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Localization of the Aronszajn-Slobodeckij norm and application to adaptive boundary element methods

Part II. The three-dimensional case

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Summary. In this paper we introduce new local a-posteriori error indicators for the Galerkin discretization of three-dimensional boundary integral equations. These error indicators are efficient and reliable for a wide class of integral operators, in particular for operators of negative order. They are based on local norms of the computable residual and can be used for controlling the adaptive refinement. The proofs of efficiency and reliability are based on the result that the Aronszajn-Slobodeckij norm $\|\cdot\|_{H^s(\Gamma)}$ (given by a double integral for a non-integer $s \in \mathbb{R}_{>0} \setminus \mathbb{N}$) is localizable for certain functions. Neither inverse estimates nor saturation properties are needed. In this paper, we extend the two-dimensional results of a previous paper to the three-dimensional case.

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1 Introduction

Many problems in physical and engineering sciences can be formulated as boundary value problems for partial differential equations in a domain $\Omega \subseteq \mathbb{R}^d$, and many boundary value problems can be translated into boundary integral equations defined on the surface $\Gamma = \partial\Omega$ (see for example [30], [9] and [21, Sect. 8]).

Boundary integral equations are considered in the abstract form

$$Au = g \qquad \text{on } \Gamma \tag{1.1}$$

with a given right-hand side g and with a bounded and bijective integral operator A of order $2\alpha \in \mathbb{R}$. For A, we distinguish the following two cases

$$A : H^{\alpha}(\Gamma) \to H^{-\alpha}(\Gamma)\,, \;\; \alpha > 0\,, \tag{1.2}$$

$$A : H^{s+2\alpha}(\Gamma) \to H^{s}(\Gamma)\,, \;\; s \geq 0\,, \quad \alpha \in \mathbb{R}\,, \tag{1.3}$$

where $H^{\mathrm{t}}(\Gamma)$ is the Sobolev space of order $t \in \mathbb{R}$.

For the Galerkin discretization of problem (1.1), we introduce a mesh $\triangle$ on Γ and a finite dimensional Galerkin trial space $\mathcal{G} = \mathcal{G}_\triangle$ consisting of piecewise polynomials associated with the mesh $\triangle$. There are three possibilities to improve the corresponding Galerkin solution $u_\mathcal{G} \in \mathcal{G}_\triangle$. The h-version improves $u_\mathcal{G}$ by refining the mesh and using piecewise polynomials with a fixed degree p. The p-version fixes the mesh and improves $u_\mathcal{G}$ by increasing the polynomial degree p in the elements. The hp-version combines both h-refinement and p-refinement. Adaptive strategies are required, if the discretization error $e := u_\mathcal{G} - u$ is not uniformly distributed over the mesh $\triangle$. In this case, we want to refine (in the sense of h- or p-refinement) only the elements with large local error.

In general, adaptive refinement is controlled by *local a posteriori error indicators* (or briefly (local) error indicators) $\{\varepsilon_\nu\}_{\nu=1}^n$, where $\{\varepsilon_\nu\}_{\nu=1}^n$ are local quantities associated with the n elements of $\triangle$. Their definition is based on the discrete solution $u_\mathcal{G}$ and they estimate

$$C^{eff} \sum_{\nu=1}^{n} \varepsilon_\nu^2 \quad \leq \quad |\!|\!| u_\mathcal{G} - u |\!|\!|^2 \quad \leq \quad C^{rel} \sum_{\nu=1}^{n} \varepsilon_\nu^2 \tag{1.4}$$

with $|\!|\!| \cdot |\!|\!| = \| \cdot \|_{H^\alpha(\Gamma)}$ in the case (1.2) and $|\!|\!| \cdot |\!|\!| = \| \cdot \|_{H^{s+2\alpha}(\Gamma)}$ in the case (1.3). Local error indicators are called *reliable* if they satisfy the upper estimate in (1.4) with a constant C^{rel} independent of u, $\triangle$ and the local polynomial degrees, and they are called *efficient* if they satisfy the lower estimate in (1.4) with a constant C^{eff} independent of u, $\triangle$ and the local polynomial degrees.

In practice, local error indicators are used in the following way for controlling the adaptive strategy:

Adaptive refinement process:

- Compute the discrete solution $u_\mathcal{G} \in \mathcal{G}$.
- Compute the error indicators $\{\varepsilon_\nu\}_{\nu=1}^n$. If they are not exactly computable, then compute approximations $\tilde{\varepsilon}_\nu$ of ε_ν.
- Stop the refinement if $\sum_{\nu=1}^n \tilde{\varepsilon}_\nu^2$ is small enough.
  Otherwise, mark all elements of $\triangle$ associated with large $\tilde{\varepsilon}_\nu$. Decide for every marked element if it is an h-element or p-element. Refine

the marked h-elements geometrically and increase the local polynomial degree in all marked p-elements. This generates a new Galerkin trial space $\hat{\mathcal{G}}$ associated with a new mesh $\hat{\triangle}$. Start this process again with the enriched Galerkin space $\hat{\mathcal{G}}$.

For finite element methods (FEM), adaptive refinement controlled by local error indicators has been the subject of many papers in recent years. However, for boundary element methods (BEM), the nonlocal character of the integral operator and the nonlocal Sobolev spaces cause difficulties in the mathematical derivation of local error indicators. Hence, only a few authors have investigated local *a posteriori* error estimates of the form (1.4): heuristically motivated error indicators and numerical results were presented in [14] for hp-methods and in [3] for h-methods. Reliable local error indicators were introduced for example in [26,27,33,34,31,7,4,5,23,8,29] for the h-version and in [6] for the hp-version.

For BEM, the proof of efficiency is problematic: [4] shows efficiency only for uniform meshes, and [23] show efficiency and reliability only for uniform meshes and under the additional assumption of a saturation condition. For the direct boundary element method, an efficient and reliable global error estimator ε (with $C^{eff}\varepsilon^2 \leq |||u_{\mathcal{G}} - u|||^2 \leq C^{rel}\varepsilon^2$) is proposed in [29] and the adaptive mesh refinement is controlled by reliable local error indicators $\{\varepsilon_\nu\}_{\nu=1}^n$ which satify $\varepsilon^2 \leq \sum_{\nu=1}^n \varepsilon_\nu^2$. The other cited papers do not have any efficiency results.

For Galerkin discretization with stable multiscale bases (e.g., wavelet bases) we refer to [11]. There, an efficient and reliable error estimate similar to (1.4) is developed but with considerably more than n error indicators.

Asymptotically exact error indicators, i.e.,

$$\frac{\sum_{\nu=1}^n \varepsilon_\nu^2}{\|u_{\mathcal{G}} - u\|_{L^2(\Gamma)}^2} \quad \longrightarrow \quad 1 \qquad \text{for } h_\triangle \to 0\,,$$

are presented in [13] for arbitrary meshes and for integral operators of the second kind $A = I - K : L^2(\Gamma) \to L^2(\Gamma)$ with a compact operator $K : L^2(\Gamma) \to L^2(\Gamma)$.

For more general operators, we presented in [15–17] local error indicators for the Galerkin discretization, which are reliable for shape regular meshes (see Remark 3.7, suitable for adaptive refinement) and efficient for arbitrary meshes. In the cited papers, the results were only formulated for the h-version, but they also hold for the hp-version. The efficiency could be shown only for integral operators $A : H^\alpha(\Gamma) \to H^{-\alpha}(\Gamma)$ with $\alpha > -\frac{1}{2}$ (i.e., in particular for the case (1.2)).

In [19], we presented two error indicators for the case (1.3) (which includes operators $A : H^\alpha(\Gamma) \to H^{-\alpha}(\Gamma)$ of negative order $2\alpha < 0$). The first indicators are efficient for arbitrary meshes and reliable for shape regular meshes, and the second indicators are efficient for shape regular meshes and reliable for arbitrary meshes. Both error indicators are based on the computable residual $r := Au_{\mathcal{G}} - g$ and can be used for controlling the adaptive refinement as described above. The results in [19] were only shown for the two-dimensional case, where $\Gamma = \partial\Omega$ is a curve in $\mathbb{R}^2$. The aim of this paper is to extend the results of [19] to the three-dimensional case.

We emphasize that inverse estimates or saturation properties are not needed, either in this paper or in [15–17,19]. To our knowledge these are the first approaches for integral operators of the first kind with n efficient and reliable local error indicators $\{\varepsilon_\nu\}_{\nu=1}^n$, which avoid inverse estimates and saturation properties.

The proofs of efficiency respectively reliability in this paper and in [19] are based on localization results for the Aronszajn-Slobodeckij norm $\|\cdot\|_{H^s(\Gamma)}$ of non-integer order s. These localization results are not only useful for adaptive boundary element methods, they also can be used to show inverse estimates for non-uniform meshes (see [12]). Moreover, they are interesting in their own right. Therefore, we present one of them in the following.

In this paper, let Γ be the Lipschitz boundary of a bounded and simply connected domain $\Omega \subseteq \mathbb{R}^3$. For $s \in (0,1)$ and $\Gamma' \subseteq \Gamma$ the Aronszajn-Slobodeckij norm is given by

$$\|v\|^2_{H^s(\Gamma')} \;=\; \|v\|^2_{L^2(\Gamma')} + |v|^2_{H^s(\Gamma')}$$

with the semi-norm

$$|v|^2_{H^s(\Gamma')} \;:=\; \int_{\Gamma'}\int_{\Gamma'} \frac{|v(\xi)-v(\eta)|^2}{|\xi-\eta|^{2+2s}}\, d\xi\, d\eta\,. \tag{1.5}$$

More details about the norm definition can be found after (3.1). We also denote $\|\cdot\|_{H^s(\Gamma')}$ as “double integral norm” because of (1.5).

On the surface Γ, we introduce a mesh $\triangle$ as a partition of Γ into closed and possibly curved triangular elements $\tau \in \triangle$, i.e., $\Gamma \subseteq \cup\{\tau \mid \tau \in \triangle\}$ (details are given in Sect. 2). The set of nodal points of the mesh $\triangle$ is denoted by $\mathcal{N}_\triangle$. For a mesh point $q \in \mathcal{N}_\triangle$, we introduce the neighbourhood

$$\omega_q \;:=\; \bigcup\{\tau \in \triangle \mid\; q \in \tau\} \;\subseteq\; \Gamma\,. \tag{1.6}$$

For $k \in \mathbb{N}_0$, the global norm $\|\cdot\|_{H^k(\Gamma)}$ is additive, i.e.,

$$\|v\|^2_{H^k(\Gamma)} \;=\; \sum_{\tau\in\triangle} \|v\|^2_{H^k(\tau)} \qquad \text{for any } v \in H^k(\Gamma)\,. \tag{1.7}$$

This property fails to hold for the double integral norm of non-integer order: more precisely, for $s \in (0,1)$ and for $v \in H^s(\Gamma)$, we have

$$\|v\|^2_{H^s(\Gamma)} \;=\; \sum_{\tau\in\triangle} \|v\|^2_{H^k(\tau)} \;+\; |v|^2_{H^s(\Gamma)}\,,$$

where

$$\begin{aligned} |v|^2_{H^s(\Gamma)} \;&=\; \sum_{\tau\in\triangle} \int_\tau\int_\Gamma \frac{|v(\xi)-v(\eta)|^2}{|\xi-\eta|^{2+2\sigma}}\,d\xi\,d\eta \\ &=\; \sum_{\tau\in\triangle} |v|^2_{H^s(\tau)} \;+\; \underbrace{\sum_{\tau\in\triangle} \int_\tau\int_{\Gamma\setminus\tau} \frac{|v(\xi)-v(\eta)|^2}{|\xi-\eta|^{2+2\sigma}}\,d\xi\,d\eta}_{=:\,p(v,\triangle)}\,. \end{aligned} \tag{1.8}$$

Unfortunately, it is not even possible to estimate the perturbing term $p(v,\triangle)$ in (1.8) in terms of $\sum_{\tau\in\triangle}\|v\|^2_{H^s(\tau)}$ (see [18, Satz 3.1]).

Nevertheless, it is possible to localize the global norm $\|r\|_{H^s(\Gamma)}$ (i.e., to estimate $\|r\|^2_{H^s(\Gamma)}$ by a sum of local norms) for certain functions $r \in H^s(\Gamma)$, if one replaces the partition $\triangle$ of Γ by the overlapping sets $\{\omega_q\}_{q\in\mathcal{N}_\triangle}$ with small overlap zones. In Sect. 3 and 4, we will show: for $s \in (0,1)\cup(1,2)$, there is a constant C such that the estimate

$$\|r\|^2_{H^s(\Gamma)} \;\leq\; C \sum_{q\in\mathcal{N}_\triangle} |r|^2_{H^s(\omega_q)} \tag{1.9}$$

holds for all shape regular meshes $\triangle$ and for any function $r \in H^s(\Gamma)$ which is orthogonal to a minimal set of finite element functions. The constant C in (1.9) depends only on s, on a lower bound κ for the angles of $\triangle$ and on the smoothness of the finite element functions. In Sect. 5 we will apply (1.9) to the Galerkin residual $r := Au_{\mathcal{G}} - g \in H^s(\Gamma)$ which is orthogonal to the Galerkin trial space $\mathcal{G}$.

