Numer. Math. (2002) 92: 197–232
Digital Object Identifier (DOI) 10.1007/s002110100365

Numerische
Mathematik

# Numerical exterior algebra and the compound matrix method

**Leanne Allen, Thomas J. Bridges**

Department of Mathematics and Statistics, University of Surrey, Guildford, GU2 7XH, UK;
e-mail: t.bridges@eim.surrey.ac.uk

Received February 2, 2001 / Revised version received May 28, 2001 /
Published online October 17, 2001 – 

**Summary.** The compound matrix method, which was first proposed for numerically integrating systems of differential equations in hydrodynamic stability on $k = 2, 3$ dimensional subspaces of $\mathbb{C}^n$, by using compound matrices as coordinates, is reformulated in a coordinate-free way using exterior algebra spaces, $\bigwedge^k(\mathbb{C}^n)$.

This formulation leads to a general framework for studying systems of differential equations on $k$-dimensional subspaces. The framework requires the development of several new ideas: the role of Hodge duality and the Hodge star operator in the construction, an efficient strategy for constructing the induced differential equations on $\bigwedge^k(\mathbb{C}^n)$, general formulation of induced boundary conditions, the role of geometric integrators for preserving the manifold of $k-$dimensional subspaces – the Grassmann manifold, $G_k(\mathbb{C}^n)$, and a formulation for induced systems on an unbounded interval.

The numerical exterior algebra framework is most advantageous for numerical solution of differential eigenvalue problems on unbounded domains, where there are significant difficulties in setting up matrix discretizations.

The formulation is presented for $k$-dimensional subspaces of systems on $\mathbb{C}^n$ with $k$ and $n$ arbitrary, and examples are given for the cases of $k = 2$ and $n = 4$, and $k = 3$ and $n = 6$, with an indication of implementation details for systems of larger dimension.

The theory is illustrated by application to four differential eigenvalue problems on unbounded intervals: hydrodynamic stablity of boundary-layer flow past a compliant surface, the eigenvalue problem associated with the stability of solitary waves, the stability of Bickley jet in oceanography, and

*Correspondence to*: T.J. Bridges

the eigenvalue problem associated with the stability of the Ekman layer in atmospheric dynamics.

*Mathematics Subject Classification (1991):* 65L99; 76E99

## 1 Introduction

Consider a linear system of ordinary differential equations

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\,\mathbf{u}\,, \quad \mathbf{u} \in \mathbb{C}^n\,, \quad \lambda \in \Lambda \subset \mathbb{C}\,, \tag{1.1}$$

where $\mathbf{A}(x,\lambda)$ is an $n \times n$ matrix depending analytically on $\lambda$, and differentiably on $x$, and $x$ lies in some interval, possibly unbounded.

Systems of this type arise in a wide range of applications. We will be most concerned here with the case where $\lambda$ is an eigenvalue parameter, and there are boundary conditions – possibly at infinity – associated with (1.1). The boundary conditions will define $k$-dimensional subspaces of $\mathbb{C}^n$, and to solve (1.1) numerically will require numerical integration of $k-$dimensional subspaces.

The case of interest is when the $x$-interval is unbounded, but for illustrative purposes, suppose $n = 4$ and $0 \leq x \leq 1$, and there are two homogeneous boundary conditions imposed at $x = 0$ and two at $x = 1$. The system (1.1) is an eigenvalue problem with eigenvalue parameter $\lambda$. The natural approach to integrating (1.1) would be to integrate the induced system

$$\mathbf{U}_x = \mathbf{A}(x,\lambda)\,\mathbf{U}\,, \quad \mathbf{U}(x)\big|_{x=0} = \mathbf{U}_0 \in \mathbb{C}^{4\times 2}\,, \tag{1.2}$$

where the columns of $\mathbf{U}_0$ span the two-dimensional subspace which satisfies the boundary conditions at $x = 0$. The system (1.2) is then integrated with a numerical method of sufficient accuracy from $x = 0$ to $x = 1$. Imposition of the boundary conditions at $x = 1$ then leads to a complex analytic function $D(\lambda)$, the characteristic determinant, whose zeros correspond to eigenvalues of (1.1).

However, for many interesting examples, systems of the form (1.1) are stiff, and therefore the columns of $\mathbf{U}(x,\lambda)$ in (1.2) will not remain linearly independent during the numerical integration. The most well-known approach to addressing this linear dependence problem is to use either discrete orthogonalization, where the Gram-Schmidt algorithm is applied to the columns of (1.2) every few time steps, or continuous orthogonalization (cf. Drazin and Reid [17] Sect. 30, Hairer and Wanner [21] and references therein).

However there are two significant disadvantages of orthogonalization applied to complex systems. When using orthogonalization, particularly continuous orthogonalization, the induced system is *nonlinear*, and therefore numerical integration is an order of magnitude more complex. Secondly, for systems like (1.1) which depend analytically on a parameter, the induced orthogonalized system is not an analytic function of $\lambda$: for example, even though a vector $\xi(\lambda) \in \mathbb{C}^n$ may depend analytically on $\lambda$, its length is not analytic. Discrete orthogonalization also results in a solution which is not analytic. So a basic property of the original system (1.1) is not preserved.

An alternative to orthogonalization – the compound matrix method – was proposed in Ng and Reid [30–32] and Davey [13]) for integrating stiff linear systems, and has been successfully applied to other problems (cf. Davey [14], Straughn and Walker [34], Nicodemus et al. [33]). In this approach *compound matrices* are used as coordinates for integrating (1.2). Let

$$\mathbf{U} = \begin{bmatrix}\mathbf{u}\ \mathbf{v}\end{bmatrix} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \\ u_4 & v_4 \end{bmatrix} \in \mathbb{C}^{4\times 2}\,, \tag{1.3}$$

and consider all possible $2\times 2$ sub-determinants of $\mathbf{U} \in \mathbb{C}^{4\times 2}$ as coordinates:

$$\begin{aligned} &y_1 = u_1v_2 - u_2v_1\,, \quad y_2 = u_1v_3 - u_3v_1\,, \quad y_3 = u_1v_4 - u_4v_1\,, \\ &y_4 = u_2v_3 - u_3v_2\,, \quad y_5 = u_2v_4 - u_4v_2\,, \quad y_6 = u_3v_4 - u_4v_3\,. \end{aligned} \tag{1.4}$$

Differentiating the compound matrix coordinates $y_1, \ldots, y_6$ and using the property that $\mathbf{u} \in \mathbb{C}^4$ and $\mathbf{v} \in \mathbb{C}^4$ satisfy the differential equation, it follows that the coordinates $\mathbf{y} \in \mathbb{C}^6$ satisfy

$$\mathbf{y}_x = \mathbf{B}(x,\lambda)\,\mathbf{y}\,, \quad \mathbf{y} \in \mathbb{C}^6\,, \tag{1.5}$$

where $\mathbf{B}(x,\lambda)$ is a $6\times 6$ matrix whose entries depend linearly on the entries of $\mathbf{A}(x,\lambda)$. The compound matrix coordinates then lead to induced boundary conditions at $x = 0$ and $x = 1$ (see Ng and Reid [30,32] and Drazin and Reid [17] Sect. 43 for full details of this derivation).

The advantage of integrating the induced system, (1.5), over the original system, is that each $2-$dimensional subspace is represented by a line in (1.5) and therefore the numerical linear dependence problem is eliminated. Moreover, the induced system is *linear*, and when $\mathbf{A}(x,\lambda)$ depends analytically on $\lambda$, $\mathbf{B}(x,\lambda)$ will also depend analytically on $\lambda$.

However there are several issues with this method that are unresolved. Implicit in the above derivation is a choice of basis for $\mathbb{C}^4$: how can this basis be changed? In principle the idea should work for any $k$ with $1 \le k \le n$, but how can this be done in a straightforward and implementable way? What about boundary conditions on infinite domains? How are the systems on

$k-$dimensional subspaces related to the system on $(n-k)-$dimensional subspaces? Is there any advantage to using particular numerical integrators?

The purpose of this paper is multifold. We will show that exterior algebra is the general setting which lies behind the construction of compound matrices and with it, every existing aspect of the compound matrix method can be illuminated and generalized. Moreover, the theory of exterior algebra suggests several new results.

The paper is outlined as follows. Presented in Sect. 2 are the required general aspects of exterior algebra along with the construction – and constructive aspects – of induced systems such as (1.5) for any $k, n$ and with any basis for $\mathbb{C}^n$.

In Sect. 2.1, an important property of $\bigwedge^k(\mathbb{C}^n)$ is discussed: the set of subspaces of $k-$dimension is a *submanifold* of $\mathbb{C}^d$, where $d$ is the dimension of the induced system. In algebraic geometry this embedding is known as the Plücker embedding of the Grassmann manifold. From a numerical point of view, it is important to preserve this submanifold, and here we appeal to results in the theory of geometric numerical integration: a class of implicit Runge-Kutta methods are ideal for preservation of this basic submanifold to machine accuracy.

In Sect. 4 we consider general aspects of induced boundary conditions using exterior algebra with generalizations of the theory of Ng and Reid as well as several new results, particularly for semi-infinite and infinite domains, where a new formulation for asymptotic boundary conditions is constructed.

A key part of the new framework is the importance of Hodge duality and the Hodge star operator. It is not at all obvious how this operator would work in the setting of compound matrices, but it arises naturally when exterior algebra is used. As far as we are aware this is the first use of Hodge duality in a numerical setting, and this theory is developed in Sect. 5.

Analyticity is preserved by the mapping of $\mathbf{A}(x,\lambda)$ to the induced system on $\bigwedge^k(\mathbb{C}^n)$, and some of the implications of this are discussed in Sect. 6.

One of the most important applications of the theory is to eigenvalue problems on an unbounded interval. On an unbounded interval, the most obvious alternative to a shooting algorithm for (1.1) is to discretize (1.1) and turn it into a matrix eigenvalue problem – this is most advantageous when $\lambda$ appears linearly in $\mathbf{A}(x,\lambda)$. However, applying matrix methods leads to problems when using correct asymptotic boundary conditions. For example, consider the Schrödinger type equation on a semi-infinite interval,

$$u_{xx} + a(x)\,u = \lambda\,u\,, \quad 0 \le x \le +\infty\,, \quad a(x) \to 0 \text{ as } x \to +\infty\,, \tag{1.6}$$

with a boundary condition at $x = 0$, say $u|_{x=0} = 0$. To approximate this equation on the bounded interval, $0 \le x \le L_\infty$, we would normally impose the *exact asymptotic boundary condition* for boundedness of the solution as $x \to \infty$,

$$u_x + \sqrt{\lambda}\, u = 0\,, \quad \text{at } x = L_\infty\,, \tag{1.7}$$

derived using Levinson's Theorem (cf. Coppel [12]). However, discretization – say by a finite difference method or a spectral method – of (1.6) with the proper boundary condition (1.7) would lead to a matrix which is a *non-linear function of* $\lambda$, and therefore matrix $QR$ or $QZ$ algorithms no longer apply. This has lead in most cases to the use of *approximate* boundary conditions at infinity (e.g. Grosch and Orszag [19], Boyd [4] Chapter 17, Beyn and Lorenz [3], and references therein). The use of approximate boundary conditions usually has a dramatic impact on the essential – i.e. continuous – spectrum, whereas shooting using numerical exterior algebra computes discrete eigenvalues with no effect on the essential spectrum.

In Sect. 7-10 we present four examples of eigenvalue problems on infinite or semi-infinite intervals. In Sect. 7 the Orr-Sommerfeld equation for boundary layer stability, on a semi-infinite interval, is considered.

In Sect. 8 it is shown how exterior algebra leads to the solution of an open problem: how to accurately compute eigenvalues associated with the linearization about solitary waves and fronts. Here the problem of stability of the Hocking-Stewartson pulse is considered, with full details to appear elsewhere (cf. Afendikov and Bridges [1]). Mathematically, the problem of stability of solitary waves is identical to the stability of jets in atmospheric dynamics, and the stability of the classical Bickley jet is considered in Sect. 9.

In higher dimension, $n > 4$ and $k > 2$, new features appear, and we sketch some work in progress in Sect. 10 where the framework is being applied to the stability of the Ekman boundary layer which is fundamental in oceanography and atmospheric dynamics, and leads to a problem of the form (1.1) with $k = 3$ and $n = 6$.

## 2 Exterior algebra spaces and differential equations on $\bigwedge^k$

The starting point is a given system of linear differential equations of the form

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\,\mathbf{u}\,, \quad \mathbf{u} \in \mathbb{C}^n\,, \quad x \in \mathbb{R}\,, \quad \lambda \in \Lambda\,, \tag{2.1}$$

where $\mathbf{A}(x,\lambda)$ is a continuously differentiable function of $x$ and an analytic function of $\lambda$ for all $\lambda \in \Lambda$, and $\Lambda$ is some specified subset of the complex plane. The trace of $\mathbf{A}(x,\lambda)$ will feature prominently in the sequel; therefore define

$$\tau(x,\lambda) = \mathrm{Tr}(\mathbf{A}(x,\lambda))\,. \tag{2.2}$$

In this section, we consider the restriction of (2.1) to $k-$dimensional subspaces of $\mathbb{C}^n$, using exterior algebra. For example, if $\xi_1, \ldots, \xi_k$ span a $k-$dimensional space, then $\xi_1 \wedge \cdots \wedge \xi_k$, where $\wedge$ is the wedge product, is a $k-$form which represents the $k-$dimensional subspace. The linear space of all $k-$forms in $\mathbb{C}^n$ creates a vector space $\bigwedge^k(\mathbb{C}^n)$. Introducing a basis enables a straightforward method for approaching constructive aspects of these vector spaces.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be an orthonormal basis for $\mathbb{C}^n$. Then the nonzero and distinct members of the set

$$\{\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k} \ : \ i_1, \ldots, i_k = 1, \ldots, n\} \tag{2.3}$$

form a basis for the vector space $\bigwedge^k(\mathbb{C}^n)$, with exactly $d = \frac{n!}{(n-k)!k!}$ (the dimension of $\bigwedge^k(\mathbb{C}^n)$) distinct elements in this set.

