The Thermodynamic Limit in Mean Field Spin Glass Models

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Abstract: We present a simple strategy in order to show the existence and uniqueness of the infinite volume limit of thermodynamic quantities, for a large class of mean field disordered models, as for example the Sherrington-Kirkpatrick model, and the Derrida p-spin model. The main argument is based on a smooth interpolation between a large system, made of N spin sites, and two similar but independent subsystems, made of N_1 and N_2 sites, respectively, with $N_1 + N_2 = N$. The quenched average of the free energy turns out to be subadditive with respect to the size of the system. This gives immediately convergence of the free energy per site, in the infinite volume limit. Moreover, a simple argument, based on concentration of measure, gives the almost sure convergence, with respect to the external noise. Similar results hold also for the ground state energy per site.

1. Introduction

The main objective of this paper is to propose a general strategy in order to control the infinite volume limit of thermodynamic quantities for a class of mean field spin glass models. For the sake of definiteness, we consider firstly in full detail the Sherrington-Kirkpatrick (SK) model, [12, 13, 8]. Then, we show how to generalize our method to similar related mean field disordered models, as for example the Derrida p-spin model, [2, 3].

It is very well known that the rigorous control of the infinite volume limit for these mean field models is very difficult, due to the effects of very large fluctuations produced by the external noise. In particular, it is very difficult to produce very effective trial states, to be exploited in variational principles. It is only for the high temperature, or high external field, regime that a satisfactory control can be reached, as shown for example in [11, 16, 6].

We will introduce a very simple strategy for the control of the infinite volume limit. The main idea is to split a large system, made of N spin sites, into two subsystems, made of N_1 and N_2 sites, respectively, where each subsystem is subject to some external noise, similar but *independent* from the noise acting on the large system. By a smooth

interpolation between the system and the subsystems, we will show subadditivity of the quenched average of the free energy, with respect to the size of the system, and, therefore, obtain complete control of the infinite volume limit.

Moreover, the well known selfaveraging of the free energy density, as shown originally by Pastur and Shcherbina in [9], extended to the estimates given by the concentration of measure, as explained in [15, 7], does allow an even more detailed control of the limit. In effect, it will turn out that the free energy per site, without quenched average, converges almost surely, with respect to the external noise. These results extend to other thermodynamic quantities, in particular to the ground state energy per site, as will be shown in the paper.

The organization of the paper is as follows. In Sect. 2 we recall the general structure of the Sherrington-Kirkpatrick mean field spin glass model, in order to define the main quantities, and fix the notations. Next, Sect. 3 contains the main results of the paper, related to the control of the infinite volume limit. In Sect. 4 we show how to extend our results to other mean field spin glass models, in particular to the Derrida p-spin model and to models with non-Gaussian couplings. Section 5 contains conclusions and outlook for future developments and extensions.

2. The Structure of the Sherrington-Kirkpatrick Model

Let us recall some basic definitions.

Ising spin variables $\sigma_i = \pm 1$, attached to each site $i = 1, 2, \ldots, N$, define the generic configuration of the mean field spin glass model. The external quenched disorder is given by the $N(N-1)/2$ independent and identical distributed random variables J_{ij}, defined for each couple of sites. For the sake of simplicity, we assume each J_{ij} to be a centered unit Gaussian with averages

$$E(J_{ij}) = 0, \quad E(J_{ij}^2) = 1.$$

The Hamiltonian of the model, in some external field of strength h, is given by

$$H_N(\sigma, h, J) = -\frac{1}{\sqrt{N}} \sum_{1 \leq i < j \leq N} J_{ij}\sigma_i\sigma_j - h \sum_{i=1}^{N} \sigma_i. \tag{1}$$

The first term in (1) is a long range random two body interaction, while the second represents the interaction of the spins with a fixed external magnetic field h.

For a given inverse temperature β, we introduce the disorder dependent partition function $Z_N(\beta, h, J)$, the quenched average of the free energy per site $f_N(\beta, h)$, the Boltzmann state ω_J, and the auxiliary function $\alpha_N(\beta, h)$, according to the definitions

$$Z_N(\beta, h, J) = \sum_{\{\sigma\}} \exp(-\beta H_N(\sigma, h, J)), \tag{2}$$

$$-\beta f_N(\beta, h) = N^{-1} E \log Z_N(\beta, h, J) = \alpha_N(\beta, h), \tag{3}$$

$$\omega_J(A) = Z_N(\beta, h, J)^{-1} \sum_{\{\sigma\}} A \exp(-\beta H_N(\sigma, h, J)), \tag{4}$$

where A is a generic function of the σ's. In the notation ω_J, we have stressed the dependence of the Boltzmann state on the external noise J, but, of course, there is also a dependence on β, h and N.

