Commun. Math. Phys. 230, 271 – 288 (2002)
Digital Object Identifier (DOI) 10.1007/s00220-002-0677-4

Communications in
Mathematical Physics

Local ν-Euler Derivations and Deligne's Characteristic Class of Fedosov Star Products and Star Products of Special Type

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Received: 28 June 1999 / Accepted: 11 April 2002
Published online: 11 September 2002 –

Dedicated to the memory of Moshé Flato

Abstract: In this paper we explicitly construct local ν-Euler derivations $\mathsf{E}_\alpha = \nu\partial_\nu + \mathcal{L}_{\xi_\alpha} + \mathsf{D}_\alpha$, where the ξ_α are local, conformally symplectic vector fields and the D_α are formal series of locally defined differential operators, for Fedosov star products on a symplectic manifold (M, ω) by means of which we are able to compute Deligne's characteristic class of these star products. We show that this class is given by $\frac{1}{\nu}[\omega] + \frac{1}{\nu}[\Omega]$, where $\Omega \in \nu Z^2_{\mathrm{dR}}(M)[[\nu]]$ is a formal series of closed two-forms on M the cohomology class of which coincides with the one introduced by Fedosov to classify his star products. Moreover, we consider star products that have additional algebraic structures and compute the effect of these structures on the corresponding characteristic classes of these star products. Specifying the constituents of Fedosov's construction we obtain star products with these special properties. Finally, we investigate equivalence transformations between such special star products and prove existence of equivalence transformations being compatible with the considered algebraic structures.

1. Introduction

Since the very beginning of deformation quantization in the pioneering articles [2] by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer there has been not only an immense interest in answering the question of existence of star products $\star$ (i.e. formal, associative deformations of the classical Poisson algebra of complex-valued functions $C^\infty(M)$ on a symplectic or, more generally, Poisson manifold M, such that in the first order of the formal parameter ν the commutator of the star product yields the Poisson bracket) positively, but also in finding a classification of the star product algebras up to isomorphy of algebras. Therefore the proofs of existence given by DeWilde and Lecomte [11,12], Fedosov [13,14] in the symplectic case and recently by Kontsevich [20] in the general case of a Poisson manifold always contained results on classification. Moreover, there have been several other results on classification up to equivalence by Nest and Tsygan [22,23], Bertelson, Cahen and Gutt [3], Weinstein and Xu [24]. Their common result is that every star product on a symplectic manifold is equivalent to a Fedosov star product. A comparison between the results of DeWilde, Lecomte and Fedosov is due to Halbout and can be found in [17]. In the case of deformation quantizations with separation of variables on Kähler manifolds Karabegov proved existence and gave a classification using a formal deformation of the Kähler form in [18, 19].

In his article [10] Deligne has introduced the notions of *intrinsic derivation-related* and *characteristic class* in order to compare the different constructions and classifications of DeWilde, Lecomte and Fedosov. In his paper Deligne uses the language of algebraic geometry to approach deformation theory and proves (cf. [10, Prop. 3.6.]) that the *relative class* $c(*) - c(*')$ of two Fedosov star products being the difference of the characteristic classes of two Fedosov star products $*$, $*'$ equals $\frac{1}{\nu}(F(*) - F(*'))$, where $F(*)$ denotes the cohomology class of the Weyl-curvature Fedosov introduced to classify his star products, that naturally arises when one constructs a star product using Fedosov's method.

Recently, Gutt and Rawnsley [15] gave an alternative approach to Deligne's various classes that avoids using methods of algebraic geometry. They also show how the classification of DeWilde and Lecomte fits into this framework (cf. [15, Sect. 7]). Using their methods we succeed in slightly generalizing Deligne's result in proving that Deligne's characteristic class equals $\frac{1}{\nu}$ times the cohomology class of the Weyl-curvature. We should like to emphasise that our proof is purely algebraic and does not use any results on sheaf cohomology except for the de Rham isomorphism relating the second Čech cohomology with the second de Rham cohomology.

The interest in the relation between the characteristic class and the Fedosov class is also motivated by the occurrence of the latter in formulas for canonical traces resp. trace densities obtained by Halbout in [16] whose results are based on investigations of invariants in the cyclic cohomology of M made by Connes, Flato and Sternheimer [9]. Moreover, he has shown that the cocycle of Connes, Flato, Sternheimer which is an invariant for closed star products on a symplectic manifold can be expressed by $\widehat{A}(TM)$ and the Fedosov class.

The paper is organized as follows: After a brief summary of Fedosov's construction of star products on symplectic manifolds we close Sect. 2 by a short review of the definitions of Deligne's various classes. Section 3 constitutes the main part of our work, where we give an explicit construction of local ν-Euler derivations for an arbitrary Fedosov star product. After these preparations it is an easy task computing Deligne's derivation-related and characteristic class in Sect. 4. As an application of the properties of Deligne's characteristic class and its relation to Fedosov's Weyl-curvature we study star products of special type in Sect. 5 that satisfy special algebraic identities with respect to complex conjugation and the mapping $\nu \mapsto -\nu$ changing the sign of the formal parameter and compute the influence on the corresponding characteristic classes. Moreover, we can show that there are always Fedosov star products satisfying these special algebraic identities the characteristic class of which coincides with a suitably given element of $\frac{[\omega]}{\nu} + H^2_{\mathrm{dR}}(M)[[\nu]]$. Considering equivalent star products satisfying the same algebraic identities with respect to the mappings mentioned above we can show that there are always equivalence transformations between these star products commuting with these mappings. In Appendix A we give a short proof of the deformed Cartan formula that is of great value for our considerations in Sect. 4, but seems to be folklore. A further Appendix B is added for completeness giving the computation of the term of the characteristic class that cannot be determined from the algebraic considerations in Sect. 4.

2. Fedosov Star Products and Deligne's Characteristic Class

In this section we shall briefly recall Fedosov's construction of a star product for a given symplectic manifold (M, ω). The notation is mainly the same as in Fedosov's book [14] and in [13]. In addition we collect the definitions as they were introduced in [10] of *Deligne's intrinsic derivation-related class* and *Deligne's characteristic class* and the relations between them. For proofs and a detailed discussion of these topics the reader is referred to the exposition [15].

Let (M, ω) be a smooth symplectic manifold and define

$$\mathcal{W} \otimes \Lambda(M) := \left(\mathsf{X}_{s=0}^{\infty} \mathbb{C} \left(\Gamma^{\infty} \left(\bigvee^{s} T^{*}M \otimes \bigwedge T^{*}M \right) \right) \right) [[\nu]]. \tag{1}$$

If there is no possibility for confusion we simply write $\mathcal{W} \otimes \Lambda$ and denote by $\mathcal{W} \otimes \Lambda^{k}$ the elements of anti-symmetric degree k and set $\mathcal{W} := \mathcal{W} \otimes \Lambda^{0}$. For two elements $a, b \in \mathcal{W} \otimes \Lambda$ one defines their pointwise product denoted by $\mu(a \otimes b) = ab$ by the symmetric $\vee$-product in the first factor and the anti-symmetric $\wedge$-product in the second factor. Then the degree-maps $\deg_{s}$ and $\deg_{a}$ with respect to the symmetric and anti-symmetric degree are derivations of this product and $(\mathcal{W} \otimes \Lambda, \mu)$ is super-commutative with respect to the anti-symmetric degree. For a vector field X we define the symmetric substitution (insertion) $i_{s}(X)$ and the anti-symmetric substitution $i_{a}(X)$ which are super-derivations of symmetric degree -1 resp. 0 and anti-symmetric degree 0 resp. -1. Following Fedosov we define $\delta := (1 \otimes dx^{i}) i_{s}(\partial_{i})$ and $\delta^{*} := (dx^{i} \otimes 1) i_{a}(\partial_{i})$, where $x^{1}, \ldots, x^{n}$ are local coordinates for M and $\partial_{i} = \partial_{x^{i}}$ denotes the corresponding coordinate vector fields. For $a \in \mathcal{W} \otimes \Lambda$ with $\deg_{s} a = ka$ and $\deg_{a} a = la$ we define $\delta^{-1} a := \frac{1}{k+l} \delta^{*} a$ if $k + l \neq 0$ and $\delta^{-1} a := 0$ if $k + l = 0$. Clearly $\delta^{2} = \delta^{*2} = 0$. Moreover, we denote by $\sigma : \mathcal{W} \otimes \Lambda \to C^{\infty}(M)[[\nu]]$ the projection onto the part of symmetric and anti-symmetric degree 0. Then one has the following 'Hodge-decomposition' for any $a \in \mathcal{W} \otimes \Lambda$ (see e.g. [13, Eq. (2.8)]): $a = \delta \delta^{-1} a + \delta^{-1} \delta a + \sigma(a)$. Now we consider the fibrewise associative deformation $\circ$ of the pointwise product having the form

$$a \circ b = \mu \circ \exp \left(\frac{\nu}{2} \Lambda^{ij} i_{s}(\partial_{i}) \otimes i_{s}(\partial_{j}) \right) (a \otimes b), \tag{2}$$

where Λ^{ij} denotes the components of the Poisson tensor corresponding to the symplectic form ω. Moreover, we define $\deg_{a}$-graded super-commutators with respect to $\circ$ and set $\mathrm{ad}(a)b := [a, b]$. Now $\deg_{s}$ is no longer a derivation of the deformed product $\circ$ but $\mathrm{Deg} := \deg_{s} + 2\deg_{\nu}$ is still a derivation and hence the algebra $(\mathcal{W} \otimes \Lambda, \circ)$ is formally Deg-graded, where $\deg_{\nu} := \nu \partial_{\nu}$. We shall refer to this degree as total degree.

