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A Hamiltonian Model for Linear Friction in a Homogeneous Medium

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Abstract: We introduce and study rigorously a Hamiltonian model of a classical particle moving through a homogeneous dissipative medium at zero temperature in such a way that it experiences an effective *linear* friction force proportional to its velocity (at small speeds). The medium consists at each point in a space of a vibration field modelling an obstacle with which the particle exchanges energy and momentum in such a way that total energy and momentum are conserved. We show that in the presence of a constant (not too large) external force, the particle reaches an asymptotic velocity proportional to this force. In a potential well, on the other hand, the particle comes exponentially fast to rest in the bottom of the well. The exponential rate is in both cases an explicit function of the model parameters and independent of the potential.

1. Introduction

Many simple microscopic or macroscopic systems obey (at zero temperature) an effective equation of motion of the type

$$m\ddot{q}(t) + \gamma\dot{q}(t) = -\nabla V(q(t)), \quad \gamma > 0. \tag{1.1}$$

Examples include the motion of electrons in a metal, or of a small particle in a viscous medium, but the coordinate q needs not always be of a geometrical nature. The energy loss due to the linear friction force $-\gamma\dot{q}$ (occurring at a rate $-\gamma\dot{q}^2$) implied by this equation leads to several well-known phenomena. First, for confining potentials V, the particle will come to a stop exponentially fast (with rate $\frac{\gamma}{2m}$ if γ is small enough) at one of the critical points of the potential. Note in particular that the decay rate $\frac{\gamma}{2m}$ does not depend on the potential V. If, on the other hand, $V(q) = -F \cdot q$, for some $F \in \mathbb{R}^d$, the particle will reach a limiting speed $v(F) = \frac{F}{\gamma}$ which is proportional to the applied field. This, in particular, is at the origin of Ohm's law. Again, the approach is exponential, but this time with rate $\frac{\gamma}{m}$. In particular, if $F = 0$, the particle comes exponentially fast to a full stop. The phenomenological friction force summarizes the reaction of the environment of the particle to its passage and the energy lost by the particle is transferred to the medium surrounding the particle by various processes (such as inelastic collisions, for example). A more fundamental, microscopic treatment of these phenomena requires therefore considering the combined system consisting of the particle and the medium. This combined system should allow for a Hamiltonian treatment in which the total energy is conserved.

Our goal in this paper is to present and study a Hamiltonian model of a system composed of a particle and a homogeneous medium. We show rigorously that the particle has the behaviour described above and analyse the physical mechanisms at the origin of the observed phenomena. We stay within the context of classical mechanics and at zero temperature, hoping to come back to other points in the $\hbar - T$ plane at a later date. In particular, at positive temperature, a fluctuating force term is to be added to (1.1), transforming the equation to the Langevin equation. Such a term is indeed produced by our model, but is much harder to analyse at positive temperatures.

The model we consider consists of one classical particle that is, on the one hand, coupled to "obstacles" represented by scalar vibration fields and on the other subjected to a time-independent external force $F = -\nabla V$. We are mostly interested in the more difficult case where F is constant (so $V = -F \cdot q$), but we will also deal with confining potentials.

More precisely, the equations of motion for the coupled system are:

$$\partial_t^2 \psi(x, y, t) - c^2 \Delta_y \psi(x, y, t) = -\rho_1(x - q(t))\sigma_2(y), \tag{1.2}$$

$$\ddot{q}(t) = -\nabla V(q(t)) - \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^n} dy \rho_1(x - q(t))\sigma_2(y)(\nabla_x \psi)(x, y, t). \tag{1.3}$$

Here ψ is the vibration field and $q \in \mathbb{R}^d$ the position of the particle. The "form factor" $\rho_1(x)\sigma_2(y)$ determines the coupling of the particle to the vibration field ψ. We shall assume:

(H1) $\rho_1(x)\sigma_2(y) \in C_0^\infty(\mathbb{R}^{d+n})$, $\rho_1\sigma_2 \neq 0$, where $\rho_1, \sigma_2 \geq 0$, are radial functions with $\rho_1(x) = 0$ if $|x| \geq R_1 > 0$ and $\sigma_2(y) = 0$ if $|y| \geq R_2 > 0$.

To obtain our main results (Sects. 4 and 5), we will need to take the propagation speed c large enough, for reasons that will be explained then. In the first part of the paper, on the other hand, it is convenient to absorb c through the scaling

$$\phi(x, y, t) = c^{\frac{n}{2}} \psi(x, cy, t) \quad \text{and} \quad \rho_2(y) = c^{\frac{n}{2}} \sigma_2(cy), \tag{1.4}$$

so that (1.2)–(1.3) is transformed into

$$\partial_t^2 \phi(x, y, t) - \Delta_y \phi(x, y, t) = -\rho_1(x - q(t))\rho_2(y), \tag{1.5}$$

$$\ddot{q}(t) = -\nabla V(q(t)) - \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^n} dy\, \rho_1(x - q(t))\rho_2(y)(\nabla_x \phi)(x, y, t). \tag{1.6}$$

Note that the field $\phi(x, y, t) \equiv \phi_t(x, y)$ plays the role of a potential for the particle. Indeed, the second term in (1.6) is $\mathcal{F}_{\phi_t}(q(t))$, where

$$\mathcal{F}_\phi(q) = -\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^n} dy\, \rho_1(x - q)\rho_2(y)(\nabla_x \phi)(x, y) \tag{1.7}$$

is the force exerted on the particle by the environment when the latter is in the state ϕ; this becomes $\mathcal{F}_\phi(q) = -\nabla\phi(q, 0)$ if we consider a point interaction so that $\rho_1(x)\rho_2(y) = \delta(x)\delta(y)$. To understand the model intuitively the following observations are helpful. First of all, the particle moves in x-space (or, more precisely, in the $y = 0$ subspace of $\mathbb{R}^{d+n}$). In fact, one can think of $\phi(x, \cdot)$ as representing, for each fixed value of x in the configuration space of the particle, an "obstacle", which has a large number of degrees of freedom, and is therefore modeled for simplicity by a vibration field $\phi(x, \cdot)$. The variables y should in other words not be interpreted geometrically and are in particular not spatial variables for the particle. To understand this, it is helpful to Fourier transform (1.5)–(1.6) in the y variable to obtain

$$\partial_t^2\hat{\phi}(x, k, t) + |k|^2\hat{\phi}(x, k, t) = -\rho_1(x - q(t))\hat{\rho}_2(k), \tag{1.8}$$

$$\ddot{q}(t) = -\nabla V(q(t)) + \int_{\mathbb{R}^n} dk\, \tilde{\mathcal{F}}_{\phi_t}(q(t), k), \tag{1.9}$$

where

$$\tilde{\mathcal{F}}_\phi(q, k) = -\int_{\mathbb{R}^d} dx \rho_1(x - q)\hat{\rho}_2(k)\nabla_x\hat{\phi}(x, k). \tag{1.10}$$

Clearly, $\hat{\phi}(x, k, t)$ is, for each value of x and k, the amplitude of a driven oscillator of frequency $\omega(k) = |k|$. All of these oscillators are decoupled and each of them contributes separately a force $-\rho_1(x - q(t))\hat{\rho}_2(k)\nabla_x\hat{\phi}(x, k, t)$ acting on the particle.

One way to get an intuitive understanding of why this model should exhibit dissipative behaviour is to imagine for a moment the particle is constrained to move in one dimension ($x \in \mathbb{R}$), and that $y \in \mathbb{R}^2$, so that one can picture $\phi(x, y)$ as describing the vibrations of an elastic membrane positioned at x, perpendicular to the axis on which the particle moves. As the particle hits the successive membranes, it creates a wake, much like a boat ploughing the surface of a lake (Fig. 1). Taking for the moment $V(q) = 0$ in (1.6) (so that there is no external field: $F = 0$) one can imagine launching the particle with an initial speed v_0, with all membranes initially at rest. In that case intuition predicts that the particle should lose all its kinetic energy into the membranes and come to a full stop. We shall prove that this intuition is correct and that the particle stops exponentially fast for arbitrary values of d, but with $n = 3$ (Theorem 3) and for c large enough.

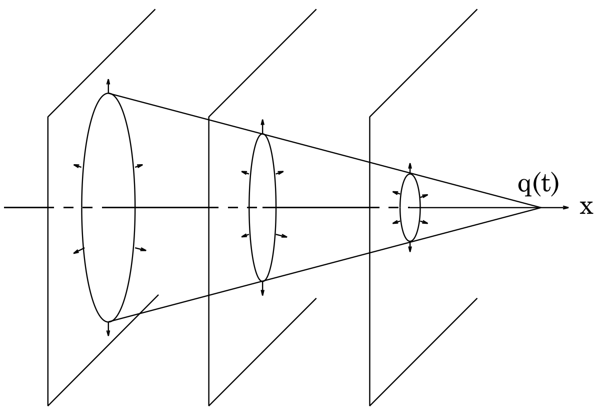

Fig. 1. Waves created by the passage of the particle through the successive membranes

The physical origin of these restrictions to the case where n equals 3 and c is large is explained in Sect. 2 and at the end of Sect. 4.

Another situation of interest is the case where V is confining. Then techniques similar to the ones used in [KKS1] allow to show the particle comes to rest at one of the equilibrium positions of the potential V. We furthermore show this approach is exponential with the expected rate (Theorem 4) provided the particle comes to rest on a non-degenerate minimum of the potential.

Our main interest is in the case where $V(q) = -F \cdot q$. In that case, we show that for a suitable class of initial conditions (and for c large enough) the particle approaches asymptotically a constant speed $v(F)$ (Theorem 2) which is linear in F for small F, as in an ohmic medium.

Various Hamiltonian models for dissipation in general and for linear friction in particular have previously been proposed in the physics literature, mostly with the purpose of deriving the classical or quantum Langevin equation (see [CEFM] and [FLO] for further references). As in the model we propose here (see (1.8)–(1.9)), they all involve the coupling of a particle to a family of independent oscillators representing the degrees of freedom of the environment. Our model has the particular feature of describing a homogeneous (i.e. translationally invariant) medium to which the particle is coupled in a translationally invariant manner (see (3.3)). The coupling is therefore non-linear in the particle position (no dipole approximation), while it is linear in the field variables. It is the only Hamiltonian model we are aware of that describes linear friction at low speeds in the presence of each of the three most commonly studied potentials: $V = 0$, $V = -F \cdot q$ and V confining.

In more realistic models, one ought to couple the oscillators at different points in space. This is easily done in the context of our models by changing the potential energy of the field into

$$\int dx\, dy\, \left(c_1^2 |\nabla_x \psi(x, y)|^2 + c_2^2 |\nabla_y \psi(x, y)|^2\right).$$

It turns out, however, that in that case the force exerted by the medium on a particle moving at constant speed v vanishes identically for all $|v| \leq c_1$. In such models, the friction force is therefore proportional to higher derivatives of q. In particular, this is the case when $c = c_1 = c_2$, as in the model for radiation damping studied in [KKS1, KKS2, KS]. This leads to some very different behaviour. For example, in that case, there exist for all $|v| < c$ constant speed solutions for the particle in absence of an external potential V. In a confining potential, the particle still converges exponentially fast to a minimum of the potential, but this time the exponential rate also depends on the shape of the potential.

The rest of the paper is organized as follows. In Sect. 2 we study in detail the friction force exerted by the medium on the particle. This allows us to discuss in some detail the intuition behind the model. The rather routine but essential question of existence and uniqueness of the solutions of (1.5)–(1.6) is settled in Sect. 3. In Sect. 4 we study the long time asymptotics of the particle behaviour for the case when $V(q) = -F \cdot q$, whereas Sect. 5 is devoted to the confined case.

2. The Friction Force

Crucial for understanding the model and for the proofs of our results is a detailed study of the reaction force of the medium defined in (1.7). Imagine we apply a constant external force F to the particle. We then look for solutions of the equations of motion (1.5)–(1.6) where the particle executes a uniform rectilinear motion $q(t) = q_0 + vt$ and the field is comoving, i.e.: $\phi_v(x, y, t) = \Phi_v(x - (q_0 + vt), y)$. Inserting this ansatz into (1.8), one easily finds the solution:

$$\hat{\Phi}_v(x, k) = -\int_0^{+\infty} ds\, \rho_1(x + vs)\hat{\rho}_2(k)\frac{\sin(|k|s)}{|k|}. \tag{2.1}$$

This is the so-called retarded solution, describing the waves created in the "membranes" by the passage of the particle. Note that it has zero initial conditions at $t = -\infty$ in the sense that, for all $(x, y) \in \mathbb{R}^{d+n}$, there exists T (depending only on x) so that $\phi_v(x, y, t) = 0$ for all $t \le T$ (Fig. 1). It is easy to see this is the unique comoving solution. This wave $\phi_v(x, y, t)$ induces a force on the particle that is easily computed from (2.1) and (1.7) using a change of variables in the integration ($x \to x + vt + q_0$):

$$\begin{aligned}\mathcal{F}_{\phi_{v,t}}(q_0 + vt) &= -\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^n} dy\, \rho_1(x - (q_0 + vt))\rho_2(y)(\nabla_x\phi_v)(x, y, t)\\ &= -\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^n} dk \int_0^{+\infty} ds \nabla\rho_1(x)\rho_1(x + vs)|\hat{\rho}_2(k)|^2\frac{\sin(|k|s)}{|k|}\\ &=: f(v),\end{aligned} \tag{2.2}$$

which is clearly independent of q_0 and t. As a result, $\phi_v(x, y, t)$ and $q(t) = vt + q_0$ will satisfy the coupled system (1.5)–(1.6) with $-\nabla V = F$ provided v satisfies the equation $f(v) = -F$. In conclusion, a comoving solution to (1.5)–(1.6) at velocity v exists provided the equation $f(v) = -F$ has at least one solution. We will see below that for F sufficiently small two such solutions exist, one at "low" and one at "high" velocity. Our main result will say that, given "any" sufficiently small initial condition and any not too large force F, the particle trajectory asymptotically converges to the corresponding constant velocity trajectory (Theorem 2).