Choose an ordering such as a standard lexical ordering and label the nonzero distinct elements in the set (2.3) by $\omega_1, \ldots, \omega_d$. Then, any element $\mathbf{U} \in \bigwedge^k(\mathbb{C}^n)$ can be represented as

$$\mathbf{U} = \sum_{j=1}^{d} U_j \, \omega_j \, .$$

The compound matrix method can be interpreted as the restriction of (2.1) to $\bigwedge^k(\mathbb{C}^n)$. The system (2.1) restricted to $\bigwedge^k(\mathbb{C}^n)$ is defined to be

$$\mathbf{U}_x = \mathbf{A}^{(k)}(x,\lambda)\, \mathbf{U} \, , \quad \mathbf{U} \in \bigwedge^k(\mathbb{C}^n) \cong \mathbb{C}^d \, , \tag{2.4}$$

where $\mathbf{A}^{(k)}(x,\lambda) \, : \, \bigwedge^k(\mathbb{C}^n) \to \bigwedge^k(\mathbb{C}^n)$ is a $d \times d$ matrix. The key to constructing the induced system is an algorithm for constructing the matrix $\mathbf{A}^{(k)}(x,\lambda)$. There is a natural way to construct the induced matrix $\mathbf{A}^{(k)}$, given $\mathbf{A} \in \mathbb{C}^{n\times n}$, using the vector space structure of the spaces $\bigwedge^k(\mathbb{C}^n)$.

An inner product on $\mathbb{C}^n$ induces an inner product on each vector space $\bigwedge^k(\mathbb{C}^n)$ as follows. Let $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ be a complex inner product with conjugation on the first element,

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{C}} = \sum_{j=1}^{n} \overline{u}_j \, v_j \, , \quad \mathbf{u}, \mathbf{v} \in \mathbb{C}^n \, .$$

To construct an inner product on $\bigwedge^k(\mathbb{C}^n)$, let

$$\mathbf{U} = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \text{ and } \mathbf{V} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k, \ \mathbf{u}_i, \mathbf{v}_j \in \mathbb{C}^n, \quad \forall \, i, j = 1, \ldots, k \, ,$$

be any decomposable $k$-forms. A $k-$form is decomposable if it can be written as a pure form: a wedge product between $k$ linearly independent vectors in $\mathbb{C}^n$ (further discussion of decomposability appears below). The inner product of $\mathbf{U}$ and $\mathbf{V}$ is defined by

$$[\![\mathbf{U},\mathbf{V}]\!]_k \stackrel{\text{def}}{=} \det \begin{bmatrix} \langle \mathbf{u}_1,\mathbf{v}_1\rangle_{\mathbb{C}} & \cdots & \langle \mathbf{u}_1,\mathbf{v}_k\rangle_{\mathbb{C}} \\ \vdots & \ddots & \vdots \\ \langle \mathbf{u}_k,\mathbf{v}_1\rangle_{\mathbb{C}} & \cdots & \langle \mathbf{u}_k,\mathbf{v}_k\rangle_{\mathbb{C}} \end{bmatrix}, \quad \mathbf{U},\mathbf{V} \in \bigwedge^k(\mathbb{C}^n)\,.$$

Since every element in $\bigwedge^k(\mathbb{C}^n)$ is a sum of decomposable elements, this definition extends by (bi)-linearity to any $k$-form.

The induced matrix $\mathbf{A}^{(k)} : \bigwedge^k(\mathbb{C}^n) \to \bigwedge^k(\mathbb{C}^n)$ is then the $d\times d$ matrix with entries

$$\{\mathbf{A}^{(k)}\}_{i,j} = [\![\omega_i, \mathbf{A}\omega_j]\!]_k\,, \quad i,j = 1,\dots,d\,, \quad d = \frac{n!}{k!(n-k)!}\,, \tag{2.5}$$

where, for any $\mathbf{U} = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \in \bigwedge^k(\mathbb{C}^n)$,

$$\mathbf{A}^{(k)}\mathbf{U} \stackrel{\text{def}}{=} \sum_{j=1}^{k} \mathbf{u}_1 \wedge \cdots \wedge \mathbf{A}\mathbf{u}_j \wedge \cdots \wedge \mathbf{u}_k\,.$$

With this definition, $\mathbf{A}^{(k)}(x,\lambda)$ is an analytic function of $\lambda$ whenever $\mathbf{A}(x,\lambda)$ is analytic. Indeed, if the basis $\mathbf{e}_1,\dots,\mathbf{e}_n$ is independent of $x$ and $\lambda$, then $\mathbf{A}^{(k)}(x,\lambda)$ inherits exactly the differentiability properties of $\mathbf{A}(x,\lambda)$. Another advantage of this definition of the induced matrix is that it is easily automated, in MAPLE, MATLAB or FORTRAN, which is essential for large $n$. For small $n$, the induced matrices can be constructed explicitly.

For example, suppose $n = 4$ and $k = 2$ and let $\mathbf{A} \in \mathbb{C}^{4\times 4}$ be an arbitrary matrix of the form

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}. \tag{2.6}$$

Take $\mathbf{e}_1,\dots,\mathbf{e}_4$ to be the standard basis for $\mathbb{C}^4$ and let $\omega_1,\dots,\omega_6$ be a basis for $\bigwedge^2(\mathbb{C}^4)$. For example, using a standard lexical ordering,

$$\begin{aligned} &\omega_1 = \mathbf{e}_1 \wedge \mathbf{e}_2\,, \quad \omega_2 = \mathbf{e}_1 \wedge \mathbf{e}_3\,, \quad \omega_3 = \mathbf{e}_1 \wedge \mathbf{e}_4\,, \\ &\omega_4 = \mathbf{e}_2 \wedge \mathbf{e}_3\,, \quad \omega_5 = \mathbf{e}_2 \wedge \mathbf{e}_4\,, \quad \omega_6 = \mathbf{e}_3 \wedge \mathbf{e}_4\,. \end{aligned} \tag{2.7}$$

The basis $\omega_1, \ldots, \omega_6$ in (2.7) is orthonormal with respect to the inner product $[\![\cdot,\cdot]\!]_2$. Therefore

$$\begin{aligned}\{\mathbf{A}^{(2)}\}_{1,1} &= [\![\omega_1, \mathbf{A}\omega_1]\!]_2 = [\![\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{A}\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{A}\mathbf{e}_2]\!]_2 \\ &= [\![\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{A}\mathbf{e}_1 \wedge \mathbf{e}_2]\!]_2 + [\![\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{A}\mathbf{e}_2]\!]_2 \\ &= \det\begin{bmatrix}\langle \mathbf{e}_1, \mathbf{A}\mathbf{e}_1\rangle_{\mathbb{C}} & \langle \mathbf{e}_1, \mathbf{e}_2\rangle_{\mathbb{C}} \\ \langle \mathbf{e}_2, \mathbf{A}\mathbf{e}_1\rangle_{\mathbb{C}} & \langle \mathbf{e}_2, \mathbf{e}_2\rangle_{\mathbb{C}}\end{bmatrix} + \det\begin{bmatrix}\langle \mathbf{e}_1, \mathbf{e}_1\rangle_{\mathbb{C}} & \langle \mathbf{e}_1, \mathbf{A}\mathbf{e}_2\rangle_{\mathbb{C}} \\ \langle \mathbf{e}_2, \mathbf{e}_1\rangle_{\mathbb{C}} & \langle \mathbf{e}_2, \mathbf{A}\mathbf{e}_2\rangle_{\mathbb{C}}\end{bmatrix} \\ &= \langle \mathbf{e}_1, \mathbf{A}\mathbf{e}_1\rangle_{\mathbb{C}} + \langle \mathbf{e}_2, \mathbf{A}\mathbf{e}_2\rangle_{\mathbb{C}} = a_{11} + a_{22}\,.\end{aligned}$$

Similarly

$$\{\mathbf{A}^{(2)}\}_{1,2} = [\![\omega_1, \mathbf{A}\omega_2]\!]_2 = \langle \mathbf{e}_2, \mathbf{A}\mathbf{e}_3\rangle_{\mathbb{C}} = a_{23}\,.$$

Continuing this way, we find

$$\mathbf{A}^{(2)} = \begin{bmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{bmatrix}. \tag{2.8}$$

The induced matrix in (2.8) is precisely the form obtained using the compound matrix method (see equation (2.11) in [32]). The advantage of exterior algebra is that it is clear how to change basis, to automate the construction, and to generalize it to any $k$ and $n$.

A simple and illuminating example showing the effect of basis change is as follows. Keep the standard basis for $\mathbb{C}^4$, but consider the following permuted basis for $\bigwedge^2(\mathbb{C}^4)$,

$$\begin{aligned} &\omega_1 = \mathbf{e}_1 \wedge \mathbf{e}_2\,, \quad \omega_2 = \mathbf{e}_1 \wedge \mathbf{e}_3\,, \quad \omega_3 = \mathbf{e}_1 \wedge \mathbf{e}_4\,, \\ &\omega_4 = \mathbf{e}_3 \wedge \mathbf{e}_4\,, \quad \omega_5 = \mathbf{e}_4 \wedge \mathbf{e}_2\,, \quad \omega_6 = \mathbf{e}_2 \wedge \mathbf{e}_3\,. \end{aligned} \tag{2.9}$$

Starting with $\mathbf{A}$ in (2.6), the induced $6 \times 6$ matrix on $\bigwedge^2(\mathbb{C}^4)$ is

$$\mathbf{A}^{(2)} = \begin{bmatrix} a_{11}+a_{22} & a_{23} & a_{24} & 0 & a_{14} & -a_{13} \\ a_{32} & a_{11}+a_{33} & a_{34} & -a_{14} & 0 & a_{12} \\ a_{42} & a_{43} & a_{11}+a_{44} & a_{13} & -a_{12} & 0 \\ 0 & -a_{41} & a_{31} & a_{33}+a_{44} & -a_{32} & -a_{42} \\ a_{41} & 0 & -a_{21} & -a_{23} & a_{22}+a_{44} & -a_{43} \\ -a_{31} & a_{21} & 0 & -a_{24} & -a_{34} & a_{22}+a_{33} \end{bmatrix}. \tag{2.10}$$

The interesting feature of this matrix is that a partition into $3 \times 3$ sub-matrices has extra structure,

$$\mathbf{A}^{(2)} = \begin{bmatrix} \mathbf{B} & \mathbf{S}_1 \\ \mathbf{S_2} & \tau \mathbf{I}_3 - \mathbf{B}^T \end{bmatrix} ,$$

where the $3 \times 3$ sub-matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ are skew-symmetric. With this basis, the ODE $\mathbf{U}_x = \mathbf{A}^{(2)}\mathbf{U}$, with $\mathbf{U} = (\mathbf{V}, \mathbf{W})$ and $\mathbf{A}^{(2)}$ in the form (2.10), can be written

$$\mathbf{V}_x = \mathbf{B}\mathbf{V} + \mathbf{S}_1\mathbf{W} \quad \text{and} \quad \mathbf{W}_x = -\mathbf{B}^T\mathbf{W} + \mathbf{S}_2\mathbf{V} + \tau\,\mathbf{W}\,. \tag{2.11}$$

## 2.1 $k$-dimensional subspaces, decomposability and Grassmannians

A $k$-form representing a $k$-dimensional subspace is in $\bigwedge^k(\mathbb{C}^n)$, but an arbitrary point $\mathbf{U} \in \bigwedge^k(\mathbb{C}^n)$ will not necessarily represent a $k$-dimensional subspace. For example, the two-form

$$\mathbf{U} = \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4 \in \bigwedge^2(\mathbb{C}^4)\,,$$

does not represent a 2-dimensional subspace of $\mathbb{C}^4$ because it can not be written as $\xi \wedge \eta$ with $\xi, \eta \in \mathbb{C}^4$. In general, if $\mathbf{U} \in \bigwedge^k(\mathbb{C}^n)$ represents a $k$-dimensional subspace then it is called *decomposable* (cf. Marcus [29]).

Therefore in order to integrate (2.4) along a path of $k$-dimensional subspaces, it has to be restricted to decomposable $k$-forms. We will give the details of this restriction for $n = 4$ and $k = 2$ and then mention aspects of the case for general $k$ and $n$.

A 2-form $\mathbf{U} \in \bigwedge^2(\mathbb{C}^4)$ is decomposable if $\mathbf{U} \wedge \mathbf{U} = 0$ (cf. Griffiths and Harris [18]). Expanding $\mathbf{U}$ in terms of the standard basis (2.7) of $\bigwedge^2(\mathbb{C}^4)$,

$$\begin{aligned} 0 = \mathbf{U} \wedge \mathbf{U} &= \sum_{i=1}^{6}\sum_{j=1}^{6} U_i U_j\, \omega_i \wedge \omega_j \\ &= (U_1U_6 - U_2U_5 + U_3U_4)\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4\,. \end{aligned} \tag{2.12}$$

Define $\mathcal{I} : \bigwedge^2(\mathbb{C}^4) \to \mathbb{C}$ by

$$\mathcal{I}(\mathbf{U}) = U_1U_6 - U_2U_5 + U_3U_4\,. \tag{2.13}$$

For a path of the equation (2.4) with $k = 2$ and $n = 4$ to be a path of 2-dimensional subspaces, the function (2.13) has to be preserved. The surface defined by $\mathcal{I}(\mathbf{U}) = 0$ is $G_2(\mathbb{C}^4)$, the Grassmannian manifold of 2-planes in $\mathbb{C}^4$ (cf. [18]).