An outline of this paper is as follows. In Sect. 2, we describe in detail the surface Γ, its parametrizations and the mesh $\triangle$. The localization of the double integral norm (1.9) will be proven for $s \in (0,1)$ in Sect. 3 and for $s \in (1,2)$ in Sect. 4. Based on the analysis in Sect. 3 and 4, we introduce in Sect. 5 reliable and efficient error indicators for the Galerkin discretization of problem (1.1) for the case (1.3).

2 The surface Γ and the finite element spaces

Throughout this paper we assume that Γ is a Lipschitz boundary (i.e., $\Gamma \in C^{0,1}$) and that Γ is parameterized by the surface of a polyhedron $\hat{\Gamma} \subseteq \mathbb{R}^3$ via a bijective mapping

$$\gamma : \hat{\Gamma} \to \Gamma \,.$$

The closed and plane polygonal faces of $\hat{\Gamma}$ are denoted by $P_1, \ldots, P_M$ (i.e., $\hat{\Gamma} = \cup_{\mu=1}^{M} P_\mu$). We assume that γ and its inverse γ^{-1} are Lipschitz continuous and that the restrictions $\gamma|_{P_\mu}$ are two times differentiable (i.e., $\gamma|_{P_\mu} \in C^2$). Then, the smooth and closed surface components $\Gamma_\mu := \gamma(P_\mu)$ have pairwise disjoint interior and satisfy $\Gamma = \cup_{\mu=1}^{M} \Gamma_\mu$. Without loss of generality, each face P_μ can be identified with a polygonal closed set in $\mathbb{R}^2$. Then, the surface integral of a measurable function $v : \Gamma \to \mathbb{R}$ is defined by

$$\int_\Gamma v(\xi)\, d\xi \;:=\; \sum_{\mu=1}^{M} \int_{P_\mu} v(\gamma(x)) \sqrt{G_\mu(x)}\, dx \,,$$

with the Gram determinant $G_\mu := \det(\langle \partial_i \gamma, \partial_j \gamma \rangle)_{i,j=1}^2$ of the differentiable function $\gamma|_{P_\mu}$. We assume that G_μ is bounded, i.e., there are constants $\underline{C}_G, \overline{C}_G > 0$ with

$$\text{(2.1)} \qquad \underline{C}_G \;\le\; \sqrt{G_\mu(x)} \;\le\; \overline{C}_G \qquad \text{for all } x \in P_\mu\,,\ \mu \in \{1, \ldots, M\}\,.$$

Furthermore, let C_Γ be a Lipschitz constant with the following property

$$\text{(2.2)} \qquad \frac{|\gamma(x) - \gamma(y)|}{|x - y|} \;\le\; C_\Gamma \quad \text{for all } \underset{\neq}{x, y} \in P_\mu\,,\ \mu \in \{1, \ldots, M\}\,.$$

For a closed and connected polygonal set $D \subseteq \mathbb{R}^2$ we introduce an admissible mesh $\triangle_D$ which is a collection of closed triangles $T \subseteq D$ satisfying the following properties: $D = \cup\{T \,|\, T \in \triangle_D\}$ and the intersection $T \cap T'$ of each distinct pair $T, T' \in \triangle_D$ is either empty or a common vertex or a common edge of both elements T and T'. The set of nodal points of the mesh $\triangle_D$ is denoted by $\mathcal{N}_{\triangle_D}$. For a mesh point $p \in \mathcal{N}_{\triangle_D}$ and for an element $T \in \triangle_D$, we introduce the neighbourhood (see Figure 2.1)

$$\text{(2.3)} \qquad \begin{aligned} \omega_p &:= \bigcup \{T' \in \triangle_D \mid\ p \in T'\} \qquad \text{and} \\ \omega_T &:= \bigcup \{T' \in \triangle_D \mid\ T' \cap T \neq \emptyset\} \end{aligned}$$

and the distance

$$\text{(2.4)} \qquad d_T \;:=\; dist(T, D \setminus \omega_T) \;>\; 0$$

For $\mu \in \{1, \ldots, M\}$, let $\triangle_\mu$ be an admissible mesh on P_μ consisting of closed planar triangles $T \subseteq P_\mu$ and let $\mathcal{N}_{\triangle_\mu}$ be the corresponding set of mesh points. We assume that the meshes $\{\triangle_\mu\}_{\mu=1}^{M}$ fit together in the following sense: for any nodal point p of $\triangle_\mu$ which satisfies $p \in \partial P_\mu \cap \partial P_l$

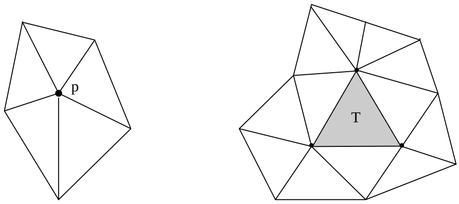

Fig. 2.1. ω_p and ω_T

($\mu, l \in \{1, \ldots, M\}$, $\mu \neq l$), we have that p is also a nodal point of $\triangle_l$. This assumption implies that

(2.5) $$\triangle \;=\; \{\tau = \gamma(T) \mid \; T \in \triangle_\mu \, , \, \mu \in \{1, \ldots, M\} \, \}$$

is an admissible mesh (consisting of curvilinear triangles) on the surface Γ . The mesh size of $\triangle$ is defined by

(2.6) $$h_\triangle \;:=\; \max_{\tau \in \triangle} \, diam(\tau) \, .$$

As in Sect. 1, let $\mathcal{N}_\triangle$ be the set of nodal points of the mesh $\triangle$, and the neighbourhood $\omega_q \subseteq \Gamma$ of $q \in \mathcal{N}_\triangle$ is defined in (1.6). For an element $\tau \in \triangle$, we introduce the neighbourhood

(2.7) $$\omega_\tau \;:=\; \bigcup \{\tau' \in \triangle \mid \; \tau' \cap \tau \neq \emptyset\} \;\subseteq\; \Gamma$$

and the distance

(2.8) $$d_\tau \;:=\; dist(\tau \, , \, \Gamma \setminus \omega_\tau) \;>\; 0$$

(where d_τ is the distance in $\mathbb{R}^3$).

The following definitions introduce piecewise polynomials and special finite element spaces $H^m_{\triangle_D}(D)$ and $H^m_\triangle(\Gamma)$ for $m \in \{0, 1, 5, 6\}$ on $D \subseteq \mathbb{R}^2$ and on the surface Γ .

Definition 2.1 *Let* $D \subseteq \mathbb{R}^2$ *be a closed and connected polygonal set and let* $\triangle_D$ *be an admissible mesh on* D *.*

a) *The space of piecewise polynomials associated with* $\triangle_D$ *is denoted by*

$$\mathbb{P}_{\triangle_D}(D) := \{v : D \to \mathbb{R} \mid \; v|_T \text{ is polynomial for all } T \in \triangle_D\}.$$

Since flexible local polynomial degrees are important for hp*-methods, we introduce on any element* $T \in \triangle_D$ *a local polynomial degree* $\delta_T \in \mathbb{N}_0$ *and define for the degree "vector"* $\delta = (\delta_T)_{T \in \triangle_D} \in \mathbb{N}_0^{\triangle_D}$ *the space*

$$\mathbb{P}^\delta_{\triangle_D}(D) \;:=\; \{v \in \mathbb{P}_{\triangle_D}(D) \mid \; v|_T \text{ has degree } \leq \delta_T \text{ for all } T \in \triangle_D\} \, .$$

For $m \in \mathbb{N}_0$*, let* $\mathbb{P}^m_{\triangle_D}(D) := \mathbb{P}^\delta_{\triangle_D}(D)$*, where* $\delta = (\delta_T)_{T\in\triangle_D}$ *is the constant degree vector with* $\delta_T = m$*.*

b) *The characteristic function of an element* $T \in \triangle_D$ *is denoted by* $\varphi^{[0]}_T : D \to \mathbb{R}$*.*

c) *For* $p \in \mathcal{N}_{\triangle_D}$*, we introduce* $\varphi^{[1]}_p : D \to \mathbb{R}$ *as the piecewise linear and continuous hat function characterized by* $\varphi^{[1]}_p(p) = 1$ *and* $\mathrm{supp}(\varphi^{[1]}_p) = \omega_p$*.*

Moreover, we introduce $\varphi^{[5]}_p : D \to \mathbb{R}$ *as the Argyris element (see [10, Theorem 2.2.11]) characterized by the following conditions:* $\varphi^{[5]}_p(p) = 1$*,* $\varphi^{[5]}_p(p')$ *vanishes in the other mesh points* $p' \in \mathcal{N}_{\triangle_D} \setminus \{p\}$*, its derivatives of order* $k \in \{1, 2\}$ *vanish in all mesh points* $p' \in \mathcal{N}_{\triangle_D}$ *and its normal derivatives also vanish in the midpoints of all edges of the mesh.*

Argyris elements are locally polynomials of degree 5 *on any element of the mesh* $\triangle_D$ *and globally* C^1*-functions. The support of* $\varphi^{[5]}_p$ *is* ω_p *(due to the above mentioned conditions).*

d) *Let* $T \in \triangle_D$ *and let* p_0, p_1, p_2 *be the vertices of the triangle* T*. Then, we define the bubble function* $\varphi^{[6]}_T : D \to \mathbb{R}$ *by* $\varphi^{[6]}_T := 3^6(\varphi^{[1]}_{p_0}\varphi^{[1]}_{p_1}\varphi^{[1]}_{p_2})^2$*, where* $\varphi^{[1]}_{p_j}$ *is the hat function introduced in c). This bubble function is locally a polynomial of degree* 6 *on any element of the mesh* $\triangle_D$ *and globally a* C^1*-functions with support in* T*.*

e) *The finite element space* $H^m_{\triangle_D}(D) \subseteq \mathbb{P}^m_{\triangle_D}(D)$ *is defined by*

$$H^m_{\triangle_D}(D) := span\{\varphi^{[m]}_T\}_{T\in\triangle_D} \qquad \textit{for } m \in \{0, 6\}$$

and

$$H^m_{\triangle_D}(D) := span\{\varphi^{[m]}_p\}_{p\in\mathcal{N}_{\triangle_D}} \qquad \textit{for } m \in \{1, 5\}.$$

Definition 2.2 *Let* $\triangle$ *be a mesh on the surface* Γ *(as in (2.5)) and let* $\triangle_\mu$ *the corresponding mesh on the planar face* P_μ *for* $\mu \in \{1, \ldots, M\}$*.*

a) *The space of piecewise polynomials associated with* $\triangle$ *is denoted by*

$$\mathbb{P}_\triangle(\Gamma) := \{v : \Gamma \to \mathbb{R} \mid v \circ \gamma|_T \text{ is polynomial for all } T \in \triangle_\mu \text{ and } \mu \in \{1, \ldots, M\}\}.$$

For a degree "vector" $\delta = (\delta_\tau)_{\tau\in\triangle} \in \mathbb{N}_0^\triangle$*, we define*

$$\mathbb{P}^\delta_\triangle(\Gamma) := \{v \in \mathbb{P}_\triangle(\Gamma) \mid v \circ \gamma|_T \text{ has degree } \leq \delta_\tau \text{ for all } \tau \in \triangle, \text{ where } \tau = \gamma(T) \text{ with } T \in \triangle_\mu,\ \mu \in \{1, \ldots, M\}\}$$

and analogously we have $\mathbb{P}^m_\triangle(\Gamma)$ *for* $m \in \mathbb{N}_0$*.*

b) *For* $\mu \in \{1, \ldots, M\}$*,* $T \in \triangle_\mu$ *and* $p \in \mathcal{N}_{\triangle_\mu}$ *let*

$$\varphi^{[m]}_T : \hat{\Gamma} \to \mathbb{R} \quad \textit{for } m \in \{0, 6\} \quad \textit{and} \quad \varphi^{[m]}_p : \hat{\Gamma} \to \mathbb{R} \quad \textit{for } m \in \{1, 5\}$$

be the finite element functions introduced in Definition 2.1. We introduce for the surface element $\tau := \gamma(T) \in \triangle$ *and the surface nodal point* $q := \gamma(p) \in \mathcal{N}_\triangle$ *the finite element functions* $\Phi_\tau^{[m]} : \Gamma \to \mathbb{R}$ *(* $m \in \{0,6\}$ *) and* $\Phi_q^{[m]} : \Gamma \to \mathbb{R}$ *(* $m \in \{1,5\}$ *) by*

$$\Phi_\tau^{[m]}(\xi) \ := \ \varphi_T^{[m]}(x) \quad \text{and} \quad \Phi_q^{[m]}(\xi) \ := \ \varphi_p^{[m]}(x) \qquad \text{for } \ \xi = \gamma(x)\,,\ x \in \hat{\Gamma}\,. \tag{2.9}$$

c) *The finite element space* $H_\triangle^m(\Gamma) \subseteq \mathbb{P}_\triangle^m(\Gamma)$ *is defined by*

$$H_\triangle^m(\Gamma) \ := \ span\{\Phi_\tau^{[m]}\}_{\tau\in\triangle} \ \subseteq \ \mathbb{P}_\triangle^m(\Gamma) \qquad \text{for } \ m \in \{0,6\}$$

and

$$H_\triangle^m(\Gamma) \ := \ span\{\Phi_q^{[m]}\}_{q\in\mathcal{N}_\triangle} \ \subseteq \ \mathbb{P}_\triangle^m(\Gamma) \qquad \text{for } \ m \in \{1,5\}\,.$$

The space $H_\triangle^0(\Gamma)$ consists of discontinuous functions, $H_\triangle^1(\Gamma)$ consists of continuous functions, and $H_\triangle^m(\Gamma)$, $m \in \{5,6\}$, consists of C^1-functions.

