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Effects of uncertainties in the domain on the solution of Dirichlet boundary value problems

Ivo Babuška[1,⋆], **Jan Chleboun**[2,⋆⋆]

[1] TICAM, The University of Texas at Austin, TX 78713, USA;
e-mail: babuska@mail.utexas.edu
[2] Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, Prague 115 67, Czech Republic; e-mail: chleb@math.cas.cz

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Summary. A domain with possibly non-Lipschitz boundary is defined as a limit of monotonically expanding or shrinking domains with Lipschitz boundary. A uniquely solvable Dirichlet boundary value problem (DBVP) is defined on each of the Lipschitz domains and the limit of these solutions is investigated. The limit function also solves a DBVP on the limit domain but the problem can depend on the sequences of domains if the limit domain is unstable with respect to the DBVP. The core of the paper consists in estimates of the difference between the respective solutions of the DBVP on two close domains, one of which is Lipschitz and the other can be unstable. Estimates for starshaped as well as rather general domains are derived. Their numerical evaluation is possible and can be done in different ways.

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1 Introduction

The paper deals with uncertain boundary in the definition of Dirichlet boundary value problems. A boundary value problem is defined by a domain, an equation in the domain, and a condition given along the boundary of the domain. It is common to assume that the three inputs are known exactly though

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our perception is uncertain to some extent and this uncertainty should be reflected in our mathematical models of the real world, cf. [10].

The geometry of a body is usually considered well defined. An inspection reveals, however, that it is not so in many cases due to inaccurate measurements and other limitations. Digital images can serve as typical examples. Their resolution is limited and all details smaller than the pixel size are indistinguishable. Moreover, interpretation of digital images depends on other parameters as, for instance, threshold values for color-based separation of domains, see [3] for black and white examples. As a consequence, a digital image should be interpreted as a representation of a whole family of bodies hidden under the umbrella of an uncertainty.

If such a family of bodies is taken into account instead of a unique domain then also one has to consider a respective family of boundary value problems and their solutions. A natural question arises whether it is possible to asses the effect of the uncertainty in the domain on the solution of the boundary value problem.

The question is addressed for two dimensional domains and a second order elliptic equation with the *Neumann* boundary condition in [3]. An uncertain domain Ω is represented by the limit of a monotone sequence of domains with Lipschitz boundary. It corresponds to a sequence of domains defined via smaller and smaller pixels in our digital image example. As the convergence in the set sense is considered only, the boundary of the limit domain can be "wild" (non-Lipschitz). This can give rise to the *unstability* phenomenon. It means that the limit of the solutions of the boundary value problems on a sequence of domains depends on the sequence though the limit domain Ω is identical for the all sequences. In detail, the limit function may not exist or different limits can exist for different sequences of domains.

If the limit domain Ω is *stable* then the limit of solutions, let us denote it u, does not depend on the sequence approaching Ω. Moreover, function u is the unique solution of a boundary value problem (BVP) naturally corresponding to the BVP on the sequence of domains. Thus the BVP solution depends continuously on the domain of definition and the problem is well-posed, cf. [7].

In [3], a simple example using domains originating in digital images of a circle shows an unstable behavior of even a noncomplicated Neumann BVP with a nonhomogeneous boundary condition given in a classical way, i.e., as the normal derivative along the boundary. To avoid such unnatural loss of stability, a reformulation of the boundary condition is proposed in [3]. As a consequence, the circle (as well as any Lipschitz domain) becomes a stable domain.

The stability issue for elliptic equations with the *Dirichlet* or the homogeneous Neumann boundary condition was already treated in [1,2], where sufficient conditions guaranteeing that the limit domain Ω is stable are given. An example can be $\partial\Omega$ is Lipschitz but this assumption can be weakened.

The stability problem is not only academic as it has connections to the well-known plate paradox [2,4], to give an example.

Exploiting digital images, we have no chance to know Ω in practice because the pixel size is always greater than a fixed positive value. In other words, we are not able to construct a sequence of domains converging to Ω.

As Ω is uncertain, we have to admit it could be unstable. Unlike other approaches, where the uncertain boundary is rather artificially made certain and piece-wise smooth (see [5,6] for some algorithms) before the BVP is solved, we wish to take the impact of uncertainty into account.

We hope or assume to have reasonable lower and upper bounds for Ω, i.e., domains Ω_{low} and Ω_{up} such that the BVP can be solved there and $\Omega_{\text{low}} \subset \Omega \subset \Omega_{\text{up}}$.

Knowing the respective solutions u_{low} and u_{up}, we wish to asses the difference $u - u_{\text{low}}$ or $u - u_{\text{up}}$ in a proper norm. The energy norm seems to be a natural choice, at least from the theoretical numerical analysis point of view. However, the choice of a domain over which the norm is to be defined is less clear. In essence, there are two basic possibilities. Having $\Omega_1 \subset \Omega_2$ and functions u_1, u_2 defined in Ω_2, we can use either $\|\cdot\|_{\Omega_1}$ or $\|\cdot\|_{\Omega_2}$. We decided for the latter approach. The former one is slightly touched in Sect. 6.

Estimates of the difference between solutions of the Neumann problem on close domains are given in [3].

The current paper focuses on analogous estimates for the Dirichlet boundary value problem, i.e., it is a continuation of [1,2] motivated by [3].

The paper is organized as follows. Basic notions as well as some known results comprise Sect. 2. In Sect. 3, first steps to estimate the difference between two solutions on two close domains are made. Assuming general coefficients of the equation and starshaped domains, Sect. 4 finishes the estimate. Its version for constant coefficients and non-starshaped domains is given in Sect. 5. Numerical examples are presented in Sect. 6.

2 The Dirichlet problem defined on a set of domains

We introduce basic notions and recall some known results in this section.

The Dirichlet boundary value problems we deal with are defined on domains $\Omega, \Omega_n \subset \mathbb{R}^d$, $d \in \{1, 2, 3, \dots\}$. Moreover, we suppose a ball $B \subset \mathbb{R}^d$ exists such that it contains closures of all domains we will consider.

For any domain Ω, $H^k(\Omega)$, $k \in \{1, 2, \dots\}$, is the standard Sobolev space of square integrable functions the generalized partial derivatives up to the order k of which are also square integrable on Ω, $L^2(\Omega) \equiv H^0(\Omega)$.

The space $H^k(\Omega)$ is equipped with the norm $\|\cdot\|_{k,\Omega}$ and the kth seminorm $|\cdot|_{k,\Omega}$, while $\|\cdot\|_{0,\Omega}$ stands for the $L^2(\Omega)$-norm. Identical symbols are used for norms and seminorms of vector functions. In this case, the square of a (semi)norm is defined as the sum of squared (semi)norms of individual components.

The subspace $H_0^k(\Omega)$ equals the closure of $C_0^\infty(\Omega)$ in the $\|\cdot\|_{k,\Omega}$ norm, where $C_0^\infty(\Omega)$ is the space of infinitely smooth functions with their support contained in Ω.

Functions continuous up to the kth derivative on the closure $\overline{\Omega}$ of Ω form the space $C^k(\overline{\Omega})$ if endowed with a proper norm. We confine ourselves to $C(\overline{\Omega}) \equiv C^0(\overline{\Omega})$ with the maximum norm $\|\cdot\|_{\infty,\Omega}$, and $C^1(\overline{\Omega})$, where the norm includes also the seminorm $|v|_{1,\infty,\Omega} = \|\nabla v\|_{\infty,\Omega}$, i.e., $\|v\|_{1,\infty,\Omega} \equiv \max\{\|v\|_{\infty,\Omega}, |v|_{1,\infty,\Omega}\}$. The maximum value of component-wise norms defines $\|w\|_{\infty,\Omega}$ for a vector function w. If a matrix $A \equiv [a_{ij}]_{i,j=1}^d$ comprises elements from $C^1(\overline{\Omega})$ then $|A|_{1,\infty,\Omega} = \max\{|a_{ij}|_{1,\infty,\Omega} : \ i,j = 1,\dots,d\}$.

For simplicity reasons, we limit ourselves almost exclusively to second order scalar equations. Comments on higher order equations and systems of equations will be given in the course of exposition.

Let $f \in L^2(B)$, $\psi \in H^1(B)$ and $\Omega \subset \overline{\Omega} \subset B$ with a Lipschitz boundary be given. The Dirichlet boundary value problem on Ω then reads: Find $u \in H^1(\Omega)$ such that

$$u - \psi|_\Omega \in H_0^1(\Omega), \tag{2.1}$$

$$a_\Omega(u,v) = F_\Omega(v) \qquad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where

$$a_\Omega(u,v) = \int_\Omega A\nabla u \cdot \nabla v + buv \, \mathrm{d}x, \qquad u,v \in H^1(\Omega), \tag{2.3}$$

$$F_\Omega(v) = \int_\Omega fv \, \mathrm{d}x, \qquad v \in H^1(\Omega), \tag{2.4}$$

and $A = [a_{ij}]_{i,j=1}^d$, $a_{ij} \in L^\infty(B)$ (bounded measurable functions), $b \in L^\infty(B)$. We suppose constants $c_{Ab}, c^{Ab} > 0$ independent of $\Omega \subset B$ exist such that

$$c_{Ab}\|v\|_{1,\Omega}^2 \le a_\Omega(v,v) \qquad \forall v \in H_0^1(\Omega), \tag{2.5}$$

$$|a_\Omega(v,w)| \le c^{Ab}\|v\|_{1,\Omega}\|w\|_{1,\Omega} \qquad \forall v,w \in H^1(\Omega). \tag{2.6}$$

On the basis of (2.5)–(2.6), we can define a new norm $\|\cdot\|_{a,\Omega} \equiv (a_\Omega(\cdot,\cdot))^{1/2}$ on $H_0^1(\Omega)$. It is equivalent to $\|\cdot\|_{1,\Omega}$.

As indicated in Sect. 1, we will pay attention to uncertain boundary of domains. More precisely, we suppose that we have a domain $\Omega \subset B$ and a sequence of *known* domains $\Omega_n \subset B$, $n \to \infty$, converging to Ω in the set sense. It means that $x \in \Omega$ implies $\exists n_x \forall n > n_x\ x \in \Omega_n$, and that if $\exists n_y\ \forall n > n_y\ y \in \Omega_n$ then $y \in \overline{\Omega}$. Obviously, the convergence does not preserve the Lipschitz property of the boundary. We assume $\partial\Omega = \partial\overline{\Omega}$, i.e., Ω with cracks is excluded.

Dirichlet problem (2.1)–(2.2) is defined on all Ω_n, $n = 1, 2, \ldots,$ and its solution is denoted u_n. By the properties of a_Ω and F_Ω, (2.5)–(2.6) and the Lax-Milgram lemma, there exists a unique $u_n \in H^1(\Omega_n)$, $u_n = \psi$ on $\partial\Omega_n$. The first goal is to determine $\lim_{n\to\infty} u_n$.

According to [1,2], it is sufficient to restrict ourselves to monotone sequences of domains because these play a crucial role in the *stability phenomenon*.

We consider expanding domains $\Omega_n \nearrow \Omega$, i.e., $\Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1} \subset B$, $n = 1, 2, \ldots$, $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, and shrinking domains $\Omega_n \searrow \Omega$, i.e., $\Omega_{n+1} \subset \overline{\Omega}_{n+1} \subset \Omega_n \subset B$, $n = 1, 2, \ldots$, $\overline{\Omega} = \bigcap_{n=1}^{\infty} \Omega_n$ (the assumption $\partial\Omega = \partial\overline{\Omega}$ is crucial here). If u_n is the solution of (2.1)–(2.2) on Ω_n, a question arises whether a limit u of u_n exists and, if yes, what equation it solves.

