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Stability of piecewise polynomial collocation for computing periodic solutions of delay differential equations

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Summary. We prove numerical stability of a class of piecewise polynomial collocation methods on nonuniform meshes for computing asymptotically stable and unstable periodic solutions of the linear delay differential equation $\dot{y}(t) = a(t)y(t) + b(t)y(t-\tau) + f(t)$ by a (periodic) boundary value approach. This equation arises, e.g., in the study of the numerical stability of collocation methods for computing periodic solutions of nonlinear delay equations. We obtain convergence results for the standard collocation algorithm and for two variants. In particular, estimates of the difference between the collocation solution and the true solution are derived. For the standard collocation scheme the convergence results are "unconditional", that is, they do not require mesh-ratio restrictions. Numerical results that support the theoretical findings are also given.

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1 Introduction

We study in this paper the stability of piecewise collocation for computing periodic solutions to linear systems of delay differential equations (DDEs),

$$\dot{y}(t) = a(t)y(t) + b(t)y(t-\tau) + f(t), \tag{1}$$

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where a, b and f are periodic with period 1 and $\tau > 0$ is a fixed delay. We rewrite (1) as

$$\dot{y}(t) = a(t)y(t) + b(t)y((t-\tau) \bmod 1) + f(t),\ t \in [0,1], \tag{2}$$

and solve for $y \in C_{\mathrm{c}}^1[0,1]$ given $a, b \in \mathbf{C}_{\mathrm{c}}^0[0,1]$ and $f \in C_{\mathrm{c}}^0[0,1]$. The "circulant" (periodic) spaces $C_{\mathrm{c}}^k[0,1]$ are defined as

$$C_{\mathrm{c}}^k[0,1] = \{y \in C^k([0,1], \mathbb{R}^n) \,|\, y^{(j)}(0) = y^{(j)}(1),\ j = 0, \ldots, k\},$$

and, for matrix-valued functions,

$$\mathbf{C}_{\mathrm{c}}^k[0,1] = \{y \in C^k([0,1], \mathbb{R}^{n\times n}) \,|\, y^{(j)}(0) = y^{(j)}(1),\ j = 0, \ldots, k\}.$$

We consider the standard collocation algorithm, and two variants. Under appropriate assumptions we prove stability of the collocation method and obtain an estimate of the difference between the collocation solution and the true solution of (2). We follow the approach in [8] (and references therein), which gives convergence results using only elementary analytical techniques. Moreover, for the standard collocation algorithm our convergence results are "unconditional", that is, they do not require mesh-ratio restrictions. Unconditionally convergent methods are desirable for "difficult" problems, where adaptive meshes are essential, and where the ratio of the largest mesh interval to the smallest mesh interval can be large. Such problems include singularly perturbed equations, relaxation oscillations, "bursting" periodic orbits, near-homoclinic periodic orbits, etc. An example of a near-homoclinic periodic orbit is given in Sect. 5.

DDEs can be seen as an intermediate step between ordinary differential equations (ODEs) and partial differential equations (PDEs). Like ODEs, DDEs are formulated in a finite dimensional space; like PDEs, DDEs inherently define infinite dimensional systems. As such, DDEs allow more powerful modelling than ODEs, yet are at the same time more tractable than PDEs. Our results (and that of others) show that DDEs indeed retain some, but not all, of the computational tractability of ordinary differential equations.

The analysis in this paper is intended to provide a theoretical support for using collocation methods in bifurcation software, specifically for the continuation of periodic solutions, as done in AUTO for ordinary (non-delay) differential equations. Hence, our practical interest is in the continuation of periodic solutions of systems of autonomous nonlinear DDEs. However, to avoid cumbersome details without loss of essential features in the proofs, we avoid the use of a phase condition. Moreover, from the general stability theory for discretizations of nonlinear operator equations, see for example [20], it follows that it is sufficient to study the numerical stability and convergence properties of the discrete method applied to the type of linear problems obtained by linearization. Furthermore, we restrict the analysis to systems of first-order DDEs with one delay. Extension to higher order systems and systems with multiple delays can easily be carried out along the same lines.

Relatively little work has been done on developing numerical continuation software for periodic solutions of DDEs, and, more generally, for functional differential equations. The software package XPPAUT [13] has some capabilities for delay equations, but this does not include continuation of periodic solutions. Certain numerical continuation schemes have been developed, see for example [15,10,28] and a first generally available continuation package has recently appeared [11]. For more progress in this direction see [14,24,12].

A number of collocation schemes for functional differential equations have been investigated for boundary value problems with *finite defect*; see [21] for a precise definition and [26,6,4] for results of this type. A solution profile of a functional differential equation is uniquely determined if one provides an initial function segment. In boundary value problems of finite defect, the initial function segment is given, up to a finite number of degrees of freedom; and the boundary condition applies in a finite dimensional space. Periodic solutions of DDEs cannot be found using such a scheme. In (2) we have circumvented the infinite dimensional initial condition (or, rather, boundary condition) using the modulo operation for the delayed argument.

Our presentation is structured as follows. We first treat the standard collocation scheme, introduced in Sect. 2. Stability and convergence estimates of this scheme are obtained in Sect. 3. In Sect. 4 we analyze two variants of the standard collocation scheme, each using a different interpolation scheme to evaluate the delayed argument. Some numerical results are presented in Sect. 5. Section 6 contains concluding remarks.

2 Piecewise polynomial collocation

Write Equation (2) as

$$
\begin{aligned}
&(Ly)(t) \equiv \dot{y}(t) - a(t)y(t) - b(t)y((t-\tau) \bmod 1) = f(t), \\
&\qquad t \in [0,1],
\end{aligned} \tag{3}
$$

where $a, b \in \mathbf{C}_{\mathrm{c}}^0[0,1]$, $f \in C_{\mathrm{c}}^0[0,1]$ and $L : C_{\mathrm{c}}^1[0,1] \to C_{\mathrm{c}}^0[0,1]$, We will also consider the homogeneous problem

$$
Ly = 0, \; y \in C_{\mathrm{c}}^1[0,1], \tag{4}
$$

which we assume to only admit the zero solution.

Introduce a mesh $h \equiv \{0 = t_0 < t_1 < \ldots < t_J = 1\}$, with $h_j \equiv t_{j+1} - t_j$ and $|h| \equiv \max_j h_j$. To each mesh point t_j, $j = 0, \ldots, J-1$, associate a polynomial p_j (or, more accurately, $p_{h,j}$, to indicate the dependence on h) of degree m or less, with coefficients in $\mathbb{R}^n$. Think of the subinterval $[t_j, t_{j+1}]$ as the domain of p_j, $j = 0, \ldots, J-1$. Let $\vec{p}_h \equiv \{p_j\}_{j=0}^{J-1}$, and for each fixed mesh h let $\vec{P}_h^m$ denote the space of all $\vec{p}_h$ satisfying the matching conditions

$$p_j(t_{j+1}) = p_{j+1}(t_{j+1}),\ j = 0, \ldots, J-2, \tag{5}$$

and

$$p_{J-1}(t_J) = p_0(t_0). \tag{6}$$

Hence $\vec{p}_h \in \vec{P}_h^m$ belongs to $C_c^0[0,1]$.