We end this section with two lemmata which will be needed for the localization of the double integral norm.

Lemma 2.3 *For* $\lambda > 0$ *and for all* $y \in \mathbb{R}^2$ *and* $\varepsilon > 0$*, we have*

$$\int_{\mathbb{R}^2\setminus B_\varepsilon(y)} |y-x|^{-2-\lambda}\,dx \ \leq \ \frac{2\pi}{\lambda}\,\varepsilon^{-\lambda}\,, \tag{2.10}$$

where $B_\varepsilon(y) \subseteq \mathbb{R}^2$ *is the ball with radius* ε *centred at* y*.*

The elementary proof of Lemma 2.3 only uses polar coordinates. A result similar to (2.10) also holds for the two-dimensional manifold Γ.

Lemma 2.4 *For* $\lambda > 0$ *there is a constant* C_λ *(only depending on* λ *and on the geometry of* Γ *) such that*

$$\int_{\Gamma\setminus B_\varepsilon(z)} |z-\xi|^{-2-\lambda}\,d\xi \ \leq \ C_\lambda\,\varepsilon^{-\lambda} \qquad \text{for all } \ z \in \mathbb{R}^3 \ \text{and all } \ \varepsilon > 0\ ,$$

where $B_\varepsilon(z)$ *is now the ball in* $\mathbb{R}^3$*.*

The proof of Lemma 2.4 can be found in [21, Lemma 8.2.4]. The only assumption needed for the proof is that the Lipschitz boundary Γ is almost everywhere differentiable.

3 Localization of the norm $\|\cdot\|_{H^s(\Gamma)}$ for $s \in (0,1)$

For $s \in (0,1)$ and $\Gamma' \subseteq \Gamma$, the double integral norm is given by

$$\|v\|^2_{H^s(\Gamma')} \;=\; \|v\|^2_{H^0(\Gamma')} + |v|^2_{H^s(\Gamma')} \tag{3.1}$$

with

$$\begin{aligned}\|v\|^2_{H^0(\Gamma')} &= \|v\|^2_{L^2(\Gamma')} = \int_{\Gamma'} |v(\xi)|^2\, d\xi \\ &= \sum_{\mu=1}^{M} \int_{\gamma^{-1}(\Gamma')\cap P_\mu} |v(\gamma(x))|^2 \sqrt{G_\mu(x)}\, dx\end{aligned}$$

and with the semi-norm

$$\begin{aligned}|v|^2_{H^s(\Gamma')} &:= \int_{\Gamma'}\int_{\Gamma'} \frac{|v(\xi)-v(\eta)|^2}{|\xi-\eta|^{2+2s}}\, d\xi\, d\eta \\ &= \sum_{\mu,m=1}^{M} \int_{P_{m,\Gamma'}}\int_{P_{\mu,\Gamma'}} \frac{|v(\gamma(x))-v(\gamma(y))|^2}{|\gamma(x)-\gamma(y)|^{2+2s}} \sqrt{G_\mu(x)}\, dx \sqrt{G_m(y)}\, dy\,.\end{aligned}$$

with $P_{\mu,\Gamma'} := \gamma^{-1}(\Gamma') \cap P_\mu$.

There are other possibilities to define the Sobolev norm of non-integer order s. One could define the global norm $\|\cdot\|_{H^s(\Gamma)}$, for $s \in \mathbb{R}_{>0} \setminus \mathbb{N}$, for example, by means of the growth of the Fourier transform or by means of interpolation theory. All these global norms are equivalent (see, e.g., [2], [22, Theorem 8.5]). The local double integral norm $\|\cdot\|_{H^s(\Gamma')}$, $\Gamma' \subset \Gamma$ (given by (3.1)), the local Fourier norm (given as the minimal norm of an extension) and the local interpolation norm are also equivalent, but with equivalence constants depending on Γ'.

The norm definition via (3.1) has an important advantage in comparison to the other definitions: the local double integral norm $\|\cdot\|_{H^s(\Gamma')}$ is approximately computable using quadrature rules, whereas the local Fourier norm and the local interpolation norm are not computable.

The aim of this section is to prove the localization of the double integral norm (1.9) for $s \in (0,1)$. The proof consists of two main steps. In a first localization step, the global norm $\|v\|_{H^s(\Gamma)}$ $(v \in H^s(\Gamma))$ is estimated by a sum of local semi-norms and weighted local L^2-norms (see Corollary 3.3). In a second step, the perturbing weighted local L^2-norms will be estimated by Poincaré-type inequalities (see Lemma 3.8).

In the estimations of this section, we try to determine the constants as exactly as possible. If these constants are known exactly then the reliability constant C^{rel} in the upper estimate of (1.4) is also known. Exact knowledge of the reliability constant C^{rel} is important for the stopping criterion in the adaptive mesh refinement process and for evaluating the quality of the Galerkin solution $u_{\mathcal{G}}$.

Lemma 3.1 *Let* $s \in (0,1)$. *Then, we have for any function* $v \in H^s(\Gamma)$ *and all meshes* $\triangle$ *on* Γ *(see (2.5)) that*

$$|v|^2_{H^s(\Gamma)} \leq \sum_{\tau\in\triangle} \Big[\int_\tau \int_{\omega_\tau} \frac{|v(\xi)-v(\eta)|^2}{|\xi-\eta|^{2+2s}}\, d\xi\, d\eta + 4\, C_{2s}\, d_\tau^{-2s} \|v\|^2_{L^2(\tau)} \Big] \tag{3.2}$$

$$\leq \sum_{q\in\mathcal{N}_\triangle} |v|^2_{H^s(\omega_q)} + 4\, C_{2s} \sum_{\tau\in\triangle} d_\tau^{-2s} \|v\|^2_{L^2(\tau)}, \tag{3.3}$$

where ω_q, ω_τ *and* d_τ *are introduced in (1.6), (2.7) and (2.8) and where* C_{2s} *is the constant introduced in Lemma 2.4 (only depending on* s *and* Γ*).*

It is essential in the proof of Lemma 3.1 that the sets $\{\omega_q\}_{q\in\mathcal{N}_\triangle}$ form an overlapping covering of Γ.

Proof. Using the abbreviations $D_\tau := \cup\{\tau' \mid \tau' \in \triangle \text{ with } \tau\cap\tau' = \emptyset\} = \Gamma \setminus inn(\omega_\tau)$ and

$$\int_{\Gamma'}\int_{\Gamma''} := \int_{\Gamma'}\int_{\Gamma''} \frac{|v(\xi)-v(\eta)|^2}{|\xi-\eta|^{2+2s}}\, d\xi\, d\eta \qquad \text{for } \Gamma', \Gamma'' \subseteq \Gamma,$$

we obtain

$$|v|^2_{H^s(\Gamma)} = \int_\Gamma\int_\Gamma = \sum_{\tau\in\triangle}\int_\tau\int_\Gamma = \sum_{\tau\in\triangle}\Big[\int_\tau\int_{\omega_\tau} + \int_\tau\int_{D_\tau}\Big], \tag{3.4}$$

where

$$\begin{aligned} \int_\tau\int_{D_\tau} &:= \int_\tau\int_{D_\tau} |v(\xi)-v(\eta)|^2 |\xi-\eta|^{-2-2s}\, d\xi\, d\eta \\ &\leq 2\int_\tau |v(\eta)|^2 \Big(\int_{D_\tau} |\xi-\eta|^{-2-2s}\, d\xi\Big) d\eta \\ &\qquad + 2\int_{D_\tau} |v(\xi)|^2 \Big(\int_\tau |\xi-\eta|^{-2-2s}\, d\eta\Big) d\xi \\ &=: 2\, J_{\tau,1} + 2\, J_{\tau,2}\,. \end{aligned} \tag{3.5}$$

First, we show that

$$\sum_{\tau\in\triangle} J_{\tau,1} \;=\; \sum_{\tau\in\triangle} J_{\tau,2}\,.$$

For that, we infer with the characteristic function χ_{D_τ} of D_τ

$$\begin{aligned}
\sum_{\tau\in\triangle} J_{\tau,2} &\overset{(3.5)}{=} \sum_{\tau\in\triangle} \int_{D_\tau} |v(\xi)|^2 \Big(\int_\tau |\xi-\eta|^{-2-2s}\, d\eta \Big) d\xi \\
&= \sum_{\tau\in\triangle} \int_\Gamma \chi_{D_\tau}(\xi)\, |v(\xi)|^2 \Big(\int_\tau |\xi-\eta|^{-2-2s}\, d\eta \Big) d\xi \\
&= \int_\Gamma |v(\xi)|^2 \, \underbrace{\Big(\sum_{\tau\in\triangle} \chi_{D_\tau}(\xi) \int_\tau |\xi-\eta|^{-2-2s}\, d\eta \Big)}_{=:\, f(\xi)} d\xi \\
&= \sum_{\tau'\in\triangle} \int_{\tau'} |v(\xi)|^2 \, f(\xi)\, d\xi\,. \qquad (3.6)
\end{aligned}$$

Let $\tau'\in\triangle$ be fixed and let ξ be an interior point of τ'. Then, we obtain for any $\tau\in\triangle$

$$\begin{aligned}
\chi_{D_\tau}(\xi) &= \begin{cases} 1 & \text{if } \xi\in D_\tau = \cup\{\tau'' \mid \tau''\in\triangle \text{ with } \tau\cap\tau''=\emptyset\,\} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \tau\cap\tau'=\emptyset \\ 0 & \text{otherwise} \end{cases} .
\end{aligned}$$

This implies for f given in (3.6) and for any interior point ξ of τ' that

$$f(\xi) \;=\; \sum_{\substack{\tau\in\triangle\\ \tau\cap\tau'=\emptyset}} \int_\tau |\xi-\eta|^{-2-2s}\, d\eta \;=\; \int_{D_{\tau'}} |\xi-\eta|^{-2-2s}\, d\eta\,.$$

Inserting this into (3.6) shows

$$\sum_{\tau\in\triangle} J_{\tau,2} \;=\; \sum_{\tau'\in\triangle} \int_{\tau'} |v(\xi)|^2 \Big(\int_{D_{\tau'}} |\xi-\eta|^{-2-2s}\, d\eta \Big) d\xi \;\overset{(3.5)}{=}\; \sum_{\tau'\in\triangle} J_{\tau',1}\,,$$

which yields together with (3.4) and (3.5) that

$$\begin{aligned}
|v|^2_{H^s(\Gamma)} &\le \sum_{\tau\in\triangle} \Big[\int_\tau\int_{\omega_\tau} + 4\, J_{\tau,1} \Big] \qquad (3.7)\\
&= \sum_{\tau\in\triangle} \Big[\int_\tau\int_{\omega_\tau} +\; 4 \int_\tau |v(\eta)|^2 \Big(\int_{D_\tau} |\xi-\eta|^{-2-2s}\, d\xi \Big) d\eta \Big]\,.
\end{aligned}$$

The definitions of D_τ and d_τ imply that $D_\tau \subseteq \Gamma \setminus B_{d_\tau}(\eta)$ for any $\eta \in \tau$. This together with Lemma 2.4 leads to

$$\text{(3.8)} \qquad \int_{D_\tau} |\xi - \eta|^{-2-2s}\, d\xi \quad \leq \quad \int_{\Gamma \setminus B_{d_\tau}(\eta)} |\xi - \eta|^{-2-2s}\, d\xi \quad \overset{L.2.4}{\leq} \quad C_{2s}\, d_\tau^{-2s}\,.$$

Hence, assertion (3.2) follows from (3.7) and (3.8). The assertion (3.3) follows from

$$\sum_{\tau \in \triangle} \int_\tau \int_{\omega_\tau} = \sum_{\substack{\tau,\tau' \in \triangle \\ \tau \cap \tau' \neq \emptyset}} \int_\tau \int_{\tau'} \leq \sum_{q \in \mathcal{N}_\triangle} \sum_{\tau,\tau' \subseteq \omega_q} \int_\tau \int_{\tau'} = \sum_{q \in \mathcal{N}_\triangle} \int_{\omega_q} \int_{\omega_q},$$

where the inequality is a direct consequence of the following fact: for any $\tau, \tau' \in \triangle$ with $\tau \cap \tau' \neq \emptyset$ there is at least one nodal point $q \in \mathcal{N}_\triangle$ with $\tau, \tau' \subseteq \omega_q$. □

Analogously to Lemma 3.1, one can estimate the semi-norm $|\cdot|_{H^s(D)}$ for $D \subseteq \mathbb{R}^2$.