The problem (with $F_\Omega = 0$) was addressed in [1,2], where a modified version of (2.1)–(2.2) was used. In that formulation, $u \in H^1(B)$ and $u = \psi$ outside Ω. It was shown that $u_n \to u_\psi$ and $u_n \to u^\psi$ in $H^1(B)$, where the first limit corresponds to $\Omega_n \nearrow \Omega$ and the second one to $\Omega_n \searrow \Omega$, respectively. Function $u_\psi \in \frac{\perp}{a} H_\Omega$ solves (2.1)–(2.2) on Ω, $\frac{\perp}{a} H_\Omega = \bigcap_{n=1}^{\infty} \frac{\perp}{a} H_{\Omega_n}$, where $\Omega_n \nearrow \Omega$ and $\frac{\perp}{a} H_{\Omega_n}$ is the orthogonal complement of $H_0^1(\Omega_n)$ in $H^1(B)$ with respect to the scalar product induced by the bilinear form $a_B(\cdot,\cdot)$ after the prolongation of functions from $H_0^1(\Omega_n)$ by zero. Let us notice that $\frac{\perp}{a} H_{\Omega_{n+1}} \subset \frac{\perp}{a} H_{\Omega_n}$ if $\Omega_n \nearrow \Omega$. On the other hand, if $\Omega_n \searrow \Omega$ then $u^\psi \in \frac{\perp}{a} \widetilde{H}_\Omega = \overline{\bigcup_{n=1}^{\infty} \frac{\perp}{a} H_{\Omega_n}}$ solves (2.1)–(2.2) on Ω for all v belonging to a certain space $\widetilde{H}_0^1(\Omega) \supset H_0^1(\Omega)$. The closure is taken in the energy norm induced by $a_B(\cdot,\cdot)$.

In general, it can be $\frac{\perp}{a} \widetilde{H}_\Omega \neq \frac{\perp}{a} H_\Omega$. If $\frac{\perp}{a} \widetilde{H}_\Omega = \frac{\perp}{a} H_\Omega$ holds we say that Ω is a *stable* domain with respect to the Dirichlet equation (D-stable). Then also $u_\psi = u^\psi$ for any $\psi \in H^1(B)$. Otherwise Ω is a D-unstable domain and we can find $\psi \in H^1(B)$ such that $u_\psi \neq u^\psi$, see [1, Sect. 3, 4]; [2, Sect. 5]. An example of a D-unstable domain for the Laplace operator and two spatial dimensions is shown in [2, Theorem 5.8]. Let us remark that the stability behavior of harmonic and l-harmonic operators is representative of the behavior of the elasticity operator and higher order elliptic scalar operators, respectively, see [2, Theorem 7.1, Theorem 5.2]. If Ω is a starshaped domain or if it has the σ-property [2, Definition 5.3] or [3, Definition 3.1] then Ω is D-stable, see [2, Theorem 5.4, Theorem 5.5]. Any Lipschitz domain has the σ-property, the opposite is not true in general.

We arrive at similar conclusions if $F_{\Omega_n} \neq 0$, i.e., $0 \neq f_n \equiv f|_{\Omega_n}$. Then the solution u_n of (2.1)–(2.2) can be represented as $u_n = u_{f_n} + w_n$, where $u_{f_n} \in H_0^1(\Omega_n)$ solves (2.2) on Ω_n:

$$a_{\Omega_n}(u_{f_n}, v) = F_{\Omega_n}(v) \qquad \forall v \in H_0^1(\Omega_n), \tag{2.7}$$

and $w_n \in H^1(\Omega_n)$ solves (2.1)–(2.2) on Ω_n with $F_{\Omega_n} = 0$. In what follows, functions defined on B will often be restricted to Ω or Ω_n. Also, functions from $H_0^1(\Omega)$, $H_0^1(\Omega_n)$ will be prolonged by zero to get functions from $H^1(B)$. For the sake of simplicity, restrictions and prolongations will be made tacitly and no new symbols will be introduced to distinguish between the original function and its restriction or prolongation.

By virtue of the previous paragraphs, $w_n \to w_\psi$ or $w_n \to w^\psi$ in $H^1(B)$ respectively, for $\Omega_n \nearrow \Omega$ or $\Omega_n \searrow \Omega$. The next lemma focuses on functions u_{f_n}.

Lemma 2.1 *Let $\Omega_n \searrow \Omega$. Then $u_{f_n} \to \widetilde{u}_f$ in $H^1(B)$, $\widetilde{u}_f \in \widetilde{H}_0^1(\Omega)$ and*

$$a_\Omega(\widetilde{u}_f, v) = F_\Omega(v) \qquad \forall v \in \widetilde{H}_0^1(\Omega), \tag{2.8}$$

where

$$\widetilde{H}_0^1(\Omega) = \bigcap_{n=1}^{\infty} H_0^1(\Omega_n). \tag{2.9}$$

Proof. We can apply the technique that acquitted itself well in convergence proofs in the theory of optimal shape design, see [8]. Let us remark that (2.5), (2.9) and the Lax-Milgram lemma imply the existence and uniqueness of $\widetilde{u}_f$.

By (2.5) and (2.7)

$$c_{Ab} \|u_{f_n}\|_{1,\Omega_n}^2 \leq a_{\Omega_n}(u_{f_n}, u_{f_n}) = F_{\Omega_n}(u_{f_n}) \leq \|f\|_{0,B} \|u_{f_n}\|_{1,\Omega_n}.$$

Realizing that $\|u_{f_n}\|_{1,\Omega_n} = \|u_{f_n}\|_{1,B}$, we get

$$\|u_{f_n}\|_{1,B} \leq C, \tag{2.10}$$

$C > 0$ is a constant independent of n.

The sequence $\{u_{f_n}\}_{n=1}^\infty$ is bounded in $H^1(B)$ so a weakly convergent subsequence $\left\{u_{f_{n_i}}\right\}_{i=1}^\infty$ exists, i.e., $u_{f_{n_i}} \rightharpoonup \widetilde{u}$ (weakly) in $H^1(B)$. The limit function $\widetilde{u}$ belongs to $\widetilde{H}_0^1(\Omega)$. Indeed, if we fix an index $m > 0$ then $u_{f_n} \in H_0^1(\Omega_m)$, for all $n \geq m$. Space $H_0^1(\Omega_m)$ is weakly closed thus $\widetilde{u} \in H_0^1(\Omega_m)$. As m is arbitrary, $\widetilde{u} \in \widetilde{H}_0^1(\Omega)$ due to (2.9).

The next step is to prove $\widetilde{u} = \widetilde{u}_f$, see (2.8). To this end, we fix a function $v \in \widetilde{H}_0^1(\Omega)$. It holds

$$a_{\Omega_{n_i}}(u_{f_{n_i}}, v) = F_{\Omega_{n_i}}(v) \tag{2.11}$$

because $v \in H_0^1(\Omega_n)$ for any n, see (2.9).

By (2.4) and $\mathrm{meas}(\Omega_n \setminus \Omega) \to 0$

$$\lim_{n\to\infty} F_{\Omega_n}(v) = F_\Omega(v). \tag{2.12}$$

By the weak convergence

$$\lim_{i\to\infty} a_{\Omega_{n_i}}(u_{f_{n_i}}, v) = \lim_{i\to\infty} a_B(u_{f_{n_i}}, v) = a_B(\widetilde{u}, v) = a_\Omega(\widetilde{u}, v), \tag{2.13}$$

the last equality is due to the fact that $\widetilde{u}, v \in H_0^1(\Omega_n)$ for any Ω_n,

$$a_B(\widetilde{u}, v) = a_{\Omega_n\setminus\Omega}(\widetilde{u}, v) + a_\Omega(\widetilde{u}, v)$$

and $\mathrm{meas}(\Omega_n \setminus \Omega)$ tends to zero if $n \to \infty$.

Applying (2.12), (2.13) to (2.11), we arrive at (2.8), i.e., $\widetilde{u} \equiv \widetilde{u}_f$. Since the weak limit of any weakly convergent subsequence of $\{u_{f_n}\}$ is equal to a unique function $\widetilde{u}_f$, we get the weak convergence of the whole sequence.

Taking into account (2.5) and $\widetilde{u}_f \in H_0^1(\Omega_n)$, we infer

$$\begin{aligned}
&c_{Ab}\,\|u_{f_n} - \widetilde{u}_f\|_{1,B}^2 \le a_{\Omega_n}(u_{f_n} - \widetilde{u}_f, u_{f_n} - \widetilde{u}_f)\\
&= F_{\Omega_n}(u_{f_n}) - F_{\Omega_n}(\widetilde{u}_f) + a_{\Omega_n}(\widetilde{u}_f, \widetilde{u}_f) - a_{\Omega_n}(\widetilde{u}_f, u_{f_n})\\
&= F_B(u_{f_n}) - F_B(\widetilde{u}_f) + a_B(\widetilde{u}_f, \widetilde{u}_f) - a_B(\widetilde{u}_f, u_{f_n}) \to 0, \quad n \to 0,
\end{aligned}$$

because u_{f_n} converges weakly to $\widetilde{u}_f$ in $H^1(B)$. □

We are placed in a similar position if $\Omega_n \nearrow \Omega$. Then $u_{f_n} \to u_f$ in $H^1(B)$, where $u_f \in H_0^1(\Omega)$ solves

$$a_\Omega(u_f, v) = F_\Omega(v) \qquad \forall v \in H_0^1(\Omega) \tag{2.14}$$

because $H_0^1(\Omega) = \overline{\bigcup_{n=1}^\infty H_0^1(\Omega_n)}$. The closure can be taken in the Sobolev norm or the energy norm or the Sobolev seminorm as all are equivalent norms on $H_0^1(B)$. To show (2.14), it is helpful to follow the proof of Lemma 2.1 with a few minor modifications.

It holds $H_0^1(\Omega) \subset \widetilde{H}_0^1(\Omega)$ and it can be $H_0^1(\Omega) \neq \widetilde{H}_0^1(\Omega)$ in general, see [2, Theorem 5.8]. The latter happens iff ${}^\perp_a\widetilde{H}_\Omega \neq {}^\perp_a H_\Omega$ because

$$\begin{aligned}
{}^\perp\left(\widetilde{H}_0^1(\Omega)\right) &= {}^\perp\left(\bigcap_{n=1}^\infty H_0^1(\Omega_n)\right)\\
&= \overline{\bigcup_{n=1}^\infty {}^\perp\left(H_0^1(\Omega_n)\right)} = {}^\perp_a\widetilde{H}_\Omega \quad \text{if} \quad \Omega_n \searrow \Omega,
\end{aligned}$$

$$
\begin{aligned}
{}^{\perp}\left(H_0^1(\Omega)\right) &= {}^{\perp}\left(\overline{\bigcup_{n=1}^{\infty} H_0^1(\Omega_n)}\right) \\
&= \bigcap_{n=1}^{\infty} {}^{\perp}\left(H_0^1(\Omega_n)\right) = {}^{\perp}_{a} H_\Omega \quad \text{if} \quad \Omega_n \nearrow \Omega.
\end{aligned}
$$

Orthogonal complements are defined respective to the inner product induced by the bilinear forms a_Ω and a_{Ω_n}.

Thus, for a D-unstable domain Ω, solutions u_n can converge to different limits respective to sequences $\Omega_n \to \Omega$.

3 Difference between two solutions – introductory steps

Our goal is to estimate the difference between two solutions of problem (2.1)–(2.2) solved on two different but close domains. One of them can be the limit domain Ω.

We suppose domains Ω_1, Ω_2, and Ω_3 are given, $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset B \subset \mathbb{R}^d$. Next, we assume boundaries $\partial\Omega_1$, $\partial\Omega_3$ are known and Lipschitz whereas Ω_2 can be D-unstable. We also suppose $\partial\Omega_2 = \partial\overline{\Omega}_2$ and A is symmetric.

As a consequence, we have to take possible nonuniqueness of the boundary value problem on Ω_2 into account because, as we saw in Sect. 2, the test and trial spaces of the limit problem can depend on the sequence of domains approaching the limit domain.

We aim at approximating a solution on Ω_2 by solutions on more explicit domains Ω_1 and Ω_3. The two respective solutions are denoted u_1 and u_3.

Problem (2.1)–(2.2) with a space $\widetilde{H}$, $H_0^1(\Omega_2) \subset \widetilde{H} \subset \widetilde{H}_0^1(\Omega_2)$, instead of $H_0^1(\Omega_2)$ is solved on Ω_2 and its solution is labeled u_2. For simplicity, we define $H_1 \equiv H_0^1(\Omega_1)$, $H_2 \equiv \widetilde{H}$, $H_3 \equiv H_0^1(\Omega_3)$.