Define

$$\|\vec{p}_h\|_k \equiv \max_{l=0,\ldots,k}\ \max_{j=0,\ldots,J-1}\ \max_{t\in[t_j,t_{j+1}]} |p_j^{(l)}(t)|. \tag{7}$$

Correspondingly, for $p \in C_c^k[0,1]$ and $\vec{p}_h \in \vec{P}_h^m$,

$$\|\vec{p}_h - p\|_k \equiv \max_{l=0,\ldots,k}\ \max_{j=0,\ldots,J-1}\ \max_{t\in[t_j,t_{j+1}]} |p_j^{(l)}(t) - p^{(l)}(t)|. \tag{8}$$

The collocation equations for $\vec{p}_h \in \vec{P}_h^m$ are,

$$\begin{aligned} &\dot{p}_j(c_{j,i}) - a(c_{j,i})p_j(c_{j,i}) - b(c_{j,i})p_{k_{j,i}}((c_{j,i} - \tau) \bmod 1) = f(c_{j,i}),\\ &\qquad\qquad i = 1, \ldots, m,\ j = 0, \ldots, J-1, \end{aligned} \tag{9}$$

where for each j the $c_{j,i}$ are distinct points in $[t_j, t_{j+1}]$, and where $k_{j,i}$ is chosen such that $(c_{j,i} - \tau) \bmod 1 \in [t_{k_{j,i}}, t_{k_{j,i}+1}]$. We shall assume that the collocation points $c_{j,i}$ are locally semi-uniform, in accordance with the definition below.

Definition 2.1 *The collocation points $c_{j,i}$ are said to be* locally semi-uniform *if $\min_{i_1\neq i_2} |c_{j,i_1} - c_{j,i_2}| \geq K_0\, h_j$ for some constant K_0 that is independent of j and h.*

This assumption is satisfied for most reasonable choices of the collocation points, for example, for Gauss points and for uniformly distributed points. Note that semi-uniformity of the collocation points does not impose any restriction on the mesh.

3 Stability results

First, we adapt a lemma from [8].

Lemma 3.1 *Let $\{h^\nu\}_{\nu=1}^\infty$ be a sequence of meshes with $|h^\nu| \to 0$ as $\nu \to \infty$. For each ν let $\vec{p}_{h^\nu} \in \vec{P}_{h^\nu}^m$, $\|\vec{p}_{h^\nu}\|_1 = 1$. Then there is a subsequence $\{\vec{p}_{h^{\nu_k}}\}_{k=1}^\infty$ and a function $p \in C_c^0[0,1]$, such that $\|\vec{p}_{h^{\nu_k}} - p\|_0 \to 0$ as $k \to \infty$.*

This lemma is similar to Lemma 2.1 in [8]. In fact, it is a simple, special case, except for the extra matching condition (6) that ensures periodicity of p. Also, in the current setting, the lemma follows almost directly from the Ascoli Theorem [27, Sect. 9.8].

Theorem 3.1 *Let the homogeneous problem (4) only admit the zero solution. Let* $a, b \in \mathbf{C}_\mathrm{c}^0[0,1]$ *and* $f \in C_\mathrm{c}^0[0,1]$. *Assume that the collocation points are locally semi-uniform. Then there exist positive constants* K *and* δ, *such that the collocation equations admit a unique solution* $\vec{p}_h \in \vec{P}_h^m$, *and such that*

$$\|\vec{p}_h\|_1 \le K \max_{j,i} |f(c_{j,i})|, \tag{10}$$

whenever $|h| \in (0, \delta]$.

Proof. The proof of this theorem consists of three parts.

(i). If (9) does not have a unique solution $\vec{p}_h \in \vec{P}_h^m$ for all small $|h|$ then, since $\dim \vec{P}_h^m = nmJ$ equals the number of equations in (9) and since (9) can be interpreted as a linear system of equations, we can find a sequence of meshes $\{h^\nu\}_{\nu=1}^\infty$ with $|h^\nu| \to 0$ as $\nu \to \infty$ and corresponding $\vec{p}_{h^\nu} \in \vec{P}_{h^\nu}^m$, with $\|\vec{p}_{h^\nu}\|_1 = 1$, such that $\vec{p}_{h^\nu}$ satisfies the homogeneous equations corresponding to (9). By Lemma 3.1 there is a subsequence $\{\vec{p}_{h^\nu}\}_{\nu=1}^\infty$ and a function $p \in C_\mathrm{c}^0[0,1]$, such that $\|\vec{p}_{h^\nu} - p\|_0 \to 0$ as $\nu \to \infty$. Using the collocation equation we will show in part (iii) that in fact $p \in C_\mathrm{c}^1[0,1]$, and that p is a nontrivial solution of the homogeneous problem (4). This contradicts the first assumption of the theorem.

(ii). Assuming for the moment the existence of a unique $\vec{p}_h$ for all sufficiently small $|h|$, there remains the problem of establishing the bound (10). If (10) does not hold then we can find a sequence of meshes $\{h^\nu\}_{\nu=1}^\infty$, with $|h^\nu| \to 0$ as $\nu \to \infty$ and for each mesh quantities $\{f^\nu(c_{j,i})\}$ with $\max_{j,i} |f^\nu(c_{j,i})| \to 0$ as $\nu \to \infty$, such that the corresponding unique solution $\vec{p}_{h^\nu} \in \vec{P}_{h^\nu}^m$, of (9) has $\|\vec{p}_h\|_1 = 1$. As in (i), using Lemma 3.1 there is a subsequence $\{\vec{p}_{h^\nu}\}_{\nu=1}^\infty$ and a function $p \in C_\mathrm{c}^0[0,1]$, such that $\|\vec{p_{h^\nu}} - p\|_0 \to 0$ as $\nu \to \infty$. Again we claim that p is nontrivial and satisfies the homogeneous equations (4).

(iii). What remains to be proved in (i) is a special case of the completion of (ii). Hence, we only give details for the latter. Let $s \in [0,1]$ and consider the subsequence and its limit found previously. For each ν let j_ν be such that $s \in [t_{j_\nu}, t_{j_\nu+1}]$. Since $\dot{p}_{j_\nu}$ is a polynomial of degree at most $m-1$ we can write

$$\dot{p}_{j_\nu}(s) = \sum_{i=1}^{m} \psi_{j_\nu,i}(s) \dot{p}_{j_\nu}(c_{j_\nu,i}), \tag{11}$$

where for each j_ν the functions $\{\psi_{j_\nu,i}(t)\}_{i=1}^m$ denote the Lagrange interpolating polynomials for the points $\{c_{j_\nu,i}\}_{i=1}^m$. Abbreviate $j \equiv j_\nu$, $\psi_i \equiv \psi_{j,i}$, $c_i \equiv c_{j,i}$, $\tilde{s} \equiv (s-\tau) \bmod 1$ and $\tilde{c}_i \equiv (c_i - \tau) \bmod 1$ where $\tilde{c}_i \in [t_k, t_{k+1}]$. Note that k depends on j_ν and i. Then