Lemma 3.2 *Let $s \in (0,1)$ and let $D \subseteq \mathbb{R}^2$ be a closed and connected polygonal set. Then, we have for any function $w \in H^s(D)$ and all admissible meshes $\triangle_D$ on D that*

$$|w|^2_{H^s(D)} \leq \sum_{T \in \triangle_D} \Big[\int_T \int_{\omega_T} \frac{|w(x) - w(y)|^2}{|x-y|^{2+2s}} dx\, dy \; + \; \frac{4\pi}{s}\, d_T^{-2s} \|w\|^2_{L^2(T)} \Big]$$

$$\text{(3.9)} \qquad \leq \sum_{p \in \mathcal{N}_{\triangle_D}} |w|^2_{H^s(\omega_p)} \; + \; \frac{4\pi}{s} \sum_{T \in \triangle_D} d_T^{-2s} \|w\|^2_{L^2(T)}\,,$$

where ω_p, ω_T and d_T are introduced in (2.3) and (2.4).

The proofs of Lemma 3.1 and 3.2 only distinguish in (3.8), where one has to use Lemma 2.3 instead of Lemma 2.4.

As a consequence of Lemma 3.1 we obtain

Corollary 3.3 *Let $s \in (0,1)$. Then, we have for any function $v \in H^s(\Gamma)$ and all meshes $\triangle$ on Γ that*

$$\text{(3.10)} \qquad \|v\|^2_{H^s(\Gamma)} \leq \sum_{\tau \in \triangle} \Bigg[\int_\tau \int_{\omega_\tau} \frac{|v(\xi) - v(\eta)|^2}{|\xi - \eta|^{2+2s}}\, d\xi\, d\eta \; + \; C_s^{loc}\, d_\tau^{-2s} \|v\|^2_{L^2(\tau)} \Bigg]$$

$$\text{(3.11)} \qquad \leq \sum_{q \in \mathcal{N}_\triangle} |v|^2_{H^s(\omega_q)} \; + \; C_s^{loc} \sum_{\tau \in \triangle} d_\tau^{-2s} \|v\|^2_{L^2(\tau)}$$

with a constant C_s^{loc} *independent of* v *and* $\triangle$. *More precisely,* C_s^{loc} *is an upper bound of* $h_\triangle^{2s} + 4C_{2s}$, *where* C_{2s} *is given by Lemma 2.4.*

For further estimates of $\|v\|_{H^s(\Gamma)}$, it would be useful if one could estimate the perturbation terms $d_\tau^{-2s}\|v\|^2_{L^2(\tau)}$ on the right-hand side of (3.10) and (3.11) in the following way

$$d_\tau^{-2s}\|v\|^2_{L^2(\tau)} \;\leq\; const\,\|v\|^2_{H^s(\tau')}\,, \tag{3.12}$$

where τ' is a small neighbourhood of τ with a constant independent of $v \in H^s(\Gamma)$ and independent of shape regular meshes $\triangle$. Unfortunately, (3.12) does not hold for arbitrary functions v: the constant function $v = 1$ is a simple counterexample since

$$\frac{d_\tau^{-2s}\|v\|^2_{L^2(\tau)}}{\|v\|^2_{H^s(\tau')}} = \frac{d_\tau^{-2s} area(\tau)}{area(\tau')} \overset{h_\triangle \to 0}{\longrightarrow} \infty\,, \quad \text{if } \frac{area(\tau')}{area(\tau)} \text{ is bounded.}$$

One can show (3.12) only for functions $v \in H^s(\Gamma)$ which satisfy some additional conditions. In the following Lemmata we will show an estimate similar to (3.12) for functions $v \in H^s(\Gamma)$ being orthogonal to certain finite element functions.

Lemma 3.4 *Let* $S \subseteq \mathbb{R}^2$ *be a polygonal domain. Then, we obtain for* $s \in (0,1)$ *and any function* $w \in H^s(S)$ *that*

(3.13)
$$\|w\|^2_{L^2(S)} \leq \frac{1}{2}\,\frac{diam(S)^{2+2s}}{area(S)}\,|w|^2_{H^s(S)} + \frac{1}{area(S)}\left|\int_S w(x)\,dx\right|^2.$$

Let $T \subseteq \mathbb{R}^2$ *be a triangle. Then, we obtain for any function* $w \in H^1(T)$

$$\|w\|_{L^2(T)} \;\leq\; \frac{1}{\pi}\,diam(T)\,|w|_{H^1(T)} \tag{3.14}$$

provided that $\int_T w(x)\,dx = 0$.

Estimate (3.14) is denoted as the Poincaré inequality in [25] and [24].

Proof. The proof of (3.14) is given in [25] for convex domains T. Using $|w(x) - w(y)|^2 = w(x)^2 + w(y)^2 - 2w(x)w(y)$ for $x, y \in S$, we obtain, taking $J := \int_S w(x)\,dx$, that

$$\begin{aligned}
&\int_S\int_S |w(x) - w(y)|^2\,dx\,dy \\
&= \int_S\int_S w(x)^2\,dx\,dy + \int_S\int_S w(y)^2\,dx\,dy - 2\int_S w(x)\underbrace{\int_S w(y)\,dy}_{=J}\,dx \\
&= 2\,area(S)\int_S w(x)^2\,dx - 2\,J^2\,.
\end{aligned}$$

Hence, (3.13) follows from

$$2\; area(S) \|w\|^2_{L^2(S)} - 2\, J^2 = \int_S \int_S \frac{|w(x)-w(y)|^2}{|x-y|^{2+2s}} \; \underbrace{|x-y|^{2+2s}}_{\leq\, diam(S)^{2+2s}} \; dx\, dy$$
$$\leq diam(S)^{2+2s}\, |w|^2_{H^s(S)}\,.$$

□

In the following let $\triangle$ be a mesh on the surface Γ (as in (2.5)) and let $\triangle_\mu$ the corresponding mesh on the planar face P_μ (identified with with a polygonal closed set in $\mathbb{R}^2$) for $\mu \in \{1,\ldots,M\}$.

Lemma 3.5 *Let* $m \in \{0,1,5,6\}$. *Then, for* $T \in \triangle_\mu$ *and* $p \in \mathcal{N}_{\triangle_\mu}$, *the finite element function* $\varphi_T^{[m]} : P_\mu \to \mathbb{R}$ *respectively* $\varphi_p^{[m]} : P_\mu \to \mathbb{R}$ *(introduced in Definition 2.1) satisfies*

$$\text{(3.15)} \quad \int_T |1-\varphi_T^{[m]}(x)|^2\, dx \;\leq\; (1-C_m^{FE})\, area(T) \quad \textit{for } m \in \{0,6\}$$

and

$$\text{(3.16)} \quad \int_{\omega_p} |1-\varphi_p^{[m]}(x)|^2\, dx \;\leq\; (1-C_m^{FE})\, area(\omega_p) \quad \textit{for } m \in \{1,5\}$$

with the constants

$$C_0^{FE} \;=\; 1\,, \quad C_1^{FE} \;=\; \frac{1}{2}\,, \quad C_5^{FE} \;=\; \frac{3}{25}\,, \quad C_6^{FE} \;=\; \frac{2}{5}\,.$$

These estimates holds for arbitrary meshes $\triangle_\mu$ *in the case* $m \in \{0,1,6\}$. *For* $m = 5$, *we have the following restriction for the mesh: the ratio of the length of neighbouring edges has to be bounded. More precisely, (3.16) holds for* $m = 5$ *if*

$$\text{(3.17)} \qquad \frac{|p-a|}{|p-b|}\,,\; \frac{|p-b|}{|p-a|} \;\leq\; 5$$

for any mesh points $p,a,b \in \mathcal{N}_{\triangle_\mu}$ *which describe an element of* $\triangle_\mu$.

Proof. **(i)** For $m=0$, (3.15) follows from $\int_T |1-\varphi_T^{[0]}(x)|^2\, dx = 0$ (since $\varphi_T^{[0]}$ is the characteristic function of T).

(ii) For $m \in \{1,6\}$, we use some properties of barycentric coordinate functions. Let $p_0,p_1,p_2 \in \mathcal{N}_{\triangle_\mu}$ be the vertices of the triangle T, then the hat functions $\varphi_{p_j}^{[1]}$ coincide with the barycentric coordinate functions λ_j. Thus, by [10, Exercise 4.1.1], we have for any $\nu = (\nu_0,\nu_1,\nu_2) \in \mathbb{N}_0^3$ that

(3.18)

$$\int_T \varphi_{p_0}^{[1]}(x)^{\nu_0}\, \varphi_{p_1}^{[1]}(x)^{\nu_1}\, \varphi_{p_2}^{[1]}(x)^{\nu_2}\, dx = \frac{2\,\nu_0!\,\nu_1!\,\nu_2!}{(2+\nu_0+\nu_1+\nu_2)!}\, area(T)\,.$$

Hence, (3.15) follows for $m = 6$ from

$$\begin{aligned}
\int_T |1 - \varphi_T^{[6]}(x)|^2 \, dx &= \int_T \left[1 - 3^6 \left(\varphi_{p_0}^{[1]} \varphi_{p_1}^{[1]} \varphi_{p_2}^{[1]} \right)^2 \right]^2 dx \\
&\overset{(3.18)}{=} \left(1 - 2 \cdot 3^6 \frac{2^4}{8!} + 3^{12} \frac{2\,(4!)^3}{14!} \right) area(T) \\
&\leq \left(1 - \frac{2}{5} \right) area(T) \, .
\end{aligned}$$

Now, let $m = 1$ and let $T' \in \triangle_\mu$ such that $T' \subseteq \omega_p$. Then, $p_0 := p$ is one vertex of T' and we have

$$\begin{aligned}
\int_{T'} |1 - \varphi_p^{[1]}|^2 dx &= \int_{T'} 1 - 2\varphi_{p_0}^{[1]} + (\varphi_{p_0}^{[1]})^2 dx \\
&\overset{(3.18)}{=} \left(1 - 2\frac{2}{3!} + \frac{2^2}{4!} \right) area(T') \\
&= \frac{1}{2} \, area(T') \, ,
\end{aligned}$$

which yields

$$\begin{aligned}
\int_{\omega_p} |1 - \varphi_p^{[1]}|^2 dx &= \sum_{\substack{T' \in \triangle_\mu \\ T' \subseteq \omega_p}} \int_{T'} |1 - \varphi_p^{[1]}|^2 dx \\
&= \frac{1}{2} \sum_{\substack{T' \in \triangle_\mu \\ T' \subseteq \omega_p}} area(T') \\
&= \left(1 - \frac{1}{2} \right) area(\omega_p) \, .
\end{aligned}$$

(iii) The proof for $m = 5$, which is relatively long and technical, is given in the appendix. □

As consequence of Lemma 3.4 and 3.5, we obtain the following Lemma.

Lemma 3.6 *Let $s \in (0,1)$, $m \in \{0, 1, 5, 6\}$ and let C_m^{FE} be the constant given by Lemma 3.5.*

a) *For $T \in \triangle_\mu$, we have*

$$\|w\|_{L^2(T)}^2 \leq \frac{1}{2\, C_m^{FE}} \frac{diam(T)^{2+2s}}{area(T)} |w|_{H^s(T)}^2 \tag{3.19}$$

for any function $w \in H^s(P_\mu)$ which satisfies $w \perp \varphi_T^{[m]}$ with $m \in \{0, 6\}$.

b) *For $p \in \mathcal{N}_{\triangle_\mu}$ we have*

$$\|w\|_{L^2(\omega_p)}^2 \leq \frac{1}{2\, C_m^{FE}} \frac{diam(\omega_p)^{2+2s}}{area(\omega_p)} |w|_{H^s(\omega_p)}^2 \tag{3.20}$$

for any function $w \in H^s(P_\mu)$ *which satisfies* $w \perp \varphi_p^{[m]}$ *with* $m \in \{1,5\}$. *For* $m = 5$, $\triangle_\mu$ *has to satisfy additionally assumption (3.17).*

Proof. In order to prove (3.19) and (3.20) in one step, we introduce

$$\varphi^{[m]} := \begin{cases} \varphi_T^{[m]} & \text{for } m \in \{0,6\} \\ \varphi_p^{[m]} & \text{for } m \in \{1,5\} \end{cases} \qquad \text{and}$$

$$S := \operatorname{supp}(\varphi^{[m]}) = \begin{cases} T & \text{for } m \in \{0,6\} \\ \omega_p & \text{for } m \in \{1,5\} \end{cases}.$$

Due to (3.13), we have

(3.21)

$$\|w\|^2_{L^2(S)} \le \frac{1}{2}\,\frac{diam(S)^{2+2s}}{area(S)}\,|w|^2_{H^s(S)} + \frac{1}{area(S)}\underbrace{\left|\int_S w(x)\,dx\,\right|^2}_{=:\,J^2}.$$

Since $w \perp \varphi^{[m]}$, we obtain

$$\begin{aligned} J^2 \;&=\; \left|\int_S w(x)(1-\varphi^{[m]}(x))\,dx\,\right|^2 \;\le\; \|1-\varphi^{[m]}\|^2_{L^2(S)}\,\|w\|^2_{L^2(S)} \\ &\overset{L.\,3.5}{\le} (1-C_m^{FE})\;area(S)\;\|w\|^2_{L^2(S)}\,. \end{aligned}$$

Inserting this into (3.21) yields

$$\|w\|^2_{L^2(S)} \;\le\; \frac{1}{2}\,\frac{diam(S)^{2+2s}}{area(S)}\,|w|^2_{H^s(S)} \;+\; (1-C_m^{FE})\,\|w\|^2_{L^2(S)}$$

and thus

$$C_m^{FE}\,\|w\|^2_{L^2(S)} \;\le\; \frac{1}{2}\,\frac{diam(S)^{2+2s}}{area(S)}\,|w|^2_{H^s(S)}\,.$$

□

Remark 3.7

a) We will consider in the following so-called *shape regular* meshes $\triangle$ with non-degenerated angles. Therefore, we introduce for $\kappa > 0$, the family of meshes

(3.22) $\mathcal{M}_\kappa(\Gamma) \;:=\; \{\triangle \mid$ $\triangle$ is a mesh on Γ as in (2.5) and κ is a lower bound of all angles of $\triangle_\mu$, $\mu \in \{1,\dots,M\}\}$.

