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Uniform boundary controllability of a semi-discrete 1-D wave equation

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Summary. A numerical scheme for the controlled semi-discrete 1-D wave equation is considered. We analyze the convergence of the boundary controls of the semi-discrete equations to a control of the continuous wave equation when the mesh size tends to zero. We prove that, if the high modes of the discrete initial data have been filtered out, there exists a sequence of uniformly bounded controls and any weak limit of this sequence is a control for the continuous problem. The number of the eliminated frequencies depends on the mesh size and the regularity of the continuous initial data. The case of the HUM controls is also discussed.

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1 Introduction

The start point of our study is the boundary controllability of the 1-D wave equation: given $T > 2$ and $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ there exists a control function $v \in L^2(0,T)$ such that the solution of the equation

$$(1)\qquad \begin{cases} u'' - u_{xx} = 0 & \text{for } x \in (0,1), \quad t > 0 \\ u(t,0) = 0 & \text{for } t > 0 \\ u(t,1) = v(t) & \text{for } t > 0 \\ u(0,x) = u^0(x) & \text{for } x \in (0,1) \\ u'(0,x) = u^1(x) & \text{for } x \in (0,1) \end{cases}$$

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satisfies

$$u(T,\cdot) = u'(T,\cdot) = 0. \tag{2}$$

By $'$ we denote the time derivative.

This problem has been studied and solved some decades ago and several approaches are now known. The moments theory is one of the oldest and most successful (see, for instance, [1] and [11]). More recent, Hilbert uniqueness method (HUM) offered a different and a very general way to solve this and multi-dimensional similar problems (see, for instance, [10]).

In the last years many works have dealt with the numerical approximations for the control problem (1)-(2). For instance, in [4], [6] and [5], by using HUM, some numerical algorithms have been proposed. In these articles a bad numerical behaviour of the approximate controls has been observed. This phenomenon is due to the high frequency components of the discrete solution and a biharmonic Tychonoff regularization procedure has been given in order to avoid it.

This paper studies a finite-difference space discretization of equation (1). As we shall see, the main problem of the numerical algorithms we have just mentioned (bad behaviour of the discrete high modes) is still a characteristic of this case. Our analysis will be based on the filtering of the high frequencies of the initial data.

Let us consider first $N \in \mathbb{N}^*$, a step $h = \frac{1}{N+1}$ and an equidistant division of the interval $(0,1)$, $0 = x_0 < x_1 < ... < x_N < x_{N+1} = 1$, with $x_j = jh$, $0 \le j \le N+1$.

We introduce the following finite-difference semi-discretization of (1):

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} = 0 & \text{for } 1 \le j \le N, \;\; t > 0 \\ u_0(t) = 0 & \text{for } t > 0 \\ u_{N+1}(t) = v_h(t) & \text{for } t > 0 \\ u_j(0) = u_j^0, \;\; u_j' = u_j^1 & \text{for } 1 \le j \le N \end{cases} \tag{3}$$

and we study the following controllability problem: given $T > 0$ and $(u_j^0, u_j^1)_{1 \le j \le N} \in \mathbb{C}^{2N}$, there exists a control function $v_h \in L^2(0,T)$ such that the solution u of (3) satisfies

$$u_j(T) = u_j'(T) = 0, \;\; \forall j = 1, 2, ..., N. \tag{4}$$

System (3) consists of N linear differential equations with N unknowns $u_1, u_2, ..., u_N$. $u_j(t)$ is an approximation of the solution u of (1) in (t, x_j), provided that $(u_j^0, u_j^1)_{1 \le j \le N}$ approximates the initial datum (u^0, u^1).

It is not difficult to see that the controllability problem we have just addressed has a positive answer. Moreover, as we shall see later on, explicit discrete controls v_h can be provided. Our interest is to study when is the sequence $(v_h)_{h>0}$ a good approximation of a control of the continuous problem (1). The first question we address is the boundedness of the sequence of the controls.

It is by now well known that, generally, the sequence $(v_h)_{h>0}$ is not bounded in $L^2(0,T)$. In order to explain the causes of this phenomenon we introduce the following Fourier decomposition of the initial datum $U^0(h) = (u_j^0, u_j^1)_{1\leq j\leq N}$ of (3):

$$U^0(h) = \sum_{\substack{|n|\leq N \\ n\neq 0}} a_n^0(h)\Phi^n(h) \tag{5}$$

where $(\Phi^n(h))_{\substack{|n|\leq N \\ n\neq 0}}$ is the family of $2N$ orthonormal eigenvectors of the matrix of the system (3). Let also $(i\lambda_n)_{1\leq|n|\leq N}$ be the family of the eigenvalues of (3). Full details will be given in Sect. 3.

As we shall prove later on in the paper, to control the high eigenmodes of $U^0(h)$, a control with an exponentially increasing L^2-norm is needed. For instance, if $U^0(h) = \Phi^N(h)$, any control v_h satisfies

$$||v_h||_{L^2(0,T)} \geq C\exp(\sqrt{N}) \tag{6}$$

where C is a constant not depending on N.

Hence, it seems impossible to find a sequence of uniformly bounded controls $(v_h)_{h>0}$ for (3) if $U^0(h)$ contains high eigenmodes.

Moreover, we shall prove that in any Sobolev space there exist initial data (u^0, u^1) such that the following natural choice of $U^0(h)$

$$u_j^0 = u^0(jh),\ u_j^1 = u^1(jh), \quad 1\leq j\leq N \tag{7}$$

does not ensure the uniformly boundedness of the controls.

As we have said before, this phenomenon is due to the fact that the numerical schema introduces spurious high frequency vibrations that are not observed in the continuous problem. More precisely, as it was pointed out in [7] (see also [5]), the differences between the discrete and the continuous systems become significant for the modes of order of N.

The choice of an appropriate approximation $(u_j^0, u_j^1)_{1\leq j\leq N}$ for the initial datum (u^0, u^1) of (1) reveals to be crucial if one wants to ensure the existence of a bounded sequence of controls $(v_h)_{h>0}$.

Since the existence of high eigenmodes in the initial datum $U^0(h)$ has this unwanted effect on the discrete controls, it seems natural to look for discrete approximations of (u^0, u^1) in which the high frequencies have been filtered out. More precisely, we shall consider that $U^0(h)$ has been chosen of the following form

$$U^0(h) = \sum_{\substack{|n|\leq M \\ n\neq 0}} a_n^0(h)\Phi^n(h) \tag{8}$$

where M depends on the size of the space step $h = \frac{1}{N+1}$ and on the regularity of the initial datum of the continuous problem.

We shall prove that, by considering $U^0(h)$ like in (8) with $M \leq \sqrt{N}$, there exists a sequence of bounded controls $(v_h)_{h>0}$ for (3). Moreover, if (u^0, u^1) has a sufficient amount of analyticity, one can choose $M = N$ and no filtering is needed. In this case, the discretization (7) can be used and guarantees the existence of a bounded sequence of discrete controls. All these results are true for control times T sufficiently large but independent of N.

In [7] it was proved that uniform boundary observability can be obtained for the adjoint homogeneous system corresponding to (3), if the short wave length components of the solutions are eliminated. This result ensures the existence of a sequence of uniformly bounded controls which led to zero the projection of the solution of (3) over a space generated by low frequency eigenvectors. Our approach is different in the sense that we eliminate from the very beginning the short wave length components of the initial datum and we prove the existence of a sequence of uniformly bounded controls for the solutions of (3). Moreover, our analysis gives more information on the behaviour of the controls corresponding to the high frequencies.

The rest of the article is organized in the following way: In Sect. 2 we prove some estimates for the biorthogonal families to the set of complex exponentials $\left(\mathrm{e}^{\mathrm{i}\lambda_j t}\right)_{\substack{|j|\leq N\\ j\neq 0}}$. These estimates will offer bounds for the discrete controls. In Sect. 3 some general results for the semi-discrete system (3) are given. A moments problem is deduced and some inequalities for the adjoint equation are proved. In Sect. 4 the main results on the boundedness of the controls are given and some convergence results are also proved. Finally, in the last section, we discuss the existence of unbounded controls and we analyze the case of HUM controls.

2 Estimates for the norm of a biorthogonal family

Let us consider the sequence $(\lambda_j)_{\substack{|j|\leq N\\ j\neq 0}}$ where $\lambda_j = \frac{2}{h}\sin\left(\frac{j\pi h}{2}\right)$. As we shall see in the following section, $(i\lambda_j)_{\substack{|j|\leq N\\ j\neq 0}}$ are the eigenvalues of the semi-discrete problem (3).

In this section we construct an explicit biorthogonal sequence $(\Theta_m)_{\substack{|m|\leq N\\ m\neq 0}}$ to the family of complex exponentials $\left(\mathrm{e}^{\mathrm{i}\lambda_j t}\right)_{\substack{|j|\leq N\\ j\neq 0}}$ in $L^2(-T,T)$ and we estimate the norm of the elements of this biorthogonal sequence.

We recall that $(\Theta_m)_{\substack{|m|\leq N\\ m\neq 0}}$ is a biorthogonal sequence to $\left(\mathrm{e}^{\mathrm{i}\lambda_j t}\right)_{\substack{|j|\leq N\\ j\neq 0}}$ in $L^2(-T,T)$ if

(9) $$\int_{-T}^{\mathrm{T}} \Theta_m(t) \mathrm{e}^{\mathrm{i}\lambda_n t} dt = \delta_{mn}, \ \forall m, n = \pm 1, \pm 2, ..., \pm N$$

(see [1] and [12]).

Since $\left(\mathrm{e}^{\mathrm{i}\lambda_j t}\right)_{\substack{|j| \leq N \\ j \neq 0}}$ is a finite family of exponential functions, it follows immediately that there are infinitely many biorthogonal families. Nevertheless, since we are interested on the dependence of these biorthogonals on N, it is not easy to give precise estimates for the norm of the elements of them.

In the next Theorem we shall construct an explicit biorthogonal and we shall evaluate the norms of its elements.

Theorem 2.1 *If $T > 0$ is sufficiently large, there exists a sequence $(\Theta_m)_{\substack{|m| \leq N \\ m \neq 0}}$, biorthogonal in $L^2(-T, T)$ to the family of complex exponentials $\left(\mathrm{e}^{\mathrm{i}\lambda_j t}\right)_{\substack{|j| \leq N \\ j \neq 0}}$, such that*

(10) $$||\Theta_m||_{L^2(-T,T)} \leq C|\lambda_m| \exp\left(\alpha \frac{|\lambda_m|^2}{N}\right), \textit{ for } m = \pm 1, \pm 2, ..., \pm N$$

where C and α are two positive constants which do not depend on m and N.

Remark 1 As we shall see, Theorem 2.1 provides a biorthogonal set for any $T > 0$. However, for the estimates (10) we need a time T sufficiently large (but independent of the discretized problem). An estimate for T can be also obtained from the proof.

Remark 2 Let us remark that Theorem 2.1 implies that there exists a biorthogonal sequence $(\Theta_m)_{\substack{|m| \leq N \\ m \neq 0}}$, such that

(11) $$||\Theta_m||_{L^2(-T,T)} \leq C'|\lambda_m|, \text{ for } m = \pm 1, \pm 2, ..., \pm \sqrt{N}$$

where C' is a constant which does not depend on N.

Remark 3 From Theorem 2.1 it follows that

(12)
$$||\Theta_m||_{L^2(-T,T)} \leq C'|\lambda_m| \exp(\alpha'|\lambda_m|), \text{ for } m = \pm 1, \pm 2, ..., \pm N$$

where C' and α' are two positive constants which do not depend on N.

Proof. Let us first define, for each m such that $\mid m \mid \leq N$ and $m \neq 0$,

$$\xi_m(z) = \left(\prod_{\substack{|n|\leq N \\ n\neq 0,m}} \frac{z-\lambda_n}{\lambda_m - \lambda_n} \right) \left(\frac{\sin \frac{T(z-\lambda_m)}{4N}}{\frac{T(z-\lambda_m)}{4N}} \right)^{2N}$$

$$\times \left(\frac{\sin \frac{T(z-\lambda_m)}{4}}{\frac{T(z-\lambda_m)}{4}} \right)^{2} . \tag{13}$$

Each function ξ_m has the following properties:

- ξ_m is an entire function
- $\xi_m(\lambda_n) = \delta_{nm}, \forall |n| \leq N, n \neq 0$
- $\xi_m(x) \in L^2(-\infty, \infty)$
- ξ_m is of the exponential type at most T, i.e. there exists a constant $A_m > 0$ such that, for all $\varepsilon > 0$, we have

$$|\xi_m(z)| \leq A_m \mathrm{e}^{(T+\varepsilon)|z|}, \quad \forall z \in \mathbb{C}.$$

We introduce now the Fourier transform of ξ_m

$$\Theta_m(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi_m(x) \mathrm{e}^{-xzi} dx. \tag{14}$$

We shall show that $\{\Theta_m\}_{\substack{|m|\leq N \\ m\neq 0}}$ is the biorthogonal sequence we are looking for.

From the properties of ξ_m, by using Paley-Wiener Theorem, it follows that $\Theta_m(t)$ has compact support in $[-T, T]$, it belongs to $L^2(-T, T)$ and

$$\int_{-T}^{\mathrm{T}} \Theta_m(t) \mathrm{e}^{\mathrm{i}\lambda_n t} dt = \xi_m(\lambda_n) = \delta_{nm}, \quad \forall |n| \leq N, \; n \neq 0.$$

It follows that $(\Theta_m)_{\substack{|m|\leq N \\ m\neq 0}}$ is a biorthogonal sequence to $\left\{\mathrm{e}^{\mathrm{i}\lambda_n t}\right\}_{\substack{|n|\leq N \\ n\neq 0}}$.

Our next objective is to estimate the norm of Θ_m. From Plancherel's Theorem we have

$$\sqrt{2\pi} \parallel \Theta_m \parallel_{L^2(-T,T)} = \parallel \xi_m \parallel_{L^2(-\infty,\infty)} . \tag{15}$$

Hence, to estimate the norm of Θ_m, we have to study the norm of ξ_m in $L^2(-\infty, \infty)$. We have

$$\parallel \xi_m \parallel^2_{L^2(-\infty,\infty)}$$

$$= \int_{-\infty}^{\infty} \left| \left(\prod_{\substack{|n|\leq N \\ n\neq 0,m}} \frac{x-\lambda_n}{\lambda_m - \lambda_n} \right) \left(\frac{\sin \frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}} \right)^{2N} \left(\frac{\sin \frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}} \right)^{2} \right|^2 dx$$

$$= \left(\prod_{\substack{|n|\le N \\ n\ne 0,m}} \frac{1}{|\lambda_m - \lambda_n|^2} \right) \int_{-\infty}^{\infty}$$

$$\times \left| \left(\prod_{\substack{|n|\le N \\ n\ne 0,m}} (x-\lambda_n) \right) \left(\frac{\sin \frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}} \right)^{2N} \left(\frac{\sin \frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}} \right)^{2} \right|^2 dx$$

Let us first evaluate the constant

$$\gamma_1(N) = \prod_{\substack{|n|\le N \\ n\ne 0,m}} \frac{1}{|\lambda_m - \lambda_n|^2}. \tag{16}$$

Lemma 2.1 *The following estimates hold:*

(i) $\gamma_1(N) = 4\cos^4\left(\frac{m\pi h}{2}\right) \prod_{\substack{|n|\le N \\ n\ne 0,m}} \frac{1}{|\lambda_k|^2}$.

