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On the Spectrum of a Class of Second Order Periodic Elliptic Differential Operators

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Abstract: Under an additional symmetry condition, we prove that the spectrum of a second order self-adjoint elliptic differential operator with periodic coefficients is purely absolutely continuous.

1. Introduction

Let

$$L = -\sum_{p,l=1}^{n} \frac{\partial}{\partial x_p} g_{pl}(x) \frac{\partial}{\partial x_l} + \frac{1}{i} \sum_{l=1}^{n} \left(a_l(x) \frac{\partial}{\partial x_l} + \frac{\partial}{\partial x_l} a_l(x) \right) + V(x) \qquad (1.1)$$

be an elliptic second order differential operator in $\mathbf{R}^n$. We assume that $g_{pl}(x) = g_{lp}(x)$, $p, l = 1, \ldots, n$, all the functions $g_{pl}(x)$, $a_l(x)$, and $V(x)$ are smooth, real-valued, and 2π-periodic in all variables. The differential expression (1.1) defines a self-adjoint operator in $L^2(\mathbf{R}^n)$. It is believed that its spectrum is always purely absolutely continuous. However, this theorem has not been proven yet. In this paper, we prove that the spectrum of L is absolutely continuous under an additional symmetry assumption on L.

Before we formulate our theorem, let us recall some previous results. In his celebrated paper, L. E. Thomas [Th] proved absolute continuity of the spectrum for a periodic Schrödinger operator. M. Sh. Birman and T. A. Suslina proved the theorem for a two-dimensional magnetic Schrödinger operator [BS], and A. Sobolev [S] proved it for a magnetic Schrödinger operator in higher dimensions. A. Morame proved in [M] the absence of singular spectrum for a two-dimensional periodic Schrödinger operator in the case of a non-constant metric (see also [KL].) There have been a number of recent publications on the subject; we are not going to review them here. If $n > 2$, all previously known results deal essentially with the situations where the leading coefficients $g_{pl}(x)$ are constant.

Our additional assumption will be that the operator L is invariant under the symmetry $x_1 \mapsto -x_1$. We will use the following notations. The indices that take values from 1 to n will be denoted by Roman letters; the indices that take values from 2 to n will be denoted by Greek letters. If $x = (x_1, x_2, \ldots x_n)$ then $x' = (x_2, \ldots, x_n)$, so $x = (x_1, x')$. In terms of the coefficients of L, our symmetry assumption means that

$$\begin{aligned} g_{11}(-x_1, x') &= g_{11}(x_1, x'), \quad g_{\alpha\beta}(-x_1, x') = g_{\alpha\beta}(x_1, x'), \\ a_\alpha(-x_1, x') &= a_\alpha(x_1, x'), \quad V(-x_1, x') = V(x_1, x'), \\ g_{1\alpha}(-x_1, x') &= -g_{1\alpha}(x_1, x'), \quad a_1(-x_1, x') = -a_1(x_1, x'). \end{aligned} \tag{1.2}$$

Theorem. *Assume that the operator L given by (1.1) is elliptic, that the functions $g_{pl}(x)$, $a_l(x)$, and $V(x)$ are smooth, real-valued, 2π-periodic in all variables, and that they satisfy (1.2). Then the spectrum of the operator L in $L^2(\mathbf{R}^n)$ is purely absolutely continuous.*

Let us recall some facts from Floquet's theory (e.g., see [Ku].) Let $k = (k_1, \ldots, k_n) \in \mathbf{R}^n$. One introduces a family of operators

$$\begin{aligned} L(k) = &- \sum_{p,l=1}^{n} \left(\frac{\partial}{\partial x_p} + ik_p \right) g_{pl}(x) \left(\frac{\partial}{\partial x_l} + ik_l \right) \\ &+ \frac{1}{i} \sum_{l=1}^{n} \left(a_l(x) \left(\frac{\partial}{\partial x_l} + ik_l \right) + \left(\frac{\partial}{\partial x_l} + ik_l \right) a_l(x) \right) + V(x). \end{aligned} \tag{1.3}$$

As a set, the L^2 spectrum of the operator L is the union of periodic spectra of operators $L(k)$ over all $k \in \mathbf{R}^n$. (Actually, one can take $k \in [0, 1)^n$.) It follows from a theorem of Thomas [Th, Ku] that the spectrum of L is not purely absolutely continuous if, for some value of λ, the equation

$$(L(k) - \lambda)u = 0 \tag{1.4}$$

has a non-trivial periodic solution for any choice of $k \in \mathbf{C}^n$. Let us emphasize that here the *quasi-momentum* k is allowed to be complex-valued. We will assume that this is the case, and our assumption will eventually lead us to a contradiction. Because $L(k) - \lambda$ is an operator of the type (1.1), with $V(x)$ replaced by $V(x) - \lambda$, we can assume that $\lambda = 0$. So, our assumption is

$$\ker(L(k)) \neq 0, \quad k \in \mathbf{C}^n. \tag{1.4}$$

Here, $L(k)$ is considered an operator acting on periodic functions.