Remark 3.1 Setting $\widetilde{H}$ in the above way, we intend to cover all cases that can happen. First, Ω_2 can be Lipschitz. It could belong to a sequence of domains converging to Ω, to give an example. Second, Ω_2 can be the D-stable limit of a sequence of Lipschitz domains. In both cases, $H_0^1(\Omega_2) = \widetilde{H}_0^1(\Omega_2)$. Third, Ω_2 can be the D-unstable limit of a sequence of Lipschitz domains. Then $H_0^1(\Omega_2) \subsetneqq \widetilde{H}_0^1(\Omega_2)$ and the limit of the BVP solutions depends on the sequence of domains. We studied two basic instances in Sect. 2, i.e., the limit belongs to $H_0^1(\Omega_2)$ or $\widetilde{H}_0^1(\Omega_2)$. If the sequence of domains is not monotone but the limit of solutions exists then the limit belongs to a space $\widetilde{H}$ between the spaces $H_0^1(\Omega_2)$ and $\widetilde{H}_0^1(\Omega_2)$. The limit function solves the Dirichlet BVP defined in Ω_2 by $\widetilde{H}$.

We wish to derive estimates dependent on Ω_2 only through the bounds Ω_1 and Ω_3. □

We can express solutions u_i as

$$u_i = u_{i0} + \psi|_{\Omega_i}, \quad u_{i0} \in H_i, \quad i = 1, 2, 3.$$

To reformulate problem (2.1)–(2.2), we define continuous linear functionals

$$F^{\psi}_{\Omega_i}(v) = F_{\Omega_i}(v) - a_{\Omega_i}(\psi, v), \quad v \in H^1(\Omega_i),\ i = 1, 2, 3,$$

as well as quadratic functionals

$$J_{\Omega_i}(v) = \frac{1}{2} a_{\Omega_i}(v, v) - F^{\psi}_{\Omega_i}(v), \quad v \in H^1(\Omega_i),\ i = 1, 2, 3.$$

Then

$$u_{i0} = \arg\min_{v \in H_i} J_{\Omega_i}(v), \quad i = 1, 2, 3. \tag{3.1}$$

The equivalent formulation is

$$a_{\Omega_i}(u_{i0}, v) = F^{\psi}_{\Omega_i}(v) \quad \forall v \in H_i, \quad i = 1, 2, 3. \tag{3.2}$$

Lemma 3.1 *Let* $u_{i0} \in H_i$ *is given by (3.1) or (3.2),* $i = 1, 2, 3$. *Then*

$$\|u_{20} - u_{10}\|^2_{a,\Omega_2} \le \|u_{30}\|^2_{a,\Omega_3} - \|u_{10}\|^2_{a,\Omega_1}, \tag{3.3}$$

$$\|u_{30} - u_{20}\|^2_{a,\Omega_3} \le \|u_{30}\|^2_{a,\Omega_3} - \|u_{10}\|^2_{a,\Omega_1}. \tag{3.4}$$

Proof. By virtue of $H_1 \subset H_2$ and (3.1)–(3.2)

$$\begin{aligned} -\|u_{20}\|^2_{a,\Omega_2} &= 2 \min_{v \in H_2} J_{\Omega_2}(v) \\ &\le 2 \min_{v \in H_1} J_{\Omega_2}(v) = 2 \min_{v \in H_1} J_{\Omega_1}(v) = -\|u_{10}\|^2_{a,\Omega_1}. \end{aligned} \tag{3.5}$$

Similarly, as $H_2 \subset H_3$,

$$\begin{aligned} -\|u_{30}\|^2_{a,\Omega_3} &= 2 \min_{v \in H_3} J_{\Omega_3}(v) \\ &\le 2 \min_{v \in H_2} J_{\Omega_3}(v) = 2 \min_{v \in H_2} J_{\Omega_2}(v) = -\|u_{20}\|^2_{a,\Omega_2}. \end{aligned} \tag{3.6}$$

Utilizing (3.2) and $H_1 \subset H_2 \subset H_3$, we also have

$$a_{\Omega_2}(u_{20}, u_{10}) = F^{\psi}_{\Omega_2}(u_{10}) = F^{\psi}_{\Omega_1}(u_{10}) = a_{\Omega_1}(u_{10}, u_{10}) = \|u_{10}\|^2_{a,\Omega_1}, \tag{3.7}$$

$$a_{\Omega_3}(u_{30}, u_{20}) = F^{\psi}_{\Omega_3}(u_{20}) = F^{\psi}_{\Omega_2}(u_{20}) = a_{\Omega_2}(u_{20}, u_{20}) = \|u_{20}\|^2_{a,\Omega_2}. \tag{3.8}$$

We infer from (3.7) and (3.6) that

$$\|u_{20} - u_{10}\|^2_{a,\Omega_2} = \|u_{20}\|^2_{a,\Omega_2} - 2a_{\Omega_2}(u_{20}, u_{10}) + \|u_{10}\|^2_{a,\Omega_2}$$

$$= \|u_{20}\|^2_{a,\Omega_2} - \|u_{10}\|^2_{a,\Omega_1}$$
$$\le \|u_{30}\|^2_{a,\Omega_3} - \|u_{10}\|^2_{a,\Omega_1} .$$

Combining (3.8) and (3.5), we get

$$\|u_{30} - u_{20}\|^2_{a,\Omega_3} = \|u_{30}\|^2_{a,\Omega_3} - \|u_{20}\|^2_{a,\Omega_2} \le \|u_{30}\|^2_{a,\Omega_3} - \|u_{10}\|^2_{a,\Omega_1} . \square$$

Remark 3.2 The right-hand sides of the inequalities in Lemma 3.1 do not depend either on Ω_2 or on the stability status of Ω_2. They are also independent of the spatial dimension of Ω_2. In the same way, analogous inequalities can be derived for systems of equations as well as for higher order Dirichlet boundary value problems defined through quadratic functionals. $\square$

Remark 3.3 Lemma 3.1 has an *a posteriori* nature. We have to know u_{30} and u_{10} to assess $u_{20} - u_{10}$ or $u_{30} - u_{20}$. It also offers a hint for estimates based on numerical computation. As domains Ω_1 and Ω_3 are known, u_{10} and u_{30} can be approximated by means of a numerical method, e.g. the finite element method (FEM), the boundary element method, etc. The error of the approximate solution can be estimated via an a posteriori error analysis in many instances. Thus a guaranteed estimate of (3.3), (3.4) can be accomplished. $\square$

Remark 3.4 It can happen that domains estimating Ω_2 from inside and outside, respectively, are not suitable for a numerical treatment of the problem. Pixel domains are typical examples. Their fine boundary structure can require very fine computational meshes, leading to expensive FEM calculations. Approximating pixel domains by more FEM-oriented domains, we can avoid this difficulty and still benefit from the approach proposed in Remark 3.3. $\square$

Remark 3.5 Let Ω be D-stable with respect to monotone sequences of domains and let $u_0^{\Omega} \equiv u_{20}$ be the limit solution of the Dirichlet BVP. According to Lemma 3.1, Ω is D-stable for *any* (i.e., even nonmonotone) sequence of Lipschitz domains converging to Ω and the limit of the respective BVP solutions $u_0^{\widehat{\Omega}_m}$ is u_0^{Ω}.

Indeed, if $\widehat{\Omega}_m \to \Omega$ then we can find Lipschitz domains $\Omega_{10}^m \subset \Omega \cap \widehat{\Omega}_m$ and $\Omega_{30}^m \supset \Omega \cup \widehat{\Omega}_m$ such that $\Omega_{10}^m \nearrow \Omega$ and $\Omega_{30}^m \searrow \Omega$, respectively (cf. [1, Theorem 4.1, Theorem 4.2]). We construct the respective solutions u_{10}^m as well as u_{30}^m, and observe that both u_0^{Ω} and $u_0^{\widehat{\Omega}_m}$ can play the role of u_{20} in Lemma 3.1. Applying (3.4) to the summands on the right-hand side of the triangle inequality

$$\left\| u_0^{\Omega} - u_0^{\widehat{\Omega}_m} \right\|_{a,\Omega_{30}^m} \le \left\| u_0^{\Omega} - u_{30}^m \right\|_{a,\Omega_{30}^m} + \left\| u_{30}^m - u_0^{\widehat{\Omega}_m} \right\|_{a,\Omega_{30}^m} ,$$

we prove the statement by virtue of the D-stability of Ω because $\|u_{30}^m\|^2_{a,\Omega_{30}^m} - \|u_{10}^m\|^2_{a,\Omega_{30}^m}$ tends to zero if $m \to \infty$. □

In the sequel, we will find useful estimates related to u_{10} and u_{30}. As u_{10} solves (3.2), $i = 1$, we get by (2.5), (2.6)

$$
\begin{aligned}
c_{Ab} \|u_{10}\|^2_{1,\Omega_1} &\leq \|u_{10}\|^2_{a,\Omega_1} \\
&= a_{\Omega_1}(u_{10}, u_{10}) \\
&= F^{\psi}_{\Omega_1}(u_{10}) \\
&\leq \left(\|f\|_{0,\Omega_1} + c^{Ab} \|\psi\|_{1,\Omega_1}\right) \|u_{10}\|_{1,\Omega_1} .
\end{aligned}
$$

Then

$$
\|u_{10}\|_{1,\Omega_1} \leq \theta_1 \equiv c_{Ab}^{-1} \left(\|f\|_{0,\Omega_1} + c^{Ab} \|\psi\|_{1,\Omega_1}\right), \tag{3.9}
$$

$$
\|u_{10}\|_{a,\Omega_1} \leq \sqrt{c_{Ab}}\theta_1. \tag{3.10}
$$

Analogously,

$$
\|u_{30}\|_{a,\Omega_3} \leq \sqrt{c_{Ab}}\theta_1', \tag{3.11}
$$

$$
\theta_1' \equiv c_{Ab}^{-1} \left(\|f\|_{0,\Omega_3} + c^{Ab} \|\psi\|_{1,\Omega_3}\right). \tag{3.12}
$$

We will further assess (3.3) and (3.4) in the next sections to get *a priori* estimates dependent solely on known input data as Ω_1, Ω_3, f, ψ, c_{Ab}, etc.

4 Estimates for starshaped domains

If the shapes of Ω_1 and Ω_3 are related with a simple rule we have a chance to find a fairly explicit estimate of the difference between u_2 and u_3 or u_1, respectively.

We suppose Ω_1 is a starshaped domain with respect to the origin, i.e., any half-line starting from the origin intersects $\partial\Omega_1$ only once. We assume Ω_3 is given such that

$$
\Omega_3 = \left\{ y \in \mathbb{R}^d : \; y/\alpha \in \Omega_1 \right\},
$$

where $\alpha > 1$ is a constant, i.e., the mapping $\varkappa(x) = \alpha x$ maps Ω_1 onto Ω_3. Assuming a function v defined on Ω_1, we can define a new function v_α on Ω_3 by $v_\alpha(y) = v_\alpha(\varkappa(x)) = v(x)$. This α-subscript convention applies to other scalar or matrix functions as f, b, or A. If v is differentiable in Ω_1 then

$$
\nabla_y v_\alpha(y) = \alpha^{-1} \nabla_x v(x), \qquad y = \alpha x, \tag{4.1}
$$

where the subscripts x and y symbolize differentiation with respect to the components of x and y, respectively. We observe that $v \in H^1(\Omega_1)$ iff $v_\alpha \in H^1(\Omega_3)$. By the substitution theorem

$$\alpha^{d-2} \|v\|^2_{1,\Omega_1} \le \|v_\alpha\|^2_{1,\Omega_3} = \alpha^{d-2} |v|^2_{1,\Omega_1} + \alpha^d \int_{\Omega_1} v^2 \,\mathrm{d}x$$

$$\le \alpha^d \|v\|^2_{1,\Omega_1} . \tag{4.2}$$

We will follow a simple idea. First, we define

$$u_{10\alpha}(y) = u_{10\alpha}(\varkappa(x)) = u_{10}(x), \quad x \in \Omega_1.$$

Next, we will find a boundary value problem $u_{10\alpha}$ solves on Ω_3. This equation together with (3.2), $i = 3$, will enable us to estimate $\|u_{30} - u_{10\alpha}\|_{a,\Omega_3}$ by means of f, ψ and α. Plugging $\pm u_{10\alpha}$ into the right-hand side of (3.3), (3.4), and using the estimate as well as (4.2), we will express estimates (3.3), (3.4) in known quantities.