According to Fedosov's construction of a star product we consider a torsion free, symplectic connection ∇ on TM that extends in the usual way to a connection ∇ on $T^{*}M$ and symmetric resp. anti-symmetric products thereof. Using this connection we define the map $\nabla : \mathcal{W} \otimes \Lambda \to \mathcal{W} \otimes \Lambda$ by $\nabla := (1 \otimes dx^{i}) \nabla_{\partial_{i}}$. Then due to the property of the connection being symplectic ∇ turns out to be a super-derivation of anti-symmetric degree 1 and symmetric and total degree 0 of the fibrewise product $\circ$. Moreover $[\delta, \nabla] = 0$ since the connection is torsion free and ∇^{2} turns out to be an inner super-derivation $\nabla^{2} = -\frac{1}{\nu} \mathrm{ad}(R)$, where $R := \frac{1}{4} \omega_{it} R^{t}_{jkl} dx^{i} \vee dx^{j} \otimes dx^{k} \wedge dx^{l} \in \mathcal{W} \otimes \Lambda^{2}$ involves the curvature of the connection. Moreover, one has $\delta R = 0 = \nabla R$ as consequences of the Bianchi identities.

Now remember the following facts which are just restatements of Fedosov's original theorems in [13, Thm. 3.2, 3.3] resp. [14, Thm. 5.3.3]:

For all $\Omega = \sum_{i=1}^{\infty} \nu^i \Omega_i \in \nu Z^2_{\mathrm{dR}}(M)[[\nu]]$ and all $s \in \mathcal{W}$ of total degree ≥ 3 with $\sigma(s) = 0$ there exists a unique element $r \in \mathcal{W}\otimes\Lambda^1$ of total degree ≥ 2 such that

$$\delta r = R + \nabla r - \frac{1}{\nu} r \circ r + 1 \otimes \Omega \quad \text{and} \quad \delta^{-1} r = s. \tag{3}$$

Moreover r satisfies the formula

$$r = \delta s + \delta^{-1}\left(R + 1 \otimes \Omega + \nabla r - \frac{1}{\nu} r \circ r\right) \tag{4}$$

from which r can be determined recursively. In this case the Fedosov derivation

$$\mathcal{D} := -\delta + \nabla - \frac{1}{\nu}\mathrm{ad}(r) \tag{5}$$

is a super-derivation of anti-symmetric degree 1 and has square zero: $\mathcal{D}^2 = 0$.

Then for any $f \in C^\infty(M)[[\nu]]$ there exists a unique element $\tau(f) \in \ker(\mathcal{D}) \cap \mathcal{W}$ such that $\sigma(\tau(f)) = f$ and $\tau : C^\infty(M)[[\nu]] \to \ker(\mathcal{D}) \cap \mathcal{W}$ is $\mathbb{C}[[\nu]]$-linear and referred to as the Fedosov-Taylor series corresponding to $\mathcal{D}$. In addition $\tau(f)$ can be obtained recursively for $f \in C^\infty(M)$ from

$$\tau(f) = f + \delta^{-1}\left(\nabla\tau(f) - \frac{1}{\nu}\mathrm{ad}(r)\tau(f)\right). \tag{6}$$

Since $\mathcal{D}$ as constructed above is a $\circ$-super-derivation $\ker(\mathcal{D}) \cap \mathcal{W}$ is a $\circ$-sub-algebra and a new associative product $*$ for $C^\infty(M)[[\nu]]$ is defined by pull-back of $\circ$ via τ, which turns out to be a star product.

Observe that in (3) we allowed for an arbitrary element $s \in \mathcal{W}$ with $\sigma(s) = 0$ that contains no terms of total degree lower than 3, as normalization condition for r, i.e. $\delta^{-1} r = s$ instead of the usually used equation $\delta^{-1} r = 0$. In the sequel we shall especially show that this more general normalization condition does not affect the equivalence class of the resulting star product. In the following we shall refer to the associative product $*$ defined above as the Fedosov star product. Moreover, we shall denote by $F(*)$ Fedosov's characteristic class of the star product $*$ as discussed in [14, Sect. 5.3] which is given by $F(*) = [\omega] + [\Omega]$.

Next we collect some basic concepts of characteristic classes for star products as they can be found in [10,15]. Deligne's characteristic class $c(\star)$ of a star product has been introduced in [10] and classifies in a functorial way the equivalence classes of star products on a symplectic manifold (M, ω). It lies in the affine space $\frac{[\omega]}{\nu} + H^2_{\mathrm{dR}}(M)[[\nu]]$ and can be calculated by methods of Čech cohomology. Let us provide some details of the calculation as far as they are needed for our purposes. At this instance we should mention that our conventions, that are as in [1], differ from those used in [15] by a sign in the Poisson bracket causing the positive sign in front of $\frac{[\omega]}{\nu}$ in $c(\star)$.

If $\star$ is a star product on the symplectic manifold (M, ω) there exists a good open cover $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ of M (i.e. all finite intersections of the $\mathcal{U}_\alpha$ are contractible) together with a family $\{\mathsf{E}_\alpha\}_{\alpha\in I}$ of local ν-*Euler derivations* of $(C^\infty(\mathcal{U}_\alpha)[[\nu]], \star)$ i.e. a family of derivations E_α of $\star$ over $\mathcal{U}_\alpha$ having the form

$$\mathsf{E}_\alpha = \nu\partial_\nu + \mathcal{L}_{\xi_\alpha} + \mathsf{D}_\alpha, \tag{7}$$

where ξ_α is conformally symplectic ($\mathcal{L}_{\xi_\alpha}\omega\big|_{\mathcal{U}_\alpha} = \omega|_{\mathcal{U}_\alpha}$) and $\mathsf{D}_\alpha = \sum_{i=1}^{\infty} \nu^i \mathsf{D}_{\alpha,i}$ is a formal series of differential operators over $\mathcal{U}_\alpha$. The existence of such ν-Euler derivations has already been shown in [15] using cohomological methods, whereas in the case of a Fedosov star product we are going to give a very direct, purely algebraic proof of this fact in the next section since for our purposes we need a quite concrete formula for the differential operators D_α. As every ν-linear derivation over a contractible, open set $\mathcal{U}$ is of the form $\frac{1}{\nu}\mathrm{ad}_\star(d)$ with $d \in C^\infty(\mathcal{U})[[\nu]]$ there exist formal functions $d_{\alpha\beta} \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)[[\nu]]$ fulfilling

$$\mathsf{E}_\alpha - \mathsf{E}_\beta = \frac{1}{\nu}\mathrm{ad}_\star(d_{\alpha\beta}) \tag{8}$$

over $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. This fact can also be seen directly from the results of the following two sections. Now, whenever $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \neq \emptyset$ the sums $d_{\alpha\beta\gamma} = d_{\beta\gamma} - d_{\alpha\gamma} + d_{\alpha\beta}$ lie in $\mathbb{C}[[\nu]]$ and define a 2-cocycle whose Čech class $[d_{\alpha\beta\gamma}] \in H^2(M, \mathbb{C})[[\nu]]$ does not depend on the choices made and the corresponding class $d(\star) \in H^2_{\mathrm{dR}}(M)[[\nu]]$ is called *Deligne's intrinsic derivation-related class.*

Definition 1 (cf. [15, Def. 6.3])**.** *Deligne's characteristic class $c(\star)$ of a star product $\star$ on (M, ω) is the element $c(\star) = \frac{[\omega]}{\nu} + \sum_{i=0}^{\infty} \nu^i c(\star)^i$ of the affine space $\frac{[\omega]}{\nu} + H^2_{\mathrm{dR}}(M)[[\nu]]$ defined by*

$$c(\star)^0 = -2C_2^{-\sharp}, \qquad \partial_\nu c(\star) = \frac{1}{\nu^2} d(\star). \tag{9}$$

Hereby $C_2^{-\sharp}$ is the image under the projection onto the second part in the decomposition $H^2_{\mathrm{Chev,nc}}(C^\infty(M), C^\infty(M)) = \mathbb{C} \oplus H^2_{\mathrm{dR}}(M)$ of the second Chevalley cohomology (null on constants, on $(C^\infty(M), \{\,,\,\})$ with respect to the adjoint representation) of the anti-symmetric part $C_2^-(f, g) = \frac{1}{2}(C_2(f, g) - C_2(g, f))$ of the bidifferential operator C_2 in the expansion of $\star$ which is a 2-cocycle with respect to this cohomology by the Jacobi-identity for star commutators.

Remark 1. Notice that for Fedosov star products $*$ we have $C_1(f, g) = \frac{1}{2}\{f, g\}$ implying (cf. [15, Rem. 6.1]) that $C_2^-(f, g) = \rho_2(X_f, X_g)$ for a closed two-form ρ_2 on M, where X_f denotes the Hamiltonian vector field with respect to ω that corresponds to $f \in C^\infty(M)$, and hence $C_2^{-\sharp} = [\rho_2]$.

3. Explicit Construction of Local ν-Euler Derivations

To simplify the notation we use the convention that whenever an equation contains indices α, β, γ this means that it is valid on the intersection of the members of the good open cover whose indices occur in it. As a first step in the construction of local ν-Euler derivations we have to find local, conformally symplectic vector fields ξ_α. Since $d\omega = 0$ we can find one-forms θ_α on each $\mathcal{U}_\alpha$ such that $\omega = -d\theta_\alpha$ by the Poincaré lemma. Using these local one-forms we can define local vector fields ξ_α by $i_{\xi_\alpha}\omega = -\theta_\alpha$ that obviously satisfy $\mathcal{L}_{\xi_\alpha}\omega = \omega$. Using these vector fields we find the following lemma:

Lemma 1. *Let $\mathcal{H}_\alpha : \mathcal{W} \otimes \Lambda(\mathcal{U}_\alpha) \to \mathcal{W} \otimes \Lambda(\mathcal{U}_\alpha)$ be defined by*

$$\mathcal{H}_\alpha := \nu\partial_\nu + \mathcal{L}_{\xi_\alpha}, \tag{10}$$

then $\mathcal{H}_\alpha$ is a local (super-)derivation with respect to the fibrewise product $\circ$ of anti-symmetric and total degree 0*, i.e.*

$$\mathcal{H}_\alpha(a \circ b) = \mathcal{H}_\alpha a \circ b + a \circ \mathcal{H}_\alpha b \tag{11}$$

for all $a, b \in \mathcal{W} \otimes \Lambda(\mathcal{U}_\alpha)$. Moreover, we have $[\mathcal{L}_{\xi_\alpha}, \delta] = [\mathcal{L}_{\xi_\alpha}, \delta^*] = 0$ and $[\mathcal{H}_\alpha, \delta] = [\mathcal{H}_\alpha, \delta^*] = 0$.

