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Integrable Nonlinear Evolution Equations on the Half-Line

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Abstract: A rigorous methodology for the analysis of initial-boundary value problems on the half-line, $0 < x < \infty$, $t > 0$, is applied to the nonlinear Schrödinger (NLS), to the sine-Gordon (sG) in laboratory coordinates, and to the Korteweg-deVries (KdV) with dominant surface tension. Decaying initial conditions as well as a smooth subset of the boundary values $\{\partial_x^l q(0,t) = g_l(t)\}_0^{n-1}$ are given, where $n = 2$ for the NLS and the sG and $n = 3$ for the KdV. For the NLS and the KdV equations, the initial condition $q(x,0) = q_0(x)$ as well as one and two boundary conditions are given respectively; for the sG equation the initial conditions $q(x,0) = q_0(x)$, $q_t(x,0) = q_1(x)$, as well as one boundary condition are given. The construction of the solution $q(x,t)$ of any of these problems involves two separate steps: (a) Given decaying initial conditions define the spectral (scattering) functions $\{a(k), b(k)\}$. Associated with the smooth functions $\{g_l(t)\}_0^{n-1}$, define the spectral functions $\{A(k), B(k)\}$. Define the function $q(x,t)$ in terms of the solution of a matrix Riemann-Hilbert problem formulated in the complex k-plane and uniquely defined in terms of the spectral functions $\{a(k), b(k), A(k), B(k)\}$. Under the *assumption* that there exist functions $\{g_l(t)\}_0^{n-1}$ such that the spectral functions satisfy a certain *global algebraic relation*, prove that the function $q(x,t)$ is defined for all $0 < x < \infty$, $t > 0$, it satisfies the given nonlinear PDE, and furthermore that $q(x,0) = q_0(x)$, $\{\partial_x^l q(0,t) = g_l(t)\}_0^{n-1}$. (b) Given a subset of the functions $\{g_l(t)\}_0^{n-1}$ as boundary conditions, prove that the above algebraic relation characterizes the unknown part of this set. In general this involves the solution of a *nonlinear* Volterra integral equation which is shown to have a global solution. For a particular class of boundary conditions, called *linearizable*, this nonlinear equation can be bypassed and $\{A(k), B(k)\}$ can be constructed using only the algebraic manipulation of the global relation. For the NLS, the sG, and the KdV, the following particular linearizable cases are solved: $q_x(0,t) - \chi q(0,t) = 0$, $q(0,t) = \chi$, $\{q(0,t) = \chi, q_{xx}(0,t) = \chi + 3\chi^2\}$, respectively, where χ is a real constant.

1. Introduction

We first review briefly the inverse scattering method, then we summarize the new method.

A. The Inverse Scattering (Spectral) Method. There exist nonlinear evolution equations in one space variable, such as the nonlinear Schrödinger (NLS), the Korteweg-deVries (KdV) and the sine-Gordon (sG) equations, which can be written as the compatibility condition of two linear eigenvalue equations. Such equations are called *integrable* and the associated linear equations are called *Lax pairs* [1]. The two equations constituting the Lax pair are usually referred to as the x part and the t part. A method for solving the *initial-value problem with decaying initial data* was discovered in 1967 [2]. This method can be thought of as a *nonlinear Fourier transform method*. However, this nonlinear Fourier transform is *not* the same for every nonlinear evolution equation, but it is constructed from the x part of the Lax pair. Furthermore, neither the direct nonlinear Fourier transform of the initial data, nor the inverse nonlinear Fourier transform can be expressed in closed form: the former involves a linear Volterra integral equation and the latter involves a matrix Riemann-Hilbert problem. It should be emphasized that the construction of this nonlinear transform is based solely on the x part of the Lax pair and it involves the spectral analysis of this eigenvalue equation; the t part (or alternatively the nonlinear PDE itself) is used only to determine the evolution of the direct nonlinear Fourier transform (see [3] for the early history and [4] for some recent developments).

A method for solving the *initial value problem with space-periodic initial data* was developed in the mid-1970s [5–7]. This method involves algebraic-geometric techniques and can be thought of as formulating a Riemann-Hilbert problem which can be solved using functions defined on a Riemann surface.

Following the solution of the initial value problem with decaying and periodic initial data, the outstanding open problem in the analysis of integrable equations became the solution of initial-boundary value problems. The simplest such problem is formulated on the half line; following the success of the nonlinearization of the Fourier transform, a natural strategy is to solve the associated linear problem by an x transform and then to nonlinearize this transform. However, this strategy fails: for the NLS the associated linear equation can be solved by either the sine or the cosine transform depending on whether $q(0,t)$ or $q_x(0,t)$ is given, but neither of these transforms nonlinearizes; for the KdV, the associated linear equation involves a third order spatial derivative and for such equations there does *not* exist an appropriate x transform. The author has emphasized that the failure of the nonlinearization of the sine and cosine transforms suggests that these transforms are *not* fundamental; the fact that they are limited to equations with second order spatial derivatives provides further support to this claim. The author has introduced recently what appears to be the fundamental transform for a linear evolution equation with *arbitrary* order spatial derivatives and this transform *can* be nonlinearized [8].

B. The New Method. A general approach to solving boundary value problems for two-dimensional integrable PDE's was announced in [9] and further developed in several publications, see the review [8]. An equation in two-dimensions (x, y) is called integrable if and only if it can be written as the condition that an appropriate differential 1-form $W(x, y, k)$, $k \in \mathbb{C}$, is closed. Examples of integrable equations are linear PDEs with constant coefficients and the usual nonlinear integrable PDEs. For the latter class of PDEs, the existence of W is a direct consequence of the existence of a Lax Pair.

The rigorous implementation of the new method for the solution of an initial-boundary value problem on the half line for a linear dispersive evolution equation with spatial derivatives of arbitrary order is presented in [10]. The rigorous implementation of the new method for the analogous problem for the NLS equation is presented in [11]. Here the method is first presented in general and then applied to the following equations in particular: The NLS equation

$$i\frac{\partial q}{\partial t}+\frac{\partial^2 q}{\partial x^2}-2\lambda|q|^2q=0, \quad \lambda=\pm 1, \tag{1.1}$$

the KdV equation with dominant surface tension

$$\frac{\partial q}{\partial t}+\frac{\partial q}{\partial x}-\frac{\partial^3 q}{\partial x^3}+6q\frac{\partial q}{\partial x}=0, \quad q \text{ real}, \tag{1.2}$$

and the sine-Gordon in laboratory coordinates

$$\frac{\partial^2 q}{\partial t^2}-\frac{\partial^2 q}{\partial x^2}+\sin q=0, \quad q \text{ real}. \tag{1.3}$$

Regarding Eq. (1.2), we recall that in the original derivation of Korteweg and deVries [12] the coefficient of q_{xxx} is given by $h^2/3 - Th/\rho g$, where h, ρ, T, denote the mean height of the water, the density of the water, and the surface tension respectively; thus for sufficiently large surface tension this coefficient is negative.

We have included the results for the NLS for the sake of completeness.

Notation. Subscripts with respect to x and to t denote partial derivatives, for example $q_t=\frac{\partial q}{\partial t}$, $q_x=\frac{\partial q}{\partial x}$, etc.

- if $f(k)$ is a function then $\overline{f(k)}$ denotes the complex conjugate of $f(k)$.
- σ_3 denotes the third Pauli's matrix, $\hat\sigma_3$ denotes the matrix commutator with σ_3; then $\exp(\hat\sigma_3)$ can be easily computed,

$$\sigma_3=\operatorname{diag}(1,-1), \quad \hat\sigma_3 A=[\sigma_3, A], \quad e^{\hat\sigma_3}A=e^{\sigma_3}Ae^{-\sigma_3}, \tag{1.4}$$

  where A is a 2×2-matrix.
- The constant ρ used throughout the paper takes values $\rho=1$ for KdV, $\rho=-1$ for sG, $\rho=\lambda$ for NLS.
- $\mathcal{S}(\mathbb{R}^+)$ denotes the space of Schwartz functions on the positive real axis.
- $\bar D$ denotes the closure of the domain D.
- $a(k)$ and $b(k)$ are the (22) and (12) elements of the matrix $s(k)$ which is uniquely defined in terms of the initial conditions.
- $A(k)$ and $B(k)$ are the (22) and (12) elements of the matrix $S(k)$ which is uniquely defined in terms of the boundary values of $x=0$; these boundary values will be denoted by $\{\partial_x^l q(0,t)=g_l(t)\}_0^{n-1}$, where $n=2$ for NLS and sG, and $n=3$ for KdV.

An exact 1-form. The starting point of the method is the construction of an exact differential 1-form $W(x,t,k)$, $k\in\mathbb{C}$. In order to present this form explicitly we assume that the given nonlinear PDE admits a Lax pair formulation of the form

$$\begin{aligned}\mu_x+if_1(k)\hat\sigma_3\mu &= Q(x,t,k)\mu,\\ \mu_t+if_2(k)\hat\sigma_3\mu &= \tilde Q(x,t,k)\mu, \quad k\in\mathbb{C},\end{aligned} \tag{1.5}$$

where the eigenfunction $\mu(x, t, k)$ is a 2×2 matrix valued function of the arguments indicated, $f_1(k)$, $f_2(k)$ are given analytic functions of k, and the 2×2 matrix valued functions Q, $\tilde{Q}$ are given analytic functions of k, of $q(x, t)$, of $\bar{q}(x, t)$, and of the derivatives of these functions. Equations (1.1)–(1.3), the modified KdV equation, as well as several other nonlinear PDEs of physical significance admit a Lax pair of the form (1.5). For example for the NLS equation (1.1) [13],

$$f_1(k) = k, \quad f_2(k) = 2k^2, \quad Q(x, t) = \begin{pmatrix} 0 & q \\ \lambda\bar{q} & 0 \end{pmatrix}, \qquad (1.6)$$
$$\tilde{Q}(x, t, k) = 2kQ - iQ_x\sigma_3 - i\lambda|q|^2\sigma_3.$$

An exact 1-form W for an equation admitting the Lax pair (1.5) is given by

$$W(x, t, k) = e^{i(f_1(k)x + f_2(k)t)\hat{\sigma}_3}\left(Q(x, t, k)\mu(x, t, k)dx + \tilde{Q}(x, t, k)\mu(x, t, k)dt\right). \qquad (1.7)$$

Indeed, the Lax pair (1.5) is equivalent to the statement that $\exp[i(f_1(k)x + f_2(k)t)\hat{\sigma}_3]$ $\mu(x, t, k)$ is a differential 0-form associated with the exact differential 1-form W,

$$d\left[e^{i(f_1(k)x+f_2(k)t)\hat{\sigma}_3}\mu(x, t, k)\right] = W(x, t, k), \quad k \in \mathbb{C}. \qquad (1.8)$$

Equations (1.5) are the coordinate form of Eq. (1.8); however, for the implementation of the new method, it is often convenient to use Eq. (1.8).

Statement of the problem. Let $q(x, t)$ satisfy a nonlinear evolution equation with spatial derivatives of order n, on the half line $0 < x < \infty$, and for $0 < t < T$, where T is a positive constant. Let $q(x, t)$ satisfy decaying initial conditions at $t = 0$, as well as appropriate boundary conditions at $x = 0$. Assume that this PDE admits a Lax pair formulation of the type (1.5), i.e. assume that this PDE is equivalent to the compatibility condition of Eqs. (1.5). Then such an initial-boundary value problem can be analyzed using the following steps.

Step 1: A RH problem formulation under the assumption of existence. Assuming that $q(x, t)$ exists, express $q(x, t)$ through the solution of a matrix Riemann-Hilbert problem uniquely defined in terms of the so-called spectral functions denoted by $\{s(k), S(k)\}$. Express $s(k)$ and $S(k)$ through the solution of linear integral equations uniquely defined in terms of the initial conditions and of the boundary values $\{g_l(t)\}_0^{n-1}$, respectively. Furthermore, derive the global algebraic relation satisfied by the spectral functions.

More specifically: Let $\mu_j(x, t, k)$, $j = 1, 2, 3$, be the 2×2 matrix valued functions defined by

$$e^{i(f_1(k)x+f_2(k)t)\hat{\sigma}_3}\mu_j(x, t, k) = I + \int_{(x_j,t_j)}^{(x,t)} W(\xi, \tau, k), \quad 0 < x < \infty, \quad 0 < t < T, \qquad (1.9)$$

where $I = \text{diag}(1, 1)$, the integral denotes a smooth curve from (x_j, t_j) to (x, t), and $(x_1, t_1) = (0, T)$, $(x_2, t_2) = 0$, $(x_3, t_3) = (\infty, t)$, see Fig. 1.1.

The fundamental theorem of calculus implies that the functions μ_j satisfy Eq. (1.8) and that, since the 1-form W is closed, μ_j are independent of the path of integration. The functions μ_1 and μ_2 are entire functions of k, while the function μ_3 is defined only

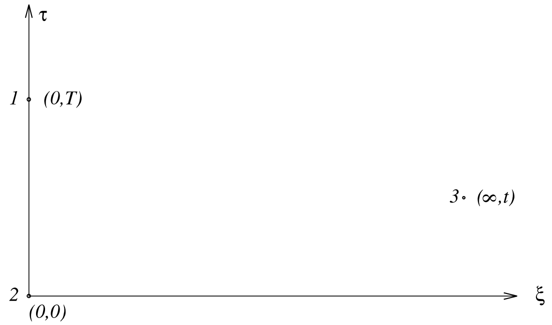

Fig. 1.1. The points 1, 2, 3 used for the definition of μ_j, $j = 1, 2, 3$

for k in some domain of the complex k-plane. The boundedness of μ_j with respect to k depends on $f_1(k)$, $f_2(k)$ and (x_j, t_j). It was shown in [14] that if (x_j, t_j) are the corners of the given polygonal domain (i.e. in our case the points denoted by 1, 2, 3), then the exponential terms appearing in Eq. (1.9) are bounded in different sectors of the complex k-plane whose union is the entire complex k-plane. Assuming that the dependence of Q and $\tilde{Q}$ on k is such that $\mu_j = I + O(1/k)$ as $k \to \infty$, it follows that the functions μ_j are the fundamental eigenfunctions needed for the formulation of a RH problem in the complex k-plane. The "jump matrix" of this RH problem is uniquely defined in terms of the 2×2-matrix valued functions

$$s(k) = \mu_3(0, 0, k), \quad S(k) = \left(e^{if_2(k)T\hat{\sigma}_3}\mu_2(0, T, k)\right)^{-1}. \tag{1.10}$$

This is a direct consequence of the fact that any two solutions of Eq. (1.9) are simply related,

$$\begin{aligned} \mu_3(x, t, k) &= \mu_2(x, t, k)e^{-i(f_1(k)x+f_2(k)t)\hat{\sigma}_3}\mu_3(0, 0, k), \\ \mu_1(x, t, k) &= \mu_2(x, t, k)e^{-i(f_1(k)x+f_2(k)t)\hat{\sigma}_3}\left(e^{if_2(k)T\hat{\sigma}_3}\mu_2(0, T, k)\right)^{-1}. \end{aligned} \tag{1.11}$$

The functions $s(k)$ and $S(k)$ follow from the evaluation at $x = 0$ and at $t = T$ of the functions $\mu_3(x, 0, k)$ and $\mu_2(0, t, k)$ respectively; these functions satisfy the following linear integral equations:

$$e^{if_1(k)x\hat{\sigma}_3}\mu_3(x, 0, k) = I - \int_x^\infty e^{if_1(k)\xi\hat{\sigma}_3}(Q\mu_3)(\xi, 0, k)d\xi, \tag{1.12}$$

$$e^{if_2(k)t\hat{\sigma}_3}\mu_2(0, t, k) = I + \int_0^t e^{if_2(k)\tau\hat{\sigma}_3}(\tilde{Q}\mu_2)(0, \tau, k)d\tau. \tag{1.13}$$

The functions $\mu_3(x, 0, k)$ and $\mu_2(0, t, k)$, and hence the functions $s(k)$ and $S(k)$, are uniquely defined in terms of $Q(x, 0, k)$ and $\tilde{Q}(0, t, k)$, i.e. in terms of the initial conditions and of the boundary values $\{g_l(t)\}_0^{n-1}$, respectively.