Lemma 4.1 *Function* $u_{10\alpha} \in H^1_0(\Omega_3)$ *solves the equation*

$$a^\alpha_{\Omega_3}(u_{10\alpha}, v) = F^{\psi,\alpha}_{\Omega_3}(v) \qquad \forall v \in H^1_0(\Omega_3), \tag{4.3}$$

where

$$a^\alpha_{\Omega_3}(u_{10\alpha}, v) \equiv \int_{\Omega_3} \left(\alpha^2 A_\alpha \nabla u_{10\alpha} \cdot \nabla v + b_\alpha u_{10\alpha} v\right) \mathrm{d}y,$$

$$F^{\psi,\alpha}_{\Omega_3}(v) \equiv \int_{\Omega_3} (f_\alpha - b_\alpha \psi_\alpha) v \,\mathrm{d}y - \alpha^2 \int_{\Omega_3} A_\alpha \nabla \psi_\alpha \cdot \nabla v \,\mathrm{d}y.$$

Proof. Let $\widetilde{v}(x) = v(\varkappa(x))$, $x \in \Omega_1$. Applying (4.1), the substitution theorem and (3.2) on the slightly modified left-hand side of (4.3), we get

$$\int_{\Omega_3} \left[A_\alpha(\alpha \nabla_y u_{10\alpha}) \cdot (\alpha \nabla_y v) + b_\alpha u_{10\alpha} v\right] \mathrm{d}y$$

$$= \alpha^d \int_{\Omega_1} \left(A \nabla_x u_{10} \cdot \nabla_x \widetilde{v} + b u_{10} \widetilde{v}\right) \mathrm{d}x$$

$$= \alpha^d \int_{\Omega_1} (f - b\psi) \widetilde{v} \,\mathrm{d}x - \alpha^d \int_{\Omega_1} A \nabla_x \psi \cdot \nabla_x \widetilde{v} \,\mathrm{d}x$$

$$= \int_{\Omega_3} (f_\alpha - b_\alpha \psi_\alpha)\, v \,\mathrm{d}y - \alpha^2 \int_{\Omega_3} A_\alpha \nabla_y \psi_\alpha \cdot \nabla_y v \,\mathrm{d}y.$$

□

It will be useful to introduce an auxiliary function $\widetilde{u}_{30} \in H^1_0(\Omega_3)$ as the solution of the equation

$$a_{\Omega_3}(\widetilde{u}_{30}, v) = F^{\psi,\alpha}_{\Omega_3}(v) \qquad \forall v \in H^1_0(\Omega_3). \tag{4.4}$$

Referring to (3.2), (4.3), (4.4), and plugging $\pm b_\alpha \psi v$, $\pm A_\alpha \nabla\psi \cdot \nabla v$, $\pm A_\alpha \nabla\psi_\alpha \cdot \nabla v$ at proper places, we can write

$$\begin{aligned}
&|a_{\Omega_3}(u_{30} - \widetilde{u}_{30}, v)| \\
&\quad = \left| F^{\psi}_{\Omega_3}(v) - F^{\psi,\alpha}_{\Omega_3}(v) \right| \\
&\quad = \left| \int_{\Omega_3} (f - f_\alpha - b\psi + b_\alpha \psi_\alpha)\, v \,\mathrm{d}y \right. \\
&\qquad \left. - \int_{\Omega_3} \left(A\nabla\psi \cdot \nabla v - \alpha^2 A_\alpha \nabla\psi_\alpha \cdot \nabla v \right) \mathrm{d}y \right| \\
&\quad \le \|f - f_\alpha\|_{0,\Omega_3} \|v\|_{0,\Omega_3} + \int_{\Omega_3} |b_\alpha - b||\psi v| \,\mathrm{d}y \\
&\qquad + \int_{\Omega_3} |b_\alpha||(\psi_\alpha - \psi) v| \,\mathrm{d}y + \int_{\Omega_3} |(A_\alpha - A)\nabla\psi \cdot \nabla v| \,\mathrm{d}y \\
&\qquad + \int_{\Omega_3} |A_\alpha \nabla(\psi_\alpha - \psi) \cdot \nabla v| \,\mathrm{d}y \\
&\qquad + (\alpha^2 - 1) \int_{\Omega_3} |A_\alpha \nabla\psi_\alpha \cdot \nabla v| \,\mathrm{d}y \\
&\quad = I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \qquad (4.5)
\end{aligned}$$

Similarly, due to (4.4) and (4.3)

$$\begin{aligned}
|a_{\Omega_3}(\widetilde{u}_{30} - u_{10\alpha}, v)| &= \left| F^{\psi,\alpha}_{\Omega_3}(v) - a_{\Omega_3}(u_{10\alpha}, v) \right| \\
&= \left| a^{\alpha}_{\Omega_3}(u_{10\alpha}, v) - a_{\Omega_3}(u_{10\alpha}, v) \right| \\
&\le \left| \int_{\Omega_3} \left(\alpha^2 A_\alpha - A \right) \nabla u_{10\alpha} \cdot \nabla v \,\mathrm{d}y \right| \\
&\quad + \left| \int_{\Omega_3} (b_\alpha - b) u_{10\alpha} v \,\mathrm{d}y \right| \\
&= I_7 + I_8. \qquad (4.6)
\end{aligned}$$

To further estimate (4.5) and (4.6), we need a few auxiliary lemmata. We start with citing [3, Lemma 4.4], where parameters $\varepsilon_0 = (\alpha - 1)r$ and $r = \sup_{x\in\Omega_1} \|x\|_{\mathbb{R}^d}$ appear. Let us notice that $\varepsilon_0 \le (\alpha - 1)\mathrm{diam}\Omega_1$.

Lemma 4.2 *Let $\varphi \in L^1(\Omega_3) \cap C(\Omega_3)$ be a nonnegative function. Then*

$$\int_{\Omega_1} \Big(\int_x^{\alpha x} \varphi(z) \,\mathrm{d}z \Big) \mathrm{d}x \le \varepsilon_0 \int_{\Omega_3} \varphi(x) \,\mathrm{d}x.$$

Proof. The proof utilizes an idea from the proof of [9, Lemma 1.4.6] and can be found in [3], where $d = 2$ is assumed. As its use is expected in Sect. 5, we reproduce the proof modifying it for a general parameter d.

Let us define the function $\gamma(x) = (\alpha - 1)\|x\|_{\mathbb{R}^d}$. Its value at x is equal to the length of the segment $(x, \alpha x)$. Then we define $x/\|x\|_{\mathbb{R}^d} = 0$ if $x = 0$ and calculate

$$\begin{aligned}\int_{\Omega_1} \Big(\int_x^{\alpha x} \varphi(z)\,\mathrm{d}z \Big)\,\mathrm{d}x &= \int_{\Omega_1} \Big(\int_0^{\gamma(x)} \varphi\big(x + t\frac{x}{\|x\|_{\mathbb{R}^d}}\big)\,\mathrm{d}t \Big)\,\mathrm{d}x \\ &\le \int_{\Omega_1} \Big(\int_0^{\varepsilon_0} \varphi\big(x + t\frac{x}{\|x\|_{\mathbb{R}^d}}\big)\,\mathrm{d}t \Big)\,\mathrm{d}x \\ &\le \int_0^{\varepsilon_0} \Big(\int_{\Omega_3} \varphi(x)\,\mathrm{d}x \Big)\,\mathrm{d}t \\ &= \varepsilon_0 \int_{\Omega_3} \varphi(x)\,\mathrm{d}x.\end{aligned}$$

We integrated over spheres with increasing diameter to get the integral over Ω_1 and to infer the last inequality, details in [3]. □

In the next three lemmata, w and w_α are functions on Ω_3, $w_\alpha(y) = w_\alpha(\varkappa(x)) = w(x)$, $x \in \Omega_1$.

Lemma 4.3 *Let* $w \in H^1(\Omega_3)$. *Then*

$$\|w - w_\alpha\|_{0,\Omega_3}^2 \le \alpha^d(\alpha - 1)^2 r^2 |w|_{1,\Omega_3}^2 .$$

Proof. We follow the idea of the proof of [3, Lemma 4.5]. Let us suppose $w \in C^\infty(\Omega_3) \cap H^1(\Omega_3)$.

By the substitution theorem, differentiability of w, the Schwarz inequality, and Lemma 4.2

$$\begin{aligned}\int_{\Omega_3} (w(y) - w_\alpha(y))^2\,\mathrm{d}y &= \int_{\Omega_1} (w(\alpha x) - w(x))^2\,\alpha^d\,\mathrm{d}x \\ &\le \alpha^d \int_{\Omega_1} \Big(\int_x^{\alpha x} |\nabla w(z)|\,\mathrm{d}z \Big)^2\,\mathrm{d}x \\ &\le \alpha^d \int_{\Omega_1} \varepsilon_0 \int_x^{\alpha x} |\nabla w(z)|^2\,\mathrm{d}z\,\mathrm{d}x \\ &\le \alpha^d(\alpha - 1)^2 r^2 \int_{\Omega_3} |\nabla w(x)|^2\,\mathrm{d}x.\end{aligned}$$

Since smooth functions are dense in $H^1(\Omega_3)$, the proof is finished. □

Lemma 4.4 *Let* $w \in H^2(\Omega_3)$. *Then*

$$|w - w_\alpha|_{1,\Omega_3}^2 \le 2\alpha^2(\alpha - 1)^2 \left(r|w|_{2,\Omega_3} + |w|_{1,\Omega_1} \right)^2 .$$

Proof. Lemma 4.4 is, in fact, [3, Lemma 4.5]. The proof is based on the idea presented in the proof of Lemma 4.3 and applied to partial derivatives of w, w_α. □

Lemma 4.5 *Let $w \in C^1(\overline{\Omega}_3)$. Then*

$$|w - w_\alpha|_{\infty,\Omega_3} \le \varepsilon_0 d^{1/2} |w|_{1,\infty,\Omega_3}.$$

Proof. Let $y = \alpha x$, $y \in \Omega_3$, $x \in \Omega_1 \subset \Omega_3$. It holds for a point ζ between x and αx

$$|w(y) - w_\alpha(y)| = |w(\alpha x) - w(x)| \le \varepsilon_0 |\nabla w(\zeta)| \le \varepsilon_0 d^{1/2} |w|_{1,\infty,\Omega_3}.$$
□

Before formulating and proving the next lemma, we recall that

$$A = [a_{ij}]_{i,j=1}^d,\ A_\alpha = [a_{ij}^\alpha]_{i,j=1}^d,\ a_{ij}^\alpha(y) = a_{ij}^\alpha(\alpha x) = a_{ij}(x),$$
$$x \in \Omega_1,\ \alpha > 1.$$

Moreover, we introduce a constant $c^A > 0$ such that

$$\text{(4.7)}\quad \int_{\Omega_3} A\nabla v \cdot \nabla w \,\mathrm{d}y \le c^A |v|_{1,\Omega_3} |w|_{1,\Omega_3} \qquad \forall v, w \in H^1(\Omega_3).$$

Lemma 4.6 *Let $\alpha, \beta > 1$ be two parameters, let $a_{ij} \in C^1(\overline{\Omega}_3)$, $i, j = 1, \dots, d$, and let $v, w \in H^1(\Omega_3)$. Then*

$$\left| \int_{\Omega_3} (\beta A_\alpha - A)\nabla v \cdot \nabla w \,\mathrm{d}y \right|$$
$$\le \left(\varepsilon_0 \beta d^{3/2} |A|_{1,\infty,\Omega_3} + (\beta - 1) c^A \right) |v|_{1,\Omega_3} |w|_{1,\Omega_3},$$

where $|A|_{1,\infty,\Omega_3} = \max\{|a_{ij}|_{1,\infty,\Omega_3} :\ i, j = 1, \dots, d\}$.