It is important for the proof of our results to understand the behaviour of the function $f(v)$ rather well, a task we now turn to. Remark that f is a functional of ρ_1 and ρ_2. In this and the following section the latter are kept fixed, so we do not explicitly indicate this dependence. In Sects. 4 and 5, we will reintroduce c explicitly via (1.4) keeping ρ_1 and σ_2 fixed: f will then be a function of v and c.

It is clear that $f \in C^\infty(\mathbb{R}^d)$. Furthermore, it is easy to see that

$$f(v) = -f_r(|v|)\frac{v}{|v|}, \quad f_r(|\,v|) > 0, \tag{2.3}$$

so that the reaction force of the medium on the particle is directed opposite to the particle velocity as expected for a friction force. To prove this, first note that the rotational invariance of ρ_1 implies that

$$\forall R \in O(d), \quad R[f(v)] = f(Rv).$$

Now, if $v = |v|e_1$, one finds, after a few changes of variables ($\lambda = |v|s$ and $\tilde{k} = \frac{k}{|v|}$):

$$f(v) = -|v|^{n-2}\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^n} d\tilde{k} \int_0^{+\infty} d\lambda \nabla\rho_1(x)\rho_1(x + \lambda e_1)|\hat{\rho}_2(|v|\tilde{k})|^2\frac{\sin(\lambda|\tilde{k}|)}{|\tilde{k}|}.$$

The rotational invariance of ρ_1 now implies that $f_i(|v|e_1) = 0$ for $i = 2, \dots, d$, so that $f(v)$ has the direction of e_1 and so in the general case ($v \neq 0$) one has indeed

$$f(v) = -f_r(|v|)\frac{v}{|v|}.$$

We need to study the asymptotic behaviour of $f(v)$ as $|v|$ goes to 0 and as $|v|$ goes to $+\infty$. For that purpose, we write (see (2.2))

$$f(v) = \int_{\mathbb{R}^n} dk\ f(v,k),$$

with (after some manipulations)

$$f(v,k) = -f_r(|v|,|k|)\frac{v}{|v|}, \tag{2.4}$$

$$f_r(|v|,|k|) = \frac{1}{|v|^2}|\hat\rho_2(k)|^2 h\left(\frac{k}{|v|}\right), \tag{2.5}$$

$$h(\xi) = \int_0^{+\infty} d\lambda \int_{\mathbb{R}^d} dx\ \partial_1\rho_1(x)\rho_1(x_1+\lambda, x_\perp)\frac{\sin\lambda|\xi|}{|\xi|} \tag{2.6}$$

$$= \pi\int_{\mathbb{R}^{d-1}} d\eta|\hat\rho_1(|\xi|,\eta)|^2, \tag{2.7}$$

where $\hat\rho_1$ is the Fourier transform of ρ_1. Here $f(v,k)$ is the force produced by the "oscillators" $\hat\phi(x,k)$ of frequency $\omega = |k|$. It follows immediately from the above that $f_r(|v|) > 0$ and that, given $\ell \in \mathbb{N}$, there exists a constant $C_\ell > 0$ so that

$$|f(v,k)| \leq C_\ell\frac{1}{|k|^2}\left(\frac{|v|}{|k|}\right)^\ell.$$

In other words, for fixed k, $f(v,k)$ vanishes to all orders in $|v|$ as $v \to 0$. So, as $v \to 0$, the force on the particle due to one of the oscillators of frequency $\omega(k) = |k|$ present at x, decreases faster than any power of $|v|$ for small v (*i.e.* when $|v| \ll |k|R_1$). Roughly speaking, the coupling of the particle to such an oscillator is extremely weak when $|v|$ is much smaller than $|k|R_1$. This corresponds to a well-known piece of physical intuition: if the particle has speed v, it interacts during a time of order $\frac{R_1}{|v|}$ with any given oscillator. For the energy transfer between the particle and the oscillator to be efficient, this interaction time has to be comparable to the period of the oscillator as an explicit computation easily confirms. Indeed, the total energy transfer ΔE (from $t = -\infty$ to $t = +\infty$) to a driven oscillator of frequency ω,

$$\ddot u(t) + \omega^2 u(t) = \sigma(t),$$

is easily computed to be $\Delta E = \pi|\hat\sigma(\omega)|^2$. Applying this to (1.8) with $q(t) = vt$, one finds $\Delta E = \frac{\pi}{|v|^2}|\hat\rho_2(k)|^2|\hat\rho_1(|k|/|v|,0)|^2$ which vanishes again to all orders in $|v|$ as $|v| \to 0$.

In particular, it is clear from this observation that when coupling the particle to a family of oscillators, all of the same fixed frequency (as in a pinball machine where each circular obstacle would be mounted on a spring), no ohmic behaviour can be expected since the friction force is not linear in v at small v in that case. As the particle slows down, it couples less and less effectively to such oscillators, leading to a friction force vanishing to all orders in $|v|$. To remedy this effect, one has to couple the particle to a family of sufficiently many oscillators of arbitrarily low-frequency. As the particle slows down, it will then transfer energy to those oscillators with which it is in resonance. In the model above, the number of low-frequency oscillators present at the point x depends on the dimension n of the y variables through the volume element $dk = |k|^{n-1}d|k|d\Omega$. Because of the factor $|k|^{n-1}$, the higher the dimension n, the fewer such oscillators are present. This reflects itself immediately in the low v behaviour of the force $f(v)$:

$$\begin{aligned} f_r(|v|) &= |v|^{n-2} \int_{\mathbb{R}^n} |\hat{\rho}_2(|v|\xi)|^2 h(\xi) d\xi \\ &= |v|^{n-2} |\hat{\rho}_2(0)|^2 \int_{\mathbb{R}^n} h(\xi) d\xi + \mathrm{o}(|v|^{n-2}). \end{aligned} \tag{2.8}$$

One notices indeed that for small $|v|$, f_r is smaller if n is higher. So, only when $n = 3$ a friction force proportional to the velocity (and hence ohmic behaviour) is obtained! More precisely, for $n = 3$,

$$f(v) = -\left[|\hat{\rho}_2(0)|^2 \int_{\mathbb{R}^n} h(\xi) d\xi \right] v + o(v) = -\gamma v + o(v),$$

where we defined

$$\gamma = |\hat{\rho}_2(0)|^2 \int_{\mathbb{R}^n} h(\xi) d\xi. \tag{2.9}$$

This shows how motion through the medium modeled here produces a friction term of the type occurring in (1.1) provided $n = 3$. Note that the friction coefficient γ is given explicitly in terms of ρ_1 and ρ_2 and is different from 0 under hypothesis (H1). Since, in this paper, we are interested in studying linear friction at low v, we will restrict ourselves to $n = 3$ in the main theorems (Sects. 4 and 5).

We now turn to the behaviour of $f_r(|v|)$ for large values of $|v|$. It is easy to see from (2.8) that $\lim_{|v| \to +\infty} f_r(|v|) = 0$. In other words, at high speeds as well, the friction force exerted by the medium on the particle is small. As one can see in Eqs. (1.8) and (2.8), this is mostly due to the fact that for high $\omega(k) = |k|$, the oscillators are only very weakly coupled to the particle due to the presence of the smooth form factor $\hat{\rho}_2$. In particular, in the presence of an external driving force F, the model can therefore only be expected to display dissipative behaviour when v is not too large. The profile for $f_r(|v|)$ when ρ is a Gaussian is given in Fig. 2.

3. Existence of Solutions

The assumptions on the potential are

(H2) $V \in C^1(\mathbb{R}^d)$ and ∇V is Lipschitz. Moreover, one of the two following assumptions holds: either ∇V is bounded (such as when $V(q) = -F \cdot q$) or V is bounded from below.

We are now ready to introduce the phase space $\mathcal{E}$ of the model. Let $\|\cdot\|_2$ denote the usual norm on $L^2(\mathbb{R}^{d+n}, dxdy)$. On $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^n)$, $\|\phi\| = \|\nabla_y \phi\|_2$ defines a norm. Let E be the completion of $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^n)$ with this norm. Actually, as a consequence

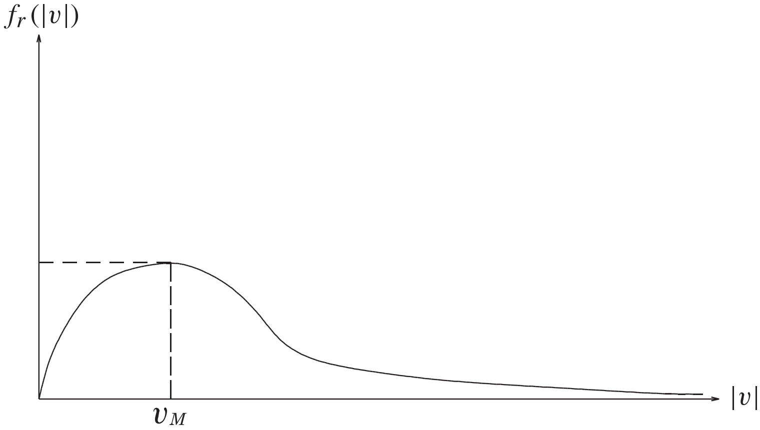

Fig. 2. Profile of $f_r(|v|)$

of the Sobolev imbedding theorems ([B], Chapter 9), E is the space $L^2(\mathbb{R}^d, D, dx)$ where

$$D = \{\phi \in L^{\frac{2n}{n-2}}(\mathbb{R}^n, dy) | \nabla_y \phi \in L^2(\mathbb{R}^n, dy)\}.$$

We then define

$$\mathcal{E} = E \times \mathbb{R}^d \times L^2(\mathbb{R}^{d+n}) \times \mathbb{R}^d$$

with the norm:

$$|Y|_{\mathcal{E}} = (\|\phi\|^2 + |q|^2 + \|\pi\|_2^2 + |p|^2)^{\frac{1}{2}} \quad \text{for } Y = (\phi, q, \pi, p).$$

With this norm, $\mathcal{E}$ is a Hilbert space.

We now write the problem (1.5)–(1.6) in a more convenient way, so as to prove the existence and uniqueness of a solution:

$$\begin{cases} \dot{Y}(t) = G(Y(t)) \\ Y(0) = Y_0 \in \mathcal{E} \end{cases}, \tag{3.1}$$

where

$$\begin{aligned} G : (\phi, q, \pi, p) \to \Bigg(& \pi, p, \Delta_y \phi - \rho_1(x-q)\rho_2(y), \\ & - \nabla V(q) + \int_{\mathbb{R}^{d+n}} dx\, dy\, \nabla \rho_1(x-q)\rho_2(y)\phi(x, y) \Bigg). \end{aligned} \tag{3.2}$$

By solution, we mean that:

$$Y(t) = Y_0 + \int_0^t G(Y(s))ds$$

in the sense of the distributions.

Theorem 1. *Let $n \geq 3$. Under the assumptions (H1) and (H2), we have:*

1. *For each Y_0 in $\mathcal{E}$, the differential equation* (3.1) *has a unique solution $Y(t)$ in $C^0(\mathbb{R}, \mathcal{E})$.*
2. *For every $t \in \mathbb{R}$, the map $W^t : Y_0 \to Y(t)$ is continuous on $\mathcal{E}$.*
3. *For every $t \in \mathbb{R}$, $H(Y(t)) = H(Y_0)$, where*

$$H(Y) = \frac{p^2}{2} + V(q) + \frac{1}{2}\int_{\mathbb{R}^{d+n}} dx\, dy\, (|\nabla_y \phi(x, y)|^2 + |\pi(x, y)|^2) + \int_{\mathbb{R}^{d+n}} dx\, dy\, \rho_1(x - q)\rho_2(y)\phi(x, y) \tag{3.3}$$

is a continuous function on $\mathcal{E}$.

Remark 1. Using the first part of the theorem and (1.6), one sees that $q(t) \in C^2(\mathbb{R}, \mathbb{R}^d)$.

For later reference, we define

$$H_0(Y) = \frac{p^2}{2} + \frac{1}{2}\int_{\mathbb{R}^{d+n}} dx\, dy\, (|\nabla_y \phi(x, y)|^2 + |\pi(x, y)|^2). \tag{3.4}$$

Note that the densely defined bilinear anti-symmetric form

$$\omega(Y_1, Y_2) = q_1 p_2 - p_1 q_2 + \int_{\mathbb{R}^{d+n}} dxdy\, (\phi_1\pi_2 - \pi_1\phi_2)$$

makes $\mathcal{E}$ a symplectic vector space. The equations of motion (1.5)–(1.6) are of course the Hamiltonian equations for the Hamiltonian H in (3.3), which is the total energy of the system. Note that the latter is not bounded from below when V is not, such as when $V = -F \cdot q$. This makes for a slight complication in the existence proof, which is otherwise standard and largely follows [KKS1].

Proof. We start by showing there is a local solution. Then we will use conservation of energy to show the solution is global.

We first look at the problem

$$\begin{cases} \dot{Y}(t) = G_0(Y(t)) \\ Y(0) = Y_0 \end{cases} \tag{3.5}$$

with

$$G_0 Y = (\pi; 0; \Delta_y \phi; 0). \tag{3.6}$$

This problem is just the free wave equation in $\mathbb{R}^n$ with a parameter $x \in \mathbb{R}^d$. It admits a unique solution: $t \in \mathbb{R} \to Y(t) \in \mathcal{E}$. Moreover, let $W_0^t : Y_0 \to Y(t)$ denote the corresponding continuous group; then W_0^t turns out to be a linear isometry with the norm $|\ |_{\mathcal{E}}$. It is also continuous on $\mathbb{R} \times \mathcal{E}$ ([LM], Chapter 3).