(ii) $\gamma_1(N) \le \frac{|\cos\left(\frac{m\pi h}{2}\right)\sin(m\pi h)|^2}{h^2 2^{4N-2}(N!)^4}$.

Proof. First of all remark that

$$|\lambda_n - \lambda_m| = \frac{4}{h}\left|\sin\left(\frac{n-m}{4}\pi h\right)\cos\left(\frac{n+m}{4}\pi h\right)\right|.$$

Let us now evaluate $\prod_{\substack{|n|\le N \\ n\ne 0,m}} |\lambda_m - \lambda_n|$.

For $1 \le m \le N$, we obtain

$$\prod_{\substack{|n|\le N \\ n\ne 0,m}} |\lambda_m - \lambda_n|$$

$$= \left(\frac{4}{h}\right)^{2N-1} \prod_{\substack{|n|\le N \\ n\ne 0,m}} \left|\sin\left(\frac{n-m}{4}\pi h\right)\right| \prod_{\substack{|n|\le N \\ n\ne 0,m}} \left|\cos\left(\frac{n+m}{4}\pi h\right)\right|.$$

But

$$\prod_{\substack{|n|\le N \\ n\ne 0,m}} \left|\sin\left(\frac{n-m}{4}\pi h\right)\right|$$

$$= \prod_{1\le n\le m-1} \left|\sin\left(\frac{n-m}{4}\pi h\right)\right| \prod_{m+1\le n\le N} \left|\sin\left(\frac{n-m}{4}\pi h\right)\right|$$

$$\prod_{-(N-m)\le n\le -1} \left|\sin\left(\frac{n-m}{4}\pi h\right)\right| \prod_{-N\le n\le -(N-m)-1} \left|\sin\left(\frac{n-m}{4}\pi h\right)\right|$$

$$= \prod_{1\le k\le m-1}\left|\sin\left(\frac{k\pi h}{4}\right)\right| \prod_{1\le k\le N-m}\left|\sin\left(\frac{k\pi h}{4}\right)\right|$$

$$\prod_{m+1\le k\le N}\left|\sin\left(\frac{k\pi h}{4}\right)\right| \prod_{-N-1+m\le k\le -N}\left|\sin\left(\frac{(m-k)\pi h}{4}\right)\right|$$

$$= \prod_{\substack{1\le k\le N\\ k\neq m}}\left|\sin\left(\frac{k\pi h}{4}\right)\right| \prod_{1\le k\le N-m}\left|\sin\left(\frac{k\pi h}{4}\right)\right|$$

$$\times \prod_{N+1-m\le k\le N}\left|\sin\left(\frac{(k+m)\pi h}{4}\right)\right|.$$

On the other hand

$$\prod_{\substack{|n|\le N\\ n\neq 0,m}}\left|\cos\left(\frac{n+m}{4}\pi h\right)\right|$$

$$= \prod_{-m\le n\le -1}\left|\cos\left(\frac{n+m}{4}\pi h\right)\right| \prod_{-N\le n\le -m-1}\left|\cos\left(\frac{n+m}{4}\pi h\right)\right|$$

$$\prod_{1\le n\le N-m}\left|\cos\left(\frac{n+m}{4}\pi h\right)\right| \prod_{N-m+1\le n\le N}\left|\cos\left(\frac{n+m}{4}\pi h\right)\right|$$

$$= \frac{1}{\left|\cos\left(\frac{m\pi h}{2}\right)\right|} \prod_{0\le k\le m-1}\left|\cos\left(\frac{k\pi h}{4}\right)\right| \prod_{1\le k\le N-m}\left|\cos\left(\frac{k\pi h}{4}\right)\right|$$

$$\prod_{m+1\le k\le N}\left|\cos\left(\frac{k\pi h}{4}\right)\right| \prod_{N+1-m\le k\le N}\left|\cos\left(\frac{(m+k)\pi h}{4}\right)\right|$$

$$= \frac{1}{\left|\cos\left(\frac{m\pi h}{2}\right)\right|} \prod_{\substack{0\le k\le N\\ k\neq m}}\left|\cos\left(\frac{k\pi h}{4}\right)\right| \prod_{1\le k\le N-m}\left|\cos\left(\frac{k\pi h}{4}\right)\right|$$

$$\times \prod_{N+1-m\le k\le N}\left|\cos\left(\frac{(k+m)\pi h}{4}\right)\right|.$$

Hence, we obtain

$$\prod_{\substack{|n|\le N\\ n\neq 0,m}} \mid \lambda_m - \lambda_n \mid = \frac{2^{2N-1}}{h^{2N-1}\left|\cos\left(\frac{m\pi h}{2}\right)\right|} \prod_{\substack{1\le k\le N\\ k\neq m}}\left|\sin\left(\frac{k\pi h}{2}\right)\right|$$

$$\prod_{1\le k\le N-m}\left|\sin\left(\frac{k\pi h}{2}\right)\right| \prod_{N+1-m\le k\le N}\left|\sin\left(\frac{(k+m)\pi h}{2}\right)\right|.$$

Let us now remark that

$$\prod_{N+1-m\leq k\leq N}\left|\sin\left(\frac{(k+m)\pi h}{2}\right)\right| = \prod_{k=N+1}^{N+m}\left|\sin\left(\frac{k\pi h}{2}\right)\right|$$
$$= \frac{1}{\left|\cos\frac{m\pi h}{2}\right|}\prod_{1\leq k\leq m}\left|\cos\left(\frac{k\pi h}{2}\right)\right|$$
$$= \frac{1}{\left|\cos\frac{m\pi h}{2}\right|}\prod_{N+1-m\leq k\leq N}\left|\sin\left(\frac{k\pi h}{2}\right)\right|.$$

It follows that, for $1 \leq m \leq N$,

$$\prod_{\substack{|n|\leq N\\ n\neq 0,m}} |\lambda_m - \lambda_n| = \frac{2^{2N-1}}{h^{2N-1}|\cos\left(\frac{m\pi h}{2}\right)\sin(m\pi h)|} \times \left(\prod_{1\leq k\leq N}\left|\sin\left(\frac{k\pi h}{2}\right)\right|\right)^2.$$

On the other hand, if $-N \leq m \leq -1$, we have

$$\prod_{\substack{|n|\leq N\\ n\neq 0,m}} |\lambda_m - \lambda_n|$$
$$= \left(\frac{4}{h}\right)^{2N-1}\prod_{\substack{|n|\leq N\\ n\neq 0,m}}\left|\sin\left(\frac{n-m}{4}\pi h\right)\right|\prod_{\substack{|n|\leq N\\ n\neq 0,m}}\left|\cos\left(\frac{n+m}{4}\pi h\right)\right|$$
$$= \left(\frac{4}{h}\right)^{2N-1}\prod_{\substack{|n|\leq N\\ n\neq 0,m}}\left|\sin\left(\frac{-n+m}{4}\pi h\right)\right|\prod_{\substack{|n|\leq N\\ n\neq 0,m}}\left|\cos\left(\frac{-n-m}{4}\pi h\right)\right|$$
$$= \left(\frac{4}{h}\right)^{2N-1}\prod_{\substack{|n|\leq N\\ n\neq 0,-m}}\left|\sin\left(\frac{n-(-m)}{4}\pi h\right)\right|$$
$$\times \prod_{\substack{|n|\leq N\\ n\neq 0,-m}}\left|\cos\left(\frac{n+(-m)}{4}\pi h\right)\right|$$
$$= \prod_{\substack{|n|\leq N\\ n\neq 0,-m}} |\lambda_{-m} - \lambda_n| = \frac{2^{2N-1}}{h^{2N-1}|\cos\left(\frac{m\pi h}{2}\right)\sin(m\pi h)|}$$
$$\times \left(\prod_{1\leq k\leq N}\left|\sin\left(\frac{k\pi h}{2}\right)\right|\right)^2.$$

It follows that, for each $|m| \leq N$, $m \neq 0$, we have

$$\prod_{\substack{|n|\leq N\\ n\neq 0,m}} |\lambda_m - \lambda_n| = \frac{2^{2N-1}}{h^{2N-1}|\cos\left(\frac{m\pi h}{2}\right)\sin(m\pi h)|} \times \left(\prod_{1\leq k\leq N}\left|\sin\left(\frac{k\pi h}{2}\right)\right|\right)^2. \tag{17}$$

By taking into account that $\lambda_k = \frac{2}{h}\sin\left(\frac{k\pi h}{2}\right)$ (i) follows immediately from (17).

On the other hand, by taking into account that $\sin\left(\frac{k\pi h}{2}\right) \geq kh$, (ii) can be obtained directly from (i) and the proof of Lemma 2.1 finishes. □

Let us now evaluate the integral

$$\gamma_2(N) = \int_{-\infty}^{\infty}\left|\left(\prod_{\substack{|n|\leq N\\ n\neq 0,m}}(x-\lambda_n)\right) \times \left(\frac{\sin\frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}}\right)^{2N}\left(\frac{\sin\frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}}\right)^2\right|^2 dx. \tag{18}$$

If $0 < \delta < 1$ is a positive sub-unitary number we have that $\gamma_2(N) = I_1 + I_2$ where

$$I_1 = \int_{|x-\lambda_m|\leq\delta N\pi}\left|\prod_{\substack{|n|\leq N\\ n\neq 0,m}}(x-\lambda_n)\right|^2\left|\frac{\sin\frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}}\right|^{4N}\left|\frac{\sin\frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}}\right|^4 dx,$$

$$I_2 = \int_{|x-\lambda_m|\geq\delta N\pi}\left|\prod_{\substack{|n|\leq N\\ n\neq 0,m}}(x-\lambda_n)\right|^2\left|\frac{\sin\frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}}\right|^{4N}\left|\frac{\sin\frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}}\right|^4 dx.$$

We shall evaluate each of the two integrals. For the second integral we have

Lemma 2.2 *For $T > 0$ sufficiently large but independent of N there exists a positive constant $C_1 > 0$, which does not dependent on N, such that*

$$\gamma_1(N)I_2 \leq C_1. \tag{19}$$

Proof.

$$I_2 = \int_{|x-\lambda_m|\geq \delta N\pi} \left| \prod_{\substack{|n|\leq N \\ n\neq 0,m}} (x-\lambda_n) \right|^2 \left| \frac{\sin \frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}} \right|^{4N} \left| \frac{\sin \frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}} \right|^{4} dx$$

$$\leq \frac{(4N)^{4N}}{T^{4N}} \int_{|x-\lambda_m|\geq \delta N\pi} \prod_{\substack{|n|\leq N \\ n\neq 0,m}} \left| \frac{x-\lambda_n}{x-\lambda_m} \right|^2 \frac{1}{|x-\lambda_m|^2} dx.$$

But, for any x such that $|x-\lambda_m| \geq \delta N\pi$, we have

$$\left| \frac{x-\lambda_n}{x-\lambda_m} \right| \leq \frac{|x-\lambda_m| + |\lambda_m - \lambda_n|}{|x-\lambda_m|} \leq 1 + \frac{2N\pi}{|x-\lambda_m|} \leq 1 + \frac{2}{\delta}.$$

It follows that,

$$I_2 \leq \frac{(4N)^{4N}}{T^{4N}} \left(1+\frac{2}{\delta}\right)^{4N-2} \int_{|x-\lambda_m|\geq \delta N\pi} \frac{1}{|x-\lambda_m|^2} dx$$

$$\leq \frac{2(4N)^{4N}}{\delta N \pi T^{4N}} \left(1+\frac{2}{\delta}\right)^{4N-2}.$$

Moreover, from the second estimate of Lemma 2.1, by using Stirling's formula, it follows that

$$\gamma_1(N) I_2 \leq \exp[\beta N - 4N \ln(T)]$$

where β is a positive constant independent of N.

Hence, for $T>0$ sufficiently large (but independent of N), there exists a positive constant C_1, independent of N, such that $\gamma_1(N)I_2 \leq C_1$ and the proof of the Lemma finishes. □

The estimates for the first integral are more laborious. Let us first remark that

$$I_1 = \int_{|x-\lambda_m|\leq \delta N\pi} \left| \prod_{\substack{|n|\leq N \\ n\neq 0,m}} (x-\lambda_n) \right|^2 \left| \frac{\sin \frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}} \right|^{4N} \left| \frac{\sin \frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}} \right|^{4} dx$$

$$= \left(\prod_{\substack{|n|\leq N \\ n\neq 0,m}} |\lambda_n|^2 \right) \int_{|x-\lambda_m|\leq \delta N\pi} \left(\prod_{\substack{|n|\leq N \\ n\neq 0,m}} \left| \frac{x-\lambda_n}{\lambda_n} \right|^2 \right)$$

$$\times \left| \frac{\sin \frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}} \right|^{4N} \left| \frac{\sin \frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}} \right|^{4} dx.$$

We denote by I_3 the integral

$$I_3 = \int_{|x-\lambda_m|\leq \delta N\pi} \left(\prod_{\substack{|n|\leq N \\ n\neq 0,m}} \left| \frac{x-\lambda_n}{\lambda_n} \right|^2 \right) \times \left| \frac{\sin \frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}} \right|^{4N} \left| \frac{\sin \frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}} \right|^{4} dx.$$

The following result holds

Lemma 2.3 *For $T > 0$ sufficiently large but independent of N there exist two positive constants C_2 and C_3, which do not dependent on N, such that*

$$I_3 \leq (C_2|\lambda_m|^2 + C_3)\mathrm{e}^{\frac{16|\lambda_m|^2}{N}}. \tag{20}$$

Proof. We evaluate first the term $\left| \frac{\sin \frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}} \right|^{4N}$.

Let us first remark that, there exists $a > \pi^2$ such that

$$\frac{\sin x}{x} \leq 1 - \frac{1}{a}x^2, \quad \forall\ |x| < \pi. \tag{21}$$

From (21) it follows that, for $\left|\frac{T(x-\lambda_m)}{4N}\right| < \pi$, we have

$$\begin{aligned}
\left| \frac{\sin \frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}} \right|^{4N} &\leq \left(1 - \frac{1}{a}\left(\frac{T(x-\lambda_m)}{4N}\right)^2\right)^{4N} \\
&= \exp\left(4N \ln\left(1 - \frac{1}{a}\left(\frac{T(x-\lambda_m)}{4N}\right)^2\right)\right) \\
&\leq \exp\left(4N\left(-\frac{1}{a}\left(\frac{T(x-\lambda_m)}{4N}\right)^2\right)\right) \\
&\leq \exp\left(-\frac{T^2(x-\lambda_m)^2}{4aN}\right).
\end{aligned}$$

Hence,

$$\left| \frac{\sin \frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}} \right|^{4N} \leq \exp\left(-\frac{T^2(x-\lambda_m)^2}{4aN}\right), \text{ if } |x-\lambda_m| < \frac{4N\pi}{T}. \tag{22}$$

On the other hand, if $\left|\frac{T(x-\lambda_m)}{4N}\right| \geq \pi$, it follows that

$$\left|\frac{\sin\frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}}\right|^{4N} \leq \frac{1}{\left|\frac{T(x-\lambda_m)}{4N}\right|^{4N}} \leq \exp(-4N\ln(\pi)).$$

Hence,

$$\left|\frac{\sin\frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}}\right|^{4N} \leq \exp(-4N\ln(\pi)), \text{ if } |x-\lambda_m| \geq \frac{4N\pi}{T}. \tag{23}$$

Let us now pass to evaluate the product $\prod_{\substack{|n|\leq N\\ n\neq 0,m}}\left|\frac{x-\lambda_n}{\lambda_n}\right|$. Since $\lambda_n = -\lambda_{-n}$ we have that

$$\prod_{\substack{|n|\leq N\\ n\neq 0,m}}\left|\frac{x-\lambda_n}{\lambda_n}\right| = \frac{|x-\lambda_{-m}|}{|\lambda_{-m}|}\prod_{n=1}^{N}{}'\left|\frac{x^2-\lambda_n^2}{\lambda_n^2}\right| = \frac{|x+\lambda_m|}{|\lambda_m|}\prod_{n=1}^{N}{}'\left|\frac{x^2-\lambda_n^2}{\lambda_n^2}\right|.$$

In the above $'$ indicates that the index $|m|$ has been skipped.

