Numer. Math. (2003) 95: 735–759
Digital Object Identifier (DOI)10.1007/s002110200395

Numerische
Mathematik

Optimal a priori estimates for interface problems

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Received February 2, 2001 / Revised version received September 28, 2001 /
Published online: March 24, 2003 – © Springer-Verlag 2003

Summary. We consider a priori estimates in weighted norms for interface problems with piecewise constant diffusion constants which do not depend on the ratio between the constants. Our result generalizes an estimate of LEMRABET to arbitrary dimensions and includes curved boundaries. Furthermore, we discuss criteria for the existence of a uniform Poincaré estimate in weighted norms. In the affirmative case we obtain a robust finite element error bound in weighted norms. Finally, we present numerical experiments including a case with no uniform Poincaré constant.

Mathematics Subject Classification (1991): 65N15, 65N20

1 Introduction

The solution of an elliptic problem with smooth coefficients in a domain Ω is smooth in the interior Ω, and for many cases all boundary singularities can be classified, cf. GRISVARD [8]. If the coefficients are piecewise smooth, regularity gets lost at the interface; nevertheless, for piecewise constant coefficients a singularity classification is possible for a wide range of problems, cf. NICAISE [13] for polygonal domains. In this context, a priori estimates for the smooth part of the solution are required. Here, we are interested in estimates for the smooth part of the solution which in addition are independent of the coefficients in appropriate weighted norms, i. e., in robust estimates, and we generalize the results to the case of nonpolygonal domains. Such robust estimates provide e. g. robust finite element error

Supported in part by the Deutsche Forschungsgemeinschaft, SFB 404
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estimates, which in particular, together with a robust smoothing property [11, 15], lead to parameter-independent multigrid convergence.

We consider the following model problem: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v)_\Omega \qquad \text{for all } v \in H_0^1(\Omega), \tag{1}$$

where $\Omega \subset \mathbf{R}^d$ is a bounded Lipschitz domain, $f \in L^2(\Omega)$, and a is the elliptic bilinear form

$$a(u, v) = \sum_{k=1}^{K} \alpha_k \int_{\Omega_k} \nabla u \cdot \nabla v \, dx, \qquad \bar{\Omega} = \bigcup_{k=1}^{K} \bar{\Omega}_k,$$

with non-overlapping Lipschitz subdomains $\Omega_k \subset \Omega$ and constant coefficients $\alpha_k > 0$.

We assume that, besides the Lipschitz property, $\partial\Omega_k$ is a piecewise smooth boundary (in the sense that some closed measure-zero subset $Z_k \subset \partial\Omega_k$ exists such that $\partial\Omega_k \setminus Z_k$ is C^2-smooth; a precise formulation of the requirements is given in c) in Sect. 2). ν_k denotes the outer unit normal on $\partial\Omega_k$. On $\partial\Omega_k \setminus Z_k$ we can define the second fundamental tensor $S_k(x)$ (with respect to ν_k), the mean curvature $H_k(x)$, and the maximal principal curvature $P_k(x)$; we assume that H_k and P_k are bounded functions.

The inner product in $L^2(\Omega)$ is denoted by $(\ ,\ )_\Omega$, $\nabla v = (D_i v)_{i=1,\dots,d}$ denotes the gradient, $D^2 v = (D_i D_j v)_{i,j=1,\dots,d}$ the Hessian matrix, $\|\ \|_\Omega$ denotes the norm in $L^2(\Omega)$, $L^2(\Omega)^d$ and $L^2(\Omega)^{d\times d}$, and for matrices we use the Frobenius norm. Moreover, we denote by $v_k := v|_{\Omega_k}$ the restriction to the subdomain Ω_k and, if it exists, its continuous extension to $\bar{\Omega}_k$. Finally, set

$$\begin{aligned} R = \{u \in H_0^1(\Omega) \mid u_k \in H^2(\Omega_k),& \\ \alpha_k \frac{\partial u_k}{\partial \nu_k} = -\alpha_j \frac{\partial u_j}{\partial \nu_j} &\text{ a. e. on } \partial\Omega_k \cap \partial\Omega_j,\ k \neq j\}. \end{aligned}$$

Note that for $u \in R$ the tangential derivatives $\frac{\partial u_k}{\partial t_k} = \nabla u_k - \frac{\partial u_k}{\partial \nu_k}\nu_k$, taken from both sides of the interface, are identical.

The main result of this paper is the following theorem.

Theorem 1 *Let $\overline{S_k} \in L^\infty(\partial\Omega_k)$ $(k = 1, \dots, K)$ be defined by*

$$\begin{aligned} \overline{S_k}(x) &:= (d-1)H_k(x) && \text{for } x \in (\partial\Omega_k \cap \partial\Omega) \setminus Z_k, \\ \overline{S_k}(x) &:= \max\{(d-1)H_k(x),\ P_k(x)\} && \text{for } x \in (\partial\Omega_k \setminus \partial\Omega) \setminus Z_k. \end{aligned}$$

Then, for all $u \in R$,

$$\sum_{k=1}^{K} \alpha_k \, \|D^2 u\|_{\Omega_k}^2 = \sum_{k=1}^{K} \alpha_k \, \|\Delta u\|_{\Omega_k}^2 +$$

$$+\sum_{k=1}^{K}\alpha_k\int_{\partial\Omega_k}\Big((d-1)H_k\left(\frac{\partial u_k}{\partial\nu_k}\right)^2+\left(\frac{\partial u_k}{\partial t_k}\right)^T S_k\frac{\partial u_k}{\partial t_k}\Big)d\sigma$$

$$\le\sum_{k=1}^{K}\alpha_k\,\|\Delta u\|_{\Omega_k}^2+\sum_{k=1}^{K}\alpha_k\int_{\partial\Omega_k}\overline{S_k}\,|\nabla u|^2 d\sigma, \tag{2}$$

where, in the last line, equality holds in the case $d = 2$ or if all Ω_k are polygonal / polyhedral (i.e. if $S_k \equiv 0$ for $k = 1, \dots K$).

This theorem generalizes a result by LEMRABET [10], where the case of two polygonal domains ($K = 2, \overline{S_1} = \overline{S_2} = 0, d = 2$) is considered.

For $u \in R$, the identities

$$\sum_{k=1}^{K}\alpha_k\int_{\partial\Omega_k}(d-1)H_k\left(\frac{\partial u_k}{\partial\nu_k}\right)^2 d\sigma$$
$$=\sum_{k=1}^{K}\alpha_k\int_{\partial\Omega_k\cap\partial\Omega}(d-1)H_k\left(\frac{\partial u_k}{\partial\nu_k}\right)^2 d\sigma$$
$$+\sum_{k<j}\int_{\partial\Omega_k\cap\partial\Omega_j}(1/\alpha_k-1/\alpha_j)(d-1)H_k\left(\alpha_k\frac{\partial u_k}{\partial\nu_k}\right)^2 d\sigma$$

and

$$\sum_{k=1}^{K}\alpha_k\int_{\partial\Omega_k}\left(\frac{\partial u_k}{\partial t_k}\right)^T S_k\frac{\partial u_k}{\partial t_k}d\sigma=\sum_{k<j}\int_{\partial\Omega_k\cap\partial\Omega_j}(\alpha_k-\alpha_j)\left(\frac{\partial u_k}{\partial t_k}\right)^T S_k\frac{\partial u_k}{\partial t_k}d\sigma$$

show that for nontrivial curvature terms (even in the simple case $d = 2$ and fibrous materials with circular inclusions, where $H_k = S_k = \text{const.}$) no decision about the sign of the boundary term in Theorem 1 can be made (except in the fully rotationally symmetric case, where $\frac{\partial u_k}{\partial t_k} = 0$, cf. the example in Sect. 3). A discussion of robustness with respect to the curvature and the number of the subdomains would require a much deeper knowledge about the relations between the normal and tangential derivatives.

Up to a finite dimensional correction, the boundary integral in the previous theorem can be removed, as shown in Theorem 2 below. The following Lemma 1 prepares this result. Here, we use the notation

$$(v,w)_\alpha=\sum_{k=1}^{K}\alpha_k\,(v,w)_{\Omega_k},\qquad \|v\|_\alpha^2=(v,v)_\alpha,\qquad v,w\in L^2(\Omega).$$

Lemma 1 *Let* $0 < \lambda_1^{(\alpha)} \le \lambda_2^{(\alpha)} \le \cdots$ *denote the eigenvalues of the problem*

$$(\nabla w, \nabla v)_\alpha = \lambda\, (w, v)_\alpha, \qquad v \in H_0^1(\Omega), \tag{3}$$

associated with a complete, $(\cdot,\cdot)_\alpha$*-orthonormal sequence* $(w_i^{(\alpha)})_{i\in\mathbf{N}}$ *of eigenfunctions in* $H_0^1(\Omega)$*. Let L denote the number of indices* $i \in \mathbf{N}$ *such that* $\lambda_i^{(\alpha)}$ *is not bounded away from zero, as* $\alpha = (\alpha_1, ..., \alpha_K)$ *varies over* $(0,\infty)^K$*. Then,*

$$L \le \#\{k \mid \mathrm{meas}(\partial\Omega_k \cap \partial\Omega) = 0\}.$$

Theorem 2 *For all* $u \in R$ *the estimate*

$$\sum_{k=1}^K \alpha_k\, \|D^2 u\|_{\Omega_k}^2 \le C_0 \sum_{k=1}^K \alpha_k\, \|\Delta u\|_{\Omega_k}^2 + C_1 \sum_{k=1}^K \alpha_k\, (P_0 u, -\Delta u)_{\Omega_k} \tag{4}$$

holds, where P_0 *is the* $(\cdot,\cdot)_\alpha$*-orthogonal projection onto* $W_0 = \mathrm{span}\{w_1^{(\alpha)}, ..., w_L^{(\alpha)}\}$.

Moreover, $C_0 = 1$ *and* $C_1 = 0$ *if the interfaces* $\partial\Omega_k \cap \partial\Omega_j$ *are polygonal / polyhedral.*

Here and in the following, $C, C_0, C_1, ...$ always denote positive constants independent of $\alpha_1, ..., \alpha_K$ and of the mesh parameter h.

An immediate consequence of Lemma 1 and Theorem 2 is the following robust a priori estimate:

Corollary 1 *If* $\mathrm{meas}(\partial\Omega_k \cap \partial\Omega) > 0$ *for all* $k = 1, ..., K$*, the estimate* (4) *holds with* $C_1 = 0$.