The above RH problem yields $\mu(x, t, k)$ in terms of $s(k)$ and $S(k)$, then either of Eqs. (1.5) yields $q(x, t)$. The function $S(k)$ involves explicitly T. However, the solution of an evolution equation cannot depend on a "future time"; indeed, it can be shown that for t such that $0 < t < T_0 < T$ the above RH problem is equivalent to a RH problem with $S(k)$ replaced by

$$S(T_0, k) = \left(e^{if_2(k)T_0\hat{\sigma}_3}\mu_2(0, T_0, k)\right)^{-1}. \quad (1.14)$$

The spectral functions $s(k)$ and $S(k)$ are *not* independent but they satisfy a simple algebraic relation, see Sect. 2:

$$-I + S(k)^{-1}s(k) + e^{if_2(k)T\hat{\sigma}_3}C(k) = 0, \quad k \in (\bar{D}_3 \cup \bar{D}_4, \quad \bar{D}_1 \cup \bar{D}_2), \quad (1.15)$$

where

$$C(k) = \int_0^\infty e^{if_1(k)x\hat{\sigma}_3}(Q\mu_3)(x, T, k)dx,$$

and $k \in (D_1, D_2)$ means that the first and second columns of the matrix equation (1.15) are valid for $k \in D_1$ and $k \in D_2$ respectively. The domains D_j, $j = 1, 2, 3, 4$, are defined by

$$\begin{aligned} D_1 &= \{k \in \mathbb{C}, Im f_1(k) > 0 \cap Im f_2(k) > 0\}, \\ D_2 &= \{k \in \mathbb{C}, Im f_1(k) > 0 \cap Im f_2(k) < 0\}, \\ D_3 &= \{k \in \mathbb{C}, Im f_1(k) < 0 \cap Im f_2(k) > 0\}, \\ D_4 &= \{k \in \mathbb{C}, Im f_1(k) < 0 \cap Im f_2(k) < 0\}. \end{aligned} \quad (1.16)$$

We emphasize that although Eq. (1.15) involves the *unknown* function $C(k)$ it *does* impose a severe restriction between $s(k)$ and $S(k)$. This becomes immediately clear if the domain is the quarter plane, $0 < x < \infty$, $0 < t < \infty$, in which case $S(k) = S(\infty, k)$ and Eq. (1.15) becomes

$$-I + S(k)^{-1}s(k) = 0, \quad k \in (\bar{D}_3, \bar{D}_1). \quad (1.15)_\infty$$

The global relation is a simple consequence of the fundamental fact that since the 1-form W is closed, its integral around the boundary $\{0 < \xi < \infty, 0 < \tau < T\}$ vanishes.

Step 2: Existence under the assumption that the spectral functions satisfy the global relation. Given $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$ use Eq. (1.12) to define $s(k)$. Assume that there exist smooth functions $\{g_l(t)\}_0^{n-1}$, such that if $S(k)$ is defined in terms of these functions by Eq. (1.13), then the spectral functions satisfy the global relation (1.15), where $C(k)$ is some function analytic and bounded for $k \in (D_3 \cup D_4, D_1 \cup D_2)$. Define $\mu(x, t, k)$ as the solution of the RH problem formulated in Step 1 and define $q(x, t)$ in terms of $\mu(x, t, k)$. Then prove that: (a) $q(x, t)$ is defined for all $0 < x < \infty, 0 < t < T$. (b) $q(x, t)$ solves the given nonlinear PDE. (c) $q(x, 0) = q_0(x)$, $0 < x < \infty$. (d) $\{\partial_x^l q(0, t) = g_l(t)\}_0^{n-1}$, $0 < t < T$.

More specifically: The global existence of $q(x, t)$ is based on the unique solvability of the associated RH problem, which in turn is based on the distinctive nature of the functions defining the jump matrix: these functions have explicit x, t dependence, in the form of $\exp[i(f_1(k)x + f_2(k)t)]$, and involve the spectral functions $s(k)$ and $S(k)$; using certain symmetry properties of the spectral functions it can be shown that in all cases (1.1)–(1.3), the associated homogeneous RH problem has only the trivial solution (i.e. there exists a so-called vanishing lemma).

The proof that $q(x, t)$ solves the given nonlinear PDE uses the standard arguments of the dressing method. The proof that $q(0, t) = q_0(x)$ is based on the fact that the RH problem satisfied by $\mu(x, 0, k)$ is equivalent to the RH problem defined by $s(k)$ which characterizes $q_0(x)$.

The proof that $\{\partial_x^l q(0, t) = g_l(t)\}_0^{n-1}$, makes crucial use of the global algebraic relation (1.15). Indeed, the RH problem satisfied by $\mu(0, t, k)$ is equivalent to the RH problem defined by $S(k)$, which characterizes $g_l(t)$, if and only if the spectral functions satisfy this global relation.

From the above discussion the crucial role played by the global relation becomes clear: it is not only a necessary but it is also a sufficient condition. Thus given $q_0(x)$ and a subset of the functions $\{g_l(t)\}_0^{n-1}$, the main problem becomes to show that the global relation characterizes the unknown part of the set $\{g_l(t)\}_0^{n-1}$.

Step 3: Analyse the global condition. (a) Identify a class of boundary conditions for which it is possible to compute explicitly $S(k)$, using only algebraic manipulations of the global relation.

More specifically: The function $S(k)$ is defined in terms of $\mu_2(0, t, k)$ which satisfies

$$\partial_t \mu_2(0, t, k) + i f_2(k)\hat{\sigma}_3 \mu_2(0, t, k) = \tilde{Q}(0, t, k)\mu_2(0, t, k). \tag{1.17}$$

Let the transformation $k \to \nu(k)$ be defined by the requirement that it leaves $f_2(k)$ invariant. The function $\mu_2(0, t, \nu(k))$ satisfies Eq. (1.17) with $\tilde{Q}(0, t, k)$ replaced by $\tilde{Q}(0, t, \nu(k))$. Suppose that there exists a nonsingular matrix $N(k)$ such that

$$N(k)^{-1}\left[i f_2(k)\sigma_3 - \tilde{Q}(0, t, \nu(k))\right] N(k) = i f_2(k)\sigma_3 - \tilde{Q}(0, t, k); \tag{1.18}$$

then if $\mu_2(0, t, k) = M(t, k)\exp[i f_2(k) t \sigma_3]$, it follows that

$$M(t, \nu(k)) = N(k) M(t, k) N(k)^{-1}. \tag{1.19}$$

This equation yields a relation between $S(k)$ and $S(\nu(k))$, and then $S(k)$ follows from the algebraic manipulation of this relation and of the global relation.

Equation (1.18) implies that a necessary condition for the existence of $N(k)$ is that the determinant of $i f_2(k)\sigma_3 - \tilde{Q}(0, t, k)$ depends on k only in the form of $f_2(k)$.

For the non-linearizable boundary conditions:

(b) Given $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$ and a subset of smooth functions $\{g_l(t)\}_0^{n-1}$, prove that the global relation yields the unknown part of the set $\{g_l(t)\}_0^{n-1}$ for all $0 < t < T$.

More specifically: Integrating the first and the second columns of the global relation around the boundary of the domains of D_3 and of D_1 respectively, and using the analyticity of the term $\exp[i f_2(k) T \hat{\sigma}_3] C(k)$, it follows that this term vanishes. Since $S(k)$ is defined in terms of $\mu_2(0, T, k)$, this equation is a relation between $s(k)$ and $\mu_2(0, T, k)$. This relation together with the definition of $\mu_2(0, t, k)$, i.e. Eq. (1.13), define a nonlinear Volterra integral equation for the unknown part of $\{g_l(t)\}_0^{n-1}$. For such an equation it is tedious but straightforward to establish solvability for small t (or equivalently for data with small norm); the challenging question is to establish solvability for all $0 < t < T$ (or equivalently *without* a small norm assumption). We emphasize that to achieve this we make crucial use of the *analyticity structure* of the global relation, see Sect. 5.

C. Outline and Main Results. In Sects. 2–4 we implement Steps 1–3a for each of Eqs. (1.1)–(1.3). In particular in Theorem 3.1 we formulate the basic RH problem; this problem has a jump matrix which is uniquely defined in terms of the scalar functions $a(k)$, $b(k)$ and $\Gamma(k)$, where $\Gamma(k)$ involves $a(k)$, $b(k)$ and $B(k)/A(k)$,

$$\overline{\Gamma(\bar{k})} = \frac{\rho \frac{B(k)}{A(k)}}{\overline{a(\bar{k})}\left[\overline{a(\bar{k})} - \rho \overline{b(\bar{k})}\frac{B(k)}{A(k)}\right]}. \tag{1.20}$$

In Sect. 4 we analyse the following concrete linearizable cases:
NLS:

$$q_x(0,t) - \chi q(0,t) = 0, \quad \chi \text{ constant}, \quad \chi \geq 0. \tag{1.21}$$

sG:

$$q(0,t) = \chi, \quad \chi \text{ constant}. \tag{1.22}$$

KdV:

$$q(0,t) = \chi, \quad q_{xx}(0,t) = \chi + 3\chi^2, \quad \chi \text{ constant}. \tag{1.23}$$

For each of these cases, B/A, and hence $\Gamma(k)$, can be explicitly given in terms of $a(k)$, $b(k)$:

$$\text{NLS}: \quad \frac{B(k)}{A(k)} = -\frac{2k + i\chi}{2k - i\chi}\frac{b(-k)}{a(-k)}, \tag{1.24}$$

$$\text{KdV, sG}: \quad \frac{B(k)}{A(k)} = \frac{f(k)b(\nu(k)) - a(\nu(k))}{f(k)a(\nu(k)) - b(\nu(k))}, \tag{1.25}$$

where for the sG,

$$\nu(k) = \frac{1}{k}, \quad f(k) = i\frac{k^2+1}{k^2-1}\frac{\sin\chi}{\cos\chi - 1}, \tag{1.26}$$

while for the KdV,

$$\nu^2 + k\nu + k^2 + \frac{1}{4} = 0, \quad f(k) = \frac{\nu + k}{\nu - k}\left(1 - \frac{4\nu k}{\chi}\right). \tag{1.27}$$

The homogeneous Neumann and the homogeneous Dirichlet cases of the NLS, i.e. $q_x(0,t) = 0$ and $q(0,t) = 0$, follow from Eq. (1.24) as $\chi \to 0$ and $\chi \to \infty$ respectively. Similarly the homogeneous case $\chi = 0$ of the sG and KdV follow from Eq. (1.26) as $\chi \to 0$, i.e. $f(k) \to \infty$.

We emphasize that since $\{a(k), b(k)\}$ are determined in terms of the initial conditions and since $B(k)/A(k)$ and therefore $\Gamma(k)$ is explicitly written in terms of $\{a(k), b(k)\}$, it follows that linearizable initial boundary value problems on the half line are solved as effectively as initial value problems on the line.

In Sect. 5 we summarize the results of [11] which, in the case of the NLS, given $g_0(t)$ establish the existence and uniqueness of $g_1(t)$. We also discuss the extension of these results to other integrable nonlinear PDEs such as the KdV and sG.

In Sect. 6 we discuss further the new method.

Remark 1.1. For simplicity of presentation we give all relevant formulae for the solitonless case. The solitons were included in [11]. Using the formula of [11] it is straightforward to add the solitonic part to the formulae for the KdV and the sG: the zeros for $0 < \arg k < \pi/2$ and $\pi/2 < \arg k < \pi$ in the NLS can occur in the domains D_1 and D_2 for the sG and the KdV. The solitons for the sG are also discussed in [16]. We note that the existence of solitons in the linearizable cases studied in Sect. 5 depends on the sign of χ; it is again straightforward to add the solitonic part.

2. A RH Problem Formulation Under the Assumption of Existence

In this section we give the details of Step 1. Let σ_j denote the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1}$$

The NLS, the KdV, and the sG equations are equivalent to Eq. (1.8), where f_1, f_2, Q, $\tilde{Q}$, are defined below.

NLS: see Eqs. (1.6).

KdV: $f_1 = -k$, $f_2 = k + 4k^3$,

$$\begin{aligned} Q(x,t,k) &= \frac{q}{2k}(\sigma_2 - i\sigma_3), \\ \tilde{Q}(x,t,k) &= -2kq\sigma_2 + q_x\sigma_1 + \frac{2q^2 + q - q_{xx}}{2k}(i\sigma_3 - \sigma_2). \end{aligned} \tag{2.2}$$

sG: $f_1 = \frac{1}{4}\left(k - \frac{1}{k}\right)$, $f_2 = \frac{1}{4}(k + \frac{1}{k})$,

$$\begin{aligned} Q(x,t,k) &= -\frac{i}{4}(q_x + q_t)\sigma_1 - \frac{i}{4k}(\sin q)\sigma_2 + \frac{i}{4k}\left((\cos q) - 1\right)\sigma_3, \\ \tilde{Q}(x,t,k) &= Q(x,t,-k). \end{aligned} \tag{2.3}$$

2.1. Analytic and Bounded Eigenfunctions. For the contours appearing in the integral of Eq. (1.9) we choose the specific contours depicted in Fig. 2.1

This choice implies the following inequalities:

$$\mu_1 : \xi - x \le 0,\ \tau - t \ge 0,$$

$$\mu_2 : \xi - x \le 0,\ \tau - t \le 0,$$

$$\mu_3 : \xi - x \ge 0.$$

The second column of the matrix equation (1.9) involves $\exp[if_1(k)(\xi - x) + if_2(k)(\tau - t)]$. Using the above inequalities it follows that this exponential is bounded in the following regions of the complex k-plane:

$$\mu_1 : \{Im f_1 \le 0 \cap Im f_2 \ge 0\},$$

$$\mu_2 : \{Im f_1 \le 0 \cap Im f_2 \le 0\},$$

$$\mu_3 : \{Im f_1 \ge 0\}.$$

The first column of the matrix equation (1.9) involves the inverse of the above exponential, which is bounded in

$$\mu_1 : \{Im f_1 \ge 0 \cap Im f_2 \le 0\},$$

$$\mu_2 : \{Im f_1 \ge 0 \cap Im f_2 \ge 0\},$$

$$\mu_3 : \{Im f_1 \le 0\}.$$

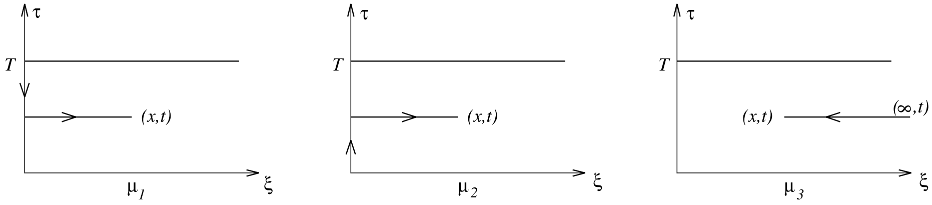

Fig. 2.1. The contours used for the definition of μ_j, $j = 1, 2, 3$

Using the definitions (1.16) of the domains $\{D_j\}_1^4$, the above discussion can be summarized schematically by:

$$\mu_1 : (\bar{D}_2, \bar{D}_3), \quad \mu_2 : (\bar{D}_1, \bar{D}_4), \quad \mu_3 : (\bar{D}_3 \cup \bar{D}_4, \bar{D}_1 \cup \bar{D}_2). \tag{2.4}$$

The definitions of the domains D_j, $j = 1, 2, 3, 4$, imply that for Eqs. (1.1)–(1.3) these domains are given below and depicted in Fig. 2.2.

NLS:

$$\begin{aligned} D_1 &= \{0 < \arg k < \tfrac{\pi}{2}\}, \quad D_2 = \{\tfrac{\pi}{2} < \arg k < \pi\}, \\ D_3 &= \{\pi < \arg k < \tfrac{3\pi}{2}\}, \; D_4 = \{\tfrac{3\pi}{2} < \arg k < 2\pi\}. \end{aligned} \tag{2.5}$$

KdV: Let the curves $l_\pm$ be defined by

$$\begin{aligned} l_+ &= \left\{k = k_R + ik_I, \; k_I > 0, \; \tfrac{1}{4} + 3k_R^2 - k_I^2 = 0\right\}, \\ l_- &= \left\{k = k_R + ik_I, \; k_I < 0, \; \tfrac{1}{4} + 3k_R^2 - k_I^2 = 0\right\}. \end{aligned}$$

Then

$$\begin{aligned} D_1 &= \{Imk < Imk_-\}, \quad D_2 = \{Imk_- < Imk < 0\}, \quad k_- \in l_-, \\ D_3 &= \{0 < Imk < Imk_+\}, \quad D_4 = \{Imk > Imk_+\}, \quad k_+ \in l_+. \end{aligned} \tag{2.6}$$

sG:

$$\begin{aligned} D_1 &= \{Imk > 0 \cap |k| > 1\}, \quad D_2 = \{Imk > 0 \cap |k| < 1\}, \\ D_3 &= \{Imk < 0 \cap |k| < 1\}, \quad D_4 = \{Imk < 0 \cap |k| > 1\}. \end{aligned} \tag{2.7}$$

For the NLS, $Q(x, t)$ is independent of k and $\tilde{Q}(x, t, k)$ depends linearly on k, thus the region in the complex k-plane where μ_j is bounded and analytic is determined completely by the associated exponential. Hence

$$\mu_1 = \left(\mu_1^{(2)}, \mu_1^{(3)}\right), \mu_2 = \left(\mu_2^{(1)}, \mu_2^{(4)}\right), \mu_3 = \left(\mu_3^{(34)}, \mu_3^{(12)}\right), \tag{2.8}$$

where the first equation means that the first and the second column vectors of the matrix μ are bounded and analytic in D_2 and in D_3 respectively, etc. We also note that μ_1 and μ_2 are entire functions of k.