In Sect. 2 we exhibit our main construction, and in Sect. 3 we prove the theorem.

2. The Main Construction

First, we restrict ourselves to quasi-momenta $k = (k_1, 0, \ldots, 0)$. With some abuse of notations, we will use k for k_1. Then, the problem of finding periodic solutions of the equation $L(k)u = 0$ is equivalent to the problem of finding solutions of the equation $Lu = 0$ that are periodic in x'-variables, and that satisfy the quasi-periodicity condition

$$u(x_1 + 2\pi, x') = e^{2\pi ik} u(x_1, x'). \tag{2.1}$$

Let $C = [-\pi, \pi] \times \mathbf{T}^{n-1}$ be a cylinder; here $\mathbf{T}^{n-1}$ is an $(n-1)$-dimensional torus. We denote $\zeta = \exp(2\pi i k) \neq 0$. Then the above problem is equivalent to the boundary value problem

$$Lu = 0 \quad \text{in } C, \quad u(\pi, x') = \zeta u(-\pi, x'), \quad \frac{\partial u}{\partial x_1}(\pi, x') = \zeta \frac{\partial u}{\partial x_1}(-\pi, x'). \tag{2.2}$$

Let $\Gamma_\pm = \{\pm\pi\} \times \mathbf{T}^{n-1}$ be the top and the bottom of the cylinder C. The symmetry assumptions (1.2) imply that $g_{1\alpha}(\pm\pi, x') = 0$, so the x_1 direction is normal to both the top and the bottom of the cylinder. It is convenient to take $\partial_\nu = \partial/\partial\nu = \pm g_{11}(\pm\pi, x')\partial/\partial x_1$ as a standard outward normal vector to $\Gamma_\pm$ at $(\pm\pi, x')$. With this convention, one has the standard Green formula

$$(Lu, v)_C = (u, Lv)_C - (\partial_\nu u, v)_\Gamma + (u, \partial_\nu v)_\Gamma, \tag{2.3}$$

where $\Gamma = \Gamma_+ \cup \Gamma_-$, and $(\cdot, \cdot)$ is the usual L^2 scalar product. Notice that $a_1(\pm\pi, x') = 0$ (see (1.2)), so there are no boundary terms that come from first order terms in (1.1).

We will make the reduction of problem (2.2) to the boundary. To make this reduction, we introduce the Dirichlet-to-Neumann operators. Let $\ker(L_D)$ be the space of solutions of the Dirichlet problem for the equation $Lu = 0$ in C. This space is finite dimensional. We introduce a space

$$\mathcal{L} = \{\phi(x') \in L^2(\mathbf{T}^{n-1}) : \ \phi(x') = \partial_\nu u(-\pi, x') \quad \text{for some } u \in \ker(L_D)\}. \tag{2.4}$$

Notice that, in this definition, one can replace $-\pi$ by π because the operator L is invariant under the reflection $x_1 \mapsto -x_1$. It is a standard fact from the elliptic theory that the boundary value problem

$$Lu = 0 \quad \text{in } C, \quad u(-\pi, x') = \psi(x'), \ u(\pi, x') = 0 \tag{2.5}$$

is solvable if and only if $\psi \perp \mathcal{L}$. If problem (2.5) is solvable then its solution is not unique, but one can find the unique solution that satisfies an additional constraint $\partial_\nu u(-\pi, x') \perp \mathcal{L}$. Such a solution will be denoted by $P\psi$. (P stands for "Poisson operator.") For a function $u(x)$ in C, we define $j_\pm u$ to be its normal derivatives on $\Gamma_\pm$. Finally we define the Dirichlet-to-Neumann operators

$$N_0\psi = j_- P\psi, \quad N_1\psi = j_+ P\psi, \quad \psi \in \mathcal{L}^\perp. \tag{2.6}$$