By the detailed investigation of an example, we show that in general no estimate (4) with $C_1 = 0$ exists, where C_0 is independent of the coefficients. Furthermore, we will see that the case where $\lambda_1^{(\alpha)}$ is not bounded away from zero really occurs, so that there is no Poincaré estimate of the form

$$\Big(\sum_{k=1}^K \alpha_k\, \|v\|_{\Omega_k}^2\Big)^{1/2} \le C_P \Big(\sum_{k=1}^K \alpha_k\, \|\nabla v\|_{\Omega_k}^2\Big)^{1/2}, \qquad v \in H_0^1(\Omega)$$

(with C_P independent of $\alpha_1, ..., \alpha_K$), but we can show that this estimate holds up to some finite dimensional "remainder" in W_0.

For these reasons, we can only expect weaker results in the general case where $C_1 > 0$. The relations

$$\begin{aligned}(P_0 u, -\Delta u)_\alpha &= \sum_{i=1}^L (u, w_i^{(\alpha)})_\alpha\, (-\Delta u, w_i^{(\alpha)})_\alpha = \sum_{i=1}^L \frac{1}{\lambda_i^{(\alpha)}}\, (-\Delta u, w_i^{(\alpha)})_\alpha^2 \\ &\le \frac{1}{\lambda_1^{(\alpha)}}\, \|P_0(\Delta u)\|_\alpha^2 \le \frac{1}{\lambda_1^{(\alpha)}}\, \|\Delta u\|_\alpha^2\end{aligned}$$

provide, as a consequence of Theorem 2, the following estimate, which is not robust if (as in general) $\lambda_1^{(\alpha)}$ is not bounded away from zero, but nevertheless gives insight into the dependence of the constants on α:

Corollary 2 *For all $u \in R$,*

$$\sum_{k=1}^{K} \alpha_k \, \|D^2 u\|_{\Omega_k}^2 \le \left(C_0 + \frac{C_1}{\lambda_1^{(\alpha)}} \right) \sum_{k=1}^{K} \alpha_k \, \|\Delta u\|_{\Omega_k}^2. \tag{5}$$

Since the a priori estimates hold for all $u \in R$, they can be applied to the solution of the elliptic problem (1), if we make the regularity assumption

$$u \in R \text{ for } f \in L^2(\Omega). \tag{6}$$

Of course, this regularity assumption is a severe restriction to the shape of $\partial\Omega_k$, and for any given $\varepsilon > 0$ it is easy to construct a polygonal example where the solution is not contained in $H^{1+\varepsilon}(\Omega)$. Nevertheless, there is an interesting class of examples modeling composite materials where this regularity result holds (e. g., for smooth boundaries see FOHT [7]).

For laminated materials or for smooth inner boundaries $\partial\Omega_k$ without cross points the regularity result is derived from a classical BERNSTEIN result [4] by BABUŠKA-CALOZ-OSBORN [1, Th. 2.2 and Th. 2.3]. There, it is not assumed that the coefficients are locally constant, but all constants in the estimates depend on the coefficients in a hardly assessable way.

Besides Theorems 1 and 2, we derive a priori bounds for finite element approximations of (1), by applying Cea's Lemma and the Aubin-Nitsche Lemma to weighted norms. Let $V_h \subset H_0^1(\Omega)$ be a conforming finite element space depending on a mesh parameter h, and let $u_h \in V_h$ be the finite element solution satisfying

$$a(u_h, v_h) = (f, v_h)_\Omega, \qquad v_h \in V_h. \tag{7}$$

Theorem 3 *Let $\Pi_h \colon R \longrightarrow V_h$ be an interpolation operator satisfying*

$$\|\nabla(v - \Pi_h(v))\|_{\Omega_k} \le C_2 \, h \, \|D^2 v\|_{\Omega_k}, \qquad v \in H^2(\Omega_k). \tag{8}$$

We assume that the regularity assumption (6) *holds.*

a) If $C_1 = 0$ is satisfied, we have for the solution $u \in H_0^1(\Omega)$ of (1) *and the finite element solution $u_h \in V_h$ of* (7) *the estimate in the energy norm*

$$\left(\sum_{k=1}^{K} \alpha_k \, \|\nabla(u - u_h)\|_{\Omega_k}^2 \right)^{1/2} \le C_3 \, h \left(\sum_{k=1}^{K} \alpha_k^{-1} \, \|f\|_{\Omega_k}^2 \right)^{1/2}, \tag{9}$$

and we have the weighted L^2-error-estimate

$$\left(\sum_{k=1}^{K} \alpha_k \, \|u - u_h\|_{\Omega_k}^2 \right)^{1/2} \le C_3^2 \, h^2 \left(\sum_{k=1}^{K} \alpha_k^{-1} \, \|f\|_{\Omega_k}^2 \right)^{1/2}. \tag{10}$$

b) *If* $C_1 > 0$, *the estimates* (9) *and* (10) *hold with the* α*-dependent constant*

$$C_2\Big(C_0 + \frac{C_1}{\lambda_1^{(\alpha)}}\Big)^{1/2}$$

in place of C_3.

We do not know in which form (9) and (10) (with α*-independent* constants) extend to more general cases. Some empirical results presented at the end of the paper indicate that (10) is indeed violated in some examples.

The results can be applied to nonhomogeneous Dirichlet data g if an explicit extension $w \in R$ with $w|_{\partial\Omega} = g$ and robust estimates for w depending on g are available (which in general will be difficult to obtain).

The paper is organized as follows. In Sect. 2 the main theorem is proved by combining classical representation formulas for the derivatives on the boundary for smooth functions with technical density arguments and an exception handling for Z_k. In Sect. 3 we prove Theorem 2 as a consequence of the main theorem, and in a detailed investigation of a radially symmetric example we comment on the limits of Theorem 2. In Sect. 4 the a priori estimates are applied to finite elements. Finally, we present a numerical experiment illustrating the finite element estimates and demonstrating robust multigrid convergence in two examples – one example where we can prove a robust approximation property (as a consequence of Theorem 3), and another example where we have robust multigrid convergence, but no robust approximation property.

2 Proof of the main theorem

The proof of Theorem 1 consists of four steps:

a) It will be advantageous to work, in the proof, with vector functions in the space

$$\begin{aligned} Q := \{q \in L^2(\Omega)^d \mid\ & q_k \in H^1(\Omega_k)^d, \\ & q_k - (q_k \cdot \nu_k)\nu_k = q_j - (q_j \cdot \nu_j)\nu_j \\ & \text{and} \quad \alpha_k q_k \cdot \nu_k + \alpha_j q_j \cdot \nu_j = 0 \text{ on } \partial\Omega_k \cap \partial\Omega_j,\ k \neq j, \\ & \text{and } q_k - (q_k \cdot \nu_k)\nu_k = 0 \text{ on } \partial\Omega_k \cap \partial\Omega\} \end{aligned}$$

so that in particular $\nabla u \in Q$ for $u \in R$.
Note that vector functions are also used for establishing a priori bounds in polygonal domains in [8, Sect. 4.3] and [10].

b) For homogeneous Neumann or Dirichlet boundary conditions, the Hessian matrix of smooth functions can be represented by the Laplacian and products of the gradient and curvature quantities from

$$\|D^2 v\|^2_{\Omega_k} = \|\Delta v\|^2_{\Omega_k} + \int\limits_{\partial\Omega_k} \left(-\Delta v \frac{\partial v}{\partial \nu_k} + \nu_k^T \cdot D^2 v \cdot \nabla v\right) d\sigma$$

(see e. g. LADYZHENSKAYA [9]). For smooth functions, Theorem 1 can be deduced directly from this formula; for our more general theorem in R, this result is transferred to vector functions.
In the first step the case of one smooth interface is discussed.

c) An additional exception handling is required in $Z := \bigcup_{k=1}^K Z_k$, where the boundaries are not smooth. Therefore, we specify the assumptions on the boundaries. For $k = 1, ..., K$ we assume that $Z_k \subset \partial\Omega_k$ is closed and has $(d-1)$-dimensional measure zero, that $\partial\Omega_k \setminus Z_k$ is a C^2-hypersurface of $\mathbf{R}^d$ with bounded principal curvatures, and moreover, that

$$\text{(11)} \qquad \operatorname{meas}\{x \in \mathbf{R}^d \mid \operatorname{dist}(x, Z) < \varepsilon\} \leq C\,\varepsilon^2$$

for all (sufficiently small) $\varepsilon > 0$.
Using a partition of unity, the results of step b) are extended to the general situation under an additional boundedness assumption.

d) Another density argument removes this boundedness assumption.

Note that the (parameter-depending) results in [1] are also derived via a partition of unity, and density arguments are already required for a priori estimates in only one polygonal domain, e. g., [8, Lem. 4.3.1.2].

Lemma 2 *Let $k \in \{1, \ldots, K\}$ and $\Omega_k^* \subset \Omega_k$ some domain. Let $\Gamma^* \subset \partial\Omega_k \cap \partial\Omega_k^*$ be a (relatively) open subset of $\partial\Omega_k \setminus Z_k$, and $p \in C^1(\overline{\Omega_k^*})^d$.*

a) If $p - (p \cdot \nu_k)\nu_k = 0$ on Γ^*, then $\nu_k^T (Dp)\nu_k - \operatorname{div} p = (d-1) H_k \cdot (p \cdot \nu_k)$ on Γ^*.

b) If $p \cdot \nu_k = 0$ on Γ^*, then $\nu_k^T (Dp)(I - \nu_k \nu_k^T) = p^T S_k (I - \nu_k \nu_k^T)$ on Γ^*.