For the KdV and the sG equations similar considerations are valid in the punctured complex k-plane, $k \in \mathbb{C} - \{0\}$. The behavior of μ_j as $k \to 0$ can be easily characterized, see Appendix A.

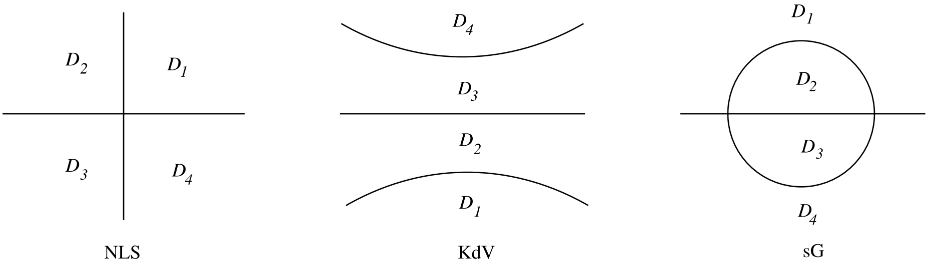

Fig. 2.2. The domains D_j, $j = 1, \ldots, 4$, for the NLS, KdV and sG equations

2.2. Other Properties of the Eigenfunctions. For the NLS, as well as for the KdV and the sG with $q(x,t)$ real, the matrices Q and $\tilde{Q}$ have certain symmetry properties. These symmetries imply the following symmetries for μ:

$$(\mu(x,t,k))_{11} = \overline{(\mu(x,t,\bar{k}))_{22}}, \quad (\mu(x,t,k))_{12} = \rho\overline{(\mu(x,t,\bar{k}))_{21}}, \tag{2.9}$$

where $\rho = \lambda$ for the NLS, $\rho = 1$ for the KdV, and $\rho = -1$ for the sG.

In addition, in the case of the KdV and of the sG equations the following symmetries are valid:

$$(\mu(x,t,k))_{11} = (\mu(x,t,-k))_{22}, \quad (\mu(x,t,k))_{12} = (\mu(x,t,-k))_{21}.$$

Integration by parts implies that in the domains where μ is bounded and analytic

$$\mu(x,t,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{2.10}$$

The fact that Q and $\tilde{Q}$ are traceless together with Eq. (2.10) imply

$$\det \mu(x,t,k) = 1. \tag{2.11}$$

2.3. The Spectral Functions. The spectral functions $s(k)$ and $S(k)$ are defined in terms of $\mu_3(x,0,k)$ and $\mu_2(0,t,k)$. The latter function and the function $\mu_1(0,t,k)$ have larger domains of analyticity

$$\begin{aligned} \mu_1(0,t,k) &= \left(\mu_1^{(24)}(0,t,k), \mu_1^{(13)}(0,t,k)\right), \\ \mu_2(0,t,k) &= \left(\mu_2^{(13)}(0,t,k), \mu_2^{(24)}(0,t,k)\right). \end{aligned} \tag{2.12}$$

The symmetry properties (2.9) imply similar symmetry properties for the spectral functions, for example if $(s(k))_{22}$ is denoted by $a(k)$, then $(s(k))_{11} = \overline{a(\bar{k})}$, etc. We will use the following notations for the spectral functions:

$$s(k) = \begin{pmatrix} \overline{a(\bar{k})} & b(k) \\ \rho\overline{b(\bar{k})} & a(k) \end{pmatrix}, \quad S(k) = \begin{pmatrix} \overline{A(\bar{k})} & B(k) \\ \rho\overline{B(\bar{k})} & A(k) \end{pmatrix}. \tag{2.13}$$

These notations and the definitions of $s(k)$ and $S(k)$, i.e. Eqs. (1.10), imply

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \mu_3^{(12)}(0,0,k), \quad \begin{pmatrix} -e^{-2if_2(k)T}B(k) \\ \overline{A(\bar{k})} \end{pmatrix} = \mu_2^{(24)}(0,T,k), \tag{2.14}$$

where the vectors $\mu_3^{(12)}(x,0,k)$ and $\mu_2^{(24)}(0,t,k)$ satisfy the following ODE's:

$$\partial_x \mu_3^{(12)}(x,0,k) + 2if_1(k)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mu_3^{(12)}(x,0,k) = Q(x,0,k)\mu_3(x,0,k),$$
$$k \in \bar{D}_1 \cup \bar{D}_2,\ 0 < x < \infty,$$

$$\lim_{x\to\infty} \mu_3^{(12)}(x,0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.15}$$

and

$$\partial_t \mu_2^{(24)}(0,t,k) + 2if_2(k)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mu_2^{(24)}(0,t,k) = \tilde{Q}(0,t,k)\mu_2(0,t,k),$$
$$k \in \bar{D}_2 \cup \bar{D}_4,\ 0 < t < T,$$

$$\mu_2^{(24)}(0,0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.16}$$

The above definitions imply the following properties:

$\mathbf{a(k), b(k)}$

- $a(k), b(k)$ are defined and are analytic for $k \in D_1 \cup D_2$.
- $|a(k)|^2 - \rho|b(k)|^2 = 1,\ k \in \mathbb{R}$.
- $a(k) = 1 + O(\frac{1}{k}),\ b(k) = O(\frac{1}{k}),\ k \to \infty$. (2.17)

$\mathbf{A(k), B(k)}$

- $A(k), B(k)$ are entire functions which are bounded for $k \in D_1 \cup D_3$; if $T = \infty$ these functions are defined and are analytic for k in this domain.
- $A(k)\overline{A(\bar{k})} - \rho B(k)\overline{B(\bar{k})} = 1,\ k \in \mathbb{C}$.
- $A(k) = 1 + O\left(\frac{1+e^{2if_2(k)T}}{k}\right),\ B(k) = O\left(\frac{1+e^{2if_2(k)T}}{k}\right),\ k \to \infty$. (2.18)

For the KdV equation the above are valid in the punctured complex k-plane, $k \in \mathbb{C} \setminus \{0\}$.

All of the above properties, except for the property that $B(k)$ is bounded for $k \in D_1 \cup D_3$, follow from the analyticity properties of $\mu_3(x,0,k)$, $\mu_2(0,t,k)$ (see Eqs. (2.8), (2.12)), from the conditions of unit determinant, and from the large k asymptotics of these eigenfunctions. Regarding $B(k)$ we note that $B(k) = B(T,k)$, where $B(t,k) = -\exp[2if_2(k)t](\mu_2^{24}(0,t,k))_1$. Equations (2.16) imply a linear Volterra integral equation for the vector $\exp(2if_2(k)t)\mu_2^{(24)}(0,t,k)$, from which it immediately follows that $B(t,k)$ is an entire function of k bounded for $k \in D_1 \cup D_3$.

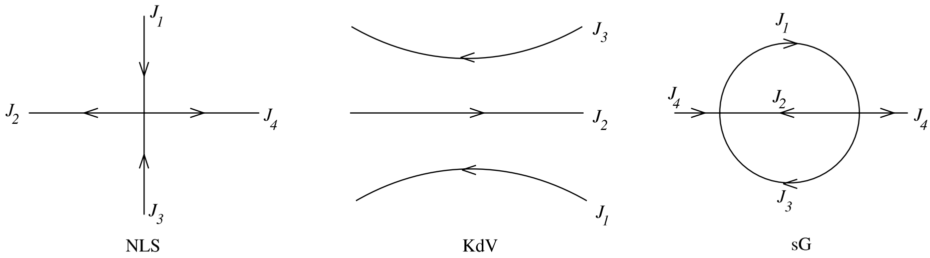

Fig. 2.3. The oriented contours $\mathcal{L}$ and the jump matrices J for the NLS, KdV and sG equations

2.4. The RH Problem. Equations (1.11) can be rewritten in the form

$$M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \quad k \in \mathcal{L}, \tag{2.19}$$

where the matrices M_-, M_+ J, and the oriented contour $\mathcal{L}$ are defined below

$$M_+ = \left(\frac{\mu_2^{(1)}}{a(k)},\ \mu_3^{(12)}\right), \quad k \in D_1; \quad M_- = \left(\frac{\mu_1^{(2)}}{d(k)},\ \mu_3^{(12)}\right), \quad k \in D_2;$$

$$M_+ = \left(\mu_3^{(34)}, \frac{\mu_1^{(3)}}{\overline{d(\bar{k})}}\right), \quad k \in D_3; \quad M_- = \left(\mu_3^{(34)}, \frac{\mu_2^{(4)}}{\overline{a(\bar{k})}}\right), \quad k \in D_4, \tag{2.20}$$

$$d(k) = a(k)\overline{A(\bar{k})} - \rho b(k)\overline{B(\bar{k})}, \tag{2.21}$$

$$J(x,t,k) = \begin{cases} J_1,\ k \in D_1 \cap D_2 \doteqdot \mathcal{L}_1 \\ J_2,\ k \in D_2 \cap D_3 \doteqdot \mathcal{L}_2 \\ J_3,\ k \in D_3 \cap D_4 \doteqdot \mathcal{L}_3 \\ J_4,\ k \in D_4 \cap D_1 \doteqdot \mathcal{L}_4, \quad J_2 = J_3 J_4^{-1} J_1, \end{cases} \tag{2.22}$$

$$J_1 = \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2i\theta} & 1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & -\rho\overline{\Gamma(\bar{k})}e^{-2i\theta} \\ 0 & 1 \end{pmatrix},$$
$$J_4 = \begin{pmatrix} 1 & -\gamma(k)e^{-2i\theta} \\ \rho\bar{\gamma}(k)e^{2i\theta} & 1-\rho|\gamma(k)|^2 \end{pmatrix}, \tag{2.23}$$

where

$$\gamma(k) = \frac{b(k)}{\overline{a(k)}},\ k \in \mathbb{R}; \quad \Gamma(k) = \frac{\rho\overline{B(\bar{k})}}{a(k)d(k)},\ k \in D_2; \quad \theta(x,t,k) = f_1(k)x + f_2(k)t. \tag{2.24}$$

In order to derive Eq. (2.19) we write Eqs. (1.11) in the form

$$\left(\mu_3^{(34)}, \mu_3^{(12)}\right) = \left(\mu_2^{(1)}, \mu_2^{(4)}\right)\begin{pmatrix} \bar{a} & be^{-2i\theta} \\ \rho\bar{b}e^{2i\theta} & a \end{pmatrix}, \tag{2.25}$$

$$\left(\mu_1^{(2)}, \mu_1^{(3)}\right) = \left(\mu_2^{(1)}, \mu_2^{(4)}\right) \begin{pmatrix} \bar{A} & Be^{-2i\theta} \\ \rho \bar{B} e^{2i\theta} & A \end{pmatrix}. \tag{2.26}$$

In order to compute J_4 we must relate those eigenfunctions which are bounded in D_1 and in D_4; thus rearranging Eq. (2.25) and using (2.17b) we find Eq. (2.19) with $J = J_4$ and M_-, M_+ given by Eqs. (2.20d), (2.20a) respectively. Similarly, in order to compute J_1 we must relate those eigenfunctions which are bounded in D_1 and in D_2; thus eliminating $\mu_2^{(4)}$ from the second column of Eq. (2.25) and from the first column of Eq. (2.26) we find (2.19) with $J = J_1$ and M_-, M_+ given by (2.20b), (2.20a) respectively. The computation of J_3 follows from the elimination of $\mu_2^{(1)}$ from the first column of Eq. (2.25) and from the second column of Eq. (2.26).

The jump condition (2.19), together with the analyticity properties and the large k behavior of μ_j, define a 2×2 matrix RH problem for the determination of the matrix $M(x, t, k)$. This is in general a meromorphic function of k in $\mathbb{C} \setminus \mathcal{L}$. The possible poles of M are generated by the zeros of $a(k)$, $k \in D_1$, of $d(k)$, $k \in D_2$, and from the complex conjugates of these zeros. For compactness of presentation we assume that no such zeros occur, see Remark 1.1.

2.5. The Global Relation. For t such that $0 < t < T_0 < T$, the following equation is valid:

$$-I + S(T_0, k)^{-1} s(k) + e^{if_2(k)T_0\hat{\sigma}_3} \int_0^\infty e^{if_1(k)\xi\hat{\sigma}_3} (Q\mu_3)(\xi, T_0, k) d\xi = 0,$$
$$k \in (\bar{D}_3 \cup \bar{D}_4, \bar{D}_1 \cup \bar{D}_2), \tag{2.27}$$

where $S(T_0, k)$ is defined by Eq. (1.14). In order to derive Eq. (2.27) we integrate the closed 1-form $W(\xi, \tau, k)$ defined by Eq. (1.7) with $\mu = \mu_3$ around the boundary of the domain $\{0 < \xi < \infty, 0 < \tau < T_0\}$:

$$\int_\infty^0 e^{if_1\xi\hat{\sigma}_3} (Q\mu_3)(\xi, 0, k) d\xi + \int_0^{T_0} e^{if_2\tau\hat{\sigma}_3} (\tilde{Q}\mu_3)(0, \tau, k) d\tau$$
$$+ e^{if_2 T_0\hat{\sigma}_3} \int_0^\infty e^{if_1\xi\hat{\sigma}_3} (Q\mu_3)(\xi, T_0, k) d\xi + \lim_{X\to\infty} e^{if_1 X\hat{\sigma}_3} \int_{T_0}^0 e^{if_2\tau\hat{\sigma}_3} (\tilde{Q}\mu_3)(X, \tau, k) d\tau = 0. \tag{2.28}$$

The definition of $s(k)$, i.e. Eq. (1.10a), implies that the first term of this equation equals $s(k) - I$. Equation (1.11a) evaluated at $x = 0$ yields

$$\mu_3(0, \tau, k) = \mu_2(0, \tau, k) e^{-if_2\tau\hat{\sigma}_3} s(k),$$

thus

$$e^{if_2\tau\hat{\sigma}_3} (\tilde{Q}\mu_3)(0, \tau, k) = \left(e^{if_2\tau\hat{\sigma}_3} (\tilde{Q}\mu_2)(0, \tau, k)\right) s(k);$$

this equation together with the definition of $\mu_2(0, t, k)$, i.e. Eq. (1.13), imply that the second term of Eq. (2.28) equals

$$\left(e^{if_2 T_0\hat{\sigma}_3} \mu_2(0, T_0, k) - I\right) s(k).$$

Hence, assuming that q has sufficient decay as $x \to \infty$, Eq. (2.28) becomes Eq. (2.27).

At $T = \infty$, $S(k)$ is defined only for $k \in (\bar{D}_3, \bar{D}_1)$, thus Eq. (2.27) with T_0 replaced by ∞ becomes Eq. $(1.15)_\infty$.

The function $S(T_0, k)$ has similar properties with those of $S(k)$. In particular if $S(T_0, k)$ is denoted by

$$S(T_0, k) = \begin{pmatrix} \overline{A(T_0, \bar{k})} & B(T_0, k) \\ \rho \overline{B(T_0, \bar{k})} & A(T_0, k) \end{pmatrix},$$

it follows that $A(T_0, k)$, $B(T_0, k)$ have similar properties with those of $A(k)$, $B(k)$.

