frac{t}{2}, t]$, given that we have here both the constraint that

$$a\int_{\frac{t}{2}}^{t} \|\nabla\omega\|^2 \le a\sum_{n=\frac{t}{2}+1}^{t} D_n(\omega) \le \frac{\kappa}{2}Rt,$$

and that $\|l_1(\frac{t}{2}) - l_2(\frac{t}{2})\|^2 = \|l(\frac{t}{2})\|^2 \le \|\omega(\frac{t}{2})\|^2 \le Rt$. Thus,

$$\left|\int \mu^t_{\omega(0)}(d\mathbf{s})(F - F_0)(1 - 1_\omega)(1 - 1_D)\right| \le \|F\|_\alpha e^{-c\alpha Rt}. \tag{106}$$

Altogether, combining (102–106), we get:

$$\left|\int \mu^t_{\omega(0)}(d\mathbf{s})(F - F_0)\right| \le C(\omega(0))\|F\|_\alpha e^{-ct}, \tag{107}$$

where $c = c(R, \alpha)$.

Returning to (101), we will finish the proof by bounding

$$\int \mu^t_{\omega_0}(d\mathbf{s})F_0 - \int \mu^t_{\omega_0'}(d\mathbf{s})F_0.$$

We insert the expansion (74) in each term and integrate over $\mathbf{s}([0, t])$; since F_0 depends only on $\mathbf{s}([\frac{t}{2}, t])$, we obtain, in each term of the sum, a formula like (77) for the factors occurring before the first K_i intersecting $[\frac{t}{2}, t]$ (and an expression depending on F_0 for the rest). Now, expand the resulting factors μ^{n_i}, after t_0, as above (see the arguments leading to (97)). As before, let $P^t_n(\omega_0, F_0)$ collect all the terms containing a factor μ_0

(after t_0 and before the first K_i intersecting $[\frac{t}{2}, t]$). Again, $P_n^t(\omega_0, F_0) = P_n^t(\omega_0', F_0)$. Now, for $R^t(\omega_0, F_0)$, we first bound F_0 by its supremum, then bound each term of the resulting expansion, using (96) for the μ factors and (85) for the other factors. The result is

$$\left| \int \mu_{\omega_0}^t(d\mathbf{s}) F_0 - \int \mu_{\omega_0'}^t(d\mathbf{s}) F_0 \right| \le C(\omega(0)) \|F_0\|_\infty (1-\delta')^{t/2T} e^{e^{-cT}\frac{t}{T}},$$

where the $(1-\delta')^{t/2T}$ factor comes from the fact that, in R, we have only the factors μ_K with $|K| \geq 2T$, Δ^m, $\lambda^p - \mu_0$, $\Delta\bar{\mu}$, $\bar{\mu}$, λ^q, with $q < p$, appearing during the time interval $[t_0, t/2]$ and we can therefore use (95), with t replaced by $t/2$ to obtain a lower bound on the number of $\lambda^p(d\mathbf{s}|\mathbf{s}') - \mu_0(d\mathbf{s})$ factors. Combining this with (107), (101), we obtain (98) with 1_E replaced by F. To finish the proof, we can now use arguments like (99–100) to get (8) in general. □

The next two sections will be devoted to the proof of, respectively, Propositions 3 and 4.

7. Proof of Proposition 3

Consider the expression

$$X(\sigma') \equiv \int |\Delta^m \mu_K(d\sigma|\sigma')| \tag{108}$$

for $K = [\tau_0, \tau]$ a T-interval. Let $\pi = (J_1, \ldots, J_{n-m})$ be a partition of K in the sum (73) and define also for $i \in [1, m] J_{n-m+i} = [\tau + (i-1)T, \tau + iT]$. Hence the J_i, for $i \in [1, n]$, form a partition of the set $\bar{K} = [\tau_0, \tau + mT]$. Let $\mathbf{k}_i \in \mathbf{N}^{J_i}$, $i = 1, \ldots, n$. Set $\mathbf{k} = (\mathbf{k}_1, \ldots, \mathbf{k}_n,) \in \mathbf{N}^{|K|+mT}$. Finally, let $\mathbf{j} = (j_3, \ldots, j_{n-m})$. Then combining the definitions (73), (75), (80) and (82), we can bound

$$\begin{aligned} X(\sigma') \le \sum_{\pi\mathbf{kj}} \int \Big| &\prod_{i=1}^{n-m} \mu_{\mathbf{k}_i, j_i}(d\mathbf{s}(J_i)|\mathbf{s}([\tau_{j_i-1}, \tau_{i-1}])) \\ \times &\prod_{i=n-m+1}^{n} \mu_{\mathbf{k}_i i-1}(d\mathbf{s}(J_i)|\mathbf{s}(J_{i-1}))\Big| 1(\mathbf{k}|\mathbf{k}'), \end{aligned} \tag{109}$$

where $\mathbf{s}([\tau_0 - T, \tau_0]) = \mathbf{s}'$ unless $0 \in K$, in which case $\mathbf{s}' \equiv \omega(0)$. We also put

$$1(\mathbf{k}|\mathbf{k}') = \prod_{i=0}^{n-m-1} 1_{J_i J_{i+1}}(\mathbf{k}_i, \mathbf{k}_{i+1}) \prod_{i=n-m}^{n-1} |1_{J_i J_{i+1}}(\mathbf{k}_i, \mathbf{k}_{i+1}) - 1| \prod_{i=1}^{n} 1_{J_i}(\mathbf{k}_i), \tag{110}$$

with $\mathbf{k}_0 = \mathbf{k}'$ and the sum over π, $\mathbf{j}$, has the constraint that the set (72) or (71) is K. Let $\mathcal{I} = \{i \mid j_i \neq i-1\} \subset \{3, \ldots, n-m\}$. For $i \in \mathcal{I}$, we rewrite $\mu_{\mathbf{k}_i, j_i}$ (see Eq. (69)) as

$$\mu_{\mathbf{k}_i, j_i} = \chi_{\mathbf{k}_i j_i} g_{i j_i} - \chi_{\mathbf{k}_i j_i+1} g_{i j_i+1} = (\delta_i \chi + \chi_{\mathbf{k}_i j_i+1} \delta_i g) g_{i j_i}$$

where

$$\delta_i \chi = \chi_{\mathbf{k}_i j_i} - \chi_{\mathbf{k}_i j_i+1} \tag{111}$$

and

$$\delta_i g = 1 - \frac{g_{i j_i+1}}{g_{i j_i}}. \tag{112}$$

Introducing the probability measures

$$\mu_{\mathbf{j}} = \prod_{i=1}^{n} g_{i j_i} \nu_{s(\tau)}^{|K|}, \tag{113}$$

where $j_i \equiv i-1$ if $i > n-m$, we can write (109) as

$$\begin{aligned} X(\sigma') &\leq \sum_{\pi \mathbf{kj}} \sum_{A \subset \mathcal{I}} \int \prod_{i \in A} |\delta_i \chi| \prod_{i \in B} \chi_{\mathbf{k}_i j_i+1} |\delta_i g| \prod_{i \in C} \chi_{\mathbf{k}_i i-1} \mu_{\mathbf{j}} 1(\mathbf{k}|\mathbf{k}') \\ &\equiv \sum_{\pi \mathbf{kj}A} \int \mathcal{R}_{\mathbf{kj}A} \mu_{\mathbf{j}} 1(\mathbf{k}|\mathbf{k}'), \end{aligned} \tag{114}$$

where $B = \mathcal{I} \setminus A$ and $C = \{1, \ldots, n\} \setminus \mathcal{I}$.

Letting

$$\delta_i f = f_{j_i+1} - f_{j_i},$$

(112) can be written as

$$\delta_i g = 1 - e^{\int_{J_i} (\delta_i f(t), \gamma^{-1}(ds(t) - f_{j_i}(t)dt)) - \frac{1}{2}\int_{J_i} (\delta_i f(t), \gamma^{-1} \delta_i f(t))dt} \equiv 1 - H_i, \tag{115}$$

and

$$\delta_i \chi = \prod_{\mathbf{n} \subset J_i} \phi_{k_n}(D_n(\omega_{j_i})) - \prod_{\mathbf{n} \subset J_i} \phi_{k_n}(D_n(\omega_{j_i+1})). \tag{116}$$

We will now undo the Girsanov transformation, i.e. change variables from s back to b. Let E denote the expectation with respect to the Brownian motion b with covariance γ on the time interval K. Then,

$$\int \mathcal{R}_{\mathbf{kj}A} \mu_{\mathbf{j}} = E \mathcal{R}_{\mathbf{kj}A}. \tag{117}$$

where $\mathcal{R}$ is given by the same expression as before, but the symbols s and ω_{j_i} have to be interpreted as follows: s is the progressively measurable function of b defined on each interval J_i as the solution of

$$ds(t) = f_{j_i}(t)dt + db(t), \tag{118}$$

where $f_j(t) = PF(\omega_j(t))$ and $\omega_j(t) = s(t) + l(t, \mathbf{s}([\tau_{j-1}, t]), 0)$, with, for $i = 1$, $\mathbf{s}([\tau_{j_1-1}, \tau_0])$ replaced by $\mathbf{s}'(J_0)$, which expresses the dependence of (117) on $\mathbf{s}'$; H_i, defined by (115), can be written:

$$H_i = e^{\int_{J_i} (\delta_i f(t), \gamma^{-1} db(t)) - \frac{1}{2}\int_{J_i} (\delta_i f(t), \gamma^{-1} \delta_i f(t))dt.} \tag{119}$$

We will call the $\omega_{j_i}(t)$ collectively by

$$\omega_{\mathbf{j}}(t) = \omega_{j_i}(t) \quad \text{for } \ t \in J_i, \tag{120}$$

and reserve the notation $\omega(t)$ for the solution of the Navier Stokes equation (4) with given $b(\bar{K})$ and with initial condition $\omega(\tau_0) = (s(\tau_0), l(\tau_0, \mathbf{s}'(J_0), 0))$ determined by the $\mathbf{s}'$ in (109).

Remark. $\omega_{\mathbf{j}}(t)$ is *not* a solution of (4) on the interval $\bar{K}$ with initial condition given at time τ. On each interval J_i it solves (4) but when moving to the next interval the l-part is possibly set equal to zero, depending on $\mathbf{j}$.

The following proposition contains the key bounds needed to estimate (117).

Proposition 5. *Let b belong to the support of $\mathcal{R}_{\mathbf{kj}A}$ in (117). Then, there exists a constant c such that*

$$\|\delta_i f(t)\| \leq e^{-c\kappa R \mathrm{dist}(J_i, J_{j_i})} \equiv \epsilon_i \tag{121}$$

and

$$\prod_{i\in A} |\delta_i \chi| \prod_{i\in B} \chi_{\mathbf{k}_i j_i+1} \prod_{i\in C} \chi_{\mathbf{k}_i i-1} \leq \prod_{i\in A} |J_i|\epsilon_i 1_{\mathbf{k}}(\omega), \tag{122}$$

where $1_{\mathbf{k}}(\omega)$ is the indicator function of the set of $b \in C(\bar{K}, H_s)$ such that, for all $\mathbf{n} \subset \bar{K}$, we have

$$D_n(\omega) \in [2^{k_n-1}R, 2^{k_n+3}R], \quad for\ k_n \neq 0; \quad D_n(\omega) \in [0, 5R], \ for\ k_n = 0, \tag{123}$$

and ω is the solution of the Navier-Stokes equation explained above.