Proof. For $x \in \Gamma^*$ given, let $U \subset \mathbf{R}^{d-1}$ denote some open neighborhood of 0, and $\phi : U \to \Gamma^*$ some local C^2-parameterization of Γ^* such that $\phi(0) = x$ and $D\phi(0)^T D\phi(0) = I_{d-1}$, i. e., the columns of $D\phi(0)$ form an orthonormal basis of the tangential space at x, the matrix $\Psi := (D\phi(0)|\nu_k(x)) \in \mathbf{R}^{d,d}$ is orthogonal, and $D\phi(0)^T \nu_k(x) = 0$. Straightforward calculations (see [14, Sect. 4]) show that the second fundamental tensor S_k (with respect to ν_k) at the point x can be written as

$$\text{(12)} \qquad S_k(x) = -D(\nu_k \circ \phi)(0) \cdot D\phi(0)^T \,.$$

To prove a), differentiate the identity $p \circ \phi \equiv [(p \circ \phi) \cdot (\nu_k \circ \phi)](\nu_k \circ \phi)$ on U to obtain

$$Dp(x)D\phi(0) = (p \cdot \nu_k)(x)D(\nu_k \circ \phi)(0) \\ +\nu_k(x)\left[p(x)^T D(\nu_k \circ \phi)(0) + \nu_k(x)^T Dp(x)D\phi(0)\right] .$$

Multiplication from the left by $D\phi(0)^T$ and taking the trace yields, by (12),

(13)
$$\begin{aligned}\operatorname{trace}\left[D\phi(0)^T Dp(x)D\phi(0)\right] &= -(p \cdot \nu_k)(x) \operatorname{trace}(S_k(x)) \\ &= -(d-1)H_k(x)(p \cdot \nu_k)(x) .\end{aligned}$$

Since the matrix Ψ is orthogonal, we obtain

$$\begin{aligned}(\operatorname{div} p)(x) &= \operatorname{trace}[Dp(x)] \\ &= \operatorname{trace}\left[\Psi^T Dp(x)\Psi\right] \\ &= \operatorname{trace}\left[D\phi(0)^T Dp(x)D\phi(0)\right] + \nu_k(x)^T Dp(x)\nu_k(x) .\end{aligned}$$

By use of (13), assertion a) follows.
To prove b), differentiate the identity $(p \circ \phi) \cdot (\nu_k \circ \phi) \equiv 0$ on U to obtain

$$\nu_k(x)^T Dp(x)D\phi(0) + p(x)^T D(\nu_k \circ \phi)(0) = 0 .$$

Multiplication by $D\phi(0)^T$ from the right yields, by $D\phi(0)^T D\phi(0) = I_{d-1}$ and (12),

$$\begin{aligned}\nu_k(x)^T Dp(x)D\phi(0)D\phi(0)^T &= -p(x)^T D(\nu_k \circ \phi)(0)D\phi(0)^T D\phi(0)D\phi(0)^T \\ &= p(x)^T S_k(x)D\phi(0)D\phi(0)^T ,\end{aligned}$$

and assertion b) follows since $D\phi(0)D\phi(0)^T = I - \nu_k(x)\nu_k(x)^T$ due to the orthogonality of the matrix Ψ. □

For the following, we choose some open ball B containing $\overline{\Omega}$ and put $\Omega_0 := B \setminus \overline{\Omega}$. Furthermore, after (possibly) enlarging the zero-sets $Z_1, \ldots, Z_k$, we may assume for $k = 1, \ldots K$ that $Z_k = \partial\Omega_k \cap Z$. This implies (together with the Lipschitz continuity of $\partial\Omega_k$), that $\Gamma_{kj} := (\partial\Omega_k \cap \partial\Omega_j) \setminus Z_k = (\partial\Omega_k \setminus Z_k) \cap (\overline{\Omega}_k \cup \overline{\Omega}_j)^0$ is an open subset of $\partial\Omega_k \setminus Z_k$ for $k = 1, ..., K$ and $j = 0, ..., K$, $k \neq j$. Let $I := \{(k, j) \in \{0, \ldots, K\}^2 \mid k \neq j,\ \Gamma_{kj} \neq \emptyset\}$ and set $\Omega_{kj} := \Omega_k \cup \Omega_j \cup \Gamma_{kj}$ for $(k, j) \in I$.

Lemma 3 *Let $p, q \in Q$, let $(k, j) \in I$, $k > 0$, and let $\varphi \in H^{1,\infty}(B)$ denote some function with compact support in Ω_{kj}. Then,* *a) in the case $j > 0$ we have*

(14)
$$\sum_{i\in\{k,j\}} \alpha_i \Bigg\{ \int_{\Omega_i} \varphi\,[\mathrm{trace}(Dp_i \cdot Dq_i) - (\mathrm{div}\, p_i)(\mathrm{div} q_i)]\,dx$$
$$+ \int_{\Omega_i} \nabla\varphi \cdot [(Dp_i)q_i - (\mathrm{div}\, p_i)q_i]\,dx$$
$$- \int_{\partial\Omega_k\cap\partial\Omega_j} \varphi\Big[(d-1)H_i(p_i\cdot\nu_i)(q_i\cdot\nu_i)$$
$$+ \Big(p_i - (p_i\cdot\nu_i)\nu_i\Big)^T S_i\Big(q_i - (q_i\cdot\nu_i)\nu_i\Big)\Big]d\sigma\Bigg\} = 0\ ;$$

b) in the case $j = 0$ we have

$$\int_{\Omega_k} \varphi\,[\mathrm{trace}(Dp_k \cdot Dq_k) - (\mathrm{div}\, p_k)(\mathrm{div} q_k)]\,dx \tag{15}$$
$$+ \int_{\Omega_k} \nabla\varphi \cdot [(Dp_k)q_k - (\mathrm{div}\, p_k)q_k]\,dx$$
$$- \int_{\partial\Omega_k\cap\partial\Omega} \varphi(d-1)H_k(p_k\cdot\nu_k)(q_k\cdot\nu_k)d\sigma = 0\ .$$

Proof. Let $(k, j) \in I$, $k > 0$, be fixed. If $j = 0$ (so that our goal is to prove b)), extend p_k and q_k, as H^1-functions, into Ω_0, and define

$$p_0 := \frac{\alpha_k}{\alpha_0} p_k,\ \ q_0 := \frac{\alpha_k}{\alpha_0} q_k \text{ on } \Omega_0\ , \tag{16}$$

where $\alpha_0 > 0$ is arbitrary for the moment. Since the tangential components $p_k - (p_k\cdot\nu_k)\nu_k$ and $q_k - (q_k\cdot\nu_k)\nu_k$ vanish on $\partial\Omega_k \cap \partial\Omega_0 = \partial\Omega_k \cap \partial\Omega$ due to the boundary conditions in Q, (16) shows that the interface conditions posed in Q also hold if $j = 0$.
Now return to general $j \in \{0, \ldots, K\}$. We choose some Lipschitz domain Ω^* such that $\Omega^* \cap \Omega_k =: \Omega_k^*$ and $\Omega^* \cap \Omega_j =: \Omega_j^*$ are both Lipschitz domains and that

$$\mathrm{supp}(\varphi) \subset \Omega^*, \qquad \overline{\Omega^*} \subset \Omega_{kj}\ . \tag{17}$$

Since Γ_{kj} is a C^2-manifold, the unit normal ν_k is in $C^1(\Gamma_{kj})^d$ and can therefore be extended to a function

$$\nu_k \in C^1(\overline{\Omega^*})^d\ . \tag{18}$$

Define a function $\widetilde{p}$ on $\overline{\Omega_k} \cup \overline{\Omega_j}$ by

$$\widetilde{p} := p_k \text{ on } \overline{\Omega}_k, \quad \widetilde{p} := p_j + \left(\frac{\alpha_j}{\alpha_k} - 1\right)(p_j \cdot \nu_k)\nu_k \text{ on } \overline{\Omega}_j \ . \tag{19}$$

The interface conditions posed for $p \in Q$ ensure that $\widetilde{p}$ is well-defined on Γ_{kj}, and that $\widetilde{p} \in H^1(\Omega_{kj})^d$. We can find a sequence $(\widetilde{p}^{(m)})_{m\in\mathbf{N}}$ in $C^1(\overline{\Omega^*})^d$ such that

$$\|\widetilde{p}^{(m)} - \widetilde{p}\|_{H^1(\Omega^*)^d} \to 0 \text{ as } m \to \infty \ , \tag{20}$$

applying the usual mollifier technique. Now, define a new sequence $(p^{(m)})_{m\in\mathbf{N}}$ of functions on $\Omega_k^* \cup \Omega_j^*$ by

$$p^{(m)} := \begin{cases} \widetilde{p}^{(m)} & \text{on } \Omega_k^* \ , \\ \widetilde{p}^{(m)} + \left(\frac{\alpha_k}{\alpha_j} - 1\right)\left(\widetilde{p}^{(m)} \cdot \nu_k\right)\nu_k & \text{on } \Omega_j^* \ , \end{cases} \tag{21}$$

and let $p_k^{(m)}$ and $p_j^{(m)}$ denote the continuous extensions of $p^{(m)}|_{\Omega_k^*}$ and $p^{(m)}|_{\Omega_j^*}$ to $\overline{\Omega_k^*}$ and $\overline{\Omega_j^*}$, respectively. By (18) and the smoothness of $\widetilde{p}^{(m)}$, (21) provides

$$p_k^{(m)} \in C^1(\overline{\Omega_k^*})^d \ , \qquad p_j^{(m)} \in C^1(\overline{\Omega_j^*})^d \ , \tag{22}$$

$$\begin{aligned} p_k^{(m)} - (p_k^{(m)} \cdot \nu_k)\nu_k &= p_j^{(m)} - (p_j^{(m)} \cdot \nu_k)\nu_k \ , \\ \alpha_k(p_k^{(m)} \cdot \nu_k) &= \alpha_j(p_j^{(m)} \cdot \nu_k) \end{aligned} \tag{23}$$

on $\partial\Omega_k^* \cap \partial\Omega_j^*$. Moreover, (18), (19), (20), (21) imply

$$\|p_k^{(m)} - p_k\|_{H^1(\Omega_k^*)^d} \to 0, \quad \|p_j^{(m)} - p_j\|_{H^1(\Omega_j^*)^d} \to 0 \ \text{ as } m \to \infty \ . \tag{24}$$

Now, let $i \in \{k, j\}$ and define $w := \varphi[(D\widehat{p})q_i - (\operatorname{div}\widehat{p})q_i]$, where $\widehat{p} \in C^\infty(\overline{\Omega_i^*})^d$ is arbitrary for the moment, and with φ specified in the lemma. Then $w \in H^1(\Omega_i^*)^d$, and by (17), $w|_{\partial\Omega_i^*\setminus\Gamma} \equiv 0$, with $\Gamma \subset \Gamma_{kj}$ denoting some (relatively) open subset satisfying $\operatorname{supp}(\varphi) \cap \partial\Omega_i^* \subset \Gamma \subset \partial\Omega_i^*$. Thus, the Divergence Theorem yields