The (12) element of Eq. (2.27) is

$$a(k)B(T_0, k) - b(k)A(T_0, k) = e^{2if_2(k)T_0} c(T_0, k), \quad k \in \bar{D}_1 \cup \bar{D}_2, \tag{2.29}$$

where the scalar function $c(T_0, k) = \int_0^\infty e^{2if_1(k)\xi} (Q\mu_3)_{12}(\xi, T_0, k) d\xi$, is defined and is analytic in k for $k \in D_1 \cup D_2$ and it is of $O(1/k)$ as $k \to \infty$. Evaluating Eq. (2.29) at $T_0 = T$ we find

$$a(k)B(k) - b(k)A(k) = e^{2if_2(k)T} c(k), \quad k \in \bar{D}_1, \tag{2.30}$$

where $c(k) = c(T, k)$ is an analytic function and for $k \in D_1 \cup D_2$, is of $O(1/k)$ as $k \to \infty$.

3. Existence Under the Assumption that the Spectral Functions Satisfy the Global Relation

In this section we implement Step 2; to this end, we first define the spectral functions.

Definition 3.1. a(k), b(k). *For the NLS and the KdV equations, let $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$; for the sG equation, let $q_0(x) - 2\pi m \in \mathcal{S}(\mathbb{R}^+)$ and $q_1(x) \in \mathcal{S}(\mathbb{R}^+)$, where m is an integer. Let the domains D_j, $j = 1, \cdots, 4$, be defined in equations (2.5)–(2.7). The map*

$$\mathbf{S}: \begin{matrix} \{q_0(x)\} \\ \text{or} \\ \{q_0(x), q_1(x)\} \end{matrix} \Longrightarrow \{a(k), b(k)\}, \tag{3.1}$$

is defined as follows:

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \varphi(0, k), \tag{3.2}$$

where the vector valued function $\varphi(x, k)$ is defined in terms of $q_0(x)$ or $\{q_0(x), q_1(x)\}$ by

$$\partial_x \varphi(x, k) + 2if_1(k) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi(x, k) = Q(x, k)\varphi(x, k), 0 < x < \infty, k \in \bar{D}_1 \cup \bar{D}_2,$$

$$\lim_{x \to \infty} \varphi(x, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.3}$$

and $Q(x, k)$ is given for the NLS, KdV, sG respectively by:

$$Q(x) = \begin{pmatrix} 0 & q_0(x) \\ \lambda \bar{q}_0(x) & 0 \end{pmatrix},$$

$$Q(x,k) = \frac{q_0(x)}{2k}(\sigma_2 - i\sigma_1),$$
$$Q(x,k) = -\frac{i}{4}\left(\frac{dq_0(x)}{dx} + q_1(x)\right)\sigma_1 - \frac{i}{4k}(\sin q_0(x))\sigma_2 + \frac{i}{4k}(\cos q_0(x) - 1)\sigma_3. \tag{3.4}$$

Properties of $a(k)$, $b(k)$.

1. $a(k), b(k)$ are analytic and bounded for $k \in D_1 \cup D_2$.
2. $|a(k)|^2 - \rho|b(k)|^2 = 1, k \in \mathbb{R}$.
3. $a(k) = 1 + O(\frac{1}{k}), b(k) = O(\frac{1}{k}), k \to \infty$.
4. The inverse of the map (3.1) denoted by $\mathbb{Q}$ can be defined for (3.4a), (3.4b), (3.4c) respectively as follows:

$$\begin{aligned} q_0(x) &= 2i \lim_{k\to\infty} (kM^{(x)}(x,k))_{12}; \\ q_0(x) &= -2i \lim_{k\to\infty} \partial_x (kM^{(x)}(x,k))_{22}; \\ \cos q_0(x) &= 1 + 2 \lim_{k\to\infty} \left\{(kM^{(x)}(x,k))_{12}^2 + 2i\partial_x (kM^{(x)}(x,k))_{22}\right\}, \\ q_1(x) &= -\frac{d}{dx}q_0(x) - 2 \lim_{k\to\infty} (kM^{(x)}(x,k))_{12}; \end{aligned} \tag{3.5}$$

where $M^{(x)}(x,k)$ is the unique solution of the following RH problem:

- $$M^{(x)}(x,k) = \begin{cases} M_+^{(x)}(x,k), \ k \in D_1 \cup D_2 \\ M_-^{(x)}(x,k), \ k \in D_3 \cup D_4, \end{cases}$$

  is a meromorphic function of k for $k \in \mathbb{C} \setminus \mathbb{R}$.
- $$M^{(x)}(x,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty.$$
- $$M_-^{(x)}(x,k) = M_+^{(x)}(x,k) J^{(x)}(x,k), \quad k \in \mathbb{R},$$

  where

  $$J^{(x)}(x,k) = \begin{pmatrix} 1 & -\frac{b(k)}{\bar{a}(k)} e^{-2if_1(k)x} \\ \rho \frac{\bar{b}(k)}{a(k)} e^{2if_1(k)x} & \frac{1}{|a|^2} \end{pmatrix}, \quad k \in \mathbb{R}. \tag{3.6}$$

- For the KdV the jump condition is on $\mathbb{R} \setminus \{0\}$. Also

  $$M_+^{(x)}(x,k) \sim \frac{\alpha(x)}{k}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad k \to 0.$$

- Appropriate residue conditions if $a(k)$ has zeros for $k \in D_1 \cup D_2$, see Remark 1.1.

5. $\mathbf{S}^{-1} = \mathbb{Q}$.
   For the KdV,

$$a(k) = \frac{i\alpha}{k} + O(1), \quad b(k) = -\frac{i\alpha}{k} + O(1), \quad k \to 0,$$

   where α is a real constant.

Proof. The definitions (3.2) and (3.3) are motivated by Eqs. (2.14a) and (2.15), with the identification $\varphi(x,k) = \mu_3^{(12)}(x,0,k)$. Actually the matrix $\mu_3(x,0,k)$ motivates the introduction of the matrix

$$\mu_3(x,k) = \begin{pmatrix} \overline{\varphi_2(x,\bar{k})} & \varphi_1(x,k) \\ \overline{\rho\varphi_1(x,\bar{k})} & \varphi_2(x,k) \end{pmatrix} k \in (\bar{D}_3 \cup \bar{D}_4, \bar{D}_1 \cup \bar{D}_2), \tag{3.7}$$

where φ_1 and φ_2 denote the first and the second components of the vector φ. This matrix satisfies the integral equation (1.12) with $\mu_3(x,0,k)$ and $Q(x,0,k)$ replaced by $\mu_3(x,k)$ and $Q(x,k)$. This integral equation is a linear Volterra integral equation. Furthermore it is equivalent to the x-part of the Lax pair evaluated at $x=0$,

$$\partial_x \mu(x,k) + if_1(k)\hat{\sigma}_3\mu(x,k) = Q(x,k)\mu(x,k), \tag{3.8}$$

and supplemented with the boundary condition $\lim_{x\to\infty}\mu(x,k) = I$. The analysis of the above linear Volterra integral equation and of the associated ODE, immediately implies properties (1)–(3).

The derivation of properties (4) and (5) is based on the spectral analysis of the ODE (3.8). This analysis uses $\mu_3(x,k)$ as well as the eigenfunction motivated by $\mu_2(x,0,k)$, i.e. the eigenfunction

$$\mu_2(x,k) = \begin{pmatrix} \psi_1(x,k) & \overline{\psi_2(x,\bar{k})} \\ \psi_2(x,k) & \overline{\rho\psi_1(x,\bar{k})} \end{pmatrix}, k \in (\bar{D}_1 \cup \bar{D}_2, \bar{D}_3 \cup \bar{D}_4),$$

defined as the unique solution of the linear Volterra integral equation satisfied by $\mu_2(x,0,k)$, i.e. by the equation

$$e^{if_1(k)x\hat{\sigma}_3}\mu_2(x,k) = I + \int_0^x e^{if_1(k)\xi\hat{\sigma}_3}(Q\mu_2)(\xi,k)d\xi.$$

Since both eigenfunctions $\mu_3(x,k)$ and $\mu_2(x,k)$ satisfy the same ODE (3.8) they are related by the equation (compare with Eq. (1.11a))

$$\mu_3(x,k) = \mu_2(x,k)e^{-if_1(k)x\hat{\sigma}_3}s(k), \quad k \in \mathbb{R}.$$

Introducing the notations

$$M_+^{(x)} = \left(\frac{\psi}{a(k)}, \varphi\right), k \in D_1 \cup D_2; \quad M_-^{(x)} = \left(\varphi^*, \frac{\psi*}{\overline{a(\bar{k})}}\right), k \in D_3 \cup D_4,$$

where

$$\varphi^*(x,k) = \left(\overline{\varphi_2(x,\bar{k})}, \overline{\rho\varphi_1(x,\bar{k})}\right)^\tau, \quad \psi^*(x,k) = \left(\overline{\psi_2(x,\bar{k})}, \overline{\rho\psi_1(x,\bar{k}}\right)^\tau,$$

the relation between $\mu_3(x,t)$ and $\mu_2(x,k)$ becomes Eq. (3.6).

The substitution of the asymptotic expansion

$$M^{(x)}(x,k) = I + \frac{m_1(x)}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty$$

into Eq. (3.8) with $Q(x)$ given by Eq. (3.4a) yields $q_0(x) = 2i(m_1(x))_{12}$, i.e. Eq. (3.5a). Similarly for Eqs. (3.5b), (3.5c).

The investigation of properties (1)-(3) is called the "direct problem" in scattering theory, while the investigation of property (4) is called the "inverse problem". There is an extensive investigation of these problems in the literature, see for example [3]. The derivation of property (5) is discussed in [11]. □

Definition 3.2. A(k), B(k). *Let* $\{g_l(t)\}_0^{n-1}$*, be smooth functions for* $0 < t < T$*, where* $n = 2$ *for NLS, sG and* $n = 3$ *for KdV. Let the domains* D_j*,* $j = 1, \cdots, 4$*, be defined in Equations (2.5)–(2.7). The map*

$$\tilde{\mathbf{S}} : \{g_l(t)\}_0^{n-1} \to \{A(k), B(k)\} \tag{3.9}$$

is defined as follows:

$$\begin{pmatrix} -e^{-2if_2(k)T} B(k) \\ \overline{A(\bar{k})} \end{pmatrix} = \Phi(T,k), \tag{3.10}$$

where the vector valued function $\Phi(t,k)$ *is defined in terms of* $\{g_l(t)\}_0^{n-1}$ *by*

$$\partial_t \Phi(t,k) + 2if_2(k)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\Phi(t,k) = \tilde{Q}(t,k)\Phi(t,k), \quad 0 < t < T, \quad k \in D_2 \cup D_4,$$

$$\Phi(0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.11}$$

and $\tilde{Q}(t,k)$ *is given for the NLS, KdV, sG respectively by:*

$$\begin{aligned}
\tilde{Q}(t,k) &= 2k\begin{pmatrix} 0 & g_0(t) \\ \lambda\bar{g}_0(t) & 0 \end{pmatrix} - i\begin{pmatrix} 0 & g_1(t) \\ \lambda\bar{g}_1(t) & 0 \end{pmatrix}\sigma_3 - i\lambda|g_0(t)|^2\sigma_3, \\
\tilde{Q}(t,k) &= -2kg_0(t)\sigma_2 + g_1(t)\sigma_1 + \frac{2g_0(t)^2 + g_0(t) - g_2(t)}{2k}(i\sigma_3 - \sigma_2), \\
\tilde{Q}(t,k) &= -\frac{i}{4}\left(\frac{dg_0(t)}{dt} + g_1(t)\right)\sigma_1 + \frac{i}{4k}(\sin g_0(t))\sigma_2 - \frac{i}{4k}((\cos g_0(t)) - 1)\sigma_3.
\end{aligned} \tag{3.12}$$

Properties of $A(k)$*,* $B(k)$*.*

1. $A(k)$, $B(k)$ are entire functions which are bounded for $k \in D_1 \cup D_3$. If $T = \infty$ they are defined and are analytic for k in this domain.
2. $A(k)\overline{A(\bar{k})} - \rho B(k)\overline{B(\bar{k})} = 1$, $k \in \mathbb{C}$.
3. $A(k) = 1 + O\left(\frac{1+e^{2if_2(k)T}}{k}\right)$, $B(k) = O\left(\frac{1+e^{2if_2(k)T}}{k}\right)$, $k \to \infty$.

4. The inverse of the map (3.9) denoted by $\tilde{\mathbb{Q}}$ can be defined for (3.12a), (3.12b), (3.12c), respectively as follows:

$$\begin{aligned}
g_0(t) &= 2i \lim_{k\to\infty} (kM^{(t)})_{12}, \\
g_1(t) &= \lim_{k\to\infty} \left\{4(k^2M^{(t)})_{12} + 2ig_0(t)kM_{22}^{(t)}\right\}; \\
g_0(t) &= 4 \lim_{k\to\infty} (k^2M^{(t)})_{12}, \\
g_1(t) &= 2i \lim_{k\to\infty} \left[4(k^3M^{(t)})_{12} - g_0(t)kM_{22}^{(t)} - 4kg_0\right], \\
g_2(t) &= g_0(t) + g_0(t)^2 + 2i\frac{d}{dt}\lim_{k\to\infty} (kM^{(t)})_{11}; \\
\cos g_0(t) &= 1 - 2\lim_{k\to\infty} \left\{(kM^{(t)}(t,k))_{12}^2 + 2i\partial_t(kM^{(t)}(t,k))_{22}\right\}, \\
g_1(t) &= -\frac{d}{dt}g_0(t) - 2\lim_{k\to\infty} (kM^{(t)}(t,k))_{12};
\end{aligned} \tag{3.13}$$

where $M^{(t)}(t,k)$ is the unique solution of the following RH problem:

- $$M^{(t)}(t,k) = \begin{cases} M_+^{(t)}(t,k), & k \in D_1 \cup D_3 \\ M_-^{(t)}(t,k), & k \in D_2 \cup D_4, \end{cases}$$

  is a meromorphic function of k for $k \in \mathbb{C} \setminus \mathcal{L}$ and $\mathcal{L}$ is defined in Sect. 2.4.
- $$M^{(t)}(t,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty.$$
- $$M_-^{(t)}(t,k) = M_+^{(t)}(t,k)J^{(t)}(t,k), \quad k \in \mathcal{L},$$

  where

  $$J^{(t)}(t,k) = \begin{pmatrix} 1 & -\frac{B(k)}{\overline{A(\bar{k})}}e^{-2if_2(k)t} \\ \frac{\rho\overline{B(\bar{k})}}{A(k)}e^{2if_2(k)t} & \frac{1}{A(k)\overline{A(\bar{k})}} \end{pmatrix}. \tag{3.14}$$
- Appropriate residue conditions if $A(k)$ has zeros for $k \in D_1 \cup D_3$, see Remark 1.1.

5. $\tilde{\mathbf{S}}^{-1} = \tilde{\mathbb{Q}}$.
   For the KdV,

$$A(k) = \frac{i\beta}{k} + O(1), \quad B(k) = -\frac{i\beta}{k} + O(1), \quad k \to 0,$$

where β is a real constant.

Proof. The definitions (3.10), (3.11) are motivated by Eqs. (2.14b) and (2.16). Actually the matrix $\mu_2(0,t,k)$ motivates the introduction of the matrix

$$\mu_2(t,k)=\begin{pmatrix}\overline{\Phi_2(t,\bar{k})} & \Phi_1(t,k)\\ \rho\overline{\Phi_1(t,\bar{k})} & \Phi_2(t,k)\end{pmatrix},\quad k\in(\bar{D}_1\cup\bar{D}_3,\bar{D}_2\cup\bar{D}_4),\tag{3.15}$$

where Φ_1, Φ_2 denote the first, second component of the vector Φ. This matrix satisfies the integral equation (1.13) with $\mu_2(0,t,k)$ and $\tilde{Q}(0,t,k)$ replaced by $\mu_2(t,k)$ and $\tilde{Q}(t,k)$. This integral equation is a linear Volterra integral equation. Furthermore, it is equivalent to the t-part of the Lax pair evaluated at $x=0$,

$$\partial_t\mu(t,k)+if_2(k)\hat{\sigma}_3\mu(t,k)=\tilde{Q}(t,k)\mu(t,k),\tag{3.16}$$

and supplemented with the boundary condition $\mu(0,k)=I$. The analysis of the above linear Volterra integral equation and of the associated ODE implies properties (1)–(3).