Let us now introduce the important concept of replicas. Consider a generic number s of independent copies of the system, characterized by the Boltzmann variables $\sigma_i^{(1)}$, $\sigma_i^{(2)}$, ..., distributed according to the product state

$$\Omega_J = \omega_J^{(1)} \omega_J^{(2)} \ldots \omega_J^{(s)},$$

where all $\omega_J^{(\alpha)}$ act on each one $\sigma_i^{(\alpha)}$'s, and are subject to the *same* sample J of the external noise. Clearly, the *Boltzmannfaktor* for the replicated system is given by

$$\exp\left(-\beta(H_N(\sigma^{(1)}, h, J) + H_N(\sigma^{(2)}, h, J) + \cdots + H_N(\sigma^{(s)}, h, J))\right). \quad (5)$$

The overlaps between any two replicas a, b are defined according to

$$q_{ab}(\sigma^{(a)}, \sigma^{(b)}) = \frac{1}{N} \sum_i \sigma_i^{(a)} \sigma_i^{(b)},$$

and they satisfy the obvious bounds

$$-1 \leq q_{ab} \leq 1.$$

For a generic smooth function F of the overlaps, we define the $\langle \rangle$ averages

$$\langle F(q_{12}, q_{13}, \ldots) \rangle = E\Omega_J\big(F(q_{12}, q_{13}, \ldots)\big),$$

where the Boltzmann averages Ω_J act on the replicated σ variables, and E is the average with respect to the external noise J.

3. Control of the Infinite Volume Limit

Let us explain the main idea behind our method. We divide the N sites into two blocks N_1, N_2 with $N_1 + N_2 = N$, and define

$$Z_N(t) = \sum_{\{\sigma\}} \exp\left(\beta\sqrt{\frac{t}{N}} \sum_{1 \leq i < j \leq N} J_{ij}\sigma_i\sigma_j + \beta\sqrt{\frac{1-t}{N_1}} \sum_{1 \leq i < j \leq N_1} J'_{ij}\sigma_i\sigma_j \right.$$

$$\left. +\beta\sqrt{\frac{1-t}{N_2}} \sum_{N_1 < i < j \leq N} J''_{ij}\sigma_i\sigma_j \right) \exp \beta h \sum_{i=1}^N \sigma_i, \quad (6)$$

with $0 \leq t \leq 1$.

The external noise is represented by the *independent* families of unit Gaussian random variables J, J' and J''. Notice that the two subsystems are subject to a different external noise, with respect to the original system. But, of course, the probability distributions are the same. The parameter t allows to interpolate between the original N spin system at $t = 1$ and a system composed of two non-interacting parts at $t = 0$, so that

$$Z_N(1) = Z_N(\beta, h, J), \quad (7)$$
$$Z_N(0) = Z_{N_1}(\beta, h, J')Z_{N_2}(\beta, h, J''). \quad (8)$$

As a consequence, by taking into account the definition in (3), we have

$$E \ln Z_N(1) = N\alpha_N(\beta, h), \tag{9}$$
$$E \ln Z_N(0) = N_1 \alpha_{N_1}(\beta, h) + N_2 \alpha_{N_2}(\beta, h). \tag{10}$$

By taking the derivative of $N^{-1} E \ln Z_N(t)$ with respect to the parameter t, we obtain

$$\frac{d}{dt} \frac{1}{N} E \ln Z_N(t) = \frac{\beta}{2N} E \left(\frac{1}{\sqrt{tN}} \sum_{1 \le i < j \le N} J_{ij} \omega_t(\sigma_i \sigma_j) \right.$$
$$- \frac{1}{\sqrt{(1-t)N_1}} \sum_{1 \le i < j \le N_1} J'_{ij} \omega_t(\sigma_i \sigma_j)$$
$$\left. - \frac{1}{\sqrt{(1-t)N_2}} \sum_{N_1 < i < j \le N} J''_{ij} \omega_t(\sigma_i \sigma_j) \right), \tag{11}$$

where $\omega_t(.)$ denotes the Gibbs state corresponding to the partition function (6). A standard integration by parts on the Gaussian noise, as done for example in [8, 5], gives