(25)
$$\begin{aligned} \int_\Gamma \varphi[(D\widehat{p})q_i - (\operatorname{div}\widehat{p})q_i] \cdot \nu_i d\sigma &= \int_{\Omega_i^*} (\operatorname{div} w)\, dx \\ &= \int_{\Omega_i^*} \varphi[\operatorname{trace}(D\widehat{p} \cdot Dq_i) - (\operatorname{div}\widehat{p})(\operatorname{div} q_i)]\, dx \\ &\quad + \int_{\Omega_i^*} \nabla\varphi \cdot [(D\widehat{p})q_i - (\operatorname{div}\widehat{p})q_i]\, dx \ . \end{aligned}$$

For each fixed $m \in \mathbf{N}$, (22) and (17) show that $p_i^{(m)}$ can be approximated on $\operatorname{supp}(\varphi) \cap \overline{\Omega_i^*}$, with uniform convergence up to the first derivatives, by C^∞-functions $\widehat{p}$. This shows that (25) also holds with $p_i^{(m)}$ in place of $\widehat{p}$. Multiplying (25) (with $p_i^{(m)}$) by α_i and adding the two equations (for $i = k$ and for $i = j$) we obtain, using $\nu_j \equiv -\nu_k$ on Γ,

$$(26)\quad \int_\Gamma \varphi\Big\{\alpha_k[(Dp_k^{(m)})q_k - (\operatorname{div} p_k^{(m)})q_k] \cdot \nu_k \\ -\alpha_j[(Dp_j^{(m)})q_j - (\operatorname{div} p_j^{(m)})q_j] \cdot \nu_k\Big\} d\sigma \\ = \sum_{i\in\{k,j\}} \alpha_i \left\{ \int_{\Omega_i^*} \varphi[\operatorname{trace}(Dp_i^{(m)} \cdot Dq_i) - (\operatorname{div} p_i^{(m)})(\operatorname{div} q_i)]\, dx \right. \\ \left. + \int_{\Omega_i^*} \nabla\varphi \cdot [(Dp_i^{(m)})q_i - (\operatorname{div} p_i^{(m)})q_i]\, dx \right\} .$$

Defining, for $i \in \{k, j\}$, $p_{i,t}^{(m)} := p_i^{(m)} - (p_i^{(m)} \cdot \nu_k)\nu_k$, $q_{i,t} := q_i - (q_i \cdot \nu_k)\nu_k$, and using that $\alpha_k(q_k \cdot \nu_k) = \alpha_j(q_j \cdot \nu_k)$, $q_{k,t} = q_{j,t}$ on Γ, we find that the term in braces on the left-hand side of (26) equals

$$(27)\quad \Big\{\Big[\nu_k^T(Dp_k^{(m)})\nu_k - \operatorname{div} p_k^{(m)}\Big] - \Big[\nu_k^T(Dp_j^{(m)})\nu_k - \operatorname{div} p_j^{(m)}\Big]\Big\}\, \alpha_k(q_k \cdot \nu_k) \\ + \Big\{\alpha_k \nu_k^T(Dp_k^{(m)}) - \alpha_j \nu_k^T(Dp_j^{(m)})\Big\}\, q_{k,t} \\ = \big\{\nu_k^T(DP^{(m)})\nu_k - \operatorname{div} P^{(m)}\big\}\, \alpha_k(q_k \cdot \nu_k) + \nu_k^T(D\widetilde{P}^{(m)})q_{k,t},$$

where $P^{(m)} := p_k^{(m)} - p_j^{(m)}$, $\widetilde{P}^{(m)} := \alpha_k p_k^{(m)} - \alpha_j p_j^{(m)}$. After extending $p_j^{(m)}$, as a C^1-function, into $\overline{\Omega_k^*}$, we may regard $P^{(m)}$ and $\widetilde{P}^{(m)}$ as functions in $C^1(\overline{\Omega_k^*})^d$ (note (22)). Moreover, (23) shows that $P^{(m)}$ and $\widetilde{P}^{(m)}$ satisfy the assumptions of parts a) and b), respectively, of Lemma 2, so that the right-hand side of (27) equals

$$(28)\quad (d-1)H_k(P^{(m)} \cdot \nu_k)\alpha_k(q_k \cdot \nu_k) + (\widetilde{P}^{(m)})^T S_k q_{k,t} \\ = \sum_{i\in\{k,j\}} \alpha_i \left\{(d-1)H_i(p_i^{(m)} \cdot \nu_i)(q_i \cdot \nu_i) + (p_{i,t}^{(m)})^T S_i q_{i,t}\right\} ,$$

where we have used that $\nu_j \equiv -\nu_k$, $H_j \equiv -H_k$, $S_j \equiv -S_k$, and that the normal component of $\widetilde{P}^{(m)}$ vanishes on Γ, so that $\widetilde{P}^{(m)} \equiv \alpha_k p_{k,t}^{(m)} - \alpha_j p_{j,t}^{(m)}$ on Γ. We now insert the right-hand side of (28) (which equals the term in braces on the left-hand side of (26)) into (26), and let m tend to ∞. Using (24) and the continuity of the trace mapping $H^1(\Omega_i^*) \to L^2(\Gamma)$ (and the fact that φ, $\nabla\varphi$, and the curvatures are bounded), we find that (26) (with (28) replacing the brace-term) also holds with p_k and p_j in place of $p_k^{(m)}$ and $p_j^{(m)}$, respectively. Moreover, (17) implies that the integration ranges Γ and Ω_i^* may now be replaced by $\partial\Omega_k \cap \partial\Omega_j$ and Ω_i, respectively. This establishes (14) and therefore, part a) of the Lemma.

To prove b), regard that (14) now also holds if $j = 0$. Observing that the tangential components of p_k and q_k vanish on $\partial\Omega_k \cap \partial\Omega_0 = \partial\Omega_k \cap \partial\Omega$ and using (16), we obtain from (14) (with $j = 0$) that the left-hand side of (15) equals

$$-\frac{\alpha_k}{\alpha_0}\left\{\int_{\Omega_0} \varphi[\mathrm{trace}(Dp_k \cdot Dq_k) - (\mathrm{div}\, p_k)(\mathrm{div} q_k)]\, dx \right.$$
$$+\int_{\Omega_0} \nabla\varphi \cdot [(Dp_k)q_k - (\mathrm{div}\, p_k)q_k]\, dx$$
$$\left. - \int_{\partial\Omega_k\cap\partial\Omega_0} \varphi(d-1)H_0(p_k\cdot\nu_0)(q_k\cdot\nu_0)\, d\sigma \right\},$$

so that (15) follows by letting α_0 tend to ∞. □

Lemma 4 *Let $p \in Q$ and $q \in Q \cap L^\infty(\Omega)^d$. Then*

$$\sum_{k=1}^{K} \alpha_k \left\{ \int_{\Omega_k} \Big[\mathrm{trace}(Dp_k \cdot Dq_k) - (\mathrm{div}\, p_k)(\mathrm{div} q_k)\Big] dx \right.$$
$$- \int_{\partial\Omega_k} \Big[(d-1)H_k(p_k\cdot\nu_k)(q_k\cdot\nu_k)$$
$$\left. + \Big(p_k - (p_k\cdot\nu_k)\nu_k\Big)^T S_k \Big(q_k - (q_k\cdot\nu_k)\nu_k\Big)\Big] d\sigma \right\} = 0\,.$$

Proof. Let $p_0 \equiv 0,\ q_0 \equiv 0,\ \alpha_0 := 1$. For $\varphi \in H^{1,\infty}(B)$ we define $F[\varphi] \in L^1(B)$ and $f_k[\varphi] \in L^1(\partial\Omega_k)$ ($k = 0, \ldots, K$) by

(29) $$F[\varphi]\big|_{\Omega_k} := \alpha_k \left\{\varphi\Big[\mathrm{trace}(Dp_k \cdot Dq_k) - (\mathrm{div}\, p_k)(\mathrm{div} q_k)\Big] + \nabla\varphi \cdot \Big[(Dp_k)q_k - (\mathrm{div}\, p_k)q_k\Big]\right\},$$

(30) $$f_k[\varphi] := \alpha_k\varphi \Big[(d-1)H_k(p_k\cdot\nu_k)(q_k\cdot\nu_k) + \Big(p_k - (p_k\cdot\nu_k)\nu_k\Big)^T S_k \Big(q_k - (q_k\cdot\nu_k)\nu_k\Big)\Big]$$

for $k = 0, \ldots, K$. Then Lemma 3 (parts a) and b)) reads

$$\int_{\Omega_k} F[\varphi]dx + \int_{\Omega_j} F[\varphi]dx = \int_{\partial\Omega_k \cap \partial\Omega_j} (f_k[\varphi] + f_j[\varphi])d\sigma \tag{31}$$

for all $(k, j) \in I$ and each $\varphi \in H^{1,\infty}(B)$ with compact support in Ω_{kj}. (If $k = 0$ or $j = 0$, regard that $F[\varphi]\big|_{\Omega_0} \equiv 0$, $f_0[\varphi] \equiv 0$, and that $q_i - (q_i \cdot \nu_i)\nu_i = 0$ on $\partial\Omega_i \cap \partial\Omega$ for $i \geq 1$.)