Alternatively, the form of the invariant (2.13) can be derived using the identity,

$$\varSigma\,\mathbf{A}^{(2)} + \left(\varSigma\,\mathbf{A}^{(2)}\right)^T = \tau\,\varSigma\,, \tag{2.14}$$

where $\mathbf{\Sigma}$ is a $6 \times 6$ symmetric orthogonal matrix associated with Hodge duality, defined in equation (5.11) in Sect. 5, where the numerics of Hodge duality is developed. The identity (2.14) can be verified by direct calculation using the explicit expression (2.8), although a more general result can be proved for any $\mathbf{A}^{(k)}$ using Hodge duality; see [6] and Sect. 5. Using $\mathbf{\Sigma}$, (2.13) becomes

$$\mathcal{I}(\mathbf{U}) = \langle \overline{\mathbf{U}}, \mathbf{\Sigma}\,\mathbf{U}\rangle_{\mathbb{C}} = \langle \mathbf{U}, \mathbf{\Sigma}\,\mathbf{U}\rangle_{\mathbb{R}} = U_1U_6 - U_2U_5 + U_3U_4\,, \tag{2.15}$$

where the $\mathbb{R}$ subscript indicates a real inner product. The expression (2.15) and the identity (2.14) can be used to prove that $\mathcal{I}(\mathbf{U})$ is an invariant manifold of (2.4) when $k = 2$ and $n = 4$,

$$\begin{aligned}
\frac{d}{dx}\mathcal{I}(\mathbf{U}) &= \frac{d}{dx}\langle \mathbf{U}, \Sigma\,\mathbf{U}\rangle_{\mathbb{R}} \\
&= 2\,\langle \mathbf{U}, \Sigma\,\mathbf{U}_x\rangle_{\mathbb{R}} \\
&= 2\,\langle \mathbf{U}, \Sigma\,\mathbf{A}^{(2)}\mathbf{U}\rangle_{\mathbb{R}}\,, \quad \text{since } \mathbf{U}_x = \mathbf{A}^{(2)}\mathbf{U} \\
&= 2\,\langle \mathbf{U}, (\tau\Sigma - (\Sigma\mathbf{A}^{(2)})^T)\mathbf{U}\rangle_{\mathbb{R}}\,, \quad \text{using (2.14)} \\
&= 2\tau\,\langle \mathbf{U}, \Sigma\,\mathbf{U}\rangle_{\mathbb{R}} - 2\,\langle \mathbf{U}, \Sigma\,\mathbf{A}^{(2)}\mathbf{U}\rangle_{\mathbb{R}} \\
&= 2\tau\,\mathcal{I} - \mathcal{I}_x
\end{aligned}$$

and so

$$\frac{d}{dx}\mathcal{I}(\mathbf{U}) = \tau\,\mathcal{I}(\mathbf{U})\,. \tag{2.16}$$

If $\mathcal{I}(\mathbf{U}) = 0$ at the starting value of $x$, then $\mathcal{I}(\mathbf{U}) = 0$ for all $x$. When $\tau = 0$, $\mathcal{I}_x = 0$ independent of the value of $\mathcal{I}(\mathbf{U})$. In this latter case, we say that the constraint manifold is a *strong* invariant. (See Leimkuhler and Reich [28] for definitions of strong and weak constraint manifolds.) The preservation of constraint manifolds of this type will be an important requirement of any numerical scheme for integrating (2.4).

When $n > 4$ and $k > 1$ the number of constraints that $\mathbf{U} \in \bigwedge^k(\mathbb{C}^n)$ must satisfy is much greater. However, the constraints are always quadratic. Abstract general formulas for the quadratic constraints are given in [18] and [22].

For the case $k = 3$ and $n = 6$, Davey [16] has worked out the complete collection of quadric surfaces and he shows that $G_3(\mathbb{C}^6)$ – the manifold of $3-$dimension subspaces of $\mathbb{C}^6$ – is the intersection of exactly 35 quadric surfaces. The preservation of all these quadrics by a numerical scheme is an interesting open problem. In [7], the case $k = 2$ and $n = 5$ is studied, where there are exactly 5 quadrics.

When the basis for $\bigwedge^k(\mathbb{C}^n)$ changes, the form of the quadrics will also change. For example, when the basis for $\bigwedge^2(\mathbb{C}^4)$ is taken to be (2.9), the function $\mathcal{I}(\mathbf{U})$ is transformed to

$$\mathcal{I}(\mathbf{U}) = \langle \mathbf{U}, \mathbf{\Sigma}\, \mathbf{U} \rangle_{\mathbb{R}} = U_1U_4 + U_2U_5 + U_3U_6 \,, \tag{2.17}$$

since, relative to the basis (2.9),

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} .$$

In the real case, an alternative basis for $\bigwedge^2(\mathbb{R}^4)$ which provides illuminating information is

$$\begin{aligned} \omega_1 &= \mathbf{e}_1 \wedge \mathbf{e}_2 - \mathbf{e}_3 \wedge \mathbf{e}_4 \,, \quad \omega_2 = \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4 \,, \\ \omega_3 &= \mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3 \,, \quad \omega_4 = \mathbf{e}_1 \wedge \mathbf{e}_4 + \mathbf{e}_2 \wedge \mathbf{e}_3 \,, \\ \omega_5 &= -\mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4 \,, \quad \omega_6 = \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4 \,. \end{aligned} \tag{2.18}$$

The mapping $\mathbf{\Sigma}$ relative to this basis simplifies to

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} ,$$

and therefore the invariant manifold $\mathcal{I}$ is transformed to

$$\mathcal{I}(\mathbf{U}) = \mathbf{V} \cdot \mathbf{V} - \mathbf{W} \cdot \mathbf{W} = V_1^2 + V_2^2 + V_3^2 - W_1^2 - W_2^2 - W_3^2 = 0 \,,$$

when $\mathbf{U} = (\mathbf{V}, \mathbf{W}) \in \bigwedge^2(\mathbb{R}^4) \cong \mathbb{R}^3 \times \mathbb{R}^3$, and so $V_1^2 + V_2^2 + V_3^2 = W_1^2 + W_2^2 + W_3^2$. By a suitable scaling of the magnitude of $\mathbf{V}$ and $\mathbf{W}$, these coordinates show that the invariant manifold $\mathcal{I}$ is the double sphere $S^2 \times S^2$, and is a way of seeing the differential geometric result, namely $G_2(\mathbb{R}^4) \cong S^2 \times S^2$ (cf. Chern et al. [10], p. 64).

## 3 Geometric numerical integration

In choosing a numerical method for integrating the induced systems on $\bigwedge^k(\mathbb{C}^n)$, accuracy is an important factor. However, it is also important to preserve the manifold of $k$-dimensional subspaces.

Numerical integration of the induced systems obtained by the compound matrix method have been integrated using explicit fourth-order Runge-Kutta algorithms (cf. [30–32]). However, explicit algorithms will not necessarily preserve the surface $\mathcal{I}(\mathbf{U})$ accurately, especially over long range integration.

Our main observation is that the natural family of integrators for these systems is the class of implicit Gauss-Legendre Runge-Kutta algorithms, because they possess the special property that strong quadratic invariants are preserved automatically to machine accuracy.

To illustrate the role played by the integrator, consider the case of $n = 4$ and $k = 2$. Then the induced system on $\bigwedge^2(\mathbb{C}^4)$ to be integrated is

$$\mathbf{U}_x = \mathbf{B}(x,\lambda)\,\mathbf{U}\,, \quad \mathbf{U}(x,\lambda)\big|_{x=a} = \xi(\lambda) \in \bigwedge^2(\mathbb{C}^4)\,, \tag{3.1}$$

where $\mathbf{B}(x,\lambda) = \mathbf{A}^{(2)}(x,\lambda)$, and $\xi(\lambda)$ is a decomposable element of $\bigwedge^2(\mathbb{C}^4)$.

The decomposability of $\xi(\lambda)$ implies that $\mathcal{I}(\xi) = 0$, and by (2.16) $\mathcal{I}(\mathbf{U}) = 0$ for all $x$ in the range of integration. Therefore if possible, the numerical method should be designed to preserve this constraint exactly.

The requirement $\mathcal{I}(\mathbf{U}) = 0$ is a quadratic constraint on the differential equation (3.1). Cooper [11] has proved that implicit Gauss-Legendre Runge-Kutta (GL-RK) methods preserve strong quadratic constraints – of linear and nonlinear systems of differential equations – to machine accuracy. Algorithms for the implementation of GL-RK methods are given in Hairer et al. [20].

If $\tau = 0$, the Grassmanian $I = 0$ is a strong invariant and therefore $\mathcal{I}^{s+1} = \mathcal{I}^s$ and the value of $\mathcal{I}$ will be preserved exactly (to machine precision) by the numerical scheme. This special case is of great interest because many examples of interest, such as the Orr-Sommerfeld equation and the linearized Ginzburg Landau equation have the property that $\tau = 0$. In the numerical results presented in Sect. 7-10, both the second order GL-RK and the fourth order GL-RK algorithm will be used.

## 4 Induced boundary conditions on $\bigwedge^k$

One way that preferred $k$-dimensional subspaces of $\mathbb{C}^n$ arise is through boundary conditions. In this section, the induced boundary conditions on $\bigwedge^k(\mathbb{C}^n)$ are derived when $k$-boundary conditions for the original system on $\mathbb{C}^n$ are specified at some point, possibly infinity.

In Sect. 4.1, the case of boundary conditions on a finite domain expressed in standard form is considered. The induced conditions in this case can be deduced using compound matrices (cf. [32]) and we show how exterior algebra approaches the problem, and how the construction depends on the chosen basis.

In the case of semi-infinite Sect. 4.2 and infinite domains Sect. 4.3, the approach using exterior algebra leads to a new and straightforward approach to asymptotic boundary conditions. When the $x-$domain is infinite, the natural way to integrate is from $x = L_\infty$ to $x = 0$ and then from $x = -L_\infty$ to $x = 0$, where $L_\infty$ is some chosen large value of $x$. The difficulty is how to match the two solutions at $x = 0$. In the exterior algebra setting, the matching condition is naturally derived using Hodge duality and the Hodge star operator. As far as we are aware this is the first use of Hodge duality in a numerical setting. The necessary details about Hodge duality are given in Sect. 5, and in Sect. 5.1, the new matching condition at $x = 0$ is derived.

### *4.1 Two-point boundary conditions – finite interval*

Suppose that the boundary conditions for the linear system

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\,\mathbf{u}\,, \quad \mathbf{u} \in \mathbb{C}^n\,, \quad \lambda \in \Lambda\,, \tag{4.1}$$

are

$$\langle \zeta_j(\lambda), \mathbf{u}(x,\lambda)\rangle_{\mathbb{R}}\big|_{x=a} = 0\,, \quad j = 1,\ldots,n-k \tag{4.2}$$

and

$$\langle \eta_j(\lambda), \mathbf{u}(x,\lambda)\rangle_{\mathbb{R}}\big|_{x=b} = 0\,, \quad j = 1,\ldots,k\,, \tag{4.3}$$

where $\{\zeta_1(\lambda),\ldots,\zeta_{n-k}(\lambda)\}$ and $\{\eta_1(\lambda),\ldots,\eta_k(\lambda)\}$ each form linearly independent sets, and depend analytically on $\lambda$. If a complex inner product is used then the conjugates of $\zeta_j(\lambda)$ and $\eta_j(\lambda)$ are used in (4.2) and (4.3).

The boundary conditions associated with the induced system on $\bigwedge^k(\mathbb{C}^n)$ are obtained as follows. The conditions (4.2) define a $k$-dimensional subspace of $\mathbb{C}^n$. Let $\{\zeta_{n-k+1}(\lambda),\cdots,\zeta_n(\lambda)\}$ be an analytic basis for this space. The $k$-form

$$\zeta_{n-k+1}(\lambda)\wedge\cdots\wedge\zeta_n(\lambda) \in \bigwedge^k(\mathbb{C}^n)$$

(or any complex multiple of it) is a characterizing form for the space. However, an expression for this form in terms of the same basis used in constructing $\mathbf{A}^{(k)}$ is needed.

Let $\omega_1,\ldots,\omega_d$ be an orthonormal basis for $\bigwedge^k(\mathbb{C}^n)$. This basis should be the same one used to construct the induced system on $\bigwedge^k(\mathbb{C}^n)$ for (4.1). Then, the above $k$-form can be expanded as

$$\zeta_{n-k+1}(\lambda)\wedge\cdots\wedge\zeta_n(\lambda) = \sum_{j=1}^{d} a_j\,\omega_j\,. \tag{4.4}$$

The $d-$dimensional vector, $\mathbf{a} = (a_1,\ldots,a_d) \in \bigwedge^k(\mathbb{C}^n)$ – or any complex multiple of $\mathbf{a}$ – is then the starting vector for the integration,

$$\frac{d}{dx}\mathbf{U} = \mathbf{A}^{(k)}(x,\lambda)\,\mathbf{U}\,, \quad \mathbf{U}(x,\lambda)\big|_{x=a} = \mathbf{a} \in \bigwedge^k(\mathbb{C}^n)\,. \tag{4.5}$$

At $x = b$ the appropriate boundary condition on $\bigwedge^k(\mathbb{C}^n)$ is deduced from (4.3). Expand,

$$\eta_1(\lambda)\wedge\cdots\wedge\eta_k(\lambda) = \sum_{j=1}^{d} b_j\omega_j\,,$$

then the boundary condition imposed on $\mathbf{U}(x,\lambda)$ at $x=b$ is

$$\langle \mathbf{b}, \mathbf{U}(x,\lambda)\rangle_{\mathbb{R}}\big|_{x=b} = 0\,,$$

where $\mathbf{b} = (b_1,\dots,b_d) \in \bigwedge^k(\mathbb{C}^n)$ and, considering $\bigwedge^k(\mathbb{C}^n)$ as a complex $d-$dimensional vector space, the inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is the standard inner product on $\mathbb{R}^d$. This suggests the introduction of a complex analytic function $D(\lambda)$,

$$D(\lambda) = \langle \mathbf{b}, \mathbf{U}(b,\lambda)\rangle_{\mathbb{R}}\,, \tag{4.6}$$

whose zeros correspond to eigenvalues of the original boundary value problem.

### *4.2 Boundary conditions at infinity*

When $b=+\infty$ and the matrix $\mathbf{A}(x,\lambda)$ is asymptotically constant (independent of $x$),

$$\lim_{x\to\infty} \mathbf{A}(x,\lambda) = \mathbf{A}_\infty(\lambda)\,, \quad \forall\lambda \in \Lambda\,, \tag{4.7}$$

asymptotically correct boundary conditions can be derived for the numerical integration.