$$\frac{d}{dt} \frac{1}{N} E \ln Z_N(t) = \frac{\beta^2}{4N^2} \sum_{i,j=1}^{N} E(1 - \omega_t^2(\sigma_i \sigma_j)) - \frac{\beta^2}{4NN_1} \sum_{i,j=1}^{N_1} E(1 - \omega_t^2(\sigma_i \sigma_j))$$
$$- \frac{\beta^2}{4NN_2} \sum_{i,j=N_1+1}^{N} E(1 - \omega_t^2(\sigma_i \sigma_j)) \tag{12}$$
$$= -\frac{\beta^2}{4} \left\langle q_{12}^2 - \frac{N_1}{N} (q_{12}^{(1)})^2 - \frac{N_2}{N} (q_{12}^{(2)})^2 \right\rangle, \tag{13}$$

where we have defined

$$N_1 q_{12}^{(1)} = \sum_{i=1}^{N_1} \sigma_i^1 \sigma_i^2, \tag{14}$$

$$N_2 q_{12}^{(2)} = \sum_{i=N_1+1}^{N} \sigma_i^1 \sigma_i^2. \tag{15}$$

Since q_{12} is a convex linear combination of $q_{12}^{(1)}$ and $q_{12}^{(2)}$ in the form

$$q_{12} = \frac{N_1}{N} q_{12}^{(1)} + \frac{N_2}{N} q_{12}^{(2)},$$

due to convexity of the function $f : x \to x^2$, we have the inequality

$$\left\langle q_{12}^2 - \frac{N_1}{N} (q_{12}^{(1)})^2 - \frac{N_2}{N} (q_{12}^{(2)})^2 \right\rangle \le 0.$$

Therefore, we can state our first preliminary result.

Lemma 1. *The quenched average of the logarithm of the interpolating partition function, defined by (6), is increasing in t, i.e.*

$$\frac{d}{dt}\frac{1}{N}E \ln Z_N(t) \geq 0. \tag{16}$$

By integrating in t and recalling the boundary conditions (9), we get the first main result.

Theorem 1. *The following superadditivity property holds*

$$N \alpha_N(\beta, h) \geq N_1 \alpha_{N_1}(\beta, h) + N_2 \alpha_{N_2}(\beta, h). \tag{17}$$

Of course, due to the minus sign in (3), we have subadditivity for the quenched average of the free energy.

The subadditivity property gives an immediate control on the infinite volume limit, as explained for example in [10]. In fact, we have

Theorem 2. *The infinite volume limit for $\alpha_N(\beta, h)$ does exist and equals its* sup

$$\lim_{N\to\infty} \alpha_N(\beta, h) = \sup_N \alpha_N(\beta, h) \equiv \alpha(\beta, h). \tag{18}$$

For finite N and a given realization J of the disorder, define the ground state energy density $-e_N(J, h)$ as

$$-e_N(J, h) = \frac{1}{N} \inf_\sigma H_N(\sigma, h, J). \tag{19}$$

Now we show, from simple thermodynamic properties, that Eq. (18) implies the existence of the thermodynamic limit for $E(e_N(h, J))$. First of all, notice that the bounds

$$e^{\beta N e_N(J,h)} \leq \sum_{\{\sigma\}} e^{-\beta H_N(\sigma,h,J)} \leq 2^N e^{\beta N e_N(J,h)} \tag{20}$$

hold for any J, N, β, h, so that

$$0 \leq \frac{\ln Z_N(\beta, h, J)}{\beta N} - e_N(J, h) \leq \frac{\ln 2}{\beta}. \tag{21}$$

The bounds (21), together with the obvious

$$\partial_\beta \frac{\ln Z_N(\beta, h, J)}{\beta} \leq 0,$$

imply that

$$\lim_{\beta\to\infty} \frac{\ln Z_N(\beta, h, J)}{\beta N} \downarrow e_N(J, h). \tag{22}$$

Of course, by taking the expectation value in (21) and defining

$$e_N(h) = E e_N(J, h),$$

one also finds

$$\lim_{\beta\to\infty} \frac{\alpha_N(\beta, h)}{\beta} \downarrow e_N(h). \tag{23}$$

Therefore, by taking into account the superadditivity (17), the inequalities (21), and the existence of the limit $\alpha(\beta, h)$ for $\alpha_N(\beta, h)$, we have from (23) the proof of the following