Proof. We have

$$\int_{\Omega_3} (\beta A_\alpha - A)\nabla v \cdot \nabla w \,\mathrm{d}y = \int_{\Omega_3} \beta (A_\alpha - A)\nabla v \cdot \nabla w \,\mathrm{d}y$$
$$+ \int_{\Omega_3} (\beta - 1) A\nabla v \cdot \nabla w \,\mathrm{d}y$$
$$= I_9 + I_{10}.$$

By Lemma 4.5

$$\forall y \in \Omega_3 \quad \left| a_{ij}^\alpha(y) - a_{ij}(y) \right| \le \varepsilon_0 \sqrt{d}\, |a_{ij}|_{1,\infty,\Omega_3} \le \varepsilon_0 \sqrt{d}\, |A|_{1,\infty,\Omega_3},$$
$$\text{(4.8)}\qquad i, j = 1, \dots, d.$$

Taking a vector $t \in \mathbb{R}^d$, we define $\widehat{t} \equiv (|t_1|, |t_2|, \dots, |t_d|)$. We also introduce a matrix $M = [m_{ij}]_{i,j=1}^d$, $m_{ij} = 1$, $i, j = 1, \dots, d$.

Applying (4.8), the Schwarz inequality and $\left(\sum_{i=1}^{d}|t_i|\right)^2 \le d\sum_{i=1}^{d} t_i^2$, we can estimate

$$|I_9| \le \beta\varepsilon_0 d^{1/2}|A|_{1,\infty,\Omega_3}\int_{\Omega_3} M\widehat{\nabla v}\cdot\widehat{\nabla w}\,\mathrm{d}y$$
$$\le \beta\varepsilon_0 d^{1/2}|A|_{1,\infty,\Omega_3}\int_{\Omega_3}\left(\sum_{i=1}^{d}\left|\frac{\partial v}{\partial y_i}\right|\right)\left(\sum_{i=1}^{d}\left|\frac{\partial w}{\partial y_i}\right|\right)\mathrm{d}y$$
$$\le \beta\varepsilon_0 d^{3/2}|A|_{1,\infty,\Omega_3}|v|_{1,\Omega_3}|w|_{1,\Omega_3}.$$

Inequality (4.7) gives an upper bound of $|I_{10}|$ and the proof is finished. □

We are ready to finish estimate (4.5).

Lemma 4.7 *Let $f, v \in H^1(\Omega_3)$, $\psi \in H^2(\Omega_3)$, $b, a_{ij} \in C^1(\overline{\Omega}_3)$, $i, j = 1, \dots, d$. Then*

$$|a_{\Omega_3}(u_{30} - \widetilde{u}_{30}, v)| \le (\alpha - 1)\left(\theta_2\,\|v\|_{0,\Omega_3} + \theta_3|v|_{1,\Omega_3}\right),$$

where

$$\theta_2 \equiv r\left(d^{1/2}|b|_{1,\infty,\Omega_3}\,\|\psi\|_{0,\Omega_3} + \alpha^{d/2}|f|_{1,\Omega_3} + \alpha^{d/2}\,\|b\|_{\infty,\Omega_1}\,|\psi|_{1,\Omega_3}\right),$$
$$\theta_3 \equiv \sqrt{2}\alpha d\,\|A\|_{\infty,\Omega_1}\,(r|\psi|_{2,\Omega_3} + |\psi|_{1,\Omega_1}) + rd^{3/2}|A|_{1,\infty,\Omega_3}|\psi|_{1,\Omega_3}$$
$$+(\alpha+1)\alpha^{(d-2)/2}c^A|\psi|_{1,\Omega_1}.$$

Proof. Focusing on (4.5) first, we get from Lemma 4.3 and Lemma 4.5

$$I_1 + I_2 \le (\alpha - 1)\left(\alpha^{d/2} r|f|_{1,\Omega_3} + rd^{1/2}|b|_{1,\infty,\Omega_3}\,\|\psi\|_{0,\Omega_3}\right)\|v\|_{0,\Omega_3}. \tag{4.9}$$

By Lemma 4.3

$$I_3 \le \|b\|_{\infty,\Omega_1}\,\|\psi_\alpha - \psi\|_{0,\Omega_3}\,\|v\|_{0,\Omega_3}$$
$$\le (\alpha - 1)\alpha^{d/2} r\,\|b\|_{\infty,\Omega_1}\,|\psi|_{1,\Omega_3}\,\|v\|_{0,\Omega_3}. \tag{4.10}$$

By Lemma 4.6 with $\beta = 1$, Lemma 4.4 and the algebraic inequality used in the proof of Lemma 4.6

$$I_4 + I_5 \le (\alpha - 1) rd^{3/2}|A|_{1,\infty,\Omega_3}|\psi|_{1,\Omega_3}|v|_{1,\Omega_3}$$
$$+d\,\|A\|_{\infty,\Omega_1}\sqrt{2}\alpha(\alpha - 1)\,(r|\psi|_{2,\Omega_3} + |\psi|_{1,\Omega_1})\,|v|_{1,\Omega_3}. \tag{4.11}$$

To estimate I_6, we introduce $\widetilde{v}(x) = v(\alpha x)$, $x \in \Omega_1$. Then the substitution theorem, (4.1), (4.7) and again the substitution theorem (cf. (4.2)) lead to

$$I_6 = (\alpha^2 - 1)\alpha^{-2}\int_{\Omega_1}|A(x)\nabla_x\psi(x)\cdot\nabla_x\widetilde{v}(x)|\,\alpha^d\,\mathrm{d}x$$

$$\leq (\alpha^2-1)\alpha^{d-2}c^A|\psi|_{1,\Omega_1}|\widetilde{v}|_{1,\Omega_1}$$
$$= (\alpha^2-1)\alpha^{(d-2)/2}c^A|\psi|_{1,\Omega_1}|v|_{1,\Omega_3}. \tag{4.12}$$

Combining (4.9)–(4.12), we complete the proof. □

The next lemma finishes estimate (4.6).

Lemma 4.8 *Let $b, a_{ij} \in C^1(\overline{\Omega}_3)$, $i,j = 1,\dots,d$, and let $v \in H^1(\Omega_3)$. Then*

$$|a_{\Omega_3}(\widetilde{u}_{30}-u_{10\alpha}, v)| \leq (\alpha-1)\theta_4 \|v\|_{1,\Omega_3},$$

where

$$\theta_4 = c_{Ab}^{-1}\alpha^{d/2}\left(r\alpha^2 d^{3/2}|A|_{1,\infty,\Omega_3} + (\alpha+1)c^A + rd^{1/2}|b|_{1,\infty,\Omega_3}\right) \times \left(\|f\|_{0,\Omega_1} + c^{Ab}|\psi|_{1,\Omega_1}\right).$$

Proof. By (4.6), Lemma 4.6 with $\beta = \alpha^2$, Lemma 4.5 and (4.2)

$$I_7 + I_8 \leq (\alpha-1)\left(r\alpha^2 d^{3/2}|A|_{1,\infty,\Omega_3} + (\alpha+1)c^A + rd^{1/2}|b|_{1,\infty,\Omega_3}\right) \times \|u_{10\alpha}\|_{1,\Omega_3}\|v\|_{1,\Omega_3}$$
$$\leq (\alpha-1)\left(r\alpha^2 d^{3/2}|A|_{1,\infty,\Omega_3} + (\alpha+1)c^A + rd^{1/2}|b|_{1,\infty,\Omega_3}\right) \times\alpha^{d/2}\|u_{10}\|_{1,\Omega_1}\|v\|_{1,\Omega_3}. \tag{4.13}$$

Finally, we plug (3.9) into (4.13). □

To follow the ideas presented just before Lemma 4.1, we now estimate the norm of the difference $u_{30} - u_{10\alpha}$.

Lemma 4.9 *Under the assumptions of Lemma 4.7,*

$$\|u_{30}-u_{10\alpha}\|_{a,\Omega_3} \leq (\alpha-1)\theta_5, \tag{4.14}$$

where $\theta_5 \equiv \sqrt{c^{Ab}}c_{Ab}^{-1}(\theta_2+\theta_3+\theta_4)$ and parameters θ_2, θ_3, θ_4 are defined in Lemmata 4.7 and 4.8.

Proof. Inequality (2.5) and Lemma 4.8 give

$$c_{Ab}\|\widetilde{u}_{30}-u_{10\alpha}\|_{1,\Omega_3}^2 \leq (\alpha-1)\theta_4\|\widetilde{u}_{30}-u_{10\alpha}\|_{1,\Omega_3}. \tag{4.15}$$

Similarly, by (2.5) and Lemma 4.7

$$c_{Ab}\|u_{30}-\widetilde{u}_{30}\|_{1,\Omega_3}^2 \leq (\alpha-1)(\theta_2+\theta_3)\|u_{30}-\widetilde{u}_{30}\|_{1,\Omega_3}. \tag{4.16}$$

Using the triangle inequality, (4.15) and (4.16) (after canceling), we infer

$$\|u_{30}-u_{10\alpha}\|_{1,\Omega_3} \leq \|u_{30}-\widetilde{u}_{30}\|_{1,\Omega_3} + \|\widetilde{u}_{30}-u_{10\alpha}\|_{1,\Omega_3}$$

$$\leq (\alpha - 1) c_{Ab}^{-1} \left(\theta_2 + \theta_3 + \theta_4\right). \tag{4.17}$$

Then (4.14) is a consequence of (2.6), i.e.,

$$(c^{Ab})^{-1/2} \left\| u_{30} - u_{10\alpha} \right\|_{a,\Omega_3} \leq \left\| u_{30} - u_{10\alpha} \right\|_{1,\Omega_3},$$

and (4.17). □

We also need a counterpart to (4.2) for $\|v_\alpha\|_{a,\Omega_3}$.

Lemma 4.10 *Let $v \in H^1(\Omega_1)$, $b, a_{ij} \in C^1(\overline{\Omega}_3)$, $i, j = 1, \ldots, d$. Then*

$$\|v_\alpha\|_{a,\Omega_3}^2 \leq \alpha^d \|v\|_{a,\Omega_1}^2 + (\alpha - 1)\theta_6 \|v\|_{a,\Omega_1}^2,$$

where

$$\theta_6 = r d^{1/2} c_{Ab}^{-1} \left(d\alpha^{d-2} |A|_{1,\infty,\Omega_3} + \alpha^d |b|_{1,\infty,\Omega_3} \right).$$

Proof. Plugging $\pm A_\alpha$, $\pm b_\alpha$ into the integral representing $\|v_\alpha\|_{a,\Omega_3}^2$, we can proceed as in Lemma 4.6 ($\beta = 1$), Lemma 4.5 and (4.2). In detail,

$$\begin{aligned}
\|v_\alpha\|_{a,\Omega_3}^2 = & \int_{\Omega_3} \Big[(A - A_\alpha) \nabla v_\alpha \cdot \nabla v_\alpha + (b - b_\alpha) v_\alpha^2 \\
& + A_\alpha \nabla v_\alpha \cdot \nabla v_\alpha + b_\alpha v_\alpha^2 \Big] \mathrm{d}x \\
\leq & \ \varepsilon_0 d^{3/2} |A|_{1,\infty,\Omega_3} |v_\alpha|_{1,\Omega_3}^2 + \varepsilon_0 d^{1/2} |b|_{1,\infty,\Omega_3} \|v_\alpha\|_{0,\Omega_3}^2 \\
& + \alpha^d \|v\|_{a,\Omega_1}^2 \\
= & \ \varepsilon_0 d^{3/2} \alpha^{d-2} |A|_{1,\infty,\Omega_3} |v|_{1,\Omega_1}^2 + \varepsilon_0 d^{1/2} \alpha^d |b|_{1,\infty,\Omega_3} \|v\|_{0,\Omega_1}^2 \\
& + \alpha^d \|v\|_{a,\Omega_1}^2 .
\end{aligned}$$

Applying (2.5), we finish the proof. □

We are at the point of upgrading Lemma 3.1.

Theorem 4.1 *Under the assumptions of Lemma 4.7,*

$$\|u_{20} - u_{10}\|_{a,\Omega_2}^2 \leq (\alpha - 1) X \quad \textit{and} \quad \|u_{30} - u_{20}\|_{a,\Omega_3}^2 \leq (\alpha - 1) X,$$

where

$$\begin{aligned}
X \equiv & \Bigg[(\alpha - 1)\theta_5^2 + 2 c_{Ab}^{1/2} \left(\alpha^d + (\alpha - 1)\theta_6 \right)^{1/2} \theta_1 \theta_5 \\
& + c_{Ab} (\alpha^{d-1} + \cdots + 1 + \theta_6) \theta_1^2 \Bigg]
\end{aligned} \tag{4.18}$$

or

$$X \equiv \left[\theta_5 + c_{Ab}^{1/2} (\alpha^{d-1} + \cdots + 1 + \theta_6) \theta_1 \right] c_{Ab}^{1/2} (\theta_1 + \theta_1'). \tag{4.19}$$

Parameters θ_1, θ_1', θ_5 and θ_6 are defined in (3.9), (3.12), Lemma 4.9, and Lemma 4.10, respectively.