Proof. The proof is a straightforward computation using that $\nu\partial_\nu$ as well as $\mathcal{L}_{\xi_\alpha}$ are derivations of the undeformed product μ and the equation $\mathcal{L}_{\xi_\alpha}\Lambda = -\Lambda$ which follows from $\mathcal{L}_{\xi_\alpha}\omega = \omega$. The commutation relations are obvious from the very definitions. □

At first sight it might be desirable to construct local derivations with respect to $*$ by restricting $\mathcal{H}_\alpha$ to $C^\infty(\mathcal{U}_\alpha)[[\nu]]$. In fact this can be done in some special cases where the connection ∇ is compatible with the Lie derivative with respect to the vector fields ξ_α. An important example for this situation are homogeneous star products on cotangent bundles that have been discussed in [5,6]. But this cannot be done in general since the failure of the connection to be compatible with the above Lie derivatives causes that the Fedosov derivation $\mathcal{D}$ does not commute with $\mathcal{H}_\alpha$ and hence $\mathcal{H}_\alpha$ does not map elements of $\ker(\mathcal{D})$ to elements of $\ker(\mathcal{D})$. So we try to extend $\mathcal{H}_\alpha$ to a $\circ$-(super-)derivation of anti-symmetric degree 0 that commutes with $\mathcal{D}$. To this end we make the ansatz

$$\mathcal{E}_\alpha = \mathcal{H}_\alpha + \frac{1}{\nu}\mathrm{ad}(h_\alpha) = \nu\partial_\nu + \mathcal{L}_{\xi_\alpha} + \frac{1}{\nu}\mathrm{ad}(h_\alpha) \tag{12}$$

with $h_\alpha \in \mathcal{W}(\mathcal{U}_\alpha)$ such that $\sigma(h_\alpha) = 0$ and compute $[\mathcal{D}, \mathcal{E}_\alpha]$.

Lemma 2. *Let $\mathcal{E}_\alpha$ be defined as above, then we have*

$$[\mathcal{D}, \mathcal{E}_\alpha] = \frac{1}{\nu}\mathrm{ad}(\mathcal{D}h_\alpha) + [\nabla, \mathcal{L}_{\xi_\alpha}] + \frac{1}{\nu}\mathrm{ad}(\mathcal{H}_\alpha r - r). \tag{13}$$

Proof. The proof of this formula relies on the fact that $\mathcal{D}$ is a super-derivation of anti-symmetric degree 1 with respect to $\circ$ and that $\mathcal{H}_\alpha$ is a (super-)derivation of anti-symmetric degree 0 with respect to $\circ$. Moreover, we used $[\delta, \mathcal{H}_\alpha] = 0$ and $[\nabla, \mathcal{H}_\alpha] = [\nabla, \mathcal{L}_{\xi_\alpha}]$. □

Now we consider the mapping $[\nabla, \mathcal{L}_{\xi_\alpha}]$ more closely. The formulas we collect in the following two lemmas are essential for the whole construction of local ν-Euler derivations.

Lemma 3. *For the locally defined vector fields ξ_α the mapping $[\nabla, \mathcal{L}_{\xi_\alpha}]$ enjoys the following properties:*

(i) *In local coordinates one has*

$$[\nabla, \mathcal{L}_{\xi_\alpha}] = (dx^j \otimes dx^i) i_s((\mathcal{L}_{\xi_\alpha}\nabla)_{\partial_i}\partial_j) = (dx^j \otimes dx^i) i_s(S_\alpha(\partial_i, \partial_j)), \tag{14}$$

where the local tensor field $S_\alpha \in \Gamma^\infty(T^*\mathcal{U}_\alpha \otimes T^*\mathcal{U}_\alpha \otimes T\mathcal{U}_\alpha)$ is defined by

$$\begin{aligned} S_\alpha(\partial_i, \partial_j) = (\mathcal{L}_{\xi_\alpha}\nabla)_{\partial_i}\partial_j &:= \mathcal{L}_{\xi_\alpha}\nabla_{\partial_i}\partial_j - \nabla_{\partial_i}\mathcal{L}_{\xi_\alpha}\partial_j - \nabla_{\mathcal{L}_{\xi_\alpha}\partial_i}\partial_j \\ &= R(\xi_\alpha, \partial_i)\partial_j + \nabla^{(2)}_{(\partial_i, \partial_j)}\xi_\alpha. \end{aligned} \tag{15}$$

(ii) *S_α as defined above is symmetric, i.e.* $S_\alpha \in \Gamma^\infty(\bigvee^2 T^*\mathcal{U}_\alpha \otimes T\mathcal{U}_\alpha)$.
(iii) *For all* $X, Y, Z \in \Gamma^\infty(T\mathcal{U}_\alpha)$ *we have* $\omega(Z, S_\alpha(X, Y)) = -\omega(S_\alpha(X, Z), Y)$.

Proof. The proof of the local expression for $[\nabla, \mathcal{L}_{\xi_\alpha}]$ is a straightforward computation. The last equality in (15) follows from the torsion freeness of the connection ∇. The fact that S_α is symmetric is a consequence from the first Bianchi identity for the connection ∇. (iii) follows from a direct computation essentially using $\nabla\omega = 0$ and $\mathcal{L}_{\xi_\alpha}\omega = \omega$. □

Now the local tensor fields S_α as defined above naturally give rise to elements T_α of $\mathcal{W}\otimes\Lambda(\mathcal{U}_\alpha)$ of symmetric degree 2 and anti-symmetric degree 1 by

$$T_\alpha(Z, Y; X) := \omega(Z, S_\alpha(X, Y)). \tag{16}$$

In local coordinates this reads $T_\alpha = \frac{1}{2}\omega_{ij} S_{\alpha}{}^{j}_{kl} dx^i \vee dx^l \otimes dx^k$, where $S_{\alpha}{}^{j}_{kl} = dx^j(S_\alpha(\partial_k, \partial_l))$ denotes the components of S_α in local coordinates.

Lemma 4. *The local tensor field T_α as defined in (16) satisfies the following equations:*

(i)

$$\frac{1}{\nu}\mathrm{ad}(T_\alpha) = [\nabla, \mathcal{L}_{\xi_\alpha}], \tag{17}$$

(ii)

$$T_\alpha = i_a(\xi_\alpha)R + \nabla\left(\frac{1}{2}D\theta_\alpha \otimes 1\right), \tag{18}$$

where the operator of symmetric covariant derivation D is defined by $D := dx^i \vee \nabla_{\partial_i}$.

(iii)

$$\delta T_\alpha = 0 \;\; \textit{and} \;\; \nabla T_\alpha = \mathcal{L}_{\xi_\alpha} R - R. \tag{19}$$

Proof. The first assertion easily follows from the properties of S_α given in Lemma 3 by a direct computation. Part (ii) can be easily proven by direct computation using (15) and the definitions of R and T_α. The equations given in (iii) follow from the super-Jacobi-identity applied to the equations $[\mathcal{H}_\alpha, [\delta, \nabla]] = 0$ and $[\mathcal{H}_\alpha, \frac{1}{2}[\nabla, \nabla]] = -[\mathcal{H}_\alpha, \frac{1}{\nu}\mathrm{ad}(R)]$. For the second equation one has to observe that R does not depend on ν and again that $\mathcal{H}_\alpha$ is a derivation with respect to $\circ$. Moreover, we used the fact that the only central elements of the Fedosov algebra $\mathcal{W}\otimes\Lambda$ with respect to $\circ$ with symmetric degree 1 resp. 2 are zero. □

Collecting our results we have shown that

$$[\mathcal{D}, \mathcal{E}_\alpha] = \frac{1}{\nu}\mathrm{ad}(\mathcal{D}h_\alpha + T_\alpha + \mathcal{H}_\alpha r - r). \tag{20}$$

Our next aim is to prove that h_α can be chosen such that $\mathcal{D}h_\alpha + T_\alpha + \mathcal{H}_\alpha r - r = 1 \otimes A_\alpha$, where A_α is a formal series of locally defined one-forms that have to be chosen suitably, since then $[\mathcal{D}, \mathcal{E}_\alpha] = 0$. The necessary condition for this equation to be solvable is $\mathcal{D}(1 \otimes A_\alpha - T_\alpha - \mathcal{H}_\alpha r + r) = 0$ since $\mathcal{D}^2 = 0$. But this is also sufficient since the $\mathcal{D}$-cohomology on elements a with positive anti-symmetric degree is trivial since one has the following homotopy formula $\mathcal{D}\mathcal{D}^{-1}a + \mathcal{D}^{-1}\mathcal{D}a = a$, where $\mathcal{D}^{-1}a := -\delta^{-1}\left(\frac{1}{\mathrm{id} - [\delta^{-1}, \nabla - \frac{1}{\nu}\mathrm{ad}(r)]}a\right)$ (cf. [14, Thm. 5.2.5]).