$$\begin{aligned}
&\left|\dot{p}_j(s) - a(s)p(s) - b(s)p(\tilde{s})\right| \\
&\quad = \left|\sum_{i=1}^m \psi_i(s)\dot{p}_j(c_i) - a(s)p(s) - b(s)p(\tilde{s})\right| \qquad \text{(using (11))} \\
&\quad = \left|\sum_{i=1}^m \psi_i(s)\left\{f^\nu(c_i) + a(c_i)p_j(c_i) + b(c_i)p_k(\tilde{c}_i)\right\}\right. \\
&\qquad \left. -a(s)p(s) - b(s)p(\tilde{s})\right| \quad \text{(using (9))} \\
&\quad \leq \left|\sum_{i=1}^m \psi_i(s)f^\nu(c_i)\right| + \left|a(s)p(s) - \sum_{i=1}^m \psi_i(s)a(c_i)p_j(c_i)\right| \\
&\qquad \left|b(s)p(\tilde{s}) - \sum_{i=1}^m \psi_i(s)b(c_i)p_k(\tilde{c}_i)\right| \\
&\quad = \left|\sum_{i=1}^m \psi_i(s)f^\nu(c_i)\right| + \left|\sum_{i=1}^m \psi_i(s)\left(a(s)p(s) - a(c_i)p_j(c_i)\right)\right| \\
&\qquad + \left|\sum_{i=1}^m \psi_i(s)\left(b(s)p(\tilde{s}) - b(c_i)p_k(\tilde{c}_i)\right)\right| \\
&\qquad\qquad \left(\text{using } \sum_{i=1}^m \psi_{j,i}(t) \equiv 1\right) \\
&\quad \leq m\, K_1 \max_{j,i} |f^\nu(c_{j,i})| + K_1 \sum_{i=1}^m |a(s)p(s) - a(c_i)p_j(c_i)| \\
&\qquad + K_1 \sum_{i=1}^m |b(s)p(\tilde{s}) - b(c_i)p_k(\tilde{c}_i)|. \qquad (12)
\end{aligned}$$

In the final inequality,

$$K_1 \equiv \frac{1}{K_0^{m-1}} \geq \max_{j,i} \max_{t\in[t_j,t_{j+1}]} |\psi_{j,i}(t)|,$$

is independent of the mesh, due to the semi-uniformity of the collocation points (cf. Definition 2.1). This final expression becomes arbitrarily small as $\nu \to \infty$. This is a consequence of the choice of

f^ν and the convergence of $\vec{p}_{h^\nu}$ to p. Hence $\dot{p}_{h^\nu,j_\nu}(s)$ converges to $a(s)p(s)+b(s)p((s-\tau) \bmod 1)$. In fact, this convergence is uniform in s. Indeed, we have,

$$\begin{aligned}&|a(s)p(s) - a(c_i)p_j(c_i)|\\ &\quad \le |a(s)|(|p(s) - p(c_i)| + |p(c_i) - p_j(c_i)|)\\ &\qquad +|p_j(c_i)||a(s) - a(c_i)|,\end{aligned}$$

and

$$\begin{aligned}&|b(s)p(\tilde{s}) - b(c_i)p_k(\tilde{c}_i)|\\ &\quad \le |p(\tilde{s})||b(s) - b(c_i)| + |b(c_i)|(|p(\tilde{s}) - p(\tilde{c}_i)|\\ &\qquad +|p(\tilde{c}_i) - p_k(\tilde{c}_i)|).\end{aligned}$$

Thus uniform convergence follows from the boundedness and continuity (and thus also the uniform continuity) of a, b, and p; from the fact $\|\vec{p}_{h^\nu}\|_1 = 1$ and from the uniform convergence of $\vec{p}_{h^\nu}$ to p (by Lemma 3.1).

Let $\dot{\vec{p}}_{h^\nu}$ denote the (at mesh points discontinuous) derivative of $\vec{p}_{h^\nu}$. Using the uniform convergence established above we have

$$\int_0^{\mathrm{t}} \dot{\vec{p}}_{h^\nu}(s)\mathrm{d}s - \int_0^{\mathrm{t}} a(s)p(s) + b(s)p((s-\tau) \bmod 1)\mathrm{d}s \to 0 \text{ as } \nu \to \infty.$$

Upon integration it follows that (because of the continuity of $\vec{p}_{h^\nu}$),

$$\vec{p}_{h^\nu}(t) - \vec{p}_{h^\nu}(0) - \int_0^{\mathrm{t}} a(s)p(s) + b(s)p((s-\tau) \bmod 1)\mathrm{d}s \to 0 \text{ as } \nu \to \infty.$$

Taking the limit we obtain

$$p(t) - p(0) - \int_0^{\mathrm{t}} a(s)p(s) + b(s)p((s-\tau) \bmod 1)\mathrm{d}s = 0.$$

This implies in particular that p is continuously differentiable on $[0,1]$. Differentiation gives

$$\dot{p}(t) - a(t)p(t) - b(t)p((t-\tau) \bmod 1) = 0, \tag{13}$$

and in particular $\dot{p}(0) = \dot{p}(1)$. Hence $Lp = 0$, $p \in C^1_{\mathrm{c}}[0,1]$. Using (13) in (12), and recalling that already $\|\vec{p}_{h^\nu} - p\|_0 \to 0$ as $\nu \to \infty$, we have

$$\|\vec{p}_{h^\nu} - p\|_1 \to 0 \text{ as } \nu \to \infty.$$

Since $\|\vec{p}_{h^\nu}\|_1 = 1$ for all ν this implies that $p \neq 0$ and a contradiction has been arrived at. $\square$

Note the importance of the fact that the interpolation formula (11) is *local*, i.e., that the derivative of the local polynomial at any t can be expressed in terms of the derivatives of the polynomial at the local collocation points. Without this property it would be difficult to establish stability; in fact such a scheme may not be stable, even on uniform meshes. In particular, the local interpolation property does not hold if the approximating spaces are required to have higher than C^0 continuity. C^1 continuity can be accommodated in the proof, provided that the mesh points are included as collocation points (using e.g. Gauss-Lobatto points), i.e., provided the smoothness arises from collocation.

Having established the existence of a unique collocation solution for sufficiently small $|h|$, we can investigate its limit as the mesh is refined. This limit is, of course, a solution of the inhomogeneous equation (3), as will be shown below. From a Fredholm Alternative principle for periodic solutions of delay equations, it follows that the inhomogeneous equation (3) has a unique solution if the homogeneous problem (4) admits only the zero solution. (See [21] for a very general result of this type.) We note that Theorem 3.1 does not actually rely on this principle. Also, as is clear from the proof, Theorem 3.1 remains valid if f is replaced by a sequence of functions f_h, in fact, if f is replaced by a sequence of pointwise values $f_h(c_{j,i})$. We exploit this fact below for a sequence of values $\tau_h(c_{j,i})$.