Theorem 3. *For the quenched average of the ground state energy we have the subaddi-tivity property*

$$N\, e_N(h) \geq N_1\, e_{N_1}(h) + N_2\, e_{N_2}(h) \tag{24}$$

and the existence of the infinite volume limit

$$\lim_{N\to\infty} e_N(h) = \sup_N e_N(h) \equiv e_0(h). \tag{25}$$

Finally, we can write the limit $e_0(h)$ in terms of $\alpha(\beta, h)$ as

$$\lim_{\beta\to\infty} \frac{\alpha(\beta, h)}{\beta} \downarrow e_0(h). \tag{26}$$

After proving the existence of the thermodynamic limit for the quenched averages, we can easily extend our results to prove that convergence holds for almost every disorder realization J. In fact, we can state

Theorem 4. *The infinite volume limits*

$$\lim_{N\to\infty} \frac{1}{N} \ln Z_N(\beta, h, J) = \alpha(\beta, h), \tag{27}$$

$$\lim_{N\to\infty} e_N(J, h) = e_0(h), \tag{28}$$

do exist J-almost surely.

For the proof, we notice that the fluctuations of the free energy per site vanish expo-nentially fast as N grows, a result strengthening the pioneering quadratic selfaveraging proven in [9]. Indeed, the following result holds [15]:

$$P\left(\left|\frac{1}{\beta N} \ln Z_N(\beta, h, J) - \frac{1}{\beta N} E \ln Z_N(\beta, h, J)\right| \geq u\right) \leq e^{-Nu^2/2}. \tag{29}$$

Since the r.h.s. of (29) is summable in N for every fixed u, the Borel-Cantelli lemma [14], and the convergence given by (18) imply (27). The same argument can be exploited for the ground state energy. In fact, by taking the $\beta \to \infty$ limit in (29), we get

$$P\left(|e_N(J, h) - e_N(h)| \geq u\right) \leq e^{-Nu^2/2}. \tag{30}$$

Again, the Borel-Cantelli lemma implies (28), and the theorem is proven.

Notice that all the results of this section hold also in the case where on each spin σ_i acts on a random magnetic field h_i, where the h_i's are i.i.d. random variables.

4. Existence of the Thermodynamic Limit for Other Mean Field Spin Glass Models

In this section, we show how the above results on the (almost sure) existence of the ther-modynamic limit for the free energy and for the ground state energy can be extended to other mean field spin glass models. In the first place, we can immediately extend the approach to p-spin models, for even p. On the other hand, we can allow the quenched disorder variables to be non-Gaussian, provided that suitable bounds are imposed on their moments.

4.1. p-spin models. The p-spin model is defined by the Hamiltonian

$$H_N^{(p)}(\sigma, h, J) = -\sqrt{\frac{p!}{2N^{p-1}}} \sum_{(i_1,\dots,i_p)} J_{i_1\dots i_p}\sigma_{i_1}\dots\sigma_{i_p} - h\sum_i \sigma_i, \qquad (31)$$

where p is an integer and $J_{i_1\dots i_p}$ are i.i.d. unit Gaussian random variables. The summation in the first term is performed on all the different p-ples of indices $i_1, \dots, i_p$. Note that for $p = 2$ this is just the SK model. The p-spin model has been proposed by Derrida in [2], and extensively studied thereafter, see for example [3, 8, 17, 18].

For the sake of simplicity, we consider the case of even p. As we did for the SK model, we define the auxiliary partition function $Z_N(t)$, in analogy with (6). By taking the t derivative, we find after integration by parts

$$\frac{d}{dt}\frac{1}{N}E \ln Z_N(t) = -\frac{\beta^2}{4}\left\langle q_{12}^p - \frac{N_1}{N}(q_{12}^{(1)})^p - \frac{N_2}{N}(q_{12}^{(2)})^p \right\rangle + O(1/N) \qquad (32)$$

$$\geq O(1/N),$$

for p even, by the same convexity argument as before. It is easy to realize the reason for the appearance of the terms $O(1/N)$. In fact, for $p = 2$, we can write

$$\frac{2}{N^2}\sum_{(i,j)}E\,(1 - \omega_t^2(\sigma_i\sigma_j)) = \frac{1}{N^2}\sum_{i,j=1}^N E\,(1 - \omega_t^2(\sigma_i\sigma_j)) = (1 - \langle q_{12}^2\rangle), \qquad (33)$$