Asymptotic conditions for integration using the compound matrix method have been derived by Ng and Reid [31] and Davey [14]. We expand on these results in several directions. First, we show that the asymptotic conditions can be derived using the induced system, and that the asymptotic matrix associated with the induced system has a *unique simple eigenvalue of largest negative real part which controls the asymptotics*. Secondly, the framework of exterior algebra shows how the induced boundary condition can be derived relative to any basis. Thirdly, when $\mathbf{A}_\infty(\lambda)$ depends analytically on $\lambda$, the asymptotic boundary conditions can always be constructed to be analytic, even when the eigenvalues of $\mathbf{A}_\infty(\lambda)$ are not analytic.

Suppose that the spectrum of $\mathbf{A}_\infty(\lambda)$ has $k$ eigenvalues with negative real part, and $n-k$ eigenvalues with positive real part, for all $\lambda\in\Lambda$. Then the space of solutions which are bounded as $x\to+\infty$ is $k$-dimensional. Let

$$\sigma^+(\lambda) = \sum_{j=1}^{k} \mu_j^+(\lambda)\,, \tag{4.8}$$

where $\mu_j^+(\lambda)$ are the eigenvalues of $\mathbf{A}_\infty(\lambda)$ with *negative* real part (the plus superscript implies that they are associated with functions bounded as $x \to +\infty$). The function $\sigma^+(\lambda)$ represents the decay rate of the entire $k$-dimensional subspace of solutions which is bounded as $x \to +\infty$.

The function $\sigma^+(\lambda)$ is an analytic function of $\lambda$ – even if some of the individual $\mu_j^+(\lambda)$'s are not analytic. This follows since $\sigma^+(\lambda)$ is a simple eigenvalue of $\mathbf{A}^{(k)}(\lambda)$. First note that,

$$\mathbf{A}^{(k)}\mathbf{U} = \sum_{j=1}^{k} \mathbf{u}_1 \wedge \cdots \wedge \mathbf{A}\mathbf{u}_j \wedge \cdots \wedge \mathbf{u}_k\,, \quad \text{for any } \mathbf{U} = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \in \textstyle\bigwedge^k(\mathbb{C}^n)\,,$$

and so the eigenvalues of $\mathbf{A}_\infty^{(k)}(\lambda)$ are the $k$-fold sums of the eigenvalues of $\mathbf{A}_\infty(\lambda)$. The combination $\sigma^+(\lambda)$ has negative real part strictly smaller than any other $k$-fold combination. Therefore it is simple, and by standard arguments it is an analytic function (cf. Kato [26] Chapter 2).

The matrix $\mathbf{A}_\infty^{(k)}(\lambda)$ can also be obtained by taking the limit as $x \to +\infty$ of $\mathbf{A}^{(k)}(x,\lambda)$. Let $\xi^+(\lambda) \in \bigwedge^k(\mathbb{C}^n)$ be the eigenvector of $\mathbf{A}_\infty^{(k)}(\lambda)$ associated with the eigenvalue $\sigma^+(\lambda)$,

$$\mathbf{A}_\infty^{(k)}(\lambda)\,\xi^+(\lambda) = \sigma^+(\lambda)\,\xi^+(\lambda)\,. \tag{4.9}$$

Since $\sigma^+(\lambda)$ is simple, the eigenvector $\xi^+(\lambda)$ can also be chosen to be an analytic function.

By standard arguments from the theory of differential equations (cf. Coppel [12]), there exists a solution of the differential equation in (4.5) which is an analytic function of $\lambda$ and satisfies

$$\lim_{x\to+\infty} e^{-\sigma^+(\lambda)x}\mathbf{U}^+(x,\lambda) = \xi^+(\lambda)\,, \tag{4.10}$$

or a complex multiple of $\xi^+(\lambda)$.

The numerical strategy to compute this solution is to integrate the differential equation in (4.5) from $x = L_\infty$ to $x = 0$ with $\xi^+(\lambda)$ as the starting vector.

If the boundary condition at $x = 0$ is of the form (4.3), then the formulation of the condition at $x = 0$ follows the argument in Sect. 4.1.

### *4.3 Doubly infinite intervals*

When the $x-$domain extends from $-\infty$ to $+\infty$, the procedure of the previous subsection can be used twice, but an additional subtlety arises when the integration is matched at $x = 0$. For simplicity suppose that

$$\lim_{x\to\pm\infty} \mathbf{A}(x,\lambda) = \mathbf{A}_\infty(\lambda)\,, \quad \forall\,\lambda \in \Lambda\,, \tag{4.11}$$

that is, the limiting matrix $\mathbf{A}_\infty(\lambda)$ is the same at $x = \pm\infty$. (It is straightforward to modify the theory for the case where $\mathbf{A}_{-\infty}(\lambda) \neq \mathbf{A}_{+\infty}(\lambda)$.)

As in the previous subsection, assume that the spectrum of $\mathbf{A}_\infty(\lambda)$ has $k$ eigenvalues with negative real part and $n-k$ eigenvalues with positive real part.

In this case we will have to integrate separate systems for $x > 0$ and $x < 0$. As $x \to +\infty$ the space of solutions which is bounded is $k$-dimensional, and therefore we can use the procedure of the previous section, and integrate

$$\mathbf{U}_x = \mathbf{A}^{(k)}(x,\lambda)\,\mathbf{U}\,, \quad \mathbf{U}(x,\lambda)\big|_{x=L_\infty} = \xi^+(\lambda) \in \textstyle\bigwedge^k(\mathbb{C}^n)\,,$$

from $x = L_\infty$ to $x = 0$.

However, as $x \to -\infty$ the space of solutions which is bounded is $(n-k)-$dimensional, and therefore it is required to integrate

$$\mathbf{V}_x = \mathbf{A}^{(n-k)}(x,\lambda)\,\mathbf{V}\,, \quad \mathbf{V}(x,\lambda)\big|_{x=-L_\infty} = \xi^-(\lambda) \in \textstyle\bigwedge^{n-k}(\mathbb{C}^n)\,,$$

from $x = -L_\infty$ to $x = 0$, where $\xi^-(\lambda)$ is the eigenvector corresponding to the eigenvalue $\sigma^-(\lambda)$ of $\mathbf{A}_\infty^{(n-k)}(\lambda)$ of largest real part, satisfying,

$$\mathbf{A}_\infty^{(n-k)}(\lambda)\xi^-(\lambda) = \sigma^-(\lambda)\,\xi^-(\lambda)\,.$$

The eigenvalue $\sigma^-(\lambda)$ will also be simple and an analytic function of $\lambda$.

A point $\lambda \in \Lambda$ will be an eigenvalue if the space of bounded solutions as $x \to +\infty$, $\mathbf{U}^+(x,\lambda)$, has a nontrivial intersection with the space of solutions which is bounded as $x \to -\infty$, $\mathbf{U}^-(x,\lambda)$. In other words, if

$$\mathbf{U}^+(x,\lambda) \wedge \mathbf{U}^-(x,\lambda) = 0\,, \quad \forall\, x \in \mathbb{R}\,.$$

Introduce the complex analytic function

$$\Delta(\lambda) = \mathrm{e}^{-\int_0^x \tau(s,\lambda)\,ds}\,\mathbf{U}^+(x,\lambda) \wedge \mathbf{U}^-(x,\lambda)\,. \tag{4.12}$$

The function $\Delta(\lambda)$ is independent of $x$ – it is essentially the Wronskian for the linear system – and vanishes if $\lambda \in \Lambda$ is an eigenvalue. However, this expression is not in a form which is useful for numerical computation. In the next section, we will develop the necessary machinery – based on the Hodge star operator – for a numerical evaluation of $\Delta(\lambda)$.

## 5 Numerics of Hodge duality and the Hodge star operator

The vector spaces $\bigwedge^k(\mathbb{C}^n)$ and $\bigwedge^{n-k}(\mathbb{C}^n)$ are isomorphic, and the mapping which takes elements of one space to the other is the Hodge star operator $\star$. In this section, explicit properties of this isomorphism are derived for use in numerics.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be an orthonormal basis for $\mathbb{C}^n$, with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$, introduced in Sect. 2, and fix a volume form for $\mathbb{C}^n$, for example,

$$\mathcal{V} = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n \, .$$

Let $\omega_1, \ldots, \omega_d$ be an orthonormal decomposable basis for $\bigwedge^k(\mathbb{C}^n)$, where $d = \dim(\bigwedge^k(\mathbb{C}^n))$,

$$[\![\omega_i, \omega_j]\!]_k = \delta_{ij} \, , \quad \text{for} \quad i,j = 1, \ldots, d \, ,$$

and let $\alpha_1, \ldots, \alpha_d$ be an orthonormal decomposable basis for $\bigwedge^{n-k}(\mathbb{C}^n)$:

$$[\![\alpha_i, \alpha_j]\!]_{n-k} = \delta_{ij} \, , \quad \text{for} \quad i,j = 1, \ldots, d \, .$$

The Hodge star operator maps an element $\mathbf{V} \in \bigwedge^{n-k}(\mathbb{C}^n)$ to an element $\star\mathbf{V} \in \bigwedge^k(\mathbb{C}^n)$, and it can be explicitly defined by its action on basis vectors

(5.1) $$\star\alpha_i \wedge \alpha_j = \delta_{ij} \, \mathcal{V} \, , \quad i,j = 1, \ldots, d \, .$$

The definition depends on the chosen inner product on $\mathbb{C}^n$ and the chosen orientation.

Since $\star\alpha_i \in \bigwedge^k(\mathbb{C}^n)$ for $i = 1, \ldots, d$, there exists a $d \times d$ matrix $\mathbf{\Sigma}$ such that

(5.2) $$\star\alpha_i = \sum_{j=1}^{d} \mathbf{\Sigma}_{ij} \, \omega_j \, , \quad i = 1, \ldots, d \, .$$

The action of $\star$ also includes complex conjugation (cf. Wells [35]). Here we will assume that the basis is real in which case $\mathbf{\Sigma}$ will be real. Moreover, the matrix $\mathbf{\Sigma}$ will be orthogonal, since it is a mapping from one orthogonal basis to another orthogonal basis. Therefore, the entries of $\mathbf{\Sigma}$ will satisfy

(5.3) $$\mathbf{\Sigma}^T \mathbf{\Sigma} = \mathbf{I}_d \quad \text{or} \quad \sum_{m=1}^{d} \Sigma_{mi} \Sigma_{mj} = \delta_{ij} \, .$$

Substitution of (5.2) into (5.1) leads to

$$\sum_{m=1}^{d} \Sigma_{im} \, \omega_m \wedge \alpha_j = \delta_{ij} \, \mathcal{V} \, .$$

Multiply this expression by $\Sigma_{ip}$ and sum over $i$,

$$\sum_{m=1}^{d} \left( \sum_{i=1}^{d} \Sigma_{ip} \Sigma_{im} \right) \omega_m \wedge \alpha_j = \left( \sum_{i=1}^{d} \delta_{ij} \, \Sigma_{ip} \right) \mathcal{V} \, .$$

Now, using (5.3) this expression reduces to

$$\omega_i \wedge \alpha_j = \Sigma_{ji}\,\mathcal{V}\,. \tag{5.4}$$

Given $\mathbf{V} = \sum_{j=1}^d V_j\alpha_j \in \bigwedge^{n-k}(\mathbb{C}^n)$, a general expression for the action of $\star$ is then

$$\star\mathbf{V} = \sum_{j=1}^d \overline{V}_j \star\alpha_j = \sum_{j=1}^d \overline{V}_j \sum_{\ell=1}^d \mathbf{\Sigma}_{j\ell}\omega_\ell = \sum_{\ell=1}^d F_\ell\,\omega_\ell$$
$$\text{with}\quad F_\ell = \sum_{j=1}^d \mathbf{\Sigma}_{j\ell}\overline{V}_j\,.$$

Let $\mathbf{F} \in \bigwedge^k(\mathbb{C}^n)$ have components $F_\ell$ as above, then an explicit expression for the star operation is

$$\star\mathbf{V} = \mathbf{F} = \mathbf{\Sigma}^T\,\overline{\mathbf{V}}\,, \tag{5.5}$$

where the components of $\mathbf{V}$ are relative to the basis $\alpha_i$ of $\bigwedge^{n-k}(\mathbb{C}^n)$ and the components of $\mathbf{F}$ are relative to the basis $\omega_j$ of $\bigwedge^k(\mathbb{C}^n)$.

These constructions can now be used to prove the general expression

$$\mathbf{U} \wedge \mathbf{V} = [\![\star\mathbf{V}, \mathbf{U}]\!]_k\,\mathcal{V} = \langle\overline{\mathbf{V}}, \mathbf{\Sigma}\mathbf{U}\rangle_{\mathbb{C}}\,\mathcal{V}\,, \tag{5.6}$$

for any $\mathbf{U} \in \bigwedge^k(\mathbb{C}^n)$ and $\mathbf{V} \in \bigwedge^{n-k}(\mathbb{C}^n)$, where $[\![\cdot,\cdot]\!]_k$ is the standard inner product on $\bigwedge^k(\mathbb{C}^n)$, and $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ is a standard inner product on $\mathbb{C}^d$. Note that inner product on $\mathbb{C}^d$ can be written

$$\langle\overline{\mathbf{V}}, \mathbf{\Sigma}\mathbf{U}\rangle_{\mathbb{C}} = \langle\mathbf{V}, \mathbf{\Sigma}\mathbf{U}\rangle_{\mathbb{R}}\,. \tag{5.7}$$

To prove (5.6), expand $\mathbf{U} \in \bigwedge^k(\mathbb{C}^n)$ and $\mathbf{V} \in \bigwedge^{n-k}(\mathbb{C}^n)$ in terms of the respective basis vectors,

$$\mathbf{U} = \sum_{i=1}^d U_i\,\omega_i \quad\text{and}\quad \mathbf{V} = \sum_{j=1}^d V_j\,\alpha_j\,.$$

Substitution into the left-hand side of (5.6) and use of the property

$$\mathbf{U} \wedge \mathbf{V} = (-1)^{pq}\,\mathbf{V} \wedge \mathbf{U},\ \text{for any } \mathbf{U} \in {\textstyle\bigwedge}^p(\mathbb{C}^n),\ \mathbf{V} \in {\textstyle\bigwedge}^q(\mathbb{C}^n)\,, \tag{5.8}$$

leads to

$$\begin{aligned}\mathbf{U}\wedge\mathbf{V} = (-1)^{k(n-k)}\mathbf{V}\wedge\mathbf{U} &= (-1)^{k(n-k)}\sum_{i=1}^{d}\sum_{j=1}^{d} U_iV_j\,\alpha_j\wedge\omega_i\,,\\ &= \sum_{i=1}^{d}\sum_{j=1}^{d} U_iV_j\,\omega_i\wedge\alpha_j\,,\\ &= \sum_{i=1}^{d}\sum_{j=1}^{d} U_iV_j\,\Sigma_{ji}\\ &= \langle\overline{\mathbf{V}},\mathbf{\Sigma}\mathbf{U}\rangle_{\mathbb{C}} = \langle\mathbf{V},\mathbf{\Sigma}\mathbf{U}\rangle_{\mathbb{R}}\,,\end{aligned}$$

using (5.4).