Proof. Let us insert $\pm u_{10\alpha}$ into the right-hand side of (3.3), apply the triangle inequality, Lemma 4.9, and Lemma 4.10

$$\begin{aligned}\|u_{20} - u_{10}\|^2_{a,\Omega_2} &\leq \left(\|u_{30} - u_{10\alpha}\|_{a,\Omega_3} + \|u_{10\alpha}\|_{a,\Omega_3}\right)^2 - \|u_{10}\|^2_{a,\Omega_1} \\ &= \|u_{30} - u_{10\alpha}\|^2_{a,\Omega_3} + 2\,\|u_{10\alpha}\|_{a,\Omega_3}\,\|u_{30} - u_{10\alpha}\|_{a,\Omega_3} \\ &\quad + \|u_{10\alpha}\|^2_{a,\Omega_3} - \|u_{10}\|^2_{a,\Omega_1} \\ &\leq (\alpha-1)^2\theta_5^2 + 2\left(\alpha^d + (\alpha-1)\theta_6\right)^{1/2}\|u_{10}\|_{a,\Omega_1} \\ &\quad \times(\alpha-1)\theta_5 + (\alpha^d - 1)\,\|u_{10}\|^2_{a,\Omega_1} \\ &\quad +(\alpha-1)\theta_6\,\|u_{10}\|^2_{a,\Omega_1}.\end{aligned}$$

Taking into account (3.10) and $(\alpha^d - 1) = (\alpha - 1)(\alpha^{d-1} + \cdots + \alpha^0)$, we derive (4.18).

To get (4.19), we modify (3.3)

$$\begin{aligned}\|u_{20} - u_{10}\|^2_{a,\Omega_2} &\leq \left(\|u_{30}\|_{a,\Omega_3} - \|u_{10}\|_{a,\Omega_1}\right)\left(\|u_{30}\|_{a,\Omega_3} + \|u_{10}\|_{a,\Omega_1}\right) \\ &\equiv I_{11}I_{12}.\end{aligned}$$

By (3.10), (3.11)

$$I_{12} \leq \sqrt{c_{Ab}}\left(\theta_1 + \theta_1'\right).$$

By the triangle inequality, Lemma 4.9, Lemma 4.10, and (3.10)

$$\begin{aligned}I_{11} &\leq \|u_{30} - u_{10\alpha}\|_{a,\Omega_3} + \|u_{10\alpha}\|_{a,\Omega_3} - \|u_{10}\|_{a,\Omega_1} \\ &\leq (\alpha-1)\theta_5 + \left[\left(\alpha^d + (\alpha-1)\theta_6\right)^{1/2} - 1\right]\|u_{10}\|_{a,\Omega_1} \\ &\leq (\alpha-1)\left[\theta_5 + (\alpha^{d-1} + \cdots + 1 + \theta_6)c_{Ab}^{1/2}\theta_1\right].\end{aligned}$$

To derive the last inequality, we also used

$$\begin{aligned}\left[\left(\alpha^d + (\alpha-1)\theta_6\right)^{1/2} - 1\right] &\leq \left[\left(\alpha^d + (\alpha-1)\theta_6\right)^{1/2} - 1\right] \\ &\quad \times\left[\left(\alpha^d + (\alpha-1)\theta_6\right)^{1/2} + 1\right] \\ &= (\alpha-1)(\alpha^{d-1} + \cdots + 1 + \theta_6).\end{aligned}$$

□

Remark 4.1 The framework presented in this section is not restricted to the second order scalar Dirichlet boundary value problems. It is applicable to systems of equations as well as to higher order problems. One can expect, however, increasing complexity of estimates. □

5 Estimates for more general domains

Our next goal is to find a parallel to Theorem 4.1 if Ω_3 is not an α-multiple of Ω_1. Then the transformation of functions and their derivatives is more complex than in Sect. 4 but we can still expect that the framework presented there will prove itself useful in achieving our purpose.

To make calculations easier and more lucid, we confine ourselves to *constant* coefficients A and b.

Let a domain Ω_1 be fixed and a parameter $\varepsilon_{\Omega_1} > 0$ be given. We assume that the boundary of Ω_1 is Lipschitz and that a family of ε-dependent one-to-one mappings $\varkappa_\varepsilon : \overline{\Omega}_1 \to \mathbb{R}^d$ and their inverses $\varkappa_\varepsilon^{-1} : \overline{\Omega}_{1\varepsilon} \to \mathbb{R}^d$, where $\Omega_{1\varepsilon} = \varkappa_\varepsilon(\Omega_1)$, exists such that

(A.1) $$\forall\, \varepsilon \in [0, \varepsilon_{\Omega_1}) \quad \varkappa_\varepsilon \in \left[C^1(\overline{\Omega}_1)\right]^d \quad \varkappa_\varepsilon^{-1} \in \left[C^1(\overline{\Omega}_{1\varepsilon})\right]^d;$$

$\forall x \in \Omega_1$ and $\varkappa_\varepsilon(x)$ can be connected by a straight segment lying in $\Omega_{1\varepsilon}$;

(A.2)

$$\forall\, \varepsilon \in [0, \varepsilon_{\Omega_1}) \quad \forall x \in \overline{\Omega}_1 \quad \varkappa_\varepsilon(x) = x + e_\varepsilon(x), \quad \|e_\varepsilon\|_{1,\infty,\Omega_1} \le \varepsilon C_e;$$

(A.3)

$$\forall\, \varepsilon \in [0, \varepsilon_{\Omega_1}) \quad \forall y \in \overline{\Omega}_{1\varepsilon} \quad \varkappa_\varepsilon^{-1}(y) = y + g_\varepsilon(y), \quad \|g_\varepsilon\|_{1,\infty,\Omega_{1\varepsilon}} \le \varepsilon C_g,$$

(A.4)

where C_e and C_g are positive constants independent of ε.

By virtue of its properties $\varkappa_\varepsilon$ transforms $H^1(\Omega_1)$ into $H^1(\Omega_{1\varepsilon})$ and $\varkappa_\varepsilon^{-1}$ transforms $H^1(\Omega_{1\varepsilon})$ into $H^1(\Omega_1)$.

We will need a generalization of Lemma 4.2 valid for rather unspecified mappings $\varkappa_\varepsilon$. To this end we assume that

(A.5) $$\int_{\Omega_1} \Big(\int_x^{\varkappa_\varepsilon(x)} \varphi(z)\, \mathrm{d}z \Big) \mathrm{d}x \le \varepsilon \sqrt{d} C_e^0 \int_{\Omega_{1\varepsilon}} \varphi(x)\, \mathrm{d}x$$

holds for any nonnegative function $\varphi \in L^1(\Omega_{1\varepsilon}) \cap C(\Omega_{1\varepsilon})$, where $C_e^0 > 0$ is a constant such that $\|e_\varepsilon\|_{\infty,\Omega_1} \le \varepsilon C_e^0$. It is $C_e^0 \le C_e$, cf. (A.3).

Remark 5.1 Mappings $\varkappa_\varepsilon$ fulfilling (A1), (A3), (A4) exist if $d = 2$ and Ω_1 has the Lipschitz boundary, see [3, Sect. 5]. □

We will transfer derivatives as well as integrals from Ω_1 to $\varkappa_\varepsilon(\Omega_1)$ and vice versa. That is why we have to pay attention to the Jacobi determinant of $\varkappa_\varepsilon^{-1}$.

Applying the chain rule onto $w(y) = v(\varkappa_\varepsilon^{-1}(y)) = v(x)$, $x \in \Omega_1$, we derive

(5.1) $$\frac{\partial w(y)}{\partial y_j} = \sum_{i=1}^{d} \frac{\partial v(x)}{\partial x_i} \frac{\partial x_i(y)}{\partial y_j}, \quad j = 1, \ldots, d,$$

a system of linear equations for an unknown vector $\nabla v(x)$.

Taking (A.4) into consideration, we see that the matrix of the system equals $I + G(y)$, where I is the $d \times d$ identity matrix and the elements of the $d \times d$ matrix $G(y)$ are equal to partial derivatives of the components of g_ε at y. The determinant $D(y)$ of $I + G(y)$ has the form

$$D(y) = 1 + \widehat{g}_\varepsilon(y), \qquad \|\widehat{g}_\varepsilon\|_{\infty,\Omega_{1\varepsilon}} \leq \varepsilon C, \tag{5.2}$$

where $\widehat{g}_\varepsilon$ is a continuous scalar function on $\Omega_{1\varepsilon}$ and C is a positive constant independent of ε.

We can employ the Cramer rule to get $\nabla v(x)$ from (5.1). Then

$$\nabla_x v(x) = (\nabla_y w(y) + M_{g_\varepsilon}(y)\nabla_y w(y))/D(y), \quad y = \varkappa_\varepsilon(x), \tag{5.3}$$

where M_{g_ε} is a $d \times d$ matrix function the elements of which consist of summed products of partial derivatives of g_ε.

We assume that ε_{Ω_1} is sufficiently small to ensure D is positive and close to 1.

Now we are ready to investigate how a_{Ω_1} changes if transferred from Ω_1 to $\Omega_{1\varepsilon}$.

Let us recall that $u_{10} \in H_1$ is the solution of (3.2), $i = 1$. We define $u_{10\varepsilon}(y) = u_{10}(\varkappa_\varepsilon^{-1}(y)) = u_{10}(x)$ and $v_\varepsilon(y) = v(\varkappa_\varepsilon^{-1}(y)) = v(x)$, $v \in H^1(\Omega_1)$. The substitution theorem, (5.3), (5.2), and $1/D = 1 + (1 - D)/D$ give

$$\begin{aligned}
&\int_{\Omega_1} (A\nabla u_{10} \cdot \nabla v + b u_{10} v)\,\mathrm{d}x \\
&\quad = \int_{\Omega_{1\varepsilon}} A(\nabla_y u_{10\varepsilon} + M_{g_\varepsilon}\nabla_y u_{10\varepsilon}) \cdot (\nabla_y v_\varepsilon + M_{g_\varepsilon}\nabla v_\varepsilon)|D|/D^2\,\mathrm{d}y \\
&\qquad + \int_{\Omega_{1\varepsilon}} b u_{10\varepsilon} v_\varepsilon |D|\,\mathrm{d}y \\
&\quad = \int_{\Omega_{1\varepsilon}} A\nabla u_{10\varepsilon} \cdot \nabla v_\varepsilon D^{-1}\,\mathrm{d}y + \hat{a}(g_\varepsilon; u_{10\varepsilon}, v_\varepsilon) \\
&\qquad + \int_{\Omega_{1\varepsilon}} b u_{10\varepsilon} v_\varepsilon (1 + \widehat{g}_\varepsilon)\,\mathrm{d}y \\
&\quad = a_{\Omega_{1\varepsilon}}(u_{10\varepsilon}, v_\varepsilon) + \int_{\Omega_{1\varepsilon}} A\nabla u_{10\varepsilon} \cdot \nabla v_\varepsilon (1 - D)/D\,\mathrm{d}y \\
&\qquad + \hat{a}(g_\varepsilon; u_{10\varepsilon}, v_\varepsilon) + \int_{\Omega_{1\varepsilon}} b u_{10\varepsilon} v_\varepsilon \widehat{g}_\varepsilon\,\mathrm{d}y \\
&\quad = a_{\Omega_{1\varepsilon}}(u_{10\varepsilon}, v_\varepsilon) + a_{g_\varepsilon}(u_{10\varepsilon}, v_\varepsilon).
\end{aligned} \tag{5.4}$$

The bilinear form $\hat{a}(g_\varepsilon; \cdot, \cdot)$ consists of terms related to M_{g_ε} and D^{-1}. The bilinear form $a_{g_\varepsilon}(\cdot, \cdot)$ contains $\hat{a}(g_\varepsilon; \cdot, \cdot)$ and terms with $\widehat{g}_\varepsilon$ and $(1 - D)/D$.