Now define $Z(t) = W_0^{-t} Y(t)$ or $Y(t) = W_0^t Z(t)$. In particular, $Z(0) = Y(0) = Y_0$. We have $\dot{Y}(t) = G_0 Y(t) + W_0^t \dot{Z}(t)$. $Y(t)$ is a solution of the problem (3.1) if and only if $Z(t)$ satisfies

$$\begin{cases} \dot{Z}(t) = W_0^{-t} G_1(W_0^t Z(t)) \\ Z(0) = Y_0 \end{cases}, \tag{3.7}$$

where

$$G_1 : (\phi; q; \pi; p) \in \mathcal{E} \to \Big(0; p; -\rho_1(x-q)\rho_2(y); -\nabla V(q) + \int_{\mathbb{R}^{d+n}} dx\, dy\, \nabla\rho_1(x-q)\rho_2(y)\phi(x,y)\Big) \in \mathcal{E}. \quad (3.8)$$

Introducing

$$\tilde{G} : (t, Z) \in \mathbb{R} \times \mathcal{E} \to W_0^{-t} G_1(W_0^t Z) \in \mathcal{E},$$

it is clear that $\tilde{G}$ is continuous on $\mathbb{R} \times \mathcal{E}$ and Lipschitz on $\mathcal{E}$ because W_0^t is an isometry and G_1 is Lipschitz. This problem satisfies all the conditions of the Cauchy-Lipschitz theorem ([H], Theorem 3.1), so it has a unique solution which is defined on an open interval. More precisely, there exists an open interval J such that $0 \in J$ and there exists a unique function $Z : t \in J \to Z(t) \in \mathcal{E}$ satisfying (3.7). Moreover, $\tilde{W}^t : Z_0 \to Z(t)$ is continuous on $\mathcal{E}$ for every $t \in J$ and so we have the same results for

$$W^t : Y_0 \in \mathcal{E} \to Y(t) = W_0^t \tilde{W}^t Y_0 \in \mathcal{E}.$$

In order to prove global existence, we now prove conservation of energy. We first prove the result for smooth initial data (*i.e.* $\phi_0, \pi_0 \in C^\infty(\mathbb{R}^{d+n})$). Let $Y_0 = (\phi_0, q_0, \pi_0, p_0)$ with $\phi_0, \pi_0 \in C_0^\infty(\mathbb{R}^{d+n})$. Then $W_0^t Y_0$ is smooth ([CH], Chapter 6) and by the integral representation:

$$Y(t) = W_0^t Y_0 + \int_0^t ds\, W_0^{t-s} G_1(Y(s)),$$

it is clear that $\phi(t), \pi(t)$ are smooth as well (in x and y). Note that $\phi(x, y, t)$ and $\pi(x, y, t)$ are also smooth in t ([LM], Chap. 3). For such initial data a simple computation then yields:

$$\frac{d}{dt} H(Y(t)) = 0,$$

so that, for smooth initial data, $H(Y(t))$ is a constant for all t in J. We now prove that H is continuous on $\mathcal{E}$. The continuity of W^t on $\mathcal{E}$ and the fact that smooth initial data are dense in $\mathcal{E}$ will then imply the result for all initial data. Since V is continuous, it only remains to show the interaction term in H is continuous. Its continuity in ϕ is immediate from the following computation:

$$\begin{aligned} &\Big| \int_{\mathbb{R}^{d+n}} dx\, dy\, \rho_1(x-q)\rho_2(y)\phi(x,y) - \int_{\mathbb{R}^{d+n}} dx\, dy\, \rho_1(x-q)\rho_2(y)\psi(x,y) \Big| \\ &= \Big| \int_{\mathbb{R}^{d+n}} dx\, dk\, \tfrac{\rho_1(x-q)\bar{\hat{\rho}}_2(k)}{|k|} (|k|\hat{\phi}(x,k) - |k|\hat{\psi}(x,k)) \Big| \\ &\le \Big\| \tfrac{\rho_1(x-q)\bar{\hat{\rho}}_2(k)}{|k|} \Big\|_{L^2} \times \Big\| \, |k|(\hat{\phi} - \hat{\psi}) \Big\|_{L^2} \\ &\le \Big\| \tfrac{\rho_1(x-q)\bar{\hat{\rho}}_2(k)}{|k|} \Big\|_{L^2} \times \|\phi - \psi\|. \end{aligned}$$

Because ρ has compact support and $n \ge 3$ the first factor of the right-hand side is finite and so H is continuous (the continuity in (q, ϕ) follows similarly).

We will furthermore need the following obvious inequality (based on $| ab | \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$):

$$\left| \int_{\mathbb{R}^{d+n}} dx\, dy\, \rho_1(x-q)\rho_2(y)\phi(x,y) \right| \leq \frac{\| \phi \|^2}{4} - \langle \rho_1\rho_2; \rho_1\Delta_y^{-1}\rho_2 \rangle. \tag{3.9}$$

Hence:

$$H(Y(t)) \geq \frac{1}{2}p(t)^2 + V(q(t)) + \frac{1}{4}\|\phi(t)\|^2 + \frac{1}{2}\|\pi(t)\|_2^2 + \langle \rho_1\rho_2; \rho_1\Delta_y^{-1}\rho_2 \rangle. \tag{3.10}$$

We are now ready to prove that $J = \mathbb{R}$. We know that J can be written $]a; b[$ with $-\infty \leq a < 0$ and $0 < b \leq +\infty$. We will show by contradiction that $b = +\infty$ (the same can be done for $a = -\infty$). If $b < +\infty$, we know by the theory of differential equations ([H], Theorem 2.1) that

$$\lim_{t \to b} |Z(t)|_{\mathcal{E}} = +\infty,$$

and the same holds for $|Y(t)|_{\mathcal{E}}$ because

$$|Y(t)|_{\mathcal{E}} = |W_0^t Z(t)|_{\mathcal{E}} = |Z(t)|_{\mathcal{E}}.$$

We consider first the (harder) case where ∇V is bounded (but V is not necessarily bounded below). For $t > 0$, we can write ϕ as:

$$\phi = \phi^r + \phi^0, \tag{3.11}$$

where ϕ^r is the solution of the wave equation with initial data equal to 0 and ϕ^0 is the solution of the homogeneous wave equation with initial data ϕ_0 and π_0 [CH] [J]. Consequently

$$\begin{aligned} \dot{p}(t) = -\nabla V(q(t)) &+ \int_{\mathbb{R}^{d+n}} dx\, dy\, \nabla\rho_1(x - q(t))\rho_2(y)\phi^r(x,y,t) \\ &+ \int_{\mathbb{R}^{d+n}} dx\, dy\, \nabla\rho_1(x - q(t))\rho_2(y)\phi^0(x,y,t). \end{aligned}$$

The first term $-\nabla V(q(t))$ is bounded by hypothesis. The second one is easily bounded using the Cauchy-Schwarz inequality and the exact form of ϕ^r given in ([CH], p.692). Using (3.9) with $\nabla\rho_1$ instead of ρ_1, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^{d+n}} dx\, dy\, \nabla\rho_1(x - q(t))\rho_2(y)\phi^0(x,y,t) \right| &\leq \frac{1}{4} \| \nabla_y\phi^0(t) \|_2^2 \\ &\quad + \| \nabla^{-1}\rho_2(y)\nabla\rho_1(x - q(t)) \|_2^2 . \end{aligned}$$

But ϕ^0 is a solution of the free wave equation with initial conditions ϕ_0 and π_0, so, by energy conservation

$$\| \nabla_y\phi^0(t) \|_2^2 + \left\| \frac{d}{dt}\phi^0(t) \right\|_2^2 = \| \nabla_y\phi_0 \|_2^2 + \| \pi_0 \|_2^2, \tag{3.12}$$

and so $\| \nabla_y\phi^0(t) \|_2^2$ is bounded too.

Finally, $\dot{p}(t)$ is bounded on J: there exists $C > 0$ such that

$$\forall t \in J, \quad t > 0 \ \mid \dot{p}(t) \mid \leq C. \tag{3.13}$$

We have supposed b to be finite, so $p(t)$ and $q(t)$ are also bounded for $t > 0, t \in J$.

By energy conservation and (3.10), $\|\phi(t)\|$ and $\|\pi(t)\|_2$ are bounded. Therefore $|Y(t)|_{\mathcal{E}}$ is bounded as well which is a contradiction with the fact that b is finite.

We finally deal with the second (easier) case, where V is bounded from below. There exists $V_0 \in \mathbb{R}$ such that for every $q \in \mathbb{R}^d$, $V(q) \geq V_0$.

Equation (3.3) implies

$$H(Y_0) \geq \frac{1}{2}p(t)^2 + V_0 + \frac{1}{4}\|\phi(t)\|^2 + \frac{1}{2}\|\pi(t)\|_2^2 + \langle \rho_1\rho_2; \rho_1\Delta_y^{-1}\rho_2\rangle. \tag{3.14}$$

So $p(t)$, $\|\phi(t)\|$ and $\|\pi(t)\|_2$ are bounded on J and because b is supposed to be finite, $q(t)$ is also bounded which is again a contradiction. □

4. Behaviour of the Solutions: Constant Force

From now on, we take $n = 3$. To prove our results we shall need to assume that the propagation speed c (see (1.2)) is large. We will comment on this condition at the end of this section. We therefore reintroduce c explicitly as in (1.4):

$$\rho_2(y) = c^{\frac{3}{2}}\sigma_2(cy). \tag{4.1}$$

In the following, ρ_1 and σ_2 are fixed and satisfy (H1); c is treated as a parameter. The force exerted by the medium on a particle moving at velocity v is defined in (2.2). One has

$$f(v) = \frac{1}{c^2}\tilde{f}\left(\frac{v}{c}\right) = -\frac{1}{c^2}\tilde{f}_r\left(\frac{|v|}{c}\right)\frac{v}{|v|}, \tag{4.2}$$

where, for all $w \in \mathbb{R}^d$,

$$\tilde{f}(w) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^n} dk \int_0^{+\infty} ds \nabla\rho_1(x)\rho_1(x+ws)|\hat{\sigma}_2(k)|^2 \frac{\sin(|k|s)}{|k|}.$$

Remark that $\tilde{f}$ and $\tilde{f}_r$ do not depend on c. The friction coefficient γ defined in (2.9) then becomes

$$\gamma = f_r'(0) = \frac{1}{c^3}\tilde{f}_r'(0) \equiv \frac{\tilde{\gamma}}{c^3} > 0,$$

where $\tilde{\gamma}$ does not depend on c:

$$\tilde{\gamma} = \left[|\hat{\sigma}_2(0)|^2 \int_{\mathbb{R}^n} h(\xi)d\xi\right] > 0. \tag{4.3}$$

We can define w_M to be the smallest zero of $\tilde{f}_r'$ and $\tilde{F}_M = \tilde{f}_r(w_M)$. For all $w < w_M$, $\tilde{f}_r$ is increasing, so for all $F \in \mathbb{R}^d$, $|F| \leq \frac{\tilde{F}_M}{c^2}$, there exists a unique $v(F) \in \mathbb{R}^d$, $|v(F)| \leq w_M c = v_M$ (see Fig. 2) such that

$$f(v(F)) = -F. \tag{4.4}$$

This defines $v(F)$.

To obtain our results, we finally need some hypothesis on the initial conditions. For that purpose, we define the following set:

Definition 1. *Let $\mathcal{D}$ be the set of all states $Y_0 = (\phi_0, q_0, \pi_0, p_0)$ in $\mathcal{E}$ such that*

$$\left[|\phi_0(x, y)| + |y|(|\nabla_y \phi_0(x, y)| + |\pi_0(x, y)|)\right] \leq \kappa(x)(1 + |y|)^{-\nu} \tag{4.5}$$

for some $\nu > 2$ and $\kappa \in L^\infty \cap L^2$.

We are now ready to state our main results.

Theorem 2. *Let ρ_1 and σ_2 satisfy (H1) and consider* (1.5)–(1.6) *with $V(q) = -F \cdot q$, $F \in \mathbb{R}^d$.*

(i) For all $F_0, K, R, \varepsilon, \eta > 0$ there exists $c_0(\rho_1, \sigma_2, \varepsilon, \eta, F_0, K, R) > 0$ such that for any $c > c_0$, for all $|F| < F_0 c^{-2-\varepsilon}$ and for all $Y_0 \in \mathcal{E}$ such that $\phi_0(x, \cdot), \pi_0(x, \cdot)$ have compact support in $B_{Rc} \subset \mathbb{R}^3$, satisfying $H_0(Y_0) < K c^{2-2\varepsilon}$, there exist $q_\infty(F, Y_0) \in \mathbb{R}^d$ and $K' > 0$ such that for all $t > 0$,

$$|q(t) - q_\infty - v(F)t| \leq K' e^{-\frac{\tilde{\gamma}(1-\eta)}{c^3} t}. \tag{4.6}$$

(ii) For all $F_0, K, \varepsilon, \eta > 0$ there exists $c_0(\rho_1, \sigma_2, \varepsilon, \eta, F_0, K) > 0$ such that for any $c > c_0$, for all $|F| < F_0 c^{-2-\varepsilon}$ and for all $Y_0 \in \mathcal{D}$ with $\|\kappa\|_\infty < Kc$ and $H_0(Y_0) < K c^{2-2\varepsilon}$, there exist $q_\infty(F, Y_0) \in \mathbb{R}^d$ and $K' > 0$ such that for all $t > 0$,

$$|q(t) - q_\infty - v(F)t| \leq K' |t|^{2-\nu}.$$

Note that, since η can be taken arbitrarily small, the exponential decay rate in (4.6) is essentially given by the friction coefficient $\gamma = \frac{\tilde{\gamma}}{c^3}$ and, in addition that

$$v(F) = \frac{F}{\gamma} + O(c^{1-2\varepsilon}) \tag{4.7}$$

uniformly for $|F| \leq F_0 c^{-2-\varepsilon}$. This shows that the solutions $q(t)$ of (1.6) do indeed have the same asymptotic behaviour as those of (1.1), as announced in the introduction. The restriction on the energy $H_0(Y_0)$ of the initial conditions in the hypothesis is related to the fact that $f(v) \to 0$ as $v \to \infty$. Indeed, it is intuitively clear that, if at some time t, $|\dot{q}(t)|$ is too large, then the reaction force of the medium will be too small to counter the driving force F and the particle will accelerate. This argument fails when $F = 0$. In that case, one can indeed omit the hypothesis on the initial energy $H_0(Y_0)$, provided one imposes an additional hypothesis on σ_2 :

(W) $\hat{\sigma}_2(k) \neq 0$ for all $k \in \mathbb{R}^3$.