Lemma 5. *Choosing local potentials $\Theta_{i\alpha}$ for the closed two-forms Ω_i on $\mathcal{U}_\alpha$, and defining*

$$A_\alpha := (\mathsf{id} - \mathcal{H}_\alpha)\Theta_\alpha = (\mathsf{id} - \mathcal{H}_\alpha)\sum_{i=1}^{\infty} \nu^i \Theta_{i\alpha}, \tag{21}$$

the equation $\mathcal{D}(1 \otimes A_\alpha - T_\alpha - \mathcal{H}_\alpha r + r) = 0$ is fulfilled.

Proof. Using Eq. (3), $[\mathcal{H}_\alpha, \delta] = 0$ and Eq. (11) as well as Lemma 4 (i), (iii) one computes

$$\begin{aligned}\mathcal{D}\left(\mathcal{H}_\alpha r - r\right) &= 1 \otimes (\Omega - \mathcal{H}_\alpha \Omega) + R - \mathcal{L}_{\xi_\alpha} R + [\nabla, \mathcal{L}_{\xi_\alpha}]r \\ &= 1 \otimes (\mathsf{id} - \mathcal{H}_\alpha)\Omega - \nabla T_\alpha + \frac{1}{\nu}\mathrm{ad}(T_\alpha)r.\end{aligned}$$

On the other hand we get from $\delta T_\alpha = 0$ and $dA_\alpha = d(\mathsf{id} - \mathcal{H}_\alpha)\Theta_\alpha = (\mathsf{id} - \mathcal{H}_\alpha)\Omega$ that

$$\mathcal{D}\left(1 \otimes A_\alpha - T_\alpha\right) = 1 \otimes (\mathsf{id} - \mathcal{H}_\alpha)\Omega - \nabla T_\alpha + \frac{1}{\nu}\mathrm{ad}(r)T_\alpha,$$

proving the lemma. □

This lemma enables us to prove the following important proposition.

Proposition 1. *There are uniquely determined elements $h_\alpha \in \mathcal{W}(\mathcal{U}_\alpha)$ such that $\mathcal{D}h_\alpha = 1 \otimes A_\alpha + r - \mathcal{H}_\alpha r - T_\alpha$ and $\sigma(h_\alpha) = 0$. Moreover h_α is explicitly given by*

$$h_\alpha = \mathcal{D}^{-1}\left(1 \otimes A_\alpha + r - \mathcal{H}_\alpha r - T_\alpha\right), \tag{22}$$

where $\mathcal{D}^{-1}a = -\delta^{-1}\left(\frac{1}{\mathsf{id} - [\delta^{-1}, \nabla - \frac{1}{\nu}\mathrm{ad}(r)]}a\right)$. With these elements h_α the fibrewise, local ν-Euler derivations $\mathcal{E}_\alpha = \nu\partial_\nu + \mathcal{L}_{\xi_\alpha} + \frac{1}{\nu}\mathrm{ad}(h_\alpha)$ commute with the Fedosov derivation $\mathcal{D}$.

Proof. Using the homotopy formula $a = \mathcal{D}\mathcal{D}^{-1}a + \mathcal{D}^{-1}\mathcal{D}a$ that is valid for elements $a \in \mathcal{W}\otimes\Lambda$ with positive anti-symmetric degree on $1 \otimes A_\alpha + r - \mathcal{H}_\alpha r - T_\alpha$ we get

$$1 \otimes A_\alpha + r - \mathcal{H}_\alpha r - T_\alpha = \mathcal{D}\mathcal{D}^{-1}(1 \otimes A_\alpha + r - \mathcal{H}_\alpha r - T_\alpha)$$

from the preceding lemma. Since we want the last expression to equal $\mathcal{D}h_\alpha$ one gets $h_\alpha = \mathcal{D}^{-1}(1 \otimes A_\alpha + r - \mathcal{H}_\alpha r - T_\alpha) + \tau(\varphi_\alpha)$ with arbitrary, locally defined formal functions $\varphi_\alpha \in C^\infty(\mathcal{U}_\alpha)[[\nu]]$. From the demand $\sigma(h_\alpha) = 0$ we get $\sigma(\tau(\varphi_\alpha)) = \varphi_\alpha = 0$ since $\mathcal{D}^{-1}$ raises the symmetric degree and the formula for h_α is proven. The fact that $\mathcal{E}_\alpha$ commutes with $\mathcal{D}$ now follows from Eq. (20). □

Using the fibrewise, local ν-Euler derivations $\mathcal{E}_\alpha$ we constructed we are in the position to define local ν-Euler derivations with respect to the Fedosov star product $*$.

Definition 2. *Let $h_\alpha \in \mathcal{W}(\mathcal{U}_\alpha)$ be given as in Eq. (22). Denoting by $\mathcal{E}_\alpha : \mathcal{W}\otimes\Lambda(\mathcal{U}_\alpha) \to \mathcal{W}\otimes\Lambda(\mathcal{U}_\alpha)$ the fibrewise local ν-Euler derivations $\mathcal{E}_\alpha = \nu\partial_\nu + \mathcal{L}_{\xi_\alpha} + \frac{1}{\nu}\mathrm{ad}(h_\alpha)$ we define the mappings $\mathsf{E}_\alpha : C^\infty(\mathcal{U}_\alpha)[[\nu]] \to C^\infty(\mathcal{U}_\alpha)[[\nu]]$ by*

$$\mathsf{E}_\alpha f := \sigma\left(\mathcal{E}_\alpha \tau(f)\right) \tag{23}$$

for $f \in C^\infty(\mathcal{U}_\alpha)[[\nu]]$.

With this definition we get the main result of this section.

Theorem 1. *The mapping* E_α *as defined in Eq. (23) is a local derivation with respect to the Fedosov star product* $*$. *Moreover* $\mathsf{E}_\alpha = \nu\partial_\nu + \mathcal{L}_{\xi_\alpha} + \mathsf{D}_\alpha$, *where* $\mathsf{D}_\alpha = \sum_{i=1}^{\infty} \nu^i \mathsf{D}_{\alpha,i}$ *is a formal series of differential operators over* $\mathcal{U}_\alpha$.

Proof. The fact that E_α is a local derivation with respect to $*$ is obvious from the fact that $\mathcal{E}_\alpha$ is a local derivation with respect to $\circ$ and the property of $\mathcal{E}_\alpha$ mapping elements in $\mathcal{W}(\mathcal{U}_\alpha) \cap \ker(\mathcal{D})$ to elements in $\mathcal{W}(\mathcal{U}_\alpha) \cap \ker(\mathcal{D})$ which was achieved by constructing $\mathcal{E}_\alpha$ such that $[\mathcal{D}, \mathcal{E}_\alpha] = 0$. The assertion about the shape of E_α follows from the fact that σ commutes with $\nu\partial_\nu$ and $\mathcal{L}_{\xi_\alpha}$ yielding $\mathsf{E}_\alpha f = \nu\partial_\nu f + \mathcal{L}_{\xi_\alpha} f + \frac{1}{\nu}\sigma\,(\mathrm{ad}(h_\alpha)\tau(f))$. The fact that the last term involving h_α and τ defines a formal series of differential operators is obvious from the properties of the Fedosov-Taylor series. The only thing one has to observe is that this formal series starts at order one in the formal parameter. But this follows from the fact that h_α only contains terms of total degree greater or equal to three, which is a consequence of $\mathcal{D}^{-1}$ raising the symmetric degree, not decreasing the ν-degree and $1 \otimes A_\alpha + r - \mathcal{H}_\alpha r - T_\alpha$ only containing terms of total degree greater or equal to two. □

4. Computation of Deligne's Characteristic Class

With the aid of the local ν-Euler derivations we constructed in the preceding section we are in the position to compute Deligne's intrinsic derivation-related class $d(*)$ and hence the characteristic class $c(*)$ for every Fedosov star product $*$ as defined in Sect. 2. To this end we have to find formal functions $d_{\alpha\beta} \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)[[\nu]]$ such that on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ we have $\mathsf{E}_\alpha - \mathsf{E}_\beta = \frac{1}{\nu}\mathrm{ad}_*(d_{\alpha\beta})$. From the definition of the ν-Euler derivations E_α and the deformed Cartan formula (cf. Appendix A) we have the following:

Lemma 6. *For* $g \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)[[\nu]]$ *we have*

$$
\begin{aligned}
&\left(\mathsf{E}_\alpha - \mathsf{E}_\beta\right)(g) \qquad (24)\\
&= \frac{1}{\nu}\sigma\left(\mathrm{ad}\left(h_\alpha - h_\beta + f_{\alpha\beta} + df_{\alpha\beta} \otimes 1 + \frac{1}{2} D df_{\alpha\beta} \otimes 1 - i_a(X_{f_{\alpha\beta}})r\right)\tau(g)\right),
\end{aligned}
$$

where $f_{\alpha\beta} \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ *satisfies* $df_{\alpha\beta} = \theta_\alpha - \theta_\beta$ *and the local one-forms* θ_α *satisfy* $d\theta_\alpha = -\omega$.