Now define, for $\Omega_\varepsilon := \{x \in B \mid \operatorname{dist}(x, Z) < \varepsilon\}$,

$$\psi_\varepsilon(x) := \left\{ \begin{array}{ll} 1 & \text{if } x \in \Omega_\varepsilon \\ 2 - \frac{1}{\varepsilon}\operatorname{dist}(x, Z) & \text{if } x \in \Omega_{2\varepsilon} \setminus \Omega_\varepsilon \\ 0 & \text{if } x \in B \setminus \Omega_{2\varepsilon} \end{array} \right\} . \tag{32}$$

Since the distance function is Lipschitz-continuous (and thus, a $H^{1,\infty}$-function), we obtain $\psi_\varepsilon \in H^{1,\infty}(B)$ and $\|\nabla\psi_\varepsilon\|_\infty \leq C/\varepsilon$ (with C independent of ε). Using (11) we obtain

$$\|\nabla\psi_\varepsilon\|_{\Omega_{2\varepsilon}} \leq \widetilde{C} \ , \ \text{ with } \widetilde{C} \text{ independent of } \varepsilon \ . \tag{33}$$

For fixed $\varepsilon > 0$, $(\Omega_{kj})_{(k,j)\in I}$ is an open covering of the compact set $\overline{\Omega} \setminus \Omega_\varepsilon$. Let $(\phi_{kj})_{(k,j)\in I}$ denote a C^∞-partition of unity subordinate to this covering (i.e. $\sum_{(k,j)\in I} \phi_{kj} \equiv 1$ on $\overline{\Omega} \setminus \Omega_\varepsilon$, $\operatorname{supp}(\phi_{kj}) \subset \Omega_{kj}$ for $(k, j) \in I$). For each $(k, j) \in I$, $\phi_{kj} + \phi_{jk} \equiv 1$ on $\Gamma_{kj} \setminus \Omega_\varepsilon$, since $\Omega_{rs} \cap \Gamma_{kj} = \emptyset$ for all $(r, s) \in I \setminus \{(k, j), (j, k)\}$. By (32), we therefore obtain

$$(1 - \psi_\varepsilon)(\phi_{kj} + \phi_{jk}) \equiv 1 - \psi_\varepsilon \text{ on } \partial\Omega_k \cap \partial\Omega_j \ . \tag{34}$$

Furthermore, $\varphi := (1 - \psi_\varepsilon)(\phi_{kj} + \phi_{jk})$ satisfies the assumptions of Lemma 3 (resp. of (31)) and vanishes in Ω_ε. Using (31), (34) we therefore obtain

$$\begin{aligned}
\int_{\Omega\setminus\Omega_\varepsilon} F[1 - \psi_\varepsilon]dx &= \sum_{(k,j)\in I} \int_{\Omega\setminus\Omega_\varepsilon} F[(1 - \psi_\varepsilon)\phi_{kj}]dx \\
&= \frac{1}{2} \sum_{(k,j)\in I} \int_{\Omega\setminus\Omega_\varepsilon} F[(1 - \psi_\varepsilon)(\phi_{kj} + \phi_{jk})]dx \\
&= \frac{1}{2} \sum_{(k,j)\in I} \Big[\int_{\Omega_k} F[(1 - \psi_\varepsilon)(\phi_{kj} + \phi_{jk})]dx \\
&\qquad + \int_{\Omega_j} F[(1 - \psi_\varepsilon)(\phi_{kj} + \phi_{jk})]dx \Big]
\end{aligned}$$

$$= \frac{1}{2} \sum_{(k,j)\in I} \int_{\partial\Omega_k \cap \partial\Omega_j} \Big(f_k[(1-\psi_\varepsilon)(\phi_{kj}+\phi_{jk})] + f_j[(1-\psi_\varepsilon)(\phi_{kj}+\phi_{jk})]\Big) d\sigma$$

$$= \frac{1}{2} \sum_{(k,j)\in I} \left[\int_{\partial\Omega_k \cap \partial\Omega_j} f_k[1-\psi_\varepsilon] d\sigma + \int_{\partial\Omega_k \cap \partial\Omega_j} f_j[1-\psi_\varepsilon] d\sigma \right]$$

$$= \sum_{(k,j)\in I} \int_{\partial\Omega_k \cap \partial\Omega_j} f_k[1-\psi_\varepsilon] d\sigma = \sum_{k=1}^{K} \int_{\partial\Omega_k} f_k[1-\psi_\varepsilon] d\sigma \ .$$

Therefore, using (32),

$$\int_\Omega F[1]dx = \int_{\Omega\setminus\Omega_\varepsilon} F[1-\psi_\varepsilon]dx + \int_{\Omega_{2\varepsilon}} F[\psi_\varepsilon]dx$$

$$= \sum_{k=1}^{K} \left[\int_{\partial\Omega_k} f_k[1]d\sigma - \int_{\partial\Omega_k\cap\Omega_{2\varepsilon}} f_k[\psi_\varepsilon]d\sigma \right] + \int_{\Omega_{2\varepsilon}} F[\psi_\varepsilon]dx \ ,$$

and our assertion

$$\int_\Omega F[1]dx = \sum_{k=1}^{K} \int_{\partial\Omega_k} f_k[1]d\sigma$$

(regard (30), (29)) follows if we can show that, for $k = 1, \ldots, K$,

$$\int_{\partial\Omega_k\cap\Omega_{2\varepsilon}} f_k[\psi_\varepsilon]d\sigma \to 0 \quad \text{as } \varepsilon \to 0 \ , \tag{35}$$

$$\int_{\Omega_k\cap\Omega_{2\varepsilon}} F[\psi_\varepsilon]dx \to 0 \text{ as } \varepsilon \to 0 \ . \tag{36}$$

The $(d-1)$-dimensional measure of $\partial\Omega_k \cap \Omega_{2\varepsilon}$ tends to 0 as $\varepsilon \to 0$, since Z_k has measure zero and is compact. Therefore, (35) immediately follows from (29), using that $0 \le \psi_\varepsilon \le 1$, $p_k|_{\partial\Omega_k}$ and $q_k|_{\partial\Omega_k}$ are in $L^2(\partial\Omega_k)^d$, and the principal curvatures are bounded. To prove (36), we use (30), (32), (33):

$$\int_{\Omega_k\cap\Omega_{2\varepsilon}} |F[\psi_\varepsilon]|dx \le \alpha_k \int_{\Omega_k\cap\Omega_{2\varepsilon}} |\text{trace}(Dp_k \cdot Dq_k) - (\text{div} p_k)(\text{div} q_k)|\, dx$$
$$+\alpha_k \widetilde{C} \|Dp_k - (\text{div} p_k) I\|_{\Omega_k\cap\Omega_{2\varepsilon}} \|q_k\|_\infty \ .$$

The right-hand side indeed tends to zero as $\varepsilon \to 0$ since $p_k \in H^1(\Omega_k)^d$, $q_k \in H^1(\Omega_k)^d \cap L^\infty(\Omega_k)^d$, and $\text{meas}(\Omega_k \cap \Omega_{2\varepsilon}) \to 0$ as $\varepsilon \to 0$. □

We intend to use Lemma 4, for given $u \in R$, with $p = q := \nabla u$, which however is not yet possible due to the restriction $q \in L^\infty(\Omega)^d$ made in Lemma 4. To remove it, we prove

Lemma 5 *$Q \cap L^\infty(\Omega)^d$ is dense in Q with respect to the norm $\|q\| := \sum_{k=1}^{K} \|q_k\|_{H^1(\Omega_k)^d}$.*

Proof. Let $q \in Q$ be given. By Calderon's Extension Theorem, each $q_k = q|_{\Omega_k}$ $(k = 1, \ldots, K)$ can be extended to a function in $H^1(\mathbf{R}^d)^d$, which we call q_k again. We define $\psi \in H^1(\Omega)$ by

$$\psi(x) := \left[\sum_{k=1}^{K} q_k(x) \cdot q_k(x)\right]^{\frac{1}{2}} \quad (x \in \Omega)$$

and, for given $M > 0$,

$$\begin{aligned} \Omega^{(M)} &:= \left\{x \in \Omega \;\middle|\; \psi(x) > M\right\}, \\ \phi^{(M)}(x) &:= \min\left\{1, \tfrac{M}{\psi(x)}\right\} \; (x \in \Omega), \\ q^{(M)} &:= \phi^{(M)} q \; . \end{aligned} \tag{37}$$

$\phi^{(M)}$ is weakly differentiable on Ω, and

$$\nabla\phi^{(M)} = \left\{ \begin{array}{ll} -\frac{M}{\psi^2}\nabla\psi = -\frac{M}{\psi^3} \cdot \sum_{k=1}^{K} (Dq_k)^T q_k & \text{on } \Omega^{(M)} \\ 0 & \text{on } \Omega \setminus \Omega^{(M)} \end{array} \right\}$$

which implies that, on $\Omega^{(M)}$,

$$\begin{aligned} \left|\nabla\phi^{(M)}\right|^2 &= \frac{M^2}{\psi^6} \sum_{s=1}^{d} \left(\sum_{k=1}^{K} \frac{\partial q_k}{\partial x_s}^T q_k\right)^2 \\ &\le \frac{1}{\psi^4} \sum_{s=1}^{d} \left(\sum_{k=1}^{K} \left|\frac{\partial q_k}{\partial x_s}\right|^2\right) \left(\sum_{k=1}^{K} |q_k|^2\right) \\ &= \frac{1}{\psi^2} \sum_{k=1}^{K} |Dq_k|^2 \ , \end{aligned} \tag{38}$$

Since $\nabla\phi^{(M)} = 0$ outside $\Omega^{(M)}$, $\psi > M$ in $\Omega^{(M)}$, and $|Dq_k| \in L^2(\Omega)$, (38) in particular yields

$$\phi^{(M)} \in H^1(\Omega) \ . \tag{39}$$

Furthermore, (38) shows that, for each $j \in \{1, \ldots, K\}$,

$$\left|q_j(\nabla\phi^{(M)})^T\right|^2 = |q_j|^2\ |\nabla\phi^{(M)}|^2 \le \sum_{k=1}^{K} |Dq_k|^2\ , \tag{40}$$

so that we find, for $q_j^{(M)} := q^{(M)}|_{\Omega_j} = \phi^{(M)} q_j$,

$$Dq_j^{(M)} = \phi^{(M)} Dq_j + q_j(\nabla\phi^{(M)})^T \in L^2(\Omega_j)^{d\times d}\ ,$$

and thus, $q_j^{(M)} \in H^1(\Omega_j)^d$. Moreover, by (37) and (40),

$$\begin{aligned}\left\|Dq_j - Dq_j^{(M)}\right\|_{\Omega_j} &\le \left\|(1-\phi^{(M)})Dq_j\right\|_{\Omega_j} + \left\|q_j\left(\nabla\phi^{(M)}\right)^T\right\|_{\Omega_j\cap\Omega^{(M)}} \\ &\le \|Dq_j\|_{\Omega_j\cap\Omega^{(M)}} + \left[\sum_{k=1}^{K}\|Dq_k\|^2_{\Omega_j\cap\Omega^{(M)}}\right]^{\frac{1}{2}}\end{aligned}$$

which tends to 0 as $M \to \infty$, since $\mathrm{meas}(\Omega_j \cap \Omega^{(M)}) \to 0$.
For the same reason,

$$\|q_j - q_j^{(M)}\|_{\Omega_j} = \|(1-\phi^{(M)})q_j\|_{\Omega_j} \le \|q_j\|_{\Omega_j\cap\Omega^{(M)}} \to 0 \text{ as } M \to \infty\ ,$$

so that we have $\|q_j - q_j^{(M)}\|_{H^1(\Omega_j)^d} \to 0$ as $M \to \infty$.