We shall consider the cases $|m| \leq \delta N$ and $|m| \geq \delta N$.

Case I: $m \leq \delta N$.

We first remark that

$$\prod^{N}{}'\left|\frac{x^2-\lambda_n^2}{\lambda_n^2}\right| = \begin{cases} \left|\frac{x^2-\lambda_{N+1-|m|}^2}{\lambda_{N+1-|m|}^2}\right| \prod^{N/2}{}'\left|\frac{x^2-\lambda_n^2}{\lambda_n^2}\right|\left|\frac{x^2-\lambda_{N+1-n}^2}{\lambda_{N+1-n}^2}\right|, & \text{if } N \text{ even} \\ \left|\frac{x^2-\lambda_{N+1-|m|}^2}{\lambda_{N+1-|m|}^2}\frac{x^2-\lambda_{(N+1)/2}^2}{\lambda_{(N+1)/2}^2}\right| \prod^{(N-1)/2}{}'\left|\frac{x^2-\lambda_n^2}{\lambda_n^2}\right|\left|\frac{x^2-\lambda_{N+1-n}^2}{\lambda_{N+1-n}^2}\right|, & \text{if } N \text{ odd.}\end{cases}$$

Since, for $|j| \geq \frac{N}{2}$, we have that $\frac{N}{2} \leq |\lambda_j| \leq N\pi$, it follows that

$$\max\left\{\left|\frac{x^2-\lambda_{N+1-|m|}^2}{\lambda_{N+1-|m|}^2}\right|\left|\frac{x^2-\lambda_{(N+1)/2}^2}{\lambda_{(N+1)/2}^2}\right|\right\} \leq \frac{(2\delta\pi N)^2+(N\pi)^2}{\left(\frac{N}{2}\right)^2} = 4\pi^2(4\delta^2+1) < 20\pi^2.$$

Hence,

$$\prod^{N}{}' \left|\frac{x^2-\lambda_n^2}{\lambda_n^2}\right| \leq (20\pi^2)^2 \prod^{[\frac{N}{2}]}{}' \left(\left|\frac{x^2-\lambda_n^2}{\lambda_n^2}\right| \left|\frac{x^2-\lambda_{N+1-n}^2}{\lambda_{N+1-n}^2}\right|\right)$$
$$= 400\pi^4 \prod^{[\frac{N}{2}]}{}' \left(\left|\frac{x^2-\frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right)}{\frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right)}\right| \left|\frac{x^2-\frac{4}{h^2}\cos^2\left(\frac{n\pi h}{2}\right)}{\frac{4}{h^2}\cos^2\left(\frac{n\pi h}{2}\right)}\right|\right)$$
$$= 400\pi^4 \prod^{[\frac{N}{2}]}{}' \left(\left|\frac{x^4-\frac{4}{h^2}x^2+\frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right)\frac{4}{h^2}\cos^2\left(\frac{n\pi h}{2}\right)}{\frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right)\frac{4}{h^2}\cos^2\left(\frac{n\pi h}{2}\right)}\right|\right).$$

Since $|x-\lambda_m| \leq \delta N\pi$ and $|\lambda_m| \leq \delta N\pi$ it follows that $|x| \leq 2\delta\pi N$. Hence, if $\delta < \frac{1}{4\pi}$, there exists $p \in \mathbb{N}^*$, $p \leq \frac{N}{4}$ and a real number z with $p \leq z < p+1$ such that $x^2 = \frac{4}{h^2}\sin^2\left(\frac{z\pi h}{2}\right)$. We obtain that

$$\prod_{\substack{|n|\ \leq N\\ n\neq 0,m}} \left|\frac{x-\lambda_n}{\lambda_n}\right|$$
$$\leq \frac{|x+\lambda_m|}{|\lambda_m|} \prod^{[\frac{N}{2}]}{}' \left|\frac{\sin^2(n\pi h)-\sin^2(z\pi h)}{\sin^2(n\pi h)}\right|$$
$$= \frac{|x+\lambda_m|}{|\lambda_m|} \prod^{p}{}' \left(\frac{\sin^2(z\pi h)}{\sin^2(n\pi h)}-1\right) \prod_{n=p+1}^{[\frac{N}{2}]}{}' \left(1-\frac{\sin^2(z\pi h)}{\sin^2(n\pi h)}\right)$$
$$\leq \frac{|x+\lambda_m|}{|\lambda_m|} \prod^{p}{}' \left(\frac{\sin^2[(p+1)\pi h]}{\sin^2(n\pi h)}-1\right) \prod_{n=p+1}^{[\frac{N}{2}]}{}' \left(1-\frac{\sin^2(p\pi h)}{\sin^2(n\pi h)}\right)$$
$$= \frac{|x+\lambda_m|}{|\lambda_m|} \prod^{p}{}' \frac{\sin[(p+1-n)\pi h]\sin[(p+1+n)\pi h]}{\sin^2(n\pi h)}$$
$$\times \prod_{n=p+1}^{[\frac{N}{2}]}{}' \frac{\sin[(n-p)\pi h]\sin[(n+p)\pi h]}{\sin^2(n\pi h)}$$
$$\leq \frac{|x+\lambda_m|}{|\lambda_m|} \frac{\sin^2(|m|\pi h)}{|\sin[(p-|m|)\pi h]\sin[(p+|m|)\pi h]|}$$
$$\times \prod_{n=1}^{p} \frac{\sin[(p+1-n)\pi h]\sin[(p+1+n)\pi h]}{\sin^2(n\pi h)}$$

$$\times \prod_{n=p+1}^{\left[\frac{N}{2}\right]} \frac{\sin[(n-p)\pi h]\sin[(n+p)\pi h]}{\sin^2(n\pi h)}$$
$$\leq \frac{|x+\lambda_m|}{2} \prod_{n=1}^{p} \frac{\sin[(p+1-n)\pi h]\sin[(p+1+n)\pi h]}{\sin^2(n\pi h)}$$
$$\times \prod_{n=p+1}^{\left[\frac{N}{2}\right]} \frac{\sin[(n-p)\pi h]\sin[(n+p)\pi h]}{\sin^2(n\pi h)}.$$

But

$$\prod_{n=1}^{p} \sin[(p+1-n)\pi h]\sin[(p+1+n)\pi h]$$
$$\times \prod_{n=p+1}^{\left[\frac{N}{2}\right]} \sin[(n-p)\pi h]\sin[(p+n)\pi h]$$
$$= \prod_{k=1}^{p} \sin(k\pi h) \prod_{k=p+2}^{2p+1} \sin(k\pi h) \prod_{k=1}^{\left[\frac{N}{2}\right]-p} \sin(k\pi h) \prod_{k=2p+1}^{\left[\frac{N}{2}\right]+p} \sin(k\pi h)$$
$$= \frac{\sin[(2p+1)\pi h]}{\sin[(p+1)\pi h]} \prod_{k=1}^{\left[\frac{N}{2}\right]} \sin(k\pi h) \prod_{k=1}^{\left[\frac{N}{2}\right]-p} \sin(k\pi h) \prod_{k=\left[\frac{N}{2}\right]+1}^{\left[\frac{N}{2}\right]+p} \sin(k\pi h)$$
$$\leq 2 \prod_{k=1}^{\left[\frac{N}{2}\right]} \sin(k\pi h) \prod_{k=1}^{\left[\frac{N}{2}\right]-p} \sin(k\pi h) \prod_{k=\left[\frac{N}{2}\right]+1}^{\left[\frac{N}{2}\right]+p} \sin(k\pi h).$$

It follows that

$$\prod_{\substack{|n|\leq N\\ n\neq 0,m}} \left|\frac{x+\lambda_n}{\lambda_n}\right| \leq 400\pi^4|x+\lambda_m| \prod_{k=\left[\frac{N}{2}\right]-p+1}^{k=\left[\frac{N}{2}\right]} \frac{1}{\sin(k\pi h)} \prod_{\left[\frac{N}{2}\right]+1}^{\left[\frac{N}{2}\right]+p} \sin(k\pi h)$$
$$= 400\pi^4|x+\lambda_m| \prod_{k=\left[\frac{N}{2}\right]-p+1}^{\left[\frac{N}{2}\right]} \frac{\sin[(p+k)\pi h]}{\sin(k\pi h)}.$$

Taking into account that $0 \leq (p+k)\pi h \leq \pi$ and since the function $h(x) = \frac{\sin x}{x}$ is decreasing on $[0,\pi]$, it follows that

$$\frac{\sin[(p+k)\pi h]}{\sin(k\pi h)} \leq \frac{p+k}{k}$$

and hence, since $p \leq \frac{N}{4}$, we obtain that

$$\begin{aligned}
\prod_{\substack{|n|\leq N\\ n\neq 0,m}} \left|\frac{x-\lambda_n}{\lambda_n}\right| &\leq 400\pi^4|x+\lambda_m| \prod_{k=\left[\frac{N}{2}\right]-p+1}^{\left[\frac{N}{2}\right]} \frac{p+k}{k} \\
&\leq 400\pi^4|x+\lambda_m| \exp\left(\sum_{k=\left[\frac{N}{2}\right]-p+1}^{\left[\frac{N}{2}\right]} \ln\left(1+\frac{p}{k}\right)\right) \\
&\leq 400\pi^4|x+\lambda_m| \exp\left(\sum_{k=\left[\frac{N}{2}\right]-p+1}^{\left[\frac{N}{2}\right]} \frac{p}{k}\right) \\
&\leq 400\pi^4|x+\lambda_m| \exp\left(\int_{\left[\frac{N}{2}\right]-p+1}^{\left[\frac{N}{2}\right]+1} \frac{p}{y}dy\right) \\
&= 400\pi^4|x+\lambda_m| \exp\left(p\ln\left(\frac{\left[\frac{N}{2}\right]+1}{\left[\frac{N}{2}\right]-p+1}\right)\right) \\
&= 400\pi^4|x+\lambda_m| \exp\left(p\ln\left(1+\frac{p}{\left[\frac{N}{2}\right]-p+1}\right)\right) \\
&\leq 400\pi^4|x+\lambda_m| \exp\left(\frac{p^2}{\left[\frac{N}{2}\right]-p+1}\right) \\
&\leq 400\pi^4|x+\lambda_m| \exp\left(\frac{4p^2}{N}\right) \\
&\leq 400\pi^4|x+\lambda_m| \exp\left(\frac{4x^2}{N}\right).
\end{aligned}$$

Hence,

$$\prod_{\substack{|n|\leq N\\ n\neq 0,m}} \left|\frac{x-\lambda_n}{\lambda_n}\right| \leq 400\pi^4|x+\lambda_m| \exp\left(\frac{4x^2}{N}\right). \tag{24}$$

From (22) and (24) it follows that

$$\begin{aligned}
&\int_{|x-\lambda_m|\leq\frac{4N\pi}{T}} \left(\prod_{\substack{|n|\leq N\\ n\neq 0,m}} \left|\frac{x-\lambda_n}{\lambda_n}\right|^2\right) \left|\frac{\sin\frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}}\right|^{4N} \left|\frac{\sin\frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}}\right|^{4} dx \\
&\quad \leq 400\pi^4 \int_{|x-\lambda_m|\leq\frac{4N\pi}{T}} |x+\lambda_m|^2 \exp\left(-\frac{T^2(x-\lambda_m)^2}{4aN}\right) \exp\left(\frac{8x^2}{N}\right)
\end{aligned}$$

$$\times \left| \frac{\sin \frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}} \right|^4 dx$$
$$\leq 400\pi^4 e^{\frac{16|\lambda_m|^2}{N}} \int_{|x-\lambda_m| \leq \frac{4N\pi}{T}} |x+\lambda_m|^2$$
$$\times \exp\left(-\frac{T^2(x-\lambda_m)^2}{4aN} + \frac{16(x-\lambda_m)^2}{N} \right) \left| \frac{\sin \frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}} \right|^4 dx.$$

Hence, for $T > 0$ sufficiently large, we get that

$$\int_{|x-\lambda_m| \leq \frac{4N\pi}{T}} \left(\prod_{\substack{|n| \leq N \\ n \neq 0,m}} \left| \frac{x-\lambda_n}{\lambda_n} \right|^2 \right) \left| \frac{\sin \frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}} \right|^{4N} \left| \frac{\sin \frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}} \right|^4 dx$$
$$\leq 400\pi^4 e^{\frac{16|\lambda_m|^2}{N}} \int_{|x-\lambda_m| \leq \frac{4N\pi}{T}} |x+\lambda_m|^2 \left| \frac{\sin \frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}} \right|^4 dx$$
$$\leq 400\pi^4 e^{\frac{16|\lambda_m|^2}{N}}$$
$$\times \left(\int_{|x-\lambda_m| \leq \frac{4N\pi}{T}} (2|x-\lambda_m|^2 + 2\lambda_m^2) \left| \frac{\sin \frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}} \right|^4 dx \right)$$
$$\leq (C_2'|\lambda_m|^2 + C_3') e^{\frac{16|\lambda_m|^2}{N}}$$

where C_2' and C_3' are two positive constants which do not depend on N and m.