For $m = 5$, we have to tighten the definition of $\mathcal{M}_\kappa(\Gamma)$ by assuming that meshes $\triangle \in \mathcal{M}_\kappa(\Gamma)$ satisfy additionally (3.17).

b) There is a constant C_κ^{shape} (only depending on κ and Γ) with the following properties: For any shape regular mesh $\triangle \in \mathcal{M}_\kappa(\Gamma)$ and the corresponding mesh $\triangle_\mu$ on P_μ ($\mu \in \{1, \ldots, M\}$), we have

$$\frac{diam(T)^2}{area(T)}\ ,\ \frac{diam(T)^2}{d_T^2}\ ,\ \frac{diam(T)^2}{d_\tau^2} \quad \leq \quad C_\kappa^{shape} \tag{3.23}$$

for all elements $T \in \triangle_\mu$ with $\tau := \gamma(T) \subseteq \Gamma$ and with d_τ introduced in (2.8). Moreover, we have

$$\frac{diam(\omega_p)^2}{area(\omega_p)}\ ,\ \frac{diam(\omega_p)^2}{d_T^2}\ ,\ \frac{diam(\omega_p)^2}{d_\tau^2} \quad \leq \quad C_\kappa^{shape} \tag{3.24}$$

for any nodal point p of $\triangle_\mu$ and for all $T \in \triangle_\mu$ with $T \subseteq \omega_p$ and $\tau := \gamma(T)$, where ω_p is introduced in (2.3). The constant C_κ^{shape} increases for decreasing κ.

c) Shape regular meshes with non-degenerated angles are suitable for adaptive mesh refinement, since they may contain small elements as well as large elements.

d) The constant C_κ^{shape} will be used in the following Lemma 3.8 (where a Poincaré type inequality similar to (3.12) will be shown for functions $v \in H^s(\Gamma)$ which are orthogonal to $H_\triangle^m(\Gamma)$, $m \in \{0, 1, 5, 6\}$) and in Theorem 3.9 (where the localization of the double integral norm will be shown). For the case $m \in \{0, 6\}$, we only need that C_κ^{shape} is characterized by (3.23), and the characterization by (3.24) is only needed for the case $m \in \{1, 5\}$. The formulations would be more precise if one would introduce two constants $C_{\kappa,0}^{shape}$ and $C_{\kappa,1}^{shape}$, where $C_{\kappa,0}^{shape}$ (respectively $C_{\kappa,1}^{shape}$) satisfies (3.23) (respectively (3.24)). Then, Lemma 3.8 and Theorem 3.9 hold for even m with $C_{\kappa,0}^{shape}$ and for odd m with $C_{\kappa,1}^{shape}$ (instead of C_κ^{shape}). Only to simplify the notation, we use one constant C_κ^{shape}.

Now, we are in position to prove a Poincaré type inequality similar to (3.12).

Lemma 3.8 *For $s \in (0,1)$, $m \in \{0, 1, 5, 6\}$ and $\kappa > 0$, there is a constant $C_{s,m,\kappa}^{poinc}$ with*

$$\sum_{\tau \in \triangle} d_\tau^{-2s} \|v\|_{L^2(\tau)}^2 \ \leq \ C_{s,m,\kappa}^{poinc} \sum_{q \in \mathcal{N}_\triangle} |v|_{H^s(\omega_q)}^2 \tag{3.25}$$

for all shape regular meshes $\triangle \in \mathcal{M}_\kappa(\Gamma)$ with sufficiently small mesh size (i.e., $h_\triangle \leq h_0$) and for any function $v \in H^s(\Gamma)$ which is orthogonal to $H_\triangle^m(\Gamma)$. The constant is explicitly given by

$$C_{s,m,\kappa}^{poinc} = \frac{2}{c_m} \, \frac{(C_\kappa^{shape} C_\Gamma^2)^{1+s}\, \overline{C}_G}{C_m^{FE}\, \underline{C}_G^2} \quad \textit{with} \quad c_m = \begin{cases} 1 \textit{ for odd } m \\ 3 \textit{ for even } m \end{cases}, \tag{3.26}$$

where C_m^{FE}, C_κ^{shape}, $\underline{C}_G$, $\overline{C}_G$ and C_Γ, are the constants given by Lemma 3.5, Remark 3.7, (2.1) and (2.2).

Proof. **(i)** First, let $m \in \{1, 5\}$. Let $\mu \in \{1, \ldots, M\}$ and let $q \in \mathcal{N}_\triangle$ be an interior mesh point of Γ_μ (i.e., $\omega_q \subseteq \Gamma_\mu$). Then, the corresponding parameter point $p := \gamma^{-1}(q) \in \mathcal{N}_{\triangle_\mu}$ is an interior point of P_μ (i.e., $p \notin \partial P_\mu$) and $\omega_p = \gamma^{-1}(\omega_q)$. By $v \perp H^m_\triangle(\Gamma)$, we have

$$
\begin{aligned}
0 &= \langle v, \Phi_q^{[m]} \rangle_{L^2(\Gamma)} \\
&= \int_{\omega_p} \underbrace{\Phi_q^{[m]}(\gamma(x))}_{\overset{(2.9)}{=}\ \varphi_p^{[m]}(x)} \ \underbrace{v(\gamma(x)) \sqrt{G_\mu(x)}}_{=:\, w(x)} \, dx \\
&= \langle w, \varphi_p^{[m]} \rangle_{L^2(P_\mu)} \,.
\end{aligned}
\tag{3.27}
$$

Since $w \perp \varphi_p^{[m]}$, we may apply (3.20) and obtain

$$
\begin{aligned}
\sum_{\substack{\tau \in \triangle \\ \tau \subseteq \omega_q}} d_\tau^{-2s} \, \|v\|^2_{L^2(\tau)} &= \sum_{\substack{\tau \in \triangle \\ \tau \subseteq \omega_q}} d_\tau^{-2s} \int_{\gamma^{-1}(\tau)} |v(\gamma(x))|^2 \sqrt{G_\mu(x)} \, dx \\
&= \sum_{\substack{\tau \in \triangle \\ \tau \subseteq \omega_q}} d_\tau^{-2s} \int_{\gamma^{-1}(\tau)} |w(x)|^2 \frac{1}{\sqrt{G_\mu(x)}} \, dx \\
&\overset{(2.1)}{\leq} \underline{C}_G^{-1} \sum_{\substack{\tau \in \triangle \\ \tau \subseteq \omega_q}} d_\tau^{-2s} \, \|w\|^2_{L^2(\gamma^{-1}(\tau))} && (3.28) \\
&\overset{(3.24)}{\leq} \underline{C}_G^{-1} \, (C_\kappa^{shape})^s \, diam(\omega_p)^{-2s} \, \|w\|^2_{L^2(\omega_p)} \\
&\overset{(3.20)}{\leq} \frac{(C_\kappa^{shape})^s}{2 C_m^{FE} \underline{C}_G} \, \frac{diam(\omega_p)^2}{area(\omega_p)} \, |w|^2_{H^s(\omega_p)} \\
&\overset{(3.24)}{\leq} C_1 \, |w|^2_{H^s(\omega_p)} && (3.29)
\end{aligned}
$$

with the constant $C_1 := \frac{(C_\kappa^{shape})^{1+s}}{2 C_m^{FE} \underline{C}_G}$. Since the Gram determinant G_μ is differentiable on P_μ, we have

$$
\max_{\mu \in \{1,\ldots,M\}} \ \max_{\substack{p \in \mathcal{N}_{\triangle_\mu} \\ p \notin \partial P_\mu}} \ \sup_{x \in \omega_p} \int_{\omega_p} \frac{|\sqrt{G_\mu(x)} - \sqrt{G_\mu(y)}|^2}{|x-y|^{2+2s}} \, dy \ \leq \ C_2
\tag{3.30}
$$

with a constant C_2 only depending on Γ and independent of $\triangle$. Hence

(3.31)

$$
\begin{aligned}
&|w|^2_{H^s(\omega_p)} = \\
&= \int_{\omega_p}\int_{\omega_p} |x-y|^{-2-2s} \underbrace{|v(\gamma(x))\sqrt{G_\mu(x)} - v(\gamma(y))\sqrt{G_\mu(y)}|^2}_{\leq 2\,|v(\gamma(x))|^2\,|\sqrt{G_\mu(x)}-\sqrt{G_\mu(y)}|^2 \,+\, 2\,G_\mu(y)\,|v(\gamma(x))-v(\gamma(y))|^2} dx\,dy \\
&\overset{(3.30)}{\leq} 2\,C_2 \int_{\omega_p} |v(\gamma(x))|^2\,dx \\
&\qquad + 2\int_{\omega_p}\int_{\omega_p} \frac{|v(\gamma(x)) - v(\gamma(y))|^2}{|x-y|^{2+2s}}\, G_\mu(y)\,dx\,dy \\
&\overset{(2.1)\,(2.2)}{\leq} \frac{2\,C_2}{\underline{C}_G}\,\|v\|^2_{L^2(\gamma(\omega_p))} + \frac{2\,\overline{C}_G C_\Gamma^{2+2s}}{\underline{C}_G} |v|^2_{H^s(\gamma(\omega_p))}\,.
\end{aligned}
$$

Combining (3.29) and (3.31), we have with the constants $C_3 := \frac{2C_1C_2}{\underline{C}_G}$ and $C_4 := \frac{2C_1C_\Gamma^{2+2s}\overline{C}_G}{\underline{C}_G}$

(3.32)
$$
\begin{aligned}
\sum_{\substack{\tau\in\triangle\\ \tau\subseteq\omega_q}} d_\tau^{-2s}\,\|v\|^2_{L^2(\tau)} &\leq C_3\,\|v\|^2_{L^2(\omega_q)} + C_4\,|v|^2_{H^s(\omega_q)} \\
&\leq C_3 h_\triangle^{2s} \sum_{\substack{\tau\in\triangle\\ \tau\subseteq\omega_q}} d_\tau^{-2s}\,\|v\|^2_{L^2(\tau)} + C_4\,|v|^2_{H^s(\omega_q)}\,.
\end{aligned}
$$

For a mesh $\triangle$ with sufficiently small mesh size (such that $C_3 h_\triangle^{2s} \leq \frac{1}{2}$), we obtain

$$
\sum_{\substack{\tau\in\triangle\\ \tau\subseteq\omega_q}} d_\tau^{-2s}\,\|v\|^2_{L^2(\tau)} \;\leq\; 2\,C_4\,|v|^2_{H^s(\omega_q)}
$$

and therefore

$$
\begin{aligned}
\sum_{\tau\in\triangle} d_\tau^{-2s}\|v\|^2_{L^2(\tau)} &\leq \sum_{\mu=1}^{M} \sum_{\substack{q\in\mathcal{N}_\triangle\\ \omega_q\subseteq\Gamma_\mu}} \sum_{\substack{\tau\in\triangle\\ \tau\subseteq\omega_q}} d_\tau^{-2s}\|v\|^2_{L^2(\tau)} \\
&\leq 2\,C_4 \sum_{q\in\mathcal{N}_\triangle} |v|^2_{H^s(\omega_q)}\,,
\end{aligned}
$$

which shows assertion (3.25) for the case $m\in\{1,5\}$.

(ii) Now, let $m\in\{0,6\}$. Let $\tau\in\triangle$, let $\mu\in\{1,\ldots,M\}$ with $\tau\subseteq\Gamma_\mu$ and let $T := \gamma^{-1}(\tau)\in\triangle_\mu$. From $v\perp H^m_\triangle(\Gamma)$ we infer (analogously to (3.27)) that $w \perp \varphi_T^{[m]}$ for the function $w := (v \circ \gamma) \cdot \sqrt{G_\mu}$. Hence, we may apply (3.19) and obtain (similar to (3.28)) that

$$\text{(3.33)} \quad d_\tau^{-2s} \|v\|^2_{L^2(\tau)} \le \underline{C}_G^{-1} d_\tau^{-2s} \|w\|^2_{L^2(\gamma^{-1}(\tau))} \overset{(3.19)}{\le} \frac{1}{2C_m^{FE}\underline{C}_G} \Big(\frac{diam(T)}{d_\tau} \Big)^{2s} \frac{diam(T)^2}{area(T)} |w|^2_{H^s(T)} \overset{(3.23)}{\le} C_1 |w|^2_{H^s(T)} C_1 |w|^2_{H^s(T)}$$

with the constant C_1 introduced after (3.29). Since the Gram determinant G_μ is differentiable on P_μ, we deduce (analogously to (3.31))

$$\text{(3.34)} \quad |w|^2_{H^s(T)} \le \frac{2\,C_2}{\underline{C}_G} \|v\|^2_{L^2(\gamma(T))} + \frac{2\,\overline{C}_G C_\Gamma^{2+2s}}{\underline{C}_G} |v|^2_{H^s(\gamma(T))} .$$

For a mesh $\triangle$ with sufficiently small mesh size (such that $C_3 h_\triangle^{2s} \le \frac{1}{2}$), we infer from (3.33) and (3.34) (analogously to (3.32)) that

$$d_\tau^{-2s} \|v\|^2_{L^2(\tau)} \le 2\,C_4 |v|^2_{H^s(\tau)}$$

and therefore

$$\sum_{\tau \in \triangle} d_\tau^{-2s} \|v\|^2_{L^2(\tau)} \le 2\,C_4 \sum_{\tau \in \triangle} |v|^2_{H^s(\tau)} \le \frac{2}{3} C_4 \sum_{q \in \mathcal{N}_\triangle} \sum_{\substack{\tau \in \triangle \\ \tau \subseteq \omega_q}} |v|^2_{H^s(\tau)} \le \frac{2}{3} C_4 \sum_{q \in \mathcal{N}_\triangle} |v|^2_{H^s(\omega_q)} .$$

□

The combination of Corollary 3.3 and Lemma 3.8 leads to the following localization of the double integral norm which is the aim of this section.