Since obviously (37) provides $q^{(M)} \in L^\infty(\Omega)^d$, the Lemma is proved if we show that $q^{(M)}$ satisfies the interface and boundary conditions required for elements of Q; for this purpose, it is sufficient to prove the trace equality

$$q_j^{(M)}|_{\partial\Omega_j} = \phi^{(M)}|_{\partial\Omega_j} \cdot q_j|_{\partial\Omega_j} \quad (j = 1, \ldots, K) \tag{41}$$

(to be understood as an equation in $L^p(\partial\Omega_j)^d$, where $p \in \left(1, \frac{d}{d-1}\right)$), since q *does* satisfy the conditions in Q, and $\phi^{(M)}$ provides the same trace from "both sides" of the boundary $\partial\Omega_j$ due to (39).

To prove (41) (for fixed $j \in \{1, \ldots, K\}$ and $M > 0$), let $(\widetilde{q}^{(m)})_{m\in\mathbf{N}}$ denote a sequence in $C^\infty(\overline{\Omega_j})^d$ converging to q_j in $H^1(\Omega_j)^d$. By Sobolev's Embedding Theorem, $\phi^{(M)}\widetilde{q}^{(m)} \to \phi^{(M)}q_j = q_j^{(M)}$ in $H^{1,p}(\Omega_j)$ for $p \in \left(1, \frac{d}{d-1}\right)$, and thus,

$$\left(\phi^{(M)}\widetilde{q}^{(m)}\right)|_{\partial\Omega_j} \to q_j^{(M)}|_{\partial\Omega_j} \text{ in } L^p(\partial\Omega_j)^d\ .$$

On the other hand, $(\phi^{(M)}\widetilde{q}^{(m)})|_{\partial\Omega_j} = \phi^{(M)}|_{\partial\Omega_j} \cdot \widetilde{q}^{(m)}|_{\partial\Omega_j}$ since $\widetilde{q}^{(m)}$ is in C^∞, and here the right-hand side converges to $\phi^{(M)}|_{\partial\Omega_j} \cdot q_j|_{\partial\Omega_j}$ in $L^p(\partial\Omega_j)^d$, due to the Sobolev embedding $H^1(\Omega_j) \hookrightarrow L^{2p}(\partial\Omega_j)$. This establishes (41). □

Collecting the results, we can now prove Theorem 1.

Proof. According to Lemma 5 and to the continuity of the trace mapping $H^1(\Omega_k)^d \to L^2(\partial\Omega_k)^d$, the assertion of Lemma 4 holds for *all* $p, q \in Q$, in particular, for $p = q := \nabla u$, with $u \in R$ given. The assertion follows since

$$(d-1)H_k\left(\frac{\partial u_k}{\partial \nu_k}\right)^2 + \left(\nabla u_k - \frac{\partial u_k}{\partial \nu_k}\nu_k\right)^T S_k\left(\nabla u_k - \frac{\partial u_k}{\partial \nu_k}\nu_k\right)$$
$$\leq \overline{S_k}\left[\left(\frac{\partial u_k}{\partial \nu_k}\right)^2 + \left|\nabla u_k - \frac{\partial u_k}{\partial \nu_k}\nu_k\right|^2\right] = \overline{S_k}|\nabla u_k|^2\,. \tag{42}$$

If $d = 2$, we have $H_k \equiv P_k$, and $S_k(x) = P_k(x)\mathrm{id}_x$ (with id_x denoting the identity on the tangential space at x). Thus, equality holds in (42) and therefore, in the assertion. The same is obviously true if $S_k \equiv 0$ for $k = 1, \ldots, K$. □

3 An a priori bound (and proof of Theorem 2)

The this section we consider estimates for the boundary integral in Theorem 1.

Lemma 6 *For $u \in R$, we have*

$$\sum_{k=1}^{K}\alpha_k\,\|D^2u\|^2_{\Omega_k} \leq C\,\sum_{k=1}^{K}\alpha_k\,\|\Delta u\|^2_{\Omega_k} + C'\sum_{k=1}^{K}\alpha_k\,\|\nabla u\|^2_{\Omega_k}.$$

Moreover, for polygonal / polyhedral interfaces $\partial\Omega_k$ we have $C = 1$ and $C' = 0$.

Proof. For $k \in \{k \mid \kappa_k > 0\}$, where $\kappa_k = \sup_{x\in\partial\Omega_k\setminus Z_k}\overline{S_k}(x)$, we apply the trace theorem in GRISVARD [8, Th. 1.5.1.10] to ∇u_k, and we obtain

$$\|\nabla u_k\|^2_{\partial\Omega_k} \leq \frac{1}{2\kappa_k}\,\|D^2u_k\|^2_{\Omega_k} + C''\,\|\nabla u_k\|^2_{\Omega_k}. \tag{43}$$

□

Remark 1 Note that the constant C'' is large for large curvature κ_k. Thus, our estimates are not robust with respect to the diameter of small fibers in a composite material.

The remaining task is to estimate $\|\nabla u_k\|_{\Omega_k}$.

Lemma 7 *With P_0 introduced in Theorem 2, we have*

$$\sum_{k=1}^{K}\alpha_k\,\|v\|^2_{\Omega_k} \leq C_6\,\sum_{k=1}^{K}\alpha_k\,\|\nabla v\|^2_{\Omega_k} + \sum_{k=1}^{K}\alpha_k\|P_0v\|^2_{\Omega_k} \quad \textit{for } v \in H^1_0(\Omega).$$

Proof of Lemma 1 Besides the eigenvalue problem (3), we consider the decoupled space

$$\tilde{V} = \{\tilde{v} \in L^2(\Omega) \mid \tilde{v}_k \in H^1(\Omega_k),\ \tilde{v}_{k|\partial\Omega_k \cap \partial\Omega} = 0\}$$

and ask for the eigenvalues $\mu_1 \le \mu_2 \le \cdots$ and eigenfunctions $\tilde{w}_1, \tilde{w}_2, ... \in \tilde{V}$ of

$$(\nabla\tilde{w}, \nabla\tilde{v})_\alpha = \mu\ (\tilde{w}, \tilde{v})_\alpha, \qquad \tilde{v} \in \tilde{V}.$$

Note that the eigenvalues μ_i are independent of $\alpha_1, ..., \alpha_K$. Furthermore, we have $\mu_1 = \cdots = \mu_{\tilde{L}} = 0$ and $\mu_{\tilde{L}+1} > 0$ for $\tilde{L} := \#\{k \mid \mathrm{meas}(\partial\Omega_k \cap \partial\Omega) = 0\}$. Since $H_0^1(\Omega) \subset \tilde{V}$, the min-max theorem (see e. g. [3]) guarantees

$$\lambda_i^{(\alpha)} \ge \mu_i, \qquad i = 1, 2, ...$$

This shows in particular that $\lambda_{\tilde{L}+1}^{(\alpha)}$ is bounded away from zero (by $\mu_{\tilde{L}+1}$). □

Proof of Lemma 7 Let $\gamma := \inf_\alpha \lambda_{L+1}^{(\alpha)}$ (γ is positive due to the choice of L). Thus we have, for $v \in H_0^1(\Omega)$ such that $(v, w_i^\alpha)_\alpha = 0$ for $i = 1, 2, ..., L$,

$$\|v\|_\alpha^2 \le \frac{1}{\lambda_{L+1}^{(\alpha)}}\ \|\nabla v\|_\alpha^2 \le \frac{1}{\gamma}\ \|\nabla v\|_\alpha^2.$$

This provides, with $P_0(v) = \sum_{i=1}^{L} (w_i^\alpha, v)_\alpha\ w_i^\alpha$, that

$$\|v\|_\alpha^2 - \|P_0 v\|_\alpha^2 = \|v - P_0 v\|_\alpha^2 \le \frac{1}{\gamma}\ \|\nabla(v - P_0 v)\|_\alpha^2 \le \frac{1}{\gamma}\ \|\nabla v\|_\alpha^2$$

for all $v \in H_0^1(\Omega)$. □

Proof of Theorem 2 Applying Lemma 7 yields

$$\begin{aligned} a(u, u) &= -(\Delta u, u)_\alpha = -(\Delta u, u - P_0 u)_\alpha - (\Delta u, P_0 u)_\alpha \\ &\le \|\Delta u\|_\alpha \|u - P_0 u\|_\alpha - (\Delta u, P_0 u)_\alpha \\ &\le \sqrt{C_6}\ \|\Delta u\|_\alpha \|\nabla(u - P_0 u)\|_\alpha - (\Delta u, P_0 u)_\alpha \\ &\le \frac{C_6}{2}\ \|\Delta u\|_\alpha^2 + \frac{1}{2}\|\nabla u\|_\alpha^2 - (\Delta u, P_0 u)_\alpha, \end{aligned}$$

and we obtain

$$\sum_{k=1}^{K} \alpha_k \|\nabla u\|_{\Omega_k}^2 \le C_6 \sum_{k=1}^{K} \alpha_k \|\Delta u\|_{\Omega_k}^2 + 2 \sum_{k=1}^{K} \alpha_k (-\Delta u, P_0 u)_{\Omega_k}. \tag{44}$$

Combining Lemma 6 and (44) gives the assertion. □

Remark 2 For comparison, we state two other a priori estimates. In the case of two subdomains and $d = 2$ with a polygonal interface, LEMRABET [10] proves

$$\alpha_1 \|u\|^2_{H^{s+2}(\Omega_1)} + \alpha_2 \|u\|^2_{H^{s+2}(\Omega_2)} \leq C_s \Big(\alpha_1 \|\Delta u\|^2_{H^s(\Omega_1)} + \alpha_2 \|\Delta u\|^2_{H^s(\Omega_2)}\Big).$$

Note that there is no restriction on the space dimension in our proof. Moreover, we consider $K \geq 2$ and curved interfaces.