Let $\eta_i(t)$ be the indicator function of the event that $\delta_i f(t)$ satisfies the bound (121). $\eta_i(t)$ is progressively measurable. Since $\eta_i = 1$ on the support of the summand in (114), we may replace $\delta_i f(t)$ there by $\eta_i(t)\delta_i f(t)$. Denote H_i, defined in (115), after this replacement, by $\bar{H}_i$. We have, using (122),

$$E\mathcal{R}_{\mathbf{kj}A} \leq \prod_{i\in A} |J_i|\epsilon_i E\left(\prod_{i\in B}(1-\bar{H}_i)1_{\mathbf{k}}\right),$$

and inserting this to (114)

$$X(\sigma') \leq \sum_{\pi \mathbf{j} A} \prod_{i\in A} |J_i|\epsilon_i E\left(\prod_{i\in B}(1-\bar{H}_i)1_\pi(\omega)\right), \tag{124}$$

where

$$1_\pi(\omega) = \sum_{\mathbf{k}} 1_{\mathbf{k}}(\omega) 1(\mathbf{k}|\mathbf{k}'). \tag{125}$$

The expectation in (124) is bounded using Schwarz' inequality by

$$\left(E\prod_{i\in B}(1-\bar{H}_i)^2\right)^{\frac{1}{2}} \left(E 1_\pi^2\right)^{\frac{1}{2}}. \tag{126}$$

To estimate the first square root renumber the intervals J_i for $i \in B$ as $J_1, \ldots, J_b$, $J_i = [\sigma_i', \sigma_i]$ with $\sigma_1 > \sigma_1' \geq \sigma_2 \ldots$. Denote expectations in the Brownian filtration $\mathcal{F}_\tau$ by E_τ. Then

$$E\prod_{i\in B}(1-\bar{H}_i)^2 = E_{\sigma_1'}\left(E_{\sigma_1}((1-\bar{H}_1)^2 \mid \mathcal{F}_{\sigma_1'})\prod_{i>1}(1-\bar{H}_i)^2\right). \tag{127}$$

Expanding $(1-\bar{H}_i)^2 = 1-2\bar{H}_i+\bar{H}_i^2$, we first bound from below, using (121) and Jensen's inequality,

$$E_{\sigma_1}(\bar{H}_1|\mathcal{F}_{\sigma_1'}) \geq \exp\left(-\tfrac{1}{2}|J_1|\epsilon_1^2\rho^{-1}\right). \tag{128}$$

For an upper bound for the expectation of $\bar{H}_1^2$, we use

Lemma 7.1. *Let $\zeta(t) \in C([0,t], H_s)$ be progressively measurable. Then*

$$Ee^{\int_0^t(\zeta,\gamma^{-1}db)+\lambda\int_0^t(\zeta,\gamma^{-1}\zeta)dt} \leq e^{2(1+\lambda)t\|\zeta\|^2\rho^{-1}} \tag{129}$$

where $\|\zeta\| = \sup_\tau \|\zeta(\tau)\|_2$.

Proof. This is just a Novikov bound: we bound the LHS, using Schwarz' inequality, by

$$\left(Ee^{\int_0^t(2\zeta,\gamma^{-1}db)-2\int_0^t(\zeta,\gamma^{-1}\zeta dt)}\right)^{\frac{1}{2}}\left(Ee^{2(1+\lambda)\int_0^t(\zeta,\gamma^{-1}\zeta)dt}\right)^{\frac{1}{2}}$$

and note that the expression inside the first square root is the expectation of a martingale and equals one. □

Applying Lemma 7.1 to $\zeta = 2\eta_i\delta_i f$ and $\lambda = -\frac{1}{4}$ we obtain

$$E_{\sigma_1}(\bar{H}_1^2|\mathcal{F}_{\sigma_1'}) \leq \exp(3|J_1|\epsilon_1^2\rho^{-1}). \tag{130}$$

This and (128) imply

$$E_{\sigma_1}((1-\bar{H}_1)^2 \mid \mathcal{F}_{\sigma_1'}) \leq C|J_1|\epsilon_1^2\rho^{-1}\exp(C|J_1|\epsilon_1^2\rho^{-1}).$$

Iterating the argument, we arrive at

$$E\prod_{i\in B}(1-\bar{H}_i)^2 \leq \prod_{i\in B} C|J_i|\epsilon_i^2\rho^{-1}\exp(C|J_i|\epsilon_i^2\rho^{-1}). \tag{131}$$

Since $\mathrm{dist}(J_i, J_{j_i}) \geq T$, by choosing $T > T(\rho)$ we may bound the i^{th} factor in (131) by ϵ_i if J_i is small (so that $|J_i| = T$) and, by $\epsilon_i e^{\delta|J_i|}$ if J_i is large, where δ can be made arbitarily small by increasing T. Thus, we may combine (124), (126) and (131), to get

$$X(\sigma') \leq \sum_{\pi\mathbf{j}A}\prod_{i\in A\cup B} e^{-c\kappa R\mathrm{dist}(J_i,J_{j_i})}\prod_{|J_i|>T} e^{\delta|J_i|}(E1_\pi^2)^{\frac{1}{2}}.$$

Writing $c = 2c_1$, the sums over $\mathbf{j}$ and A are controlled by

$$\sum_{\mathbf{j}}\prod_{i\in A\cup B} e^{-c_1\kappa R\mathrm{dist}(J_i,J_{j_i})} \leq e^{-c_2\kappa RT|A\cup B|}$$

(since $\mathrm{dist}(J_i, J_{j_i}) \geq T$) and

$$\sum_{A\subset K} e^{-c''\kappa RT|A|} < 2^{|K|}$$

and the last expectation by

Lemma 7.2. *Under the assumptions of Proposition 3,*

$$E 1_{\pi}^{2} \leq C_K(\omega(0)) C^{|K|+mT} e^{-c\beta' mT} \prod_{J_i \ large} e^{-c\beta'|J_i|},$$

where $C_K(\omega(0))$ has the same form as in Proposition 3 (with another c').

We are thus left with the bound

$$X(\sigma') \leq C_K(\omega(0)) C^{|K|+mT} e^{-c\beta' mT} \sum_{\pi} \prod_{J_i \text{ large}} e^{-c\beta'|J_i|} \sup_{\mathbf{j}} \prod_{i \in \mathcal{I}} e^{-c_1 \kappa R \operatorname{dist}(J_i, J_{j_i})}, \tag{132}$$

where we recall that $\mathcal{I} = A \cup B = \{i \mid j_i \neq i-1\}$.

Let first $0 \notin K$. Then K is the union of the sets on the LHS of (71). Each small J is either a subset of $[\tau_{j_i - 1}, \tau_i]$ or a J_{i+1} for J_i large. Thus the summand in (132) is smaller than $e^{-c\beta'|K|}$, for $\kappa R \geq \beta'$. For $0 \in K$, we have a similar bound, except that J_1, J_2 may be small and not in any $[\tau_{j_i - 1}, \tau_i]$ so that $|K|$ is replaced by $|K| - 2T$; but the $2T$ may be absorbed to the $c'\beta' T$ in $C_K(\omega_0)$ (see (86)). The sum over π is a sum over partitions of K into T-intervals (with labels for the intervals of length T), and thus is bounded by $C^{|K|}$. Thus the claim (85) follows for β' large enough. $\square$

Proof of Proposition 5. Let us start with the proof of (122). For that, we need to have a bound on the difference $|D_n(\omega_{j_i}) - D_n(\omega_{j_{i+1}})|$, which is the difference between the arguments of the two χ functions in (111) (see (67)). For that, we need some lemmas. Remember the definition $\omega_j(t) = \omega(t, \mathbf{s}([\tau_{j-1}, t]), 0)$. We have

Lemma 7.3. *Let $n > m \geq \tau_i$, $i > j$. Then*

$$|D_n(\omega_i) - D_n(\omega_j)| \leq e^{-\kappa R(n-m-1) + a\sum_{p=m+1}^{n} D_p(\omega_i)} (\|\delta l(m)\| + \|\delta l(m)\|^2), \tag{133}$$

where $\delta l = l_i - l_j$.

Proof. By definition,

$$\begin{aligned} |D_n(\omega_i) - D_n(\omega_j)| \ &\leq \tfrac{1}{2} |\sup_t \|l_i(t)\|^2 - \sup_t \|l_j(t)\|^2| \\ &\quad + |\int_{\mathbf{n}} \|\nabla l_i(t)\|^2 dt - \int_{\mathbf{n}} \|\nabla l_j(t)\|^2 dt|. \end{aligned} \tag{134}$$

The second term is bounded by

$$\int_{\mathbf{n}} \|\nabla \delta l(t)\| (2\|\nabla \omega_i(t)\| + \|\nabla \delta l(t)\|) dt. \tag{135}$$

Remembering the calculation in Proposition 1, (16), (17), we have:

$$\int_{\mathbf{n}} \|\nabla \delta l(t)\|^2 dt \leq \|\delta l(n-1)\|^2 + a \int_{\mathbf{n}} \|\delta l(t)\|^2 \|\nabla \omega_i(t)\|^2 dt. \tag{136}$$

Using (12), the second term is bounded by

$$a \int_{\mathbf{n}} \|\nabla \omega_i(t)\|^2 e^{2a \int_{n-1}^{t} \|\nabla \omega_i(\tau)\|^2} \|\delta l(n-1)\|^2 \leq (e^{2a D_n(\omega_i)} - 1) \|\delta l(n-1)\|^2 \tag{137}$$

which, together with (136), yields

$$\int_{\mathbf{n}} \|\nabla \delta l(t)\|^2 dt \le e^{2aD_n(\omega_i)} \|\delta l(n-1)\|^2.$$

Now, using this for the second term on the RHS of (135), and Schwarz' inequality to bound the first one, we get

$$(135) \le 2D_n(\omega_i)^{\frac{1}{2}} e^{aD_n(\omega_i)} \|\delta l(n-1)\| + e^{2aD_n(\omega_i)} \|\delta l(n-1)\|^2, \tag{138}$$

since $\int_{\mathbf{n}} \|\nabla \omega_i(t)\|^2 dt \le D_n(\omega_i)$.