Proof. We have $(\mathsf{E}_\alpha - \mathsf{E}_\beta)(g) = \sigma\left(\left(\mathcal{L}_{\xi_\alpha - \xi_\beta} + \frac{1}{\nu}\mathrm{ad}(h_\alpha - h_\beta)\right)\tau(g)\right)$ by definition of E_α. Now on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ we have $-d\theta_\alpha = \omega = -d\theta_\beta$ and hence by the Poincaré lemma we can find locally defined functions $f_{\alpha\beta}$ such that $df_{\alpha\beta} = \theta_\alpha - \theta_\beta$. Now by definition of the local vector fields ξ_α we get $d(-f_{\alpha\beta}) = i_{\xi_\alpha - \xi_\beta}\omega$ implying that $\xi_\alpha - \xi_\beta = X_{-f_{\alpha\beta}}$ is the Hamiltonian vector field of the function $-f_{\alpha\beta}$. Thus we can apply the deformed Cartan formula (40) proven in Proposition 5 and immediately obtain the statement of the lemma since $\mathcal{D}\tau(g) = 0$ and $i_a(X_{-f_{\alpha\beta}})\tau(g) = 0$. □

Now we are to show that the term occurring in the argument of ad in Eq. (24) can be extended by adding a locally defined formal function $a_{\alpha\beta} \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)[[\nu]]$ (clearly satisfying $\mathrm{ad}(a_{\alpha\beta}) = 0$) such that the whole argument is the Fedosov–Taylor series $\tau(f_{\alpha\beta} - \sigma(i_a(X_{f_{\alpha\beta}})r) + a_{\alpha\beta})$ of the local formal function $d_{\alpha\beta} := f_{\alpha\beta} - \sigma(i_a(X_{f_{\alpha\beta}})r) + a_{\alpha\beta}$. If we succeed to find such a local function, Eq. (24) yields $\mathsf{E}_\alpha - \mathsf{E}_\beta = \frac{1}{\nu}\mathrm{ad}_*(d_{\alpha\beta})$

enabling us to give an expression for Deligne's intrinsic derivation-related class $d(*)$ of $*$. We thus have to show that $a_{\alpha\beta}$ can be chosen such that

$$\mathcal{D}\left(h_\alpha - h_\beta + a_{\alpha\beta} + f_{\alpha\beta} + df_{\alpha\beta} \otimes 1 + \frac{1}{2} Ddf_{\alpha\beta} \otimes 1 - i_a(X_{f_{\alpha\beta}})r\right) = 0. \tag{25}$$

Lemma 7. *With the notations from above we have*

$$\mathcal{D}\left(h_\alpha - h_\beta + f_{\alpha\beta} + df_{\alpha\beta} \otimes 1 + \frac{1}{2} Ddf_{\alpha\beta} \otimes 1 - i_a(X_{f_{\alpha\beta}})r\right) \\ = 1 \otimes ((i_{\xi_\alpha}\Omega + A_\alpha) - (i_{\xi_\beta}\Omega + A_\beta)), \tag{26}$$

where A_α is given as in Lemma 5.

Proof. From the construction of the elements $h_\alpha \in \mathcal{W}(\mathcal{U}_\alpha)$ we gave in the preceding section (cf. Proposition 1) we get $\mathcal{D}\left(h_\alpha - h_\beta\right) = 1 \otimes (A_\alpha - A_\beta) - T_\alpha + T_\beta - \mathcal{L}_{\xi_\alpha - \xi_\beta} r$. Another straightforward calculation yields $\mathcal{D}\left(f_{\alpha\beta} + df_{\alpha\beta} \otimes 1 + \frac{1}{2} Ddf_{\alpha\beta} \otimes 1\right) = -\frac{1}{\nu}\mathrm{ad}(r)\left(df_{\alpha\beta} \otimes 1 + \frac{1}{2} Ddf_{\alpha\beta} \otimes 1\right) + \frac{1}{2}\nabla\left(Ddf_{\alpha\beta} \otimes 1\right)$. Using the deformed Cartan formula once again combined with Eq. (3) and the definition of $\mathcal{D}$ we get

$$\begin{aligned}
&\mathcal{D}(-i_a(X_{f_{\alpha\beta}})r) \\
&= -(\mathcal{D} i_a(X_{f_{\alpha\beta}})r + i_a(X_{f_{\alpha\beta}})\mathcal{D}r) + i_a(X_{f_{\alpha\beta}})\left(-\delta r + \nabla r - \frac{1}{\nu}\mathrm{ad}(r) r\right) \\
&= \mathcal{L}_{\xi_\alpha - \xi_\beta} r + \frac{1}{\nu}\mathrm{ad}\left(-df_{\alpha\beta} \otimes 1 - \frac{1}{2} Ddf_{\alpha\beta} \otimes 1 + i_a(X_{f_{\alpha\beta}})r\right) r \\
&\quad + i_a(\xi_\alpha - \xi_\beta)\left(R + 1 \otimes \Omega + \frac{1}{\nu} r \circ r\right) \\
&= \mathcal{L}_{\xi_\alpha - \xi_\beta} r + \frac{1}{\nu}\mathrm{ad}\left(-df_{\alpha\beta} \otimes 1 - \frac{1}{2} Ddf_{\alpha\beta} \otimes 1\right) r + i_a(\xi_\alpha - \xi_\beta) R + 1 \otimes i_{\xi_\alpha - \xi_\beta}\Omega,
\end{aligned}$$

since $\frac{1}{\nu} i_a(\xi_\alpha - \xi_\beta)(r \circ r) = -\frac{1}{\nu}\mathrm{ad}(i_a(X_{f_{\alpha\beta}})r)r$. All these results together with Eq. (18) and $df_{\alpha\beta} = \theta_\alpha - \theta_\beta$ prove the statement of the lemma. □

After these preparations we are able to formulate the following proposition.

Proposition 2. *There are locally defined formal functions $d_{\alpha\beta} \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)[[\nu]]$ such that $\mathsf{E}_\alpha - \mathsf{E}_\beta = \frac{1}{\nu}\mathrm{ad}_*(d_{\alpha\beta})$. Moreover, these formal functions satisfy $dd_{\alpha\beta} = df_{\alpha\beta} - d(\sigma(i_a(X_{f_{\alpha\beta}})r)) + da_{\alpha\beta} = \theta_\alpha - \theta_\beta + d(\sigma(i_a(\xi_\alpha)r)) - d(\sigma(i_a(\xi_\beta)r)) - ((A_\alpha + i_{\xi_\alpha}\Omega) - (A_\beta + i_{\xi_\beta}\Omega))$. Thus they define a 2-cocycle and the image of the corresponding Čech class under the de Rham isomorphism, which is just Deligne's intrinsic derivation-related class, is given by $d(*) = -[\omega] - [\Omega - \nu\partial_\nu\Omega]$.*

Proof. From Lemma 7 we get that $a_{\alpha\beta}$ has to satisfy the equation $da_{\alpha\beta} = -\left((A_\alpha + i_{\xi_\alpha}\Omega) - (A_\beta + i_{\xi_\beta}\Omega)\right)$ so that (25) is fulfilled. From the definition of A_α we get that the right-hand side of this equation is closed since $d(A_\alpha + i_{\xi_\alpha}\Omega) = \Omega - \nu\partial_\nu\Omega$. Therefore the existence of $a_{\alpha\beta} \in C^\infty(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)[[\nu]]$ as desired is guaranteed by the Poincaré lemma. Now we have $\sigma(h_\alpha - h_\beta + a_{\alpha\beta} + f_{\alpha\beta} + df_{\alpha\beta} \otimes 1 + \frac{1}{2} Ddf_{\alpha\beta} \otimes 1 - i_a(X_{f_{\alpha\beta}})r) = f_{\alpha\beta} - \sigma(i_a(X_{f_{\alpha\beta}})r) + a_{\alpha\beta}$ and Eq. (25) is fulfilled implying $(\mathsf{E}_\alpha - \mathsf{E}_\beta)(g) =$

$\frac{1}{\nu}\sigma(\mathrm{ad}(\tau(f_{\alpha\beta} + \sigma(i_a(\xi_\alpha)r) - \sigma(i_a(\xi_\beta)r) + a_{\alpha\beta}))\tau(g)) = \frac{1}{\nu}\mathrm{ad}_*(d_{\alpha\beta})g$ by Lemma 6. The assertion about the corresponding de Rham class is obvious from the properties of $f_{\alpha\beta}$ and $a_{\alpha\beta}$ we have already proven, namely $d(\theta_\alpha + d(\sigma(i_a(\xi_\alpha)r)) - (A_\alpha + i_{\xi_\alpha}\Omega)) = -(\omega + \Omega - \nu\partial_\nu\Omega)$. □

From this proposition and from the computation of $C_2^{-\sharp}$ in Appendix B we obtain our final result.

Theorem 2. *Deligne's characteristic class $c(*)$ of a (slightly generalized) Fedosov star product $*$ as constructed in Sect. 2 is given by*

$$c(*) = \frac{1}{\nu}[\omega] + \frac{1}{\nu}[\Omega] = \frac{1}{\nu}F(*), \tag{27}$$

where $F(*)$ denotes Fedosov's characteristic class of the star product $*$.