For the right-hand side of (5.6),

$$\begin{aligned}[\![\star\mathbf{V},\mathbf{U}]\!]_k &= [\![\sum_{i=1}^{d}\overline{V}_i\,\star\alpha_i, \sum_{j=1}^{d} U_j\,\omega_j]\!]_k\\ &= \sum_{i=1}^{d}\sum_{j=1}^{d} V_iU_j\,[\![\star\alpha_i,\omega_j]\!]_k\\ &= \sum_{i=1}^{d}\sum_{j=1}^{d} V_iU_j\,\sum_{m=1}^{d}\mathbf{\Sigma}_{im}\,[\![\omega_m,\,\omega_j]\!]_k\\ &= \sum_{i=1}^{d}\sum_{j=1}^{d} V_iU_j\mathbf{\Sigma}_{ij}\\ &= \langle\mathbf{V},\mathbf{\Sigma}\mathbf{U}\rangle_{\mathbb{R}}\,.\end{aligned}$$

Comparing this expression with that obtained for the left-hand side of (5.6) completes the proof of the expression (5.6). For the numerics the useful formula deduced from (5.6) is

(5.9) $$\mathbf{U}\wedge\mathbf{V} = \langle\mathbf{V},\mathbf{\Sigma}\mathbf{U}\rangle_{\mathbb{R}},\ \text{for any }\mathbf{U}\in{\textstyle\bigwedge}^k(\mathbb{C}^n),\ \mathbf{V}\in{\textstyle\bigwedge}^{n-k}(\mathbb{C}^n)\,.$$

For particular $k$ and $n$, the elements of $\mathbf{\Sigma}$ are easily constructed using the formula (5.1). For example, suppose $n = 4$ and $k = 2$. Then $d = 6$, $\bigwedge^k(\mathbb{C}^n) = \bigwedge^2(\mathbb{C}^4)$ and $\bigwedge^{n-k}(\mathbb{C}^n) = \bigwedge^2(\mathbb{C}^4)$. Therefore, take the same basis for each space: take $\alpha_i = \omega_i$, $i = 1,\ldots,6$, with

(5.10) $$\begin{aligned}&\omega_1 = \mathbf{e}_1\wedge\mathbf{e}_2\,, \quad \omega_2 = \mathbf{e}_1\wedge\mathbf{e}_3\,, \quad \omega_3 = \mathbf{e}_1\wedge\mathbf{e}_4\,,\\ &\omega_4 = \mathbf{e}_2\wedge\mathbf{e}_3\,, \quad \omega_5 = \mathbf{e}_2\wedge\mathbf{e}_4\,, \quad \omega_6 = \mathbf{e}_3\wedge\mathbf{e}_4\,.\end{aligned}$$

Now, apply formula (5.1) to each basis vector,

$$\begin{aligned}\star\alpha_1 &= \omega_6 \quad \text{since} \quad \alpha_1\wedge\omega_6 = \mathcal{V}\,,\\ \star\alpha_2 &= -\omega_5 \quad \text{since} \quad \alpha_2\wedge\omega_5 = -\mathcal{V}\,,\\ \star\alpha_3 &= \omega_4 \quad \text{since} \quad \alpha_3\wedge\omega_4 = \mathcal{V}\,.\end{aligned}$$

Using the formula (5.8), it follows that $\star\alpha_4 = \omega_3$, $\star\alpha_5 = -\omega_2$, $\star\alpha_6 = \omega_1$, and therefore, when $k = 2$ and $n = 4$, and the basis (5.10) is used,

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}. \tag{5.11}$$

In this case $\boldsymbol{\Sigma}$ is orthogonal and symmetric. The formula (5.9) reduces to

$$\begin{aligned}\mathbf{U}\wedge\mathbf{V} = \langle\overline{\mathbf{V}},\boldsymbol{\Sigma}\mathbf{U}\rangle_{\mathbb{C}} = \langle\mathbf{U},\boldsymbol{\Sigma}\mathbf{V}\rangle_{\mathbb{R}}\,,\\ \text{for any} \quad \mathbf{U}\in{\textstyle\bigwedge}^2(\mathbb{C}^4),\ \mathbf{V}\in{\textstyle\bigwedge}^2(\mathbb{C}^4)\,,\end{aligned} \tag{5.12}$$

where $\mathbf{U} = (U_1,\ldots,U_6)$ and $\mathbf{V} = (V_1,\ldots,V_6)$ are the components of $\mathbf{U}$ and $\mathbf{V}$ in the expansions $\mathbf{U} = \sum_{i=1}^6 U_i\,\omega_i$ and $\mathbf{V} = \sum_{j=1}^6 V_j\,\omega_j$.

Other useful formulae which follow from the above constructions, for any $k$ and $n$, are

$$\begin{aligned}\star\omega_i &= (-1)^{k(n-k)}\sum_{j=1}^{d}\Sigma_{ji}\,\alpha_j\,, \quad i = 1,\ldots,d\,,\\ \text{and} \quad \star\star\alpha_i &= (-1)^{k(n-k)}\,\alpha_i\,.\end{aligned} \tag{5.13}$$

Using these expressions and (5.5),

$$**\,\mathbf{V} = \Sigma^T\,\Sigma^T\,\mathbf{V} = (-1)^{k(n-k)}\mathbf{V}\,.$$

Therefore, $\Sigma$ is symmetric (respectively skew-symmetric) when $k(n-k)$ is even (respectively odd).

### *5.1 Interior matching condition for infinite domains*

The formula (4.12) is not useful for numerics. A formula convenient for numerics can be derived using Hodge duality.

Since $\mathbf{U}^+\in\bigwedge^k(\mathbb{C}^n)$ and $\mathbf{U}^-\in\bigwedge^{n-k}(\mathbb{C}^n)$, the function (4.12) is the product of a complex function times the volume form on $\mathbb{C}^n$. Fix the volume form to be $\mathcal{V} = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is a fixed orthonormal basis. The expression (4.12) then reduces to

$$\Delta(\lambda) = D(\lambda)\,\mathcal{V}\,. \tag{5.14}$$

An explicit expression for $D(\lambda)$ will be derived using Hodge duality. The vector spaces $\bigwedge^k(\mathbb{C}^n)$ and $\bigwedge^{n-k}(\mathbb{C}^n)$ are isomorphic and the isomorphism is the Hodge star operator $\star$. In particular, in (5.9) it is shown that, for any pair of forms $\mathbf{U}^+ \in \bigwedge^k(\mathbb{C}^n)$ and $\mathbf{U}^- \in \bigwedge^{n-k}(\mathbb{C}^n)$,

$$\mathbf{U}^+(x,\lambda) \wedge \mathbf{U}^-(x,\lambda) = \langle \mathbf{U}^-, \mathbf{\Sigma}\,\mathbf{U}^+\rangle_{\mathbb{R}}\,.$$

An algorithm for constructing the $d \times d$ matrix $\mathbf{\Sigma}$ is given in Sect. 5, and in the right-hand side of this formula, the components of $\mathbf{U}^+ \in \bigwedge^k(\mathbb{C}^n)$ are with respect to the basis $\omega_1, \ldots, \omega_d$. Similarly for $\mathbf{U}^-$, whose components are the coordinates of $\mathbf{U}^- \in \bigwedge^{n-k}(\mathbb{C}^n)$ with respect to the basis $\alpha_1, \ldots, \alpha_d$.

Therefore the matching function $D(\lambda)$ takes the explicit form,

$$D(\lambda) = \mathrm{e}^{-\int_0^x \tau(s,\lambda)\,ds}\,\langle \mathbf{U}^-, \mathbf{\Sigma}\mathbf{U}^+\rangle_{\mathbb{R}}\,. \tag{5.15}$$

When $\tau = 0$ the above formula simplifies. For the Orr-Sommerfeld equation on an infinite domain, the case relevant for the stability of jets, wakes and mixing layers for example, the above formula simplifies further. In this case $k = 2$, $n = 4$, $\tau(x,\lambda) = 0$ and $\mathbf{\Sigma}^T = \mathbf{\Sigma}$ and so

$$D(\lambda) = \langle \mathbf{U}^+, \mathbf{\Sigma}\,\mathbf{U}^-\rangle_{\mathbb{R}}\,, \tag{5.16}$$

where $\Sigma$ is given in (5.11).

## 6 Intermezzo: holomorphic systems and analytic subspaces

The function $D(\lambda)$, associated with either the finite or infinite interval case, whose roots are eigenvalues, is a complex analytic function, and so application of Newton's method to find roots is straightforward. Given a suitable first guess $\lambda_0$ the Newton sequence

$$\lambda_{m+1} = \lambda_m - \frac{D(\lambda_m)}{D'(\lambda_m)}\,, \quad m = 0, \ldots\,,$$

can be computed. When

$$D(\lambda) = \langle \mathbf{b}(\lambda), \mathbf{U}(0,\lambda)\rangle_{\mathbb{R}}\,,$$

the derivative $D'(\lambda)$ is

$$D'(\lambda) = \langle \mathbf{b}'(\lambda), \mathbf{U}(0,\lambda)\rangle_{\mathbb{R}} + \langle \mathbf{b}(\lambda), \partial_\lambda \mathbf{U}(0,\lambda)\rangle_{\mathbb{R}}\,.$$

The vector $\mathbf{b}(\lambda)$ is known explicitly and therefore its derivative can be computed analytically. The derivative of $\mathbf{U}(0,\lambda)$ can be computed by appending a differential equation for $\partial_\lambda \mathbf{U}(x,\lambda)$ to the basic ODE,

$$\frac{d}{dx}\begin{pmatrix}\mathbf{U}\\ \partial_\lambda \mathbf{U}\end{pmatrix} = \begin{bmatrix}\mathbf{A}^{(k)}(x,\lambda) & \mathbf{0}\\ \partial_\lambda \mathbf{A}^{(k)}(x,\lambda) & \mathbf{A}^{(k)}(x,\lambda)\end{bmatrix}\begin{pmatrix}\mathbf{U}\\ \partial_\lambda \mathbf{U}\end{pmatrix}. \tag{6.1}$$

The initial condition for (6.1) is the initial condition for $\mathbf{U}(x,\lambda)$ and the derivative of this initial condition. When the domain is finite this approach is straightforward.

When the domain is infinite or semi-infinite, this approach is still satisfactory although computing the starting values is not as straightforward. This construction is needed for example when computing the stability for 3D rotating flows (cf. Sect. 10). An algorithm for computing the starting vector and its derivative is constructed as follows.

To simplify notation, let $\mathbf{B}(\lambda) = \mathbf{A}_\infty^{(k)}(\lambda)$. The problem is to find the eigenvalue of $\mathbf{B}(\lambda)$ of largest positive (or negative) real part, its eigenvector and the derivative with respect to $\lambda$ of its eigenvector. For definiteness, assume it is the eigenvalue with largest positive real part that is desired, and denote it by $\sigma(\lambda)$ and denote its eigenvector by $\xi(\lambda)$.

It follows from the theory in Sect. 4 that the eigenvalue of largest real part of $\mathbf{B}(\lambda)$ is simple and therefore analytic. The eigenvalue equation, $\mathbf{B}(\lambda)\xi(\lambda) = \sigma(\lambda)\xi(\lambda)$, is analytic and when differentiated with respect to $\lambda$,

$$(\mathbf{B}(\lambda) - \sigma(\lambda)\mathbf{I})\frac{d}{d\lambda}\xi(\lambda) = -\mathbf{B}'(\lambda)\xi(\lambda) + \sigma'(\lambda)\,\xi(\lambda)\,. \tag{6.2}$$

To obtain $\frac{d}{d\lambda}\xi(\lambda)$ it is necessary to solve this system, but the matrix $(\mathbf{B}(\lambda)-\sigma(\lambda))$ is singular since $\sigma(\lambda)$ is an eigenvalue. The following coupled bordered system on $\mathbb{C}^{d+1}$ is solved,

$$\left[\begin{pmatrix}[\mathbf{B}(\lambda) - \sigma(\lambda)\mathbf{I}] & -\xi(\lambda)\\ -\eta(\lambda)^* & 0\end{pmatrix}\right]\begin{pmatrix}\xi'(\lambda)\\ \sigma'(\lambda)\end{pmatrix} = \begin{pmatrix}-\mathbf{B}'(\lambda)\,\xi(\lambda)\\ 0\end{pmatrix}. \tag{6.3}$$

It is straightforward to show that the bordered matrix is invertible.

In summary, for fixed $\lambda$, the eigenvalue (and associated eigenvector) of $\mathbf{A}_\infty^{(k)}(\lambda)$, of largest positive (or negative) real part is obtained numerically. Since this eigenvalue is simple, and has real part farther from the origin than any other eigenvalue of positive (respectively negative) real part, this numerical construction will be robust. The augmented system (6.3) is then solved for $\xi'(\lambda)$, and the starting vector for (6.1) is then $(\xi(\lambda),\xi'(\lambda)) \in \mathbb{C}^d \times \mathbb{C}^d$.

There is another more subtle way that analyticity enters the analysis when the domain of integration is infinite.