On the other hand (23) implies that

$$\int_{\frac{4N\pi}{T} \leq |x-\lambda_m| \leq \delta N\pi} \left(\prod_{\substack{|n| \leq N \\ n \neq 0,m}} \left| \frac{x-\lambda_n}{\lambda_n} \right|^2 \right)$$
$$\times \left| \frac{\sin \frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}} \right|^{4N} \left| \frac{\sin \frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}} \right|^4 dx$$
$$\leq 400\pi^4 \int_{\frac{4N\pi}{T} \leq |x-\lambda_m| \leq \delta N\pi} |x+\lambda_m|^2$$
$$\times \exp\left(-4N\ln(\pi)\right) \exp\left(\frac{8x^2}{N} \right) \left| \frac{\sin \frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}} \right|^4 dx$$
$$\leq 400\pi^4 e^{\frac{16|\lambda_m|^2}{N}} \int_{\frac{4N\pi}{T} \leq |x-\lambda_m| \leq \delta N\pi} |x+\lambda_m|^2$$

$$\times \exp\left(-4N\ln(\pi) + \frac{16(x-\lambda_m)^2}{N}\right)\left|\frac{\sin\frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}}\right|^4 dx$$

$$\leq 400\pi^4 \mathrm{e}^{\frac{16|\lambda_m|^2}{N}} \int_{\frac{4N\pi}{T}\leq|x-\lambda_m|\leq\delta N\pi} |x+\lambda_m|^2$$

$$\times \exp\left(-4N\ln(\pi) + 16\delta^2\pi^2 N\right)\left|\frac{\sin\frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}}\right|^4 dx.$$

Since δ can be made arbitrarily small we obtain that

$$\int_{\frac{4N\pi}{T}\leq|x-\lambda_m|\leq\delta N\pi} \left(\prod_{\substack{|n|\leq N\\ n\neq 0,m}}\left|\frac{x-\lambda_n}{\lambda_n}\right|^2\right)$$

$$\times \left|\frac{\sin\frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}}\right|^{4N}\left|\frac{\sin\frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}}\right|^4 dx$$

$$\leq 400\pi^4 \mathrm{e}^{\frac{16|\lambda_m|^2}{N}} \int_{\frac{4N\pi}{T}\leq|x-\lambda_m|\leq\delta N\pi} |x+\lambda_m|^2 \left|\frac{\sin\frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}}\right|^4 dx$$

$$\leq (C"_2|\lambda_m|^2 + C"_3)\mathrm{e}^{\frac{16|\lambda_m|^2}{N}}$$

where $C"_2$ and $C"_3$ are two positive constants which do not depend on N and m.

It follows that

$$I_3 = \int_{\frac{4N\pi}{T}\leq|x-\lambda_m|\leq\delta N\pi} \left(\prod_{\substack{|n|\leq N\\ n\neq 0,m}}\left|\frac{x-\lambda_n}{\lambda_n}\right|^2\right)$$

$$\times \left|\frac{\sin\frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}}\right|^{4N}\left|\frac{\sin\frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}}\right|^4 dx$$

$$+ \int_{|x-\lambda_m|\leq\frac{4N\pi}{T}} \left(\prod_{\substack{|n|\leq N\\ n\neq 0,m}}\left|\frac{x-\lambda_n}{\lambda_n}\right|^2\right)$$

$$\times \left|\frac{\sin\frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}}\right|^{4N}\left|\frac{\sin\frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}}\right|^4 dx$$

$$\leq (C_2|\lambda_m|^2 + C_3)\mathrm{e}^{\frac{16|\lambda_m|^2}{N}}$$

where C_2 and C_3 are two positive constants which do not depend on N and m.

Case II: $m \geq \delta N$.

This case is much simpler than the previous one since we have that

$$\begin{aligned}\prod_{\substack{|n|\leq N\\ n\neq 0,m}}\left|\frac{x-\lambda_n}{\lambda_n}\right| &\leq \prod_{\substack{|n|\leq N\\ n\neq 0,m}}\frac{|x-\lambda_m|+|\lambda_m-\lambda_n|}{|\lambda_n|}\\ &\leq \prod_{\substack{|n|\leq N\\ n\neq 0,m}}\frac{\delta N\pi+4(N+1)}{2|n|}\\ &\leq (4N)^{2N-1}\frac{m}{(N!)^2}\\ &\leq |\lambda_m|\mathrm{e}^{32N}\\ &\leq |\lambda_m|\mathrm{e}^{\frac{8|\lambda_m|^2}{N}}.\end{aligned}$$

It follows that

$$\begin{aligned}I_3 = \int_{|x-\lambda_m|\leq\delta N\pi}&\left(\prod_{\substack{|n|\leq N\\ n\neq 0,m}}\left|\frac{x-\lambda_n}{\lambda_n}\right|^2\right)\\ &\times\left|\frac{\sin\frac{T(x-\lambda_m)}{4N}}{\frac{T(x-\lambda_m)}{4N}}\right|^{4N}\left|\frac{\sin\frac{T(x-\lambda_m)}{4}}{\frac{T(x-\lambda_m)}{4}}\right|^{4}dx\\ &\leq 2\delta N\pi|\lambda_m|^2\mathrm{e}^{\frac{16|\lambda_m|^2}{N}}\\ &\leq (C_2|\lambda_m|^2+C_3)\mathrm{e}^{\frac{16|\lambda_m|^2}{N}}\end{aligned}$$

where C_2 and C_3 are two positive constants which do not depend on N and m.

The proof of the Lemma is now complete. □

We are able now to conclude the proof of Theorem 2.1. Indeed we have

$$\begin{aligned}2\pi||\Theta_m||^2_{L^2(-T,T)} &= ||\xi_m||^2_{L^2(-\infty,\infty)}\\ &= \gamma_1(N)I_2+\gamma_1(N)I_3\left(\prod_{\substack{|n|\leq N\\ n\neq 0,m}}|\lambda_n|^2\right).\end{aligned}$$

Relation (i) from Lemma 2.1 implies that

$$\gamma_1(N)\left(\prod_{\substack{|n|\le N\\ n\neq 0,m}}|\lambda_n|^2\right)\le 4$$

and the proof of the Theorem finishes by taking into account the estimates from Lemmas 2.2 and 2.3. ■

Remark 4 The explicit biorthogonal family $(\Theta_m)_{\substack{|m|\le N\\ m\neq 0}}$ may not have minimal norm. Of course, there exists an unique biorthogonal of minimal norm which belongs to the space generated by $\left(\mathrm{e}^{\mathrm{i}\lambda_j t}\right)_{\substack{|j|\le N\\ j\neq 0}}$. However, it is not easy to evaluate its norm.

Theorem 2.1 gives a biorthogonal sequence, $(\Theta_m)_{\substack{|m|\le N\\ m\neq 0}}$, to the family of complex exponentials $\left(\mathrm{e}^{\mathrm{i}\lambda_j t}\right)_{\substack{|j|\le N\\ j\neq 0}}$ with the property that the norms of the elements $(\Theta_m)_{\substack{|m|\le\sqrt{N}\\ m\neq 0}}$ increase polynomially with m. Note also that the norms of all the elements of the biorthogonal family depend of m but do not depend explicitly of N. Nevertheless, for m large, these norms can have an exponential growth. In the following Theorem we show that in any biorthogonal family there are elements with exponentially big norms.

Theorem 2.2 *Let $(\psi_m)_{\substack{|m|\le N\\ m\neq 0}}$ biorthogonal to $\{\mathrm{e}^{\mathrm{i}\lambda_n t}\}_{\substack{|n|\le N\\ n\neq 0}}$ in $L^2(-T,T)$. Then there exists a positive constants C_2 not depending on N, such that*

$$\| \psi_N \|_{L^2(-T,T)}\ge C_2\mathrm{e}^{\sqrt{N}}. \tag{25}$$

Proof. In order to prove the theorem some arguments from [3] will be used. We shall give the proof in several steps.

Step 1: Let us define the following sequence of functions

$$\tau_m(z)=\int_{-T}^{T}\psi_m(t)\mathrm{e}^{\mathrm{i}tz}dt,\quad |\, m\,|\le N,\quad m\neq 0. \tag{26}$$

From Paley-Wiener Theorem it follows that τ_m is an entire function of exponential type at most T. Moreover,

$$|\,\tau_m(x)\,|\le\sqrt{2T}\,\|\,\psi_m\,\|_{L^2(-T,T)},\quad \forall x\in\mathbb{R}. \tag{27}$$

Since τ_m is a function of exponential type it follows from Hadamard's Factorization Theorem that

$$\tau_m(z)=az^p\mathrm{e}^{bz}\prod_{z_k\in E}\left(1-\frac{z}{z_k}\right)\mathrm{e}^{z/z_k} \tag{28}$$

where E is the set of the zeros z_k of τ_m with $z_k \neq 0$, $E = \{z_k \in \mathbb{C} \mid \tau_m(z_k) = 0, \quad z_k \neq 0\}$.

From the definition of the function τ_m it follows that $\tau_m(\lambda_n) = \delta_{m,n}$. Therefore $\{\lambda_n : \mid n \mid\leq N, \, n \neq 0, \, n \neq m\} \subseteq E$. Let $E' = \{\lambda_n : \mid n \mid\leq N, \, n \neq 0, \, n \neq \pm m\}$ and define the polynomial function

$$P_m(z) = \prod_{\substack{|n|\leq N \\ n\neq 0, \pm m}} \frac{z-\lambda_n}{\lambda_m - \lambda_n}. \tag{29}$$

Let us now define function $\phi_m(z)$ by

$$\phi_m(z) = \frac{\tau_m(z)}{P_m(z)}. \tag{30}$$

The function ϕ_m has the following properties:

- is an entire function of exponential type at most T
- $\phi_m(\lambda_m) = 1$
- $\tau_m(z) = P_m(z)\phi_m(z)$

Let us define $\varphi_N : \mathbb{C} \to \mathbb{C}, \quad \varphi_N(z) = \phi_N(\lambda_N - z)$. Evidently, φ_N is an entire function such that $\varphi_N(0) = 1$.

Step 2: In this step we shall give some estimates for $\mid P_N(\lambda_N - z) \mid$.

$$P_N(\lambda_N - z) = \prod_{\substack{|n|\leq N-1 \\ n\neq 0}} \frac{\lambda_N - z - \lambda_n}{\lambda_N - \lambda_n} = \prod_{\substack{|n|\leq N-1 \\ n\neq 0}} \frac{\mu_n - z}{\mu_n},$$

where

$$\mu_n = \lambda_n - \lambda_N = \frac{4}{h}\cos\left(\frac{N+n}{4}\pi h\right)\sin\left(\frac{N-n}{4}\pi h\right), \quad 0 < |n| < N. \tag{31}$$

Let us now denote by $\nu_j = \mu_{N-j} = \frac{4}{h}\sin\left(\frac{j\pi h}{4}\right)\sin\left(\frac{(j+2)\pi h}{4}\right)$ for $1 \leq j \leq 2N-1$, $j \neq N$ and put $\nu_0 = 0$. Evidently, the sequence $(\nu_j)_{\substack{1\leq j\leq 2N-1 \\ j\neq N}}$ is increasing and

$$P_N(\lambda_N - z) = \prod_{\substack{1\leq j\leq 2N-1 \\ j\neq N}} \frac{\nu_j - z}{\nu_j}.$$

Now, if $z \in \mathbb{C}$ is such that $\mid z \mid\leq N$, there exists $p \in \{0, 1, ..., \left[\frac{N}{2}\right]\}$ such that $|z| \in [\nu_p, \nu_{p+1}]$.

We obtain that

$$
\begin{aligned}
|P_N(\lambda_N - z)| &= \prod_{1\le j\le p} \frac{|\nu_j - z|}{|\nu_j|} \prod_{\substack{p+1\le j\le 2N-1\\ j\neq N}} \frac{|\nu_j - z|}{|\nu_j|} \\
&\ge \prod_{1\le j\le p} \frac{|z| - \nu_j}{\nu_j} \prod_{\substack{p+1\le j\le 2N-1\\ j\neq N}} \frac{\nu_j - |z|}{\nu_j} \\
&\ge \left(\prod_{1\le j\le p-1} \frac{\nu_p - \nu_j}{\nu_j} \prod_{\substack{p+2\le j\le 2N-1\\ j\neq N}} \frac{\nu_j - \nu_{p+1}}{\nu_j} \right) \\
&\quad \times \left(\frac{|z| - \nu_p}{\nu_p} \frac{\nu_{p+1} - |z|}{\nu_{p+1}} \right).
\end{aligned}
$$

But

$$
\begin{aligned}
\prod_{1\le j\le p-1} \frac{\nu_p - \nu_j}{\nu_j} &= \prod_{1\le j\le p-1} \frac{\lambda_{N-j} - \lambda_{N-p}}{\lambda_N - \lambda_{N-j}} \\
&= \prod_{1\le j\le p-1} \frac{\sin\left(\frac{(p-j)\pi h}{4}\right)\cos\left(\frac{(2N-j-p)\pi h}{4}\right)}{\sin\left(\frac{j\pi h}{4}\right)\sin\left(\frac{(j+2)\pi h}{4}\right)} \\
&= \prod_{1\le j\le p-1} \frac{\sin\left(\frac{(p+j+2)\pi h}{4}\right)}{\sin\left(\frac{(j+2)\pi h}{4}\right)} \\
&= \frac{\prod_{k=p+3}^{2p+1} \sin\left(\frac{k\pi h}{4}\right)}{\prod_{k=3}^{p+1} \sin\left(\frac{k\pi h}{4}\right)}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\prod_{\substack{p+2\le j\le 2N-1\\ j\neq N}} \frac{\nu_j - \nu_{p+1}}{\nu_j} &= \prod_{\substack{p+2\le j\le 2N-1\\ j\neq N}} \frac{\lambda_{N-p-1} - \lambda_{N-j}}{\lambda_N - \lambda_{N-j}} \\
&= \prod_{\substack{j=p+2\\ j\neq N}}^{2N-1} \frac{\sin\left(\frac{(j-p-1)\pi h}{4}\right)\cos\left(\frac{(2N-j-p-1)\pi h}{4}\right)}{\sin\left(\frac{j\pi h}{4}\right)\sin\left(\frac{(j+2)\pi h}{4}\right)} \\
&= \prod_{\substack{j=p+2\\ j\neq N}}^{2N-1} \frac{\sin\left(\frac{(j-p-1)\pi h}{4}\right)\sin\left(\frac{(j+p+3)\pi h}{4}\right)}{\sin\left(\frac{j\pi h}{4}\right)\sin\left(\frac{(j+2)\pi h}{4}\right)}
\end{aligned}
$$

$$= \frac{\sin\left(\frac{N\pi h}{4}\right)\sin\left(\frac{(N+2)\pi h}{4}\right)}{\sin\left(\frac{(N-p-1)\pi h}{4}\right)\sin\left(\frac{(N+p+3)\pi h}{4}\right)} \times \frac{\prod_{k=1}^{p+1}\sin\left(\frac{k\pi h}{4}\right)\prod_{k=2N+2}^{2N+p+2}\sin\left(\frac{k\pi h}{4}\right)}{\prod_{k=p+4}^{2p+4}\sin\left(\frac{k\pi h}{4}\right)\prod_{k=2N-p-1}^{2N-1}\sin\left(\frac{k\pi h}{4}\right)}.$$

We obtain that

$$\prod_{1\le j\le p-1}\frac{\nu_p-\nu_j}{\nu_j}\prod_{\substack{p+2\le j\le 2N-1\\ j\neq N}}\frac{\nu_j-\nu_{p+1}}{\nu_j}$$
$$= \frac{\sin\left(\frac{N\pi h}{4}\right)\sin\left(\frac{(N+2)\pi h}{4}\right)}{\sin\left(\frac{(N-p-1)\pi h}{4}\right)\sin\left(\frac{(N+p+3)\pi h}{4}\right)} \times \frac{\sin\left(\frac{\pi h}{4}\right)\sin\left(\frac{2\pi h}{4}\right)\sin\left(\frac{(p+3)\pi h}{4}\right)}{\sin\left(\frac{(2p+2)\pi h}{4}\right)\sin\left(\frac{(2p+3)\pi h}{4}\right)\sin\left(\frac{(2p+4)\pi h}{4}\right)} \times \prod_{k=2N+2}^{2N+p+2}\frac{\sin\left(\frac{k\pi h}{4}\right)}{\sin\left(\frac{(k-p-3)\pi h}{4}\right)}$$
$$\ge \frac{16}{\pi^5}\frac{N(N+2)}{(N-p-1)(N+p+3)}\frac{p+3}{(p+1)(p+2)(2p+3)}.$$

It follows that

$$|P_N(\lambda_N-z)| \ge \frac{16}{\pi^5}\frac{N(N+2)}{(N-p-1)(N+p+3)}\frac{p+3}{(p+1)(p+2)(2p+3)} \times \left(\frac{|z|-\nu_p}{\nu_p}\frac{\nu_{p+1}-|z|}{\nu_{p+1}}\right).$$

Now, since $p \le \frac{N}{2}$ and since $p\nu_p \le \pi$, we obtain that there exists a positive constant $C > 0$ independent of N such that

(32)

$$|P_N(\lambda_N-z)| \ge C\left[\min_{\substack{1\le j\le 2N-1\\ j\neq N}}\{||z|-\nu_j|\}\right]^2, \quad \forall z\in\mathbb{C} \text{ with } |z|\le N.$$

Since $|\nu_{p+1}-\nu_p| = \frac{4}{h}\sin\left(\frac{k\pi h}{4}\right)\sin\left(\frac{(2p+3)\pi h}{4}\right) \ge 2ph$ we can chose $r\in[\nu_p,\nu_{p+1}]$ such that

$$(|z|-\nu_p)(\nu_{p+1}-|z|) \ge (ph)^2, \quad \forall z\in\mathbb{C} \text{ with } |z| = r.$$

Step 3: From (26) and (32) we obtain that

$$| \varphi_N(z) | = \frac{| \tau_N(\lambda_N - z) |}{| P_N(\lambda_N - z) |} \leq \frac{\sqrt{2T} \mathrm{e}^{T \operatorname{Im} z} \parallel \psi_N \parallel_{L^2(-T,T)}}{C \left[\min_{\substack{1 \leq j \leq 2N-1 \\ j \neq N}} \{ ||z| - \nu_j| \} \right]^2},$$

$$\forall z \in \mathbb{C}, \ |z| \leq N. \tag{33}$$

We shall show that (33) is not possible unless $\parallel \psi_N \parallel$ grows rapidly with N.