In words, one takes the solution $u(x)$ of (2.5) that satisfies the additional condition $\partial_\nu u(-\pi, x') \perp \mathcal{L}$; then $N_0\psi = \partial_\nu u(-\pi, x')$ and $N_1\psi = \partial_\nu u(\pi, x')$. Clearly, N_0 maps $\mathcal{L}^\perp$ into $\mathcal{L}^\perp$. Because the operator L is invariant under the symmetry $x_1 \mapsto -x_1$, one can interchange Γ_+ and Γ_-. It means that if $u(x)$ is the solution of $Lu = 0$ such that $u(-\pi, x') = 0$, $u(\pi, x') = \psi$, and $\partial_\nu u(\pi, x') \in \mathcal{L}^\perp$ then $\partial_\nu u(\pi, x') = N_0\psi$ and $\partial_\nu u(-\pi, x') = N_1\psi$. This is actually the main reason why the symmetry assumption is helpful.

It is known that N_0 is an elliptic pseudo-differential operator of order 1, its principal symbol is positive; so the number of its non-positive eigenvalues is finite. The fact that it is defined not on the whole Sobolev space $H^1(\mathbf{T}^{n-1})$ but only on its subspace of finite codimension is not essential. The operator N_1 is a smoothing operator because the Schwarz kernel of the Poisson operator P is smooth outside of Γ_-.

To make the reduction of problem (2.2) to the boundary, we set $u(-\pi, x') = \psi(x')$, and solve the equation $Lu = 0$, together with the first boundary condition in (2.2); then the second boundary condition will give us an equation for $\psi(x)$. We start from the following proposition.

Proposition 1. *Let $\zeta \neq \pm 1$. Then the problem*

$$Lf = 0 \quad \text{in } C, \quad f(-\pi, x') = \psi(x'), \quad f(\pi, x') = \zeta\psi(x') \tag{2.7}$$

is solvable if and only if $\psi(x') \in \mathcal{L}^{\perp}$.

Proof. Let

$$\tilde{\mathcal{L}} = \left\{ \begin{pmatrix} \partial_\nu u(-\pi, x') \\ \partial_\nu u(\pi, x') \end{pmatrix} : \ u(x) \in \ker(L_D) \right\}.$$

Recall that L_D is the operator in C given by the differential expression (1.1), with the Dirichlet boundary conditions. The problem (2.7) is solvable if and only if

$$\begin{pmatrix} \psi \\ \zeta\psi \end{pmatrix} \perp \tilde{\mathcal{L}}. \tag{2.8}$$

The operator L is invariant under the reflection $x_1 \mapsto -x_1$, so the kernel of L_D splits into the direct sum of even solutions and odd solutions of $Lu = 0$,

$$\ker(L_D) = (\ker(L_D))^{\text{ev}} \oplus (\ker(L_D))^{\text{odd}}.$$

This splitting gives rise to the splitting $\tilde{\mathcal{L}} = (\tilde{\mathcal{L}})^{\text{ev}} + (\tilde{\mathcal{L}})^{\text{odd}}$. Denote by $\mathcal{L}^{\text{ev(odd)}}$ the space of first components from $\tilde{\mathcal{L}}^{\text{ev(odd)}}$. Then $\mathcal{L} = \mathcal{L}^{\text{ev}} + \mathcal{L}^{\text{odd}}$. Notice that

$$\tilde{\mathcal{L}}^{\text{ev}} = \left\{ \begin{pmatrix} \phi^{\text{ev}} \\ \phi^{\text{ev}} \end{pmatrix} : \ \phi^{\text{ev}} \in \mathcal{L}^{\text{ev}} \right\} \quad \text{and} \quad \tilde{\mathcal{L}}^{\text{odd}} = \left\{ \begin{pmatrix} \phi^{\text{odd}} \\ -\phi^{\text{odd}} \end{pmatrix} : \ \phi^{\text{odd}} \in \mathcal{L}^{\text{odd}} \right\}.$$

Now (2.8) holds if and only if $(1 + \zeta)(\psi, \phi^{\text{ev}}) = 0$ for every $\phi^{\text{ev}} \in \mathcal{L}^{\text{ev}}$ and $(1 - \zeta)(\psi, \phi^{\text{odd}}) = 0$ for every $\phi^{\text{odd}} \in \mathcal{L}^{\text{odd}}$. Our assumption $\zeta \neq \pm 1$, together with $\mathcal{L} = \mathcal{L}^{\text{ev}} + \mathcal{L}^{\text{odd}}$, implies that this is equivalent to $\psi \perp \mathcal{L}$. □