Proof. From the differential equation $\partial_\nu c(*) = \frac{1}{\nu^2}d(*)$ that relates the derivation-related class to the characteristic class and from the preceding proposition we get $c(*) = \frac{1}{\nu}[\omega] + c(*)^0 + \frac{1}{\nu}\sum_{i=2}^{\infty}\nu^i[\Omega_i]$. By the result of Proposition 6 we get $c(*)^0 = [\Omega_1]$, proving the theorem. □

As an immediate corollary which originally is due to Fedosov (cf. [14, Cor. 5.5.4]) we find:

Corollary 1. *Two Fedosov star products $*$ and $*'$ for (M, ω) constructed from the data (∇, Ω, s) and (∇', Ω', s') as in Sect. 2 are equivalent if and only if $[\Omega] = [\Omega']$.*

5. Star Products of Special Type, Their Characteristic Classes and Equivalence Transformations

In this section we consider star products that have additional algebraic properties and compute their characteristic classes showing that these properties give rise to restrictions on this class. Moreover, we can show that for every characteristic class satisfying the necessary condition for a star product of this class to have the desired algebraic properties there are always Fedosov star products with suitably chosen data Ω, s having these properties. Although the following results might be known they do nevertheless not seem to have appeared in the literature except for the special case $\Omega = 0$ and $s = 0$ considered in [7, Lemma 3.3]. In this section $\mathrm{C} : \mathcal{W}\otimes\Lambda \to \mathcal{W}\otimes\Lambda$ shall always denote the complex conjugation, where we define $\mathrm{C}\nu := -\nu$ in view of our convention for the formal parameter being considered as purely imaginary. By $\mathrm{P} : \mathcal{W}\otimes\Lambda \to \mathcal{W}\otimes\Lambda$ with $\mathrm{P} := (-1)^{\deg_\nu}$ we denote the so-called ν-parity operator. Using these maps fulfilling $\mathrm{C}^2 = \mathrm{P}^2 = \mathsf{id}$ we can define special types of star products:

Definition 3. (i) *For a given star product $\star$ for (M, ω) we define the star products $\star_{\mathrm{opp}}$, $\star_{\mathrm{C}}$, $\star_{\mathrm{P}}$ for $(M, -\omega)$ by*

$$f \star_{\mathrm{opp}} g := g \star f, \tag{28}$$

$$f \star_{\mathrm{C}} g := \mathrm{C}\left((\mathrm{C}f) \star (\mathrm{C}g)\right), \tag{29}$$

$$f \star_{\mathrm{P}} g := \mathrm{P}\left((\mathrm{P}f) \star (\mathrm{P}g)\right) = f \star_{-\nu} g = \sum_{i=0}^{\infty}(-\nu)^i C_i(f, g), \tag{30}$$

where $f, g \in C^\infty(M)[[\nu]]$ and the bidifferential operators C_i describe the star product $\star$ by $f \star g = \sum_{i=0}^{\infty}\nu^i C_i(f, g)$.

(ii) *A star product $\star$ is said to have the ν-parity property if* P *is an anti-automorphism of $\star$, i.e.*

$$f \star_{\mathrm{P}} g = f \star_{\mathrm{opp}} g \quad \forall f, g \in C^{\infty}(M)[[\nu]]. \tag{31}$$

(iii) *A star product $\star$ is said to have a *-structure incorporated by complex conjugation if* C *is an anti-automorphism of $\star$, i.e.*

$$f \star_{\mathrm{C}} g = f \star_{\mathrm{opp}} g \quad \forall f, g \in C^{\infty}(M)[[\nu]]. \tag{32}$$

(iv) *A star product $\star$ is called of Weyl type if it has the ν-parity property and has a *-structure incorporated by complex conjugation.*

Using these definitions we find:

Lemma 8. (i) *The characteristic classes of $\star_{\mathrm{opp}}, \star_{\mathrm{C}}, \star_{\mathrm{P}}$ are related to the characteristic class $c(\star)$ of $\star$ by the following equations:*

$$c(\star_{\mathrm{opp}}) = -c(\star), \tag{33}$$

$$c(\star_{\mathrm{C}}) = \mathrm{C}c(\star), \tag{34}$$

$$c(\star_{\mathrm{P}})(\nu) = c(\star_{-\nu})(\nu) = c(\star)(-\nu) = \mathrm{P}\left(c(\star)(\nu)\right). \tag{35}$$

(ii) *The characteristic class of a star product $\star$ that has the ν-parity property satisfies*

$$\mathrm{P}c(\star) = -c(\star), \tag{36}$$

and hence $c(\star) = \frac{[\omega]}{\nu} + \sum_{l=0}^{\infty} \nu^{2l+1} c(\star)^{2l+1}$, i.e. $c(\star)^{2l} = [0]$ for all $l \in \mathbb{N}$.

(iii) *The characteristic class of a star product $\star$ that has* C *as *-structure satisfies*

$$\mathrm{C}c(\star) = -c(\star), \tag{37}$$

and hence $c(\star)^{2l} = -\mathrm{C}c(\star)^{2l}$ and $c(\star)^{2l+1} = \mathrm{C}c(\star)^{2l+1}$ for all $l \in \mathbb{N}$.

(iv) *The characteristic class of a star product $\star$ that is of Weyl type satisfies*

$$\mathrm{P}c(\star) = -c(\star) \quad \textit{and} \quad \mathrm{C}c(\star) = -c(\star), \tag{38}$$

and hence $c(\star)^{2l} = [0]$ and $c(\star)^{2l+1} = \mathrm{C}c(\star)^{2l+1}$ for all $l \in \mathbb{N}$.

Proof. The proof of part (i) relies on the observation that local ν-Euler derivations E_α of $\star$ yield such derivations for $\star_{\mathrm{opp}}, \star_{\mathrm{C}}$ and $\star_{\mathrm{P}}$ given by E_α, $\mathrm{C}\mathsf{E}_\alpha\mathrm{C}$ and $\mathrm{P}\mathsf{E}_\alpha\mathrm{P}$. With these derivations one easily finds $d(\star_{\mathrm{opp}}) = -d(\star)$, $d(\star_{\mathrm{C}}) = -\mathrm{C}d(\star)$ and $d(\star_{\mathrm{P}}) = -\mathrm{P}d(\star)$. From the definition of the characteristic class relating the derivation-related class d with c and the obvious observations that $c(\star_{\mathrm{opp}})^0 = -c(\star)^0$, $c(\star_{\mathrm{C}})^0 = \mathrm{C}c(\star)^0$ and $c(\star_{\mathrm{P}})^0 = c(\star)^0$ one gets the asserted statements. The assertions (ii), (iii) and (iv) are obvious from part (i) and Definition 3 (ii), (iii) and (iv). □

The statement (ii) of the lemma is the deep reason for the fact that when building a star product recursively by constructing bidifferential operators C_i DeWilde, Lecomte in [11] only have the choice of a closed two-form in case i is odd as the ν-parity property is included in their definition of a star product. The preceding lemma states that in general there are equivalence classes of star products corresponding to the characteristic classes $c(\star)$ that contain no representatives (i.e. star products $\star$ with this characteristic class) satisfying the conditions (31) resp. (32), namely those whose characteristic classes do not satisfy Eqs. (36) resp. (37). Vice versa the following proposition states that for every class $c \in \frac{[\omega]}{\nu} + H^2_{\mathrm{dR}}(M)[[\nu]]$ enjoying the properties $\mathrm{C}c = -c$ resp. $\mathrm{P}c = -c$ one can find even Fedosov star products having the characteristic class c and satisfying the conditions (32) resp. (31).

Proposition 3. (i) *For all* $c \in \frac{[\omega]}{\nu} + H^2_{\mathrm{dR}}(M)[[\nu]]$ *with* $\mathrm{P}c = -c$ *there are Fedosov star products* $*$ *for* (M, ω) *with*

$$c(*) = c \quad \text{and} \quad \mathrm{P}\left((\mathrm{P}f) * (\mathrm{P}g)\right) = g * f \quad \text{for all } f, g \in C^\infty(M)[[\nu]].$$

(ii) *For all* $c \in \frac{[\omega]}{\nu} + H^2_{\mathrm{dR}}(M)[[\nu]]$ *with* $\mathrm{C}c = -c$ *there are Fedosov star products* $*$ *for* (M, ω) *with*

$$c(*) = c \quad \text{and} \quad \mathrm{C}\left((\mathrm{C}f) * (\mathrm{C}g)\right) = g * f \quad \text{for all } f, g \in C^\infty(M)[[\nu]].$$

(iii) *For all* $c \in \frac{[\omega]}{\nu} + H^2_{\mathrm{dR}}(M)[[\nu]]$ *with* $\mathrm{P}c = -c = \mathrm{C}c$ *there are Fedosov star products* $*$ *for* (M, ω) *with*

$$c(*) = c \quad \text{and} \quad \mathrm{P}\left((\mathrm{P}f) * (\mathrm{P}g)\right) = g * f = \mathrm{C}\left((\mathrm{C}f) * (\mathrm{C}g)\right)$$

for all $f, g \in C^\infty(M)[[\nu]]$.

Proof. For the proof we first observe that the fibrewise product $\circ$ satisfies $\mathrm{C}((\mathrm{C}a) \circ (\mathrm{C}b)) = \mathrm{P}((\mathrm{P}a) \circ (\mathrm{P}b)) = (-1)^{kl} b \circ a$ for all $a, b \in \mathcal{W} \otimes \Lambda$ with $\deg_a a = ka$ and $\deg_a b = lb$. Now let $c \in \frac{[\omega]}{\nu} + H^2_{\mathrm{dR}}(M)[[\nu]]$ be written as $c = \frac{[\omega]}{\nu} + \sum_{i=0}^\infty \nu^i c^i$. For the proof of (i) we choose closed two-forms Ω_i such that $\Omega_{2l+1} = 0$ (to achieve $[\Omega_{2l+1}] = c^{2l} = [0]$) and $[\Omega_{2l+2}] = c^{2l+1}$ for all $l \in \mathbb{N}$ yielding $\mathrm{P}\Omega = \Omega$. Moreover, we choose $s = \sum_{k=3}^\infty s^{(k)} \in \mathcal{W}$ with $\sigma(s) = 0$ and $\mathrm{P}s = s$. Under these preconditions one easily proves that $\mathrm{P}r$ satisfies Eqs. (3) implying $\mathrm{P}r = r$ by uniqueness of the solution of (3). With such an element $r \in \mathcal{W} \otimes \Lambda^1$ the Fedosov derivation $\mathcal{D}$ obviously commutes with P implying that $\mathrm{P}\tau(f) = \tau(\mathrm{P}f)$ for all $f \in C^\infty(M)[[\nu]]$ since P obviously commutes with σ. Using this equation and the definition of $*$ together with $\mathrm{P}((\mathrm{P}a) \circ (\mathrm{P}b)) = (-1)^{kl} b \circ a$ and observing that $\deg_a \tau(f) = 0$ one gets the asserted property of $*$ under the mapping P. From Theorem 2 we get $c(*) = \frac{[\omega]}{\nu} + \frac{1}{\nu}[\Omega] = c$. For (ii) one proceeds quite analogously. The only difference lies in other suitable choices of Ω and s, i.e. we choose closed two-forms Ω_i such that $\mathrm{C}\Omega_{2l+2} = \Omega_{2l+2}$, $[\Omega_{2l+2}] = c^{2l+1}$ and $\mathrm{C}\Omega_{2l+1} = -\Omega_{2l+1}$, $[\Omega_{2l+1}] = c^{2l}$ for all $l \in \mathbb{N}$ implying $\mathrm{C}\Omega = \Omega$. Moreover, we choose $s \in \mathcal{W}$ such that $\mathrm{C}s = s$. As in the proof of (i) one gets that $\mathrm{C}r = r$ yielding the desired behaviour of the corresponding star product $*$ under the mapping C as in the proof of part (i). The fact that $c(*) = c$ again follows from Theorem 2 and the choice of Ω. For the proof of part (iii) one just has to bring into line the choices made for (i) with the ones made for (ii), i.e. choose s with $\mathrm{C}s = s = \mathrm{P}s$ and closed two-forms Ω_i with $\mathrm{C}\Omega_{2l+2} = \Omega_{2l+2}$ and $\Omega_{2l+1} = 0$ such that $[\Omega_{2l+2}] = c^{2l+1}$ and $[\Omega_{2l+1}] = c^{2l} = [0]$ for all $l \in \mathbb{N}$. Then the argument as in (i) and (ii) yields the stated result. □