Let us first recall the following result (see [9], p.21):

Theorem A: *Let $f(z)$ be holomorphic in the circle $| z | \leq 2eR$ $(R > 0)$ with $f(0) = 1$ and let $\eta \in \left(0, \frac{3e}{2}\right)$. Then inside the circle $| z | \leq R$, but outside of a family of excluded circles the sum of whose radii is not greater than $4\eta R$, we have*

$$\ln(| f(z) |) > -\left(2 + \ln\left(\frac{3e}{2\eta}\right)\right) \ln(M_f(2eR)) \tag{34}$$

where $M_f(2eR) = \max_{|z|=2eR} | f(z) |$.

We apply this result to the function φ_N which satisfies the hypothesis of Theorem A. It follows that, for all $R > 0$ and $\eta \in \left(0, \frac{3e}{2}\right)$,

$$\ln(| \varphi_N(z) |) > -2\left(2 + \ln\left(\frac{3e}{2\eta}\right)\right) \ln(M_{\varphi_m}(2eR)),$$

$$\forall z \in \mathbb{C}, \ | z | \leq N \tag{35}$$

outside of a set of circles the sum of whose radii is not greater than $4\eta R$.

Let us denote by $\delta = 2\left(2 + \ln\left(\frac{3e}{2\eta}\right)\right) > 1$ and chose $\eta \in (0, \frac{1}{16})$.

From Theorem A we obtain that there exists $x_0 \in \left[-R, -\frac{R}{4}\right]$ such that

$$\ln(| \varphi_N(x_0) |) > -\delta \ln\left(M_{\varphi_m}(2eR)\right). \tag{36}$$

On the other hand, from (33),

$$\begin{aligned} |\varphi_N(x_0)| &= |\phi_N(\lambda_N - x_0)| \\ &\leq \sqrt{2T} \parallel \psi_N \parallel_{L^2(-T,T)} \frac{1}{| P_N(\lambda_N - x_0) |}. \end{aligned} \tag{37}$$

But, since x_0 is a negative number,

$$|P_N(\lambda_N - x_0)| = \prod_{\substack{1 \leq j \leq 2N-1 \\ j \neq N}} \frac{\nu_j + |x_0|}{\nu_j} \geq \prod_{j=1}^{[\sqrt{N}]} \frac{\nu_j + |x_0|}{\nu_j} > |x_0|^{[\sqrt{N}]}.$$

Hence, from (36) and (37), we obtain that there exists $x_0 \in \left[-R, -\frac{R}{4}\right]$ such that

$$\ln\left(\sqrt{2T} \parallel \psi_N \parallel_{L^2(-T,T)}\right) > [\sqrt{N}] \ln(|x_0|) - \delta \ln\left(M_{\varphi_m}(2eR)\right). \tag{38}$$

We chose now R. If $S = \left[N^{\frac{3}{4}}\right]$ we consider $R > 0$ such that

$$2eR = \frac{\nu_{S+1} + \nu_S}{2}. \tag{39}$$

It follows that

(i) $\min_{\substack{1 \le j \le 2N-1 \\ j \ne N}} \{|2eR - \nu_j|\} = \frac{\nu_{S+1} - \nu_S}{2} \ge \frac{1}{N}$

(ii) $\sqrt{N} \le S(S+2)h \le R \le \frac{\pi^2}{4} S(S+2)h \le \frac{\pi^2}{4}\sqrt{N}$

Hence,

$$M_{\varphi_N}(2eR) = \max_{|z|=2eR} | \varphi_N(z) | \le \frac{\sqrt{2T} N^2 \mathrm{e}^{2eRT} ||\psi_N||_{L^2(-T,T)}}{C}. \tag{40}$$

From (40) and (38) the following estimate is obtained

$$\begin{aligned}
&(1+\delta)\ln\left(\sqrt{2T} \parallel \psi_N \parallel_{L^2(-T,T)}\right) \\
&\quad > [\sqrt{N}] \ln(|x_0|) - 2e\delta T R - \ln\left(\frac{N^2}{C}\right) \\
&\quad > [\sqrt{N}] \left(\ln\left(\frac{R}{4}\right) - 2e\delta T \frac{R}{[\sqrt{N}]} - \frac{1}{[\sqrt{N}]} \ln\left(\frac{N^2}{C}\right)\right).
\end{aligned}$$

Taking into account that $\sqrt{N} \le R \le \frac{\pi^2}{4}\sqrt{N}$ the proof finishes. ■

3 Controllability results

In this section we consider a sequence of semi-discrete systems corresponding to a continuous wave equation and we study some of the controllability properties of these systems.

The starting point of our study is the following boundary control problem: given $T > 2$ and $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ there exists a control function $v \in L^2(0,T)$ such that the solution of the equation

$$\begin{cases} u'' - u_{xx} = 0 & \text{for } x \in (0,1), \quad t > 0 \\ u(t,0) = 0 & \text{for } t > 0 \\ u(t,1) = v(t) & \text{for } t > 0 \\ u(0,x) = u^0(x), \ u'(0,x) = u^1(x) & \text{for } x \in (0,1) \end{cases} \tag{41}$$

satisfies

$$u(T,\cdot) = u'(T,\cdot) = 0. \tag{42}$$

It is well known that this problem has a positive answer (see, for instance, [10]).

We approximate (41) by a sequence of semi-discrete problems. Let $N \in \mathbb{N}^*$, a step $h = \frac{1}{N+1}$ and an equidistant division of the interval $(0,1)$, $0 = x_0 < x_1 < ... < x_N < x_{N+1} = 1$ with $x_j = jh$, $0 \le j \le N+1$.

Now, we consider the following problem of controllability: given $T > 0$ and $(u_j^0, u_j^1)_{1\le j\le N} \in \mathbb{C}^{2N}$, there exists a control function $v_h \in L^2(0,T)$ such that the solution u of

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} = 0 & \text{for } 1 \le j \le N, \ \ t > 0 \\ u_0(t) = 0 & \text{for } t > 0 \\ u_{N+1}(t) = v_h(t) & \text{for } t > 0 \\ u_j(0) = u_j^0, \ u_j' = u_j^1 & \text{for } 1 \le j \le N \end{cases} \tag{43}$$

satisfies

$$u_j(T) = u_j'(T) = 0, \ \forall j = 1, 2, ..., N. \tag{44}$$

System (43) consists of N linear differential equations with N unknowns $u_1, u_2, ..., u_N$. $u_j(t)$ is an approximation for $u(t, x_j)$, the solution of (1), provided that $(u_j^0, u_j^1)_{1\le j\le N}$ are an approximation for the initial datum in (1).

The choice of an appropriate approximation $(u_j^0, u_j^1)_{1\le j\le N}$ for the initial datum $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ of (1) is very important if we want to ensure the existence of a bounded sequence of controls $(v_h)_{h>0}$. This problem will be studied in detail in Sect. 4. The next two paragraphs are devoted to the study of the elementary properties of (43).

3.1 Analysis of the homogeneous problem

The controllability of (43) is directly related to the properties of the corresponding homogeneous adjoin problem. Therefore we introduce now the following system

$$\begin{cases} w_j''(t) - \frac{w_{j+1}(t)+w_{j-1}(t)-2w_j(t)}{h^2} = 0 & \text{for } 1 \le j \le N, \ \ t > 0 \\ w_0(t) = w_{N+1}(t) = 0 & \text{for } t > 0 \\ w_j(0) = w_j^0, \ w_j' = w_j^1 & \text{for } 1 \le j \le N \end{cases} \tag{45}$$

which represents the adjoint of (43).

In order to write (45) in an abstract Cauchy form, we define the matrix $A_h \in \mathcal{M}_{N\times N}(\mathbb{R})$

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & ..... & 0 & 0 \\ -1 & 2 & -1 & 0 & ..... & 0 & 0 \\ 0 & -1 & 2 & -1 & ..... & 0 & 0 \\ ... & ... & ... & ... & ..... & ... & ... \\ 0 & 0 & 0 & 0 & ..... & 2 & -1 \\ 0 & 0 & 0 & 0 & ..... & -1 & 2 \end{pmatrix}.$$

If we denote by $W(t) = (w_1(t), w_2(t), ..., w_N(t))^{\mathrm{T}}$, system (45) can be written as

$$\text{(46)} \qquad \begin{cases} W''(t) + A_h W(t) = 0, & \text{for } t > 0 \\ W(0) = W^0, \ W'(0) = W^1, \end{cases}$$

where $W^0 = (w_j^0)_{1\le j\le N}$ and $W^1 = (w_j^1)_{1\le j\le N}$.

Finally, if we put $Z = (W, W')^{\mathrm{T}}$, we obtain that (45) is equivalent to

$$\text{(47)} \qquad \begin{cases} Z'(t) + L_h Z(t) = 0, & \text{for } t > 0 \\ Z(0) = Z^0 = (W^0, W^1)^{\mathrm{T}} \end{cases}$$

where $L_h \in \mathcal{M}_{2N\times 2N}(\mathbb{R})$,

$$L_h = \begin{pmatrix} 0 & -I \\ A_h & 0 \end{pmatrix}.$$

It follows that (47) is a linear system of $2N$ differential equations of order one and has an unique solution $Z \in \mathcal{C}^{\omega}\left([0,\infty), \mathbb{C}^{2N}\right)$. Hence, (45) has a unique solution $W \in \mathcal{C}^{\omega}\left([0,\infty), \mathbb{C}^{N}\right)$.

The energy of (45) can be defined as

$$\text{(48)} \qquad E_h(t) = \frac{h}{2}\sum_{j=0}^{N}\left[\left|\frac{w_{j+1}(t) - w_j(t)}{h}\right|^2 + |w_j'(t)|^2\right]$$

and represents a discretization of the continuous energy corresponding to (1)

$$\text{(49)} \qquad E(t) = \frac{1}{2}\int_0^1 \left[|u'(t)|^2 + |u_x(t)|^2\right] dx.$$

It is easy to show that equation (45) is conservative, i.e.

$$\text{(50)} \qquad \frac{dE_h}{dt}(t) = 0, \forall t > 0.$$

Let us now define in $\mathbb{C}^{2N}$ the following inner product

$$(f,g) = h\left[\sum_{k=1}^{N-1} \frac{f_{k+1}-f_k}{h}\frac{\overline{g}_{k+1}-\overline{g}_k}{h} + \frac{1}{h^2}(f_1\overline{g}_1 + f_N\overline{g}_N)\right]$$

$$+h\sum_{k=1}^{N} f_{N+k}\overline{g}_{N+k} \tag{51}$$

where $f=(f_k)_{1\le k\le 2N}$ y $g=(g_k)_{1\le k\le 2N}$ are two vectors from $\mathbb{C}^{2N}$.

Energy (48) can be expressed in terms of the inner product we have just introduced. In fact we have that

$$E_h(t) = \frac{1}{2}\left(Z(t), Z(t)\right). \tag{52}$$

In order to give a Fourier decomposition of the solutions of (45) a spectral analysis of the operator L_h must be done.

It is well known that the eigenvalues of A_h are $\nu_j(h) = \frac{4}{h^2}\sin^2(\frac{j\pi h}{2}), 1\le j\le N$, and the corresponding eigenvectors are $\varphi^j(h) = (\sin(j\pi hk))_{1\le k\le N} \in \mathbb{R}^N$, $1\le j\le N$, (see [8]).

It follows that the eigenvalues of L_h are $i\,\lambda_n(h)$, where

$$\lambda_n(h) = \frac{2}{h}\sin(\frac{n\pi h}{2}), \quad -N\le n\le N,\ n\ne 0,$$

and the corresponding eigenvectors are

$$\Phi^n(h) = \begin{pmatrix} \frac{h}{2i\,\sin(\frac{n\pi h}{2})}\varphi^n(h) \\ -\varphi^n(h)\end{pmatrix} = \begin{pmatrix} \frac{1}{i\lambda_n}\varphi^n(h) \\ -\varphi^n(h)\end{pmatrix},$$
$$-N\le n\le N,\ n\ne 0.$$

We have that

Proposition 3.1 *The set of vectors* $(\Phi^n(h))_{\substack{|n|\le N\\ n\ne 0}} \subset \mathbb{C}^{2N}$ *forms an orthonormal base in* $\mathbb{C}^{2N}$.