Let $\vec{p}_h$ be the solution of the collocation equations (9). Let $\vec{\rho}_h \equiv \{\rho_j\}_{j=0}^{J-1}$, where ρ_j is a polynomial of degree m which interpolates the exact solution $y(x)$ of (3) at the $m+1$ points $\{t_{j+\frac{i}{m}}\}_{i=0}^{m}$ in $[t_j, t_{j+1}]$. Note that $i=0$ and $i=m$ correspond to interval end points. The additional points $t_{j+\frac{i}{m}}$, $i=1,\ldots,m-1$ are distinct points (unrelated to the collocation points) in (t_j, t_{j+1}). Then $\vec{\rho}_h$ satisfies the matching conditions (5), (6) and, from Lagrange interpolation,

$$
\begin{aligned}
&y(t)-\rho_j(t) = r_j(t)d_j(t) = r_j(t)\frac{y^{(m+1)}(\xi(t))}{(m+1)!}, \quad \text{for some} \\
&\xi(t) \in (t_j, t_{j+1}),
\end{aligned} \tag{14}
$$

with $r_j(t) \equiv \prod_{i=0}^{m}(t - t_{j+\frac{i}{m}})$ and $t \in [t_j, t_{j+1}]$. If y is $m+1+k$ times continuously differentiable then $d_j(t)$ in (14) is k times continuously differentiable, as follows from the Newton divided difference representation of r_j; see [19].

Let $\vec{\xi}_h = \vec{p}_h - \vec{\rho}_h \in \vec{P}_h^m$. Then we define the *local truncation errors* as the values of the collocation equations (9) applied to $\vec{\xi}_h$, i.e.,

$$
\tau_h(c_{j,i}) = \dot{\xi}_j(c_{j,i}) - a(c_{j,i})\xi_j(c_{j,i}) - b(c_{j,i})\xi_{k_{j,i}}((c_{j,i}-\tau) \bmod 1). \tag{15}
$$

Using the collocation equations (9) and the DDE (3), we have

$$\begin{aligned}\tau_h(c_{j,i}) &= f(c_{j,i}) - \big(\dot\rho_j(c_{j,i}) - a(c_{j,i})\rho_j(c_{j,i})\\ &\quad -b(c_{j,i})\rho_{k_{j,i}}((c_{j,i}-\tau) \bmod 1)\big)\\ &= \dot y(c_{j,i}) - \dot\rho_j(c_{j,i}) - a(c_{j,i})(y(c_{j,i}) - \rho_j(c_{j,i}))\\ &\quad -b(c_{j,i})(y((c_{j,i}-\tau) \bmod 1) - \rho_{k_{j,i}}((c_{j,i}-\tau) \bmod 1)).\end{aligned} \tag{16}$$

If y is sufficiently differentiable then it follows from elementary estimates of the error in an interpolation polynomial and its derivative (cf. (14)) that $\max_{j,i}|\tau_h(c_{j,i})| = \mathcal{O}(|h|^m)$, regardless of the choice of the collocation points.

The local truncation errors will be used to derive an estimate on the error $y(t) - \vec p_h(t)$. Note that the approximate solution $\vec p_h$ of (9), and hence the error $y(t) - \vec p_h(t)$, does not depend in any way on the points $t_{j+\frac{i}{m}}$, $i = 1,\dots,m-1$. However, the $t_{j+\frac{i}{m}}$ can be used as a device to show that for a class of special choices of the collocation points the estimate for $|\tau_h(c_{j,i})|$ can be somewhat improved. This class of collocation points can be characterized as the m points in (t_j, t_{j+1}) where $\dot r_j$ vanishes, for any choice of $t_{j+\frac{i}{m}}$, $i = 1,2,\dots,m-1$. For example, if $m = 2$, taking $[-1,1]$ as "reference interval", we have $r(t) = (t+1)(t-t_{1/2})(t-1)$. To get symmetrically placed collocation points set $t_{1/2} = 0$. We find that $\dot r(c) = 0$ if $c = \pm 1/\sqrt{3}$, i.e., the Gauss points. Note, however, that choices other than $t_{1/2} = 0$, and its corresponding collocation points, can be used to get the extra order of accuracy in the local truncation error. Below we verify in general that the special class of collocation points that gives higher order accuracy includes Gauss points.

Theorem 3.2 *Assume $y \in C_{\mathrm{c}}^{m+2}[0,1]$. If the $c_{j,i}$ are the roots of the m-th degree Gauss-Legendre orthogonal polynomial with respect to $[t_j, t_{j+1}]$, then it is possible to choose the $t_{j+\frac{i}{m}}$, $i = 1,\dots,m-1$ in the definition of ρ_j such that $\max_{j,i}|\tau_h(c_{j,i})| = \mathcal{O}(|h|^{m+1})$.*

Proof. We know that the divided difference $d_j(t)$ is smooth if $y \in C_{\mathrm{c}}^{m+2}$ $[0,1]$. From (14) and (16) it then follows that

$$\begin{aligned}\max_{j,i}|\tau_h(c_{j,i})| &= \max_{j,i}|\dot y(c_{j,i}) - \dot\rho_j(c_{j,i})| + \mathcal{O}(|h|^{m+1})\\ &= \max_{j,i}|\dot r_j(c_{j,i})d_j(c_{j,i}) + r_j(c_{j,i})\dot d_j(c_{j,i})|\\ &\qquad +\mathcal{O}(|h|^{m+1}),\\ &= \max_{j,i}|\dot r_j(c_{j,i})d_j(c_{j,i})| + \mathcal{O}(|h|^{m+1}).\end{aligned} \tag{17}$$

By differentiating the generating function and the fundamental recurrence formula, it is easy to obtain the following formula for the m-th degree Gauss-Legendre polynomial P_m (cf. [19]),

$$P_m(t) = \frac{1}{2m+1}(\dot{P}_{m+1}(t) - \dot{P}_{m-1}(t)). \tag{18}$$

In our case the interval under consideration is $[t_j, t_{j+1}]$. The polynomial $P_{m+1}(t) - P_{m-1}(t)$ then has roots at t_j, t_{j+1}, and at $m-1$ distinct points in (t_j, t_{j+1}), since it is orthogonal to $P_0, \ldots, P_{m-2}$. Thus, if we choose the additional points $t_{j+\frac{i}{m}}, i = 1, \ldots, m-1$ as the $m-1$ roots of $P_{m+1}(t) - P_{m-1}(t)$ then $r_j(t)$ is, in fact, a scalar multiple of $P_{m+1}(t) - P_{m-1}(t)$. Hence $\dot{r}_j$ is a scalar multiple of $P_m(t)$, and therefore $\dot{r}_j$ is zero at the roots of $P_m(t)$, i.e. at the Gauss collocation points. From (17) we then have $\max_{j,i} |\tau_h(c_{j,i})| = \mathcal{O}(|h|^{m+1})$. □

We can now use Theorem 3.1 and Theorem 3.2 to derive the following convergence result.