Introducing a weighted norm $\| \ \|_L$ via local transformations for functions $u \in H^1(\Omega)$ such that $u_k \in H^2(\Omega_k)$, the estimate proved by BABUŠKA-CALOZ-OSBORN [1] has the form

$$\|u\|_L \leq C(\alpha_1, ..., \alpha_K) \ \|f\|_\Omega.$$

There, an unidirectional result for strongly varying coefficients is transformed to more general situations by piecewise smooth local maps; note that our approach covers more general interfaces since we require piecewise smooth boundaries $\partial\Omega_k$ only. In addition, our constants are independent of $\alpha_1, ..., \alpha_K$ in many cases.

Example. We show that neither in Lemma 7 nor in Theorem 2 the second term (involving P_0) on the respective right-hand sides can be omitted, i. e. that Theorem 2 is optimal in a certain sense. For this purpose, we consider the radially symmetric example $\Omega = B_2 := \{x \in \mathbf{R}^2 \mid |x| < 2\}$, $\Omega_2 = B_1$, $\Omega_1 = \Omega \setminus \bar{\Omega}_2$, and $\alpha_1 = 1$, $\alpha_2 = \alpha$.

The Poincaré constant. Defining $v(x) = 1 - \max\{0, |x| - 1\}^2$ in Ω, we obtain $v \in R$, $\|v\|^2_\alpha = \|v\|^2_{\Omega_1} + \alpha \ \|v\|^2_{\Omega_2}$ and $\|\nabla v\|^2_\alpha = \|\nabla v\|^2_{\Omega_1}$. Thus, $\dfrac{\|\nabla v\|^2_\alpha}{\|v\|^2_\alpha} \leq \dfrac{C}{\alpha}$, which shows that no Poincaré constant independent of α exists (i. e. Lemma 7 does not hold without the P_0-term), and moreover, that the smallest eigenvalue $\lambda_1^{(\alpha)} =: \lambda^{(\alpha)}$ of problem (3) satisfies

$$\lambda^{(\alpha)} \leq \frac{C}{\alpha}, \qquad \alpha \in (0, \infty). \tag{45}$$

In particular, $\lambda^{(\alpha)}$ is not bounded away from zero.

The limit $\alpha \longrightarrow \infty$. Let $\tilde{w}^{(\alpha)} \in H^1_0(\Omega)$ denote an eigenfunction of (3) associated with $\lambda^{(\alpha)}$, normalized by $\|\tilde{w}^{(\alpha)}\|_\Omega = 1$. Now, (45) implies

$$-\Delta \tilde{w}_i^{(\alpha)} = \lambda^{(\alpha)} \tilde{w}_i^{(\alpha)} \longrightarrow 0 \text{ as } \alpha \longrightarrow \infty \quad \text{in } L^2(\Omega_i) \quad (i = 1, 2). \tag{46}$$

Since $\tilde{w}^{(\alpha)}$ is radially symmetric, its value $\gamma^{(\alpha)}$ on $\partial\Omega_2$ is a single real number. Clearly $\gamma^{(\alpha)}$ is bounded away from 0 as $\alpha \longrightarrow \infty$, because otherwise some sequence $\alpha_n \longrightarrow \infty$ would exist such that $\tilde{w}^{(\alpha_n)}|_{\partial\Omega_2} = \gamma^{(\alpha_n)} \longrightarrow 0$ as $n \longrightarrow \infty$, which together with (46) would imply $\tilde{w}_i^{(\alpha_n)} \longrightarrow 0$ in $H^2(\Omega_i)$ $(i = 1, 2)$, contradicting the normalization of $\tilde{w}^{(\alpha)}$. Consequently, the renormalized eigenfunction $w^{(\alpha)} := \frac{1}{\gamma^{(\alpha)}}\tilde{w}^{(\alpha)}$ still satisfies (46), and moreover $w^{(\alpha)} = 1$ on $\partial\Omega_2$. Thus, in $H^2(\Omega_1)$ and in $H^2(\Omega_2)$, $w^{(\alpha)}$ tends (as $\alpha \longrightarrow \infty$) to harmonic functions $w_1^{(\infty)}$ and $w_2^{(\infty)}$, respectively, which satisfy the corresponding boundary conditions (0 for $|x| = 2$, 1 for $|x| = 1$); an explicit calculation gives $w_1^{(\infty)}(x) = 1 - \log|x|/\log 2$ and $w_2^{(\infty)} \equiv 1$. In particular,

$$(47)\quad \frac{\partial w_1^{(\alpha)}}{\partial \nu_1} = -\alpha \frac{\partial w_2^{(\alpha)}}{\partial \nu_2} \longrightarrow \frac{\partial w_1^{(\infty)}}{\partial \nu_1} = \frac{1}{\log 2} \text{ as } \alpha \longrightarrow \infty \text{ on } \partial\Omega_2,$$

which implies $\alpha \int\limits_{\partial\Omega_2} \left|\frac{\partial w_2^{(\alpha)}}{\partial \nu_2}\right|^2 d\sigma \longrightarrow 0$ as $\alpha \longrightarrow \infty$. Using the radial symmetry and (47) again, we obtain that (for $u = w^{(\alpha)}$) the second term on the right-hand side in (2) tends, as $\alpha \longrightarrow \infty$, to

$$\int\limits_{\partial\Omega_1} \overline{S_1} \left|\frac{\partial w_1^{(\infty)}}{\partial \nu_1}\right|^2 d\sigma = \int\limits_{|x|=1} \left|\frac{\partial w_1^{(\infty)}}{\partial \nu_1}\right|^2 d\sigma - \frac{1}{2}\int\limits_{|x|=2} \left|\frac{\partial w_1^{(\infty)}}{\partial \nu_1}\right|^2 d\sigma$$
$$= \frac{3\pi}{2(\log 2)^2}.$$

On the other hand, the first term on the right-hand side of (2) equals

$$\|\Delta w_1^{(\alpha)}\|_{\Omega_1}^2 + \alpha\|\Delta w_2^{(\alpha)}\|_{\Omega_2}^2 = (\lambda^{(\alpha)})^2 \, \|w^{(\alpha)}\|_\alpha^2$$
$$= \frac{\alpha(\lambda^{(\alpha)})^2}{(\gamma^{(\alpha)})^2} \left(\frac{1}{\alpha}\|\tilde{w}_1^{(\alpha)}\|_{\Omega_1}^2 + \|\tilde{w}_2^{(\alpha)}\|_{\Omega_2}^2\right),$$

which tends to 0 as $\alpha \longrightarrow \infty$, due to (45).

Thus, the second term on the right-hand side of (2) cannot be bounded against the first one; since equality holds in (2) for $d = 2$, this shows that Theorem 2, in general, does not hold with $C_1 = 0$.

Improving (45), a more precise asymptotic result for $\lambda^{(\alpha)}$ can be obtained. We have

$$\int\limits_{\partial\Omega_2} w_2^{(\alpha)} \frac{\partial w_2^{(\alpha)}}{\partial \nu_2} d\sigma = \int\limits_{\partial\Omega_2} \frac{\partial w_2^{(\alpha)}}{\partial \nu_2} d\sigma = \int\limits_{\Omega_2} \Delta w_2^{(\alpha)} \, dx = -\lambda^{(\alpha)} \int\limits_{\Omega_2} w_2^{(\alpha)} \, dx,$$

which implies

$$\alpha \, \|\nabla w_2^{(\alpha)}\|_{\Omega_2}^2 = \alpha \int\limits_{\partial\Omega_2} w_2^{(\alpha)} \frac{\partial w_2^{(\alpha)}}{\partial \nu_2} d\sigma - \alpha \int\limits_{\Omega_2} w_2^{(\alpha)} \Delta w_2^{(\alpha)} \, dx$$

$$= \alpha\lambda^{(\alpha)} \int_{\Omega_2} \left(-w_2^{(\alpha)} + (w_2^{(\alpha)})^2\right) dx \longrightarrow 0 \text{ as } \alpha \longrightarrow \infty,$$

due to (45) and $w_2^{(\alpha)} \longrightarrow 1$. This gives

$$\text{(48)} \quad \alpha\lambda^{(\alpha)} = \frac{\|\nabla w_1^{(\alpha)}\|_{\Omega_1}^2 + \alpha\|\nabla w_2^{(\alpha)}\|_{\Omega_2}^2}{\frac{1}{\alpha}\|w_1^{(\alpha)}\|_{\Omega_1}^2 + \|w_2^{(\alpha)}\|_{\Omega_2}^2} \longrightarrow \frac{\|\nabla w_1^{(\infty)}\|_{\Omega_1}^2}{\|w_2^{(\infty)}\|_{\Omega_2}^2} = \frac{2}{\log 2}.$$

The limit $\alpha \longrightarrow 0$. While, as shown above, the smallest eigenvalue $\lambda^{(\alpha)}$ tends to 0 as $\alpha \longrightarrow \infty$, it is bounded away from zero as $\alpha \longrightarrow 0$, so that a uniform Poincaré estimate is at hand for $\alpha \longrightarrow 0$: assuming that $\lambda^{(\alpha)} \longrightarrow 0$ as $\alpha \longrightarrow 0$ (respectively along some sequence $\alpha_n \longrightarrow 0$), we obtain the corresponding statements (46) and (47) as before (with $\alpha \longrightarrow \infty$ replaced by $\alpha \longrightarrow 0$); but now (47) contradicts $\partial w_2^{(\alpha)}/\partial\nu_2 \longrightarrow 0$ (as $\alpha \longrightarrow 0$). This shows that Corollary 2 and Theorem 3 b) may still provide "semi-robust" estimates, even if $L \geq 1$ resp. $C_1 > 0$.

4 Application to finite elements (proof of Theorem 3)

Proof. The Galerkin orthogonality, (8) and Corollary 2 give

$$\begin{aligned}
a(u-u_h, u-u_h) &= a(u-u_h, u-\Pi_h(u)) \\
&\leq \Big(\sum_{k=1}^K \alpha_k \,\|\nabla u - \nabla u_h\|_{\Omega_k}^2\Big)^{1/2} \Big(\sum_{k=1}^K \alpha_k \,\|\nabla u - \nabla\Pi_h(u)\|_{\Omega_k}^2\Big)^{1/2} \\
&\leq \Big(\sum_{k=1}^K \alpha_k \,\|\nabla u - \nabla u_h\|_{\Omega_k}^2\Big)^{1/2} C_2\, h \Big(\sum_{k=1}^K \alpha_k \,\|D^2 u\|_{\Omega_k}^2\Big)^{1/2} \\
&\leq \|\nabla u - \nabla u_h\|_\alpha \, C_2 \sqrt{C_0 + \frac{C_1}{\lambda_1^{(\alpha)}}}\, h \Big(\sum_{k=1}^K \alpha_k^{-1} \,\|f\|_{\Omega_k}^2\Big)^{1/2},
\end{aligned}$$

which implies (9) with $C_3 = C_2\sqrt{C_0}$, and the corresponding statement in part b).