For the first term of (134), use $\|l_j(t)\|^2 = \|l_i(t)\|^2 + 2(\delta l(t), l_i(t)) + \|\delta l(t)\|^2$ to bound it by $2\sup_{t\in\mathbf{n}} |(\delta l(t), l_i(t))| + \sup_{t\in\mathbf{n}} \|\delta l(t)\|^2$, which, by Schwarz' inequality, $\sup_{t\in\mathbf{n}} \|l_i(t))\| \le D_n(\omega_i)$, and (12), leads again to the bound (138). This yields our claim if we use (12) to bound $\|\delta l(n-1)\|$. □

To be able to apply this lemma, we need to bound $D_p(\omega_i)$ in the exponent of (133); Note that the functions χ in (114) put constraints (to be in the interval $[2^{k_p}R, 2^{k_p+2}R]$, for $p \in J_i$), but the latter apply to $D_p(\omega_{j_i})$ or $D_p(\omega_{j_i+1})$, not directly to $D_p(\omega_i)$. So, we need to compare those different D_p's. This will be done in Lemma 7.5 below, whose proof will use

Lemma 7.4. *Suppose that* $\mathbf{u} \subset J_q$ *and* $p \le q-1$. *Then*

$$\sum_{\tau_{p-1}<l\le u} 2^{k_l} \le 2\beta(u-\tau_{p-1}). \tag{139}$$

Proof. Let $q \le n-m$. Then $L = [\tau_{p-1}, u]$ cannot satisfy (56) (otherwise $[\tau_{p-1}, u]$ would be inside the same large interval) and so,

$$\gamma_L \le \beta(u-\tau_{p-1}) \tag{140}$$

and the claim is true. So, suppose that $n-m < q$. The interval $L = [\tau_{p-1}, \tau_{n-m-1}+1]$ (which is empty if $n-m < p$) cannot satisfy (56) either and so,

$$\gamma_L \le \beta(|L|+1). \tag{141}$$

The intervals J_i are small if $i \ge n-m$ (J_{n-m} is small since the last interval in K is small) and thus $\gamma_{J_i} \le \beta|J_i| = \beta(\tau_i - \tau_{i-1})$. Hence, (141) holds for $L = [\tau_{p-1}, \tau_{q-1}]$. Altogether, we get

$$\sum_{\tau_{p-1}<l\le u} 2^{k_l} \le \beta(\tau_{q-1}-\tau_{p-1}+1) + \sum_{\tau_{q-1}<l\le u} 2^{k_l}. \tag{142}$$

The last term in (142) is bounded by $\beta\max\{\frac{1}{2}T, u-\tau_{q-1}\} \le \beta(u-\tau_{q-1}+\frac{1}{2}T)$, since the small J_q cannot contain an L satisfying (56). Hence, (141) is bounded by $\beta(u-\tau_{p-1}+1+\frac{1}{2}T)$ which in turn is bounded by (139) since $u-\tau_{p-1} \ge T$. □

Lemma 7.5. *Let s be in the support of the measure in the summand of (114). Let $q \ge 2$ if $0 \notin K$, $q \ge 3$ if $0 \in K$. Let $\mathbf{n} \subset J_q$ and $i, j \le q-1$. Then,*

$$|D_n(\omega_i) - D_n(\omega_j)| \le e^{-\frac{\kappa}{2}TR}. \tag{143}$$

Proof. We perform an induction in q. Suppose that the claim holds up to $q-1$. Let $\mathbf{n} \subset J_q$. In Eq. (114), because of the functions χ, the measure is supported on configurations where, for each p, either, $\forall \mathbf{m} \subset J_p$, $D_m(\omega_{j_p})$ is constrained to be in the interval $[2^{k_m}R, 2^{k_m+2}R]$, or, $\forall \mathbf{m} \subset J_p$, $D_m(\omega_{j_p+1})$ is constrained to be in that interval; remember that both j_p and j_p+1 are less than or equal to $p-1$. Let us consider, for each p, an arbitrary choice between j_p and j_p+1 and call it j^p. Thus, to repeat,

$$D_m(\omega_{j^p}) \in [2^{k_m}R, 2^{k_m+2}R] \quad \text{for} \quad \mathbf{m} \subset J_p \ , \ j^p \leq p-1. \tag{144}$$

Since the support of the measure in (114) contains only configurations such that (144) holds for some choice of the function j^p, it is enough to bound $|D_n(\omega_i) - D_n(\omega_{j^q})|$ and $|D_n(\omega_j) - D_n(\omega_{j^q})|$ for an arbitrary function j^p, assuming that (144) holds.

From Lemma 7.3, we get, for $\mathbf{n} \subset J_q$,

$$\begin{aligned} &|D_n(\omega_i) - D_n(\omega_{j^q})| \\ &\leq e^{-\kappa R(n-\tau_{q-1}-1)+a\sum_{m=\tau_{q-1}+1}^{n} D_m(\omega_{j^q})}(\|\delta l(\tau_{q-1})\| + \|\delta l(\tau_{q-1})\|^2), \end{aligned} \tag{145}$$

where $\delta l = l_i - l_{j^q}$. We need to estimate $\|\delta l(\tau_{q-1})\|$. For this to be nonzero, i and j^q cannot be equal, and they are both less than or equal to $q-1$. So, let us say that $i < q-1$. Then, by Proposition 1,

$$\|\delta l(\tau_{q-1})\| \leq e^{-\kappa R(\tau_{q-1}-\tau_{q-2})+a\sum_{m=\tau_{q-2}+1}^{\tau_{q-1}} D_m(\omega_i)}\|\delta l(\tau_{q-2})\|. \tag{146}$$

Now, use (144) for $p = q$, to get that, in (145), $D_m(\omega_{j^q}) \leq 2^{k_m+2}R$; in (146), we note that, since $\mathbf{m} \subset J_{q-1}$, and since both i and j^{q-1} are less than or equal to $q-2$, we have, by the induction hypothesis,

$$|D_m(\omega_i) - D_m(\omega_{j^{q-1}})| \leq e^{-\frac{\kappa}{2}TR}$$

and so, $D_m(\omega_i) \leq 2^{k_m+2}R + e^{-\frac{\kappa}{2}TR} < 2^{k_m+3}R$. Since we shall show below that $\|\delta l(\tau_{q-1})\| \leq 1$ which implies $\|\delta l(\tau_{q-1})\| + \|\delta l(\tau_{q-1})\|^2 \leq 2\|\delta l(\tau_{q-1})\|$ we obtain, by combining (145) and (146),

$$(145) \leq 2e^{-\kappa R(n-\tau_{q-2}-1)+aR\sum_{m=\tau_{q-2}+1}^{n} 2^{k_m+3}}\|\delta l(\tau_{q-2})\|. \tag{147}$$

Now, remember that $i \leq q-2$. If $j^q = q-1$, then $l_{j^q}(\tau_{q-2}) = 0$ and

$$\|\delta l(\tau_{q-2})\| = \|l_i(\tau_{q-2})\| \leq \|l_i(\tau_{q-2}) - l_{j^{q-1}}(\tau_{q-2})\| + \|l_{j^{q-1}}(\tau_{q-2})\|.$$

Since both i and j^{q-1} are less than or equal to $q-2$, the first term is bounded by $e^{-\frac{\kappa}{2}TR}$, using the inductive hypothesis. For the second term, we use $\|l_{j^{q-1}}(\tau_{q-2})\|^2 \leq \|\omega_{j^{q-1}}(\tau_{q-2})\|^2 \leq 8\beta RT$, see (60). Thus, altogether,

$$\|\delta l(\tau_{q-2})\| \leq (C\beta RT)^{\frac{1}{2}}.$$

If $j^q < q-1$ then, by induction, $\|\delta l(\tau_{q-2})\| \leq e^{-\frac{\kappa}{2}TR}$. By Lemma 7.4 , Eq. (139),

$$\sum_{m=\tau_{q-2}+1}^{n} 2^{k_m} \leq 2\beta(n-\tau_{q-2}).$$

Combining these observations,

$$(145) \le e^{-(\kappa - C\beta)TR}(C\beta RT)^{\frac{1}{2}}$$

since $n - \tau_{q-2} \ge T$. The same result holds for i replaced by j and hence also for $D_n(\omega_i) - D_n(\omega_j)$. Thus, the inductive claim (143) follows provided $\kappa > \kappa(\beta)$.

To start the induction, we need to distinguish the cases $0 \in K$ and $0 \notin K$. Start with the latter. Let $J_0 = [\tau_0 - T, \tau_0]$. We have $q = 2$, $\{i, j\} = \{0, 1\}$ (since the case $i = j$ is trivial). Now j^1 must be 0 or 1, i.e. we may assume $\omega_i - \omega_j = \omega_i - \omega_{j^1}$. Since for $\mathbf{m} \subset J_1$, $D_m(\omega_{j^1}) \le 2^{k_m+2}R$, Proposition 1 implies, with $\delta l = l_i - l_j$,

$$\|\delta l(\tau_1)\| \le e^{-\kappa R(\tau_1 - \tau_0 - 1) + aR\sum_{m=\tau_0+1}^{\tau_1} 2^{k_m+2}} \|\delta l(\tau_0)\|. \tag{148}$$

Thus we need to estimate $\|\delta l(\tau_0)\|$. Since $l_1(\tau_0) = 0$, this equals $\|l_0(\tau_0)\|$. But this in turn is, by (60), bounded by $\|\omega_0(\tau_0)\| \le (8\beta RT)^{\frac{1}{2}}$. We may now proceed as above, using (145) to obtain the claim for $q = 2$.

Let finally $0 \in K$. Now we start the induction from $q = 3$, and may assume $i = 1$ and $j = 2$. We should also remember that now, $\omega_1(t) = (s(t), l(t, \mathbf{s}([0, t]), l(0))$ for $t \in [0, \tau_2]$, where $l(0) = (1 - P)\omega(0)$. By contrast, $\omega_2(t) = (s(t), l(t, \mathbf{s}([\tau_1, t]), 0)$ for $t \in [\tau_1, \tau_2]$. For $\mathbf{m} \subset J_1 \cup J_2$ we have $D_m(\omega_1) \le 2^{k_m+2}R$ since no decoupling was done on those intervals. Proceeding as in the previous case we obtain

$$\|\delta l(\tau_2)\| \le e^{-\kappa R(\tau_2 - \tau_1 - 1) + aR\sum_{m=\tau_1+1}^{\tau_2} 2^{k_m+2}} \|l_1(\tau_1)\|. \tag{149}$$

Now, again by (60), $\|l_1(\tau_1)\|^2 \le \|\omega_1(\tau_1)\|^2 \le 8\beta RT$. We complete the proof for $q = 3$ again by using (145). □

Returning to the proof of Proposition 5 and, combining Lemmas 7.3 and 7.5, we deduce that, for $n \in J_i$,

$$\begin{aligned} &|D_n(\omega_{j_i}) - D_n(\omega_{j_i+1})| \\ &\quad \le e^{-\kappa R(n - \tau_{j_i} - 1) + aR\sum_{m=\tau_{j_i}+1}^{n} 2^{k_p+3}} (\|l_{j_i}(\tau_{j_i})\| + \|l_{j_i}(\tau_{j_i})\|^2). \end{aligned} \tag{150}$$

By (139) the exponent is bounded from above by $-c\kappa R \mathrm{dist}(J_i, J_{j_i})$, for $\kappa \ge \kappa(\beta)$. Since $\mathrm{dist}(J_i, J_{j_i}) \ge T$ and $\|l_{j_i}(\tau_{j_i})\|^2 \le CR\beta T$ (using (60) and (143)), (150) is bounded by ϵ_i. Then,

$$|\delta_i \chi| = |\prod_{\mathbf{n} \subset J_i} \phi_k(D_n(\omega_{j_i})) - \prod_{\mathbf{n} \subset J_i} \phi_k(D_n(\omega_{j_i+1}))| \le |J_i| \epsilon_i 1_{\mathbf{k}}|_{J_i}$$

since we may choose ϕ_k such that its derivative is uniformly bounded in k. We also used the fact that ϕ_k is supported on $[2^k, 2^{k+2}]$ combined with Lemma 7.5 to bound by $1_{\mathbf{k}}$ that has a larger support (the latter is much larger than what is needed, but our choice is notationally convenient). Similarly, Lemma 7.5 allows us to bound $\chi_{\mathbf{k}_i i-1}$ and $\chi_{\mathbf{k}_i j_i+1}$ by $1_{\mathbf{k}}$. These observations lead to (122).