Theorem 3.3 *Let the homogeneous problem (4) only admit the zero solution. Let y be the (unique) solution of (3), and assume that a, b and f are sufficiently smooth, so that $y \in C_{\mathrm{c}}^{m+1}[0,1]$. Also assume that the collocation points are locally semi-uniform. Then there exists positive constants C and δ such that, whenever $|h| \in (0, \delta]$,*

$$\max_{t\in[0,1]} |y(t) - \vec{p}_h(t)| \leq C|h|^m.$$

Moreover, if $y \in C_{\mathrm{c}}^{m+2}[0,1]$ and if the collocation points are chosen to be Gauss points then

$$\max_{t\in[0,1]} |y(t) - \vec{p}_h(t)| \leq C|h|^{m+1}.$$

Proof. We have $\max_{j,i} |\tau_h(c_{j,i})| \leq C_2|h|^{m+\sigma}$, where $\sigma = 0$ in general, but $\sigma = 1$ for special choices of the collocation points, including Gauss points (cf. Theorem 3.2). The constant C_2 does not depend on j and h, when $|h|$ is small enough. Now

$$|y(t) - \vec{p}_h(t)| \leq |y(t) - \vec{\rho}_h(t)| + |\vec{\rho}_h(t) - \vec{p}_h(t)|. \tag{19}$$

The first term can be estimated from the local interpolation error. The second term can be estimated using Theorem 3.1, with τ_h as a sequence of right-hand-side functions f_h. (We can think of the $\tau_h(c_{j,i})$ in (15) as defining a piecewise linear, hence continuous, function.)

We have

$$\max_j \max_{t\in[t_j,t_{j+1}]} |y(t)-\vec{p}_h(t)| \le C_3|h|^{m+1} + K\max_{j,i}|\tau_h(c_{j,i})|$$
$$\le C_3|h|^{m+1} + KC_2|h|^{m+\sigma}. \tag{20}$$

This gives $\mathcal{O}(|h|^m)$ or $\mathcal{O}(|h|^{m+1})$ convergence depending on σ. □

4 Collocation variants

Two variants of collocation scheme (9) can be found in the literature on initial value problems for delay differential equations; see [12, Sect. 4] or [5, 17, 23, 16]. One motivation to study different representations of the delayed argument is the well-known superconvergence phenomenon for ordinary differential equations, which gives higher order accuracy at the mesh points; see [7]. Superconvergence is generally lost for DDEs. A second motivation arises from stiff initial value systems of DDEs, see [17, 18]. The delayed argument is, for both variants, obtained by interpolation of $\vec{p}_h$ over several intervals. This representation is not local, in the sense mentioned earlier. However, we can still prove stability, provided we introduce local mesh-ratio restrictions (as introduced in [18]). Note that no mesh-ratio restriction are required for the standard collocation scheme (9).

The collocation scheme (9) is replaced by the following equation,

$$\dot{p}_j(c_{j,i}) - a(c_{j,i})p_j(c_{j,i}) - b(c_{j,i})q_{j,i}((c_{j,i}-\tau) \bmod 1) = f(c_{j,i}),$$
$$i=1,\dots,m,\ j=0,\dots,J-1, \tag{21}$$

where $q_{j,i}$ is a polynomial that approximates the delayed argument in terms of appropriate values of $\vec{p}_h$ (two specific choices will be given below). Let $\vec{q}_h$ represent the set of polynomials $q_{j,i}$, $i=1,\dots,m$, $j=0,\dots,J-1$. We require the following property of the polynomials $\vec{q}_h$.

Definition 4.1 *Let $\{h^\nu\}_{\nu=1}^\infty$ be a sequence of meshes with $|h^\nu|\to 0$ as $\nu\to\infty$. Let $\vec{p}_{h^\nu}\in \vec{P}_{h^\nu}^m$ be a sequence of piecewise polynomial functions on these meshes. Then the corresponding $\vec{q}_{h^\nu}$ (obtained from $\vec{p}_{h^\nu}$) are said to be a* consistent representation of the delayed argument *if and only if whenever $\|\vec{p}_{h^\nu}-p\|_0\to 0$ for some $p\in C_{\mathrm{c}}^0[0,1]$ with uniformly bounded derivatives, i.e., $\|\vec{p}_{h^\nu}\|_1\le Q$, $\forall\nu$, for some Q, then*

$$\max_{j,i}|q_{h^\nu,j,i}((c_{h^\nu,j,i}-\tau) \bmod 1) - p((c_{h^\nu,j,i}-\tau) \bmod 1)| \to 0$$

as $\nu\to\infty$. In other words, the delayed representation converges uniformly over the collocation points to the delayed limit of $\vec{p}_{h^\nu}\in\vec{P}_{h^\nu}^m$.

Corollary 4.1 *Theorem 3.1 remains valid for collocation variants of the type (21) provided the polynomials $\vec{q}_h$ form a consistent representation of the delayed argument.*

Proof. Part (i) and Part (ii) of the proof of Theorem 3.1 remain unchanged (the polynomials $q_{h,j,i}$ are uniquely defined in terms of the value of $\vec{p}_h$ at an appropriate number of points). For Part (iii) we find that (12) is replaced by,

$$\begin{aligned} &|\dot{p}_j(s) - a(s)p(s) - b(s)p(\tilde{s})| \\ &\quad \leq m\, K_1 \max_{j,i} |f^\nu(c_{j,i})| + K_1 \sum_{i=1}^{m} |a(s)p(s) - a(c_i)p_j(c_i)| \\ &\quad +K_1 \sum_{i=1}^{m} |b(s)p(\tilde{s}) - b(c_i)q_{j,i}(\tilde{c}_i)|\,. \end{aligned} \tag{22}$$

which convergences uniformly to zero under the above additional assumption of consistent representation of the delayed argument. The remainder of the proof remains unchanged. □

Note that, in some sense, stability of the scheme remains valid because the noncompact interpolation does not appear in the highest derivative. However, in order to meet the assumption of consistent representation of the delayed argument we will need local mesh-ratio restrictions.

First, we state two specific possibilities for $q_{j,i}$. Let $(c_{j,i} - \tau) \bmod 1 \in [t_k, t_{k+1}]$.

– Interpolation through mesh points: $q_{j,i}$ interpolates $\vec{p}$ at

$$t_{k-r}, t_{k-r+1}, \ldots, t_{k+l-1}, t_{k+l} \tag{23}$$

with $r + l = m_q$ and $r > 0$, $l > 0$ chosen such that $|t_{k+l} - t_{k-r}|$ is minimal.

– Equistage interpolation: $q_{j,i}$ interpolates $\vec{p}$ at

$$c_{k-r,i}, c_{k-r+1,i}, \ldots, c_{k+l-1,i}, c_{k+l,i} \tag{24}$$

with $r + l = m_q$ and $r > 0$, $l > 0$ chosen such that $|c_{k+l,i} - c_{k-r,i}|$ is minimal.

Here $q_{j,i}$ is allowed to interpolate $\vec{p}$ outside [0,1], using periodic extension of the mesh points, collocation points and $\vec{p}$ itself.

For equistage interpolation, the second index of the collocation points, i, corresponds to the second index of q. In terms of the associated Runge-Kutta method this means that, in each equation, only stage values of the same index are used. In view of this correspondence, we require that, for this case, the collocation points are determined from a set of distinct *collocation parameters* $\{c_i\}_{i=1}^m$ in $[0, 1]$,

(25) $$c_{j,i} = t_j + c_i(t_{j+1} - t_j),\ i = 1, \ldots, m,\ j = 0, \ldots, L-1.$$

and corresponding $K_0 = \min_{i \neq j} |c_i - c_j|$. This (natural) restriction is necessary for technical reasons further on.