This yields:

Theorem 3. *Let ρ_1, σ_2 satisfy (H1) and let σ_2 satisfy (W). We consider* (1.5)–(1.6) *with $V \equiv 0$,*

(i) For all $\eta > 0$ there exists $c_0(\rho_1, \sigma_2, \eta) > 0$ such that for any $c > c_0$ and for all $Y_0 \in \mathcal{E}$ such that $\phi_0(x, \cdot), \pi_0(x, \cdot)$ have compact support in the y direction for each x, there exist $q_\infty(Y_0) \in \mathbb{R}^d$ and $K' > 0$ such that for all $t > 0$,

$$|q(t) - q_\infty| \leq K' e^{-\frac{\tilde{\gamma}(1-\eta)}{c^3} t}.$$

(ii) For all $\eta > 0$ there exists $c_0(\rho_1, \sigma_2, \eta) > 0$ such that for any $c > c_0$ and for all $Y_0 \in \mathcal{D}$, there exist $q_\infty(Y_0) \in \mathbb{R}^d$ and $K' > 0$ such that for all $t > 0$,

$$|q(t) - q_\infty| \leq K' |t|^{2-\nu}.$$

The proof of Theorem 3, which uses techniques of this section and the following one, is given at the end of Sect. 5.

We now prove Theorem 2. We introduce some notation which will frequently appear. We denote by $Df(v)$ the differential of the function $f(v)$. One can see that in any orthonormal basis $(e_1, \dots, e_d)$, where $e_1 = \frac{v}{|v|}$ we have

$$Df(v) = \mathrm{diag}\left(-f_r'(|v|), -\frac{f_r(|v|)}{|v|}, \dots, -\frac{f_r(|v|)}{|v|}\right)$$

for $v \neq 0$ and

$$Df(0) = -\gamma Id.$$

We define for $w \in \mathbb{R}^d$,

$$\tilde{\theta}_*(w) = \max\left(\tilde{f}_r'(|w|), \frac{\tilde{f}_r(|w|)}{|w|}\right), \quad \tilde{\gamma}_*(w) = \min\left(\tilde{f}_r'(|w|), \frac{\tilde{f}_r(|w|)}{|w|}\right). \tag{4.8}$$

In view of the definition of w_M and (4.2), it is clear that $\tilde{\theta}_*(w)$ and $\tilde{\gamma}_*(w)$ are strictly positive provided $|w| < w_M$ and that $\|Df(v)\| = \frac{1}{c^3}\tilde{\theta}_*(\frac{v}{c})$. Clearly,

$$\lim_{w\to 0} \tilde{\theta}_*(w) = \lim_{w\to 0} \tilde{\gamma}_*(w) = \tilde{\gamma}, \tag{4.9}$$

where $\tilde{\gamma}$ is defined in (4.3). For simplicity, we will write Df_F, $\tilde{\gamma}_F$ and $\tilde{\theta}_F$ for $Df(v(F))$, $\tilde{\gamma}_*(\frac{v(F)}{c})$ and $\tilde{\theta}_*(\frac{v(F)}{c})$.

Since we expect to prove $\dot{q}(t) \to v(F)$, it is convenient to introduce $h(t) = q(t) - v(F)t$. For the proof of Theorem 2, we need the following lemma.

Lemma 1. *Under the hypothesis of Theorem 2(i) (resp. Theorem 2(ii)), there exist* $c_0 > 0$ *and* $\beta > 0$ *such that*

$$\sup_{t\geq 0} |\dot{h}(t)| \leq \beta c^{1-\varepsilon}$$

for all $c > c_0$ *and for all initial conditions as in Theorem 2(i) (resp. Theorem 2(ii)).*

The proof of Lemma 1 will be given below.

Proof of Theorem 2. During the proof, many estimates will be done in terms of c, so one shall remember that ρ_2 depends on c via (4.1). On the other hand, the different constants will only depend on $\rho_1, \sigma_2, \eta, \varepsilon, F_0, K, R$, but not on c, F, or on the initial conditions. We first fix c large enough so that $F_0 c^{-2-\varepsilon} < \tilde{F}_M c^{-2}$, which implies that $v(F)$ is well defined (see (4.4)), and we consider (1.5)–(1.6) for some $F \in \mathbb{R}^d$, $|F| \leq F_0 c^{-2-\varepsilon}$ and $Y_0 \in \mathcal{E}$.

The first part of the proof consists of a rather straightforward but somewhat lengthy computation leading from (1.5)–(1.6) to an effective integro-differential equation for $h(t) = q(t) - v(F)t$ obtained in (4.20).

Solving (1.5) yields, according to (3.11),

$$\phi(x, y, t) = \phi^r(x, y, t) + \phi^0(x, y, t),$$

where in the 3-dimensional case we deal with here:

$$\phi^r(x, y, t) = -\frac{1}{4\pi}\int_{|z|\leq t} dz\, \frac{\rho_2(y-z)}{|z|}\rho_1\,(x - q(t - |z|)), \tag{4.10}$$

$$\phi^0(x,y,t) = \frac{1}{4\pi t^2}\int_{S_t(y)} [\phi_0(x,\sigma) + \sigma\cdot\nabla_y\phi_0(x,\sigma) + t\pi_0(x,\sigma)]d\sigma, \tag{4.11}$$

and $S_t(y)$ is the sphere of radius t centered at y ([J], Chap. 3). Inserting this in (1.6) leads to the following integro-differential equation for $q(t)$:

$$\begin{aligned}\ddot{q}(t) = F - \frac{1}{4\pi}\iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\rho_1(x-q(t-|z|))\\ \times\nabla\rho_1(x-q(t)) + A_0(t),\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}A_0(t) = \frac{1}{4\pi t^2}\iint dx\,dy\int_{S_t(y)} d\sigma\,[\phi_0(x,\sigma) + \sigma\cdot\nabla_y\phi_0(x,\sigma)\\ + t\pi_0(x,\sigma)]\rho_2(y)\nabla\rho_1(x-q(t)).\end{aligned} \tag{4.13}$$

Since $n = 3$, it is not difficult to see that $f(v)$, defined in (2.2), can be rewritten as follows:

$$f(v) = -\frac{1}{4\pi}\int_{\mathbb{R}^d}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\rho_1(x+v|z|)\nabla\rho_1(x). \tag{4.14}$$

Using this expression to replace F in (4.12) by $-f(v(F))$ we find

$$\begin{aligned}\ddot{q}(t) = \frac{1}{4\pi}\iiint dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\rho_1(x+v(F)|z|)\nabla\rho_1(x)\\ -\frac{1}{4\pi}\iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\rho_1(x-q(t-|z|))\nabla\rho_1(x-q(t))\\ + A_0(t).\end{aligned}$$

To alleviate the notation, we shall from now write $v = v(F)$. We now divide the first integral in two parts:

$$\int_{\mathbb{R}^d} dx\int_{\mathbb{R}^3} dy\int_{\mathbb{R}^3} dz = \int_{\mathbb{R}^d} dx\int_{\mathbb{R}^3} dy\int_{|z|\le t} dz + \int_{\mathbb{R}^d} dx\int_{\mathbb{R}^3} dy\int_{|z|\ge t} dz.$$

We denote by $\tilde{f}(t)$ the second one of these two terms, i.e.

$$\tilde{f}(t) = \frac{1}{4\pi}\iiint_{|z|\ge t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\rho_1(x+v|z|)\nabla\rho_1(x). \tag{4.15}$$

Now, remark that for $|z| \ge 2\frac{R_2}{c}$, $\rho_2(y)\rho_2(y-z) = 0$ because

$$|y-z| \ge |z| - |y| \ge 2\frac{R_2}{c} - |y|$$

and $\rho_2(y) = 0$ for $|y| \ge \frac{R_2}{c}$. So $\tilde{f}(t)$ vanishes if $t \ge \frac{2R_2}{c}$. Finally, let

$$A_1(t) = A_0(t) + \tilde{f}(t). \tag{4.16}$$

This leads to

$$\ddot{q}(t) = \frac{1}{4\pi}\iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\,[-\rho_1\,(x-q(t-|z|))$$
$$\times\nabla\rho_1(x-q(t)) + \rho_1\,(x+v|z|)\,\nabla\rho_1(x)] + A_1(t). \tag{4.17}$$

Inserting

$$q(t-|z|) = q(t) - \int_{t-|z|}^{t}\dot{q}(s)ds$$

in (4.17) and using translation invariance, we find:

$$\ddot{q}(t) = \frac{1}{4\pi}\iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}$$
$$\times\left[\rho_1\,(x+v|z|) - \rho_1\left(x+\int_{t-|z|}^{t}\dot{q}(s)ds\right)\right]\nabla\rho_1(x) + A_1(t). \tag{4.18}$$

We are now ready to introduce $h(t) = q(t) - vt$, in terms of which (4.18) becomes

$$\ddot{h}(t) = \frac{1}{4\pi}\iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}$$
$$\times\left[\rho_1\,(x+v|z|) - \rho_1\left(x+v|z|+\int_{t-|z|}^{t}\dot{h}(s)ds\right)\right]\nabla\rho_1(x) + A_1(t).$$

We can write

$$\rho_1\left(x+v|z|+\int_{t-|z|}^{t}\dot{h}(s)ds\right) - \rho_1\,(x+v|z|)$$
$$= \int_{t-|z|}^{t}\dot{h}(s)\cdot\nabla\rho_1\,(x+v|z|)\,ds$$
$$+\frac{1}{2}\left\langle \mathrm{Hess}\rho_1(\tilde{x}_{t,|z|})\int_{t-|z|}^{t}\dot{h}(s)ds;\int_{t-|z|}^{t}\dot{h}(s)ds\right\rangle$$

for some $\tilde{x}_{t,|z|}$ belonging to the segment $[x+v|z|; x+v|z|+\int_{t-|z|}^{t}\dot{h}(s)ds]$. In addition, an integration by parts yields:

$$\int_{t-|z|}^{t}\dot{h}(s)ds = |z|\dot{h}(t) + \int_{t-|z|}^{t}(t-|z|-s)\ddot{h}(s)ds.$$

As a result

$$
\begin{aligned}
\ddot{h}(t) = &-\frac{1}{4\pi}\iiint_{|z|\le t} dx\,dy\,dz\,\rho_2(y-z)\rho_2(y)\left[\dot{h}(t)\cdot\nabla\rho_1(x+v|z|)\right]\nabla\rho_1(x)\\
&-\frac{1}{4\pi}\iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\\
&\times\left[\left(\int_{t-|z|}^{t}(t-|z|-s)\ddot{h}(s)ds\right)\cdot\nabla\rho_1(x+v|z|)\right]\nabla\rho_1(x)\\
&-\frac{1}{4\pi}\iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\\
&\times\frac{1}{2}\left\langle \mathrm{Hess}\rho_1(\tilde{x}_{t,|z|})\int_{t-|z|}^{t}\dot{h}(s)ds;\int_{t-|z|}^{t}\dot{h}(s)ds\right\rangle\nabla\rho_1(x)+A_1(t).
\end{aligned}
$$

Once again we rewrite the first integral $\int_{|z|\le t}=\int_{\mathbb{R}^3}-\int_{|z|\ge t}$. It is easily seen from (4.14) that the first term then equals $Df_F\cdot\dot{h}(t)$ whereas the second one is once again vanishing for $t\ge\frac{2R_2}{c}$. We define

$$
\begin{aligned}
A_2(t)=A_1(t)+\frac{1}{4\pi}&\iiint_{|z|\ge t} dx\,dy\,dz\,\rho_2(y-z)\rho_2(y)\\
&\times\left[\dot{h}(t)\cdot\nabla\rho_1(x+v|z|)\right]\nabla\rho_1(x),
\end{aligned}
\tag{4.19}
$$

and we finally obtain the following convenient form of the integro-differential equation for $h(t)=q(t)-vt$:

$$
\begin{aligned}
\ddot{h}(t)=Df_F\cdot\dot{h}(t)&-\frac{1}{4\pi}\iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\\
&\times\left[\left(\int_{t-|z|}^{t}(t-|z|-s)\ddot{h}(s)ds\right)\cdot\nabla\rho_1(x+v|z|)\right]\nabla\rho_1(x)\\
&-\frac{1}{4\pi}\iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\\
&\times\frac{1}{2}\left\langle \mathrm{Hess}\rho_1(\tilde{x}_{t,|z|})\int_{t-|z|}^{t}\dot{h}(s)ds;\int_{t-|z|}^{t}\dot{h}(s)ds\right\rangle\nabla\rho_1(x)+A_2(t),
\end{aligned}
\tag{4.20}
$$

where $A_2(t)$ is defined via (4.13)–(4.15)–(4.16)–(4.19). One recognizes here, in the first two terms, Eq. (1.1) with $V=-F\cdot q$.