Theorem 3.9 *For $s \in (0,1)$, $m \in \{0,1,5,6\}$ and $\kappa > 0$ there is a constant $C = C(s,m,\kappa)$ with*

$$\|v\|^2_{H^s(\Gamma)} \le C \sum_{q \in \mathcal{N}_\triangle} |v|^2_{H^s(\omega_q)}$$

for all shape regular meshes $\triangle \in \mathcal{M}_\kappa(\Gamma)$ and for any function $v \in H^s(\Gamma)$ which is orthogonal to $H^m_\triangle(\Gamma)$. The constant C is explicitly given by

$$C = 1 + C_s^{loc} C_{s,m,\kappa}^{poinc}, \quad \textit{where } C_s^{loc} \textit{ is an upper bound of } h_\triangle^{2s} + 4C_{2s},$$

and where $C_{s,m,\kappa}^{poinc}$ and C_{2s} are the constants given by (3.26) and Lemma 2.4.

Analogously to Theorem 3.9, one can localize the norm $\|\cdot\|_{H^s(D)}$ for $D \subseteq \mathbb{R}^2$.

Theorem 3.10 *Let $s \in (0,1)$, $m \in \{0,1,5,6\}$ and $\kappa > 0$. Let $D \subseteq \mathbb{R}^2$ be a closed and connected polygonal set. Then, we have for all shape regular meshes $\triangle_D \in \mathcal{M}_\kappa(D)$ on D (where $\mathcal{M}_\kappa(D)$ is defined analogously to (3.22) by a minimal angle condition) and for any function $w \in H^s(D)$ which is orthogonal to $H^m_{\triangle_D}(D)$ that*

$$\sum_{T\in\triangle_D} d_T^{-2s}\|w\|^2_{L^2(T)} \;\le\; \frac{(C_\kappa^{shape})^{1+s}}{6\,C_m^{FE}} \sum_{p\in\mathcal{N}_{\triangle_D}} |w|^2_{H^s(\omega_p)} \tag{3.35}$$

and

$$\|w\|^2_{H^s(D)} \;\le\; \left(1 + \frac{2\,\pi\,(C_\kappa^{shape})^{1+s}}{3\,s\,C_m^{FE}}\right) \sum_{p\in\mathcal{N}_{\triangle_D}} |w|^2_{H^s(\omega_p)} \tag{3.36}$$

with the constants C_m^{FE} and C_κ^{shape} given in Lemma 3.5 and Remark 3.7.

Proof. The proof of (3.35) is considerably simpler then the proof of (3.25) since no Gram determinants has to be treated. (3.36) is a direct consequence of (3.9) and (3.35). □

4 Localization of the norm $\|\cdot\|_{H^s(\Gamma)}$ for $s \in (1,2)$

For $s \in [1,2)$, decomposed as $s = 1+\sigma$ with $\sigma \in [0,1)$, and for $D \subseteq \mathbb{R}^2$, the norm $\|\cdot\|_{H^s(D)}$ is defined by

$$\|w\|^2_{H^s(D)} \;:=\; \sum_{k\in\{0,1\}\cup\{s\}} |w|^2_{H^k(D)} \tag{4.1}$$

with the semi-norms $|w|_{H^0(D)} := \|w\|_{L^2(D)}$ and

$$|w|^2_{H^1(D)} \;:=\; \|\partial_1 w\|^2_{L^2(D)} + \|\partial_2 w\|^2_{L^2(D)}$$

(where $\partial_j w$ is the partial derivative $\frac{\partial w}{\partial x_j}$) and for $\sigma > 0$ with

$$\begin{aligned} |w|^2_{H^s(D)} \;&:=\; \sum_{j=1}^{2} |\partial_j w|^2_{H^\sigma(D)} \\ &=\; \sum_{j=1}^{2} \int_D\int_D \frac{|(\partial_j w)(x) - (\partial_j w)(y)|^2}{|x-y|^{2+2\sigma}}\,dx\,dy\,. \end{aligned}$$

The norm $\|\cdot\|_{H^s(D)}$ is also for $s \in (1,2)$ localizable.

Theorem 4.1 *Let $s = 1 + \sigma \in (1, 2)$, let $\kappa > 0$ and let $D \subseteq \mathbb{R}^2$ be a closed and connected polygonal set. Then, we have for all shape regular meshes $\triangle_D \in \mathcal{M}_\kappa(D)$ on D and for any function $w \in H^s(D)$ which is orthogonal to $H^0_{\triangle_D}(D)$ that*

$$\|w\|^2_{H^s(D)} \;\le\; C \sum_{p\in\mathcal{N}_{\triangle_D}} |w|^2_{H^s(\omega_p)} \tag{4.2}$$

with the constant

$$C \;=\; C(s,\kappa) \;=\; \Big(1 \,+\, \frac{h^2_{\triangle_D}}{\pi^2}\Big)\Big(1 \,+\, \frac{4\,\pi\,(C^{shape}_\kappa)^{1+\sigma}}{3\,\sigma}\Big)\,,$$

where C^{shape}_κ is given in Remark 3.7.

Proof. Because of $w \perp H^0_{\triangle_D}(D)$, we have $\int_T w(x)\,dx = 0$ for all $T \in \triangle_D$ and obtain therefore $\|w\|_{L^2(T)} \le \frac{1}{\pi}\,diam(T)|w|_{H^1(T)}$ (see (3.14)). Hence

$$\|w\|^2_{H^s(D)} \;\le\; \Big(1 \,+\, \frac{h^2_{\triangle_D}}{\pi^2}\Big)(\|\partial_1 w\|^2_{H^\sigma(D)} \,+\, \|\partial_2 w\|^2_{H^\sigma(D)})\,. \tag{4.3}$$

From $w \perp H^0_{\triangle_D}(D)$ we infer by partial integration that $\partial_j w \perp H^1_{\triangle_D}(D)$. Applying (3.36) to the right-hand side of (4.3) (with $C^{FE}_1 = \frac{1}{2}$) shows assertion (4.2). □

This localization result also holds for the norm $\|\cdot\|_{H^s(\Gamma)}$ on the surface Γ.

Theorem 4.2 *For $s = 1 + \sigma \in (1, 2)$ and $\kappa > 0$, there is a constant $C = C(s, \kappa)$ with*

$$\|v\|^2_{H^s(\Gamma)} \;\le\; C \sum_{q\in\mathcal{N}_\triangle} |v|^2_{H^s(\omega_q)} \tag{4.4}$$

for all shape regular meshes $\triangle \in \mathcal{M}_\kappa(\Gamma)$ and for any function $v \in H^s(\Gamma)$ which is orthogonal to $H^0_\triangle(\Gamma)$.

The definition of the norm $\|\cdot\|_{H^s(\Gamma)}$ on a closed and smooth surface $\Gamma = \partial\Omega \in C^{1,0}$ is rather involved for $s > 1$, since one needs a system of overlapping parametrizations for Γ and a subordinated partition of unity (see [32, Chap. 4.2] for the details). The proof of (4.4) in [18] needs special parametrizations which take into account that the meshes on the overlapped parameterized parts of Γ have to fit together. Therefore, this proof is considerably more complicated then the proof of (4.2). Hence, we omit here the proof of (4.4) and refer to [18].

5 Local error indicators

In this section, we will apply the localization of the double integral norm to develop reliable and efficient error indicators for the Galerkin discretization of boundary integral equations.

The global norm $\|\cdot\|_{H^s(\Gamma)}$, for $s \geq 0$, is abbreviated by $\|\cdot\|_s$. For negative $s < 0$, we define $H^s(\Gamma)$ to be the dual space of $H^{-s}(\Gamma)$ with the dual norm

$$\|v\|_s \;=\; \|v\|_{H^s(\Gamma)} \;:=\; \|v\|_{H^{-s}(\Gamma)'} \;=\; \sup_{\substack{w\in H^{-s}(\Gamma)\\ w\neq 0}} \frac{|v(w)|}{\|w\|_{-s}} \,.$$

For $s \in \mathbb{R}$, the $L^2(\Gamma)$-scalar product $\langle .,.\rangle_0 : L^2(\Gamma)\times L^2(\Gamma)\to\mathbb{R}$ can be extended to the dual form $\langle .,.\rangle_0 : H^s(\Gamma)\times H^{-s}(\Gamma)\to\mathbb{R}$ by

$$\langle v,w\rangle_0 \;:=\; \langle v,w\rangle_{L^2(\Gamma)} \;=\; \begin{cases} w(v) & \text{if } s\geq 0\\ v(w) & \text{if } s<0\end{cases}$$

for $v\in H^s(\Gamma), w\in H^{-s}(\Gamma)$.

In the following, we consider a bijective and continuous operator

$$\text{(5.1)} \qquad A : H^{s+2\alpha}(\Gamma)\to H^s(\Gamma)\,, \qquad s\in[0,2]\,,\ \alpha\in\mathbb{R}\,,$$

of order 2α. For a given right-hand side $g\in H^s(\Gamma)$, we search for the solution $u\in H^{s+2\alpha}(\Gamma)$ of the equation

$$\text{(5.2)} \qquad Au \;=\; g \qquad \text{on } \Gamma\,.$$

An important example for (5.2) is an integral equation with the operator

$$\text{(5.3)} \qquad Au(x) \;=\; \frac{1}{4\pi}\int_\Gamma \frac{1}{|x-\xi|}\,u(\xi)\,d\xi \qquad \text{for } x\in\Gamma\,.$$

The corresponding integral equation is related to the interior and exterior 3-dimensional Laplace problem in Ω with Dirichlet boundary condition. The operator A in (5.3) is the single-layer potential of the Laplacian and it satisfies

$$A : H^{-\frac12}(\Gamma)\to H^{\frac12}(\Gamma)\,,$$

i.e., we are in the situation of (5.1) with $s=\frac12$ and $\alpha=-\frac12$ (see, e.g., [9]).

The abstract problem (5.2) is equivalent to the following variational problem: find $u\in H^{s+2\alpha}(\Gamma)$ such that

$$\text{(5.4)} \qquad \langle Au,v\rangle_{L^2(\Gamma)} \;=\; \langle g,v\rangle_{L^2(\Gamma)} \qquad \text{for all } v\in H^{-s}(\Gamma)\,.$$

For the discretization of problem (5.4), let $\triangle$ be a mesh on Γ (see (2.5)) and let $\mathcal{G} = \mathcal{G}_\triangle \subseteq H^{s+2\alpha}(\Gamma)$ be a finite dimensional Galerkin trial space associated with $\triangle$. Then, the Galerkin formulation of problem (5.4) reads: find an approximate solution $u_\mathcal{G} \in \mathcal{G}$ such that

$$(5.5) \qquad \langle A u_\mathcal{G}, v_\mathcal{G} \rangle_{L^2(\Gamma)} \;=\; \langle g, v_\mathcal{G} \rangle_{L^2(\Gamma)} \qquad \text{for all } v_\mathcal{G} \in \mathcal{G} .$$

In the following, let $u \in H^{s+2\alpha}(\Gamma)$ be the solution of problem (5.2) (and (5.4)), and let $u_\mathcal{G} \in \mathcal{G}$ be the Galerkin solution of problem (5.5). Our aim is to estimate the unknown discretization error

$$e \;:=\; u_\mathcal{G} - u \;\in\; H^{s+2\alpha}(\Gamma)$$

by computable local quantities. The residual

$$r \;:=\; Ae \;=\; Au_\mathcal{G} - g \;\in\; H^s(\Gamma)$$

is computable after the calculation of $u_\mathcal{G} \in \mathcal{G}$ and local quantities of r will be used to control the adaptive refinement.