It holds

$$|a_{g_\varepsilon}(w,\omega)| \le \varepsilon C \, \|w\|_{1,\Omega_{1\varepsilon}} \, \|\omega\|_{1,\Omega_{1\varepsilon}} \quad \forall w,\omega \in H^1(\Omega_{1\varepsilon}), \tag{5.5}$$

$C > 0$ does not depend on ε, w, and ω.

The chain of equalities (5.4) offers a hint how to estimate $\|v_\varepsilon\|_{1,\Omega_{1\varepsilon}}$. We start with $\int_{\Omega_{1\varepsilon}} \left(|\nabla v_\varepsilon|^2 + v_\varepsilon^2\right) \mathrm{d}y$ and transfer the integral onto Ω_1 using formulae analogous to (5.1)–(5.3), i.e., formulae based not on $\varkappa_\varepsilon^{-1}$ but $\varkappa_\varepsilon$. A short calculation (cf. (5.4)) and (A.3) lead us to the inequality

$$\|v_\varepsilon\|^2_{1,\Omega_{1\varepsilon}} \le \|v\|^2_{1,\Omega_1} + \varepsilon C_1 \, \|v\|^2_{1,\Omega_1} \, , \tag{5.6}$$

where the constant $C_1 > 0$ does not depend on ε and v.

The next step is to transfer the right-hand side of (3.2), $i = 1$, onto $\Omega_{1\varepsilon}$. We will need $\psi_\varepsilon(y) = \psi(\varkappa_\varepsilon^{-1}(y)) = \psi(x)$ and $f_\varepsilon(y) = f(\varkappa_\varepsilon^{-1}(y)) = f(x)$, $y \in \Omega_{1\varepsilon}$. On the basis of (5.4) and (5.2)

$$\begin{aligned} F_{\Omega_1}(v) - a_{\Omega_1}(\psi,v) &= \int_{\Omega_{1\varepsilon}} f_\varepsilon v_\varepsilon D \, \mathrm{d}y - a_{\Omega_{1\varepsilon}}(\psi_\varepsilon, v_\varepsilon) - a_{g_\varepsilon}(\psi_\varepsilon, v_\varepsilon) \\ &= \int_{\Omega_{1\varepsilon}} f_\varepsilon v_\varepsilon \, \mathrm{d}y - a_{\Omega_{1\varepsilon}}(\psi_\varepsilon, v_\varepsilon) \\ &\quad + \int_{\Omega_{1\varepsilon}} \widehat{g}_\varepsilon f_\varepsilon v_\varepsilon \, \mathrm{d}y - a_{g_\varepsilon}(\psi_\varepsilon, v_\varepsilon). \end{aligned} \tag{5.7}$$

Let us suppose that $\Omega_3 = \varkappa_\varepsilon(\Omega_1)$ for some ε, $0 < \varepsilon \le \varepsilon_{\Omega_1}$. We fix this particular ε and use Ω_3 instead of $\Omega_{1\varepsilon}$ from now on. We summarize (3.2), (5.4) and (5.7)

$$\begin{aligned} a_{\Omega_3}(u_{10\varepsilon}, w) = \int_{\Omega_3} f_\varepsilon w \, \mathrm{d}y - a_{\Omega_3}(\psi_\varepsilon, w) - a_{g_\varepsilon}(u_{10\varepsilon} + \psi_\varepsilon, w) \\ + \int_{\Omega_3} \widehat{g}_\varepsilon f_\varepsilon w \, \mathrm{d}y \qquad \forall w \in H_3. \end{aligned} \tag{5.8}$$

We subtract (3.2), $i = 3$, from (5.8)

$$\begin{aligned} a_{\Omega_3}(u_{10\varepsilon} - u_{30}, w) &= \int_{\Omega_3} (f_\varepsilon - f) w \, \mathrm{d}y - a_{\Omega_3}(\psi_\varepsilon - \psi, w) \\ &\quad - a_{g_\varepsilon}(u_{10\varepsilon} + \psi_\varepsilon, w) + \int_{\Omega_3} \widehat{g}_\varepsilon f_\varepsilon w \, \mathrm{d}y \\ &= I_{13} + I_{14} + I_{15} + I_{16}. \end{aligned} \tag{5.9}$$

To estimate (5.9), we need generalizations of Lemmata 4.3–4.4.

Lemma 5.1 *Let $f \in H^1(\Omega_3)$. Then*

$$\|f - f_\varepsilon\|_{0,\Omega_3} \le \varepsilon C_2 |f|_{1,\Omega_3},$$

where $C_2 > 0$ is a constant independent of f and ε.

Proof. We can follow the proof of Lemma 4.3 supposing first $f \in C^\infty(\Omega_3) \cap H^1(\Omega_3)$. Due to (A.3), the Jacobi determinant $\widehat{D}$ of the mapping $\varkappa_\varepsilon(x)$ is bounded from above by a constant. We know even more. If ε_{Ω_1} is sufficiently small $0 < \widehat{D} \leq 1 + \varepsilon \widehat{C}$. Estimating the distance between x and $\varkappa_\varepsilon(x)$ by $\varepsilon \sqrt{d} C_e^0$ (cf. (A.3) and (A.5)), we use the Schwarz inequality and (A.5)

$$
\begin{aligned}
\int_{\Omega_3} (f(y) - f_\varepsilon(y))^2 \, \mathrm{d}y &\leq \int_{\Omega_1} (f(\varkappa_\varepsilon(x)) - f(x))^2 \, |\widehat{D}| \, \mathrm{d}x \\
&\leq \int_{\Omega_1} \Big(\int_x^{\varkappa_\varepsilon(x)} |\nabla f(z)| \, \mathrm{d}z \Big)^2 \widehat{D} \, \mathrm{d}x \\
&\leq \varepsilon (1 + \varepsilon \widehat{C}) \sqrt{d} C_e^0 \int_{\Omega_1} \int_x^{\varkappa_\varepsilon(x)} |\nabla f(z)|^2 \, \mathrm{d}z \, \mathrm{d}x \\
&\leq \varepsilon^2 (1 + \varepsilon \widehat{C}) d (C_e^0)^2 \int_{\Omega_3} |\nabla f|^2 \, \mathrm{d}x.
\end{aligned}
$$

□

Lemma 5.2 *Let* $\psi \in H^2(\Omega_3)$. *Then*

$$
\| \psi - \psi_\varepsilon \|_{1,\Omega_3} \leq \varepsilon C_3 \left(|\psi|_{1,\Omega_3} + |\psi|_{2,\Omega_3} \right),
$$

where $C_3 > 0$ *is a constant independent of* ε *and* ψ.

Proof. If $d = 2$ the estimate

$$
|\psi - \psi_\varepsilon|_{1,\Omega_3} \leq \varepsilon C \left(|\psi|_{1,\Omega_1} + |\psi|_{2,\Omega_3} \right), \quad C > 0,
$$

is proven as Lemma 4.11 in [3]. The idea of the proof is identical to that of Lemma 5.1 but applied to the partial derivatives of $\psi - \psi_\varepsilon$. Therefore a generalization to a general d is rather straightforward and the estimate remains unchanged. Combining this estimate with Lemma 5.1, we arrive at the statement. □

The next lemma is a consequence of Lemmata 5.1–5.2 and finishes the estimate (5.9).

Lemma 5.3 *Let* $f, w \in H^1(\Omega_3)$ *and* $\psi \in H^2(\Omega_3)$. *Then*

$$
|a_{\Omega_3}(u_{10\varepsilon} - u_{30}, w)| \leq \varepsilon \theta_7 \, \|w\|_{1,\Omega_3},
$$

where $\theta_7 = C_4 \left(\|f\|_{1,\Omega_3} + \|\psi\|_{2,\Omega_3} \right)$ *and the constant* $C_4 > 0$ *does not depend on* ε, f, ψ, *and* w.

Proof. We estimate the right-hand side of (5.9).

By Lemma 5.1

$$|I_{13}| \le \|f - f_\varepsilon\|_{0,\Omega_3} \|w\|_{0,\Omega_3} \le \varepsilon C_2 |f|_{1,\Omega_3} \|w\|_{0,\Omega_3} . \tag{5.10}$$

Referring to (2.6) and Lemma 5.2, we get

$$|I_{14}| \le \varepsilon c^{Ab} C_3 \left(|\psi|_{1,\Omega_3} + |\psi|_{2,\Omega_3}\right) \|w\|_{1,\Omega_3} . \tag{5.11}$$

According to (A.4), $\|g_\varepsilon\|_{1,\infty,\Omega_3}$ has the order ε. Since $\|u_{10\varepsilon} + \psi_\varepsilon\|_{1,\Omega_3}$ is bounded by $(1 + \varepsilon C_1)^{1/2} \|u_{10} + \psi\|_{1,\Omega_1}$, see (5.6), we have

$$|I_{15}| \le \varepsilon C' \left(\|u_{10}\|_{1,\Omega_1} + \|\psi\|_{1,\Omega_1}\right) \|w\|_{1,\Omega_3} , \tag{5.12}$$

where $C' > 0$ is independent of ε, u_{10}, ψ, and w. By (5.2) the order of $\|\widehat{g}_\varepsilon\|_{\infty,\Omega_3}$ is ε. Then an analogy to (5.6) for $\|f_\varepsilon\|_{0,\Omega_{1\varepsilon}}$ implies

$$|I_{16}| \le \varepsilon C'' \|f\|_{0,\Omega_1} \|w\|_{0,\Omega_3} , \tag{5.13}$$

$C'' > 0$ does not depend on ε, f and w.

Summing up (5.10)–(5.13) and taking (3.9) into account, we finish the proof. □

Theorem 5.1 *Let $f \in H^1(\Omega_3)$ and $\psi \in H^2(\Omega_3)$. Then*

$$\|u_{20} - u_{10}\|^2_{a,\Omega_2} \le \varepsilon C \quad \text{and} \quad \|u_{30} - u_{20}\|^2_{a,\Omega_3} \le \varepsilon C,$$

where $C > 0$ is a constant dependent on $\|\psi\|_{2,\Omega_3}$ and $\|f\|_{1,\Omega_3}$ but independent of ε.

Proof. First, let us substitute $u_{10\varepsilon} - u_{30}$ for w in Lemma 5.3 and recall (2.5). Then

$$\|u_{10\varepsilon} - u_{30}\|_{a,\Omega_3} \le \varepsilon c_{Ab}^{-1/2} \theta_7 . \tag{5.14}$$

Next, we follow the framework of the proof of Theorem 4.1, i.e., we estimate (3.3) plugging $\pm u_{10\varepsilon}$ into $\|u_{30}\|_{a,\Omega_3}$. By the triangle inequality, (5.14), (2.6), (5.6), (5.4), (3.9), (5.5), and again (5.6), (3.9)

$$\begin{aligned} \|u_{20} - u_{10}\|^2_{a,\Omega_2} &\le \left(\|u_{30} - u_{10\varepsilon}\|_{a,\Omega_3} + \|u_{10\varepsilon}\|_{a,\Omega_3}\right)^2 - \|u_{10}\|^2_{a,\Omega_1} \\ &\le \varepsilon^2 c_{Ab}^{-1} \theta_7^2 + 2\sqrt{c^{Ab}(1 + \varepsilon C_1)}\, \|u_{10}\|_{1,\Omega_1} \varepsilon c_{Ab}^{-1/2} \theta_7 \\ &\quad - a_{g_\varepsilon}(u_{10\varepsilon}, u_{10\varepsilon}) \\ &\le \varepsilon^2 c_{Ab}^{-1} \theta_7^2 + \varepsilon c_{Ab}^{-1/2} 2\sqrt{c^{Ab}(1 + \varepsilon C_1)}\theta_1 \theta_7 \\ &\quad + \varepsilon C (1 + \varepsilon C_1) \theta_1 . \end{aligned}$$

□

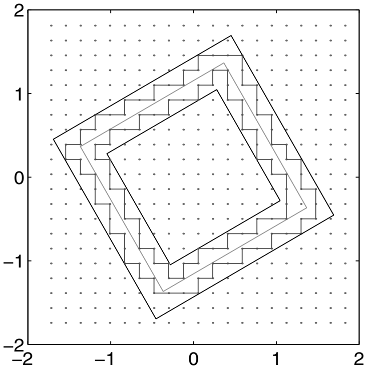

Fig. 1. Boundaries $\partial\Omega^{\mathrm{sq}}_{\mathrm{low}}$ (innermost), $\partial\Omega^{\mathrm{pix}}_{\mathrm{low}}$, $\partial\Omega$, $\partial\Omega^{\mathrm{pix}}_{\mathrm{up}}$, $\partial\Omega^{\mathrm{sq}}_{\mathrm{up}}$

6 Numerical example

To illustrate the estimates presented in Lemma 3.1 and Theorem 4.1, we investigated an uncertain boundary value problem defined via a digital image simulation.