The derivation of properties (4) and (5) is based on the spectral analysis of the ODE (3.16). This analysis uses $\mu_2(t,k)$ as well as the eigenfunction motivated by $\mu_1(0,t,k)$, i.e. the eigenfunction

$$\mu_1(t,k)=\begin{pmatrix}\overline{\Psi_2(t,\bar{k})} & \Psi_1(t,k)\\ \rho\overline{\Psi_1(t,\bar{k})} & \Psi_2(t,k)\end{pmatrix},\quad k\in(\bar{D}_2\cup\bar{D}_4,\bar{D}_1\cup\bar{D}_3)$$

defined as the unique solution of the linear Volterra integral equation satisfied by $\mu_1(0,t,k)$, i.e. by the equation

$$e^{if_2(k)t\hat{\sigma}_3}\mu_1(t,k)=I-\int_t^T e^{if_2(k)\tau\hat{\sigma}_3}(\tilde{Q}\mu_1)(\tau,k)d\tau.$$

The eigenfunctions $\mu_2(t,k)$ and $\mu_1(t,k)$ are related by the equation (compare with Eq. (1.11b))

$$\mu_1(t,k)=\mu_2(t,k)e^{-if_2(k)t\hat{\sigma}_3}S(k),\quad k\in\mathcal{L}.$$

Using the notations

$$M_-^{(t)}(t,k)=\left(\frac{\Phi^*}{A(k)},\Psi\right),\ k\in D_1\cup D_3;\quad M_+^{(t)}(t,k)=\left(\Psi^*,\frac{\Phi}{\overline{\overline{A(\bar{k})}}}\right),\ k\in D_2\cup D_4,$$

where Φ^*, Ψ^* are defined as ϕ, ψ with x replaced by t, the relation between $\mu_1(t,k)$ and $\mu_2(t,k)$ becomes Eq. (3.14).

The substitution of the asymptotic expansion

$$M^{(t)}(t,k)=I+\frac{m_1(t)}{k}+\frac{m_2(t)}{k^2}+O\left(\frac{1}{k^3}\right),\quad k\to\infty$$

into Eq. (3.16) with $\tilde{Q}(t,k)$ given by Eq. (3.12a) yields

$$g_0(t)=2i(m_1(t))_{12},\quad g_1(t)=4(m_2(t))_{12}+2ig_0(t)(m_1(t))_{22},$$

which imply Eqs. (3.13a) and (3.13b). Similarly for Eqs. (3.13c)–(3.13e). □

Theorem 3.1. *For the NLS and the KdV let $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$, for the sG let $q_0(x) - 2\pi m \in \mathcal{S}(\mathbb{R}^+)$ and $q_1(x) \in \mathcal{S}(\mathbb{R}^+)$, where m is an integer. Given these functions define $\{a(k), b(k)\}$ according to Definition 3.1. Suppose that there exist smooth functions $\{g_l(t)\}_0^{n-1}$ satisfying $\{g_l(0) = \partial_x^l q_0(0)\}_0^{n-1}$, such that the functions $\{A(k), B(k)\}$ which are defined from $g_l(t)$ according to Definition 3.2 satisfy the global relation*

$$a(k)B(k) - b(k)A(k) = e^{2if_2(k)T}c(k), \quad k \in \bar{D}_1 \cup \bar{D}_2, \tag{3.17}$$

where $c(k)$ is analytic and bounded for $k \in D_1 \cup D_2$ and is of $O(1/k)$, $k \to \infty$.

Define $M(x, t, k)$ as the solution of the following 2×2 matrix RH problem:

- M is meromorphic for k in $\mathbb{C} \setminus \mathcal{L}$, where $\mathcal{L}$ is defined in Sect. 2.4.
- $$M_-(x, t, k) = M_+(x, t, k)J(x, t, k), \quad k \in \mathcal{L}, \tag{3.18}$$
  where M is M_- for $k \in D_2 \cup D_4$, M is M_+ for $k \in D_1 \cup D_3$, and J is defined in terms of a, b, A, B in Sect. 2.4.
- $$M(x, t, k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{3.19}$$
- Appropriate residue conditions if $a(k)$ has zeros in $D_1 \cup D_2$ and/or $d(k)$ has zeros in D_2.
- In the case of the KdV, $M(x, t, k)$ has a pole at $k = 0$ satisfying
  $$M(x, t, k) \sim \frac{i\alpha(x, t)}{k} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad k \to 0.$$

Then $M(x, t, k)$ exists and is unique.

Define $q(x, t)$ for the NLS, KdV, sG respectively by

$$\begin{aligned} q(x, t) &= 2i \lim_{k\to\infty} (kM(x, t, k))_{12}, \\ q(x, t) &= -2i \lim_{k\to\infty} \partial_x (kM(x, t, k))_{22}, \\ \cos q(x, t) &= 1 + 2 \lim_{k\to\infty} \left\{(kM(x, t, k))_{12}^2 + 2i\partial_x (kM(x, t, k))_{22}\right\}. \end{aligned} \tag{3.20}$$

Then $q(x, t)$ solves the NLS, the KdV and the sG respectively. Furthermore

$$q(x, 0) = q_0(x), \quad \{\partial_x^l q(0, t) = g_l(t)\}_0^{n-1}$$

and for the sG $q_t(x, 0) = q_1(x)$.

Proof. In the absence of poles the unique solvability of the RH problem is a consequence of a "vanishing lemma", i.e. the RH problem obtained from the above RH by replacing Eq. (3.19) with $M = O(1/k)$ as $k \to \infty$, has only the trivial solution. The vanishing lemma can be established using the symmetry properties of J; for the NLS and sG the details are given in [15] and [16], for the KdV the proof is similar. In the presence of poles $M(x, t, k)$ is a meromorphic function of k. In this case the RH problem can be mapped to a RH problem without poles coupled with a system of algebraic equations, see [15, 16]. The unique solvability of the relevant algebraic equations is also based on the symmetry properties of J, see [15, 16].

The proof that the solution of certain RH problems can be used to solve certain linear PDEs is the basic idea of the so-called dressing method [17]. In this method $q(x,t)$ and its derivatives are defined in terms of the coefficients of the asymptotic expansion of M for large k, and then it is shown that M and these functions solve the x and the t parts of the Lax pair. The proof, which is essentially algebraic, makes crucial use of the explicit x and t dependence of the jump matrix and it is *independent* of the particular choice of the contour $\mathcal{L}$. Since the x,t dependence of the jump matrix J is identical with the x,t dependence of the jump matrix that appears in the usual inverse scattering method, the proof that $q(x,t)$ solves the given PDE follows immediately from the corresponding proof for the Cauchy problem on the line. □

Proof that $q(x,0) = q_0(x)$. For the NLS $q(x,0)$ is determined by $q(x,0) = 2i \lim_{k\to\infty} (kM(x,0,k))_{12}$ in terms of $M(x,0,k)$; similarly for the KdV and the sG. The function $M(x,0,k)$ satisfies the jump condition

$$M_-(x,0,k) = M_+(x,0,k)J(x,0,k), \quad k \in \mathcal{L}. \tag{3.21}$$

$q_0(x)$ is determined by a similar formula in terms of $M^{(x)}(x,k)$, which satisfies the jump condition, see Eq. (3.6),

$$M_-^{(x)}(x,k) = M_+^{(x)}(x,k)J^{(x)}(x,k), \quad k \in \mathbb{R}.$$

We will show that it is possible to map Eq. (3.21) into the above equation. Let $M^{(x)}(x,k)$ be defined in terms of $M(x,0,k)$ by

$$M^{(x)} = \begin{cases} M(x,0,k), & k \in D_1 \cup D_4 \\ M(x,0,k)J_1^{-1}(x,0,k), & k \in D_2 \\ M(x,0,k)J_3(x,0,k), & k \in D_3. \end{cases} \tag{3.22}$$

The function $M^{(x)}$ is a sectionally meromorphic function. Indeed, $J_1(x,0,k)$ involves $\Gamma(k)\exp(2if_1(k)x)$; $\Gamma(k)$ is bounded and analytic for $k \in D_2$, while $\exp(2if_1(k)x)$ is bounded and analytic for $k \in D_1 \cup D_2$, thus $J_1(x,0,k)$ is bounded and analytic for $k \in D_2$. Similarly for $J_3(x,0,k)$, $k \in D_3$.

The definition (3.22) implies that the jump conditions of $M^{(x)}$ can be computed in terms of $J_j(x,0,k)$, $j = 1,\cdots,4$. Indeed, let us introduce the following notations: $M_j(x,0,k)$ and $M_j^{(x)}(x,k)$ denote $M(x,0,k)$ and $M^{(x)}(x,k)$ for $k \in D_j$, $j = 1,\cdots,4$. Using these notations Eqs. (3.21) and (3.22) can be written as

$$\begin{aligned} M_2(x,0,k) &= M_1(x,0,k)J_1(x,0,k), \quad M_2(x,0,k) = M_3(x,0,k)J_2(x,0,k), \\ M_4(x,0,k) &= M_3(x,0,k)J_3(x,0,k), \quad M_4(x,0,k) = M_1(x,0,k)J_4(x,0,k), \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} M_1^{(x)} &= M_1(x,0,k), \quad M_2^{(x)} = M_2(x,0,k)J_1^{-1}(x,0,k), \\ M_3^{(x)} &= M_3(x,0,k)J_3(x,0,k), \quad M_4^{(x)} = M_4(x,0,k), \end{aligned} \tag{3.24}$$

where Eqs. (3.23) are valid on the respective parts of the contour $\mathcal{L}$, i.e. on $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$, respectively. Substituting Eqs. (3.24) into (3.23) we find

$$\begin{gathered} M_2^{(x)}J_1(x,0,k) = M_1^{(x)}J_1(x,0,k), \quad M_2^{(x)}J_1(x,0,k) = M_3^{(x)}(J_3^{-1}J_2)(x,0,k), \\ M_4^{(x)} = M_3^{(x)}, \quad M_4^{(x)} = M_1^{(x)}J_4(x,0,k). \end{gathered} \tag{3.24a}$$

Hence, $M^{(x)}$ does not have a jump along $\mathcal{L}_1$ and $\mathcal{L}_3$. Furthermore, recalling that $J_2 = J_3 J_4^{-1} J_1$, noting $J_4(x, 0, k) = J^{(x)}(x, k)$, and using the notation $M^{(x)} = M_+^{(x)}$ for $k \in D_1 \cup D_2$, $M^{(x)} = M_-^{(x)}$ for $k \in D_3 \cup D_4$, Eqs. (3.24) become Eq. (3.6).

The proof that these transformations preserve the residue conditions is given in [11] for the NLS; for the KdV and sG see Remark 1.1.

Proof that $\partial_x^l q(0, t) = g_l(t)$. In the dressing method, as mentioned earlier one first obtains expressions for $\partial_x^l q(x, t)$ in terms of $M(x, t, k)$. These expressions evaluated at $t = 0$ are Eqs. (3.13) with $M^{(t)}$ replaced by $M(0, t, k)$. Thus since $\partial_x^l q(0, t)$ are determined by $M(0, t, k)$, while $g_l(t)$ are determined by $M^{(t)}(t, k)$ we must establish a relation between $M(0, t, k)$ and $M^{(t)}(t, k)$. The former satisfies the jump condition (3.18) evaluated at $x = 0$, i.e.

$$\begin{aligned} &M_2(0,t,k) = M_1(0,t,k)J_1(0,t,k), \quad M_2(0,t,k) = M_3(0,t,k)J_2(0,t,k),\\ &M_4(0,t,k) = M_3(0,t,k)J_3(0,t,k), \quad M_4(0,t,k) = M_1(0,t,k)J_4(0,t,k), \end{aligned} \tag{3.25}$$

on the respective parts of the contour $\mathcal{L}$, where we have used the notations $M_j(0, t, k) = M(0, t, k)$ for $k \in D_j$, $j = 1, \cdots, 4$. The function $M^{(t)}(t, k)$ satisfies the jump condition (3.14). We will show that Eqs. (3.25) imply Eq. (3.14) if and only if the spectral functions satisfy the global relation (3.17). Let us define $M^{(t)}(t, k)$ by

$$M_j^{(t)}(t,k) = M_j(0,t,k)F_j(t,k), \quad k \in D_j,\ j = 1, \cdots, 4. \tag{3.26}$$

In order for $M^{(t)}(t, k)$ to satisfy the RH problem with the jump condition (3.14), F_j must have the following properties: be bounded and analytic in the domains of their definition, tend to I as $k \to \infty$, and satisfy the relations

$$\begin{aligned} J_1(0,t,k)F_2(t,k) &= F_1(t,k)J^{(t)}(t,k), \quad k \in \mathcal{L}_1,\\ J_3(0,t,k)F_4(t,k) &= F_3(t,k)J^{(t)}(t,k), \quad k \in \mathcal{L}_3,\\ J_4(0,t,k)F_4(t,k) &= F_1(t,k)J^{(t)}(t,k), \quad k \in \mathcal{L}_4. \end{aligned} \tag{3.27}$$

Indeed, substituting Eqs. (3.26) into Eqs. (3.25) and using Eqs. (3.27), Eqs. (3.25) become Eqs. (3.14).

We will show that such $F_j(t, k)$ are the matrices

$$\begin{aligned} &F_1 = \begin{pmatrix} \frac{a(k)}{A(k)} & c(k)e^{2if_2(T-t)} \\ 0 & \frac{A(k)}{a(k)} \end{pmatrix}, \quad F_4 = \begin{pmatrix} \frac{\overline{A(\bar{k})}}{\overline{a(\bar{k})}} & 0 \\ \rho\overline{c(\bar{k})}e^{-2if_2(T-t)} & \frac{\overline{a(\bar{k})}}{\overline{A(\bar{k})}} \end{pmatrix},\\ &F_2 = \begin{pmatrix} d(k) & \frac{-b(k)e^{-2if_2t}}{\overline{A(\bar{k})}} \\ 0 & \frac{1}{d(k)} \end{pmatrix}, \quad F_3 = \begin{pmatrix} \frac{1}{\overline{d(\bar{k})}} & 0 \\ \frac{-\rho\overline{b(\bar{k})}}{A(k)}e^{2if_2t} & \overline{d(\bar{k})} \end{pmatrix}. \end{aligned} \tag{3.28}$$

We verify the first of Eqs. (3.27): The (12) element is proportional to the global relation; the (21) and (22) elements are satisfied identically. The (11) element is satisfied iff

$$d = \frac{a}{A} + \frac{\rho\bar{B}}{A}ce^{2if_2T}. \tag{3.29}$$

Using $A\bar{A} - \lambda B\bar{B} = 1$, we find

$$d = \frac{a}{A} A\bar{A} - \rho b\bar{B} = \frac{a}{A}(1 + \rho B\bar{B}) - \rho b\bar{B} = \frac{a}{A} + \frac{\rho\bar{B}}{A}(aB - bA),$$

which equals the rhs of Eq. (3.29) in view of the global relation.

The second of Eqs. (3.27) follows from the first one and the symmetry relations. The third of Eqs. (3.27) can be verified in a way similar to the first equation. The proof that these transformations preserve the residue conditions is given in [11] for the NLS; for the KdV and the sG see Remark 1.1.

Proposition 3.1 (Independence of T). *Let* $0 < t < T_0 < T$*; let* $A(T_0, k)$*,* $B(T_0, k)$ *be defined as in Definition 3.2 with Eq. (3.10) replaced by*

$$\begin{pmatrix} -e^{-2if_2(k)T_0} B(T_0, k) \\ \overline{A(T_0, \bar{k})} \end{pmatrix} = \Phi(T_0, k). \tag{3.30}$$

Let $\tilde{J}_1(x, T_0, k)$*,* $\tilde{J}_3(x, T_0, k)$*, denote the jump matrices obtained from* $J_1(x, T_0, k)$*,* $J_3(x, T_0, k)$ *by replacing* $A(k)$*,* $B(k)$ *with* $A(T_0, k)$*,* $B(T_0, k)$*. Let* $\tilde{J}_2 = \tilde{J}_3 J_4^{-1} \tilde{J}_1$*. Let* $\tilde{M}(x, t, k)$ *satisfy a RH problem similar to that of* $M(x, t, k)$ *but with jump matrices* $\tilde{J}_1$*,* $\tilde{J}_2$*,* $\tilde{J}_3$*,* J_4*. Let* $M_j(x, t, k) = M(x, t, k)$*,* $\tilde{M}_j(x, t, k) = \tilde{M}(x, t, k)$ *for* $k \in D_j$*,* $j = 1, 2, 3, 4$*. Then*

$$M_1 = \tilde{M}_1, \quad M_4 = \tilde{M}_4, \quad M_2 = \tilde{M}_2 \tilde{J}_1^{-1} J_1, \quad M_3 = \tilde{M}_3 \tilde{J}_3 J_3^{-1}. \tag{3.31}$$

Proof. Using Eqs. (3.31) it is straightforward to verify that the jump conditions for M, i.e. Eq. (3.18), yield similar jump conditions for $\tilde{M}$ with J_1, J_3 replaced by $\tilde{J}_1$, $\tilde{J}_3$. Assuming the solitonless case it remains to show that the functions $\tilde{J}_1^{-1} J_1$ and $\tilde{J}_3 J_3^{-1}$ are analytic and bounded for $k \in D_2$ and $k \in D_3$ respectively. We will show this fact for the function $\tilde{J}_3^{-1} J_3$, the proof for $\tilde{J}_1^{-1} J_1$ follows from symmetry considerations.