The bound (121) follows from (151) below, using (12) to bound $\|\delta l\|$, $\sup_{t \in \mathbf{n}} \|\omega(t)\| \le (2D_n)^{\frac{1}{2}}$, and Lemmas 7.4, 7.5 to bound the exponent in (12).

Lemma 7.6. *Let $f(\omega) = PF(\omega)$ and $\omega = s + l$, $\omega' = s + l'$. Then,*

$$\|f(\omega) - f(\omega')\| \leq C(R)(2\|\omega\|\|\delta l\| + \|\delta l\|^2) \tag{151}$$

with $\delta l = l - l'$. Moreover

$$\|f(\omega)\| \leq C(R)(\|\omega\| + \|\omega\|^2). \tag{152}$$

Proof. We have

$$|f_k(\omega) - f_k(\omega')| \leq \sum_p |\omega_{k-p}\omega_p - \omega'_{\kappa-p}\omega'_p| \frac{|k|}{|p|}$$

which, since $|k| \leq \sqrt{\kappa R}$ is bounded by

$$\sqrt{\kappa R} \sum_p |s_{k-p}\delta l_p + s_p \delta l_{k-p} + l_p l_{k-p} - l'_p l'_{k-p}|. \tag{153}$$

Writing $l_p l_{k-p} - l'_p l'_{k-p} = l_p \delta l_{k-p} + l_{k-p}\delta l_p - \delta l_p \delta l_{k-p}$ and using Schwarz' inequality, we get

$$(153) \leq \sqrt{\kappa R}(2\|\omega\|\|\delta l\|\| + \|\delta l\|^2)$$

which proves (151), since $f_k \neq 0$ only for $k \leq \kappa R$. The proof of (152) is similar. □

To finish this section, we have only to give the

Proof of Lemma 7.2. From (123) we see that a given ω can belong to the support of at most $5^{|\bar{K}|}$ different $1_{\mathbf{k}}$. Furthermore, if $1_\pi(\omega) \neq 0$ then ω must satisfy the following conditions (remember that $\beta \geq C\beta'$):
a. For each J_i, $i \leq n - m$ such that J_i is large and $0 \notin J_i$, using (58) in Lemma 4.1 and (123), we get

$$\|\omega(\tau_{i-1})\|^2 \leq 16\beta' T R, \tag{154}$$

(since J_{i-1} is small), and, writing $J_i = [\tau_{i-1}, \tau_{i-1}+T] \cup \bar{J}_i$, either $[\tau_{i-1}, \tau_{i-1}+T] \subset J''_i$ (see Lemma 4.1, b, for the definition of J'') and so

$$D_{\tau_{i-1}+T} \geq \tfrac{1}{2}\beta' T R \quad \text{and} \quad \sum_{\mathbf{n}\subset \bar{J}_i} D_n(\omega) > \tfrac{1}{2}\beta' R|\bar{J}_i|, \tag{155}$$

(where the second bound always hold for large intervals since we have β' on the RHS; note, however, that $\bar{J}_i$ could be empty) or $J'_i \cap [\tau_{i-1}, \tau_{i-1} + T] \neq \emptyset$ and so, in particular, $J'_i \neq \emptyset$ and by (59) and (123),

$$\sum_{\mathbf{n}\subset J_i} D_n(\omega) > \tfrac{1}{8}\beta R T + \tfrac{1}{2}\beta' R(|J_i| - T). \tag{156}$$

Let l_i be the event (155) and L_i the event (156).
b. For $0 \in J_1$, if J_1 is large, then

$$\sum_{\mathbf{n}\subset J_1} D_n > \tfrac{1}{2}\beta' R|J_1|. \tag{157}$$

Let l_1 be this event.

c. For the set $K' = \bar{K} \setminus K = [\tau_{n-m}, \tau_n]$, (154) holds for $i - 1 = n - m$, because, by construction, the last interval in K is small and

$$\sum_{\mathbf{n} \subset K'} D_n(\omega) > \tfrac{1}{8} \beta R(m-1)T \tag{158}$$

(since nearest neighbour J_i's have to be intersected by L with $\gamma_L > \frac{1}{2}T$). Let L' be the event (158).

Let B be the ball in H of radius $16\beta' RT$ and define

$$\begin{aligned} \eta_i &= \sup_{\omega(\tau_{i-1}) \in B} P(l_i |\, \omega(\tau_{i-1})), \\ \varepsilon_i &= \sup_{\omega(\tau_{i-1}) \in B} P(L_i |\, \omega(\tau_{i-1})), \\ \epsilon' &= \sup_{\omega(\tau_{n-m}) \in B} P(L' |\, \omega(\tau_{n-m})), \end{aligned}$$

and in the case of $0 \in K$,

$$\eta_1 = P(l_1 | \omega(0)).$$

Then we have, for $0 \notin K$,

$$E 1_\pi^2 \leq 5^{2|\bar{K}|} \epsilon' \prod_{J_i \text{ large}} (\eta_i + \varepsilon_i)$$

and, if $0 \in K$, we have η_1 for $i = 1$ replacing $\eta_1 + \varepsilon_1$. We estimate the ϵ's and η using Proposition 2.

For η_i, $0 \notin J_i$, apply Proposition 2 with 0 replaced by τ_{i-1} (where we use (154)) and t by $\tau_{i-1} + T - 1$, and (155):

$$\eta_i \leq e^{c_1 e^{-cT} \beta' T - c_2 \beta' |J_i|} \leq e^{-c\beta' |J_i|},$$

for T large, using also $|J_i| = |\bar{J}_i| + T$;

For ε_i, Proposition 2, with 0 replaced by τ_{i-1} and (156) give:

$$\varepsilon_i \leq e^{c'\beta' T - c_2 \beta T - c_3 \beta' |J_i|} \leq e^{-c\beta' |J_i|},$$

which holds for $\beta > C\beta'$; for ϵ', Proposition 2, with 0 replaced by τ_{n-m}, and (158) give

$$\epsilon' \leq e^{c\beta' T - c' \beta (m-1) T} \leq e^{-c\beta (m-1) T} \tag{159}$$

using $\beta > C\beta'$, and provided $m > 1$. For $0 \in J_1$, J_1 large, Proposition 2 and (157) give

$$\eta_1 \leq \min \left\{ e^{\frac{c}{R} \|\omega(0)\|^2 - c'\beta' |J_1|}, 1 \right\} \leq e^{\frac{\delta}{R} \|\omega(0)\|^2 - c(\delta) \beta' |J_1|}$$

for any $c > \delta > 0$ with $c(\delta) = \delta \frac{c'}{c}$ (write $c = c - \delta + \delta$, and use the fact that $\frac{c-\delta}{R} \|\omega(0)\|^2 \leq (c' - c(\delta)) \beta' |J_1|$, whenever $\frac{c}{R} \|\omega(0)\|^2 - c'\beta' |J_1| \leq 0$). We take $\delta = \frac{1}{8}$ (we can always assume that c is larger than that). Hence, altogether, if $0 \in K$,

$$E 1_\pi^2 \leq e^{\frac{1}{8R} \|\omega(0)\|^2} e^{c\beta' T} e^{c|\bar{K}|} e^{-c\beta' (\sum_{J_i \text{ large}} |J_i| + mT)},$$

where $e^{c\beta' T}$ allows one to replace $m-1$ by m. Finally, if $0 \notin K$,

$$E 1_{\pi}^{2} \leq e^{c|\bar{K}|} e^{-c\beta'(\sum_{J_i \text{ large}} |J_i| + (m-1)T)}.$$

These inequalities give the claim (since $|K'| = |K| + mT$) except in one case: no large J_i, $0 \notin K$ and $m = 1$. In that case, J_{n-m+1} and J_{n-m} and J_{n-m-1} are all small (J_{n-m-1} is included in K, unless $|K| = T$, in which case the supremum in (85) is taken over $\mathbf{s}' \in C_s$, with J_0 in (84) equal to J_{n-m-1}). Hence, (85) holds for $i-1 = n-m-1$. We may then apply Proposition 2 with 0 replaced by τ_{n-m-1}, use the fact that $m = 1$ means that there is an interval L, where (61) is violated, intersecting both J_{n-m+1} and J_{n-m}, and get (159) with $m-1$ replaced by $1(= m)$. □

8. Markov Chain Estimates

The goal of this section is to prove Proposition 4. Although $\lambda(d\mathbf{s}|\mathbf{s}')$ defined in (88) does not define a Markov chain, because of the indicator function $\sum_{\mathbf{k}} \chi_{\mathbf{k}}(\mathbf{s}, \mathbf{s}')$, it is close to one, at least up to the time p in which we are interested, and the proof will be based essentially on Markov chain ideas. To see how close λ is to a Markov chain, compare it with $P(d\mathbf{s}|\mathbf{s}') = g_J(\omega)\nu^T_{s(\tau)}(d\mathbf{s})$ (see (78)), which is thus like λ, but without the $\sum_{\mathbf{k}} \chi_{\mathbf{k}}(\mathbf{s}, \mathbf{s}')$; the function $1 - \sum_{\mathbf{k}} \chi_{\mathbf{k}}(\mathbf{s}, \mathbf{s}')$ is supported on $\mathbf{k}$'s such that J is a large interval. For $\mathbf{s}' \in C_s$, we have $\|\omega(\tau)\|^2 \leq 2\beta' T R$ and we can use Proposition 2 to show that there exists a $c > 0$, such that, $\forall \mathbf{s}' \in C_s$, $\forall B \subset C_s$ (note that the support of λ is included in C_s),

$$|\lambda(B|\mathbf{s}') - P(B|\mathbf{s}')| \leq e^{-cT}, \tag{160}$$

and

$$P(C_s|\mathbf{s}') \geq 1 - e^{-cT}. \tag{161}$$

Indeed, if J is large, either there is an interval $L \subset J$ where (56) holds, and we use (28) for that interval, with 0 replaced by τ, $\|\omega(\tau)\|^2 \leq 2\beta' T R$ and $\beta \geq C\beta'$. Or (57) holds, i.e. $D_{\tau+T} \geq \beta' T R$, and we can use (28) with 0 replaced by τ and $t = t' - 1$ replaced by $\tau + T$. Now, we state the main result of this section:

Proposition 6. *There exists a constant $\delta > 0$, $\delta = \delta(R, \rho)$ but independent of T, such that $\forall \mathbf{s_1}, \mathbf{s_2} \in C_s$ and $\forall B \subset C_s$,*

$$\lambda^2(B|\mathbf{s_1}) + \lambda^2(B^c|\mathbf{s_2}) \geq \delta. \tag{162}$$

Remark. The important point in this proposition is that δ is independent of T. The same will be true about the constants δ_1, δ_2, used in the proof (see (174), (176)).