As discussed in Sect. 4, even though individual eigenvalues may not be analytic functions, their $k$-fold sum, which appears as an eigenvalue of the induced system is always analytic. A simple example from hydrodynamic stability which illustrates this point about analyticity is the case of the asymptotic suction profile (cf. Drazin and Reid [17], p. 227, Hocking [25], Ng and Reid [31], Herron [24]).

The governing equation for stability is a modification of the Orr-Sommerfeld equation,

$$\begin{aligned}\phi'''' + \phi''' + (-2\alpha^2 - \lambda R - i\alpha R U(y))\phi'' - \alpha^2\phi' \\ +(\alpha^4 + i\alpha R U''(y) + i\alpha^3 R U(y) + \lambda R\alpha^2)\phi = 0\,,\end{aligned}$$

where $U(y) = 1 - e^{-y}$. This system can be written in the form (2.1), and $\mathbf{A}(y,\lambda)$ goes to a well-defined limit as $y \to \infty$.

The characteristic equation for $\mathbf{A}_\infty(\lambda)$ is

$$\begin{aligned}\det[\mu\mathbf{I} - \mathbf{A}_\infty(\lambda)] &= \mu^4 + \mu^3 + (-2\alpha^2 - \lambda R - i\alpha R)\mu^2 \\ &\quad -\alpha^2\mu + (\alpha^4 + i\alpha^3 R + \lambda R\alpha^2) \\ &= 0\,,\end{aligned}$$

Unlike the Orr-Sommerfeld equation, this asymptotic system has a value of $\lambda$ with positive real part, denoted $\lambda_0$, with

$$\lambda_0 = \frac{\alpha}{R} - i\alpha\,,$$

where the two $\mu-$eigenvalues with negative real part coalesce (cf. Herron [23] and Herron [24], p. 602). This coalescence can give rise to a branch point in the complex $\lambda$ plane, and individual solutions of (6.4) will not be analytic for all $\lambda$ in $\mathbb{C}_+$, the right-half complex plane.

On the other hand, the eigenvalue $\sigma^+(\lambda)$ – the sum of the eigenvalues of negative real part – will be simple and an analytic function of $\lambda$, and therefore an eigenvalue relation for (6.4) constructed by restricting (6.4) to $\bigwedge^2(\mathbb{C}^4)$ will be analytic for all $\lambda \in \mathbb{C}_+$.

## 7 Example 1. Boundary layer interacting with a compliant surface

The study of the boundary-layer flow past a flexible surface has two primary motivations. It is a fundamental model for the fluid flow past a dolphin and other aquatic species (cf. Kramer [27]). Secondly, coating surfaces of man-made waterborne vehicles with a compliant surface has been proposed as a mechanism for delaying transition and drag reduction (cf. Carpenter [8]).

In this section the theory of Sect. 2-6 will be applied to the model proposed by Carpenter and Garrad [9] for the stability of two-dimensional boundary-layer flow past a Kramer compliant surface.

The governing equations for the fluid are the two-dimensional Navier-Stokes equations, linearized about the Blasius boundary layer, and the governing equation for the wall is a beam equation forced by the fluid pressure at the wall. After nondimensionalization, the problem can be reduced to the Orr-Sommerfeld equation, a fourth-order complex ODE, coupled to boundary conditions at the wall. The Orr-Sommerfeld equation for the vertical velocity perturbation $\phi(y)$ takes the form

$$(i\alpha R)^{-1}\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\phi = (U(y)-i\lambda/\alpha)\left(\frac{d^2}{dy^2}-\alpha^2\right)\phi - U''(y)\phi\,, \quad 0 \le y < +\infty\,, \tag{7.1}$$

where $U(y)$ is the Blasius velocity profile, $\alpha$ is the streamwise wavenumber, $R$ is the Reynolds number, and $\lambda$, which is associated with the wavespeed of the perturbation, is the eigenvalue.

The two boundary conditions at the compliant surface, $y = 0$, are

$$\alpha^2 R\,(\lambda^2\,C_m + C_B\alpha^4 + C_{KE})\,\phi(0) + \lambda(\phi'''(0) - \alpha^2\phi'(0)) = 0\,. \tag{7.2}$$

and

$$\alpha\,U'(0)\phi(0) + i\lambda\,\phi'(0) = 0\,, \tag{7.3}$$

where $U'(0)$ is the derivative of the Blasius velocity at the wall, and $C_m$, $C_B$ and $C_{KE}$ are dimensionless parameters representative of wall properties, taking the form

$$C_m = \frac{24226.420899}{R}\,, \quad C_B = 6078227.413\frac{E}{R^3} \quad \text{and} \quad C_{KE} = 2.291813\times 10^{-13}(230ER)\,. \tag{7.4}$$

The main parameter is $E$, which represents wall rigidity and is input in units of $Nm^{-2}$; an increase in $E$ represents an increase in wall rigidity, with $E=\infty$ corresponding to a rigid wall. Details of the derivation of the above system can be found in [9].

This equation can be written in the form (2.1) with $n = 4$ (and the symbol $x$ replaced by $y$) by taking

$$\mathbf{u} = \begin{pmatrix}\phi\\ \phi'\\ \phi''\\ \phi'''\end{pmatrix} \quad \text{and} \quad \mathbf{A}(y,\lambda) = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \gamma_1(y,\lambda) & 0 & \gamma_2(y,\lambda) & 0\end{bmatrix}. \tag{7.5}$$

where

$$\gamma_1(y,\lambda) = -\alpha^4 - i\alpha^3 R\,U(y) - i\alpha R U''(y) - \alpha^2\lambda R \tag{7.6}$$
$$\text{and}\quad \gamma_2(y,\lambda) = 2\alpha^2 + i\alpha R\,U(y) + \lambda R\,.$$

Note that the trace of $\mathbf{A}(y,\lambda)$ is zero.

The boundary conditions at $y = 0$ can be written in the form (4.3),

$$\langle \eta_1(\lambda), \mathbf{u}(0,\lambda)\rangle_{\mathbb{R}} = \langle \eta_2(\lambda), \mathbf{u}(0,\lambda)\rangle_{\mathbb{R}} = 0\,, \tag{7.7}$$

by taking

$$\eta_1(\lambda) = \begin{pmatrix} \alpha U'(0) \\ i\lambda \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \eta_2(\lambda) = \begin{pmatrix} \theta_1(\lambda) \\ \alpha\lambda \\ 0 \\ -\lambda/\alpha \end{pmatrix}.$$

where $\theta_1(\lambda) = -\alpha R(\lambda^2 C_m + \alpha^4 C_B + C_{KE})$.

### *7.1 Equation and boundary conditions on* $\bigwedge^2(\mathbb{C}^4)$

The equations and boundary conditions are now in standard form to apply the theory of Sect. 2-4. By fixing the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_4$ for $\mathbb{C}^4$, the induced differential equation on $\bigwedge^2(\mathbb{C}^4)$ associated with the Orr-Sommerfeld equation is

$$\mathbf{U}_y = \mathbf{A}^{(2)}(y,\lambda)\,\mathbf{U}\,, \quad \mathbf{U} \in \bigwedge^2(\mathbb{C}^4)\,, \tag{7.8}$$

with

$$\mathbf{A}^{(2)}(y,\lambda) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & \gamma_2(y,\lambda) & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\gamma_1(y,\lambda) & 0 & 0 & \gamma_2(y,\lambda) & 0 & 1 \\ 0 & -\gamma_1(y,\lambda) & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7.9}$$

The limit as $y \to \infty$ of $\mathbf{A}(y,\lambda)$ exists and therefore the theory of Sect. 4.2 applies. Working directly with (7.9), we find

$$\lim_{y\to+\infty} \mathbf{A}^{(2)}_\infty(\lambda) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & \gamma_2^\infty(\lambda) & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\gamma_1^\infty(\lambda) & 0 & 0 & \gamma_2^\infty(\lambda) & 0 & 1 \\ 0 & -\gamma_1^\infty(\lambda) & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{7.10}$$

where

(7.11) $\gamma_1^\infty(\lambda) = -\alpha^4 - i\alpha^3 R - \alpha^2\lambda R$ and $\gamma_2^\infty(\lambda) = 2\alpha^2 + i\alpha R + \lambda R.$

The eigenvalue of $\mathbf{A}_\infty^{(2)}(\lambda)$ of largest negative real part is easily calculated to be

$$\sigma^+(\lambda) = -(\alpha+\beta)\,, \quad \text{where} \quad \beta^2 = \alpha^2 + \lambda R + i\alpha R\,,$$

and the square root with positive real part is taken. The eigenvector associated with $\sigma^+(\lambda)$ is

(7.12)
$$\xi^+(\lambda) = \begin{pmatrix} 1 \\ \sigma^+(\lambda) \\ \alpha^2+\alpha\beta+\beta^2 \\ \alpha\beta \\ \alpha\beta\sigma^+(\lambda) \\ \alpha^2\beta^2 \end{pmatrix}.$$

At $y = 0$ the induced boundary conditions are determined using the theory in Sect. 4.1. The condition for an eigenvalue is $D(\lambda) = 0$ with

(7.13)
$$D(\lambda) = \langle \mathbf{b}(\lambda), \mathbf{U}(0,\lambda)\rangle_{\mathbb{R}}\,,$$

with the components of $\mathbf{b}(\lambda)$ deduced from

$$\mathbf{b}(\lambda) = \sum_{j=1}^{6} b_j\,\omega_j = \eta_1(\lambda)\wedge\eta_2(\lambda)\,.$$

But writing $\eta_1(\lambda)$ and $\eta_2(\lambda)$ with respect to the standard basis,

$$\begin{aligned}\eta_1(\lambda)\wedge\eta_2(\lambda) &= (\alpha U'(0)\mathbf{e}_1 + i\lambda\mathbf{e}_2)\wedge\left(\theta_1(\lambda)\,\mathbf{e}_1 + \alpha\lambda\,\mathbf{e}_2 - \frac{\lambda}{\alpha}\,\mathbf{e}_4\right)\\ &= \left(\alpha^2\lambda U'(0) - i\lambda\theta_1(\lambda)\right)\mathbf{e}_1\wedge\mathbf{e}_2\\ &\quad -\lambda U'(0)\mathbf{e}_1\wedge\mathbf{e}_4 - i\frac{\lambda^2}{\alpha}\mathbf{e}_2\wedge\mathbf{e}_4\\ &= \left(\alpha^2\lambda U'(0) - i\lambda\theta_1(\lambda)\right)\omega_1 - \lambda U'(0)\omega_3 - i\frac{\lambda^2}{\alpha}\omega_5\,,\end{aligned}$$

hence

$$\mathbf{b}(\lambda) = \begin{pmatrix} \alpha^2\lambda U'(0) - i\lambda\theta_1(\lambda) \\ 0 \\ -\lambda U'(0) \\ 0 \\ -i\lambda^2/\alpha \\ 0 \end{pmatrix}.$$

Therefore, the proposed algorithm is to fix values for $R$, $\alpha$, the wall parameters and $\lambda \in \Lambda$ and integrate (7.8) from $y = L_\infty$ to $y = 0$ using an implicit GL-RK method, with starting vector $\xi^+(\lambda)$ in (7.12). The results presented here are computed using $L_\infty = 10.0$ and the fourth-order implicit GL-RK method. A value $\lambda \in \Lambda$ is an eigenvalue if $D(\lambda) = 0$ with $D(\lambda)$ defined in (7.13). Roots of $D(\lambda)$ are then refined using Newton's method as discussed in Sect. 6.

### *7.2 Computing neutral curves*

Of interest in applications are curves of *neutral stability* which correspond to curves in the $\alpha - R$ plane where $\mathrm{Re}(\lambda) = 0$. Inside the curve corresponds to instability.

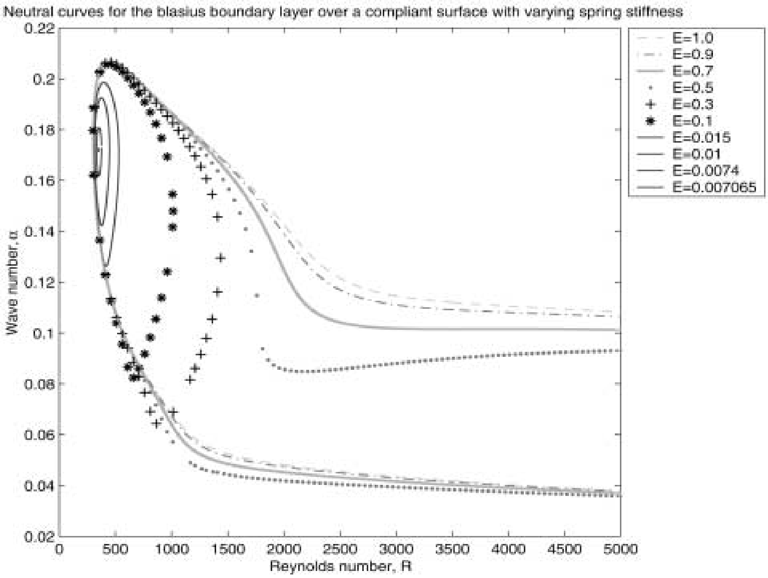


**Fig. 1.** Effect of $E$ on the neutral curves, plotted in the $\alpha - R$ plane: values inside a curve correspond to instability $\mathrm{Re}(\lambda) > 0$

In Fig. 1, the computed effect of wall rigidity on stability is shown. When $E$ is very large, the neutral curve for the Blasius boundary layer is recovered (cf. Drazin and Reid [17], Sect. 31.5). Figure 2 shows a blowup of the region near the nose of the neutral curve as $E$ approaches $E_c$. The point $E_c$, which we compute to be $E_c = 0.007065$, is the point where the neutral curve collapses to a point. The point $E_c$ is important in applications because for $E < E_c$ the flow is extraordinarily stable: the transition Reynolds number has been increased dramatically. This effect suggests that compliant surfaces could reduce drag by delaying transition to turbulence. The results in Fig. 1 show excellent qualitative agreement with Fig. 11 of [9].