The limit domain $\Omega \equiv \Omega_2$ (which is supposed to be virtually unknown in the paper) was defined as the square $(-1,1)^2$ rotated through an angle $\pi/6$. Then a regular grid of square pixels with sides parallel to the coordinate axes was imposed on Ω and its neighborhood. The union of pixels fully inside Ω formed the domain $\Omega^{\mathrm{pix}}_{\mathrm{low}} \equiv \Omega_1$. The union of all pixels with nonempty intersection with Ω formed the domain $\Omega^{\mathrm{pix}}_{\mathrm{up}} \equiv \Omega_3$.

For simplicity, the setting of the Dirichlet boundary value problem (see (2.1)–(2.4)) was given by $\psi = x_1^2 + 2x_2^2$, $A = I$ (the identity matrix), $b = 1$, $f = 3$. As a consequence, the constants c_{Ab}, c^{Ab}, c^A were equal to 1, see (2.5), (2.6), (4.7), respectively.

The estimate (3.3) was checked first. As solutions u_{20}, u_{10}, and u_{30} were not available, we approximated them by means of the finite element method with continuous piece-wise linear test and trial functions, i.e, u^{FE}_{20}, u^{FE}_{10}, and u^{FE}_{30} were computed. Gridding Ω and subdividing each square into four identical triangles, we created a finite element mesh $\mathcal{T}_\Omega$ comprising 360 000 triangles and 180 601 nodes. We considered the mesh sufficiently fine to produce an overkill solution u^{FE}_{20}. We also got $\left\|u^{\mathrm{FE}}_{20}\right\|_{\infty,\Omega} \approx 2.063$, $\left\|u^{\mathrm{FE}}_{20}\right\|_{0,\Omega} \approx 2.314$, $|u^{\mathrm{FE}}_{20}|_{1,\Omega} \approx 5.259$, and $\left\|u^{\mathrm{FE}}_{20}\right\|_{1,\Omega} \approx 5.746$.

To mesh $\Omega^{\mathrm{pix}}_{\mathrm{low}}$ and $\Omega^{\mathrm{pix}}_{\mathrm{up}}$, we simply took already defined pixels and, again, divided each of them into four identical triangles. Figure 1 depicts the boundary $\partial\Omega^{\mathrm{pix}}_{\mathrm{low}}$, $\partial\Omega$, $\partial\Omega^{\mathrm{pix}}_{\mathrm{up}}$ together with the vertices of a coarse pixel grid.

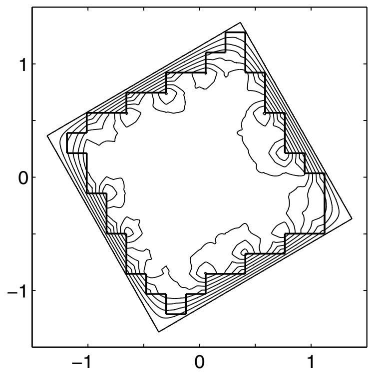

Fig. 2. The boundary $\partial\Omega^{\text{pix}}_{\text{low}}$ and contour lines of $u_{20} - u_{10}$

Table 1. Estimates for pixel approximate domains

Pixel size	Mesh size	$\|u^{\text{FE}}_{20} - u^{\text{FE}}_{10}\|_{1,\Omega}$	Estimate (3.3)
1.776×10^{-1}	396	3.938	4.965
7.855×10^{-2}	2 320	2.768	3.381
3.671×10^{-2}	11 284	1.925	2.340
1.772×10^{-2}	49 752	1.333	1.634

We extended u^{FE}_{10} by zero outside Ω_1 and replaced the unavailable value $\|u_{20} - u_{10}\|_{1,\Omega}$ by $\|u^{\text{FE}}_{20} - u^{\text{FE}}_{10}\|_{1,\Omega}$, supposing that the error caused by the approximation is not significant if compared with the influence the difference between domains Ω_1 and Ω_2 has on the value of $\|u_{20} - u_{10}\|_{1,\Omega}$.

Some features of the difference $u^{\text{FE}}_{20} - u^{\text{FE}}_{10}$ can be inferred from Fig. 2 showing $\partial\Omega^{\text{pix}}_{\text{low}}$ corresponding to a coarse pixel grid, and contour lines at fixed levels. The difference value between two levels equals 0.1.

Table 1 presents the approximation of $\|u_{20} - u_{10}\|_{1,\Omega}$ and the estimate according to Lemma 3.1, i.e., the square root of the right-hand side of (3.3). Four grids stemming from different pixel size are considered as indicated in the first column. The number of triangles forming the respective meshes on $\Omega^{\text{pix}}_{\text{low}}$ is given in the second column.

We observe that the values in the third and fourth column are simply correlated with the pixel size. In detail, the values of $\|u^{\text{FE}}_{20} - u^{\text{FE}}_{10}\|_{1,\Omega}$ and of the estimate (3.3) are equal to a multiple of the square root of the pixel size. The multiplicative parameter roughly equals 10 and 12, respectively.

Unlike Lemma 3.1, the estimates in Theorem 4.1 need only analytical work in the case of simple input data. As $\Omega^{\text{pix}}_{\text{low}}$ and $\Omega^{\text{pix}}_{\text{up}}$ are not starshaped, we constructed simply shaped domains $\Omega^{\text{sq}}_{\text{low}}$ and $\Omega^{\text{sq}}_{\text{up}}$, see Fig. 1. The former is the largest square in $\Omega^{\text{pix}}_{\text{low}}$ the sides of which are parallel to those of Ω. The latter is the smallest multiple of $\Omega^{\text{sq}}_{\text{low}}$ containing $\Omega^{\text{pix}}_{\text{up}}$, i.e., $\Omega^{\text{sq}}_{\text{up}} = \alpha\Omega^{\text{sq}}_{\text{low}}$.

Table 2. Estimates for non-pixel approximate domains

Pixel size	α	$\|u_{20}^{\mathrm{FE}} - u_{10}^{\mathrm{FE}}\|_{1,\Omega}$	Est. (3.3)	Est. (4.18)	Est. (4.19)
1.776×10^{-1}	1.618	4.476	7.288	84.809	45.640
7.855×10^{-2}	1.238	3.256	4.866	34.025	25.926
3.671×10^{-2}	1.105	2.301	3.340	18.562	16.732
1.772×10^{-2}	1.050	1.602	2.322	11.577	11.325

Solutions u_{20}, u_{20}^{FE} remain unchanged, but u_{10} and u_{30} are now approximated by the finite element solutions to the boundary value problem in $\Omega_{\mathrm{low}}^{\mathrm{sq}}$ and $\Omega_{\mathrm{up}}^{\mathrm{sq}}$, respectively. To get u_{10}^{FE} and u_{30}^{FE}, regular meshes (see the construction of $\mathcal{T}_\Omega$) with 40 000 triangles and 20 201 nodes were introduced in $\Omega_{\mathrm{low}}^{\mathrm{sq}}$ and $\Omega_{\mathrm{up}}^{\mathrm{sq}}$. The number of nodes and triangles did not depend on the pixel size and, consequently, on α. Thus the difference $\|u_{20} - u_{10}\|_{1,\Omega}$ was again approximated through the overkill solution on the mesh $\mathcal{T}_\Omega$ and the finite element solution on the domain Ω_1 equivalent to $\Omega_{\mathrm{low}}^{\mathrm{sq}}$ in this case.

Table 2 displays the size of pixels, the corresponding parameter α, the approximation of $\|u_{20}-u_{10}\|_{1,\Omega}$, estimates based on (3.3), (4.18) and (4.19). Let us recall that u_{10}^{FE} corresponds to $\Omega_{\mathrm{low}}^{\mathrm{sq}}$ and was calculated on meshes with the number of nodes and triangles independent of the pixel grids.

Again, we can infer similar correlation between the square root of the pixel size and column values as in Table 1. The respective parameters are equal to approximately 12 and 17 now. Also, the values of (4.19) are correlated with the square root of the pixel size (the ratio roughly equals 90) but the correlation is weaker for (4.18).

As a consequence, it is evident that the column values depend linearly on $\sqrt{\alpha - 1}$, too. The respective multiplicative constants equal approximately 7, 10, 60, and 50. This result is in line with the theory, see Theorem 4.1.

Observing the values of $\|u_{20}^{\mathrm{FE}} - u_{10}^{\mathrm{FE}}\|_{1,\Omega}$, we can also consider the thickness of the layer between $\partial\Omega$ and $\partial\Omega_{\mathrm{low}}^{\mathrm{sq}}$, i.e., the distance between relevant parallel sides, as an independent variable. Doing this, we again get a proportion to the square root of the thickness.

The simple estimate (3.3) is superior to estimates (4.18), (4.19). This is quite obvious as (4.18) and (4.19) stem from (3.3) by means of the chain of other estimates. The estimate (4.19) gives better results than (4.18). An inspection of the proof of Theorem 4.1 reveals that a triangle inequality is used to infer (4.18) and (4.19). To get (4.18), the inequality is squared, however. In our opinion, this is the cause of the poor performance of (4.18).

The magnitude of $u_{20}^{\mathrm{FE}} - u_{10}^{\mathrm{FE}}$ seems to be relatively large if compared with $\left\|u_{20}^{\mathrm{FE}}\right\|_{1,\Omega} \approx 5.746$. Though depicting a pixel subdomain, Fig. 2 suggests a reason valid also for a square subdomain. The difference $u_{20}^{\mathrm{FE}} - u_{10}^{\mathrm{FE}}$ has a considerable slope near the boundary $\partial\Omega$. Thus a boundary layer contributes much to the seminorm $|u_{20}^{\mathrm{FE}} - u_{10}^{\mathrm{FE}}|_{1,\Omega}$.

We have always intended to measure the difference between solutions on Ω, and the theoretical analysis aims at this goal but the aforesaid observation invokes a question whether $u_{20}^{\mathrm{FE}} - u_{10}^{\mathrm{FE}}$ restricted to a subdomain $\Omega_{\mathrm{test}} \subset \Omega_{\mathrm{low}}^{\mathrm{sq}}$ would exhibit behavior different from the above-mentioned one. For Ω_{test} fixed, numerical experiments suggest that $\|u_{20}^{\mathrm{FE}} - u_{10}^{\mathrm{FE}}\|_{1,\Omega_{\mathrm{test}}}$ is rather proportional to the pixel size (or α or the boundary layer thickness) than its square root. More complicated behavior is observed if $\Omega_{\mathrm{test}} = \Omega_{\mathrm{low}}^{\mathrm{sq}}$. In that case, both u_{10}^{FE} and $\|\cdot\|_{1,\Omega_{\mathrm{test}}}$ depend on the pixel size. In both cases, $\|u_{20}^{\mathrm{FE}} - u_{10}^{\mathrm{FE}}\|_{1,\Omega_{\mathrm{test}}}$ is significantly smaller than $\|u_{20}^{\mathrm{FE}} - u_{10}^{\mathrm{FE}}\|_{1,\Omega}$.

On the condition that functions f and ψ do not behave wildly in the uncertain layer, the example gives a hint for a computational analysis based on geometrical input data delivered by digital imaging. Taking appropriate Ω_1 and Ω_3 and evaluating (3.3), we judge whether u_{10} or u_{30} are satisfactorily close to unreachable u_{20}. If not, digital data with finer resolution are necessary. To guess how fine pixels should be taken, we employ the proportion of the estimate value to the square root of the pixel size.

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