We can now show $\dot{h}(t)\to 0$ and control the rate of convergence. We first define $g(t)=e^{\frac{\tilde{\theta}_F}{c^3}t}\dot{h}(t)$. We have

$$
\dot{h}(t)=e^{-\frac{\tilde{\theta}_F}{c^3}t}g(t),\quad \ddot{h}(t)=-\frac{\tilde{\theta}_F}{c^3}e^{-\frac{\tilde{\theta}_F}{c^3}t}g(t)+e^{-\frac{\tilde{\theta}_F}{c^3}t}\dot{g}(t)=-\frac{\tilde{\theta}_F}{c^3}\dot{h}(t)+e^{-\frac{\tilde{\theta}_F}{c^3}t}\dot{g}(t),
$$

so that (4.20) becomes

$$
\begin{aligned}
\dot g(t) = &\left[\frac{\tilde\theta_F}{c^3} Id + Df_F\right]\cdot g(t)\\
&+\frac{\tilde\theta_F}{4\pi c^3}\iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\\
&\times\left[\left(\int_{t-|z|}^t (t-|z|-s)e^{\frac{\tilde\theta_F}{c^3}(t-s)} g(s)ds\right)\cdot\nabla\rho_1(x+v|z|)\right]\nabla\rho_1(x)\\
&-\frac{1}{4\pi}\iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\nabla\rho_1(x)\\
&\times\frac12\left\langle \mathrm{Hess}\rho_1(\tilde x_{t,|z|})\int_{t-|z|}^t e^{\frac{\tilde\theta_F}{c^3}(t-s)} g(s)ds;\int_{t-|z|}^t \dot h(s)ds\right\rangle\\
&-\frac{1}{4\pi}\iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\\
&\times\left[\left(\int_{t-|z|}^t (t-|z|-s)e^{\frac{\tilde\theta_F}{c^3}(t-s)} \dot g(s)ds\right)\cdot\nabla\rho_1(x+v|z|)\right]\nabla\rho_1(x)\\
&+e^{\frac{\tilde\theta_F}{c^3}t}A_2(t).
\end{aligned}
\tag{4.21}
$$

Note that in the third term of the right-hand side we only replaced one factor $\dot h(s)$ by $e^{-\frac{\tilde\theta_F}{c^3}s}g(s)$. We will use Lemma 1 to control the other one. We define

$$
M(t)=\sup_{0\le s\le t}|g(s)| \quad\text{and}\quad N(t)=\sup_{0\le s\le t}|\dot g(s)|.
$$

Writing $R(t)$ for the right-hand side of (4.21) and using Lemma 1 to control its third term, we easily find (remembering (4.1)) there exist constants $D_1, D_2, D_3 > 0$ depending on ρ_1, σ_2 so that

$$
\begin{aligned}
|R(t)| \le &\left(\frac{\tilde\theta_F}{c^3}-\frac{\tilde\gamma_F}{c^3}\right)M(t)+\frac{\tilde\theta_F}{c^7}D_1 e^{2\tilde\theta_F\frac{R_2}{c^4}}M(t)+\frac{D_2 e^{2\tilde\theta_F\frac{R_2}{c^4}}}{c^{3+\varepsilon}}M(t)\\
&+\frac{D_3}{c^4}e^{2R_2\frac{\tilde\theta_F}{c^4}}N(t)+e^{\frac{\tilde\theta_F}{c^3}t}|A_2(t)|.
\end{aligned}
$$

Here, R_2 is the radius of the support of σ_2 (see (H1)). Taking c large enough (depending on $D_1, D_2, \tilde\theta_F$) we obtain for all $s\ge 0$,

$$
|\dot g(s)|\le\left(\frac{\tilde\theta_F}{c^3}-\frac{\tilde\gamma_F}{c^3}+\frac{D_4 e^{2\tilde\theta_F\frac{R_2}{c^4}}}{c^{3+\varepsilon}}\right)M(s)+N(s)\frac{D_3}{c^4}e^{2R_2\frac{\tilde\theta_F}{c^4}}+e^{\frac{\tilde\theta_F}{c^3}s}|A_2(s)|. \tag{4.22}
$$

Taking the supremum over all $s\in[0,t]$, first in the right-hand side and then in the left-hand side of this inequality, we obtain

$$
N(t)\left[1-\frac{D_3}{c^4}e^{2R_2\frac{\tilde\theta_F}{c^4}}\right]\le\left(\frac{\tilde\theta_F}{c^3}-\frac{\tilde\gamma_F}{c^3}+\frac{D_4 e^{2\tilde\theta_F\frac{R_2}{c^4}}}{c^{3+\varepsilon}}\right)M(t)+\sup_{0\le s\le t}\left(e^{\frac{\tilde\theta_F}{c^3}s}|A_2(s)|\right).
$$

We denote by k_c the inverse of the factor of $N(t)$. Remark that $k_c \sim 1 + \frac{k}{c^4}$. Hence

$$N(t) \leq k_c \left(\frac{\tilde{\theta}_F}{c^3} - \frac{\tilde{\gamma}_F}{c^3} + \frac{D_4 e^{2\tilde{\theta}_F \frac{R_2}{c^4}}}{c^{3+\varepsilon}} \right) M(t) + k_c \sup_{0 \leq s \leq t} \left(e^{\frac{\tilde{\theta}_F}{c^3} s} |A_2(s)| \right).$$

Remark that, in view of (4.7), uniformly for all $F \in \mathbb{R}^d$ so that $0 \leq |F| \leq F_0 c^{-2-\varepsilon}$, $\lim_{c \to +\infty} \frac{v(F)}{c} = 0$. Hence, from (4.9)

$$\lim_{c \to +\infty} \tilde{\gamma}_F = \lim_{c \to +\infty} \tilde{\theta}_F = \tilde{\gamma}. \tag{4.23}$$

Using this, it is now easy to see that for all $\eta > 0$, there exists $c_0(\rho_1, \sigma_2, K, F_0, \epsilon, \eta)$ so that, for all $c > c_0$ one has

$$0 < k_c \left(\frac{\tilde{\theta}_F}{c^3} - \frac{\tilde{\gamma}_F}{c^3} + \frac{D_4}{c^{3+\varepsilon}} e^{2\tilde{\theta}_F \frac{R_2}{c^4}} \right) \leq (\tilde{\theta}_F - \tilde{\gamma}(1-\eta)) \frac{1}{c^3}.$$

We obtain then:

$$N(t) \leq \frac{1}{c^3} (\tilde{\theta}_F - \tilde{\gamma}(1-\eta)) M(t) + k_c \sup_{0 \leq s \leq t} \left(e^{\frac{\tilde{\theta}_F}{c^3} s} |A_2(s)| \right). \tag{4.24}$$

To control the last term in this inequality, we now need to use the hypotheses on the initial conditions $Y_0 \in \mathcal{E}$. We treat Theorem 2 (ii) first. Recall that $A_2 - A_0$ is a function of compact support, vanishing for $t \geq \frac{2R_2}{c}$. Hence, there exists $B_c(Y_0)$ so that, for all $t \geq 0$:

$$k_c \sup_{0 \leq s \leq t} |e^{\frac{\tilde{\theta}_F}{c^3} s} (A_2(s) - A_0(s))| \leq B_c(Y_0) < +\infty. \tag{4.25}$$

On the other hand, since for $|y| \geq \frac{R_2}{c}$, $\rho_2(y) = 0$ (see (4.1)), we have

$$A_0(t) = \frac{1}{4\pi t^2} \int dx \int_{|y| \leq \frac{R_2}{c}} dy \int_{S_t(y)} d\sigma \, [\phi_0(x, \sigma) + \sigma \cdot \nabla_y \phi_0(x, \sigma)$$
$$+ t\pi_0(x, \sigma)] \rho_2(y) \nabla \rho_1(x - q(t)).$$

If $|y| \leq \frac{R_2}{c}$, then $|\sigma| \geq |t - \frac{R_2}{c}|$. According to (4.5), and the hypothesis on ϕ_0, π_0,

$$|\phi_0(x, \sigma)| \leq Kc \left| t - \frac{R_2}{c} \right|^{-\nu}, \quad |\sigma \cdot \nabla_y \phi_0(x, \sigma)| \leq Kc \left| t - \frac{R_2}{c} \right|^{-\nu},$$

$$|t\pi_0(x, \sigma)| \leq Kct \left| t - \frac{R_2}{c} \right|^{-\nu-1},$$

uniformly in the x variable. So we have, for some constant A,

$$| A_0(t) | \leq \frac{A}{c^{\frac{1}{2}} (1+t)^{\nu}}. \tag{4.26}$$

A simple computation then shows there exists $t_* \left(\frac{\tilde{\theta}_F}{c^3} \right) > 0$ so that

$$\left.\begin{aligned}\sup_{0\le s\le t}\left[\frac{e^{\frac{\tilde\theta_F}{c^3}s}}{(1+s)^\nu}\right] &= \frac{e^{\frac{\tilde\theta_F}{c^3}t}}{(1+t)^\nu} \quad \forall t \ge t_*\left(\frac{\tilde\theta_F}{c^3}\right),\\ &= 1 \qquad\quad \forall t \le t_*\left(\frac{\tilde\theta_F}{c^3}\right).\end{aligned}\right\} \tag{4.27}$$

We shall write $t_* = t_*\left(\frac{\tilde\theta_F}{c^3}\right)$. Now we have, for all $0 \le s \le t$,

$$|g(s)| \le |g(0)| + \int_0^s |\dot g(u)|du \le |g(0)| + \int_0^s N(u)du \le |g(0)| + \int_0^t N(u)du,$$

so that, using (4.24), (4.25) and (4.27), we find

$$\begin{aligned} M(t) &\le |M(0)| + \int_0^t N(u)du\\ &\le |g(0)| + \frac{1}{c^3}(\tilde\theta_F - \tilde\gamma(1-\eta))\int_0^t M(u)du\\ &\quad + k_cAc^{-\frac12}\int_0^t \frac{e^{\frac{\tilde\theta_F}{c^3}u}}{(1+u)^\nu}du + k_cAc^{-\frac12}t_* + B_ct.\end{aligned}$$

We can use the Gronwall lemma ([H], Lemma 6.2) to obtain

$$\begin{aligned} |g(t)| \le M(t) &\le \left(|g(0)| + k_cAc^{-\frac12}t_* + \frac{B_cc^3}{\tilde\theta_F - \tilde\gamma(1-\eta)}\right)e^{(\tilde\theta_F - \tilde\gamma(1-\eta))\frac{t}{c^3}}\\ &\quad + k_cAc^{-\frac12}\left(\int_0^t \frac{e^{\frac{\tilde\gamma}{c^3}(1-\eta)s}}{(1+s)^\nu}ds\right)e^{(\tilde\theta_F - \tilde\gamma(1-\eta))\frac{t}{c^3}}. \end{aligned}\tag{4.28}$$

We define $h_\infty = |g(0)| + k_cAc^{-\frac12}t_* + \frac{B_cc^3}{\tilde\theta_F - \tilde\gamma(1-\eta)}$. Remembering that $\dot h(t) = e^{-\frac{\tilde\theta_F}{c^3}t}g(t)$, this yields

$$\begin{aligned} |\dot h(t)| &\le h_\infty e^{-\frac{\tilde\gamma}{c^3}(1-\eta)t} + k_cAc^{-\frac12}\left(\int_0^t \frac{e^{\frac{\tilde\gamma}{c^3}(1-\eta)s}}{(1+s)^\nu}ds\right)e^{-\frac{\tilde\gamma}{c^3}(1-\eta)t}\\ &\le h_\infty e^{-\frac{\tilde\gamma}{c^3}(1-\eta)t} + k_cAc^{-\frac12}\left(\int_0^{\frac t2} \frac{e^{\frac{\tilde\gamma}{c^3}(1-\eta)s}}{(1+s)^\nu}ds\right)e^{-\frac{\tilde\gamma}{c^3}(1-\eta)t}\\ &\quad + k_cAc^{-\frac12}\left(\int_{\frac t2}^t \frac{e^{\frac{\tilde\gamma}{c^3}(1-\eta)s}}{(1+s)^\nu}ds\right)e^{-\frac{\tilde\gamma}{c^3}(1-\eta)t}\\ &\le h_\infty e^{-\frac{\tilde\gamma}{c^3}(1-\eta)t} + k_cAc^{-\frac12}\frac{e^{-\frac{\tilde\gamma}{c^3}(1-\eta)t}}{\frac{\tilde\gamma}{c^3}(1-\eta)}\left[e^{\frac{\tilde\gamma}{c^3}(1-\eta)\frac t2} - 1\right]\\ &\quad + k_cAc^{-\frac12}\int_{\frac t2}^t \frac{ds}{(1+s)^\nu}\\ &\le h_\infty e^{-\frac{\tilde\gamma}{c^3}(1-\eta)t} + \frac{k_c}{\tilde\gamma(1-\eta)}Ac^{\frac52}e^{-\frac{\tilde\gamma}{c^3}(1-\eta)\frac t2} + \frac{k_cAc^{-\frac12}}{(\nu-1)\left(1+\frac t2\right)^{\nu-1}}. \end{aligned}$$

Consequently

$$\dot{h}(t) = O(t^{1-\nu}),$$

so that we can conclude that there exists $q_\infty(Y_0, K, F, \varepsilon) \in \mathbb{R}^d$ with the property that

$$q(t) = q_\infty + v(F)t + O(t^{2-\nu}),$$

which proves the second part of the theorem.