For the variants (23), (24) we prove the following corollary.

Corollary 4.2 *Let* $\{h^\nu\}_{\nu=1}^\infty$ *be a sequence of meshes with* $|h^\nu| \to 0$ *as* $\nu \to \infty$ *and with*

(26) $$\frac{1}{H} \le \frac{h_j^\nu}{h_{j+1}^\nu} \le H, \qquad j = 0, \ldots, J-2; \quad \frac{1}{H} \le \frac{h_{J-1}^\nu}{h_0^\nu} \le H,$$

where H *is independent of* ν. *Let* $\vec{p}_{h^\nu} \in \vec{P}_{h^\nu}^m$ *be a sequence of piecewise polynomial functions on these meshes. Then, the corresponding* $\vec{q}_{h^\nu}$ *(obtained from* $\vec{p}_{h^\nu}$*) form a consistent representation of the delayed argument if it is defined using interpolation through interval points (23), or, using equistage interpolation (24) with (25).*

Proof. Suppose $\|\vec{p}_{h^\nu} - p\|_0 \to 0$ for some $p \in C_{\mathrm{c}}^0[0, 1]$ with uniformly bounded derivatives,

$$\|\vec{p}_{h^\nu}\|_1 \le Q, \quad \forall \nu,\ \text{for some } Q.$$

Let s_l, $l = -r, \ldots, s$ denote the interpolation points of $q_{h,j,i}$ and let ψ_l, $l = -r, \ldots, s$ denote the associated Lagrange interpolation polynomials. Abbreviate $\tilde{c} = (c_{j,i} - \tau) \bmod 1$. We then have

$$\begin{aligned} \vec{q}_h(\tilde{c}) &= \sum_{l=-r}^{s} \psi_l(\tilde{c}) \vec{p}_h(s_l) \\ &= \sum_{l=-r}^{s} \psi_l(\tilde{c})(\vec{p}_h(\tilde{c}) + \chi_l) \\ &= \vec{p}_h(\tilde{c}) + \sum_{l=-r}^{s} \psi_l(\tilde{c}) \chi_l \end{aligned} \tag{27}$$

with

(28) $$|\chi_l| \le Q(m_q + 1)|h|.$$

Furthermore, in the case of interpolation through interval points, we have, for some k,

$$
\begin{aligned}
|\psi_l(\tilde{c})| &= \left(\prod_{j=-r,j\neq l}^{s} |\tilde{c}-t_{k+j}|\right) \Bigg/ \left(\prod_{j=-r,j\neq l}^{s} |t_{k+l}-t_{k+j}|\right)\\
&\leq \left(\sum_{j=-r}^{s-1} h_{k+j}\right)^{m_q+1} \Bigg/ \left(\min_{j=-r,\ldots,s-1} h_{k+j}\right)^{m_q+1}\\
&\leq (1+H+H^2+\ldots+H^{r+s-1})^{m_q+1}. \qquad (29)
\end{aligned}
$$

Similarly, for equistage interpolation,

$$
\begin{aligned}
|\psi_l(\tilde{c})| &= \left(\prod_{j=-r,j\neq l}^{s} |\tilde{c}-c_{k+j,i}|\right) \Bigg/ \left(\prod_{j=-r,j\neq l}^{s} |c_{k+l,i}-c_{k+j,i}|\right)\\
&\leq \left(\sum_{j=-r}^{s-1} h_{k+j}\right)^{m_q+1} \Bigg/ \left(\min_{j=-r,\ldots,s-1} h_{k+j}K_0\right)^{m_q+1}\\
&\leq (1+H+H^2+\ldots+H^{r+s-1})^{m_q+1}/K_0^{m_q+1}. \qquad (30)
\end{aligned}
$$

The first of these inequalities follows when $m>1$, because, in this case, there always exists at least one collocation point in between collocation points with the same index i due to the ordering (25). If $m=1$, then (30) holds using $K_0=\frac{1}{2}$.

Using the bounds (27), (29), (30) and (28), which are independent of h, except for the condition (26), it follows that

$$
\max_{j,i} |q_{h^\nu,j,i}((c_{h^\nu,j,i}-\tau) \bmod 1) - p((c_{h^\nu,j,i}-\tau) \bmod 1)| \to 0
$$

as $\nu\to\infty$. □

For the local truncation errors we have

$$
\begin{aligned}
\tau_h(c_{j,i}) &= \dot{y}(c_{j,i}) - \dot{\rho}_j(c_{j,i}) - a(c_{j,i})(y(c_{j,i})-\rho_j(c_{j,i}))\\
&\quad -b(c_{j,i})(y((c_{j,i}-\tau) \bmod 1) - q^{\rho}_{j,i}((c_{j,i}-\tau) \bmod 1)), \qquad (31)
\end{aligned}
$$

where $\vec{\rho}_h$ is defined as before and where $q^{\rho}_{j,i}$ is a polynomial obtained from $\vec{\rho}_h$ as $q_{j,i}$ was from $\vec{p}_h$.

The polynomial $q^{\rho}_{j,i}$ interpolates $\vec{\rho}_h$ over several intervals at the boundaries of which $\vec{\rho}_h$ is not continuously differentiable. However, we observe that (using the notation of Corollary 4.2),

$$q^{\rho}_{j,i}(\tilde{c}) = \sum_l \psi_l(\tilde{c})\vec{\rho}_h(s_l) \;=\; \sum_l \psi_l(\tilde{c})(y(s_l) + \epsilon_l)$$
$$= \sum_l \psi_l(\tilde{c})y(s_l) + \sum_l \psi_l(\tilde{c})\epsilon_l$$

The first of these terms is $\mathcal{O}(|h|^{m_q+1})$ and $\epsilon_l = \mathcal{O}(|h|^{m+1})$ when y is sufficiently differentiable (as discussed in Sect. 3). Hence, because the Lagrange interpolation polynomials ψ_l are uniformly bounded in j, i and h under the mesh-ratio restriction (26) (as shown in the proof of Corollary 4.2), we obtain

(32) $$y(\tilde{c}) - q^{\rho}_{j,i}(\tilde{c}) = \mathcal{O}(|h|^{m_q+1}) + \mathcal{O}(|h|^{m+1})$$

uniform in j, i. We can now state the extension of Theorem 3.3.