Now, let $z \in H_0^1(\Omega)$ be the dual solution, i. e. the solution of

$$\text{(49)} \qquad a(v,z) = \sum_{k=1}^K (v, \alpha_k(u-u_h))_{\Omega_k}, \qquad v \in H_0^1(\Omega),$$

and let $z_h \in V_h$ be the corresponding discrete dual solution. This gives the dual estimate from

$$\sum_{k=1}^K \alpha_k \,\|u-u_h\|_{\Omega_k}^2 = a(u-u_h, z) = a(u-u_h, z-z_h)$$

$$\leq C_2^2 \Big(C_0 + \frac{C_1}{\lambda_1^{(\alpha)}}\Big) h^2 \Big(\sum_{k=1}^{K} \alpha_k^{-1} \, \|f\|_{\Omega_k}^2\Big)^{1/2}$$
$$\times \Big(\sum_{k=1}^{K} \alpha_k^{-1} \, \|\alpha_k (u - u_h)\|_{\Omega_k}^2\Big)^{1/2}$$

by inserting the energy error estimate (9) for u_h and z_h; thus, we have (10) and the corresponding statement in part b). □

5 A numerical experiment

In Theorem 3 we could obtain robust finite element estimates only under additional assumptions. In this section we want to illustrate by numerical experiments that the robust finite element estimates cannot be extended to all cases, i. e., that also Theorem 3 is optimal in a certain sense.

We consider two examples: in Example 1 we set $\Omega = B_1 \setminus B_{0.42}$ and $\Omega_2 = \{x \in \Omega \mid |x - (0.1, 0.2)| < 0.7\}$, in Example 2 we set $\Omega = B_1$ and $\Omega_2 = \{x \in \Omega \mid |x - (0.1, 0.2)| < 0.7\}$; in both cases we define $\Omega_1 = \{x \in \Omega \mid |x - (0.1, 0.2)| > 0.7\}$ (see Fig. 1). We consider (1) with right-hand side $f = 0$ in Ω_1 and $f = 1$ in Ω_2, and coefficients $\alpha_1 = 1$ and $\alpha_2 = \alpha$.

The experiments are realized in the software system *UG* [2]. We use a mesh with 131072 resp. 149504 linear triangular elements and maximal edge length $h = 0.0117$.

Although we required in Theorem 3 conforming finite elements in $H_0^1(\Omega)$, we use polygonal approximations Ω_h in our numerical experiments. Using the techniques developed in [12], this can be analyzed by applying Theorem 3

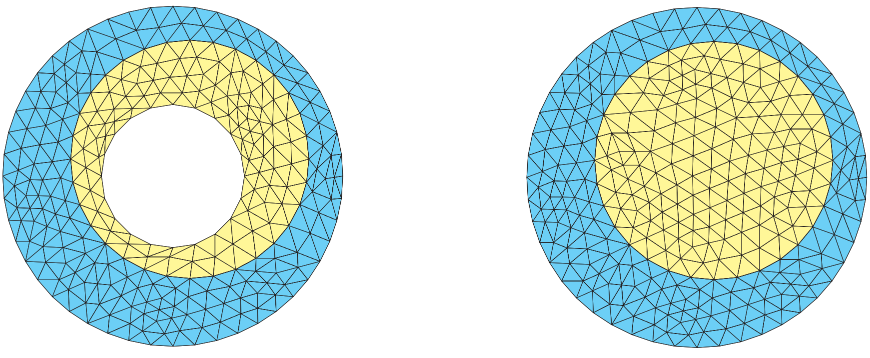

Fig. 1. Geometries and coarse mesh for Example 1 (left) and Example 2 (right). We obtain Ω_h (resp. Ω_{2h}) by 4 (resp. 3) uniform refinement steps of the coarse mesh, where the new boundary vertices are projected on the curved boundaries

Table 1. Numerical results for Examples 1 and 2 for various coefficients α and a fixed mesh parameter h

α	0.0001	0.01	1.0	100.0	10000.0
Example 1					
$q^{(\alpha)}(u_h, u_{2h}, f, h)$	0.0000218	0.0000111	0.0000123	0.0000146	0.0000147
$\lambda_{1,h}^{(\alpha)}$	18.33	18.53	28.83	13.60	13.99
$C_P^{(\alpha)}$	0.2335	0.2322	0.1862	0.2711	0.2673
Example 2					
$q^{(\alpha)}(u_h, u_{2h}, f, h)$	0.0000409	0.0000143	0.0000193	0.0004704	0.0453968
$\lambda_{1,h}^{(\alpha)}$	12.85	11.70	5.79	0.1278	0.001295
$\lambda_{2,h}^{(\alpha)}$	18.52	18.64	14.77	7.097	6.930
$C_P^{(\alpha)}$	0.2789	0.2923	0.4155	2.796	27.77

to curved finite elements (which are conforming on curved boundaries); for our experiments we assume that the consistency error due to the polygonal approximations is small.

Since all boundaries are smooth, the solution is in R, and the a priori estimate in Theorem 2 holds with $L = 0$ (and therefore $C_1 = 0$) in Example 1, and in Example 2 we have $L = \dim(W_0) \leq 1$. From Theorem 3, we expect at least for Example 1 that the quantities

$$q^{(\alpha)}(u_h, u_{2h}, f, h) = \frac{1}{h^2}\sqrt{\frac{\|u_h - u_{2h}\|_{\Omega_1}^2 + \alpha\,\|u_h - u_{2h}\|_{\Omega_2}^2}{\|f\|_{\Omega_1}^2 + \alpha^{-1}\|f\|_{\Omega_2}^2}}$$

are bounded independently of α: inserting the continuous solution u gives

$$\begin{aligned}\|u_h - u_{2h}\|_\alpha &\leq \|u - u_h\|_\alpha + \|u - u_{2h}\|_\alpha \\ &\leq C_3^2(h^2 + (2h)^2)\left(\|f\|_{\Omega_1}^2 + \alpha^{-1}\|f\|_{\Omega_2}^2\right)^{1/2},\end{aligned}$$

which yields $q^{(\alpha)}(u_h, u_{2h}, f, h) \leq 5\,C_3^2$. On the other hand, in Example 2 the quotients $q^{(\alpha)}$ can deteriorate because of $L = 1$. Since the configuration is similar to the example in Sect. 3, we can expect according to Theorem 3 b) that the best estimate is of the form $q^{(\alpha)}(u_h, u_{2h}, f, h) \leq C\,\alpha$.

We present in Tab. 1 numerical approximations of the quotients $q^{(\alpha)}$, the smallest eigenvalue, and the Poincaré constants $C_P^{(\alpha)} = 1\Big/\sqrt{\lambda_1^{(\alpha)}}$.

The results confirm clearly the assertion in Lemma 1, which predicts a stable Poincaré constant in Example 1, whereas the Poincaré constant deteriorates in Example 2: for large α we observe the asymptotic behaviour $C_P^{(\alpha)} \sim \sqrt{\alpha}$ for $\alpha \longrightarrow \infty$, cf. (48). Moreover, the second eigenvalue is bounded from below (independent of α), as prediced by the lemma. In accordance with Theorem 3, we obtain robust bounds for the quotients $q^{(\alpha)}$ in the first example (up to effects which may be caused by the consistency error due to the polygonal approximation of the interface for $\alpha = 0.0001$). On the other hand, we observe the expected deterioration of $q^{(\alpha)}$ in Example 2 (note that this effect relies on the special choice of the right-hand side). Nevertheless, in accordance with the example in Sect. 3 we observe a stable behaviour for $\alpha \longrightarrow 0$.

The first example illustrates a robust finite element estimate in a very special case; this can be stated as follows for the more general case that (4) holds with $C_1 = 0$, as an immediate consequence of Theorem 3.

Corollary 3 *If the assumptions of case a) in Theorem 3 are satisfied, a robust approximation estimate holds in the form*

$$\Big(\sum_{k=1}^{K} \alpha_k \, \|u_h - u_{2h}\|_{\Omega_k}^2\Big)^{1/2} \le C_8 \, h^2 \, \Big(\sum_{k=1}^{K} \alpha_k^{-1} \, \|f\|_{\Omega_k}^2\Big)^{1/2}. \tag{50}$$

Multigrid convergence. In the examples, the arising linear systems are solved with a multigrid method using a V(1,1)-cycle with symmetric Gauß-Seidel smoother. The paper was motivated by the observation of robust multigrid convergence rates for interface problems without singularities, cf. Tab. 2.

The "classical" multigrid analysis of the interface problem requires no regularity and no comparison with the exact continuous solution; robust multigrid convergence estimates (with respect to α_k) can be found in Bramble-Pasciak-Wang-Xu [5]. Note that this type of multigrid analysis uses only arguments in discrete spaces, and it does not require that the finite element solution converges robustly to the solution of the continuous problem; these results are restricted to polygonal domains and nested discretizations.

The extension of a *robust multigrid analysis* to more general cases requires a *robust approximation property*, which is provided (in case a) of Theorem 3) by Corollary 3. This result is the first step for proving robust multigrid convergence in nonnested spaces and for varying forms, where the analysis [5] cannot be applied. Furthermore, it can be used for interface problems with singularities, if the singularities are subtracted from the solution with the technique introduced in [6].

Table 2. Asymptotic multigrid convergence rate (= spectral radius of the iteration matrix) for Examples 1 and 2 for various coefficients α

α	0.0001	0.01	1.0	100.0	10000.0
Example 1	0.55378	0.55378	0.55378	0.55378	0.55378
Example 2	0.55378	0.55378	0.55378	0.55378	0.55378

On the other hand, the second example shows that one has to distinguish clearly between the robust convergence of the finite elements and the robust convergence of the multigrid iteration. Nevertheless, the approximation property fails only for large α and only with respect to the "small" space W_0. It remains an open question whether this information can be used for a robust multigrid analysis which includes variational crimes.

Acknowledgements. The authors are grateful to the anonymous referee for helpful comments.

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