Since $A : H^{s+2\alpha}(\Gamma) \to H^s(\Gamma)$ is assumed to be an isomorphism, we find constants $C_1^{op}, C_2^{op} > 0$ such that for all $v \in H^{s+2\alpha}(\Gamma)$ the estimate

$$(5.6) \qquad C_1^{op} \, \|Av\|^2_{H^s(\Gamma)} \;\le\; \|v\|^2_{H^{s+2\alpha}(\Gamma)} \;\le\; C_2^{op} \, \|Av\|^2_{H^s(\Gamma)}$$

is satisfied. The constants are given by $(C_1^{op})^{-1} = \|A\|^2_{H^s(\Gamma) \leftarrow H^{s+2\alpha}(\Gamma)}$ and $C_2^{op} = \|A^{-1}\|^2_{H^{s+2\alpha}(\Gamma) \leftarrow H^s(\Gamma)}$. By (5.6), we obtain

$$(5.7) \qquad C_1^{op} \, \|r\|^2_{H^s(\Gamma)} \;\le\; \|e\|^2_{H^{s+2\alpha}(\Gamma)} \;\le\; C_2^{op} \, \|r\|^2_{H^s(\Gamma)} \,.$$

For $s = k \in \mathbb{N}_0$, the global norm $\|r\|_{H^k(\Gamma)}$ is additive (see (1.7)) and therefore we have

$$(5.8) \quad C_1^{op} \sum_{\tau \in \triangle} \|r\|^2_{H^k(\tau)} \;\le\; \|e\|^2_{H^{s+2\alpha}(\Gamma)} \;\le\; C_2^{op} \sum_{\tau \in \triangle} \|r\|^2_{H^k(\tau)} \,.$$

The estimate (5.8) means that the local quantities $\{\|r\|_{H^k(\tau)}\}_{\tau \in \triangle}$ of the residual r are reliable and efficient error indicators in the case $s = k \in \mathbb{N}_0$. Such an approach was treated in [28].

In [20], the following approach was proposed for $s \in (0,1)$: the double integral norm is not additive but satisfies

$$\|r\|^2_{H^s(\Gamma)} \;=\; \sum_{\tau \in \triangle} \lambda_\tau^2$$

with

$$\lambda_\tau^2 \;:=\; \int_\tau |r(\xi)|^2 \, d\xi \;+\; \int_\tau \int_\Gamma \frac{|r(\xi) - r(\eta)|^2}{|\xi - \eta|^{1+2s}} \, d\xi \, d\eta$$

for $s \in (0,1)$. This shows

$$C_1^{op} \sum_{\tau \in \triangle} \lambda_\tau^2 \;\leq\; \|e\|^2_{H^{s+2\alpha}(\Gamma)} \;\leq\; C_2^{op} \sum_{\tau \in \triangle} \lambda_\tau^2 \, ,$$

i.e., the quantities $\{\lambda_\tau\}_{\tau \in \triangle}$ are reliable and efficient error indicators in the case $s \in (0,1)$. Unfortunately, λ_τ has only partially local character.

We are able to improve this approach using the localization of the double integral norm presented in Theorem 3.9 and 4.2.

Theorem 5.1 *Let $s = k + \sigma$ with $k \in \{0,1\}$ and $\sigma \in (0,1)$. Let $\alpha \in \mathbb{R}$ and let $A : H^{s+2\alpha}(\Gamma) \to H^s(\Gamma)$ be an isomorphism. Let $\mathcal{G} = \mathcal{G}_\triangle \subseteq H^{s+2\alpha}(\Gamma)$ be a finite dimensional trial space associated with a mesh $\triangle$ satisfying $H^m_\triangle(\Gamma) \subseteq \mathcal{G} \subseteq \mathbb{P}_\triangle(\Gamma)$ with $m \in \{0,1,5,6\}$ in the case $s \in (0,1)$ and with $m = 0$ in the case $s \in (1,2)$. (Examples for $\mathcal{G}$ are given below in Remark 5.3.)*
Then, we obtain for any solution $u \in H^{s+2\alpha}(\Gamma)$ and for all meshes $\triangle$ with the corresponding set of mesh points $\mathcal{N}_\triangle$ the following estimate for the Galerkin error $e \in H^{s+2\alpha}(\Gamma)$:

$$(5.9) \qquad C^{eff} \sum_{q \in \mathcal{N}_\triangle} \varepsilon_q^2 \;\leq\; \|e\|^2_{H^{s+2\alpha}(\Gamma)} \;\leq\; C^{rel} \sum_{q \in \mathcal{N}_\triangle} \varepsilon_q^2 \, ,$$

where $\varepsilon_q := |r|_{H^s(\omega_q)}$ is a local double integral semi-norm of the residual $r \in H^s(\Gamma)$. The efficiency of the error indicators $\{\varepsilon_q\}_{q \in \mathcal{N}_\triangle}$ (i.e., the lower estimate in (5.9)) holds for arbitrary meshes, and the reliability (i.e., the upper estimate in (5.9)) holds for shape regular meshes $\triangle \in \mathcal{M}_\kappa(\Gamma)$ with $\kappa > 0$. The constants C^{eff}, C^{rel} in (5.9) are independent of u, $\triangle$ and $\mathcal{G}$, they are explicitly given by

$$C^{eff} \;=\; \frac{1}{3} C_1^{op} \quad \textit{and} \quad C^{rel} \;=\; C^{rel}(s,m,\kappa) \;=\; C_2^{op}\,(1 + C_s^{loc}\, C_{s,m,\kappa}^{poinc}) \, ,$$

where C_1^{op}, C_2^{op}, C_s^{loc} and $C_{s,m,\kappa}^{poinc}$ are the constants given by (5.6), Theorem 3.9 and Lemma 3.8. ε_q has local character, since for $s \in (0,1)$, we have

$$\varepsilon_q^2 \;=\; |r|^2_{H^s(\omega_q)} \;=\; \int_{\omega_q} \int_{\omega_q} \frac{|r(\xi) - r(\eta)|^2}{|\xi - \eta|^{2+2\sigma}} \, d\xi \, d\eta \, .$$

Proof. We show (5.9) for $s \in (0,1)$, the proof for $s \in (1,2)$ is similar (and uses Theorem 4.2 instead of Theorem 3.9 for the upper estimate). With the abbreviation $\int_{\Gamma'}\int_{\Gamma''} := \int_{\Gamma'}\int_{\Gamma''} \frac{|r(\xi)-r(\eta)|^2}{|\xi-\eta|^{2+2\sigma}}\, d\xi\, d\eta$, for $\Gamma', \Gamma'' \subseteq \Gamma$, we obtain

$$\text{(5.10)} \quad \sum_{q\in\mathcal{N}_\triangle} |r|^2_{H^s(\omega_q)} \;\le\; \sum_{q\in\mathcal{N}_\triangle} \sum_{\substack{\tau\in\triangle \\ \tau\subseteq\omega_q}} \int_\tau\int_{\omega_q} \;\le\; 3\sum_{\tau\in\triangle}\int_\tau\int_\Gamma$$

$$\le\; 3\,\|r\|^2_{H^s(\Gamma)} \;\overset{(5.7)}{\le}\; \frac{3}{C_1^{op}}\,\|e\|^2_{H^{s+2\alpha}(\Gamma)}\,,$$

which shows the lower estimate in (5.9). For the proof of the upper estimate in (5.9), let $\triangle \in \mathcal{M}_\kappa(\Gamma)$ be shape regular mesh. The Galerkin residual $r = Au_{\mathcal{G}} - g$ is orthogonal to $\mathcal{G}$ (because of (5.5)). This implies, together with $H^m_\triangle(\Gamma) \subseteq \mathcal{G}$, that $r \perp H^m_\triangle(\Gamma)$. Hence, we may apply Theorem 3.9 to r and obtain

$$\text{(5.11)} \qquad \|r\|^2_{H^s(\Gamma)} \;\le\; (1 + C_s^{loc}\, C^{poinc}_{s,m,\kappa}) \sum_{\nu=1}^{n} |r|^2_{H^s(\omega_q)}\,.$$

The upper estimate in (5.9) follows from (5.11) and (5.7). □

In [19], we present numerical experiments for two-dimensional problems where $\Gamma = \partial\Omega$ is a curve in $\mathbb{R}^2$. These numerical results confirm the theoretical results of Theorem 5.1 that the so-called efficiency index $\sqrt{\sum_{q\in\mathcal{N}_\triangle}\varepsilon_q^2}\Big/\|e\|_{H^{s+\alpha}(\Gamma)}$ is bounded from above and below by constants independent of u and $\triangle$. They also demonstrate that the error indicators $\{\varepsilon_q\}_{q\in\mathcal{N}_\triangle}$ are a proper tool to control the adaptive mesh refinement since the discretization errors decrease in a very efficient way.

In the case (1.3) (i.e., $A : H^{s+2\alpha}(\Gamma) \to H^s(\Gamma)$, $\alpha \in \mathbb{R}$, $s \ge 0$), the residual $r \in H^s(\Gamma)$ is a function and Theorem 5.1 shows that local H^s semi-norms of r are efficient and reliable error indicators.

In the case (1.2) (i.e., $A : H^\alpha(\Gamma) \to H^{-\alpha}(\Gamma)$, $\alpha > 0$), the operator A can be interpreted in a broader sense as a differential operator and the residual $r := Au_{\mathcal{G}} - g \in H^{-\alpha}(\Gamma) = (H^\alpha(\Gamma))'$ is a functional. For this case, we considered in [15–17] error indicators, that were introduced and investigated by Babuška and Rheinboldt in [1] for FEM. We showed in [15–17] that these Babuška-Rheinboldt error indicators (BR error indicators) are also efficient and reliable for BEM in the case (1.2). The BR error indicators are local quantities of r and they can be interpreted as local $H^{-\alpha}$ norms of the residual $r \in H^{-\alpha}(\Gamma)$.

Hence, in both cases local (semi-)norms of the residual are efficient and reliable error indicators for Galerkin boundary element methods.

The error indicators $\{\varepsilon_q\}_{q\in\mathcal{N}_\triangle}$ introduced in Theorem 5.1 are efficient for arbitrary meshes and reliable for shape regular meshes. Shape regular meshes are suitable for adaptive mesh refinement, since they may contain small elements as well as large; the minimal angle condition only restricts the shape of an element not its size.

For error indicators, the reliability is more important then the efficiency. Due to our analysis in Sect. 3 and 4, we can also introduce error indicators $\{\hat{\varepsilon}_\tau\}_{\tau\in\triangle}$ which are reliable for arbitrary meshes and efficient for shape regular meshes.

Theorem 5.2 *Let $s \in (0,1)$. Using the same assumptions and notation as in Theorem 5.1, then we obtain for any solution $u \in H^{s+2\alpha}(\Gamma)$ and for all meshes $\triangle$ the following estimate for the Galerkin error $e \in H^{s+2\alpha}(\Gamma)$:*

$$(5.12) \qquad \hat{C}^{eff} \sum_{\tau\in\triangle} \hat{\varepsilon}_\tau^2 \;\le\; \|e\|^2_{H^{s+2\alpha}(\Gamma)} \;\le\; \hat{C}^{rel} \sum_{\tau\in\triangle} \hat{\varepsilon}_\tau^2 \,,$$

with

$$\hat{\varepsilon}_\tau^2 \;:=\; \int_\tau \int_{\omega_\tau} \frac{|r(\xi) - r(\eta)|^2}{|\xi - \eta|^{2+2\sigma}} \, d\xi \, d\eta \;+\; d_\tau^{-2\sigma} \|r\|^2_{L^2(\tau)} \,.$$

$\hat{\varepsilon}_\tau$ is a local quantity the residual r. The reliability of the error indicators $\{\hat{\varepsilon}_\tau\}_{\tau\in\triangle}$ holds for arbitrary meshes and the efficiency holds for shape regular meshes $\triangle \in \mathcal{M}_\kappa(\Gamma)$ with $\kappa > 0$. The constants $\hat{C}^{eff}$ and $\hat{C}^{rel}$ in (5.12) are independent of u, $\triangle$ and $\mathcal{G}$, they are explicitly given by

$$\hat{C}^{eff} \;=\; \frac{C_1^{op}}{3(1 + C^{poinc}_{s,m,\kappa})} \qquad \textit{and} \qquad \hat{C}^{rel} \;:=\; C_2^{op} \, \max\{1, C_s^{loc}\} \,.$$

The reliability constant $\hat{C}^{rel}$ of the error indicators $\{\hat{\varepsilon}_\tau\}_{\tau\in\triangle}$ is better than the reliability constant C^{rel} of the indicators $\{\varepsilon_q\}_{q\in\mathcal{N}_\triangle}$ introduced in Theorem 5.1.