Theorem 4.1 *Let the homogeneous problem (4) only admit the zero solution. Let y be the (unique) solution of (3), and assume that a, b and f are sufficiently smooth, so that $y \in C^{m+1}_{\mathrm{c}}[0,1]$. Also assume that the collocation points are locally semi-uniform and that the mesh-ratio restriction (26) holds. Then there exists positive constants C and δ such that, whenever $|h| \in (0, \delta]$,*

$$\max_{t\in[0,1]} |y(t) - \vec{p}_h(t)| \le C|h|^{\min\{m, m_q+1\}},$$

where $\vec{p}_h$ is the collocation solution of (23) or (24), (25). Moreover, if $y \in C^{m+2}_{\mathrm{c}}[0,1]$ and if the collocation points are chosen to be Gauss points then

$$\max_{t\in[0,1]} |y(t) - \vec{p}_h(t)| \le C|h|^{\min\{m+1, m_q+1\}}.$$

Proof. From (32) it follows that $\max_{j,i} |\tau_h(c_{j,i})| \le C_2|h|^{\{m+\sigma, m_q+1\}}$, where $\sigma \in \{0, 1\}$ depends on the choice of collocation points and, in particular, Gauss points imply $\sigma = 1$. Indeed, Theorem 3.2 concentrates on $\dot{r}$ which is independent of $q_{j,i}$. Now

$$|y(t) - \vec{p}_h(t)| \le |y(t) - \vec{\rho}_h(t)| + |\vec{\rho}_h(t) - \vec{p}_h(t)|.$$

The first term is again a local interpolation error. The second term can be estimated using Corollaries 4.1 and 4.2.

We have

$$\max_j \max_{t\in[t_j, t_{j+1}]} |y(t) - \vec{p}_h(t)| \le C_3|h|^{m+1} + K \max_{j,i} |\tau_h(c_{j,i})|$$
(33) $$\le C_3|h|^{m+1} + KC_2|h|^{\{m+\sigma, m_q+1\}}.$$

Assuming $m_q \ge m$, this gives $\mathcal{O}(|h|^m)$ or $\mathcal{O}(|h|^{m+1})$ convergence depending on σ. □

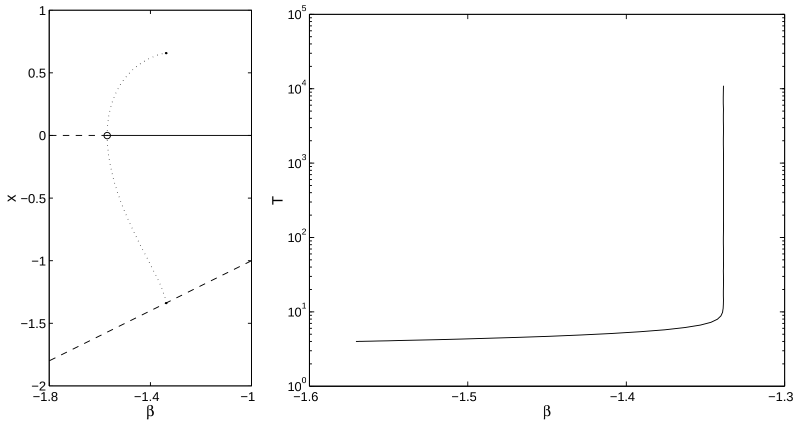

Fig. 1. Left: Bifurcation diagram of (34). Stable (—) and unstable $(--)$ branches of steady state solutions. A Hopf bifurcation (o) and the min and max of the emanating branch of periodic solutions $(\cdots)$. Right: Period along the branch of periodic solutions

5 Numerical results

In [12] an extensive set of numerical tests was used to investigate the behaviour of the collocation method (9) and the variants described in the previous section. The global error results there are consistent with the theorems in the current paper. Here we present numerical results using an example in which a homoclinic orbit in a DDE is approximated by a large-period periodic solution. This example illustrates the importance of unconditional mesh-ratio results.

Consider the following scalar DDE,

$$\dot{x}(t) = \beta x(t-1) - x^2(t). \tag{34}$$

A bifurcation diagram of (34) is shown in Fig. 1 (left) as a function of the parameter β. There are two steady state solution branches, namely, $x(t) \equiv 0$ and $x(t) \equiv \beta$. The zero steady state branch has a Hopf bifurcation at $\beta \approx -1.5708$. The emanating branch of periodic solutions approaches a limiting orbit of infinite period, which is homoclinic to the nonzero steady state solution at $\beta \approx -1.3387$. (Note that while homoclinic solutions and periodic solutions do not exist in scalar autonomous ordinary differential equations, they can exist in scalar autonomous DDEs.)

The branch of periodic solutions was computed using the first collocation variant described above, i.e., using the collocation polynomial to represent the delayed argument. The number of intervals was set to $J = 20$ and the degree to $m = 3$. Adaptive mesh selection was used as described below (see [12] for implementation details). The computed periodic solutions were

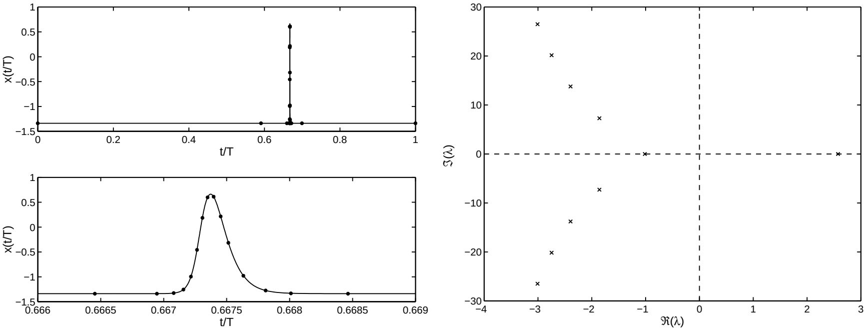

Fig. 2. Left: Profile of the periodic solution, computed using $J = 20$, $m = 3$, at $\beta \approx -1.3387$, $T = 10^4$, as a function of scaled time (upper left) with a blow-up (lower left). Dots indicate the location of the mesh points. Right: Rightmost characteristic roots of the nonzero steady state solution for the same parameter values

found to be accurate up to periods of about $T = 10^4$. (Accuracy was determined by comparison to results on much finer meshes.) The period along the branch is shown in Fig. 1 (right). The computed solution with period $T = 10^4$ at $\beta \approx -1.3387$ is shown in Fig. 2 (left). The exponential decay towards and growth away from the steady state on either side of the pulse were found to be in agreement (up to numerical accuracy) with the leading characteristic roots of the nonzero steady state solution at the same parameter values, as visualized in Fig. 2 (right). The mesh used to compute this solution is highly nonuniform. Specifically, the ratio of the smallest over the largest subinterval of the mesh used in Fig. 2 (left) is

$$\frac{\min_{j=0,\dots,J-1} h_j}{\max_{j=0,\dots,J-1} h_j} = 7.1 \times 10^{-5}.$$

Figure 3 shows the observed dependence of the error,

$$E_h = \max_{t\in[0,1]} |x(Tt) - \vec{p}_h(t)|, \tag{35}$$

on $|h|$, for different approximations to the periodic solution at $\beta = -1.34$ with period $T \approx 8.9809$. The results were obtained for the three collocation variants described above, using equidistant meshes, Gauss-Legendre collocation points, and $m_q = m = 3$. The figure indicates that the three variants have indeed the same order of convergence, $\mathcal{O}(|h|^{\min\{m+1,m_q+1\}})$, yet clearly different error constants. The numerically observed orders of convergence, based on results for $J = 100, \dots, 150$, are given in Table 1.