Before proving this proposition, we use it to give the

Proof of Proposition 4. We shall use the previous proposition and a slightly modified version of an argument taken from [3], pp. 197–198. Let, for $B \subset C_s$,

$$\underline{\lambda}(n, B) = \inf_{\mathbf{s} \in C_s} \lambda^n(B|\mathbf{s}), \quad \overline{\lambda}(n, B) = \sup_{\mathbf{s} \in C_s} \lambda^n(B|\mathbf{s}).$$

Fix $\mathbf{s_1}, \mathbf{s_2} \in C_s$ and consider the function defined on subsets $B \subset C_s$:

$$\psi_{\mathbf{s_1},\mathbf{s_2}}(B) = \lambda^2(B|\mathbf{s_1}) - \lambda^2(B|\mathbf{s_2}).$$

Let S^+ be the set such that $\psi_{\mathbf{s_1},\mathbf{s_2}}(B) \geq 0$ for $B \subset S^+$ and $\psi_{\mathbf{s_1},\mathbf{s_2}}(B) \leq 0$ for $B \subset C_s \backslash S^+ \equiv S^-$ ($S^\pm$ depend on $\mathbf{s_1}, \mathbf{s_2}$, but we suppress this dependence). Observe that, by (160, 161), we have, $\forall \mathbf{s} \in C_s$,

$$1 - e^{-cT} \leq \lambda^2(C_s|\mathbf{s}) \leq 1 \tag{163}$$

(with a smaller c than in (160, 161)). Then,

$$|\psi_{\mathbf{s_1},\mathbf{s_2}}(S^+) + \psi_{\mathbf{s_1},\mathbf{s_2}}(S^-)| = |\lambda^2(C_s|\mathbf{s_1}) - \lambda^2(C_s|\mathbf{s_2})| \leq e^{-cT}. \tag{164}$$

Moreover, using (163, 162), and $S^+ \cup S^- = C_s$,

$$\begin{aligned}\psi_{\mathbf{s_1},\mathbf{s_2}}(S^+) &= \lambda^2(S^+|\mathbf{s_1}) - \lambda^2(S^+|\mathbf{s_2}) \\ &\leq 1 - (\lambda^2(S^-|\mathbf{s_1}) + \lambda^2(S^+|\mathbf{s_2})) \leq 1 - \delta. \end{aligned} \tag{165}$$

Thus,

$$\begin{aligned}\overline{\lambda}(t+2, B) - \underline{\lambda}(t+2, B) &= \sup_{\mathbf{s_1},\mathbf{s_2}} \int (\lambda^2(d\mathbf{s}|\mathbf{s_1}) - \lambda^2(d\mathbf{s}|\mathbf{s_2}))\lambda^t(B|\mathbf{s}) \\ &= \sup_{\mathbf{s_1},\mathbf{s_2}} \int \psi_{\mathbf{s_1},\mathbf{s_2}}(d\mathbf{s})\lambda^t(B|\mathbf{s}) \\ &\leq \sup_{\mathbf{s_1},\mathbf{s_2}} (\psi_{\mathbf{s_1},\mathbf{s_2}}(S^+)\overline{\lambda}(t, B) + \psi_{\mathbf{s_1},\mathbf{s_2}}(S^-)\underline{\lambda}(t, B)) \\ &= \sup_{\mathbf{s_1},\mathbf{s_2}} (\psi_{\mathbf{s_1},\mathbf{s_2}}(S^+)(\overline{\lambda}(t, B) - \underline{\lambda}(t, B)) + (\psi_{\mathbf{s_1},\mathbf{s_2}}(S^+) \\ &\quad + \psi_{\mathbf{s_1},\mathbf{s_2}}(S^-))\underline{\lambda}(t, B)) \\ &\leq (1-\delta)(\overline{\lambda}(t, B) - \underline{\lambda}(t, B)) + e^{-cT},\end{aligned}$$

where, to get the last inequality, we used (165) and (164) and $\underline{\lambda}(t, B) \leq 1$. We conclude that $\forall \mathbf{s}' \in C_s$,

$$|\lambda^{2n}(B|\mathbf{s}') - \lambda^{2n}(B|0)| \leq \overline{\lambda}(2n, B) - \underline{\lambda}(2n, B) \leq (1-\delta)^{n-1} + \frac{e^{-cT}}{\delta}.$$

Now, choose first n sufficiently large so that $(1-\delta)^{n-1} \leq \frac{1-\bar{\delta}}{4}$, for some $\bar{\delta} > 0$ and then T sufficiently large so that $\frac{e^{-cT}}{\delta} \leq \frac{1-\bar{\delta}}{4}$. Since, with $p = 2n$,

$$\int |\lambda^p(d\mathbf{s}|\mathbf{s}') - \lambda^p(d\mathbf{s}|0)| \leq 2 \sup_B |\lambda^{2n}(B|\mathbf{s}') - \lambda^{2n}(B|0)|,$$

(89) follows, with δ in that equation equal to $\bar{\delta}$ here. $\square$

Proof of Proposition 6. First of all, observe that it is enough to prove (162) with λ replaced by P:

$$P^2(B|\mathbf{s_1}) + P^2(B^c|\mathbf{s_2}) \geq \delta \tag{166}$$

since we can then use (160) and choose T large enough to obtain the same result for λ, since δ is independent of T.

It will be convenient to write $P^2(d\mathbf{s}_+|\mathbf{s_1}) = \int P(d\mathbf{s}_+|\mathbf{s})P(d\mathbf{s}|\mathbf{s_1})$, where we write $\mathbf{s}_+ \in C_s^+$ meaning $\mathbf{s}_+ \in C_s \subset C([0,T], H_s)$ (see (84)), and similarly $\mathbf{s_1} \in C_s^- \subset C([-2T, -T], H_s)$, $\mathbf{s} \in C_s^0 \subset C([-T, 0], H_s)$, which is the variable over which we integrate.

Turning to the proof, we first get a lower bound on (166) by replacing B, B^c by $B \cap V^+$, $B^c \cap V^+$, where V^+ is defined by

$$V^+ = \left\{ \mathbf{s}_+ \in C_s^+ \,\middle|\, \sum_{n=1}^{t} D_n(\omega_0) < \zeta R t, \qquad \forall t \in [1, T] \right\}, \tag{167}$$

where ζ will be chosen large enough below and $\omega_0(t) = s(t) + l(t, \mathbf{s}_+([0,t]), 0)$. To simplify the notation, we shall assume, from now on, that $B \subset V^+$ and $B^c \equiv V^+ \backslash B$.

Next, we obtain also a lower bound on $P^2(B|\mathbf{s_1}) = \int P(B|\mathbf{s})P(d\mathbf{s}|\mathbf{s_1})$ and on $P^2(B^c|\mathbf{s_2})$ by restricting the integrations over $\mathbf{s}$, so that we have:

$$(166) \geq \int P(B|\mathbf{s})1(\mathbf{s}|\mathbf{s_1})P(d\mathbf{s}|\mathbf{s_1}) + \int P(B^c|\mathbf{s})1(\mathbf{s}|\mathbf{s_2})P(d\mathbf{s}|\mathbf{s_2}), \tag{168}$$

where $1(\mathbf{s}|\mathbf{s}') = 1_0 1_{[-1,0]} 1_{\leq -1}$ with

$$\begin{aligned} 1_0(s(0)) &= 1(\|s(0)\|^2 \leq 3\zeta' R), \\ 1_{[-1,0]}(\mathbf{s}([-1,0])) &= 1\left(\sup_{t\in[-1,0]} \|s(t)\|^2 \leq \zeta R\right), \\ 1_{\leq -1}(\mathbf{s}([-T,-1])|\mathbf{s}') &= 1(\|\omega(-1)\|^2 \leq \zeta' R), \end{aligned} \tag{169}$$

where $\omega(-1) = s(-1) + l(-1, \mathbf{s}([-T,-1]), l(-T))$, with $l(-T) = l(-T, \mathbf{s}'([-2T, -T]), 0)$, and ζ, ζ' are constants that will be chosen large enough below, but with $\zeta' \leq C\zeta$ for C large (ζ, ζ' play a role somewhat similar to β, β' in the previous sections, but they are not necessarily equal to the latter).

Before proceeding further, let us explain the basic idea of the proof. To prove (166), it would be enough to bound $\frac{P^2(B|\mathbf{s_1})}{P^2(B|\mathbf{s_2})} \geq \delta$. We do not quite do that, but first give, in Lemma 8.1 below, a lower bound on $\frac{P(B|\mathbf{s})}{P(B|\mathbf{s}')}$ for $\mathbf{s}$, $\mathbf{s}'$ in a "good" set of configurations, i.e. in the support of the indicator functions that we just introduced. Good here means that the "interaction" (or, to be more precise, the analogue of what is called in Statistical Mechanics the relative Hamiltonian), expressed through the Girsanov formula (see e.g. (179)), between the paths in C_s^0 that are in the support of those indicator functions and those in V^+ is, in some sense, bounded. This relies on Lemma 8.5, which itself follows from the results of the previous section. Next, we show that the probability of reaching that good set, does not depend very much on whether we start from $\mathbf{s_1}$ or $\mathbf{s_2}$ in C_s^- (see Lemma 8.2). This is rather straightforward, but depends on standard estimates on the Brownian bridge (see Lemmas 8.6 and 8.7) that we give in detail, for the sake of completeness. Finally, we need to show that the probability of the set of good configurations, as well as the one of V^+, is bounded from below; this is done in Lemma 8.3. Remember that all the bounds here have to be T-independent, since this was used in an essential way in the proof of the Theorem (Sect. 6).

Now, we shall state and use the lemmas that we need and that will be proven below. Let

$$W = \cup_{\bar{\mathbf{s}} \in C_s^-} \mathrm{supp}(1(\cdot|\bar{\mathbf{s}})).$$

Lemma 8.1. $\exists c = c(R, \rho) > 0$*, such that,* $\forall B \subset V^+$, $\forall \mathbf{s}, \mathbf{s}' \in W$ *with* $s(0) = s'(0)$ *and* $P(B|\mathbf{s}') \neq 0$*:*

$$\frac{P(B|\mathbf{s})}{P(B|\mathbf{s}')} \geq e^{-\frac{c(R,\rho)}{P(B|\mathbf{s}')}}. \tag{170}$$

Defining

$$h(B, s_0) = \sup_{\mathbf{s} \in W:\, s(0)=s_0} P(B|\mathbf{s}) \tag{171}$$

we conclude from the lemma that for all $\mathbf{s} \in W$ such that $s(0) = s_0$,

$$P(B|\mathbf{s}) \geq \ell_B(s_0) \equiv h(B, s_0) e^{-\frac{c(R,\rho)}{h(B,s_0)}}, \tag{172}$$

where both sides vanish if $h(B, s_0) = 0$. Hence, applying the same argument to $P(B^c|\mathbf{s})$, we get:

$$(168) \geq E(\ell_B 1(\cdot|\mathbf{s_1})|\mathbf{s_1}) + E(\ell_{B^c} 1(\cdot|\mathbf{s_2})|\mathbf{s_2}), \tag{173}$$

where here ℓ_B, ℓ_{B^c} are functions of $s(0)$ and E is the (conditional) expectation.