In part (i) of the theorem, ϕ_0 and π_0 are compactly supported. Hence A_2 is compactly supported as well. In that case, (4.24) becomes

$$N(t) \leq \frac{1}{c^3}(\tilde{\theta}_F - \tilde{\gamma}(1-\eta))M(t) + \tilde{N},$$

where $\tilde{N}$ is a constant which depends on everything except t, yielding instead of (4.28),

$$|g(t)| \leq |g(0)| e^{(\tilde{\theta}_F - \tilde{\gamma}(1-\eta))\frac{t}{c^3}} + \tilde{N} e^{(\tilde{\theta}_F - \tilde{\gamma}(1-\eta))\frac{t}{c^3}},$$

and hence

$$|\dot{h}(t)| \leq (|\dot{h}(0)| + \tilde{N}) e^{-\frac{\tilde{\gamma}}{c^3}(1-\eta)t}.$$

From this, the announced behaviour of $q(t)$ follows again. □

Proof of Lemma 1. First note that in the case considered here ($V = -F \cdot q$) the Hamiltonian is not bounded below (unless $F = 0$), so that there is no a priori reason why $\dot{h}$ should be bounded. We start by controlling $\ddot{h}(t)$. Using (4.1), (4.12) and $\ddot{h}(t) = \ddot{q}(t)$ we have:

$$|\ddot{h}(t)| \leq |F| + \frac{\tilde{K}}{c^2} + |A_0(t)|.$$

But

$$\begin{aligned} |A_0(t)| &= \left| \int \int dx\, dy\, \rho_2(y) \nabla \rho_1(x - q(t)) \phi^0(x, y, t) \right| \\ &\leq \|\nabla_y \phi^0(t)\|_2 \times \|\nabla_y^{-1} \rho_2(y) \nabla \rho_1(x - q(t))\|_2, \end{aligned}$$

and using (3.12) together with the hypothesis $H_0(Y_0) \leq Kc^{2-2\varepsilon}$ and the form of ρ_2 we have

$$|A_0(t)| \leq Ac^{-\varepsilon}. \tag{4.29}$$

Then remember that $|F| \leq F_0 c^{-2-\varepsilon} \leq F_M(c) = \frac{\tilde{F}_M}{c^2}$, so finally:

$$|\ddot{h}(t)| \leq K_0 c^{-\varepsilon}. \tag{4.30}$$

We now turn to the bound on $\dot{h}(t)$ which is obtained in (4.35). To alleviate the notation, we shall write Γ_F for Df_F. Multiplying (4.20) by $e^{-\Gamma_F t}$ and integrating between 0 and T, we obtain after some rewriting:

$$\dot h(T) = e^{\Gamma_F T}\dot h(0) - \frac{1}{4\pi}\int_0^T dt \iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}$$
$$\times \left[\left(\int_{t-|z|}^{t}(t-|z|-s)\ddot h(s)ds\right)\cdot\nabla\rho_1(x+v|z|)\right]e^{-\Gamma_F(t-T)}\nabla\rho_1(x)$$
$$-\frac{1}{4\pi}\int_0^T dt\iiint_{|z|\le t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}$$
$$\times\frac{1}{2}\left\langle \mathrm{Hess}\rho_1(\tilde x_{t,|z|})\int_{t-\frac{|z|}{c}}^{t}\dot h(s)ds;\int_{t-|z|}^{t}\dot h(s)ds\right\rangle e^{-\Gamma_F(t-T)}\nabla\rho_1(x)$$
$$+\int_0^T dt\, e^{-\Gamma_F(t-T)}A_2(t).$$

Defining $B(t)=\sup_{0\le s\le t}|\dot h(s)|$ and using (4.1) and (4.30), we find for all $t\ge 0$ and for some $K_1, K_2>0$:

$$|\dot h(t)|\le|\dot h(0)|+\frac{K_1}{c^{4+\varepsilon}}\int_0^t\|e^{-\Gamma_F(s-t)}\|ds+\frac{K_2}{c^4}\int_0^t\|e^{-\Gamma_F(s-t)}\|B^2(s)ds$$
$$+\int_0^t\|e^{-\Gamma_F(s-t)}A_2(s)\|ds.$$

Then, with the notations introduced after Theorem 3 (see (4.8)–(4.9))

$$|\dot h(t)|\le|\dot h(0)|+\frac{K_1}{c^{4+\varepsilon}}\int_0^t e^{\frac{\tilde\gamma_F}{c^3}(s-t)}ds+\frac{K_2}{c^4}B^2(t)\int_0^t e^{\frac{\tilde\gamma_F}{c^3}(s-t)}ds$$
$$+\int_0^t e^{\frac{\tilde\gamma_F}{c^3}(s-t)}|A_2(s)|ds,$$

and consequently

$$|\dot h(t)|\le|\dot h(0)|+\frac{K_1}{\tilde\gamma_F c^{1+\varepsilon}}+\frac{K_2}{\tilde\gamma_F c}B^2(t)+\int_0^t e^{\frac{\tilde\gamma_F}{c^3}(s-t)}|A_2(s)|ds.$$

We now control the last term of this inequality. Under the hypothesis of part (i) of Theorem 2, $A_0(s)$ has compact support. One should in addition remember that A_2 differs from A_0 by terms which have compact support in the ball af radius $\frac{2R_2}{c}$. Moreover one of these terms is bounded by $\frac{1}{c^2}$ and the other by $\frac{|\dot h(t)|}{c^3}$. Therefore, using (4.29), the last integral can be bounded as follows:

$$\begin{aligned}\int_0^t e^{\frac{\tilde\gamma_F}{c^3}(s-t)}|A_2(s)|ds &\le \int_0^\alpha e^{\frac{\tilde\gamma_F}{c^3}(s-t)}|A_0(s)|ds+\int_0^{\frac{2R_2}{c}}e^{\frac{\tilde\gamma_F}{c^3}(s-t)}|A_2(s)-A_0(s)|ds\\ &\le Ac^{-\varepsilon}\int_0^\alpha e^{\frac{\tilde\gamma_F}{c^3}(s-t)}ds+k(B(t)c^{-3}+c^{-2})\int_0^{\frac{2R_2}{c}}e^{\frac{\tilde\gamma_F}{c^3}(s-t)}ds\\ &\le A'\alpha c^{-\varepsilon}+k'c^{-3}+k'B(t)\alpha c^{-4},\end{aligned}\tag{4.31}$$

where α is such that $A_0(s)=0$ for $s>\alpha$ (note that $\alpha<\sup\{\frac{2R_2}{c},\frac{R_2}{c}+Rc\}$) and provided c is large enough. And so we have for all $t\ge 0$,

$$|\dot h(t)|\le|\dot h(0)|+K_3c^{1-\varepsilon}+\frac{K_2}{\tilde\gamma_F c}B^2(t)+k'B(t)c^{-4}.\tag{4.32}$$

If we are now under the hypothesis of part (ii), we use (4.26) to control $\int_0^t e^{\frac{\tilde{\gamma}_F}{c^3}(s-t)}$ $|A_0(s)|ds$. We obtain then

$$\int_0^t e^{\frac{\tilde{\gamma}_F}{c^3}(s-t)}|A_0(s)|ds \leq \frac{A}{c^{\frac{1}{2}}}\int_0^t \frac{e^{\frac{\tilde{\gamma}_F}{c^3}(s-t)}}{(1+s)^\nu}ds$$
$$\leq \int_0^t \frac{A}{c^{\frac{1}{2}}}\frac{1}{(1+s)^\nu}ds \leq A''c^{-\frac{1}{2}}$$

because $\nu > 2$. Finally we have once again (4.32) for all $t \geq 0$.

We can now conclude as follows. Since $B(t)$ is increasing, we have, for all $0 \leq t \leq T$,

$$|\dot{h}(t)| \leq |\dot{h}(0)| + K_3c^{1-\varepsilon} + \frac{K_2}{\tilde{\gamma}_F c}B^2(T) + k'B(T)c^{-4}.$$

So, taking the supremum over t, we have the following inequality for all $T \geq 0$:

$$B(T) - k'B(T)c^{-4} \leq |\dot{h}(0)| + K_3c^{1-\varepsilon} + \frac{K_2}{\tilde{\gamma}_F c}B^2(T). \tag{4.33}$$

Using the hypothesis on F and $H_0(Y_0)$, one has $|\dot{h}(0)| < Kc^{1-\varepsilon}$, so, for c large enough:

$$B(T) \leq 2(K + K_3)c^{1-\varepsilon} + \frac{2K_2}{\tilde{\gamma}_F c}B^2(T). \tag{4.34}$$

An easy computation and the continuity of $B(t)$ tell us that:

$$B(t) \leq B_- \qquad \forall t \geq 0 \quad \text{or} \quad B(t) \geq B_+ \qquad \forall t \geq 0$$

with

$$B_\pm = \frac{\tilde{\gamma}_F c}{4K_2}\left(1 \pm \sqrt{1 - \frac{16K_2(K+K_3)}{\tilde{\gamma}_F c^\varepsilon}}\right).$$

We will now take c large enough so that there exists two constants β and β' such that:

$$B_- \leq \beta c^{1-\varepsilon}, \quad B_+ \geq \beta' c > Kc^{1-\varepsilon}.$$

Note now that

$$B(0) =| \dot{h}(0) |\leq Kc^{1-\varepsilon} < B_+,$$

and so we have finally the following bound for $\dot{h}(t)$:

$$|\dot{h}(t)| \leq B(t) \leq B_- \leq \beta c^{1-\varepsilon} \qquad \forall t \geq 0. \tag{4.35}$$

□

The condition "c large" is certainly essential in our proofs. Whether the results can also be obtained without this condition is not clear. On an intuitive level, the condition can be understood as follows. Remark that the model contains three intrinsic time scales that are functions of ρ_1, σ_2 and c:

(i) the relaxation time $\tau_1 \equiv \gamma^{-1} = c^3\tilde{\gamma}^{-1}$ defined in (4.3),
(ii) the time $\tau_2 \equiv \frac{2R_1}{v_M}$ the particle needs to cross its own diameter when moving at speed v_M,
(iii) the time $\tau_3 \equiv \frac{2R_2}{c}$ the signals in the membranes need to cross the particle.

One has therefore two dimensionless parameters:

$$\frac{\tau_1}{\tau_3} = c^4 \frac{1}{2R_2\tilde{\gamma}}; \quad \frac{\tau_1}{\tau_2} = c^4 \frac{w_M}{2R_1\tilde{\gamma}}.$$

Taking c large is therefore equivalent to $\tau_1 \gg \tau_3$, which expresses the idea that the membranes evacuate the energy deposited by the particle "quickly". Alternatively, c large is equivalent to $v_M \frac{1}{\gamma} \gg 2R_1$, which is saying that the distance travelled by a particle moving at the characteristic speed v_M during a time τ_1 is much larger than the particle diameter.

5. The Confined Case

We turn to the case of a confining potential. We make the following assumptions on V and σ_2:

(C) $\lim_{q\to\infty} V(q) = +\infty$;

(W) $\hat{\sigma}_2(k) \neq 0$ for all $k \in \mathbb{R}^3$.

Then ρ_2 is defined as in (4.1). Let $S = \{q^* \in \mathbb{R}^d | \nabla V(q^*) = 0\}$ be the set of critical points of V. We suppose that S is discrete. For all $q \in \mathbb{R}^d$, we denote by ϕ_q the unique solution of $-\Delta_y \phi(x, y) = -\rho_1(x - q)\rho_2(y)$ decaying at infinity. Therefore, $\{(\phi_q, q, 0, 0) | q \in S\}$ is the set of equilibrium points for the dynamics.

Theorem 4. *Suppose that* (H_1), (H_2), (C) *and* (W) *are satisfied and* $n = 3$. *Denote by* $Y(t) = (\phi(t), q(t), \pi(t), p(t))$ *the solution of* (1.5)–(1.6). *For all* $Y_0 \in \mathcal{D}$, *there exists* $q^* \in S$ *such that:*

$$\lim_{t\to+\infty} q(t) = q^* \quad \textit{and} \quad \lim_{t\to+\infty} \dot{q}(t) = 0. \tag{5.1}$$

If moreover, q^* *is a non-degenerate minimum for* V, *then for all* $\eta > 0$ *there exists a* $c_0 > 0$ *such that for any* $c > c_0$ *and for all* ϕ_0 *and* π_0 *with compact support, we have for all* $t > 0$,

$$|q(t) - q^*| \leq K e^{\frac{-(1-\eta)\tilde{\gamma}}{2c^3} t}. \tag{5.2}$$

A similar result holds for $\dot{q}(t)$.

One could also, as in [KKS1], study the convergence of $\phi(t)$ to ϕ_{q^*}, but we shall not do this here. Note that the first part of the theorem does not require c to be large. In fact, using the linearization method of [KKS1], one could prove the convergence is exponential for all c as well. In this way, we do not, however, obtain a very explicit expression for the exponential rate of decay. Our method here shows it to be equal to $\frac{\tilde{\gamma}}{2c^3}$, confirming the solutions in this model behave very much like those of the phenomenological equation (1.1).

Proof. In order to prove (5.1), i.e. the convergence of $q(t)$ and $\dot{q}(t)$, we follow the method of [KKS1]. The exponential rate (5.2) will then be obtained by the same techniques as in the case $V(q) = -F \cdot q$.