The diagonal elements of $\tilde{J}_3^{-1} J_3$ are 1, its (21) element is 0, and its (12) element equals

$$\rho(\overline{\Gamma(\bar{k})} - \overline{\Gamma(T_0, \bar{k})})e^{-2i\theta} = \frac{B(k)A(T_0, k) - A(k)B(T_0, k)}{\overline{d(\bar{k})}\;\overline{d(T_0, \bar{k})}} \times e^{-2if_2(k)T_0} e^{-2if_1(k)x + 2if_2(k)(T_0 - t)}, \tag{3.32}$$

where the rhs of this equation follows from the lhs using the definitions of $\Gamma(k)$, of $\Gamma(T_0, k)$, and the notation

$$d(T_0, k) = a(k)\overline{A(T_0, \bar{k})} - \rho b(k)\overline{B(T_0, \bar{k})}. \tag{3.33}$$

The definition (3.30) implies that $A(T_0, k)$ and $B(T_0, k)$ have the same properties with $A(k)$, $B(k)$, where T is replaced by T_0 in the third property. Thus since $\overline{d(\bar{k})}$ is bounded and analytic for $k \in D_3$ the same is true for $\overline{d(T_0, \bar{k})}$. Definition (3.30) implies

$$\begin{aligned} &\left[B(k)A(T_0, k) - B(T_0, k)A(k)\right]e^{-2if_2(k)T_0} \\ &\quad = \overline{\Phi_2(\bar{k})}\Phi_1(T_0, k) - \Phi_1(k)\overline{\Phi_2(T_0, \bar{k})}e^{2if_2(k)(T - T_0)}. \end{aligned} \tag{3.34}$$

We will show that the r.h.s. of Eq. (3.34) is bounded and analytic for $k \in D_1 \cup D_3$; this result together with the fact that $\exp[-2if_1(k)x + 2if_2(k)(T_0 - t)]$ is bounded for $k \in D_3$ imply that the r.h.s. of Eq. (3.33) is bounded and analytic for $k \in D_3$.

In order to prove that the r.h.s. of (3.34) is bounded and analytic for $k \in D_1 \cup D_3$ we introduce the notations

$$\begin{aligned}\chi_1(t,k) &= \overline{\Phi_2(\bar{k})}\Phi_1(t,k) - \Phi_1(k)\overline{\Phi_2(t,\bar{k})}e^{2if_2(k)(T-t)},\\ \chi_2(t,k) &= \overline{\Phi_2(\bar{k})}\Phi_2(t,k) - \rho\Phi_1(k)\overline{\Phi_1(t,\bar{k})}e^{2if_2(k)(T-t)}.\end{aligned} \tag{3.35}$$

We will prove that the functions χ_1 and χ_2 satisfy the following system of linear integral equations:

$$\begin{aligned}\chi_1(t,k) &= -\int_t^T \left[\tilde{Q}_{11}(\tau,k)\chi_1(\tau,k) + \tilde{Q}_{12}(\tau,k)\chi_2(\tau,k)\right]e^{2if_2(\tau-t)}\,d\tau,\\ \chi_2(t,k) &= 1 - \int_t^T \left[\tilde{Q}_{22}(\tau,k)\chi_2(\tau,k) + \tilde{Q}_{21}(\tau,k)\chi_1(\tau,k)\right]d\tau,\end{aligned} \tag{3.36}$$

where $\tilde{Q}_{ij}$ denote the entries of the matrix $\tilde{Q}(t,k)$. Indeed, the symmetry properties of $\tilde{Q}(t,k)$ imply that if the vector $\Phi(t,k)$ with the two components Φ_1 and Φ_2 satisfies Eq. (3.11), then the vector

$$\begin{pmatrix}\overline{\Phi_2(t,\bar{k})}\\ \overline{\Phi_1(t,\bar{k})}\end{pmatrix}e^{-2if_2(k)t}$$

also satisfies the same equation. Hence the vector $\chi(t,k)$ with the two components χ_1 and χ_2 defined by Eqs. (3.35) satisfies Eq. (3.11). Furthermore,

$$\begin{aligned}\chi_1(T,k) &= \overline{\Phi_2(\bar{k})}\Phi_1(k) - \Phi_1(k)\overline{\Phi_2(\bar{k})} = 0,\\ \chi_2(T,k) &= \overline{\Phi_2(\bar{k})}\Phi_2(k) - \rho\Phi_1(k)\overline{\Phi_1(\bar{k})} = 1.\end{aligned}$$

The unique solution of Eq. (3.11) with the boundary condition $\{\chi_1(T,k) = 0,\ \chi_2(T,k) = 1\}$ satisfies Eqs. (3.36).

Equations (3.36) imply that $\chi_1(t,k)$ is bounded and analytic for $k \in D_1 \cup D_3$ for all $0 < t < T$. Since the r.h.s. of Eq. (3.34) equals $\chi_1(T_0,k)$, $T_0 < T$, it follows that the r.h.s. of Eq. (3.34) is also bounded and analytic for $k \in D_1 \cup D_3$. □

4. Linearizable Boundary Conditions

The spectral functions $A(k)$, $B(k)$ are defined in terms of the matrix eigenfunction $\mu_2(t,k)$, see Eq. (3.15), which satisfies the ODE (3.16) and the initial condition $\mu_2(0,k) = I$.

Let $M(t,k) = \mu_2(t,k)\exp[-if_2t\sigma_3]$, i.e.

$$M(t,k) = \begin{pmatrix}\overline{M_2(t,\bar{k})} & M_1(t,k)\\ \rho\overline{M_1(t,\bar{k})} & M_2(t,k)\end{pmatrix}, \quad M_1 = \Phi_1 e^{if_2t}, \quad M_2 = \Phi_2 e^{if_2t}. \tag{4.1}$$

$M(t,k)$ satisfies

$$M_t + if_2(k)\sigma_3 M = \tilde{Q}(t,k)M, \quad M(0,k) = I. \tag{4.2}$$

Let $\nu(k)$ be defined by

$$f_2(\nu(k)) = f_2(k). \tag{4.3}$$

Suppose that given a subset of the boundary values $\{g_l(t)\}_0^{n-1}$, we can compute a non-singular matrix $N(k)$ such that

$$(if_2(k)\sigma_3 - \tilde{Q}(t,\nu(k)))N(k) = N(k)(if_2(k)\sigma_3 - \tilde{Q}(t,k)). \tag{4.4}$$

Then

$$M(t,\nu(k)) = N(k)M(t,k)N(k)^{-1}. \tag{4.5}$$

Indeed, Eq. (4.4) implies a relation of the form (4.5) and furthermore Eq. (4.5) is satisfied identically at $t = 0$.

Equations (4.1) and the definition of $A(k)$, $B(k)$ imply

$$A(k) = \overline{M_2(T,\bar{k})}e^{if_2(k)T}, \quad B(k) = -M_1(T,k)e^{if_2(k)T}. \tag{4.6}$$

Equation (4.4) implies that a necessary condition for the existence of $N(k)$ is that the determinant of the matrix

$$U(t,k) = \tilde{Q}(t,k) - if_2(k)\sigma_3, \tag{4.7}$$

depends on k only in the form of $f_2(k)$.

We will use the notations

$$N_1 = N_{11}, \quad N_2 = N_{12}, \quad N_3 = N_{21}, \quad N_4 = N_{22}. \tag{4.8}$$

4.1. The NLS. The determinant of U depends only on k^2 iff

$$g_0(t)\bar{g}_1(t) - \bar{g}_0(t)g_1(t) = 0. \tag{4.9}$$

The invariance of f_2 yields $\nu(k) = -k$. A particular case of boundary conditions satisfying Eq. (4.9) is given by Eq. (1.21). In this case, if $N_2 = N_3 = 0$, then the diagonal part of Eq. (4.4) is satisfied identically, and the off diagonal elements yield

$$(2k - i\chi)N_4 + (2k + i\chi)N_1 = 0.$$

Using this equation, the second column vector of Eq. (4.5) yields

$$M_2(t,k) = M_2(t,-k),\, M_1(t,k) = -\frac{1}{f(k)}M_1(t,-k),\, f(k) = \frac{2k - i\chi}{2k + i\chi}. \tag{4.10}$$

Thus the spectral functions $A(k)$, $B(k)$ satisfy the symmetry relations

$$A(k) = A(-k), \quad B(k) = -\frac{B(-k)}{f(k)}, \quad k \in \mathbb{C}. \tag{4.11}$$

We will now show that the global relation (3.17) supplemented by these symmetry conditions yields $\Gamma(k)$ *explicitly* in terms of $a(k)$, $b(k)$. Indeed, letting $k \to -k$ in the definition of $\overline{d(\bar{k})}$ and using Eqs. (4.11) we find

$$A(k)\overline{a(-\bar{k})} + \lambda f(k)B(k)\overline{b(-\bar{k})} = \overline{d(-\bar{k})}, \quad k \in D_1. \tag{4.12}$$

This equation and the global relation (3.17), which is also valid for $k \in D_1$, can be thought of as two algebraic equations for the unknown functions $A(k)$, $B(k)$. Their solution yields

$$\begin{aligned} A(k) &= \frac{a(k)\overline{d(-\bar{k})}}{\Delta(k)} - \frac{\lambda f(k)\overline{b(-\bar{k})}e^{4ik^2T}c(k)}{\Delta(k)}, \\ B(k) &= \frac{b(k)\overline{d(-\bar{k})}}{\Delta(k)} + \frac{\overline{a(-\bar{k})}e^{4ik^2T}c(k)}{\Delta(k)}, \quad k \in D_1, \end{aligned} \tag{4.13}$$

where

$$\Delta(k) = a(k)\overline{a(-\bar{k})} + \lambda f(k)b(k)\overline{b(-\bar{k})}.$$

Equations (4.13) express $\{A(k), B(k)\}$ for $k \in D_1$, then the transformation $k \to -k$ and Eqs. (4.11) yield these functions for $k \in D_3$,

$$\begin{aligned} A(k) &= \frac{a(-k)\overline{d(\bar{k})}}{\Delta(-k)} - \frac{\lambda f(-k)\overline{b(\bar{k})}e^{4ik^2T}c(-k)}{\Delta(-k)}, \\ B(k) &= -\frac{f(-k)b(-k)\overline{d(\bar{k})}}{\Delta(-k)} - \frac{f(-k)\overline{a(\bar{k})}e^{4ik^2T}c(-k)}{\Delta(-k)}, \quad k \in D_3. \end{aligned} \tag{4.14}$$

In Eqs. (4.13), (4.14), the functions $c(k)$ and $d(k)$ are unknown, however because of the distinctive features of the RH problem of Theorem 3.1, these unknown functions do *not* contribute to $M(x,t,k)$. Indeed, we first show that the functions (4.13), (4.14) can be replaced by

$$\tilde{A}(k) = \begin{bmatrix} \frac{a(k)\overline{d(-\bar{k})}}{\Delta(k)} \\ \\ \frac{a(-k)\overline{d(\bar{k})}}{\Delta(-k)}, \end{bmatrix} \quad \tilde{B}(k) = \begin{bmatrix} \frac{b(k)\overline{d(-\bar{k})}}{\Delta(k)}, & k \in D_1, \\ \\ \frac{-f(-k)b(-k)\overline{d(\bar{k})}}{\Delta(-k)}, & k \in D_3. \end{bmatrix} \tag{4.15}$$

The proof of this fact is similar to the proof of Proposition 3.1: Let $\tilde{J}_1$, $\tilde{J}_3$ denote the matrices obtained from J_1, J_3 by replacing A, B with $\tilde{A}$, $\tilde{B}$; let $\tilde{J}_2 = \tilde{J}_3 J_4^{-1} \tilde{J}_1$. Let $\tilde{M}$ be defined in terms of M by Eqs. (3.31). Then $\tilde{M}$ satisfies a RH problem similar to M with J_1, J_2, J_3 replaced by $\tilde{J}_1$, $\tilde{J}_2$, $\tilde{J}_3$. The derivation of this result in the solitonless case is based on the fact that the functions $\tilde{J}_1^{-1}J_1$ and $\tilde{J}_3 J_3^{-1}$ are bounded and analytic for $k \in D_2$ and $k \in D_3$ respectively. The proof of the latter fact involves (compare with Eq. (3.32))

$$\rho\left(\overline{\Gamma(\bar{k})} - \overline{\tilde{\Gamma}(\bar{k})}\right)e^{-2i\theta} = \frac{B\tilde{A} - A\tilde{B}}{\overline{d(\bar{k})}\overline{\tilde{d}(\bar{k})}}e^{-4ik^2t-2ikx}, \quad k \in D_3.$$

Writing A, B in terms of $\tilde{A}$, $\tilde{B}$ and the terms involving c, we find

$$(B\tilde{A} - A\tilde{B})e^{-4ik^2t} = \frac{f(-k)}{\Delta(-k)}(\lambda\overline{b(\bar{k})}\tilde{B}(k) - \overline{a(\bar{k})}\tilde{A}(k))c(-k)e^{4ik^2(T-t)},$$

which is analytic for $k \in D_3$.

The above discussion indicates that we can use Eqs. (4.15) instead of Eqs. (4.13), (4.14); Eqs. (4.15) imply Eq. (1.24).

4.2. The sG. The determinant of the matrix U is proportional to

$$\left(k+\frac{1}{k}\right)^2+2\left[(\cos g_0(t))-1\right]+\left(\frac{dg_0(t)}{dt}+g_1(t)\right)^2.$$

Thus the necessary condition for linearizable boundary conditions is always satisfied. The invariance of $f_2(k)$ yields

$$\nu(k)=\frac{1}{k}.$$

Let $N_3=N_2$, $N_4=N_1$, then both the (11) and the (22) elements of Eq. (4.4) yield Eq. (4.16a) below, while both the (12) and (21) elements of Eq. (4.4) yield (4.16b),

$$(k^2-1)(\cos\chi-1)N_1=i(k^2+1)(\sin\chi)N_2, \tag{4.16a}$$

$$i(k^2-1)(\sin\chi)N_1=(k^2+1)(\cos\chi+1)N_2. \tag{4.16b}$$

These equations are equivalent. Using either of these equations, Eq. (4.5) yields

$$\begin{aligned}
M_1\left(\frac{1}{k}\right)&=\frac{1}{f-\frac{1}{f}}\left[\left(M_2(k)-\frac{\rho}{f}\overline{M_1(\bar{k})}\right)-\left(\overline{M_2(\bar{k})}-fM_1(k)\right)\right],\\
f(k)&=i\frac{k^2+1}{k^2-1}\frac{\sin\chi}{\cos\chi-1},\\
M_2\left(\frac{1}{k}\right)&=\frac{1}{f-\frac{1}{f}}\left[f\left(M_2(k)-\frac{\rho}{f}\overline{M_1(\bar{k})}\right)-\frac{1}{f}\left(\overline{M_2(\bar{k})}-fM_1(k)\right)\right],\quad \rho=-1,
\end{aligned}\tag{4.17}$$

where for convenience of notation we have suppressed the t dependence.