The next lemma controls the dependence on the past in (173):

Lemma 8.2. $\exists \delta_1 > 0$*,* $\delta_1 = \delta_1(R, \rho)$ *such that* $\forall \mathbf{s_1}, \mathbf{s_2} \in C_s^-$, $\forall B \subset V^+$,

$$\frac{E(\ell_{B^c} 1(\cdot|\mathbf{s_2})|\mathbf{s_2})}{E(\ell_{B^c} 1(\cdot|\mathbf{s_1})|\mathbf{s_1})} \geq \delta_1, \tag{174}$$

provided that, in (167), (169), ζ' *is large enough and* $\zeta \geq C\zeta'$ *for* C *large.*

Then, since any δ_1 satisfying (174) must be less than 1,

$$(173) \geq \delta_1 (E(\ell_B 1(\cdot|\mathbf{s_1})|\mathbf{s_1}) + E(\ell_{B^c} 1(\cdot|\mathbf{s_1})|\mathbf{s_1})). \tag{175}$$

But, we also have:

Lemma 8.3. $\exists \delta_2 > 0$*,* $\delta_2 = \delta_2(R, \rho)$*, such that,* $\forall \mathbf{s}' \in C_s^-$,

$$\int P(V^+|\mathbf{s}) 1(\mathbf{s}|\mathbf{s}') P(d\mathbf{s}|\mathbf{s}') \geq \delta_2, \tag{176}$$

and

$$\int 1(\mathbf{s}|\mathbf{s}') P(d\mathbf{s}|\mathbf{s}') \geq \tfrac{1}{2}, \tag{177}$$

provided that, in (167), (169), ζ' *is large enough and* $\zeta \geq C\zeta'$ *for* C *large.*

Remark. The important point here is that δ_2 is independent of T; to show this, we will use the fact that, in (167), the condition on $\sum_1^t$ increases sufficiently fast in time, so that, see below, (217) is finite (however, it should not grow too fast because, to prove Lemma 8.1, we need that it does not grow faster than linearly, so that (183) below holds, leading to the finiteness of (185)).

By definition (171) of h_B, and using this lemma, we have

$$\begin{aligned} &E(h_B 1(\cdot|\mathbf{s_1})|\mathbf{s_1}) + E(h_{B^c} 1(\cdot|\mathbf{s_1})|\mathbf{s_1}) \\ &\geq \int P(B|\mathbf{s}) 1(\mathbf{s}|\mathbf{s_1}) P(d\mathbf{s}|\mathbf{s_1}) + \int P(B^c|\mathbf{s}) 1(\mathbf{s}|\mathbf{s_1}) P(d\mathbf{s}|\mathbf{s_1}) \\ &= \int P(V^+|\mathbf{s}) 1(\mathbf{s}|\mathbf{s_1}) P(d\mathbf{s}|\mathbf{s_1}) \geq \delta_2, \end{aligned} \tag{178}$$

since $B \cup B^c = V^+$. Now, we need the following straightforward consequence of Jensen's inequality:

Lemma 8.4. *For any probability measure* P,

$$E(\ell 1) \geq E(h1) \exp\left(-\frac{cE(1)}{E(h1)}\right) \geq E(h1) \exp\left(-\frac{c}{E(h1)}\right),$$

where E is the expectation with respect to P, $\ell = he^{-\frac{c}{h}}$, *the functions* h, 1, *satisfy* $0 \leq h$, $0 \leq 1 \leq 1$, *h is integrable, and* $c \in \mathbf{R}_+$.

From (178), we may assume $E(h_B 1(\cdot|\mathbf{s_1})|\mathbf{s_1}) \geq \frac{\delta_2}{2}$ (if not, exchange B and B^c). Hence, applying Lemma 8.4 to $E(\ell_B 1(\cdot|\mathbf{s_1})|\mathbf{s_1})$, we get

$$E(\ell_B 1(\cdot|\mathbf{s_1})|\mathbf{s_1}) \geq \frac{\delta_2}{2} \exp\left(-\frac{2c(R,\rho)}{\delta_2}\right).$$

So, combining this with (168), (173), (175), we get:

$$(166) \geq \frac{\delta_1\delta_2}{2} \exp\left(-\frac{2c(R,\rho)}{\delta_2}\right)$$

which finishes the proof of the proposition. □

Now, we still have to prove Lemmas 8.1, 8.2, 8.3.

Proof of Lemma 8.1. Recalling (78) we have

$$P(B|\mathbf{s}) = \int e^{\int_0^T (f,\gamma^{-1}(ds_+ - \frac{1}{2} f dt))} 1_B \nu_{s(0)}(d\mathbf{s}_+) \equiv \int g 1_B \nu_{s(0)}(d\mathbf{s}_+), \tag{179}$$

where $f(t) = f(t, \mathbf{s}_+([0,t]), l(0))$, with $l(0) = l(0, \mathbf{s}([-T,0]), 0)$, is a function of $\mathbf{s}_+ \vee \mathbf{s}$ (the symbol $\vee$ was defined after Eq. (78)), and $\nu_{s(0)}(d\mathbf{s}_+)$ is the Wiener measure with covariance γ, on paths starting at $s(0)$. $P(B|\mathbf{s}')$ is defined similarly with $f'(t) = f(t, \mathbf{s}_+([0,t]), l'(0))$ and $l'(0) = l(0, \mathbf{s}'([-T,0]), 0)$. The corresponding Girsanov factor is denoted g'. Since $s(0) = s'(0)$, we can write

$$\begin{aligned} \frac{P(B|\mathbf{s})}{P(B|\mathbf{s}')} &= Ee^{\int_0^T (f,\gamma^{-1}(ds_+ - \frac{1}{2} f dt)) - (f',\gamma^{-1}(ds_+ - \frac{1}{2} f' dt))} \\ &= Ee^{\int_0^T (f-f',\gamma^{-1}(ds_+ - f' dt)) - \frac{1}{2}\int (f-f',\gamma^{-1}(f-f'))dt}, \end{aligned} \tag{180}$$

where the expectation is taken with respect to the normalized measure

$$\frac{1_B g' \nu_{s(0)}(d\mathbf{s}_+)}{\int 1_B g' \nu_{s(0)}(d\mathbf{s}_+)} = \frac{1_B g' \nu_{s(0)}(d\mathbf{s}_+)}{P(B|\mathbf{s}')}. \tag{181}$$

By Jensen's inequality,

$$(180) \geq e^{E(\int_0^T (f-f',\gamma^{-1}(ds_+ - f'dt)) - \frac{1}{2}\int (f-f',\gamma^{-1}(f-f'))dt)}, \tag{182}$$

We will bound the argument of the exponential. For that, we need some estimates that follow from the results of the previous section:

Lemma 8.5. *$\forall \mathbf{s}, \mathbf{s}' \in W$ and $\forall \mathbf{s}_+ \in V^+$,*

$$\|f(t) - f'(t)\| \leq C(R)e^{-ctR}. \tag{183}$$

The proof of this lemma will be given at the end of this section. Returning to the proof of Lemma 8.1,

$$\left| E\int_0^T (f - f', \gamma^{-1}(f - f'))dt \right| \leq c(R, \rho), \tag{184}$$

since $\gamma_k \geq \rho$ and, by (183),

$$\int_0^\infty \|f(t) - f'(t)\|^2 dt \leq C(R). \tag{185}$$

To bound the stochastic integral in (182) we proceed as in Sect. 7 by defining

$$\eta(t) = 1\Big(\|f(t) - f'(t)\| \leq C(R)e^{-ctR}\Big)$$

with c, $C(R)$ as in (183). Since the measure with respect to which the expectation E is taken has support in $B \subset V^+$ and since (183) holds in V^+, we can write, see (181),

$$\begin{aligned} &\left| E\left(\int_0^T (f - f', \gamma^{-1}(ds_+ - f'dt)\right)\right| \\ &= \left| \int g' d\nu_{s(0)} \left(\int_0^T (\eta(f - f'), \gamma^{-1}(ds_+ - f'dt))\right) 1_B \right| \\ &\leq \frac{(E_b(\int_0^T (\eta(f - f'), \gamma^{-1}db))^2)^{\frac{1}{2}} (\int g' d\nu_{s(0)} 1_B^2)^{\frac{1}{2}}}{P(B|\mathbf{s}')}, \end{aligned} \tag{186}$$

where we changed variables: $ds_+ - f'dt = db$, using Girsanov's formula (backwards), and where E_b denotes the expectation with respect to Brownian motion with covariance γ. Finally, using (185) on the support of η and the fact that $\int g' d\nu_{s(0)} = 1$, we get:

$$(186) \leq \frac{c(R, \rho)}{P(B|\mathbf{s}')}.$$

Combining this, (184) and (182), we conclude

$$\frac{P(B|\mathbf{s})}{P(B|\mathbf{s}')} \geq e^{-\frac{c(R,\rho)}{P(B|\mathbf{s}')}},$$

which proves the lemma. □

Let us turn to Lemma 8.2. It will be useful to study in some detail the paths over the interval $[-1, 0]$. Let $\nu_{s_{-1}s_0}(d\mathbf{s})$ be the (unnormalized) measure defined by the Brownian bridge going from s_{-1} at time -1 to s_0 at time 0, whose total mass is:

$$M(s_0, s_{-1}) = \prod_k \frac{1}{2\pi\gamma_k} \exp\left(-\frac{|s_{0k} - s_{-1k}|^2}{2\gamma_k}\right), \tag{187}$$

where the product runs over k such that $|k|^2 \le \kappa R$. Define

$$P(s_0, s_{-1}|\mathbf{s} \vee \mathbf{s_1}) = \int e^{\int_{-1}^{0}(f,\gamma^{-1}(ds(t)-\frac{1}{2}f dt))} 1_{[-1,0]}(\mathbf{s}) \nu_{s_{-1}s_0}(d\mathbf{s}), \tag{188}$$

where $f(t) = f(t, \mathbf{s}[-1, t], l(-1))$, with $l(-1) = l(-1, \mathbf{s} \vee \mathbf{s_1}([-2T, -1]), 0)$ and similarly

$$P_{s_{-1}}(d\mathbf{s}|\mathbf{s_1}) = e^{\int_{-T}^{-1}(f,\gamma^{-1}(ds(t)-\frac{1}{2}f dt))} \nu_{s_1(-T)s_{-1}}(d\mathbf{s}). \tag{189}$$

Then we can write

$$E(1(\cdot|\mathbf{s_1})\ell_{B^c}|\mathbf{s_1}) = \int \ell_{B^c}(s_0) 1_0(s_0) P(s_0, s_{-1}|\mathbf{s} \vee \mathbf{s_1}) 1_{\le -1}(\mathbf{s}|\mathbf{s_1}) P_{s_{-1}}(d\mathbf{s}|\mathbf{s_1}) ds_0 ds_{-1}. \tag{190}$$

We shall need

Lemma 8.6. *$\exists C_1, C_2$, $C_i = C_i(R, \rho)$, $i = 1, 2$, such that $\forall \bar{\mathbf{s}} \in C_s^-$, and $\forall s_0, s_{-1}, \mathbf{s} \in$ supp$(1(\cdot|\bar{\mathbf{s}}))$:*

$$C_1 \le P(s_0, s_{-1}|\mathbf{s} \vee \bar{\mathbf{s}}) \le C_2, \tag{191}$$

provided that, in (167), (169), ζ' is large enough and $\zeta \ge C\zeta'$ for C large.