as already exploited in (12). On the other hand, for $p > 2$ one has

$$\frac{p!}{N^p}\sum_{(i_1,\dots,i_p)}E\,(1 - \omega_t^2(\sigma_{i_1}\dots\sigma_{i_p})) = \frac{1}{N^p}\sum_{i_1,\dots,i_p=1}^N E\,(1 - \omega_t^2(\sigma_{i_1}\dots\sigma_{i_p})) + O(1/N)$$

$$= (1 - \langle q_{12}^p\rangle) + O(1/N). \qquad (34)$$

From (32) one finds, as in the previous section, the existence of the infinite volume limits

$$\lim_{N\to\infty} \alpha_N^{(p)}(\beta, h) \equiv \alpha^{(p)}(\beta, h), \qquad (35)$$

$$\lim_{N\to\infty} e_N^{(p)}(h) \equiv e_0^{(p)}(h) = \lim_{\beta\to\infty}\frac{\alpha^{(p)}(\beta, h)}{\beta}. \qquad (36)$$

Moreover, the estimate (29) holds also in this case, since it is based only on the fact that $1/N\beta \ln Z_N(\beta, h, J)$, as a function of the variables J, has a Lipshitz constant of order $C/\sqrt{N}$, where C is a constant independent of β [15]. Therefore, also in this case we have almost sure convergence for the free energy and for the ground state energy.

4.2. Non-Gaussian couplings. The method developed in the previous sections allows to prove the existence of the thermodynamic limit for $\alpha_N(\beta, h)$ and for $e_N(h)$ also for SK (or p-spin) models with non-Gaussian couplings, provided that the variables J_{ij} (respectively, $J_{i_1,\dots,i_p}$) are i.i.d. symmetric random variables with finite fourth moment, i.e., $P(J_{ij}) = P(-J_{ij})$ and $EJ_{ij}^4 < \infty$. A similar condition has been also exploited for the study of the model, at zero external field and high temperature, see for example

[1] and [4]. Consider for instance the SK case. The integration by parts on the disorder variables in (11) can be performed by use of the formula

$$E \, \eta F(\eta) = E \, \eta^2 F'(\eta) - \frac{1}{4} E \, |\eta| \int_{-|\eta|}^{|\eta|} (\eta^2 - t^2) F'''(t) \, dt, \qquad (37)$$

which holds for any symmetric random variable η and for sufficiently regular functions F, as a simple direct calculation shows. A similar expression has been exploited by Talagrand in [19], for dichotomic variables. By applying this formula to the various terms in (11), one finds that

$$\frac{d}{dt} \frac{1}{N} E \, \ln Z_N(t) = -\frac{\beta^2 J^2}{4} \left\langle q_{12}^2 - \frac{N_1}{N} (q_{12}^{(1)})^2 - \frac{N_2}{N} (q_{12}^{(2)})^2 \right\rangle + O(N^{-1/2}), \qquad (38)$$

where

$$J^2 = E \, J_{ij}^2.$$

The error terms arise from the estimates

$$\partial_{J_{ij}}^3 \omega_t(\sigma_i \sigma_j) = O(N^{-3/2})$$

and

$$E \, J_{ij}^2 \omega_t^2(\sigma_i \sigma_j) = J^2 E \, \omega_t^2(\sigma_i \sigma_j) + O(N^{-1/2}).$$

The existence of the thermodynamic limit for the quenched averages of the free energy and the ground state energy then follows.

In order to prove J-almost sure convergence, for the sake of simplicity, let us consider the case where the random variables J_{ij} are bounded. Then, the estimate (29) still holds, with the r.h.s. modified into $\exp(-N \, K \, u^2)$, where K does not depend on β. This can be proved, for instance, by using Theorem 6.6 of [20]. Almost sure convergence for the free energy and for the ground state energy then follows immediately from the Borel-Cantelli lemma. In particular, this includes the important case where $J_{ij} = \pm 1$ with equal probability. The extension to more general cases, where only the condition $E \, J_{ij}^4 < \infty$ is required, is possible, and will be reported in future work.

5. Conclusions and Outlook

We have seen that the rigorous control of the infinite volume limit for mean field spin glass models can be obtained through a simple strategy, by a smooth interpolation between a large system and its splitting into subsystems, provided the external noises are taken independent.

The extension of our methods to the important cases of diluted models, and other models of the neural network type, will be taken into account in future work.

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