Remark 2. The interest in such special star products from the viewpoint of physics is based on the interpretation of the star product algebra $(C^\infty(M)[[\nu]], \star)$ as the algebra of observables of the quantized system corresponding to the classical system described by the symplectic manifold M, and hence the existence of a *-structure incorporated by complex conjugation (the *-structure of the algebra of classical observables) is strongly recommended. Moreover, the Weyl–Moyal product on $T^*\mathbb{R}^n$ giving a correct description of the quantization of observables that are polynomials in the coordinates is of Weyl type, motivating the general interest in such star products (cf. [5,6] for further details). In addition there is the possibility of constructing *-representations for star products with C as *-structure under the precondition of having defined a formally positive functional on a suitable twosided ideal in $C^\infty(M)[[\nu]]$ that is stable under C by a formal analogue of the GNS construction (cf. [8] for details).

To conclude this section we shall discuss the question of existence of special equivalence transformations between equivalent star products satisfying Eqs. (31) and (32). The following proposition states that for two equivalent star products enjoying these additional algebraic properties there are always equivalence transformations being compatible with the mappings C and P.

Proposition 4. *Let $(C^\infty(M)[[\nu]], \star_1)$ and $(C^\infty(M)[[\nu]], \star_2)$ denote equivalent star product algebras.*

(i) *In case $\star_1$ and $\star_2$ have* C *incorporated as *-structure, then $(C^\infty(M)[[\nu]], \star_1)$ and $(C^\infty(M)[[\nu]], \star_2)$ are equivalent as *-algebras (resp.* C*-equivalent), i.e. there is an equivalence transformation $\mathcal{S}$ between them satisfying $\mathrm{C}\mathcal{S}\mathrm{C} = \mathcal{S}$.*
(ii) *In case $\star_1$ and $\star_2$ have the ν-parity property then $(C^\infty(M)[[\nu]], \star_1)$ and $(C^\infty(M)[[\nu]], \star_2)$ are* P*-equivalent, i.e. there is an equivalence transformation $\mathcal{S}$ between them satisfying $\mathrm{P}\mathcal{S}\mathrm{P} = \mathcal{S}$.*
(iii) *In case $\star_1$ and $\star_2$ are of Weyl type then $(C^\infty(M)[[\nu]], \star_1)$ and $(C^\infty(M)[[\nu]], \star_2)$ are Weyl-equivalent, i.e. there is an equivalence transformation $\mathcal{S}$ between them satisfying $\mathrm{C}\mathcal{S}\mathrm{C} = \mathcal{S}$ and $\mathrm{P}\mathcal{S}\mathrm{P} = \mathcal{S}$.*

Proof. For the proof of part (i) we consider some equivalence transformation $\mathcal{T}$ between $\star_1$ and $\star_2$ satisfying $\mathcal{T}(f \star_1 g) = (\mathcal{T}f) \star_2 (\mathcal{T}g)$ for all $f, g \in C^\infty(M)[[\nu]]$. Obviously $\mathrm{C}\mathcal{T}\mathrm{C}$ is also an equivalence transformation between $\star_1$ and $\star_2$ and hence there is an automorphism $\mathcal{A}$ of $\star_1$ such that $\mathrm{C}\mathcal{T}\mathrm{C} = \mathcal{T}\mathcal{A}$. Conjugating this equation with C and using $\mathrm{C}^2 = \mathsf{id}$ we obtain $\mathcal{T} = \mathrm{C}\mathcal{T}\mathrm{C}\mathrm{C}\mathcal{A}\mathrm{C} = \mathcal{T}\mathcal{A}\mathrm{C}\mathcal{A}\mathrm{C}$ yielding $\mathcal{A}\mathrm{C}\mathcal{A}\mathrm{C} = \mathsf{id}$. Since any automorphism of $\star_1$ starting with id has the shape $\mathcal{A} = \exp(\nu \mathsf{D})$, where D is a derivation of $\star_1$ we get $\mathsf{id} = \exp(\nu\mathsf{D})\exp(-\nu\mathrm{C}\mathsf{D}\mathrm{C})$ implying $\mathrm{C}\mathsf{D}\mathrm{C} = \mathsf{D}$. For $t \in \mathbb{R}$ we consider the automorphisms $\mathcal{A}^t := \exp(t\nu\mathsf{D})$ of $\star_1$ satisfying $\mathrm{C}\mathcal{A}^t\mathrm{C} = (\mathcal{A}^t)^{-1} = \mathcal{A}^{-t}$. Now $\mathcal{S}_t := \mathcal{T}\mathcal{A}^t$ obviously is an equivalence between $\star_1$ and $\star_2$ for all $t \in \mathbb{R}$ and we have $\mathrm{C}\mathcal{S}_t\mathrm{C} = \mathrm{C}\mathcal{T}\mathrm{C}\mathcal{A}^{-t} = \mathcal{T}\mathcal{A}^{1-t} = \mathcal{S}_{1-t}$. Therefore $\mathcal{S} := \mathcal{S}_{1/2}$ satisfies $\mathcal{S}(f \star_1 g) = (\mathcal{S}f) \star_2 (\mathcal{S}g)$ and $\mathrm{C}\mathcal{S}\mathrm{C} = \mathcal{S}$ proving part (i) of the proposition. For the proof of part (ii) one proceeds completely analogously replacing C by P in the above argumentation. For the proof of part (iii) we consider some equivalence transformation $\mathcal{T}$ between $\star_1$ and $\star_2$ and use the results of part (i) and part (ii) to obtain two further equivalence transformations $\mathcal{S}_1 = \mathcal{T}\mathcal{A}_1^{1/2}$ and $\mathcal{S}_2 = \mathcal{T}\mathcal{A}_2^{1/2}$ satisfying $\mathrm{C}\mathcal{S}_1\mathrm{C} = \mathcal{S}_1$ and $\mathrm{P}\mathcal{S}_2\mathrm{P} = \mathcal{S}_2$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are automorphisms of $\star_1$ given by $\mathrm{C}\mathcal{T}\mathrm{C} = \mathcal{T}\mathcal{A}_1$ and $\mathrm{P}\mathcal{T}\mathrm{P} = \mathcal{T}\mathcal{A}_2$. In general $\mathcal{S}_1$ fails to satisfy $\mathrm{P}\mathcal{S}_1\mathrm{P} = \mathcal{S}_1$ as well as $\mathcal{S}_2$ fails to commute with C, but by an analogous procedure as for the proofs of the statements (i) and (ii) $\mathcal{S}_1$ and $\mathcal{S}_2$ can be modified such that the resulting equivalence transformations have the desired properties. Since P commutes with C we have $\mathrm{C}\mathrm{P}\mathcal{T}\mathrm{P}\mathrm{C} = \mathrm{P}\mathrm{C}\mathcal{T}\mathrm{C}\mathrm{P}$ implying the crucial equation $\mathcal{A}_1\mathrm{C}\mathcal{A}_2\mathrm{C} = \mathcal{A}_2\mathrm{P}\mathcal{A}_1\mathrm{P}$ by the definitions of $\mathcal{A}_1$ and $\mathcal{A}_2$. Now we compute $\mathrm{C}\mathcal{S}_2\mathrm{C} = \mathcal{S}_2\mathcal{A}_2^{-1/2}\mathcal{A}_1\mathrm{C}\mathcal{A}_2^{1/2}\mathrm{C} = \mathcal{S}_2\mathcal{F}_2$, where $\mathcal{F}_2 := \mathcal{A}_2^{-1/2}\mathcal{A}_1\mathrm{C}\mathcal{A}_2^{1/2}\mathrm{C}$ is an automorphism of $\star_1$ starting with id and hence $\mathcal{F}_2 = \exp(\nu\mathsf{D}_2)$ with a derivation D_2 of $\star_1$. As in (i) one gets $\mathcal{F}_2\mathrm{C}\mathcal{F}_2\mathrm{C} = \mathsf{id}$ and $\mathcal{R}_2 := \mathcal{S}_2\mathcal{F}_2^{1/2}$ with $\mathcal{F}_2^{1/2} := \exp(\frac{\nu}{2}\mathsf{D}_2)$ is an equivalence transformation between $\star_1$ and $\star_2$ satisfying $\mathrm{C}\mathcal{R}_2\mathrm{C} = \mathcal{R}_2$. It remains to show that $\mathcal{R}_2$ satisfies $\mathrm{P}\mathcal{R}_2\mathrm{P} = \mathcal{R}_2$. To this end we compute $\mathrm{P}\mathcal{F}_2\mathrm{P}$ using $\mathcal{A}_1\mathrm{C} = \mathcal{A}_2\mathrm{P}\mathcal{A}_1\mathrm{P}\mathrm{C}\mathcal{A}_2^{-1}$,

$$\mathrm{P}\mathcal{F}_2\mathrm{P} = \mathrm{P}\mathcal{A}_2^{1/2}\mathrm{P}\mathcal{A}_1\mathrm{P}\mathrm{C}\mathcal{A}_2^{-1/2}\mathrm{C}\mathrm{P} = \mathcal{A}_2^{-1/2}\mathcal{A}_1\mathrm{C}\mathcal{A}_2^{1/2}\mathrm{C} = \mathcal{F}_2.$$