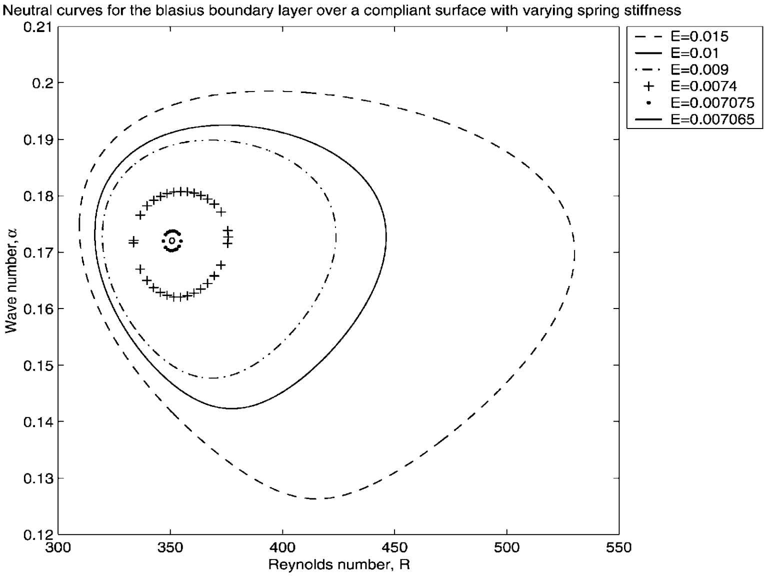


**Fig. 2.** Blowup of the nose of the neutral curve in Fig. 1 near the critical value of $E$

Further results on this model are reported in [2], including a range of new results on the effect of wall damping and wall tension.

## 8 Example 2. Eigenvalue problem associated with the Hocking-Stewartson pulse

A particularly difficult eigenvalue problem is that associated with the linearization about a solitary wave or pulse. In this section we give an example of how the framework of this paper give a new robust algorithm for studying the stability of solitary waves.

The complex Ginzburg-Landau (cGL) equation can be written in the scaled form

$$\rho\, e^{i\psi}\, A_t = A_{xx} - (1+i\omega)^2\, A + (1+i\omega)(2+i\omega)\, |A|^2 A\,, \tag{8.1}$$

where $A(x,t)$ is complex valued and $\rho > 0$, $\omega$, $\psi$ are specified real parameters. There is an exact solution – the Hocking-Stewartson (HS) pulse –

which can be explicitly determined,

$$A(x,t) \stackrel{\text{def}}{=} \widehat{A}(x) = (\cosh x)^{-1-i\omega}\,. \tag{8.2}$$

The spectral problem is obtained by linearizing the real form of (8.1) about the HS pulse (8.2), and looking for solutions proportional to $e^{\lambda t}$. Then the problem, with $\lambda \in \mathbb{C}$ as the spectral parameter, is formulated in an equivalent way as a system of the form

$$\mathbf{v}_x = \mathbf{A}(x,\lambda)\,\mathbf{v}\,,\quad x \in \mathbb{R}\,,\quad \mathbf{v} \in \mathbb{C}^4\,, \tag{8.3}$$

with $\mathrm{Trace}(\mathbf{A}(x,\lambda)) = 0$.

A discrete eigenvalue of this problem is a value of $\lambda \in \mathbb{C}$ for which (8.3) has a solution which decays exponentially as $x \to \pm\infty$. For example, $\lambda = 0$ is always an eigenvalue; in fact, of multiplicity at least two, due to the rotation and translation symmetries of (8.2).

The system is now in standard form to apply the theory of Sect. 2 and Sect. 5.1. Let

$$\mathbf{A}_\infty(\lambda) = \lim_{x\to\pm\infty} \mathbf{A}(x,\lambda)\,,$$

then it is straightforward to show that for parameter values of interest, and with $\mathrm{Re}(\lambda) > 0$, the matrix $\mathbf{A}_\infty(\lambda)$ has exactly two eigenvalues with positive real part and two with negative. Therefore, it is natural to study the ODE on the space $\bigwedge^2(\mathbb{C}^4)$.

Let $\mathbf{U}^+(x,\lambda) \in \bigwedge^2(\mathbb{C}^4)$ represent the two-dimensional space of solutions which are bounded as $x \to +\infty$, and let $\mathbf{U}^-(x,\lambda) \in \bigwedge^2(\mathbb{C}^4)$ represent the two-dimensional space of solutions which are bounded as $x \to -\infty$. Then, since the trace of $\mathbf{A}(x,\lambda)$ is zero, the complex function whose zeros correspond to eigenvalues is

$$D(\lambda) = \langle \mathbf{U}^+(x,\lambda), \varSigma\,\mathbf{U}^-(x,\lambda)\rangle_{\mathbb{R}}\,. \tag{8.4}$$

where $\varSigma$ is as defined in (5.11), and no conjugation is used in the inner product (a complex inner product could be used, and then conjugation would be applied *before* putting $\mathbf{U}^\pm$ into the inner product).

The numerical computation proceeds as follows. The infinite $x-$domain is truncated to $-L_\infty < x < L_\infty$ with $L_\infty$ suitably chosen. Then, the main part of the algorithm involves numerical integration of the following two systems with $\lambda$ fixed

$$\frac{d}{dx}\mathbf{U}^+ = \mathbf{A}^{(2)}(x,\lambda)\,\mathbf{U}^+\,,\ \mathbf{U}^+(x,\lambda)\big|_{x=L_\infty} = \xi^+(\lambda)\,, \quad \text{for}\quad L_\infty > x > 0\,, \tag{8.5}$$

and

$$\frac{d}{dx}\mathbf{U}^- = \mathbf{A}^{(2)}(x,\lambda)\,\mathbf{U}^-\,,\ \mathbf{U}^-(x,\lambda)\big|_{x=-L_\infty} = \xi^-(\lambda)\,, \quad \text{for} \quad -L_\infty < x < 0\,. \tag{8.6}$$

These two systems are integrated using the second-order implicit Gauss-Legendre Runge-Kutta (GL-RK) method (higher-order GL-RK methods could also be easily used but didn't appear to be necessary).

Numerical results using this algorithm showed for the first time that the Hocking-Stewartson pulse is unstable for the range of parameter values associated with cGL as a model for plane Poiseuille flow. Details of the numerical results are reported in [1], including comparison with Beyn and Lorenz [3] who studied the stability problem using a matrix discretization with approximate boundary conditions.

Further work on the use of numerical exterior algebra to compute the stability exponents associated with the linearization about solitary waves is in progress. For example, Bridges et al. [7] use this framework to study numerically the stability of solitary waves of the 5th-order KdV, which leads to a problem with $n = 5$ and $k = 2$.

## 9 Example 3. Instability of the Bickley jet

The stability of flows in unbounded domains, such as jets, wakes and mixing layers, is often studied using the Orr-Sommerfeld equation (cf. Drazin and Reid [17], Sect. 31, Herron [24]). In this section, the algorithm for infinite domains developed in Sect. 5.1 is illustrated by application to the Bickley jet. Mathematically, the stability problem for the Bickley jet is identical to the stability problem for a solitary wave such as the HS pulse studied in Sect. 8

In scaled variables, the horizontal velocity field for the Bickley jet takes the form

$$U(x) = \text{sech}^2 x\,, \quad -\infty < x < \infty\,. \tag{9.1}$$

The standard basis for $\mathbb{C}^4$ and $\bigwedge^2(\mathbb{C}^4)$ are chosen so that the Orr-Sommerfeld equation on $\bigwedge^2(\mathbb{C}^4)$ takes the form,

$$\mathbf{U}_x = \mathbf{A}^{(2)}(x,\lambda)\,\mathbf{U}\,, \tag{9.2}$$

with $\mathbf{A}^{(2)}(x,\lambda)$ given by (7.9).

For the jet, $\mathbf{A}_\infty(\lambda)$ is the same for both $\pm\infty$. When $\alpha \neq 0$ and $R > 0$, there are exactly two eigenvalues of $\mathbf{A}_\infty(\lambda)$ with positive real part for all

$\lambda \in \Lambda$, when $\Lambda = \{\, \mathbb{C} \,:\, \mathrm{Re}(\lambda) > 0\,\}$. The eigenvalues of $\mathbf{A}_\infty^{(2)}(\lambda)$ with largest positive and negative real part respectively are

$$\sigma^\pm(\lambda) = \mp(\alpha+\beta)\,, \tag{9.3}$$

where $\alpha > 0$ is the wavenumber, and $\beta$ is the root of $\beta^2 = \alpha^2 + \lambda R$, $\lambda \in \Lambda$, with positive real part. The connection between $\lambda$ and the wave speed $c$ is $\lambda = -i\alpha c$. The eigenvectors of $\mathbf{A}_\infty^{(2)}(\lambda)$ associated with $\sigma^\pm(\lambda)$ are

$$\xi^\pm(\lambda) = \begin{pmatrix} 1 \\ \sigma^\pm(\lambda) \\ \alpha^2+\alpha\beta+\beta^2 \\ \alpha\beta \\ \alpha\beta\,\sigma^\pm(\lambda) \\ \alpha^2\beta^2 \end{pmatrix} . \tag{9.4}$$

The system (9.2) is integrated from $x = L_\infty$ to $x = 0$ with starting vector $\xi^+(\lambda)$, and $L_\infty$ is taken to be $L_\infty = 10$ in the results reported here. Call this solution $\mathbf{U}^+(x,\lambda)$. The system (9.2) is then integrated from $x = -L_\infty$ to $x = 0$ with starting vector $\xi^-(\lambda)$. Call this solution $\mathbf{U}^-(x,\lambda)$.

A value of $\lambda \in \mathbb{C}$ is an eigenvalue if $\mathbf{U}^+(x,\lambda) \wedge \mathbf{U}^-(x,\lambda) = 0$ for all $x \in \mathbb{R}$. Using the theory in Sect. 5.1 this condition is satisfied if $D(\lambda) = 0$ where

$$D(\lambda) = \langle \mathbf{U}^+, \Sigma\,\mathbf{U}^- \rangle_{\mathbb{R}}\,, \tag{9.5}$$

where $\mathbf{\Sigma}$ is defined in (5.11).

Numerical calculation of the neutral curve for the Bickley jet using the above algorithm is shown in Fig. 9, and the curve agrees to graphical accuracy with the neutral curve due to Silcock and reported in Fig. 4.26 by Drazin and Reid [17]. Newton's method, as described in Sect. 8, and continuation were used to compute the points on the neutral curve.

The calculations were done using the implicit midpoint rule, which is only second-order accurate, but is clearly adequate for graphical accuracy.

## 10 Example 4. ODEs on $\bigwedge^6$ and the rotating Ekman layer

The stability problem for the Ekman boundary layer – which appears in atmospheric dynamics and oceanography – can be reduced to a sixth order complex ODE of the following form, which is a generalization of the Orr-Sommerfeld equation,

$$\phi'''' - b(x)\phi'' - a(x)\phi + 2\psi' = 0\,, \quad 0 \le x < +\infty \tag{10.1}$$

$$\psi'' + (\gamma^2 - b(x))\psi - i\gamma R\tilde{U}'\phi - 2\phi' = 0\,, \quad 0 \le x < +\infty\,. \tag{10.2}$$

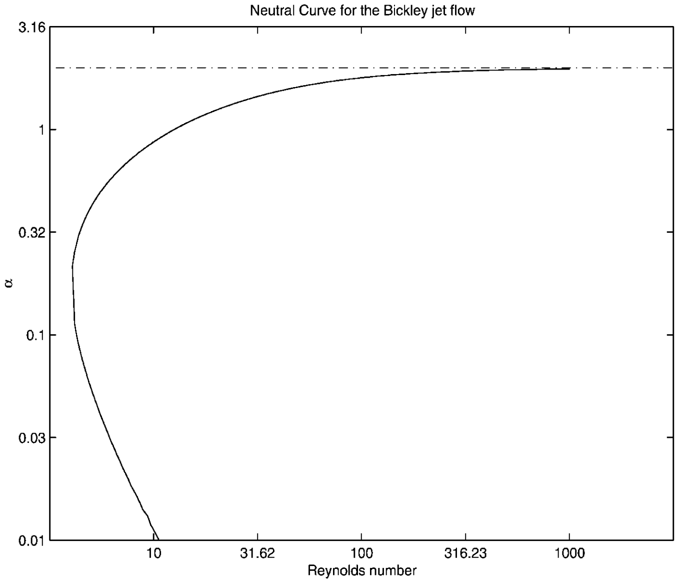


**Fig. 3.** Computed neutral curve for the Bickley jet

where

$$a(x) = -\gamma^4 - i\gamma^3 R(\tilde{V}(x) - c) - i\gamma R\tilde{V}_{xx}$$
$$b(x) = 2\gamma^2 + i\gamma R(\tilde{V}(x) - c)\,.$$

In this system, $\gamma$ is the modulus of the wavenumber, $R$ the Reynolds number, and $c = i\lambda/\gamma$ with $\lambda$ the eigenvalue. The functions $\tilde{U}$ and $\tilde{V}$ are the components of the the Ekman velocity field, and explicit expression in terms of elementary functions can be given but are not needed here (see [2]). The coordinate $x$ represents the vertical direction in a physical problem. The first equation (10.1) reduces to the Orr-Sommerfeld equation when $\phi' = 0$.

The boundary conditions associated with a rigid surface at $x = 0$ are

(10.3) $$\phi(0) = \phi'(0) = \psi(0) = 0\,,$$

It is straightforward to now transform this system into the standard form of Sect. 2. The system of ODEs can be expressed as a linear system of the form

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\mathbf{u}\,, \quad \mathbf{u} \in \mathbb{C}^6\,,$$

with three boundary conditions

$$\langle \mathbf{e}_1, \mathbf{u}(0,\lambda)\rangle = \langle \mathbf{e}_2, \mathbf{u}(0,\lambda)\rangle = \langle \mathbf{e}_5, \mathbf{u}(0,\lambda)\rangle = 0\,,$$

where $\mathbf{e}_j$ is the standard unit vector in $\mathbb{C}^6$.