Proof. We have

$$(\Phi^l(h), \Phi^j(h))$$
$$= h\left[\sum_{k=1}^{N-1}\cos\frac{j\pi h(2k+1)}{2}\cos\frac{l\pi h(2k+1)}{2} + \cos\frac{j\pi h}{2}\cos\frac{l\pi h}{2}\right.$$
$$\left. +(-1)^{j+l}\cos\frac{j\pi h}{2}\cos\frac{l\pi h}{2} + \sum_{k=0}^{N}\sin(j\pi hk)\sin(l\pi hk)\right]$$

$$= h\left[\sum_{k=1}^{N}\left(\cos\frac{j\pi h(2k+1)}{2}\cos\frac{l\pi h(2k+1)}{2}+\sin(j\pi hk)\sin(l\pi hk)\right)\right]$$
$$= \frac{h}{2}\left[\sum_{k=1}^{2N+1}\left(\cos\frac{(l-j)\pi hk}{2}-(-1)^k\cos\frac{(l+j)\pi hk}{2}\right)\right].$$

But

$$\sum_{k=0}^{2N+1}\cos\frac{q\pi hk}{2} = \begin{cases} 2N+2 & \text{if } q=0 \\ \frac{1-(-1)^q}{2} & \text{if } q\in I\!N^* \end{cases},$$
$$\sum_{k=0}^{2N+1}(-1)^q\cos\frac{q\pi hk}{2} = \begin{cases} 0 & \text{if } q=0 \\ \frac{1-(-1)^q}{2} & \text{if } q\in I\!N^* \end{cases}.$$

Finally it follows that $(\Phi^l(h),\Phi^j(h)) = \delta_{lj}$ and the proof finishes. ∎

We can now give a Fourier decomposition of the solutions of (47) in terms of the eigenfunctions of the operator L_h. Hence, if the initial datum $Z^0 = (W^0, W^1)$ of (47) is such that

$$Z^0 = \sum_{\substack{|n|\le N\\ n\neq 0}} a_n^0 \Phi^n(h) \tag{53}$$

then the corresponding solution, $Z(t)$, is

$$Z(t) = \sum_{\substack{|n|\le N\\ n\neq 0}} a_n^0 \mathrm{e}^{\mathrm{i}\,\lambda_n t}\Phi^n(h). \tag{54}$$

Let us now prove the following direct inequality for the solutions of (45):

Proposition 3.2 *Let $T>0$ and w be the solution of (45) in $(0,T)$. Then*

$$\int_0^T \left|\frac{w_N(t)}{h}\right|^2 dt \le 2(T+2)E_h(0) \tag{55}$$

Proof. The following identity is obtained in [7] by using multiplier techniques:

$$TE_h(0) - \frac{h}{4}\sum_{j=0}^{N}\int_0^T |w_j'(t)-w_{j+1}'(t)|^2 dt + X_h(t)\Big|_0^T$$
$$= \frac{1}{2}\int_0^T \left|\frac{w_N(t)}{h}\right|^2 dt, \tag{56}$$

where

$$X_h(t) = h\sum_{j=1}^{N} j\left(\frac{w_{j+1}(t)-w_{j-1}(t)}{2}\right) w_j'(t).$$

Let us evaluate now

$$\begin{aligned}|X_h| &\le h\left(\sum_{j=1}^{N}|w_j'|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{N}\left(j\frac{w_{j+1}-w_{j-1}}{2}\right)^2\right)^{\frac{1}{2}}\\ &\le h\left(\sum_{j=1}^{N}|w_j'|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{N}\frac{j^2}{2}\left(|w_{j+1}-w_j|^2+|w_j-w_{j-1}|^2\right)\right)^{\frac{1}{2}}\\ &\le h\left(\sum_{j=1}^{N}|w_j'|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{N}\left|\frac{w_{j+1}-w_j}{h}\right|^2+\left|\frac{w_1}{h}\right|^2+\left|\frac{w_N}{h}\right|^2\right)^{\frac{1}{2}}\\ &\le E_h(t).\end{aligned}$$

From (56) it follows that

$$\frac{1}{2}\int_0^{\mathrm{T}}\left|\frac{w_N(t)}{h}\right|^2 dt \le TE_h(0)+2E_h(t)$$

and the proof finishes. ■

The following inverse inequality for the solutions of (45) will be also used in the control problems:

Proposition 3.3 *Let $T > 0$ and (w, w_t) be the solution of (45) in $(0, T)$ with the initial data given by (53). Then*

(57) $$\sum_{\substack{|n|\le N\\ n\neq 0}}\rho_n|a_n^0|^2 \le C\int_0^{\mathrm{T}}\left|\frac{w_N(t)}{h}\right|^2 dt,$$

where $(\rho_m)_{\substack{|m|\le N\\ m\neq 0}}$ are any positive weights such that, if $(\Theta_m)_{\substack{|m|\le N\\ m\neq 0}}$ is biorthogonal in $L^2(0,T)$ to the family of complex exponentials $\left(\mathrm{e}^{\mathrm{i}\lambda_m t}\right)_{\substack{|m|\le N\\ m\neq 0}}$, then

(58) $$\sum_{\substack{|m|\le N\\ m\neq 0}}\rho_m\frac{||\Theta_m||^2}{\cos^2\left(\frac{m\pi h}{2}\right)} \le C.$$

Proof. Let us first remark that

$$\int_0^T \left|\frac{w_N(t)}{h}\right|^2 dt = \int_0^T \left|\sum_{\substack{|n|\le N\\ n\neq 0}} a_n^0 \frac{e^{i\lambda_n t}}{2i\sin\left(\frac{n\pi h}{2}\right)}\sin(n\pi h)\right|^2 dt$$
$$= \int_0^T \left|\sum_{\substack{|n|\le N\\ n\neq 0}} b_n^0 e^{i\lambda_n t}\right|^2 dt,$$

where $b_n^0 = -i\cos\left(\frac{n\pi h}{2}\right) a_n^0$.

Let now $(\Theta_m)_{\substack{|m|\le N\\ m\neq 0}}$ be a biorthogonal in $L^2(0,T)$ to the family of complex exponentials $\left(e^{i\lambda_j t}\right)_{\substack{|j|\le N\\ j\neq 0}}$. We have that

$$|b_m^0|^2 = \left|\int_0^T \Theta_m(t)\left(\sum_{\substack{|n|\le N\\ n\neq 0}} b_n^0 e^{i\lambda_n t}\right) dt\right|^2$$
$$\le ||\Theta_m||^2_{L^2(0,T)} \int_0^T \left|\frac{w_N(t)}{h}\right|^2 dt.$$

Let now $(\rho_n)_{\substack{|n|\le N\\ n\neq 0}}$ be positive weights such that (58) is satisfied. It follows that

$$\sum_{\substack{|n|\le N\\ n\neq 0}} |a_n^0|^2\rho_n = \sum_{\substack{|n|\le N\\ n\neq 0}} |b_n^0|^2\rho_n \frac{1}{\cos^2\left(\frac{n\pi h}{2}\right)} \le C\int_0^T \left|\frac{w_N(t)}{h}\right|^2 dt.$$

■

Remark 5 If $(\Theta_n)_{\substack{|n|\le N\\ n\neq 0}}$ is the explicit biorthogonal given by Theorem 2.1, we obtain that (57) is true for any $(\rho_n)_{\substack{|n|\le N\\ n\neq 0}}$ such that

$$\sum_{\substack{|n|\le N\\ n\neq 0}} \rho_n \frac{|\lambda_n|^2}{\cos(\frac{n\pi h}{2})} e^{\frac{|\lambda_n|^2}{N}}. \tag{59}$$

It follows that the weights from (57) corresponding to the low frequencies have a polynomial decay whereas the ones corresponding to the high frequencies have to have an exponential decay.

3.2 A moments problem

We go back now to the controllability problem mentioned in the Introduction.

Let $N \in \mathbb{N}^*$, a step $h = \frac{1}{N+1}$ and an equidistant division of the interval $(0,1)$, $0 = x_0 < x_1 < ... < x_N < x_{N+1} = 1$ with $x_j = jh$, $0 \leq j \leq N+1$.

We study the following problem of controllability: given $T > 0$ and $(u_j^0, u_j^1)_{1\leq j\leq N} \in \mathbb{C}^{2N}$, there exists a control function $v_h \in L^2(0,T)$ such that the solution u of

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} = 0 & \text{for } 1 \leq j \leq N, \ \ t > 0 \\ u_0(t) = 0 & \text{for } t > 0 \\ u_{N+1}(t) = v_h(t) & \text{for } t > 0 \\ u_j(0) = u_j^0, \ u_j' = u_j^1 & \text{for } 1 \leq j \leq N \end{cases} \tag{60}$$

satisfies

$$u_j(T) = u_j'(T) = 0, \ \forall j = 1, 2, ..., N. \tag{61}$$

By using the notations of the previous Sect. (60) can be written in the following matriceal form

$$\begin{cases} Z'(t) + L_h Z(t) = B_h(v_h(t)), \ \text{for } t > 0 \\ Z(0) = Z^0 \end{cases} \tag{62}$$

where $Z^0 = (u_j^0, u_j^1)_{1\leq j\leq N}$ is the initial datum and $B_h(v_h(t)) = \frac{1}{h^2}(0, ..., 0, v_h(t))^{\mathrm{T}} \in \mathbb{C}^{2N}$.

First of all we have the following characterization of the controllability of (60):

Proposition 3.4 *Problem (60) is controllable iff for any initial datum* $Z^0 = (u_j^0, u_j^1)_{1\leq j\leq N} \in \mathbb{C}^{2N}$ *there exists* $v_h \in L^2(0,T)$ *such that*

$$h \sum_{1\leq j\leq N} (u_j^0 \overline{w}_j^1 - u_j^1 \overline{w}_j^0) = \frac{1}{h} \int_0^{\mathrm{T}} v(t) \overline{w}_N(t) dt \tag{63}$$

for any vector $(w_j^0, w_j^1)_{1\leq j\leq N} \in \mathbb{C}^{2N}$ *and* w *solution of the homogeneous adjoint system (45).*

Proof. Indeed, let us consider $(w_j^0, w_j^1)_{1\leq j\leq N} \in \mathbb{C}^{2N}$ and let w be the solution of the homogeneous adjoint system (45). By multiplying the j-th equation of (60) by $\overline{w}_j$, $1 \leq j \leq N$, integrating by parts and adding all the relations we get that

$$\sum_{1\leq j\leq N} \int_0^{\mathrm{T}} \left(u_j''(t)\overline{w}_j(t)dt - \frac{u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)}{h^2} \overline{w}_j(t) \right) dt$$

$$
\begin{aligned}
&= 0 \Leftrightarrow \sum_{1\le j\le N} \left(u_j'(t)\overline{w}_j(t) - u_j(t)\overline{w}_j'(t)\right)\Big|_0^{\mathrm{T}} \\
&\quad + \sum_{1\le j\le N} \int_0^{\mathrm{T}} \left(u_j(t)\overline{w}_j''(t)dt - \frac{\overline{w}_{j+1}(t) + \overline{w}_{j-1}(t) - 2\overline{w}_j(t)}{h^2} u_j(t)dt \right) \\
&= 0 \Leftrightarrow \sum_{1\le j\le N} \left(u_j'(t)\overline{w}_j(t) - u_j(t)\overline{w}_j'(t)\right)\Big|_0^{\mathrm{T}} - \frac{1}{h^2}\int_0^{\mathrm{T}} v_h(t)\overline{w}_N(t)dt \\
&= 0.
\end{aligned}
$$

Hence, (60) is controllable iff $\sum_{1\le j\le N}(u_j'(T)\overline{w}_j(T) - u_j(T)\overline{w}_j'(T)) = 0$ for all $(w_j^0, w_j^1)_{1\le j\le N} \in \mathbb{C}^{2N}$ and w solution of the homogeneous adjoint system (45). The proof finishes. ■

Let us introduce now the following notation

$$
< U^0, W^0 >= h \sum_{1\le j\le N} (u_j^0\overline{w}_j^1 - u_j^1\overline{w}_j^0)
$$

where $U^0 = (u_j^0, u_j^1)_{1\le j\le N} \in \mathbb{C}^{2N}$ and $W^0 = (w_j^0, w_j^1)_{1\le j\le N} \in \mathbb{C}^{2N}$.

Remark 6 If $\Phi^n(h)$ are the eigenvectors of the operator L_h a straightforward calculation gives that that

$$
< \Phi^k(h), \Phi^n(h) >= \frac{ih}{2\sin\left(\frac{n\pi h}{2}\right)}.
$$

Indeed we have

$$
\begin{aligned}
&\left\langle \Phi^k(h), \Phi^n(h) \right\rangle \\
&= h\sum_{1\le j\le N}\left(-\frac{1}{i\lambda_k}\sin(k\pi jh)\sin(n\pi jh) - \frac{1}{i\lambda_n}\sin(k\pi jh)\sin(n\pi jh) \right).
\end{aligned}
$$

But, since

$$
\begin{aligned}
&\sum_{1\le j\le N} \sin(k\pi jh)\sin(n\pi jh) \\
&\quad = \frac{1}{2}\sum_{1\le j\le N} \left(\cos\left((k-n)j\pi h\right) - \cos\left((k+n)j\pi h\right)\right)
\end{aligned}
$$

and, for $q \in \mathbb{Z}^*$,

$$
\sum_{1\le j\le N} \cos\left(qj\pi h\right) = \begin{cases} 0 & \text{if } q \text{ odd} \\ -1 & \text{if } q \text{ even} \end{cases}
$$

the result follows immediately. ■

Our aim is to transform the initial control problem in to an equivalent moments problem. This can be done easily by using Proposition 3.4 and the Fourier decomposition of the solutions of (45).

Proposition 3.5 *System (60) is controllable iff for any initial datum* $U^0 = \sum_{\substack{|n|\leq N\\ n\neq 0}} \beta_n \Phi^n(h)$ *of (60) there exists* $v_h \in L^2(0,T)$ *such that*

$$\int_0^{\mathrm{T}} v_h(t)\mathrm{e}^{-\mathrm{i}\,\lambda_n t} dt = \frac{(-1)^n h}{\sin(n\pi h)}\beta_n, \quad -N \leq n \leq N,\ n \neq 0. \tag{64}$$

Proof. From Proposition 3.4 it follows that (60) is controllable iff for any initial datum U^0 there exists $v_h \in L^2(0,T)$ such that (63) holds for $(w_j^0, w_j^1)_{1\leq j\leq N} = Z^0 = \Phi^n(h)$, $-N \leq n \leq N,\ n \neq 0$.

Remark that, if $Z^0 = \Phi^n(h)$, then the corresponding solution of (47) is $Z(t) = \mathrm{e}^{\mathrm{i}\lambda_n t}\Phi^n(h)$ and therefore $w_N = (-1)^{n+1}\frac{i}{\lambda_n}\mathrm{e}^{\mathrm{i}\lambda_n t}\sin(n\pi h)$.

Moreover, if $U^0 = \sum_{0<|n|\leq N} \beta_n \Phi^n(h)$, by using Remark 6, it follows that

$$\left\langle U^0, W^0 \right\rangle = \sum_{\substack{|k|\leq N\\ k\neq 0}} \beta_n \left\langle \Phi^k(h), \Phi^n(h) \right\rangle = \frac{\beta_n\, i}{2\sin\left(\frac{n\pi h}{2}\right)} = \frac{i}{h\lambda_n}\beta_n.$$

Hence, given $U^0 = \sum_{0<|n|\leq N} \beta_n \Phi^n(h)$, there exists $v_h \in L^2(0,T)$ such that (63) holds for $(w_j^0, w_j^1)_{1\leq j\leq N} = Z^0 = \Phi^n(h)$, $-N \leq n \leq N,\ n \neq 0$ if and only if there exists $v_h \in L^2(0,T)$ such that (64) is satisfied and the proof finishes. ■

Remark 7 From the previuos Proposition it follows that one necessary and sufficient condition for the controllability of the initial datum $U^0 = \Phi^m(h)$ is to find a control function $v_h^m \in L^2(0,T)$ such that

$$\int_0^{\mathrm{T}} v_h^m(t)\mathrm{e}^{-\mathrm{i}\,\lambda_n t} dt = \begin{cases} 0 & \text{if } n \neq m \\ \frac{(-1)^m h}{\sin(m\pi h)} & \text{if } n = m. \end{cases} \tag{65}$$

Remark that the control v_h^m is orthogonal in $L^2(0,T)$ to the family of complex exponentials $\left(\mathrm{e}^{-\mathrm{i}\lambda_n t}\right)_{\substack{|n|\leq N\\ n\neq 0,m}}$ and it is not orthogonal to $\mathrm{e}^{-\mathrm{i}\lambda_m t}$. ■

In the view of the previous remark, a control can be easily obtained by constructing a biorthogonal sequence $(\Theta_m)_{\substack{|m|\leq N\\ m\neq 0}}$ to the sequence of the complex exponentials $\left(\mathrm{e}^{\mathrm{i}\lambda_j t}\right)_{\substack{|j|\leq N\\ j\neq 0}}$ in $L^2(0,T)$.