Thus we find $\mathrm{P}\mathcal{R}_2\mathrm{P} = \mathrm{P}\mathcal{S}_2\mathrm{P}\mathrm{P}\mathcal{F}_2^{1/2}\mathrm{P} = \mathcal{S}_2\mathcal{F}_2^{1/2} = \mathcal{R}_2$ proving part (iii). One can also modify $\mathcal{S}_1$ to obtain another equivalence transformation $\mathcal{R}_1 = \mathcal{S}_1\mathcal{F}_1^{1/2}$ having the desired properties where $\mathcal{F}_1 := \mathcal{A}_1^{-1/2}\mathcal{A}_2\mathrm{P}\mathcal{A}_1^{1/2}\mathrm{P}$ again is an automorphism of $\star_1$. □

Remark 3. The assertion about the existence of equivalence transformations between equivalent star products with a *-structure incorporated by C that commute with C has an important consequence for the GNS representations one can construct for these star product algebras, namely that such an equivalence transformation induces a unitary map between the GNS Hilbert spaces obtained by the GNS construction relating the corresponding GNS representations (cf. [6, Prop. 5.1]).

A. The Deformed Cartan Formula

The aim of this section is to prove the deformed Cartan formula that was very useful for our computations in Sect. 4. This formula and the proof of it which we shall give already appeared in [4, Lemma 4.6.]. A similar result has also been derived in [21, Prop. 4.3.] where the vector field with respect to which the Lie derivative is computed is assumed to be affine with respect to the symplectic connection ∇.

Proposition 5. *For all vector fields $X \in \Gamma^\infty(TM)$ the Lie derivative $\mathcal{L}_X : \mathcal{W}\otimes\Lambda \to \mathcal{W}\otimes\Lambda$ can be expressed in the following manner:*

$$\mathcal{L}_X = \mathcal{D}i_a(X) + i_a(X)\mathcal{D} + i_s(X) + (dx^i \otimes 1)i_s(\nabla_{\partial_i}X) + \frac{1}{\nu}\mathrm{ad}(i_a(X)r). \tag{39}$$

In case $X = X_f$ is the Hamiltonian vector field of a function $f \in C^\infty(M)$, i.e. $i_{X_f}\omega = df$ this formula takes the following form:

$$\mathcal{L}_{X_f} = \mathcal{D}i_a(X_f) + i_a(X_f)\mathcal{D} - \frac{1}{\nu}\mathrm{ad}\left(f + df \otimes 1 + \frac{1}{2}Ddf \otimes 1 - i_a(X_f)r\right), \tag{40}$$

where $D = dx^i \vee \nabla_{\partial_i}$ denotes the operator of symmetric covariant derivation.

Proof. The proof of formula (39) is obtained by collecting the following formulas, the proofs of which are all straightforward computations just using the definitions of the involved mappings and applying them to factorized sections $a = A \otimes \alpha \in \mathcal{W}\otimes\Lambda$:

$$\delta i_a(X) + i_a(X)\delta = i_s(X), \tag{41}$$

$$\frac{1}{\nu}\left(\mathrm{ad}(r)i_a(X) + i_a(X)\mathrm{ad}(r)\right) = \frac{1}{\nu}\mathrm{ad}(i_a(X)r), \tag{42}$$

$$(\nabla i_a(X) + i_a(X)\nabla)(A \otimes \alpha) = \nabla_X A \otimes \alpha + A \otimes \mathcal{L}_X\alpha. \tag{43}$$

For a symmetric one-form A it is easy to see that $\nabla_X A = \mathcal{L}_X A - dx^i \vee i_s(\nabla_{\partial_i} X)A$. Together with the observation that the operators on both sides of this equation are derivations with respect to the $\vee$-product, this and (43) imply

$$\nabla i_a(X) + i_a(X)\nabla = \mathcal{L}_X - (dx^i \otimes 1)i_s(\nabla_{\partial_i} X). \tag{44}$$

Combining (41), (42) and (44) we get the first statement of the proposition. For the second statement one just has to observe that $\mathrm{ad}(f) = 0$ and that

$$i_s(X_f) = -\frac{1}{\nu}\mathrm{ad}(df \otimes 1) \qquad (dx^i \otimes 1)i_s(\nabla_{\partial_i} X_f) = -\frac{1}{\nu}\mathrm{ad}\left(\frac{1}{2}Ddf \otimes 1\right), \tag{45}$$

which is again a straightforward computation in local coordinates using the explicit shape of the deformed product $\circ$. Using these equations combined with (39) finishes the proof of (40). □

B. Computation of $C_2^{-\sharp}$

This section just gives a sketch of the computations that are necessary to determine the anti-symmetric part of the bidifferential operator C_2 that occurs in the expression of the Fedosov star product $f * g = fg + \nu C_1(f, g) + \nu^2 C_2(f, g) + \ldots$ of two functions $f, g \in C^\infty(M)$.

Proposition 6. *The anti-symmetric part C_2^- of the bidifferential operator C_2 is given by*

$$C_2^-(f, g) = \frac{1}{2}\left(C_2(f, g) - C_2(g, f)\right) = -\frac{1}{2}\left(\Omega_1 + ds_1^{(3)}\right)(X_f, X_g) = \rho_2(X_f, X_g), \tag{46}$$

where $f, g \in C^\infty(M)$ and X_f resp. X_g denote the corresponding Hamiltonian vector fields with respect to ω and $s_1^{(3)} \in \Gamma^\infty(T^*M)$ denotes the one-form occurring in the first order of ν in $s^{(3)} = (s_3^{(3)} + \nu s_1^{(3)}) \otimes 1$, where $s_3^{(3)} \in \Gamma^\infty(\bigvee^3 T^*M)$, that comes up from the normalization condition $\delta^{-1} r = s$ (cf. Eq. (3)). Thus we have

$$c(*)^0 = -2C_2^{-\sharp} = -2[\rho_2] = [\Omega_1]. \tag{47}$$

Proof. Using the shape of the fibrewise product $\circ$ we obtain $f * g - g * f = \nu\sigma(\Lambda^{rs} i_s(\partial_r)\tau(f)i_s(\partial_s)\tau(g)) + O(\nu^3)$. To compute the terms of order less than or equal to two in ν we thus only have to know $\tau(f)$ and $\tau(g)$ except for terms of symmetric degree and ν-degree greater than one. Hence it is enough to look at $\tau(f)^{(0)}, \ldots, \tau(f)^{(3)}$, since for $\tau(f)^{(k)}$ with $k \geq 4$ either the symmetric degree or the ν-degree of the occurring terms are greater than one. Looking at the recursion formula (6) we thus see that the only terms of r that are needed are given by $r^{(2)} = \delta s^{(3)}$ and $r^{(3)} = \delta s^{(4)} + \delta^{-1}(R + \nu\Omega_1 + \nabla r^{(2)} - \frac{1}{\nu} r^{(2)} \circ r^{(2)})$ which is obtained from (4) by writing down the terms of total degree 2 resp. 3. Writing $\approx$ for equations holding modulo terms of symmetric degree resp. ν-degree greater than one, one gets by lengthy but obvious computation that

$$\begin{aligned}
\tau(f)^{(0)} &= f,\\
\tau(f)^{(1)} &= df \otimes 1,\\
\tau(f)^{(2)} &= \frac{1}{2}Ddf \otimes 1 - i_s(X_f)s_3^{(3)} \otimes 1 \approx 0,\\
\tau(f)^{(3)} &\approx -\delta^{-1}(i_s(X_f)r^{(3)}) \approx -\nu\left(i_s(X_f)s_2^{(4)} + \frac{1}{2}i_{X_f}\left(\Omega_1 + ds_1^{(3)}\right)\right) \otimes 1,
\end{aligned}$$

where we have written $s^{(4)} = (s_4^{(4)} + \nu s_2^{(4)}) \otimes 1$ with $s_k^{(4)} \in \Gamma^\infty(\bigvee^k T^*M)$. Inserting these results into $f * g - g * f$ as given above the terms involving $s_2^{(4)} \in \Gamma^\infty(\bigvee^2 T^*M)$ cancel because of their symmetry and one gets

$$f * g - g * f = \nu\{f, g\} - \nu^2(\Omega_1 + ds_1^{(3)})(X_f, X_g) + O(\nu^3)$$

proving the proposition. □

One should observe that this is the only instance of our proof of Theorem 2 where the modified normalization condition on r enters our considerations, whereas the other terms of $c(*)$ could be computed without making use of it.

Acknowledgement. I would like to thank Martin Bordemann for many useful discussions and suggestions, in particular for pointing out reference [4] containing the deformed Cartan formula. Moreover, I would like to thank Stefan Waldmann for many valuable discussions via e-mail during his stay at the Math Department of UC Berkeley. Finally, I should like to thank Alexander V. Karabegov for the inspiring discussion during his visit at the seminar on fundamental interactions at the University of Freiburg.

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Communicated by A. Connes