The natural space to integrate this system is $\bigwedge^3(\mathbb{C}^6)$ which has dimension 20. We proceed by introducing a standard lexically-ordered basis for $\bigwedge^3(\mathbb{C}^6)$. We can then construct the induced ODE

$$\mathbf{U}_x^+ = \mathbf{A}^{(3)}(x,\lambda)\mathbf{U}^+ \,, \quad \mathbf{U}^+ \in \bigwedge^3(\mathbb{C}^6)\,. \tag{10.4}$$

The induced boundary condition at $x = 0$ is that the component – called $D(\lambda)$ – of $\mathbf{U}^+$ in the direction $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_5$ should be zero.

The most difficult part of this example is constructing the starting values at $y = L_\infty$. Let

$$\mathbf{A}_\infty(\lambda) = \lim_{x\to\infty} \mathbf{A}(x,\lambda)\,.$$

The characteristic polynomial for $\mathbf{A}_\infty(\lambda)$ takes the form

$$\det[\mu\mathbf{I} - \mathbf{A}_\infty(\lambda)] = \mu^6 - f_1(\lambda)\mu^4 + f_2(\lambda)\mu^2 - f_3(\lambda)\,,$$

where $f_1$, $f_2$, and $f_3$ are analytic functions of $\lambda$ with explicit expressions. It is straightforward to prove that when $\mathrm{Re}(\lambda) > 0$ there are exactly three roots with negative real part and three with positive. However explicit expressions are difficult to work with, and therefore we used the numerical algorithm proposed in Sect. 6 to construct the starting vectors.

Preliminary numerical results show that the algorithm is robust and the results show impressive accuracy even with the second-order implicit midpoint method. Details of this problem, the numerical results, and extension to the case where the wall at $y = 0$ is compliant and so its dynamics are coupled to the fluid are given in [2].

## Appendix

### A Hodge duality and adjoint systems

In Ng and Reid [32], it is shown for the cases $n = 2k$ with $k = 2, 3$ that the solutions of the adjoint systems obtained from the compound matrix systems can be related without calculation to the solutions of the basic system. In this appendix, we give a new proof of this result and generalize it to arbitrary $n = 2k$. In fact we show that this result is due to Hodge duality: the mapping from the adjoint system is related to the Hodge star operator, and therefore a coordinate-free characterization can be given.

The basic question is the following. Given the induced system

$$\mathbf{U}_x = \mathbf{A}^{(k)}(x,\lambda)\,\mathbf{U}\,, \quad \mathbf{U} \in \bigwedge^k(\mathbb{C}^n)\,, \tag{A.1}$$

how are the solutions of the adjoint of (A.1),

$$\mathbf{U}^\dagger{}_x = -[\mathbf{A}^{(k)}(x,\lambda)]^*\,\mathbf{U}^\dagger\,, \tag{A.2}$$

related to solutions of (A.1). The superscript $*$ indicates complex conjugate transpose.

First we prove a more general result. Consider the system complementary to (A.1),

$$\mathbf{V}_x = \mathbf{A}^{(n-k)}(x,\lambda)\,\mathbf{V}\,, \quad \mathbf{V} \in \textstyle\bigwedge^{n-k}(\mathbb{C}^n)\,. \tag{A.3}$$

Suppose $\mathbf{U}$ and $\mathbf{V}$ are decomposable and complementary, and let $\Phi(x,\lambda)$ be an $n \times n$ matrix whose columns are the decomposable vectors which make up $\mathbf{U}$ and $\mathbf{V}$. Then $\Phi(x,\lambda)$ is a fundamental matrix solution of (1.1) and

$$\mathbf{U} \wedge \mathbf{V} = \det[\Phi(x,\lambda)]\,\mathcal{V}\,,$$

where $\mathcal{V}$ is a suitably chosen volume form (without loss of generality, assume the standard one), and so

$$\begin{aligned} &\frac{d}{dx}\mathbf{U} \wedge \mathbf{V} = \tau(x,\lambda)\,\mathbf{U} \wedge \mathbf{V} \\ &\text{since} \quad \frac{d}{dx}\det[\Phi(x,\lambda)] = \tau(x,\lambda)\det[\Phi(x,\lambda)]\,, \end{aligned} \tag{A.4}$$

by the Abel-Liouville Theorem for linear systems.

Now differentiate the identity (5.6)

$$\frac{d}{dx}\mathbf{U} \wedge \mathbf{V} = \langle \overline{\mathbf{V}}_x, \mathbf{\Sigma}\mathbf{U}\rangle_{\mathbb{C}}\,\mathcal{V} + \langle \overline{\mathbf{V}}, \mathbf{\Sigma}\mathbf{U}_x\rangle_{\mathbb{C}}\,\mathcal{V}\,. \tag{A.5}$$

The left-hand side can be transformed using (A.4) and (5.6).

$$\tau\,\langle \overline{\mathbf{V}}, \mathbf{\Sigma}\mathbf{U}\rangle_{\mathbb{C}}\,\mathcal{V} = \langle \overline{\mathbf{V}}_x, \mathbf{\Sigma}\mathbf{U}\rangle_{\mathbb{C}}\,\mathcal{V} + \langle \overline{\mathbf{V}}, \mathbf{\Sigma}\mathbf{U}_x\rangle_{\mathbb{C}}\,\mathcal{V}\,. \tag{A.6}$$

Now substitute for $\mathbf{U}_x$ and $\mathbf{V}_x$,

$$\tau\,\langle \overline{\mathbf{V}}, \mathbf{\Sigma}\mathbf{U}\rangle_{\mathbb{C}} = \langle \overline{\mathbf{A}^{(n-k)}\mathbf{V}}, \mathbf{\Sigma}\mathbf{U}\rangle_{\mathbb{C}} + \langle \overline{\mathbf{V}}, \mathbf{\Sigma}\mathbf{A}^{(k)}\mathbf{U}\rangle_{\mathbb{C}}\,. \tag{A.7}$$

Since this identity holds for all $\mathbf{U} \in \bigwedge^k(\mathbb{C}^n)$ and $\mathbf{V} \in \bigwedge^{n-k}(\mathbb{C}^n)$, it follows that

$$\mathbf{\Sigma}\,\mathbf{A}^{(k)} + [\mathbf{A}^{(n-k)}]^T\,\mathbf{\Sigma} = \tau\,\mathbf{\Sigma}\,. \tag{A.8}$$

Now consider the special case $n = 2k$ and take the same basis for $\bigwedge^k(\mathbb{C}^n)$ and $\bigwedge^{n-k}(\mathbb{C}^n)$. In this case $k = n-k$, and

$$\mathbf{\Sigma}\,\mathbf{A}^{(k)} + [\mathbf{A}^{(k)}]^T\,\mathbf{\Sigma} = \tau\,\mathbf{\Sigma}\,. \tag{A.9}$$

Now, define

$$\mathbf{U}^\dagger = e^{-\int_0^x \overline{\tau(x,\lambda)}\,ds}\mathbf{\Sigma}^T\,\overline{\mathbf{U}}\,, \quad \text{with} \quad \mathbf{U} \in \textstyle\bigwedge^k(\mathbb{C}^n)\,, \tag{A.10}$$

with $\mathbf{U}$ satisfying (A.1). Then

$$\begin{aligned}\frac{d}{dx}\mathbf{U}^\dagger &= e^{-\int_0^x \overline{\tau(s,\lambda)}\,ds}(-\overline{\tau(x,\lambda)}\boldsymbol{\Sigma}^T\,\overline{\mathbf{U}} + \boldsymbol{\Sigma}^T\,\overline{\mathbf{U}}_x)\\ &= e^{-\int_0^x \overline{\tau(s,\lambda)}\,ds}(-\overline{\tau(x,\lambda)}\boldsymbol{\Sigma}^T + \boldsymbol{\Sigma}^T\,\overline{\mathbf{A}^{(k)}})\overline{\mathbf{U}}\\ &= e^{-\int_0^x \overline{\tau(s,\lambda)}\,ds}(-[\mathbf{A}^{(k)}]^*\,\boldsymbol{\Sigma}^T\,\overline{\mathbf{U}}) \quad \text{using (A.9)}\\ &= -[\mathbf{A}^{(k)}]^*\,\mathbf{U}^\dagger\,,\end{aligned}$$

showing that $\mathbf{U}^\dagger$ in (A.10) is indeed the adjoint function, and is given explicitly in terms of $\mathbf{U}$ using the Hodge star isomorphism $\boldsymbol{\Sigma}$, and a scalar multiplier when the trace of $\mathbf{A}(x,\lambda)$ is nonzero. Another way to write (A.10) is

$$\mathbf{U}^\dagger = \star(e^{-\int_0^x \tau(x,\lambda)\,ds}\,\mathbf{U}) = e^{-\int_0^x \overline{\tau(x,\lambda)}\,ds}\star\mathbf{U}\,. \tag{A.11}$$

The result (A.10) is the generalization of equation (3.8) and (3.17) in [32], and for the cases $k = 2$ and $k = 3$ it agrees with [32], when the standard basis is chosen: compare the definition of $\mathbf{T}$ in equation (3.9) of [32] with $\boldsymbol{\Sigma}$ in (5.11) in Appendix A.

*Acknowledgements.* The authors thank Peter Carpenter (Warwick), Tony Davey (Newcastle) and Sebastian Reich (Imperial) for helpful discussions and suggestions.

## References

1. A.L. Afendikov, T.J. Bridges, Instability of the Hocking-Stewartson pulse and its implications for three-dimensional Poiseuille flow. Proc. Roy. Soc. Lond. A **457** (2001), 257–272
2. L. Allen, T.J. Bridges, Two-dimensional and three-dimensional boundary layers interacting with a compliant surface. in preparation
3. W.-J. Beyn, J. Lorenz, Stability of traveling waves: dichotomies and eigenvalue conditions on finite intervals. Numer. Func. Anal., Optim. **20** (1999), 201–244
4. J.P. Boyd, Chebyshev and Fourier Spectral Methods. Springer (1989)
5. T.J. Bridges, The Orr-Sommerfeld equation on a manifold. Proc. Roy. Soc. Lond. A **455** (1999), 3019–3040
6. T.J. Bridges, G. Derks, Hodge duality and the Evans function. Phys. Lett. A **251** (1999), 363–372
7. T.J. Bridges et al., Linear stability of solitary wave states of 5th-order KdV. Preprint, University of Surrey (2001)
8. P.W. Carpenter, Status of transition delay using compliant walls. In: Viscous Drag Reduction in Boundary Layers (eds. D.M. Bushnell, J.N. Hefner), AIAA (1990), 79–113
9. P.W. Carpenter, A.D. Garrad, The hydrodynamic stability of flow over Kramer-type compliant surfaces. Part 1. Tollmien-Schlichting instabilites. J. Fluid Mech. **155** (1985), 465–510
10. S.S. Chern et al., Lectures on differential geometry. World Scientific: Singapore (1999)

11. G.J. Cooper, Stability of Runge-Kutta methods for trajectory problems. IMA J. Num. Anal. **7** (1987), 1–13
12. W.A. Coppel, Stability and asymptotic behavour of differential equations. D.C. Heath and Company, (1965)
13. A. Davey, On the removal of the singularities from the Riccati method. J. Comp. Phys. **30** (1979), 137–144
14. A. Davey, A difficult numerical calculation concerning the stability of the Blasius boundary layer. in Stability in the mechanics of continua, Springer-Verlag, (ed. F.H. Schröder, 1982), pp. 365–372
15. A. Davey, An automatic orthonormalization method for solving stiff boundary-value problems. J. Comp. Phys. **51** (1983), 343–356
16. A. Davey, How to use projective geometry to transform a boundary-value problem into an initial value problem. Unpublished notes, University of Newcastle
17. P.G. Drazin, W.H. Reid, Hydrodynamic Stability. Cambridge University Press (1981)
18. P. Griffiths, J. Harris, Principles of Algebraic Geometry. Wiley: New York (1978)
19. C.E. Grosch, S.A. Orszag, Numerical solution of problems in unbounded regions: coordinate transformations. J. Comp. Phys. **25** (1977), 273–296
20. E. Hairer et al., Solving Ordinary Differential Equations I. Nonstiff Problems. Springer Series in Comp. Maths **8** (1987)
21. E. Hairer, G. Wanner, Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems. Springer Series in Comp. Maths **14** (1991)
22. J. Harris, Algebraic Geometry. Springer (1992)
23. I.H. Herron, Expansion problems in the linear stability of boundary layer flows. Adv. Appl. Math. **4** (1983), 260–297
24. I.H. Herron, The Orr-Sommerfeld equation on infinite intervals. SIAM Rev. **29** (1987), 597–620
25. L.M. Hocking, Non-linear instability of the asymptotic suction velocity profile. Quart. J. Mech. Appl. Math. **28** (1975), 341–353
26. T. Kato, Perturbation Theory for Linear Operators. 2nd Edition, Springer (1984)
27. M.O. Kramer, Hydrodynamics of the dolphin. Adv. in Hydrosci. **2** (1965), 111–130
28. B. Leimkuhler, S. Reich, Symplectic methods for constrained Hamiltonian systems. Math. Comp. **63** (1994), 589-605
29. M. Marcus, Finite Dimensional Multilinear Algebra, Part II. Marcel Dekker: New York (1975)
30. B.S. Ng, W.H. Reid, An initial-value method for eigenvalue problems using compound matrices. J. Comp. Phys. **30** (1979), 125–136
31. B.S. Ng, W.H. Reid, On the numerical solution of the Orr-Sommerfeld problem: asymptotic initial conditions for shooting methods. J. Comp. Phys. **38** (1980), 275–293
32. B.S. Ng, W.H. Reid, The compound matrix method for ordinary differential systems. J. Comp. Phys. **58** (1985), 209–228
33. R. Nicodemus et al., Variational bound on energy dissipation in plane Couette flow. Eur. Phys. J. B **10** (1999), 385–396
34. B. Straughn, D.W. Walker, Two very accurate and efficient methods for computing eigenvalues and eigenfunctions in porous convection problems. J. Comp. Phys. **127** (1996), 128–141
35. R.O. Wells, Differential Analysis on Complex Manifolds. Springer, New York (1979)