Proof. The upper estimate in (5.12) follows from

$$\begin{aligned} \|e\|^2_{H^{s+2\alpha}(\Gamma)} \;&\overset{(5.7)}{\le}\; C_2^{op} \, \|r\|^2_{H^s(\Gamma)} \;\le \\ &\overset{(3.10)}{\le}\; C_2^{op} \sum_{\tau\in\triangle} \Big[\int_\tau \int_{\omega_\tau} \frac{|r(\xi) - r(\eta)|^2}{|\xi - \eta|^{2+2\sigma}} \, d\xi \, d\eta \;+\; C_s^{loc} \, d_\tau^{-2\sigma} \|r\|^2_{L^2(\tau)} \Big] \\ &\le\; C_2^{op} \, \max\{1, C_s^{loc}\} \sum_{\tau\in\triangle} \hat{\varepsilon}_\tau^2 \,. \end{aligned}$$

For the proof of the lower estimate in (5.12), let $\triangle \in \mathcal{M}_\kappa(\Gamma)$ be a shape regular mesh. The Galerkin residual $r = Au_\mathcal{G} - g$ is orthogonal to $\mathcal{G}$ and consequently orthogonal to $H^m_\triangle(\varGamma)$. Hence, we may apply Lemma 3.8 to r and obtain

$$\begin{aligned}\sum_{\tau\in\triangle}\hat{\varepsilon}_\tau^2 &\overset{(3.11)}{\leq} \sum_{q\in\mathcal{N}_\triangle}|r|^2_{H^\sigma(\omega_q)} + \sum_{\tau\in\triangle} d_\tau^{-2\sigma}\|r\|^2_{L^2(\tau)}\,,\\ &\overset{L.3.8}{\leq} (1+C^{poinc}_{s,m,\kappa})\sum_{q\in\mathcal{N}_\triangle}|r|^2_{H^s(\omega_q)}\\ &\overset{(5.10)}{\leq} (1+C^{poinc}_{s,m,\kappa})\,\frac{3}{C_1^{op}}\,\|e\|^2_{H^{s+2\alpha}(\varGamma)}\,.\end{aligned}$$

□

Remark 5.3 In Theorems 5.1 and 5.2, we assume for the Galerkin trial space $\mathcal{G}$ that

(5.13) $$H^m_\triangle(\varGamma)\ \subseteq\ \mathcal{G}\ \subseteq\ \mathbb{P}_\triangle(\varGamma)\quad\text{with}\quad m\in\{0,1,5,6\}\,.$$

For $m\in\{0,1\}$, $\mathcal{G}$ might be for example a space of piecewise polynomials with inter-element smoothness of order $m-1$, i.e., $\mathcal{G}:=\mathbb{P}^\delta_\triangle(\varGamma)\cap C^{m-1}(\varGamma)$ ($C^{-1}(\varGamma):=L^2(\varGamma)$) with an arbitrary local degree vector $\delta=(\delta_\tau)_{\tau\in\triangle}\in\mathbb{N}_0^\triangle$ (see Definition 2.2). For $m\in\{5,6\}$, $\mathcal{G}:=\mathbb{P}^\delta_\triangle(\varGamma)\cap C^1(\varGamma)$ is also an example. The reliability constant in Theorem 5.1 and the efficiency constant in Theorem 5.2 depend on m in (5.13). This is no restriction for the hp-method since m describes only the inter-element smoothness of $\mathcal{G}$, the local polynomial degrees in $\delta\in\mathbb{N}_0^\triangle$ are not restricted by m.

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A Appendix

In this appendix, we prove Lemma 3.5 for $m=5$. For that, let $T\in\triangle_\mu$ with $T\subseteq\omega_p$. Then, p is one vertex of T, let a,b be the other vertices. In order to prove (3.16) for $m=5$, it is sufficient to show that

(A.1) $$\int_T|1-\varphi_p^{[5]}|^2dx\ \leq\ \Big(1-\frac{3}{25}\Big)area(T)\,,$$

provided that (3.17) holds. Since Argyris elements are independent of translation, we assume without loss of generality that $p=0$, i.e., $\binom{0}{0}$, a,b are the vertices of T. Let $\hat{T}$ be the reference triangle with vertices $\binom{0}{0}$, $\binom{1}{0}$, $\binom{0}{1}$ and consider the linear and bijective transformation $\varPsi:T\to\hat{T}$,

$$\varPsi(x)\ :=\ \frac{1}{a_1b_2-a_2b_1}\begin{pmatrix}b_2 & -b_1\\ -a_2 & a_1\end{pmatrix}x\ =\ \frac{1}{\langle a,Qb\rangle}\begin{pmatrix}(Qb)^{\mathrm{T}}\\ -(Qa)^{\mathrm{T}}\end{pmatrix}x$$

with the rotation $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then, $\Psi(a) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\Psi(b) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. For the construction of the Argyris element $\varphi_p^{[5]}$ on T, we compute the corresponding finite element function on $\hat{T}$, i.e., we compute $\hat{\varphi} : \hat{T} \to \mathbb{R}$ such that $\varphi_p^{[5]}(x) = \hat{\varphi}(\Psi(x))$ for all $x \in T$. Then, we obtain

$$\text{(A.2)} \qquad \begin{aligned} \int_T |1 - \varphi_p^{[5]}(x)|^2 dx &= \int_T |1 - \hat{\varphi}(\Psi(x))|^2 dx \\ &= \frac{1}{|\det(\Psi')|} \int_{\hat{T}} |1 - \hat{\varphi}(x)|^2 dx \\ &= |\langle a, Qb \rangle| \int_{\hat{T}} |1 - \hat{\varphi}(x)|^2 dx \\ &= 2\, area(T) \int_{\hat{T}} |1 - \hat{\varphi}(x)|^2 dx \,. \end{aligned}$$

Since normal derivatives change under affine transformations (see [10, page 85]), $\hat{\varphi}$ does not coincide with the Argyris element on $\hat{T}$. But the 21 conditions for the computation of $\varphi_p^{[5]}$ on T can be transformed into conditions for $\hat{\varphi}$. The following 18 conditions for $\varphi_p^{[5]}$

$$\begin{aligned} &\varphi_p^{[5]}(p) = \varphi_p^{[5]}(0) = 1, \quad \varphi_p^{[5]}(a) = \varphi_p^{[5]}(b) = 0, \\ &(\partial_1^{\nu_1} \partial_2^{\nu_2} \varphi_p^{[5]})(x) = 0 \quad \text{for all } x \in \{p, a, b\}\,,\ \nu \in \mathbb{N}_0^2 \text{ with } |\nu| \in \{1, 2\} \end{aligned}$$

can be transformed directly into the following conditions for $\hat{\varphi}$

$$\text{(A.3)} \qquad \begin{aligned} &\hat{\varphi}(0) = 1, \quad \hat{\varphi}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{\varphi}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \\ &(\partial_1^{\nu_1} \partial_2^{\nu_2} \hat{\varphi})(x) = 0 \quad \text{for all } x \in \left\{0, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}, \\ &\qquad\qquad \nu \in \mathbb{N}_0^2 \text{ with } |\nu| \in \{1, 2\}\,. \end{aligned}$$

The condition for the normal derivative in the midpoint of the edge $[a, b]$ of T is transformed in the following way: since the outer normal direction $n : \partial T \to \mathbb{R}^2$ coincides on $[a, b]$ with $c\,Q(b - a)$, where $c \in \mathbb{R}$ is a constant, we obtain for the normal derivative

$$\begin{aligned} &(\partial_n \varphi_p^{[5]})(\tfrac{1}{2}(a + b)) \\ &\quad = (\varphi_p^{[5]})'(\tfrac{1}{2}(a + b)) \cdot n(\tfrac{1}{2}(a + b)) \\ &\quad = c\,(\hat{\varphi} \circ \Psi)'(\tfrac{1}{2}(a + b)) \cdot Q(b - a) \\ &\quad = c\,\hat{\varphi}'(\Psi(\tfrac{1}{2}(a + b))) \cdot \underbrace{\Psi'(\tfrac{1}{2}(a + b))}_{= \frac{1}{\langle a, Qb \rangle} \begin{pmatrix} (Qb)^{\mathrm{T}} \\ -(Qa)^{\mathrm{T}} \end{pmatrix}} \cdot Q(b - a) \end{aligned}$$

$$
\begin{aligned}
&= \frac{c}{\langle a, Qb\rangle}\,\hat\varphi'(\tfrac12\mathbf{1})\cdot\begin{pmatrix}\langle Qb, Q(b-a)\rangle\\ -\langle Qa, Q(b-a)\rangle\end{pmatrix} \qquad \text{with } \mathbf{1} := \binom{1}{1}\\
&= \frac{c}{\langle a, Qb\rangle}\,\hat\varphi'(\tfrac12\mathbf{1})\cdot\begin{pmatrix}|b|^2-\langle a,b\rangle\\ |a|^2-\langle a,b\rangle\end{pmatrix}\\
&= \frac{c}{\langle a, Qb\rangle}\Big[[\,|b|^2-\langle a,b\rangle](\partial_1\hat\varphi)(\tfrac12\mathbf{1}) + [\,|a|^2-\langle a,b\rangle](\partial_2\hat\varphi)(\tfrac12\mathbf{1})\Big].
\end{aligned}
$$

Consequently, the condition $(\partial_n\varphi_p^{[5]})(\frac12(a+b)) = 0$ is equivalent to

$$
\text{(A.4)}\quad [\,|b|^2-\langle a,b\rangle](\partial_1\hat\varphi)(\tfrac12\mathbf{1}) + [\,|a|^2-\langle a,b\rangle](\partial_2\hat\varphi)(\tfrac12\mathbf{1}) \;=\; 0\,.
$$

Analogously, the condition for the normal derivative in the midpoint of the edge $[p,a] = [0,a]$ of T, i.e., $(\partial_n\varphi_p^{[5]})(\frac12 a) = 0$, is equivalent to

$$
\text{(A.5)}\qquad \langle a,b\rangle(\partial_1\hat\varphi)(\tfrac12\tbinom{1}{0}) - |a|^2(\partial_2\hat\varphi)(\tfrac12\tbinom{1}{0}) \;=\; 0
$$

and the condition for the normal derivative in the midpoint of the edge $[0,b]$ of T, i.e., $(\partial_n\varphi_p^{[5]})(\frac12 b) = 0$, is equivalent to

$$
\text{(A.6)}\qquad |b|^2(\partial_1\hat\varphi)(\tfrac12\tbinom{0}{1}) - \langle a,b\rangle(\partial_2\hat\varphi)(\tfrac12\tbinom{0}{1}) \;=\; 0\,.
$$

The 21 coefficients c_ν, $\nu = (\nu_1,\nu_2) \in \mathbb{N}_0^2$ with $|\nu| \le 5$, of the local polynomial

$$
\hat\varphi(x) \;=\; \sum_{\substack{\nu\in\mathbb{N}_0^2\\ |\nu|\le 5}} c_\nu\, x_1^{\nu_1}x_2^{\nu_2} \qquad \text{for } x\in\hat T
$$

can be computed by means of the 21 equations in (A.3), (A.4), (A.5), (A.6). Solving this equation system with coefficients depending on $|a|^2$, $|b|^2$ and $\langle a,b\rangle$ (by MAPLE) shows that

$$
\begin{aligned}
\hat\varphi(x) \;=\; & 1 - 10x_1^3 - 30\tfrac{\langle a,b\rangle}{|a|^2}x_1^2x_2 - 30\tfrac{\langle a,b\rangle}{|b|^2}x_1x_2^2 - 10x_2^3 + 15x_1^4\\
&+ 60\tfrac{\langle a,b\rangle}{|a|^2}x_1^3x_2 + 30\tfrac{2|a|^2\langle a,b\rangle+2|b|^2\langle a,b\rangle-|a|^2|b|^2}{|a|^2|b|^2}x_1^2x_2^2 + 60\tfrac{\langle a,b\rangle}{|b|^2}x_1x_2^3\\
&+ 15x_2^4 - 6x_1^5 - 30\tfrac{\langle a,b\rangle}{|a|^2}x_1^4x_2 - 30\tfrac{|a|^2\langle a,b\rangle+2|b|^2\langle a,b\rangle-|a|^2|b|^2}{|a|^2|b|^2}x_1^3x_2^2\\
&- 30\tfrac{2|a|^2\langle a,b\rangle+|b|^2\langle a,b\rangle-|a|^2|b|^2}{|a|^2|b|^2}x_1^2x_2^3 - 30\tfrac{\langle a,b\rangle}{|b|^2}x_1x_2^4 - 6x_2^5\,.
\end{aligned}
$$

MAPLE calculations also show that

$$
\begin{aligned}
&\int_{\hat T}|1-\hat\varphi(x)|^2dx\\
&= \frac{1}{2772}\Big(70\frac{\langle a,b\rangle}{|a|^2} + 70\frac{\langle a,b\rangle}{|b|^2} + 6\frac{\langle a,b\rangle^2}{|a|^4} + 6\frac{\langle a,b\rangle^2}{|b|^4} + 9\frac{\langle a,b\rangle^2}{|a|^2|b|^2} + 679\Big).
\end{aligned}
$$

Without loss of generality, let $|b| \leq |a|$ and let $\kappa \in [0,1]$ such that $|b| = \kappa|a|$. Then, we obtain (using also $\langle a,b \rangle \leq |a|\,|b|$)

$$\text{(A.7)} \quad \int_{\hat{T}} |1 - \hat{\varphi}(x)|^2 dx \leq \frac{1}{2772}\Big(70\Big(\kappa + \frac{1}{\kappa}\Big) + 6\Big(\kappa^2 + \frac{1}{\kappa^2}\Big) + 688\Big) =: I(\kappa)\,.$$

Assumption (3.17) implies that $\frac{1}{5} \leq \kappa$. Since $I(t)$ decreases monotone in $t \in [0,1]$, we obtain $I(\kappa) \leq I(\frac{1}{5})$. This together with (A.2) and (A.7) yields

$$\int_T |1 - \varphi_p^{[5]}(x)|^2 dx \;\leq\; 2\, area(T)\, I(\tfrac{1}{5}) \;\leq\; (1 - \tfrac{3}{25})\, area(T)\,.$$

This shows (A.1) and completes the proof of Lemma 3.5 for $m = 5$. □

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