From this, Lemma 8.2 follows easily:

Proof of Lemma 8.2. Using (190), we have:

$$\begin{aligned}\frac{E(\ell_{B^c}1(\cdot|\mathbf{s_2})|\mathbf{s_2})}{E(\ell_{B^c}1(\cdot|\mathbf{s_1})|\mathbf{s_1})} &= \frac{\int \ell_{B^c}(s_0)1_0(s_0)P(s_0, s_{-1}|\mathbf{s} \vee \mathbf{s_2})1_{\le -1}(\mathbf{s}|\mathbf{s_2})P_{s_{-1}}(d\mathbf{s}|\mathbf{s_2})ds_0 ds_{-1}}{\int \ell_{B^c}(s_0)1_0(s_0)P(s_0, s_{-1}|\mathbf{s} \vee \mathbf{s_1})1_{\le -1}(\mathbf{s}|\mathbf{s_1})P_{s_{-1}}(d\mathbf{s}|\mathbf{s_1})ds_0 ds_{-1}} \\ &\ge \inf_{s_0} \frac{\int P(s_0, s_{-1}|\mathbf{s} \vee \mathbf{s_2})1_{\le -1}(\mathbf{s}|\mathbf{s_2})P_{s_{-1}}(d\mathbf{s}|\mathbf{s_2})ds_{-1}}{\int P(s_0, s_{-1}|\mathbf{s} \vee \mathbf{s_1})1_{\le -1}(\mathbf{s}|\mathbf{s_1})P_{s_{-1}}(d\mathbf{s}|\mathbf{s_1})ds_{-1}},\end{aligned} \tag{192}$$

where the infimum is taken over $s_0 \in$ supp(1_0). Now use (191) and

$$\int 1_{\le -1}(\mathbf{s}|\mathbf{s_i})P_{s_{-1}}(d\mathbf{s}|\mathbf{s_i})ds_{-1} = \int 1_{\le -1}(\mathbf{s}|\mathbf{s_i})P(d\mathbf{s}|\mathbf{s_i}),$$

for $i = 1, 2$ to bound from below (192) by

$$(192) \ge \frac{C_1 \int 1_{\le -1}(\mathbf{s}|\mathbf{s_2})P(d\mathbf{s}|\mathbf{s_2})}{C_2 \int 1_{\le -1}(\mathbf{s}|\mathbf{s_1})P(d\mathbf{s}|\mathbf{s_1})} \ge \frac{C_1}{C_2} \int 1_{\le -1}(\mathbf{s}|\mathbf{s_2})P(d\mathbf{s}|\mathbf{s_2}) \ge \frac{C_1}{2C_2} = \delta_1.$$

where in the last inequality, we used:

$$\int 1_{\le -1}(\mathbf{s}|\mathbf{s_2})P(d\mathbf{s}|\mathbf{s_2}) \ge \int 1(\mathbf{s}|\mathbf{s_2})P(d\mathbf{s}|\mathbf{s_2}) \ge \frac{1}{2}, \tag{193}$$

where the first inequality is trivial, see (169), and the second follows from (177) in Lemma 8.3. $\square$

Now, we will prove Lemma 8.6, to complete the proof of Lemma 8.2, before proving Lemma 8.3.

Proof of Lemma 8.6. We write, for $t \in [-1, 0]$:

$$s(t) = (1+t)s_0 - ts_{-1} + \alpha(t), \tag{194}$$

where $\alpha(\cdot)$ is the Brownian bridge with covariance γ, going from 0 at time -1 to 0 at time 0, i.e. the Gaussian process with covariance:

$$E(\alpha_k(t')\alpha_p(t)) = \delta_{k,-p}\gamma_k(1+t')(-t) \quad (-1 \le t' \le t \le 0) \tag{195}$$

for k^2, $p^2 \le \kappa R$. Substituting (194) into (188), we get:

$$P(s_0, s_{-1}|\mathbf{s} \vee \bar{\mathbf{s}}) = M(s_0, s_{-1}) \int e^{\int_{-1}^{0}(f, \gamma^{-1}(d\alpha(t)+(s_0-s_{-1}-\frac{1}{2}f)dt))} 1_{[-1,0]}(\mathbf{s})\nu(d\alpha), \tag{196}$$

where ν is the probability distribution of the Brownian bridge α.

To bound $M(s_0, s_{-1})$ remember, from (169), that, for s_0, s_{-1} in the support of $1(\cdot|\bar{\mathbf{s}})$, we have

$$\|s_0\|^2 \le 3\zeta' R, \tag{197}$$

and

$$\|s_{-1}\|^2 \le \|\omega(-1)\|^2 \le \zeta' R. \tag{198}$$

These bounds, combined with the definition (187) of $M(s_0, s_{-1})$ imply that, for s_0, s_{-1} in the support of $1(\cdot|\bar{\mathbf{s}})$,

$$C_2(R, \rho) \le M(s_0, s_{-1}) \le C_1(R, \rho). \tag{199}$$

Thus, to prove (191), we need only to bound from above and from below the integral

$$\int e^{\int_{-1}^{0}(f, \gamma^{-1}(d\alpha(t)+(s_0-s_{-1}-\frac{1}{2}f)dt))} 1_{[-1,0]}(\mathbf{s})\nu(d\alpha) \tag{200}$$

by a constant depending only on R and ρ. For this, some elementary facts about the Brownian bridge will be needed:

Lemma 8.7. *Let α be the Brownian Bridge on* $[-1, 0]$ *with covariance γ. Then*
(a) *There exists a constant $c(R) > 0$ such that*

$$\int 1\left(\sup_{\tau \in [-1,0]} \|\alpha(\tau)\|^2 \le \zeta' R\right) \nu(d\alpha) \ge c(R). \tag{201}$$

(b) *Let $g(t)$ be progressively measurable with* $\sup_{t\in[-1,0]}\|g(t)\| \le A$. *Then*

$$\int e^{\int_{-1}^{0}(g, d\alpha)} \nu(d\alpha) \le C(A, R, \rho), \tag{202}$$

and

$$\int \left(\int_{-1}^{0}(g, d\alpha)\right)^2 \nu(d\alpha) \le C(A, R, \rho). \tag{203}$$

Continuing with (200), we need some bounds on $\|f(t)\|$ for $\mathbf{s}$ in the support of $1(\cdot|\bar{\mathbf{s}})$. First, we have, $\forall \mathbf{s} \in \mathrm{supp}(1(\cdot|\bar{\mathbf{s}}))$,

$$\sup_{t\in[-1,0]} \|l(t)\| \leq C(R), \tag{204}$$

where $l(t) = l(t, \mathbf{s}([-1,t]), l(-1))$, which holds combining (11) in Proposition 1, and the fact that, on $\mathrm{supp}(1(\cdot|\bar{\mathbf{s}}))$ (see (169)), both $\omega(-1)$ and $\sup_{t\in[-1,0]} \|s(t)\|^2$ are of order R. This and $\sup_{t\in[-1,0]} \|s(t)\|^2 \leq \zeta R$ on $\mathrm{supp}(1(\cdot|\bar{\mathbf{s}}))$ imply that $\|\omega(t)\|$ also satisfies (204). Then, using (152), we get:

$$\sup_{t\in[-1,0]} \|f(t)\| \leq C(R). \tag{205}$$

Consider now the lower bound on (200). By Jensen's inequality,

$$(200) \geq C(R,\rho)\left[\int 1_{[-1,0]}(\mathbf{s})\nu(d\alpha)\right]\exp\left[\frac{\int(\int_{-1}^{0}(f,\gamma^{-1}d\alpha)1_{[-1,0]}(\mathbf{s})\nu(d\alpha)}{\int 1_{[-1,0]}(\mathbf{s})\nu(d\alpha)}\right], \tag{206}$$

where $C(R,\rho)$ is a lower bound on $\exp(\int_{-1}^{0}(f,\gamma^{-1}(s_0 - s_{-1} - \frac{1}{2}f)dt)$ (which holds because of (205) and (197), (198)).

Using (197), (198), we obtain from (194) that, for s_0, s_{-1} in the support of $1(\cdot|\bar{\mathbf{s}})$, if $\sup_{t\in[-1,0]} \|\alpha(t)\|^2 \leq \zeta' R$, then $\sup_{t\in[-1,0]} \|s(t)\|^2 \leq C\zeta' R$ for $\zeta \geq C\zeta'$; hence,

$$1_{[-1,0]}(\mathbf{s}) \geq 1\left(\sup_{\tau\in[-1,0]} \|\alpha(\tau)\|^2 \leq \zeta' R\right).$$

Combining this with (206) and (201), we get

$$(200) \geq c(R)C(R,\rho)e^{-c(R)^{-1}\int|\int_{-1}^{0}(f,\gamma^{-1}d\alpha)|1_{[-1,0]}(\mathbf{s})\nu(d\alpha)}, \tag{207}$$

Now, let $g = f\gamma^{-1}$ and use Schwarz' inequality to get the upper bound

$$\int\left|\int_{-1}^{0}(f,\gamma^{-1}d\alpha)\right|1_{[-1,0]}(\mathbf{s})\nu(d\alpha) \leq \left(\int\left(\int_{-1}^{0}(g,d\alpha)\right)^2\nu(d\alpha)\right)^{\frac{1}{2}}, \tag{208}$$

(since $\int \nu(d\alpha) = 1$). Using (205) and $\gamma_k \geq \rho$ we obtain that $\sup_{t\in[-1,0]} \|g(t)\| \leq A = C(R,\rho)$ and so (203) leads to an upper bound $C(R,\rho)$ for (208) and thus, a lower bound $C(R,\rho)$ for (200).

Finally, we bound from above (200) by $C(R,\rho)$, using $1_{[-1,0]}(\mathbf{s}) \leq 1$ and then combining (205), (197), (198), and (202) with $g = f\gamma^{-1}$. $\square$

Proof of Lemma 8.7. (a) Observe that $\alpha(\tau)$ has the same distribution as $(-\tau)b(-\frac{1+\tau}{\tau})$, where $b(\cdot)$ is the Brownian motion starting at 0, with covariance γ. So that, with $t = -\frac{1+\tau}{\tau}$, (201) translates into:

$$\int 1\left(\sup_{t\in[0,\infty[}\left(\frac{\|b(t)\|}{1+t}\right)^2 \leq \zeta' R\right)\nu_0(db) \geq c(R).$$

This is readily proven, since $b(t) \in \mathbf{R}^d$ with $d = d(R)$ and the covariance γ has an R-dependent upper bound.