Using the conservation of energy and the hypothesis on V, one concludes immediately that $q(t)$, $\dot{q}(t)$ and $\ddot{q}(t)$ are bounded. Let $B_R \subset \mathbb{R}^3$ be the ball of radius R centered at 0. We define:

$$E_R(t) = \frac{p^2(t)}{2} + V(q(t)) + \frac{1}{2}\int_{\mathbb{R}^d} dx \int_{B_R} dy\, (|\nabla_y \phi(x,y,t)|^2 + |\pi(x,y,t)|^2) + \int_{\mathbb{R}^{d+n}} dx\, dy\, \rho_1(x - q(t))\rho_2(y)\phi(x,y,t). \tag{5.3}$$

We take $R \geq \frac{R_2}{c}$. Using (3.11), we can write ϕ as $\phi^r + \phi^0$ and in a similar way, $\pi = \pi^r + \pi^0$, where $\pi^r(x,y,t) = \dot{\phi}^r(x,y,t)$ and $\pi^0(x,y,t) = \dot{\phi}^0(x,y,t)$. Using this decomposition and the regularity of ϕ_0 and π_0, we see that $\phi(x,y,t)$ and $\pi(x,y,t)$ are differentiable. Let us write $n(y) = \frac{y}{|y|}$ and let $d\sigma$ be the surface area element of ∂B_R. Then differentiating (5.3), we have

$$\begin{aligned}
\frac{d}{dt}E_R(t) &= \frac{d}{dt}\left(H(Y(t)) - \frac{1}{2}\int_{\mathbb{R}^d} dx \int_{|y|>R} dy\, (|\nabla_y \phi(x,y,t)|^2 + |\pi(x,y,t)|^2)\right) \\
&= \int_{\mathbb{R}^d} dx \int_{\partial B_R} d\sigma(y)\, n(y) \cdot \nabla_y \phi(x,y,t)\pi(x,y,t) \\
&= \int_{\mathbb{R}^d} dx \int_{\partial B_R} d\sigma(y)\, n(y) \cdot (\nabla_y \phi^r(x,y,t)\pi^r(x,y,t) + \nabla_y \phi^r(x,y,t) \\
&\quad \times \pi^0(x,y,t) + \nabla_y \phi^0(x,y,t)\pi^r(x,y,t) + \nabla_y \phi^0(x,y,t)\pi^0(x,y,t)).
\end{aligned}$$

We bound the three last terms by the Young inequality, and we then integrate in t. Hence, for all $T > \frac{R_2}{c}$,

$$\begin{aligned}
&E_R(T+R) - E_R\left(R + \frac{R_2}{c}\right) \\
&\quad \leq \int_{R+\frac{R_2}{c}}^{T+R} dt \int_{\mathbb{R}^d} dx \int_{\partial B_R} d\sigma(y) \left(n(y) \cdot \nabla_y \phi^r(x,y,t)\pi^r(x,y,t)\right. \\
&\qquad \left. + \frac{1}{4}(|\nabla_y \phi^r(x,y,t)|^2 + |\pi^r(x,y,t)|^2) + 2(|\nabla_y \phi^0(x,y,t)|^2 + |\pi^0(x,y,t)|^2)\right).
\end{aligned}$$

We know that $E_R(R + \frac{R_2}{c}) \leq H(Y(R + \frac{R_2}{c})) = H(Y_0)$, and the hypothesis on the potential and (3.14) tell us that:

$$\begin{aligned}
E_R(t) &\geq H(Y(t)) - \frac{1}{2}(\|\pi(t)\|_2^2 + \|\phi(t)\|^2) \\
&\geq -H(Y_0) + 2V_0 + 2\langle \rho_1\rho_2; \rho_1\Delta^{-1}\rho_2\rangle,
\end{aligned}$$

where V_0 is the infimum of V. So we have

$$E_R(T+R) - E_R\left(R + \frac{R_2}{c}\right) \geq -C,$$

where C is a constant not depending on R, T. Hence,

$$-\int_{R+\frac{R_2}{c}}^{T+R} dt \int_{\mathbb{R}^d} dx \int_{\partial B_R} d\sigma(y)\, n(y)\cdot\nabla_y\phi^r(x,y,t)\pi^r(x,y,t) + \frac{1}{4}(|\nabla_y\phi^r|^2 + |\pi^r|^2)$$

$$\leq C + 2\int_{R+\frac{R_2}{c}}^{T+R} dt \int_{\mathbb{R}^d} dx \int_{\partial B_R} d\sigma(y)(|\nabla_y\phi^0(x,y,t)|^2 + |\pi^0(x,y,t)|^2). \tag{5.4}$$

We first need to bound the right-hand side. This follows from Lemma 3.3 of [KKS1] and the fact that $\kappa \in L^2$ (recall that κ is defined in Definition 1):

$$\int_{R+\frac{R_2}{c}}^{T+R} dt \int_{\mathbb{R}^d} dx \int_{\partial B_R} d\sigma(y)(|\nabla_y\phi^0(x,y,t)|^2 + |\pi^0(x,y,t)|^2) \leq I_0$$

uniformly in R and T.

Then, using (4.10), we have

$$\phi^r(x,y,t) = -\frac{1}{4\pi}\int_{|y-z|\leq t} dz\, \frac{\rho_2(z)}{|y-z|}\rho_1\left(x - q(t-|y-z|)\right).$$

If $|y| = R$, because $\rho_2(z) = 0$ for $|z| \geq \frac{R_2}{c}$, we have for $t > R + \frac{R_2}{c}$,

$$\phi^r(x,y,t) = -\frac{1}{4\pi}\int_{|z|<\frac{R_2}{c}} dz\, \frac{\rho_2(z)}{|y-z|}\rho_1\left(x - q(t-|y-z|c)\right).$$

Consequently, still for $|y| = R$ and $t > R + \frac{R_2}{c}$,

$$\pi^r(x,y,t) = \dot{\phi}^r(x,y,t) = -\frac{1}{4\pi}\int_{|z|<\frac{R_2}{c}} dz\, \frac{\rho_2(z)}{|y-z|}\frac{\partial}{\partial t}\rho_1\left(x - q(t-|y-z|)\right) \tag{5.5}$$

and

$$\begin{aligned}\nabla_y\phi^r(x,y,t) = {}& \frac{1}{4\pi}\int_{|z|<\frac{R_2}{c}} dz\, \frac{\rho_2(z)}{|y-z|}\frac{\partial}{\partial t}\rho_1\left(x - q(t-|y-z|)\right) n(y)\\ &+\frac{1}{4\pi}\int_{|z|<\frac{R_2}{c}} dz\, \frac{\rho_2(z)}{|y-z|^2}\rho_1\left(x - q(t-|y-z|)\right) n(y-z)\\ &+\frac{1}{4\pi}\int_{|z|<\frac{R_2}{c}} dz\, \frac{\rho_2(z)}{|y-z|}\frac{\partial}{\partial t}\rho_1\left(x - q(t-|y-z|)\right)\\ &\times(n(y-z) - n(y)).\end{aligned}$$

The last two integrals are bounded by KR^{-2} because $\dot{q}$ is bounded. Hence

$$\nabla_y\phi^r(x,y,t) = -\pi^r(x,y,t)n(y) + \mathcal{O}(|y|^{-2}).$$

Since we know that $q(t)$ is bounded by some constant Q_0, for $|x| > Q_0 + R_1 = \tilde{R}_1$, we have $\phi^r(x,y,t) = \pi^r(x,y,t) = 0$. So (5.4) becomes

$$\int_{R+\frac{R_2}{c}}^{T+R} dt \int_{|x|<Q_0+R_1} dx \int_{\partial B_R} d\sigma(y)\, |\pi^r(x,y,t)|^2 \leq K + T\mathcal{O}(R^{-2}), \tag{5.6}$$

and using once again (5.5) we have

$$\int_{R+\frac{R_2}{c}}^{T+\frac{R}{c}} dt \int_{B_{\tilde{R}_1}} dx \int_{\partial B_R} d\sigma(y) \left| \int_{B_{\frac{R_2}{c}}} dz \frac{\rho_2(z)}{|y-z|} \frac{\partial}{\partial t} \rho_1(x - q(t - |y-z|c)) \right|^2$$
$$\leq K + T\mathcal{O}(R^{-2}).$$

But $|y-z| \sim R$ and $t + R - |y-z| = t + n(y) \cdot z + \mathcal{O}(R^{-1})$, so

$$\int_{\frac{R_2}{c}}^{T} dt \int_{B_{\tilde{R}_1}} dx \int_{\partial B_R} d\sigma(y) \left| \int_{B_{\frac{R_2}{c}}} dz \frac{\rho_2(z)}{|y-z|} \frac{\partial}{\partial t} \rho_1 \left(x - q(t + n(y) \cdot z + \mathcal{O}(R^{-1})) \right) \right|^2$$
$$\leq K + T\mathcal{O}(R^{-2}).$$

After the change of variable $y = R\sigma$, we now take the limit $R \to +\infty$ and so

$$\int_{\frac{R_2}{c}}^{T} dt \int_{B_{\tilde{R}_1}} dx \int_{S^2} d\sigma \left| \int_{B_{\frac{R_2}{c}}} dz\, \rho_2(z) \frac{\partial}{\partial t} \rho_1\left(x - q(t + \sigma \cdot z)\right)) \right|^2 \leq K.$$

This bounds holds for all T and so

$$\int_{0}^{+\infty} dt \left[\int_{B_{\tilde{R}_1}} dx \int_{S^2} d\sigma \left| \int_{B_{\frac{R_2}{c}}} dz \rho_2(z) \nabla \rho_1\left(x - q(t + \sigma \cdot z)\right) \cdot \dot{q}(t + \sigma \cdot z) \right|^2 \right] < +\infty. \tag{5.7}$$

We define the function

$$I(x, \sigma, t) = \left| \int_{B_{\frac{R_2}{c}}} dz\, \rho_2(z) \nabla \rho_1\left(x - q(t + \sigma \cdot z)\right) \cdot \dot{q}(t + \sigma \cdot z) \right|^2$$

which is differentiable in x, σ and uniformly Lipschitz in t because $\dot{q}$ and $\ddot{q}$ are bounded. As a result

$$\lim_{t \to +\infty} I(x, \sigma, t) = 0 \tag{5.8}$$

uniformly in $x \in B_{\tilde{R}_1}$ and $\sigma \in S^2$. We fix σ and x. We take a basis of $\mathbb{R}^3$ such that $\sigma = e_1$, and we define

$$\bar{\rho}_2(z_1) = \int dz_2\, dz_3 \rho_2(z_1, z_2, z_3)$$

and $s = \sigma \cdot z$. Then we have

$$\begin{aligned} I(x, \sigma, t) &= \left| \int ds\, \bar{\rho}_2(s) \nabla \rho_1\left(x - q(t+s)\right)) \cdot \dot{q}(t+s) \right|^2 \\ &= \left| \int ds\, \bar{\rho}_2(t-s) \nabla \rho_1\left(x - q(s)\right)) \cdot \dot{q}(s) \right|^2 \\ &= |\bar{\rho}_2 \star (\nabla \rho_1(x-q) \cdot \dot{q})(t)|^2. \end{aligned}$$

Then (5.8) leads to

$$\lim_{t\to+\infty} \bar{\rho}_2 \star (\nabla\rho_1(x-q)\cdot\dot{q})(t) = 0.$$

Hence (W) and Pitt's extension to Wiener's Tauberian theorem [R] implies

$$\lim_{t\to+\infty} \nabla\rho_1(x-q(t))\cdot\dot{q}(t) = 0$$

uniformly in x. So we have

$$\begin{aligned} 0 &= \lim_{t\to+\infty} \sup_{x\in B_{\tilde{R}_1}} \nabla\rho_1(x-q(t))\cdot\dot{q}(t) \\ &= \lim_{t\to+\infty} \sup_{x\in\mathbb{R}^d} \nabla\rho_1(x-q(t))\cdot\dot{q}(t) \end{aligned}$$

because $\nabla\rho_1(x-q(t)) = 0$ if $|x| > \tilde{R}_1$. Hence

$$\lim_{t\to+\infty} \nabla\rho_1(x)\cdot\dot{q}(t) = 0 \quad \forall x\in\mathbb{R}^d,$$

which proves that $\dot{q}(t)$ tends to zero (one can take $x = re_i$, where r is such that $\rho_1'(re_i) \neq 0$ and $(e_1, ..., e_d)$ is any orthonormal basis). It remains to show that $q(t)$ converges to some q^* which satisfies $\nabla V(q^*) = 0$. Remember that ϕ_q is the stationary solution of (1.5) corresponding to $q(t) = q$. Let $\mathcal{A} = \{Y_q = (\phi_q, q, 0, 0) \quad q\in\mathbb{R}^d,\ |q| \leq Q_0\}$. $\mathcal{A}$ is compact in $\mathcal{E}$. Finally, we denote by $\|.\|_R$ the L^2 norm restricted to the ball of radius R and $|Y|^2_{\mathcal{E},R} = \|\nabla_y\phi\|^2_R + |q|^2 + \|\pi\|^2_R + |p|^2$. We first prove

$$\begin{aligned} \inf_{Y_q\in\mathcal{A}} |Y(t)-Y_q|^2_{\mathcal{E},R} = |p(t)|^2 + \|\pi(t)\|^2_R + \inf_{|q|\leq Q_0} (\|\nabla_y(\phi(t)-\phi_q)\|^2_R \\ +|q(t)-q|^2) \to_{t\to+\infty} 0. \end{aligned} \tag{5.9}$$

We know that $|p(t)| \to 0$ as $t\to+\infty$. Then (5.5) implies that $\|\pi^r(t)\|_R \to 0$ as $t\to+\infty$. The bound on $\|\pi^0(t)\|_R$ (see Lemma 3.3 of [KKS1]) then shows that the same result holds for $\|\pi(t)\|_R$. To estimate the infimum over q in (5.9) we take $q(t)$ for q. Then the last term vanishes and we have to control

$$\begin{aligned} &\nabla_y(\phi^r(x,y,t) - \phi_{q(t)}(x,y)) \\ &\quad = \nabla_y\left[\frac{-1}{4\pi}\int dz \frac{\rho_2(y-z)}{|z|}(\rho_1(x-q(t-|z|)) - \rho_1(x-q(t)))\right] \end{aligned}$$

for $|y| \leq R$, the term with $\nabla_y\phi^0$ being controlled using once again Lemma 3.3 of [KKS1]. The difference $\rho_1(x-q(t-|z|)) - \rho_1(x-q(t))$ can be written using an integral depending only on $\dot{q}(s)$ for $s\in[t-(R+\frac{R_2}{c}), t]$ which tends to zero as t goes to infinity uniformly in $(x,y)\in B_R$. All this proves (5.9).