For any initial datum U^0 a control v_h can be constructed with the aid of the biorthogonal sequence from Sect. 2 and the control problem (60) has a positive answer. Nevertheless, we are interested on the behaviour of the norm of these controls when N goes to infinity.

The estimates for the norm of the biorthogonal sequence we have constructed in Sect. 2 will give bounds for the norm of the controls v_h. In this way we shall be able to say when an uniformly bounded sequence of controls $(v_h)_{h>0}$ can be obtained.

4 Convergence results

The aim of this section is to show the conditions in which one can obtain controls for the continuous system (41) as limits of controls of the corresponding semi-discrete problems (43). In this section the control time T will be considered sufficiently large (but independent of the discretized problem) such that estimate (10) from Theorem 2.1 holds.

Let $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ be the initial datum of (41). We consider that (u^0, u^1) has a Fourier decomposition

$$(u^0, u^1) = \sum_{n \neq 0} a_n \Phi^n \tag{66}$$

where Φ^n are the eigenfunctions of (41)

$$\Phi^n(x) = \begin{pmatrix} \frac{1}{n\pi\, i} \sin(n\pi x) \\ -\sin(n\pi x) \end{pmatrix}.$$

Since Φ^n are orthonormal in $H_0^1(0,1) \times L^2(0,1)$, it follows that $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ iff $\sum_{n\neq 0} \frac{|a_n|^2}{|n|^2\pi^2} < \infty$.

The next step is to chose the initial datum of the semi-discrete problem (43) as an approximation of (u^0, u^1) in such way to ensure the boundedness of the sequence of controls $(v_h)_{h>0}$. In order to do this, let us first cut-off the Fourier series (66) at the range $M \in \mathbb{N}$:

$$(u_M^0, u_M^1) = \sum_{\substack{|n|\leq M \\ n\neq 0}} a_n \Phi^n$$

The number $M = M(N)$ depends on N (in fact $\lim_{N\to\infty} M = \infty$) and also depends on the regularity of the initial datum we have considered. As we shall see later on, if the initial data are analytic we can chose $M = N$ (Theorem 4.2). If the initial data are less regular, for instance in $H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1)$, we have to chose $M \leq \left[\sqrt{N}\right]$ (Theorem 4.1).

Let us now denote by $\tilde{\Phi}^n(h) \in \mathbb{C}^{2N}$ the discretization of the eigenfunction Φ^n:

$$\tilde{\Phi}^n(h) = \left(\frac{1}{n\pi\, i} \sin(n\pi jh),\ -\sin(n\pi jh) \right)^{\mathrm{T}}_{1\leq j\leq N}. \tag{67}$$

We consider now as initial datum for the semi-discrete equation (43) the following discretization of (u_M^0, u_M^1):

$$(u_j^0, u_j^1)_{1\leq j\leq N} = U_h^0 = \sum_{\substack{|n|\leq M \\ n\neq 0}} a_n \tilde{\Phi}^n(h). \tag{68}$$

Note that, if $M \to \infty$ when $N \to \infty$, we have that $U_h^0 \to (u^0, u^1)$.

Remark that $\tilde{\Phi}^n(h)$ is similar to the eigenvectors of the operator L_h but it is not identical ($\frac{1}{n\pi\, i}$ has been replaced by $\frac{1}{\lambda_n\, i}$). We consider this approximation since in some cases (initial data with a finite number of modes, for instance) it is easier to obtain. Nevertheless, since $(u_j^0, u_j^1)_{1\leq j\leq N} \in \mathbb{C}^{2N}$, we can write it in the following way

$$(u_j^0, u_j^1)_{1\leq j\leq N} = \sum_{\substack{|n|\leq M \\ n\neq 0}} a_n(h) \Phi^n(h), \tag{69}$$

where $\Phi^n(h)$ are the eigenvectors of the operator L_h which form an orthonormal basis in $\mathbb{C}^{2N}$.

It is easy to show that

$$a_n(h) = \begin{cases} 0 & \text{if } |n| > M \\ \frac{1}{2}\left(\frac{\lambda_n}{n\pi} + 1\right) a_n + \frac{1}{2}\left(\frac{\lambda_n}{n\pi} - 1\right) a_{-n} & \text{if } |n| \leq M. \end{cases} \tag{70}$$

In the next paragraph we discuss the boundedness of the sequence of controls corresponding to (43).

4.1 Uniformly bounded controls

We can prove now the existence of a bounded sequence of controls for the semi-discrete problem.

Theorem 4.1 *Let us suppose that the initial datum of (41) is such that*

$$\sum_{n\neq 0} |a_n| < \infty \tag{71}$$

and let us consider $(u_j^0, u_j^1)_{1\leq j\leq N}$ given by (68) with $M = \left[\sqrt{N}\right]$. There exists a control v_h of the semi-discrete problem (43) such that the sequence $(v_h)_{h>0}$ is bounded in $L^2(0,T)$.

Proof. Let $(\Theta_m)_{\substack{|m|\leq N\\ m\neq 0}}$ be the biorthogonal sequence in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ to $\left\{e^{i\lambda_n t}\right\}_{\substack{|n|\leq N\\ n\neq 0}}$ constructed in Theorem 2.1. From Proposition 3.5 we obtain that that

$$v_h(t)=\sum_{\substack{|n|\leq M\\ n\neq 0}}\frac{(-1)^{n+1}h}{\sin(n\pi h)}e^{-i\lambda_n\frac{T}{2}}a_n(h)\Theta_n(t-\frac{T}{2}) \tag{72}$$

is a control for (43). It follows that

$$\|v_h\|_{L^2(0,T)}\leq\sum_{\substack{|n|\leq M\\ n\neq 0}}\frac{h}{|\sin(n\pi h)|}|a_n(h)|\,\|\Theta_n\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)}. \tag{73}$$

From the estimates for the norm of Θ_n given by Theorem 2.1 it follows that there exists a constant C independent of N such that

$$\|v_h\|_{L^2(0,T)}\leq C\sum_{\substack{|n|\leq M\\ n\neq 0}}|a_n(h)|$$

and the boundedness of the controls is proved. ■

For sufficiently analytic initial data the following result holds:

Theorem 4.2 *Let us suppose that the initial datum of (41) is such that*

$$\sum_{n\neq 0}\frac{|a_n|}{\cos\left(\frac{n\pi h}{2}\right)}e^{n\pi^2}<\infty \tag{74}$$

and let us consider $(u_j^0,u_j^1)_{1\leq j\leq N}$ *given by (68) with* $M=N$. *There exists a control* v_h *of the semi-discrete problem (43) such that the sequence* $(v_h)_{h>0}$ *is bounded in* $L^2(0,T)$.

Proof. Let v_h be the control given by (72). From (73) and the estimates for the norm of Θ_m given by Theorem 2.1 it follows that

$$\begin{aligned}\|v_h\|_{L^2(0,T)}&\leq\sum_{\substack{|n|\leq N\\ n\neq 0}}\frac{h}{|\sin(n\pi h)|}|a_n(h)|\,\|\Theta_n\|_{L^2}\\ &\leq\sum_{\substack{|n|\leq N\\ n\neq 0}}|a_n(h)|\,\frac{1}{\cos\left(\frac{n\pi h}{2}\right)}e^{\frac{|\lambda_n|^2}{N}}\,.\end{aligned}$$

The proof finishes by taking into account that (74) holds. ■

Remark 8 Theorem 4.2 shows that sufficiently analytic data have always uniformly bounded controls. Hence, in this case, it is not necessary to filter the high frequencies like in Theorem 4.1.

Since the sequence of controls $(v_h)_h$ given by Theorem 4.1 or Theorem 4.2 is bounded in $L^2(0,T)$ there exists a subsequence, denoted in the same way, and $v \in L^2(0,T)$ such that $v_h \rightharpoonup v$ in $L^2(0,T)$ when $h \to 0$. In the next Theorem we show that v is a control for the corresponding continuous problem.

Theorem 4.3 *If $v \in L^2(0,T)$ is a weak limit of the sequence of bounded controls $(v_h)_h$ given by Theorem 4.1 then v is a control for the continuous problem (41).*

Proof. Indeed, like in the case of the semi-discrete problem, it is easy to prove that $v \in L^2(0,T)$ is a control for (41) iff

$$\int_0^{\mathrm{T}} v(t)\overline{w}_x(t,1)dt = \left\langle u^1, \overline{w}^0 \right\rangle_{H^{-1},H_0^1} - \int_0^1 u^0\overline{w}^1 \tag{75}$$

for any $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$ and w the solution of the adjoint equation

$$\begin{cases} w'' - w_{xx} = 0 & \text{for } x \in (0,1), \quad t > 0 \\ w(t,0) = w(t,1) = 0 & \text{for } t > 0 \\ w(0,x) = w^0(x), \ w'(0,x) = w^1(x) & \text{for } x \in (0,1). \end{cases} \tag{76}$$

Let us remark that it is sufficient to show that (75) is verified only for $(w^0, w^1) = \Phi^n$, $n \in \mathbb{Z}^*$, the eigenfunctions of the wave operator.

Indeed, from the continuity of the linear form $\Lambda : H_0^1(0,1) \times L^2(0,1) \to \mathbb{C}$,

$$\Lambda(w^0, w^1) = \int_0^{\mathrm{T}} v(t)\overline{w}_x(t,1)dt - \left\langle u^1, \overline{w}^0 \right\rangle_{H^{-1},H_0^1} + \int_0^1 u^0\overline{w}^1$$

it follows that (75) holds for any $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$ iff it is verified by a basis of the space $H_0^1(0,1) \times L^2(0,1)$.

By considering $(w^0, w^1) = \Phi^n$, we obtain that v is a control for (41) iff

$$\int_0^{\mathrm{T}} v(t)\mathrm{e}^{-\mathrm{i}n\pi t}dt = \frac{(-1)^{n+1}}{n\pi}a_n, \quad \forall n \neq 0. \tag{77}$$

Note that this is the moments problem for the continuous system (41) similar to (64) from Proposition 3.5.

From the fact that v_h is a control for the discrete problem we have from Proposition 3.5 that

$$\int_0^{\mathrm{T}} v_h(t)\mathrm{e}^{-\mathrm{i}\lambda_n t}dt = \frac{(-1)^n h}{\sin(n\pi h)}a_n(h), \quad -N \leq n \leq N, \ n \neq 0. \tag{78}$$

Taking into account that, for each $n \in \mathbb{Z}$,

$$\mathrm{e}^{\mathrm{i}\lambda_n(h)t} \to \mathrm{e}^{\mathrm{i}n\pi t} \text{ in } L^2(0,T)$$

and

$$\frac{h}{\sin(n\pi h)} a_n(h) \to \frac{1}{n\pi} a_n$$

when h tends to zero, by passing to limit in (78), it follows that v satisfies (77).

Hence, the limit v is a control for the problem (41) and the proof finishes. ■

In the next Theorem we also prove that the solutions of the discrete problem converges to a solution of the continuous problem.

Let us first define, for each $U = (u_1, u_2, ..., u_{2N})$, $W = (w_1, w_2, ..., w_{2N})$ from $\mathbb{C}^{2N}$, the duality product

$$\langle U, W\rangle = h \sum_{j=1}^{N} (u_j \overline{w}_{N+j} - u_{N+j}\overline{w}_j).$$

Remark that $\langle \cdot, \cdot \rangle$ is a bilinear and continuous application and

$$\|U\|_{-1} = \sup_{\substack{W \in \mathbb{C}^{2N} \\ \|W\|=1}} |\langle U, W\rangle|$$

defines a norm in $\mathbb{C}^{2N}$ (the dual norm of $\| \cdot \|$).

Remark 9 From Remark 6 we obtain that

$$\left\langle \Phi^j, \Phi^l \right\rangle = -\frac{1}{\lambda_j}\delta_{jl}.$$

Hence,

$$\|\Phi^j\|_{-1} = \sup_{\substack{W \in \mathbb{C}^{2N} \\ \|W\|=1}} \left|\left\langle \Phi^j, W \right\rangle\right| = \frac{1}{|\lambda_j|}.$$

Let now $U(t) = (u_1(t), ..., u_N(t), u_1'(t), ..., u_N'(t))$ be the solution of the non homogeneous wave equation

$$\text{(79)} \qquad \begin{cases} U' + L_h U = B_h(v) \\ U(0) = U^0. \end{cases}$$

The following property of the solution of (79) is a direct consequence of Proposition 3.2.

Proposition 4.1 *If U is the solution of (79) then there exists a positive constant C, independent of N, such that*

$$\|U(t)\|_{-1} \leq C(\|U^0\|_{-1} + \|v\|_{L^2(0,T)}), \quad \forall t \in [0,T]. \tag{80}$$

Proof. Indeed, like in Proposition 3.4, it easy to show that U is solution of (79) iff

$$\left\langle U^0, W^0 \right\rangle - \langle U(t), W(t) \rangle = \frac{1}{h} \int_0^t v(t)\overline{w}_N(t)dt \tag{81}$$

for any vector $W^0 \in \mathbb{C}^{2N}$ and W solution of the homogeneous adjoint system (45).

It follows that

$$|\langle U(t), W(t) \rangle| \leq \left(\|U^0\|_{-1} \|W^0\| + \|v\|_{L^2(0,T)} \left(\int_0^T \left| \frac{w_N}{h} \right|^2 \right)^{\frac{1}{2}} \right).$$

Now, taking into account the direct inequality (55) we obtain that there exists a positive constant C, not depending on N, such that

$$|\langle U(t), W(t) \rangle| \leq \left(\|U^0\|_{-1} + \|v\|_{L^2(0,T)} \right) \|W^0\|$$

for any vector $W^0 \in \mathbb{C}^{2N}$ and W solution of the homogeneous adjoint system (45).

Inequality (80) follows now from the definition of the norm $\| \cdot \|$ and the conservation of the energy of solutions of (45). ■

We consider now that, from the sequence of bounded controls, $(v_h)_{h>0}$, we have extracted a subsequence, denoted in the same way, such that $v_h \rightharpoonup v$ when $h \to 0$ in $L^2(0,T)$.

Let also $(U^0(h))_{h>0}$ be the sequence of initial data which approximate U^0, the initial datum of the continuous problem, and $(U(h,t))_{h>0}$ the solutions of the discrete problem (79) with initial datum $U^0(h)$ and control v_h. We have that

$$U(t) = \sum_{n \neq 0} a_n(t) \Phi^n$$
$$U(h,t) = \sum_{\substack{|n| \leq N \\ n \neq 0}} a_n(h,t) \Phi^n(h).$$

The following theorem gives a result of convergence of $(U(h,\cdot))_{h>0}$ to $U(\cdot)$.