Let $\zeta \neq \pm 1, 0$. For $\psi \in \mathcal{L}^{\perp}$, the general solution of problem (2.7) is

$$f(x) = (P\psi)(x_1, x') + \zeta(P\psi)(-x_1, x') + v(x), \quad v \in \ker(L_D).$$

Here, once again, we used the invariance of L under the reflection $x_1 \mapsto -x_1$. The last boundary condition from (2.2) is equivalent to

$$\frac{\partial f}{\partial \nu}(\pi, x') + \zeta \frac{\partial f}{\partial \nu}(-\pi, x') = 0.$$

In terms of the Dirichlet-to-Neumann operators, the last equality can be rewritten as

$$2\zeta N_0\psi + (1 + \zeta^2)N_1\psi + \frac{\partial v}{\partial \nu}(\pi, x') + \zeta \frac{\partial v}{\partial \nu}(-\pi, x') = 0. \tag{2.9}$$

In particular,

$$2\zeta N_0\psi + (1 + \zeta^2)N_1\psi \in \mathcal{L}. \tag{2.10}$$

Proposition 2. *Let $\zeta \neq \pm 1, 0$. The problem (2.2), with $u(-\pi, x') = \psi(x')$ is solvable if and only if $\psi \in \mathcal{L}^{\perp}$, and (2.10) holds.*

Proof. The "only if" part has already been proven. Let us do the "if" part. Assume that (2.10) holds. Denote

$$-\phi(x') = 2\zeta N_0\psi + (1+\zeta^2)N_1\psi \in \mathcal{L}.$$

We decompose ϕ as a sum $\phi^{\text{ev}} + \phi^{\text{odd}}$. (See the proof of Proposition 1.) Let $v^{\text{ev(odd)}}$ be the even (odd) solution of the Dirichlet problem for $Lv = 0$ such that $\partial_\nu v^{\text{ev(odd)}}(-\pi, x') = \phi^{\text{ev(odd)}}(x')$. Then, (2.9) is satisfied for the function

$$v(x) = \frac{v^{\text{ev}}(x)}{1+\zeta} + \frac{v^{\text{odd}}(x)}{1-\zeta}. \qquad \square$$

We conclude that assumption (1.5) implies that the inclusion (2.10) has a non-trivial solution $\psi \in \mathcal{L}^\perp$ for every $\zeta \neq \pm 1, 0$. In the next section we will show that this can not happen.

3. Proof of the Theorem

Let Q be the orthogonal projection onto the space $\mathcal{L}^\perp$, and let $z = (\zeta^2+1)/2\zeta$. Then (2.10) can be rewritten as

$$N_0\psi + z\tilde{N}_1\psi = 0, \quad \psi \in \mathcal{L}^\perp, \tag{3.1}$$

where $\tilde{N}_1 = QN_1$. The assumption (1.5) implies that Eq. (3.1) has a non-trivial solution for every $z \neq \pm 1$. First, we establish some simple properties of the operators N_0 and $\tilde{N}_1$.

Proposition 3. (i) *Operators N_0 and $\tilde{N}_1$ are self-adjoint in $\mathcal{L}^\perp \subset L^2(\mathbf{T}^{n-1})$.*
(ii) $\ker(\tilde{N}_1) = \{0\}$.

Proof. Let $\psi_1, \psi_2 \in \mathcal{L}^\perp$, and let $u_j(x_1, x') = P\psi_j(x')$, $j = 1, 2$. One applies Green's formula (2.3) to u_1 and u_2 to get

$$(N_0\psi_1, \psi_2) = (\partial_\nu u_1, u_2)_{\Gamma_-} = (u_1, \partial_\nu u_2)_{\Gamma_-} = (\psi_1, N_0\psi_2).$$

This means that the operator N_0 is symmetric, and, if one takes $H^1(\mathbf{T}^{n-1}) \cap \mathcal{L}^\perp$ as its domain, then it becomes self-adjoint.