We note that rhs of the equations for $M_1(1/k)$ and $M_2(1/k)$ involve only the two expressions appearing in the two parentheses; solving for these expressions and using the definitions of $A(k)$, $B(k)$, i.e. Eqs. (4.6), we find

$$fe^{-if_2T}B\left(\frac{1}{k}\right)+e^{if_2T}\overline{A\left(\frac{1}{\bar{k}}\right)}=e^{-if_2T}(A(k)+fB(k)),$$

$$e^{-if_2T}B\left(\frac{1}{k}\right)+fe^{if_2T}\overline{A\left(\frac{1}{\bar{k}}\right)}=e^{if_2T}(f\overline{A(\bar{k})}+\rho\overline{B(\bar{k})}.$$

Solving the first equation for $B(1/k)$, taking the complex conjugate of the second equation and then solving the resulting equation for $A(1/\bar{k})$ we obtain (using $\overline{f(\bar{k})}=\rho f(k)$),

$$\begin{aligned}
B\left(\frac{1}{k}\right)&=-\frac{e^{2if_2T}}{f}\overline{A\left(\frac{1}{\bar{k}}\right)}+\frac{A(k)}{f}+B(k),\\
A\left(\frac{1}{k}\right)&=-\rho\frac{e^{2if_2T}}{f}\overline{B\left(\frac{1}{\bar{k}}\right)}+A(k)+\frac{B(k)}{f}.
\end{aligned}\tag{4.18}$$

Letting $k\to\frac{1}{k}$ in the definition of $\overline{d(\bar{k})}$ and then replacing $A(\frac{1}{k})$ and $B(\frac{1}{k})$ in the resulting expression by the rhs of Eqs. (4.18) we find

$$\begin{aligned}
&\left(\overline{a(1/\bar{k})}-\frac{\rho}{f}\overline{b(1/\bar{k})}\right)A(k)+\left(-\rho\overline{b(1/\bar{k})}+\frac{1}{f}\overline{a(1/\bar{k})}\right)B(k)\\
&\quad=\overline{d(1/\bar{k})}+\frac{\rho}{f}\overline{c(1/\bar{k})},\quad k\in D_1.
\end{aligned}\tag{4.19}$$

For the derivation of this equation we have used the remarkable fact that the terms $\exp[2if_2T]$, $\bar{A}$, and $\bar{B}$ are eliminated because of the global relation (3.17).

Solving Eq. (4.19) and the global relation (which is also valid for $k \in D_1$) for $A(k)$ and $B(k)$, we find

$$A(k) = \frac{a(k)\left(\overline{d\left(\frac{1}{\bar{k}}\right)} + \frac{\rho}{f}\overline{c\left(\frac{1}{\bar{k}}\right)}\right)}{\Delta(k)} + \frac{e^{2if_2T}\left(\rho\overline{b\left(\frac{1}{\bar{k}}\right)} - \frac{1}{f}\overline{a\left(\frac{1}{\bar{k}}\right)}\right)c(k)}{\Delta(k)},$$
$$B(k) = \frac{b(k)\left(\overline{d\left(\frac{1}{\bar{k}}\right)} + \frac{\rho}{f}\overline{c\left(\frac{1}{\bar{k}}\right)}\right)}{\Delta(k)} + \frac{e^{2if_2T}\left(\overline{a\left(\frac{1}{\bar{k}}\right)} - \frac{\rho}{f}\overline{b\left(\frac{1}{\bar{k}}\right)}\right)c(k)}{\Delta(k)}, \quad k \in D_1, \tag{4.20}$$

where

$$\Delta(k) = a(k)\overline{a\left(\frac{1}{\bar{k}}\right)} - \rho b(k)\overline{b\left(\frac{1}{\bar{k}}\right)} + \frac{1}{f}\left(-\rho a(k)\overline{b\left(\frac{1}{\bar{k}}\right)} + b(k)\overline{a\left(\frac{1}{\bar{k}}\right)}\right).$$

In order to obtain $A(k)$ and $B(k)$ for $k \in D_3$ we let $k \to \frac{1}{k}$ in the global relation and then replace $A(1/k)$ and $B(1/k)$ in the resulting expression by the rhs of Eqs. (4.18),

$$\left(\frac{1}{f}a\left(\frac{1}{k}\right) - b\left(\frac{1}{k}\right)\right)A(k) + \left(a\left(\frac{1}{k}\right) - \frac{1}{f}b\left(\frac{1}{k}\right)\right)B(k)$$
$$= e^{2if_2T}\left(c\left(\frac{1}{k}\right) + \frac{1}{f}d\left(\frac{1}{k}\right)\right), \quad k \in D_3. \tag{4.21}$$

This time the terms $\bar{A}$ and $\bar{B}$ are eliminated using the definition of $d(\frac{1}{k})$.

Solving Eq. (4.21) and the equation defining $\overline{d(\bar{k})}$ (which is also valid for $k \in D_3$) we find

$$A(k) = \frac{\left(a\left(\frac{1}{k}\right) - \frac{1}{f}b\left(\frac{1}{k}\right)\right)\overline{d(\bar{k})}}{\overline{\Delta(\bar{k})}} + \frac{\rho e^{2if_2T}\overline{b(\bar{k})}\left(c\left(\frac{1}{k}\right) + \frac{1}{f}d\left(\frac{1}{k}\right)\right)}{\overline{\Delta(\bar{k})}},$$
$$B(k) = \frac{\left(b\left(\frac{1}{k}\right) - \frac{1}{f}a\left(\frac{1}{k}\right)\right)\overline{d(\bar{k})}}{\overline{\Delta(\bar{k})}} + \frac{e^{2if_2T}\overline{a(\bar{k})}\left(c\left(\frac{1}{k}\right) + \frac{1}{f}d\left(\frac{1}{k}\right)\right)}{\overline{\Delta(\bar{k})}}, \quad k \in D_3. \tag{4.22}$$

Following arguments very similar with those used in Sect. 4.1, it can be shown that $A(k)$, $B(k)$ can be replaced by the expressions obtained from the rhs of Eqs. (4.20) and (4.22) after deleting the terms involving $\exp(2f_2T)$. Then the ratios B/A yield Eqs. (1.25).

4.3. The KdV. The determinant of the matrix U equals

$$(k + 4k^3)^2 - \left[g_1(t)^2 + (2 + 4g_0(t))V(t) + 4k^2(g_0^2(t) + 2V(t))\right],$$
$$V(t) = g_0^2 + \frac{1}{2}g_0 - \frac{1}{2}g_2. \tag{4.23}$$

The condition that the coefficient of k^2 vanishes yields

$$3g_0^2(t) + g_0(t) - g_2(t) = 0. \tag{4.24}$$

A particular case of boundary conditions satisfying this condition is given by Eqs. (1.23). The invariance of $f_2(k)$ yields

$$\nu^2 + k\nu + k^2 + \frac{1}{4} = 0. \tag{4.25}$$

We first discuss the case of $\chi = 0$, i.e. $g_0 = g_2 = 0$. In this case the matrix $if_2\sigma_3 - \tilde{Q}(t,k)$ depends on k only through $f_2(k)$, thus $N = I$, and

$$A(k) = A(\nu(k)), \quad B(k) = B(\nu(k)), \quad k \in \mathbb{C}. \tag{4.26}$$

These equations following the arguments used in Sect. 4.1 imply equations (1.25).

If $\chi \neq 0$, we let $N_3 = N_2$, $N_4 = N_1$, then both the (11) and the (22) elements of Eq. (4.4) yield Eq. (4.27a) below, while both the (12) and (21) elements of Eq. (4.4) yield (4.27b),

$$(\nu - k)VN_1 = (\nu + k)(V + 2\chi k\nu)N_2, \tag{4.27a}$$

$$(\nu - k)(V - 2\chi k\nu)N_1 = [V(\nu + k) - 2k\nu(k + 4k^3)]N_2. \tag{4.27b}$$

These two equations are equivalent; indeed their ratio yields

$$\frac{k + 4k^3}{k + \nu} = \frac{2\chi^2}{V}k\nu,$$

and since the definition of V and the boundary conditions imply $V = -\chi^2/2$, the above equation becomes Eq. (4.25).

Since the form of N is the same as the one used in the sG, Eq. (4.5) implies that $M_1(\nu(k))$, $M_2(\nu(k))$ are given by the rhs of Eqs. (4.17), where $f(k) = N_1/N_2$ is now defined by

$$f(k) = \frac{\nu + k}{\nu - k}\left(1 - \frac{4k\nu}{\chi}\right). \tag{4.28}$$

Then the derivation of B/A is identical to that for the sG except that $1/k$ is replaced by $\nu(k)$ which is defined by Eq. (4.25).

5. A Nonlinear Volterra Integral Equation

The functions $A(k)$, $B(k)$ are defined by Eqs. (3.10),

$$A(k) = \overline{\Phi_2(T, \bar{k})}, \quad B(k) = -e^{2if_2(k)T}\Phi_1(T,k), \tag{5.1}$$

where the vector $\Phi(t,k) = (\Phi_1, \Phi_2)^\tau$ satisfies Eqs. (3.11), and is uniquely defined in terms of $\{g_l(t)\}_0^{n-1}$.

For the analysis of the global relation associated with the NLS is convenient to introduce the functions $\varphi(t,k)$, $\psi(t,k)$ and $f_0(t)$, $f_1(t)$, instead of the functions $\Phi_1(t,k)$, $\Phi_2(t,k)$ and $g_0(t)$, $g_1(t)$:

$$\varphi(t,k) = e^{4ik^2t + i\lambda\int_0^t |g_0(\tau)|^2 d\tau}\Phi_1(t,k), \quad \psi(t,k) = e^{4ik^2t - i\lambda\int_0^t |g_0|^2\tau}\Phi_2(t,k), \tag{5.2}$$

$$f_0(t) = g_0(t)e^{2i\lambda\int_0^t |g_0(\tau)|^2 d\tau}, \quad f_1(t) = g_1(t)e^{2i\lambda\int_0^t |g_0(\tau)|^2 d\tau}. \tag{5.3}$$

Using these notations it follows that Eq. (3.11) can be written as the following system of linear Volterra integral equations:

$$\varphi(t,k) = \int_0^t [2kf_0(t') + if_1(t')]\psi(t',k)dt', \tag{5.4a}$$

$$\psi(t,k) = e^{4ik^2t} + \lambda\int_0^t e^{4ik^2(t-t')}[2k\overline{f_0(t')} - i\overline{f_1(t')}]\varphi(t',k)dt'. \tag{5.4b}$$

The definitions (5.1) and the notations (5.2) imply that the global relation (3.17) can be written in the form

$$a(k)\varphi(T,k) + b(k)e^{4ik^2T}\overline{\psi(T,\bar{k})} = -e^{i\lambda\int_0^T |q_0|^2 d\tau}e^{4ik^2T}c(k).$$

Multiplying this equation by $4ke^{-4ik^2t}/\pi$ and integrating around ∂D_1, i.e. the first quadrant of the complex k-plane, we find

$$\frac{1}{2\pi}\int_{\partial D_1} 8ke^{-4ik^2t}\left(a(k)\varphi(T,k) + b(k)e^{4ik^2T}\overline{\psi(T,\bar{k})}\right)dk = 0. \tag{5.5}$$

Equations (5.4) and (5.5) can be used to obtain the missing boundary value. In order to illustrate the basic ideas with a minimum of algebra we consider the particular case of zero initial data and Dirichlet boundary conditions, i.e.

$$q_0(x) = 0, \quad q(0,t) = g_0(t). \tag{5.6}$$

We assume that $g_0(0) = 0$ and that $g_0(t)$ is a smooth function in $(0,T)$.

In this case $a = 1$ and $b = 0$, thus replacing $\varphi(T,k)$ in Eq. (5.5) by the rhs of Eq. (5.4a) we find

$$\frac{1}{2\pi}\int_{\partial D_1} 8ke^{-4ik^2t}\left(\int_0^T (2kf_0(t') + if_1(t'))\psi(t',k)dt'\right)dk = 0. \tag{5.7}$$

The most tedious step in the analysis of Eqs. (5.4) and (5.7) involves the rigorous estimates of the large k behavior of $\psi(t,k)$,

$$\psi(t,k) = e^{4ik^2t}\left(e^{-i\lambda\int_0^t |q_0(\tau)|^2 d\tau} + \sum_1^3 \frac{\chi_j(t)}{k^j}\right) + \psi_4(t,k). \tag{5.8}$$

The derivation of appropriate estimates for $\chi_j(t)$ and $\psi_4(t,k)$ in terms of the $\mathbf{H}^1$ norm of f_0 and f_1 is based entirely on the analysis of the linear equations (5.4) and uses the investigation of certain linear Volterra integral equations satisfied by these functions, see Sect. 5 of [11].

Writing the term $\psi(t,k)$ in Eq. (5.7) in the form

$$\psi(t,k) = \left(\psi(t,k) - e^{4ik^2t - i\lambda\int_0^t |q_0(\tau)|^2 d\tau}\right) + e^{4ik^2t - i\lambda\int_0^t |q_0(\tau)|^2 d\tau},$$

and using the inverse Fourier transform to compute the term

$$ke^{4ik^2(t'-t)}\left(f_1(t')e^{-i\lambda\int_0^{t'} |q_0(\tau)|^2 d\tau}\right)$$

we find

$$f_1(t)e^{-i\lambda\int_0^t |g_0(\tau)|^2 d\tau} = \frac{8}{\pi}\int_{\partial D_1} k^2 \left(\int_0^T e^{4ik^2(t'-t)} f_0(t')e^{-i\lambda\int_0^{t'}|g_0(\tau)|^2 d\tau} dt'\right) dk$$
$$+\frac{1}{2\pi}\int_{\partial D_1} 8ke^{-4ik^2 t}\left\{\int_0^T \left[2kf_0(t') + if_1(t')\right]\right.$$
$$\left.\times\left[\psi(t',k) - e^{4ik^2t'}e^{-i\lambda\int_0^{t'}|q_0(\tau)|^2 d\tau}\right]dt'\right\}dk. \tag{5.9}$$

The first term of the rhs of this equation is known; using Eq. (5.8), it follows that the second term involves $f_1(t)$ as well as the functions $\chi_j(t)$, $j = 1, 2, 3$ and ψ_4. Having apriori estimates of these functions in terms of $f_1(t)$, it is possible to establish the existence of $f_1(t)$ for T sufficiently small, using a standard fixed point argument. The details of this analysis can be found in Sect. 5 of [11].

The above analysis establishes the existence and uniqueness of $g_1(t)$ on $(0, T_*)$, where T_* is a sufficiently small positive number. In order to extend g_1 beyond T_*, we write the global relation at $T_* + \tilde{T}$, and split the relevant integrals from 0 to T_* and from T_* to $T_* + \tilde{T}$.

The integrals involving the first term of the rhs of Eq. (5.9) as well as the integral from 0 to T_* of the second term of the rhs of Eq. (5.9) are known. Hence, following exactly the same analysis as for Eq. (5.9) it follows that $g_1(t)$ exists for $T_* < t < T_* + \tilde{T}$. Also using the analyticity properties of $\psi(t,k)$ it follows that for $0 < t < T_*$ the integrals from T_* to $T_* + \tilde{T}$ vanish and hence the equation becomes precisely Eq. (5.9) (which has a solution for $0 < t < T_*$). These facts together imply existence and uniqueness for $0 < t < T_* + \tilde{T}$.

In summary, if $q_0(x) = 0$, $g_1(t)$ is sufficiently smooth in $(0, T)$ and $g_1(0) = 0$, then $g_1(t)$ exists, is unique and is smooth in $(0, T)$.

The extension of this result to the case that $q_0(x) \neq 0$, follows precisely the same logical steps; the only difference is that the fixed-point theorem proof of the existence of the solution of the analogue of Eq. (5.9) is slightly more complicated.

In order to extend this result to other integrable nonlinear PDEs, such as the sG and the KdV, one must first obtain rigorous estimates for the large k behavior of the eigenfunction satisfying the t-part of the Lax pair. Actually the derivation of these estimates is part of the investigation of the "direct problem" of this eigenvalue equation. Although such eigenvalue equations have already been used in scattering theory, to our knowledge the derivation of these estimates has not been carried out. This is in principle possible, but the actual derivation can be cumbersome.

6. Conclusions

After many years of intense investigation it appears that there exists now an elegant, rigorous method for solving the half line problem for integrable nonlinear PDE's. An effort has been made to present this new method in a form that will be accessible to a wide audience. This is important, since it is hoped that researchers will consider using this method to solve a large class of physically important boundary value problems which remain open.

We now discuss some of the features of this method as well as its impact on other areas of mathematics such as the theory of linear elliptic PDEs and the study of the Ehrenpreis principle.

(1) The method is simple to implement. Indeed, both the construction of the *basic RH problem* as well as the derivation of the *global algebraic relation* follow from the existence of the exact differential 1-form $W(x,t,k)$. The fundamental properties of an exact form $W(x,t,k)$ are the existence of a 0-form, and the vanishing of the integral of W around a closed contour. The spectral analysis of the associate 0-form gives rise to the basic RH problem, while the vanishing of the integral of W around the boundary of the domain gives rise to the global relation.