Theorem 4.4 *Let $(U(h,t))_{h>0}$ be the family of solutions of the discrete problem (79) with initial datum $U^0(h)$ and control v_h. Then, by extracting a suitable sequence $h \to 0$, we may guarantee that*

$$\left(\frac{a_n(h,\cdot)}{\lambda_n}\right)_{\substack{|n|\le N\\ n\neq 0}} \rightharpoonup \left(\frac{a_n(\cdot)}{n\pi}\right)_{n\neq 0} \quad \text{in } L^\infty(0,T;\ell^2). \tag{82}$$

Proof. First of all let us remark that, if $U^0(h) = \sum_{n\neq 0} a_n^0(h)\Phi^n(h)$, then

$$\begin{aligned}
\|U^0(h)\|_{-1}^2 &= \sum_{\substack{|n|\le N\\ n\neq 0}} \left|\frac{a_n^0(h)}{\lambda_n}\right|^2 \\
&= \frac{1}{4}\sum_{\substack{|n|\le M\\ n\neq 0}} \left|\left(\frac{1}{n\pi}+\frac{1}{\lambda_n}\right)a_n^0 + \left(\frac{1}{n\pi}-\frac{1}{\lambda_n}\right)a_{-n}^0\right|^2 \\
&\le C\sum_{n\neq 0}\frac{|a_n^0|^2}{n^2} < \infty.
\end{aligned}$$

Hence, the sequence $\left(U^0(h)\right)_{h>0}$ is uniformly bounded in the norm $\|\cdot\|_{-1}$ and $(v_h)_{h>0}$ is also uniformly bounded in $L^2(0,T)$. From Proposition 4.1, it follows that $(U(h,\cdot))_{h>0}$ is uniformly bounded in the norm $\|\cdot\|_{-1}$ and consequently $\left(\left(\frac{a_n(h,\cdot)}{\lambda_n}\right)_{\substack{|n|\le N\\ n\neq 0}}\right)_{h>0}$ is uniformly bounded in $L^\infty(0,T;\ell^2)$. We obtain that there exists a subsequence, denoted in the same way, and $(\beta_n)_{n\neq 0} \in L^\infty(0,T;\ell^2)$ such that

$$\left(\frac{a_n(h,\cdot)}{\lambda_n}\right)_{\substack{|n|\le N\\ n\neq 0}} \rightharpoonup (\beta_n(\cdot))_{n\neq 0} \ \text{ in } L^\infty(0,T;\ell^2)$$

when $h \to 0$.

We have only to show that $\beta_n(t) = \frac{a_n(\cdot)}{n\pi}$, for each $n \neq 0$, and the proof finishes.

Let define $Z(t) = \sum_{n\neq 0}\beta_n(t)n\pi\Phi^n$. We have that $Z \in L^\infty(0,T;L^2 \times H^{-1})$ and we shall show that $Z(t) = U(t)$.

In order to do this, let us remark that, since $U(h,\cdot)$ is solution of (79), we have that

$$\langle U^0, W^0\rangle - \langle U(t), W(t)\rangle = \frac{1}{h}\int_0^{\mathrm{t}} v(t)\overline{w}_N(t)dt$$

for any vector $W^0 \in \mathbb{C}^{2N}$ and W solution of the homogeneous adjoint system (45).

Taking $W^0 = \Phi^n(h)$ we obtain that

$$\frac{a_n(h,t)}{\lambda_n} - \frac{a_n^0(h)}{\lambda_n} = \frac{1}{h}\int_0^t v_h \overline{\varphi_N^n}$$

is true for any $|n| \leq N$, $n \neq 0$.

Passing to the limit one obtains that

$$\beta_n(t) - \beta_n(0) = \int_0^t (-1)^{n+1} i\, v$$

which is equivalent to

$$\left\langle Z^0, \Phi^n \right\rangle - \langle Z(t), \Phi^n \rangle = \int_0^t (-1)^{n+1} i\, v(s) ds$$

for any $|n| \leq N$, $n \neq 0$.

It follows that $Z(t)$ is the solution of the controlled continuous problem and the proof finishes. ■

5 HUM controls

The construction of a sequence of uniformly bounded controls in the previous sections was done by using the explicit biorthogonal sequence $(\Theta_m)_{\substack{|m|\leq N \\ m\neq 0}}$ for the family of complex exponentials $(\mathrm{e}^{\mathrm{i}\lambda_m t})_{\substack{|m|\leq N \\ m\neq 0}}$ given by Theorem 2.1.

Hilbert Uniqueness Method (HUM) provides another possibility to obtain controls, based on the use of the solutions of the adjoint problem.

For each discrete problem (43) (with initial datum U^0) let ζ_h denote the HUM control.

We recall that $\zeta_h = \frac{z_N}{h}$ where $Z = (z_j(t), z_j'(t))_{1\leq j\leq N}$ is the solution of the homogeneous adjoint problem (45) with initial data Z^0, where $Z^0 \in \mathbb{C}^{2N}$ minimizes the coercive, continuous and strictly convex functional

$$\mathcal{J}_h(W^0) = \int_0^T \left|\frac{w_N}{h}(t)\right|^2 dt - \left\langle U^0, W^0 \right\rangle \tag{83}$$

where $W^0 = (w_j^0, w_j^1)_{1\leq j\leq N} \in \mathbb{C}^{2N}$ and $W = (w, w')$ is the solution of (45).

Moreover, ζ_h is characterized by the following two properties:

(i) $\zeta_h = \frac{z_N}{h}$ where $Z = (z_j(t), z_j'(t))_{1\leq j\leq N}$ is a solution of (45).

(ii) ζ_h satisfies

$$h \sum_{1\leq j\leq N} (u_j^0\overline{w}_j^1 - u_j^1\overline{w}_j^0) = \frac{1}{h}\int_0^T \zeta_h(t)\overline{w}_N(t)dt \tag{84}$$

for any vector $W^0 = (w_j^0, w_j^1)_{1\leq j\leq N} \in \mathbb{C}^{2N}$ and w solution of (45).

It is well known that the HUM control ζ_h is of minimal L^2-norm (see [10]). From the uniform boundedness results we have obtained in the previous section we obtain that, under the hypothesis of Theorem 4.1 or of Theorem 4.2, the sequence of the HUM controls for the semi-discrete problems is uniformly bounded.

In the next subsection we show that, in any Sobolev space, there exists initial data such that, if the hypothesis of Theorem 4.1 are not satisfied, the sequence of HUM controls is not bounded.

5.1 Unbounded controls

Let us first define the following space of initial data

$$\mathcal{V} = \left\{ U^0 = (u^0, u^1) = \sum_{n\neq 0} a_n \Phi^n \ : \ \sum_{n\neq 0} |a_n|^2 e^{\varepsilon\sqrt{n}} < \infty \right\}$$

where ε is a positive number sufficiently small.

In $\mathcal{V}$ we define the norm

$$|| \sum_{n\neq 0} a_n \Phi^n || = \left(\sum_{n\neq 0} |a_n|^2 e^{\varepsilon\sqrt{n}} \right)^{\frac{1}{2}} .$$

$\mathcal{V}$ is a normed vector space and $\mathcal{V} \subset H^m(0,1) \times H^{m-1}(0,1)$ for any $m \geq 1$.

Theorem 5.1 *There exists at least one element (u^0, u^1) in $\mathcal{V}$, such that the sequence $(\zeta_h)_{h>0}$ of the HUM controls for the discrete equation (43) with initial data $(u_j^0, u_j^1)_{1\leq j\leq N}$ given by (68) with $M = N$ is unbounded in $L^2(0,T)$.*

Remark 10 The main difference between Theorems 5.1 and 4.1 is that in the former the high frequencies of the initial data are not filtered (we consider that $M = N$). In this case there exists regular initial data which do not have a sequence of discrete controls uniformly bounded in $L^2(0,T)$.

Proof. Suppose that any initial data $U^0 = (u^0, u^1)$ from $\mathcal{V}$ has the property that the sequence $(\zeta_h)_{h>0}$ of the HUM controls for the discrete equations (43) with initial data $(u_j^0, u_j^1)_{1\leq j\leq N}$ given by (68) with $M = N$ is bounded in $L^2(0,T)$.

For each $N \in \mathbb{N}^*$ we define the operator $T_N : \mathcal{V} \longrightarrow L^2(0,T)$ such that

$$T_N(U^0) = \zeta_h$$

where $h = \frac{1}{N+1}$ and ζ_h is the HUM control for the discrete equation (43) with initial data $(u_j^0, u_j^1)_{1\leq j\leq N}$ given by (68) with $M = N$.

It is easy to see that $(T_N)_{N\geq 1}$ is a sequence of linear and continuous operators. Moreover, for each $U^0 \in \mathcal{V}$, we have that

$$||T_N(U^0)||_{L^2(0,T)} = ||\zeta_h||_{L^2(0,T)} < \infty, \quad \forall N \geq 1.$$

From the Banach-Steinhaus Theorem it follows that the operators T_N are uniformly bounded. Hence, there exists a constant $C > 0$, not depending on N, such that

$$||T_N||_{\mathcal{L}(\mathcal{V},L^2(0,T))} \leq C, \quad \forall N \geq 1. \tag{85}$$

For each $N \in \mathbb{N}$ let us now consider $U^0 = \Phi^N$, the $N-$th eigenfunction of the wave operator. The discrete initial data are:

$$U^0(h) = \frac{1}{2}\left(\frac{\lambda_N}{N\pi} + 1\right)\Phi^N(h).$$

By taking into account the characterization of any control given by Proposition 3.5, we obtain that the HUM control $\zeta_h(t) = \alpha_N \psi_N(t - \frac{T}{2})$ where $(\psi_n)_{1<|n|<N}$ is a biorthogonal sequence for $(\mathrm{e}^{\mathrm{i}\lambda_n t})_{1<|n|<N}$ in $L^2(-\frac{T}{2}, \frac{T}{2})$ and $\alpha_N = \frac{(-1)^N h}{2\sin(h\pi)}\left(\frac{\lambda_N}{N\pi} + 1\right)\mathrm{e}^{\frac{i\lambda_N T}{2}}$.

The estimate from Theorem 2.2 and (85) give that

$$\begin{aligned} C_2 \mathrm{e}^{\sqrt{N}} &\leq ||\psi_N||_{L^2(-\frac{T}{2},\frac{T}{2})} \\ &\leq 4||\zeta_h||_{L^2(0,T)} = ||T_N(U^0)||_{L^2(0,T)} \leq C\mathrm{e}^{\varepsilon\sqrt{N}}. \end{aligned}$$

This is impossible if ε is small enough. We have obtained a contradiction and the proof finishes. ∎

5.2 Convergence of the HUM controls

Let $U^0 = (u^0, u^1)$ be an initial data for (41) and let us now suppose that the hypothesis of Theorem 4.1 or of Theorem 4.2 are fulfilled.

Since, in this case, there exists a sequence of uniformly bounded discrete controls, it follows that the sequence of the HUM controls, $(\zeta_h)_{h>0}$, is also bounded. Hence, there exists a subsequence, denoted in the same way, which converges weakly in $L^2(0,T)$ to an element $\zeta \in L^2(0,T)$.

Let $Z^0(h)$ be the initial datum which gives the HUM control ζ_h (i.e. $\zeta_h = \frac{z_N}{h}$, where $Z = (z, z')$ is the solution of the adjoint system (45) with the initial datum $Z^0(h)$).

We have that

Theorem 5.2 *The function $\zeta \in L^2(0,T)$ is the HUM control of the continuous wave equation with the initial data U^0.*

Proof. First of all let us remark that, like in Theorem 4.3, it follows that ζ is a control for the continuous problem (41) with initial datum U^0. Hence, it satisfies

$$\int_0^{\mathrm{T}} \zeta(t)\overline{w}_x(t,1)dt = \left\langle u^1, \overline{w}^0 \right\rangle_{H^{-1},H_0^1} - \int_0^1 u^0\overline{w}^1 \tag{86}$$

for any $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$ and w the solution of the adjoint equation

$$\begin{cases} w'' - w_{xx} = 0 & \text{for } x \in (0,1), \quad t > 0 \\ w(t,0) = w(t,1) = 0 & \text{for } t > 0 \\ w(0,x) = w^0(x), \; w'(0,x) = w^1(x) & \text{for } x \in (0,1). \end{cases} \tag{87}$$

Let us now remark that, if $p_x(t,1)$ is the HUM control for the continuous wave equation (41) (which corresponds to an initial datum $P^0 \in H_0^1(0,1) \times L^2(0,1)$) , then

$$\int_0^{\mathrm{T}} (\zeta(t) - p_x(t))\overline{w}_x(t,1)dt = 0 \tag{88}$$

for any $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$ and w the solution of the adjoint equation (87).

Let us now remark that, from the observation inequality given by Proposition 3.3, the sequence $(Z^0(h))_{h>0}$ converges in a weak norm (with exponential weighs) to an element Z^0. The sequence $(\zeta_h)_{h>0}$ will converge in the sense of distributions to the normal derivative $z_x(t,1)$ of the solution of system (41) with initial datum Z^0.

It follows that $\zeta(t) = z_x(t,1)$. Since $\zeta \in L^2(0,T)$ we obtain that $z_x(\cdot,1) \in L^2(0,T)$ and, consequently, $Z^0 = P^0 \in H_0^1(0,1) \times L^2(0,1)$.

Now, from (88), by taking $w = z - p$, it results that $\zeta = p_x(t)$ and the proof finishes. ∎

References

1. Avdonin, S.A. Ivanov, S.A.: Families of exponentials. The method of moments in controllability problems for distributed parameter systems. Cambridge University Press, 1995.
2. Castro C.: Boundary controllability of the one-dimensional wave equation with rapidly oscillating density, Asymptotic Analysis **20**, 317–350 (1999)
3. Fattorini H.O.: Estimates for sequences biorthogonal to certain complex exponentials and boundary control of the wave equation. Lecture Notes in Control and Inform. Sci. **2**, 111–124 (1979)
4. Glowinski R.: Ensuring well-posedness by analogy; Stokes problem and boundary control for the wave equation. J. Comput. Phys. **103**, 189–221 (1992)
5. Glowinski R., Li C. H., Lions J. L.: A numerical approach to the exact boundary controllability of the wave equation (I). Dirichlet controls: Description of the numerical methods. Jap. J. Appl. Math. **7**, 1–76 (1990)
6. Glowinski R., Lions J. L.: Exact and approximate controllability for distributed parameter systems. Acta Numerica, pp 159–333 (1996)
7. Infante J. A., Zuazua E.: Boundary observability for the space semi-discretization of the 1-D wave equation. M2AN. **33**(2), 407–438 (1999)
8. Isaakson E., Keller H. B.: Analysis of Numerical Methods. John Wiley & Sons, 1996
9. Levin, B.J.: Distribution of zeros of entire functions. Translation of Math. Monographs. **5**, AMS, 1950
10. Lions J.-L.: Contrôlabilité exacte perturbations et stabilisation de systèmes distribués, Tome 1, Masson, Paris, 1988
11. Russell D. L.: Controllability and stabilization theory for linear partial differential equations: recent progress and open questions. SIAM Review. **20**(4), 639–737 (1978)
12. Young R.M.: An introduction to nonharmonic Fourier series. New York: Academic Press, 1980