(b) Let

$$M_t = e^{2\int_{-1}^{t}(g,d\alpha-(\frac{\alpha}{\tau}+\gamma g)d\tau)}.$$

It is easy to see that M_t is a martingale (see e.g. [9] p. 158), and that, therefore, $\forall t \in [-1,0]$, $E(M_t) = 1$, where E is the expectation with respect to $\nu(d\alpha)$. So, write

$$e^{\int_{-1}^{0}(g,d\alpha)} = M_0^{\frac{1}{2}} e^{\int_{-1}^{0}(g,(\frac{\alpha}{\tau}+\gamma g))d\tau}$$

and use Schwarz' inequality and $E(M_0) = 1$ to get

$$\begin{aligned}\int e^{\int_{-1}^{0}(g,d\alpha)}\nu(d\alpha) &\le \left(\int e^{2\int_{-1}^{0}(g,(\frac{\alpha}{\tau}+\gamma g))d\tau}\right)^{\frac{1}{2}}\\ &\le C(A,R,\rho)\left(\int e^{2\int_{-1}^{0}\frac{(g,\alpha)}{\tau}d\tau}\nu(d\alpha)\right)^{\frac{1}{2}},\end{aligned}$$

where $C(A,R,\rho)$ is an upper bound on $\exp\Big(\int_{-1}^{0}(g,\gamma g)d\tau\Big)$. Applying Jensen's inequality to $e^{2\int_{-1}^{0}\frac{(g,\alpha)}{\tau}d\tau}$, with $\frac{d\tau}{2\sqrt{|\tau|}}$ as probability measure on $[-1,0]$, we may bound the RHS by

$$C(A,R,\rho)\left(\int_{-1}^{0}\frac{d\tau}{2\sqrt{|\tau|}}\int e^{4A\|\alpha(\tau)\|\,|\tau|^{-\frac{1}{2}}}\nu(d\alpha)\right)^{\frac{1}{2}},$$

where $\sup_{\tau\in[-1,0]}\|g(\tau)\| \le A$ was used. To finish the proof, observe that

$$\int e^{4A\|\alpha(\tau)\|\,|\tau|^{-\frac{1}{2}}}\nu(d\alpha) \le C(A,R,\rho)$$

since $\|\alpha\| = \left(\sum_k|\alpha_k|^2\right)^{\frac{1}{2}} \le \sum_k|\alpha_k|$, and $\alpha_k(\tau)$ is a Gaussian random variable with variance (see (195)) $\gamma_k(1+\tau)(-\tau)$. Equation (203) is an easy consequence of (202). □

This completes the proof of Lemma 8.6, hence of Lemma 8.2; so, we turn to the

Proof of Lemma 8.3. First, writing

$$\begin{aligned}&\int P(V^+|\mathbf{s})1(\mathbf{s}|\mathbf{s}')P(d\mathbf{s}|\mathbf{s}')\\ &=\int ds_0 P(V^+|(s_0,l(0,\mathbf{s}))1(\mathbf{s}|\mathbf{s}')e^{\int_{-T}^{0}(f,\gamma^{-1}(ds-\frac{1}{2}fdt))}\nu_{s'(-T)s_0}(d\mathbf{s}),\end{aligned}\tag{209}$$

where $l(0,\mathbf{s}) = l(0,\mathbf{s}([-T,0]),0)$ and $f(t) = f(t,\mathbf{s}\vee\mathbf{s}'([-2T,t]),0)$, we obtain the lower bound:

$$\int P(V^+|\mathbf{s})1(\mathbf{s}|\mathbf{s}')P(d\mathbf{s}|\mathbf{s}') \ge I_1 I_2\int 1(\mathbf{s}|\mathbf{s}')P(d\mathbf{s}|\mathbf{s}'),\tag{210}$$

where

$$I_1 = \inf_{\mathbf{s}} \frac{P(V^+|(s_0, l(0,\mathbf{s}))}{P(V^+|(s_0, 0))}, \tag{211}$$

$$I_2 = \inf_{s(0)} P(V^+|(s(0), 0)). \tag{212}$$

and the infimum in (211) is taken over $\mathbf{s} \in \mathrm{supp}(1(\cdot|\mathbf{s}'))$ with $s(0) = s_0$, while in (212) it is taken over $s(0) \in \mathrm{supp}(1_0)$. Now, Lemma 8.1 implies that

$$I_1 \geq \exp\Big(-\frac{c(R,\rho)}{I_2}\Big), \tag{213}$$

provided $I_2 \neq 0$, which we shall show now. Since $\omega_0(t)$ in terms of which V^+ was defined (see (167)) satisfies $\omega_0(0) = (s(0), 0)$, we can write:

$$I_2 = 1 - \sup_{s(0)}(E(1 - 1_{V^+}|\omega_0(0)). \tag{214}$$

To bound $E(1 - 1_{V^+}|\omega_0(0))$ we use the probabilistic estimates (28):

$$P\left(\sum_{n=1}^{t} D_n(\omega) > \zeta Rt|\omega(0)\right) \leq Ce^{-c\zeta t} \tag{215}$$

which hold for any t, $1 \leq t \leq T$, and any $\omega(0)$ with $\|\omega(0)\|^2 \leq 3\zeta' R$, provided ζ is large enough. Note that this condition on $\omega(0)$ holds for $\omega(0) = \omega_0(0) = (s(0), 0)$ and $s(0) \in \mathrm{supp}(1_0)$ (see (169)).

Thus, since $1 - 1_{V^+}$ is the indicator function of the event that

$$\sum_{n=1}^{t} D_n(\omega_0) \geq \zeta Rt \tag{216}$$

for some $t \geq 1$, (215) applied to ω_0 implies

$$E(1 - 1_{V^+}|\omega_0(0)) \leq \sum_{t=1}^{\infty} Ce^{-c\zeta t} \leq Ce^{-c\zeta}, \tag{217}$$

and, by (214),

$$I_2 \geq 1 - Ce^{-c\zeta}. \tag{218}$$

This and (213) implies:

$$I_1 \geq \exp(-c'(R,\rho)). \tag{219}$$

Finally, consider the last factor in (210); let us write

$$\int 1(\mathbf{s}|\mathbf{s}')P(d\mathbf{s}|\mathbf{s}') = 1 - E((1 - 1(\cdot|\mathbf{s}'))|\mathbf{s}'), \tag{220}$$

and let us bound from above $E((1-1(\cdot|\mathbf{s}'))|\mathbf{s}')$; remember that, by (169), $1(\cdot|\mathbf{s}')) = 1_0 1_{[-1,0]} 1_{\leq -1}$. We have

$$\begin{aligned} 1-1(\cdot|\mathbf{s}')) &= 1-1_{\leq -1}+(1-1_0 1_{[-1,0]})1_{\leq -1} \\ &\leq 1-1_{\leq -1}+1\Big(\sup_{t\in[-1,0]}\|s(t)\|^2 \geq 3\zeta' R\Big)1_{\leq -1}, \end{aligned} \tag{221}$$

where we bounded $\zeta R \geq 3\zeta' R$, in the argument of $1_{[-1,0]}$; So,

$$\begin{aligned} &E\Big((1-1(\cdot|\mathbf{s}'))\Big|\mathbf{s}'\Big) \\ &\quad\leq E\Big(1(\|\omega(-1)\|^2 \geq \zeta' R)\Big|\mathbf{s}'\Big)+\sup E\Big(1(\sup_{t\in[-1,0]}\|s(t)\|^2 \geq 3\zeta' R)\Big|\omega(-1)\Big) \\ &\quad= E\Big(1(\|\omega(-1)\|^2 > \zeta' R)\Big|\omega'(-T)\Big)+\sup E\Big(1(\sup_{t\in[-1,0]}\|s(t)\|^2 \geq 3\zeta' R)\Big|\omega(-1)\Big), \end{aligned} \tag{222}$$

where the last term comes from

$$\begin{aligned} &E\Big(1(\sup_{t\in[-1,0]}\|s(t)\|^2 \geq 3\zeta' R)1_{\leq -1}(\cdot|\mathbf{s}')\Big|\mathbf{s}'\Big) \\ &\quad\leq \sup E\Big(1(\sup_{t\in[-1,0]}\|s(t)\|^2 \geq 3\zeta' R)\Big|\omega(-1)\Big), \end{aligned}$$

and the supremum is taken over all $\mathbf{s} \in \operatorname{supp}(1_{\leq -1}(\cdot|\mathbf{s}'))$, i.e. so that $\omega(-1)$ satisfies $\|\omega(-1)\|^2 \leq \zeta' R$.

The first term of (222) is bounded by

$$E\Big(1(\|\omega(-1)\|^2 > \zeta' R)\Big|\omega'(-T)\Big) \leq C\exp(-c\zeta'), \tag{223}$$

for T large: this follows from (30), with 0 replaced by $-T$, t by -1 and the fact that, since $\mathbf{s}' \in C_s$, $\omega'(-T)$ satisfies, by (84),

$$\|\omega'(-T)\| \leq 4\beta' RT. \tag{224}$$

For the second term of (222), we use $D_t(\omega) \geq \frac{1}{2}\|s(t)\|^2$, $\|\omega(-1)\|^2 \leq \zeta' R$ and (34) to bound it also by $C\exp(-c\zeta')$. So, we have

$$E\Big((1-1(\cdot|\mathbf{s}'))\Big|\mathbf{s}'\Big) \leq C\exp(-c\zeta'). \tag{225}$$

So, combining (210), (219), (218) and (220, 225), we get that the LHS of

$$(176) \geq \exp(-c'(R,\rho))(1-C\exp(-c\zeta))(1-C\exp(-c\zeta')) = \delta_2 > 0$$

for ζ, ζ' large enough; obviously (177) follows from (220, 225), for ζ' large enough; this proves the lemma. □

We are left with the

Proof of Lemma 8.5. To prove (183), bound its LHS by

$$\|f(t) - f_0(t)\| + \|f_0(t) - f'(t)\|, \tag{226}$$

where $f_0(t) = f(t, \mathbf{s}_+([0,t]), 0)$ corresponds to ω_0. Now, to bound each term in (226) by $C(R)e^{-cRt}$, use (151), with ω there replaced by ω_0 here, to get:

$$\|f(t) - f_0(t)\| \le C(R)(\|\omega_0(t)\|\|\delta l(t)\| + \|\delta l(t)\|^2) \tag{227}$$

with $\delta l(t) = \omega(t) - \omega_0(t)$. We have, for $t \ge 1$, the bound:

$$\|\delta l(t)\| \le \exp(-cRt)\|\delta l(0)\| = \exp(-cRt)\|l(0)\|, \tag{228}$$

where the equality holds since $\omega_0(0) = (s(0), 0)$, and the inequality follows from (12) (with ω_1 replaced by ω_0) and using the bound, which holds for $t \ge 1$ and where $[t]$ is the integer part of t:

$$a\int_0^t \|\nabla\omega_0\|^2 \le a\sum_{n=1}^{[t]+1} D_n(\omega_0) \le a\zeta R([t]+1) \le \frac{\kappa Rt}{2}$$

for $t \ge 1$ and κ large. For $t \le 1$, (12) yields: $\|\delta l(t)\| \le C(R)\|\delta l(0)\| = C(R)\|l(0)\|$, since, by definition (167) of V^+, $\int_0^t \|\nabla\omega_0\|^2 \le D_1(\omega_0) \le \zeta R$. Finally, $\|\omega_0(t)\|$ in (227) is bounded by $\|\omega_0(t)\|^2 \le D_{[t]+1}(\omega_0) \le \zeta R([t]+1)$, which also follows from the definition of V^+ and which we can write as $\zeta R([t]+1) \le C(R)\exp(\frac{cRt}{2})$. Combining this with (228, 227) gives

$$\|f(t) - f_0(t)\| \le C(R)\exp\left(-\frac{cRt}{2}\right)(\|l(0)\| + \|l(0)\|^2), \tag{229}$$

and a similar bound on $\|f_0(t) - f'(t)\|$ with $l(0)$ replaced by $l'(0)$. Now, on the support of $1(\cdot|\bar{\mathbf{s}})$, for any $\bar{\mathbf{s}} \in C_s^-$, i.e. in W, we have $\|l(0)\| \le C(R)$, $\|l'(0)\| \le C(R)$ (see (204)), which finishes the proof of (183). □

References

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