Let $v(x) \in \ker(L_D)$ be such a function that

$$\partial_\nu(u_2(-x_1, x') + v(x)) \in \mathcal{L}^\perp \quad \text{when } x_1 = -\pi.$$

Let $w(x) = u_2(-x_1, x') + v(x)$. The invariance of L under the reflection $x_1 \mapsto -x_1$ implies $Lw = 0$. In addition,

$$\partial_\nu w(-\pi, x') = \tilde{N}_1\psi_2, \quad w(\pi, x') = \psi_2(x').$$

We apply Green's formula (2.3) to u_1 and w to get

$$(\tilde{N}_1\psi_1, \psi_2) = (\partial_\nu u_1, w)_{\Gamma_+} = (u_1, \partial_\nu w)_{\Gamma_-} = (\psi_1, \tilde{N}_1\psi_2).$$

This equation shows that the operator $\tilde{N}_1$ is self-adjoint. (The operator $\tilde{N}_1$ is bounded, so one does not have to worry about its domain.)

Finally, suppose that $\tilde{N}_1\psi = 0$. Let $u(x) = P\psi$. One has $Lu = 0$, $u(-\pi, x') = \psi(x')$, $u(\pi, x') = 0$, and $\partial_\nu u(\pi, x') \in \mathcal{L}$. Let $v(x) \in \ker(L_D)$, and $\partial_\nu v(\pi, x') = \partial_\nu u(\pi, x')$. Then the function $w = u - v$ is a solution of the equation $Lw = 0$, and, on Γ_+, both w and its normal derivative vanish. Therefore, $w(x) = 0$, and $\psi(x') = u(-\pi, x') = v(-\pi, x') = 0$. $\square$

It follows from the theory of analytic families of operators (e.g., see [Ka]) that $\dim\ker(N_0 + z\tilde{N}_1) = \text{const}$ for all complex numbers z, outside a discrete set $E \subset \mathbf{C}$. Our assumption that (3.1) has a non-trivial solution for all $z \neq \pm 1$ implies that this constant is positive. Moreover, the Riesz projections onto $\ker(N_0 + z\tilde{N}_1)$ depend on z analytically in $\mathbf{C} \setminus E$. In particular, one can construct a family of functions $\psi(z)$, $z \in \mathbf{R} \setminus E$, such that $\|\psi(z)\| = 1$, $\psi(z)$ solves (3.1), and $\psi(z)$ is continuous in z. We restrict z to the real axis as a matter of convenience. Let us show that

$$(\tilde{N}_1\psi(z_1), \psi(z_2)) = 0 \tag{3.2}$$

for any $z_1, z_2 \in \mathbf{R} \setminus E$. If $z_1 \neq z_2$ then

$$\begin{aligned} 0 &= ((N_0 + z_1\tilde{N}_1)\psi(z_1), \psi(z_2)) = (\psi(z_1), (N_0 + z_2\tilde{N}_1)\psi(z_2)) \\ &\quad + (z_1 - z_2)(\tilde{N}_1\psi(z_1), \psi(z_2)) = (z_1 - z_2)(\tilde{N}_1\psi(z_1), \psi(z_2)), \end{aligned}$$

and (3.2) follows immediately. If $z_2 = z_1$ then we take the limit $z_2 \to z_1$ in (3.2). Equation (3.2) implies

$$(N_0\psi(z_1), \psi(z_2)) = 0, \quad z_1, z_2 \in \mathbf{R} \setminus E. \tag{3.3}$$

Let $\mathcal{M}$ be the linear span of all functions $\psi(z)$, $z \in \mathbf{R} \setminus E$. It follows from (3.3) that $(N_0\psi, \psi) = 0$ for every $\psi \in \mathcal{M}$. Let us recall that N_0 is a self-adjoint, bounded from below operator with discrete spectrum. Therefore, $\dim\mathcal{M} < \infty$. Now, let $z_j \in \mathbf{R} \setminus E$ be a sequence such that $z_j \to \infty$, and let $\psi_j = \psi(z_j)$. The functions ψ_j lie on a finite-dimensional sphere in $\mathcal{L}^\perp$, so one can assume that $\psi_j \to \psi$. Note that $\|\psi\| = 1$, so $\psi \neq 0$. One has

$$\tilde{N}_1\psi_j = -\frac{1}{z_j}N_0\psi_j. \tag{3.4}$$

The operator $\tilde{N}_1$ is bounded, and the restriction of N_0 to $\mathcal{M}$ is bounded. ($\mathcal{M}$ is finite-dimensional!) By taking the limit $j \to \infty$ in (3.4), one gets $\tilde{N}_1\psi = 0$. This contradicts statement (ii) of Proposition 3. $\square$

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