Given a solution $Y(t)$ of (3.1), we call $\mathcal{B}$ the set of all $\bar{Y}\in\mathcal{E}$ such that there exists some sequence $t_n\to+\infty$ with $Y(t_n)\to\bar{Y}$ in the semi-norm $|.|_{\mathcal{E},R}$ for all R. The continuity of W^t tells us that $\mathcal{B}$ is an invariant set. Then, (5.9) tells us that $\mathcal{B}\subset\mathcal{A}$. So, for $\bar{Y}\in\mathcal{B}$ there exists a C^2 curve $t\to\tilde{q}(t)\in\mathbb{R}^d$ such that $W^t\bar{Y} = Y_{\tilde{q}(t)}$. But $W^t\bar{Y}$ is a solution of (3.1) so we must have $\dot{\tilde{q}}(t) = 0$, hence $\tilde{q}(t) = q^*$ with $\nabla V(q^*) = 0$ and $q^*\in S$. Therefore $\bar{Y} = Y_{q^*}$ and $\mathcal{B}\subset\{Y_q,\ q\in S\}$.

We now prove that $q(t) \to q^*$. Suppose there exist $R_0, \epsilon > 0$ and a sequence $t_n \to +\infty$ such that

$$\inf_{q\in S} |Y(t_n) - Y_q|_{\mathcal{E},R_0} \geq \epsilon. \tag{5.10}$$

But (5.9) and the compactness of $\mathcal{A}$ imply that there exists $\bar{Y} \in \mathcal{A}$ and a subsequence t_n' such that $Y(t_n') \to \bar{Y}$ in the norm $|.|_{\mathcal{E},R}$ for all R, where $\bar{Y} \in \mathcal{A}$. Then, by definition, $\bar{Y} \in \mathcal{B}$. But (5.10) is then a contradiction to $\mathcal{B} \subset \{Y_q,\ q \in S\}$. So

$$\inf_{q\in S} |q(t) - q| \to 0,$$

and because S is discrete, there exists $q^* \in S$ such that $q(t) \to q^*$. We have therefore proven (5.1).

To prove (5.2), we now suppose that q^* is a non-degenerate minimum for V. Because of the translational invariance of the interaction term, we can suppose that $q^* = 0$.

Now, the computation leading up to (4.20) in the particular case $v = 0$, so that $h(t) = q(t)$, yields:

$$\begin{aligned}
\ddot{q}(t) = -\nabla V(q(t)) - \frac{\tilde{\gamma}}{c^3}\dot{q}(t) - \frac{1}{4\pi}\iiint_{|z|\leq t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|} \\
\times \left[\left(\int_{t-|z|}^{t} (t-|z|-s)\ddot{q}(s)ds\right)\cdot\nabla\rho_1(x)\right]\nabla\rho_1(x) \\
-\frac{1}{4\pi}\iiint_{|z|\leq t} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|} \\
\times\frac{1}{2}\left\langle \mathrm{Hess}\rho_1(\tilde{x}_{t,|z|})\int_{t-|z|}^{t}\dot{q}(s)ds;\int_{t-|z|}^{t}\dot{q}(s)ds\right\rangle\nabla\rho_1(x) + A_2(t).
\end{aligned} \tag{5.11}$$

Moreover, $\nabla V(q(t)) = W\cdot q(t) + r(q(t))$, where W is the Hessian matrix of V at $q = 0$ and $r(q) = o(|q|)$. Remark that $A_2(t)$ has compact support. We now define $Q(t) = (q(t), \dot{q}(t)) \in \mathbb{R}^{2d}$ and $\tilde{W}$ the $2d \times 2d$ matrix:

$$\tilde{W} = \begin{pmatrix} O & I \\ -W & -\frac{\tilde{\gamma}}{c^3}I \end{pmatrix}.$$

Since 0 is a non-degenerate minimum for V, W is a diagonalizable positive definite matrix. One should remark that $\tilde{W}$ is diagonalizable as well with eigenvalues $\lambda_k = -\frac{\tilde{\gamma}}{2c^3} + i\alpha_k$, and so for all t in $\mathbb{R}$, $\|e^{\tilde{W}t}\| = e^{-\frac{\tilde{\gamma}}{2c^3}t}$. We rewrite (5.11) as

$$\dot{Q}(t) = \tilde{W}Q(t) + \psi(t, Q, \dot{Q}),$$

where ψ is a function we will control in terms of $|\ddot{q}(t)|$, $|\dot{q}(t)|$ and $|q(t)|$. Defining $X(t) = e^{-\tilde{W}t}Q(t)$ we have:

$$\dot{X}(t) = e^{-\tilde{W}t}\psi\left(t, e^{\tilde{W}t}X(t), \tilde{W}e^{\tilde{W}t}X(t) + e^{\tilde{W}t}\dot{X}(t)\right),$$

$$
\begin{aligned}
|\dot{X}(t)| \leq \; & \frac{1}{4\pi} \iiint_{|z|\leq t} dx\, dy\, dz\, \frac{\rho_2(y-z)\rho_2(y)}{|z|} \left(\int_{t-|z|}^{t} (t-|z|-s) \right. \\
& \left. \times\, e^{\frac{\tilde{\gamma}}{2c^3}(t-s)} (|\tilde{W} X(s)| + |\dot{X}(s)|) ds \right) |\nabla \rho_1(x)|^2 \\
& + \frac{1}{4\pi} \iiint_{|z|\leq t} dx\, dy\, dz\, \frac{\rho_2(y-z)\rho_2(y)}{|z|} \times \frac{1}{2} |\mathrm{Hess}\rho_1(\tilde{x}_{t,|z|})| \\
& \times \left(\int_{t-|z|}^{t} e^{\frac{\tilde{\gamma}}{2c^3}(t-s)} |X(s)| ds \right) \left(\int_{t-|z|}^{t} |Q(s)| ds \right) |\nabla \rho_1(x)| \\
& + e^{\frac{\tilde{\gamma}}{2c^3} t} |\tilde{A}(t)| + e^{\frac{\tilde{\gamma}}{2c^3} t} |r(q(t))|.
\end{aligned} \tag{5.12}
$$

Let $\varepsilon > 0$. Since $r(q) = o(|q|)$, there exists $\delta > 0$ such that for all $|q| < \delta, \quad |r(q)| < \varepsilon |q| < \varepsilon |Q|$. We define $\eta = \min(\delta, \varepsilon)$. Moreover, we already know that $Q(t) \to 0$, so there exists T such that for all $t \geq T$, $|Q(t)| \leq \eta$. Using the fact that the evolution of the solution $Y(t)$ of (3.1) is given by a continuous linear group and that $Y(t)$ satisfies the same conditions as Y_0, we can suppose $T = 0$. So we have

$$
\forall t \geq 0, |Q(t)| < \eta < \varepsilon \qquad \text{and} \qquad |r(q)| < \varepsilon |Q|. \tag{5.13}
$$

As in Sect. 4, we define

$$
M(t) = \sup_{0\leq s\leq t} |X(s)| \quad \text{and} \quad N(t) = \sup_{0\leq s\leq t} |\dot{X}(s)|.
$$

Remembering once more that ρ_2 depends on c via (4.1) and using (5.12) and (5.13), we have for all $0 \leq s \leq t$,

$$
\begin{aligned}
|\dot{X}(s)| \leq \; & \frac{K_1}{c^4} e^{\frac{\tilde{\gamma} R_2}{c^4}} \|\tilde{W}\| M(t) + \frac{\varepsilon K_2}{c^4} e^{\frac{\tilde{\gamma} R_2}{c^4}} M(t) + \frac{K_1}{c^4} e^{\frac{\tilde{\gamma} R_2}{c^4}} N(t) \\
& + \sup_{0\leq\tau\leq t} \left(\varepsilon e^{\frac{\tilde{\gamma}}{2c^3}\tau} |Q(\tau)| \right) + \sup_{0\leq\tau\leq t} \left(e^{\frac{\tilde{\gamma}}{2c^3}\tau} |\tilde{A}(\tau)| \right).
\end{aligned}
$$

Remark that $e^{\frac{\tilde{\gamma}}{2c^3}s} |Q(s)| = |e^{-\tilde{W}s} Q(s)| = |X(s)|$ and $\sup_{0\leq\tau\leq t} \left(e^{\frac{\tilde{\gamma}}{2c^3}\tau} |\tilde{A}(\tau)| \right) \leq K_3$, so taking the supremum over all $s \in [0, t]$ in the left hand side, we have

$$
N(t) \leq \left(\frac{K_1}{c^4} \|\tilde{W}\| e^{\frac{\tilde{\gamma} R_2}{c^4}} + \frac{\varepsilon K_2}{c^4} e^{\frac{\tilde{\gamma} R_2}{c^4}} + \varepsilon \right) M(t) + \frac{K_1}{c^4} e^{\frac{\tilde{\gamma} R_2}{c^4}} N(t) + K_3,
$$

and so

$$
\left(1 - \frac{K_1}{c^4} e^{\frac{\tilde{\gamma} R_2}{c^4}} \right) N(t) \leq \left(\frac{K_1}{c^4} \|\tilde{W}\| e^{\frac{\tilde{\gamma} R_2}{c^4}} + \frac{\varepsilon K_2}{c^4} e^{\frac{\tilde{\gamma} R_2}{c^4}} + \varepsilon \right) M(t) + K_3.
$$

We call $(k_c')^{-1}$ the factor of $N(t)$. We can choose ε as small as we want, so the factor of $M(t)$ can be bounded by $\frac{K'}{c^4}$. Then, the same computation as in the last part of the proof of Theorem 2 leads to

$$
M(t) \leq \left(M(0) + \frac{K_3 c^4}{K'} \right) e^{\frac{k_c K'}{c^4} t},
$$

and finally we have

$$|Q(t)| \leq \left(M(0) + \frac{K_3 c^4}{K'} \right) e^{\left(\frac{k_c K'}{c^4} - \frac{\tilde{\gamma}}{2c^3} \right) t},$$

which is the annouced result. □

Proof of Theorem 3. In the first part, we will follow the proof of Theorem 4 in order to prove that $\dot{q}(t) \to 0$. The only thing we have to worry about in the present case is that, unlike in the case of Theorem 4, $q(t)$ is not a priori bounded. However, $\dot{q}(t)$ is bounded because $V \equiv 0$. In order to obtain the exponential decay rate, we will then make the same computations as in the proof of Theorem 2, except that we will not use Lemma 1, but the fact that we already know that $\dot{q}(t) \to 0$.

We first prove that $\dot{q}(t) \to 0$. We follow the computation of the proof of Theorem 4. Since $\dot{q}(t)$ is bounded, if t belongs to $\left[R + \frac{R_2}{c} ; R + T \right]$, if $|y| = R$ and $|z| \leq \frac{R_2}{c}$, we have

$$|q(t - |y - z|)| \leq C \left(T + \frac{R_2}{c} \right)$$

for some constant $C > 0$. With that estimate, (5.6) clearly becomes

$$\int_{R+\frac{R_2}{c}}^{R+T} dt \int_{|x|<C\left(T+\frac{R_2}{c}\right)+R_1} dx \int_{\partial B_R} d\sigma(y)\, |\pi^r(x, y, t)|^2 \leq K + T^{d+1} \mathcal{O}(R^{-2}).$$

Then, (5.7) becomes

$$\int_0^{+\infty} dt \left[\int_{\mathbb{R}^d} dx \int_{S^2} d\sigma \left| \int_{B_{\frac{R_2}{c}}} dz \rho_2(z) \nabla \rho_1\, (x - q(t + \sigma \cdot z)))\, \dot{q}(t + \sigma \cdot z) \right|^2 \right] < +\infty,$$

and the end of the proof follows identically.

Now that we know that $\dot{q}(t) \to 0$, we can control $|q(t) - q_\infty|$ in exactly the same way as in the proof of Theorem 2, but instead of using Lemma 1, we remark that there exists $T > 0$ such that for all $t \geq T, \quad |\dot{q}(t)| < 1$. Using the fact that the evolution of the solution $Y(t)$ of (3.1) is given by a continuous linear group and that $Y(T)$ satisfies the same conditions as Y_0, we can suppose $T = 0$. So, with the notations of Sect. 4 one has, instead of (4.24),

$$N(t) \leq \eta \frac{\tilde{\gamma}}{c^3} M(t) + k_c \sup_{0 \leq s \leq t} \left(e^{\frac{\tilde{\gamma}}{c^3} s} |A_2(s)| \right).$$

The end of the proof is then similar. □

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References

[B] Brezis, H.: *Analyse fonctionnelle. Théorie et applications*. Paris: Masson, 1993

[CEFM] Castella, F., Erdös, L., Frommlet, F., Markovich, P.A.: Fokker-Planck equations as scaling limits of reversible quantum systems. J. Stat. Phys. **100**(3–4), 543–601 (2000)

[CH] Courant, R., Hilbert, D.: Methods of mathematical physics (vol 2), Interscience (1962)

[FLO] Ford, G.W., Lewis, J.T., O'Connell, R.F.: Independent oscillator model of a heat bath: exact diagonalization of the Hamiltonian. J. Stat. Phys. **53**(1/2), 439–455 (1988)

[H] Hale, J.: *Ordinary Differential Equations*. Robert E. Krieger Publishing Company, 1969

[J] John, F.: *Partial Differential Equations*. Berlin-Heidelberg-New York: Springer-Verlag, 1971

[KKS1] Komech, A., Kunze, M., Spohn, H.: Long-time asymptotics for a classical particle interacting with a scalar wave field. Comm. Partial Differential Equations **22**, 307–335 (1997)

[KKS2] Komech, A., Kunze, M., Spohn, H.: Effective dynamics for a mechanical particle coupled to a wave field. Commun. Math. Phys. **203**, 1–19 (1999)

[KS] Komech, A., Spohn, H.: Soliton-like asymptotics for a classical particle interacting with a scalar wave field. Nonlinear Anal. **33**, 13–24 (1998)

[LM] Lions, J.L., Magenes, E.: *Problèmes aux limites non homogènes* (vol 1), Dunod, 1968

[R] Rudin, W.: *Functional Analysis*. New York: McGraw Hill, 1973

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