(2) The "jump matrix" of the RH problem has *explicit* x, t dependence of the form $\exp[if_1(k)x + if_2(k)t]$, and it depends on the scalar functions $\{a(k), b(k), A(k), B(k)\}$. This means that the associated expression for $q(x,t)$ provides the proper *spectral representation* of the solution. This representation involves the direct and the inverse map between the values of $q(x,t)$ on the boundary, i.e. $\{q(x,0), \{\partial_x^l q(0,t)\}_0^{n-1}\}$ and the spectral functions $\{a(k), b(k), A(k), B(k)\}$. We emphasize that for a proper spectral decomposition (since the values of $q(x,t)$ on the boundary are functions of one variable only) the spectral functions must be functions of only one variable.

(3) Precisely because the solution is given in the above spectral representation form, it is possible to study effectively the asymptotic properties of the solution, such as its long t behavior. For the NLS, sG and KdV equations on the half line this has been done in [15, 16, 18], respectively. The relevant analysis is based on the basic RH problem and on the Deift-Zhou method [19–20]. The latter method is an elegant nonlinearization of the steepest descent method and it yields rigorous asymptotic results for RH problems with exponential x, t dependence. In our opinion this result is one of the most important developments in the theory of integrable systems in particular and in the theory of RH problems in general, thus it is quite satisfying that the new method gives rise to RH problems precisely of the type that can be analyzed by the Deift-Zhou method. We also note that recently a highly nontrivial generalization of the Deift-Zhou method has been developed which is able to analyze the zero-dispersion limit of the Cauchy problem on the line [21]. Since this method is also based on the analysis of a RH problem with exponential x, t dependence, we expect that the method of [21] applied to our RH problem will yield an effective description of the zero dispersion limit of initial-boundary value problems on the half-line.

(4) The new method is not only able to identify the linearizable class of boundary conditions, but what is more important, it is able to solve this class of boundary value problems as effectively as the usual class of initial value problems on the line.

(5) It is the author's opinion that the most remarkable fact about boundary value problems for integrable nonlinear PDEs is the *simplicity of the global algebraic relation*. Indeed, although the relation between the initial and the boundary values of q is very complicated, this relation takes a simple algebraic form in the k-space, see Eq. (3.17). The simplicity of the global relation has two important consequences: (a) Under the assumption that there exist spectral functions satisfying this relation, it is straightforward to prove that the associated $q(x,t)$ exists, satisfies the given nonlinear PDE, and $q(x,0) = q_0(x)$, $\{\partial_x^l q(0,t) = g_l(t)\}_0^{n-1}$. (b) Given initial conditions and a subset of $\{g_l(t)\}_0^{n-1}$ it is possible to prove the *global* existence of the remaining part of this set. We emphasize that the global relation is a simple algebraic relation between the two components of an eigensolution of the t-part of the Lax pair evaluated at $x = 0$. Thus since these components satisfy a *linear* eigenvalue equation, the derivation of appropriate estimates for their large k behavior is based on the analysis of a linear problem. Thus, although the global relation is a nonlinear equation its rigorous investigation involves mostly the analysis of a linear equation.

(6) The linear limit of the method yields a *new* method for solving the half line problem for linear evolution equations [10]. This method appears to be the best existing method for solving linear boundary value equations, and is able to solve half line problems as effectively as the Cauchy problem on the line. There exist two basic differences between the linear and the nonlinear problems: (a) The RH problem in the linear case is additive and hence can be solved in closed form. Thus $q(x,t)$ (rather than be given through the solution of a matrix RH problem) can be expressed through an explicit integral involving $\exp[if_1(k)x + if_2(k)t]$ and the spectral functions $\hat{q}_0(k)$ and $\hat{Q}(k)$. These functions are the linear limit of $b(k)$ and of $B(k)$ respectively (the linear limits of $a(k)$ and $A(k)$ are unity). (b) The global relation can always be solved using algebraic manipulations. This is a consequence of the fact that the unknown part of $\hat{Q}(k)$ is *invariant* under the transformation $k \to \nu(k)$. Unfortunately in the nonlinear case $\{A(k), B(k)\}$ involve $M(t,k)$ which in general is *not* invariant under this transformation. The linearizable cases are precisely those cases for which it is possible to find a simple relation between $M(t, \nu(k))$ and $M(t,k)$, see Eq. (4.5).

The new method for solving linear boundary value problems is also able to analyze linear elliptic boundary value problems in an arbitrary convex polygon with n sides [22–26]. The global relation again plays a crucial role, but for general elliptic equations this equation cannot be solved algebraically but *can* be formulated as an $(n-1)$-matrix RH problem [26]. For very simple polygons and simple boundary conditions this problem degenerates into a Wiener-Hopf factorization problem, which explains the ubiquitous role played by the latter problem in previous works. This shows that ideas from the theory of integrable equations have led to a completely new and powerful method for solving boundary value problems for linear elliptic PDEs. Furthermore, it is interesting that this new method gives rise to a new numerical method for solving linear elliptic boundary value problems (this method is based on the numerical solution of the global relation [27]).

(7) The expression of $q(x,y)$ for linear equations has explicit exponential x, y dependence consistent with the Euler-Palamodov-Ehrenpreis [28–30] representation. The expression of $q(x,t)$ for nonlinear equations involves a RH problem whose jump matrix has an explicit exponential x, t dependence. Thus the new method provides the concrete implementation as well as the nonlinearization of this fundamental representation. It is quite interesting that ideas from the theory of integrable equations have an impact on this important field of mathematics.

(8) In recent years there have been important developments in the analysis of boundary value problems of nonlinear PDEs using PDE techniques [31, 32]. It is remarkable that some of these techniques yield *global* results. It is satisfying that there exist now a rigorous theory using the integrability machinery, so that it is possible to make comparisons between these different approaches. Although at the moment the PDE results are proven in less restrictive functional spaces, the advantage of our method is that it yields rigorous asymptotic results. We reiterate that this is a consequence of the Deift-Zhou theory and of our simple RH problem.

We conclude with some historical remarks concerning integrable nonlinear PDEs on the half-line. The first such problem to be solved was the NLS with either $q(0,t) = 0$ or $q_x(0,t) = 0$ in the work of Ablowitz and Segur [33]. These authors made crucial use of an even and an odd extension to the full line. We recall that in the linear case the inhomogeneous versions of these problems can be solved by either the sine or the cosine transform. This motivated the author to construct a nonlinearization of these transforms [34]. However, this effort, in our opinion, was a failure. Indeed, the associated "nonlinear sine" and "nonlinear cosine" transforms denoted by $B(t, k)$ satisfy a highly nonlinear nonlocal equation, which becomes linear *only* in the particular cases of either $q(0, t) = 0$ or $q_x(0, t) = 0$. The situation is similar for the nonlinearization of the transform associated with the boundary condition $q_x(0, t) - \chi q(0, t) = f(t)$, only the case $f(t) = 0$ can be solved using this approach. We emphasize that since $B(t, k)$ depends on time, it is not possible to obtain long time results, furthermore since the equation for $B(t, k)$ is highly complicated the task of obtaining rigorous results is prohibitingly complicated. Other works devoted to linearizable cases include [35–38].

The new method is the result of several developments: It was first realized by the author [39] that, in addition to analyzing the x-part of the Lax pair, it is now necessary to analyze the t-part of the Lax pair; this yields $q(x, t)$ in terms of *two* RH problems. These two problems were combined into *one* basic RH problem in the works of A.R. Its and the author [15, 16, 18]. These basic RH problems for the NLS and the sG equations are identical to the ones presented in Theorem 3.1. This made it possible, using the Deift-Zhou theory, to obtain long time asymptotic results. However, since the global relation was not analyzed at that time, these results were based on the apriori assumption of existence; furthermore the derivation of the basic RH problem was based on the *separate* spectral analysis of the x and t parts of the Lax pair and was very complicated. This derivation was greatly simplified in [9] using the *simultaneous* spectral analysis of the Lax pair. This derivation was further simplified in [8] where it was realized that the best way to implement the simultaneous spectral analysis is to use the formulation of the exact 1-form presented in Sect. 2. Furthermore, this formulation yields the global algebraic relation in a straightforward manner, see Sect. 2.5. The proof that the global relation is not only a necessary but also a sufficient condition for existence, as well as the rigorous investigation of the global relation was presented in [11].

Several authors have identified linearizable boundary conditions using the existence of infinitely many symmetries and conservation laws, see for example [40–42].

The important gauge transformation (4.5) for the analysis of the linearizable cases of the NLS was first used in [43]. However, the global relation was *not* used in [43], hence an attempt was made to compute the unknown boundary values instead of the unknown spectral functions. Thus instead of the algebraic manipulation used in Sect. 4.1, the approach of [43] involves the formulation of several formal matrix RH problems whose solution is *not* established.

The physical significance of the sG equation with $q(0, t) = \chi$ as well as several approaches for the analysis of this problem can be found in [44–46].

A rigorous investigation of the spectral functions $\{a, b, A, B\}$ for the NLS equation is presented in [47].

The extension of this method to linear and integrable nonlinear PDEs on the finite integral is presented in [48] and [49] respectively. Its extension to moving boundary value problems is presented in [50] and [51].

An interesting alternative approach to boundary value problems for integrable nonlinear evolution equations was recently introduced by Sabatier [52].

Remark 6.1. In this paper all relevant formulae are given in terms of the basic domains $D_1, \cdots, D_4$ defined in Eqs. (1.16). These domains are defined explicitly in terms of the functions $f_1(k)$ and $f_2(k)$ appearing in the Lax pair. This, we hope, will make it convenient for other researchers to apply this method to other nonlinear PDE's since these basic domains will be immediately known.

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Appendix A

A.1. The eigenfunctions associated with the sG equation as $k \to 0$. The functions $\{a(k), b(k)\}$ are defined in terms of $\varphi(0, k)$, see Eq. (3.2). The vector $\varphi(x, k)$ is the second column vector of the matrix $\mu_3(x, k)$ (see Eq. (3.7)) which satisfies the ODE (3.8). We will show that

$$\mu_3(x, k) = (-1)^m \left[\cos\left(\frac{q_0(x)}{2}\right) I - i \sin\left(\frac{q_0(x)}{2}\right) \sigma_1 + o(1) \right], \quad k \to 0, \quad \text{(A.1)}$$

where $q_0(x) \to 2\pi m$ as $x \to \infty$, and $I = \text{diag}(1, 1)$.

Indeed, let

$$\mu_3(x, k) = \psi(x, k) E(x, k), \quad E(x, k) = e^{\frac{i}{4} x (k - \frac{1}{k}) \sigma_3}.$$

Then $\psi(x, k)$ satisfies

$$\psi_x + \frac{i}{4}\left(k - \frac{1}{k}\right) \sigma_3 \psi = Q(x, k), \quad \lim_{x \to \infty} \psi(x, k) E(x, k) = I,$$

where $Q(x, k)$ is defined in (3.4c). Thus

$$\psi_x = \frac{i}{4k}\left[\cos(q_0(x))\, \sigma_3 - \sin(q_0(x))\, \sigma_2\right] \psi + \mathcal{O}(1), \quad k \to 0.$$

Noting that

$$\cos(q_0(x))\, \sigma_3 - \sin(q_0(x))\, \sigma_2 = f \sigma_3 f^{-1}, \quad f = \cos\left(\frac{q_0(x)}{2}\right) I - i \sin\left(\frac{q_0(x)}{2}\right) \sigma_1,$$

it follows that

$$(f^{-1}\psi)_x = \frac{i}{4k} \sigma_3 (f^{-1}\psi) + \mathcal{O}(1), \quad k \to 0.$$

Solving this equation and using the boundary condition

$$f^{-1}\psi \to (-1)^m \exp\left[\frac{i}{4k} x \sigma_3\right],$$

we find

$$\psi(x, k) = (-1)^m \left[\cos\left(\frac{q_0(x)}{2}\right) I - i \sin\left(\frac{q_0(x)}{2}\right) \sigma_1 + o(1) \right] e^{\frac{i}{4k} x \sigma_3}, \quad k \to 0,$$

which yields Eq. (A.1).

The functions $\{A(k), B(k)\}$ are defined in terms of $\Phi(T, k)$, see Eq. (3.10). The vector $\Phi(T, k)$ is the second column vector of the matrix $\mu_2(t, k)$ (see Eq. (3.15)) which satisfies the ODE (3.16). We will show that

$$\mu_2(t, k) = \left[\cos\left(\frac{g_0(t)}{2}\right) I - i \sin\left(\frac{g_0(t)}{2}\right) \sigma_1 + o(1) \right] e^{-\frac{i}{4k} t \hat{\sigma}_3} \tilde{f}_0^{-1},$$

$$\tilde{f}_0 = \cos\left(\frac{g_0(0)}{2}\right) I - i \sin\left(\frac{g_0(0)}{2}\right) \sigma_1. \quad \text{(A.2)}$$

Indeed, let

$$\mu_2(t,k) = \Psi(t,k)\tilde{E}(t,k), \quad \tilde{E}(t,k) = e^{\frac{i}{4}(k+\frac{1}{k})t\sigma_3}.$$

Then $\Psi(t,k)$ satisfies

$$\Psi_t + \frac{i}{4}\left(k + \frac{1}{k}\right)\sigma_3\Psi = \tilde{Q}(t,k), \quad \Psi(0,k) = I,$$

where $\tilde{Q}(t,k)$ is defined in (3.12). Thus

$$\Psi_t = \frac{i}{4k}\left[\cos(g_0(t))\,\sigma_3 - \sin(g_0(t))\,\sigma_2\right]\Psi + \mathcal{O}(1), \quad k \to 0.$$

The bracket in the above equation can be written as $\tilde{f}\sigma_3\tilde{f}^{-1}$,

$$\tilde{f} = \cos\left(\frac{g_0(t)}{2}\right) I - i\sin\left(\frac{g_0(t)}{2}\right)\sigma_1,$$

thus

$$(\tilde{f}^{-1}\Psi)_t = \frac{i}{4k}\sigma_3(\tilde{f}^{-1}\Psi) + \mathcal{O}(1), \quad k \to 0.$$

This equation together with the boundary condition $(\tilde{f}^{-1}\Psi)(0,k) = \tilde{f}_0^{-1}$ yields (A.2).

A.2. The eigenfunctions associated with the KdV equation as $k \to 0$. Let $\mu(x,t,k)$ satisfy Eq. (1.5), where $f_1(k)$, $f_2(k)$, $Q(x,t,k)$, $\tilde{Q}(x,t,k)$ are defined by Eqs. (2.2). Then

$$\mu(x,t,k) = i\frac{\alpha(x,t)}{k}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \mathcal{O}(1), \quad k \to 0, \quad \alpha(x,t) \quad \text{real}. \tag{A.3}$$

Indeed, the coefficient of $1/k$ in both Eqs. (1.5) involves the matrix $\sigma_2 - i\sigma_3$. This suggests that

$$\mu(x,t,k) = \frac{1}{k}\begin{pmatrix} \alpha_1(x,t) & \alpha_2(x,t) \\ -\alpha_1(x,t) & -\alpha_2(x,t) \end{pmatrix} + \mathcal{O}(1), \quad k \to 0.$$

The symmetry condition with respect to $k \mapsto -k$ (see Sect. 2.2) implies that $\alpha_2(x,t) = \alpha_1(x,t)$. Furthermore, the symmetry condition with respect to complex conjugation (see Eqs. (2.9)) imply that $\alpha_1(x,t)$ is purely imaginary.

Equation (A.3) suggests that

$$\mu_3(x,k) = i\frac{\alpha(x)}{k}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \mathcal{O}(1), \quad k \to 0, \quad \alpha(x) \quad \text{real}, \tag{A.4}$$

and

$$\mu_2(t,k) = i\frac{\beta(t)}{k}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \mathcal{O}(1), \quad k \to 0, \quad \beta(t) \quad \text{real}. \tag{A.5}$$

These equations can be rigorously justified using the associated linear integral equations. The evaluation of Eqs. (A.4) and (A.5) at $x = 0$ and $t = T$ determines the behavior of $\{a(k), b(k)\}$ and of $\{A(k), B(k)\}$ as $k \to 0$.

The ODE (3.8) associated with the KdV equation is the time-independent Schrödinger equation. The scattering data $\{a(k), b(k)\}$ for this equation have been studied extensively in the literature; for a comprehensive recent review see [53].

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