Adaptive mesh selection was achieved through equidistributing the integral

$$\int_0^1 |x^{(m+1)}(t)|^{\frac{1}{m+1}}\, dt, \tag{36}$$

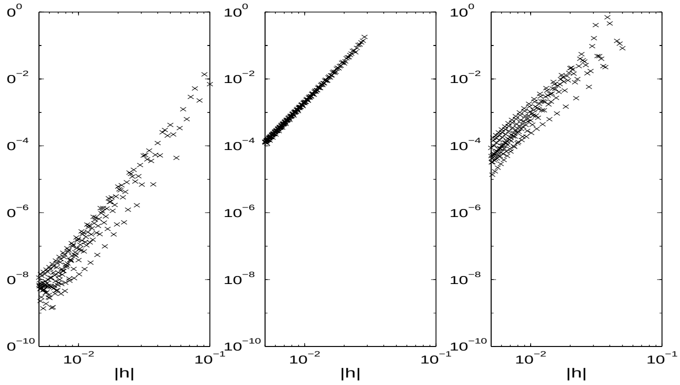

Fig. 3. Convergence of the collocation solution: the maximal error (35) versus $|h|$, using equidistant meshes, $m = m_q = 3$, and Gauss-Legendre collocation points. Representation of the delayed argument: collocation polynomial in the past (left), interpolation through mesh points (middle), and equistage interpolation (right)

Table 1. Observed orders of convergence for different representations of the delayed argument, based on computations with $J = 100, \ldots, 150$, using equidistant meshes, and $m_q = m$. Representation of the delayed argument: collocation polynomial in the past (a), interpolation through mesh points (b), equistage interpolation (c). The upper right number does not represent a good approximation of the order of convergence due to large variations in the error for these values of m, m_q and L

$m = m_q$	(a)	(b)	(c)
2	3.2	3.1	4.9
3	4.0	4.0	4.0
4	5.3	5.0	5.1

over the mesh intervals just as in the case of ordinary differential equations (ODEs) [2]. This strategy was investigated using numerical experiments in [12] and was found to be quite effective. For ODEs, this approach is based on the local error estimate

$$|x(Tt) - \vec{p}_h(t)| = C h_j^{m+1} \max_{s \in [t_j, t_{j+1}]} |x^{(m+1)}(Ts)| + \mathcal{O}(|h|^{m+2}),$$

$$t \in [t_j, t_{j+1}]. \tag{37}$$

Below we show results that indicate that formula (37) does not, in general, hold for DDEs (as was suggested in [12]). By combining equations (19) and (16) in the case of the standard collocation variant and when using Gauss points, we obtain the following bound on the error,

$$\max_{t \in [0,1]} |y(t) - \vec{p}_h(t)| \leq \left(1 + K \max_{t \in [0,1]} (|a(t)| + |b(t)|)\right) \max_{t \in [0,1]} |y(t) - \vec{\rho}_h(t)|,$$

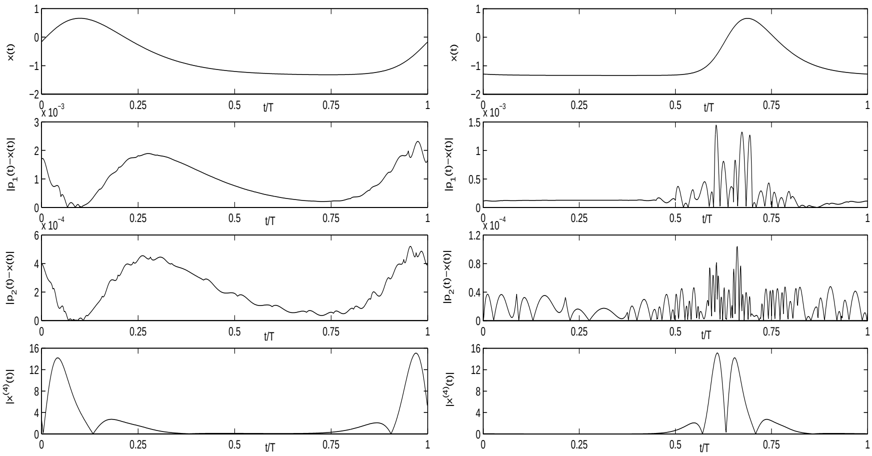

Fig. 4. Left: $\beta = 1.34$, $T \approx 8.98$. Right: $T = 15$, $\beta \approx -1.3387$. From top to bottom: Profile of the periodic solution; error profile based on an equidistant mesh, $m = 3$ and Gauss points; error profile based on an adapted mesh, $m = 3$ and Gauss points; fourth derivative of the solution profile

which, using (14), can be rewritten as,

(38)

$$\max_{t\in[0,1]} |y(t) - \vec{p}_h(t)| \le \hat{K} \max_{j=0,\dots,J-1} \left(\frac{h_j^{m+1}}{(m+1)!} \max_{\xi\in[t_j,t_{j+1}]} |y^{(m+1)}(\xi)| \right)$$

where $\hat{K} = 1 + K \max_{t\in[0,1]} (|a(t)| + |b(t)|)$. The above strategy (36) therefore optimizes with respect to the (possibly nonstrict) error bound (38).

Figure 4 compares the error profile obtained with an equidistant and with an adapted mesh for two different solution profiles using the same number of intervals $J = 20$, $m = 3$ and Gauss-Legendre collocation points. If a local error estimate of the kind (37) would be valid, the error profile on an equidistant mesh would be proportional to the $(m+1)$-derivative. In Fig. 4 (left) there is no such obvious correspondence. Adaptive mesh selection reduces the maximal error by a factor 4.5. The profile of Fig. 4 (right) is slightly more difficult. Here, a correspondence between the error profile on an equidistant mesh and the $(m+1)$-derivative seems to exist (indicating perhaps that the first term in (19) dominates). The mesh selection reduces the maximal error by a factor 13.9. This factor grows as the periodic solutions approach the homoclinic orbit. In fact, when using an equidistant mesh to compute the branch shown in Fig. 1 (right), accuracy already breaks down (rather abruptly) at periods of about 30.

6 Conclusion

Delay differential equations arise in many applications. Examples include the modeling of delayed feedback loops in control, memory effects in visco-elastic fluids, communication with finite transmission times, population dynamics, physiological delays, etc. Numerical methods for simulation of DDEs have been studied quite extensively, and there are a number of publicly available packages; see, e.g., [25,29]. Numerical bifurcation analysis of DDEs by continuation methods is not yet in such an advanced state. A first continuation package for the bifurcation analysis of steady state and periodic solutions of DDEs (which implements the standard collocation variant studied in this paper) has only recently appeared [11].

In this paper we have investigated the convergence of piecewise polynomial collocation methods for the computation of periodic solutions of DDEs with fixed, discrete delays. Periodicity is imposed *a priori*, so that we deal with (periodic) boundary value problems, rather than initial value problems. Collocation methods have been very successful in the numerical bifurcation analysis of periodic solutions in ordinary differential equations, and are implemented in, e.g., the packages AUTO, CONTENT, COLSYS and COLDAE; see [9,22,1,3]. In this paper we have shown that the global convergence properties of collocation extend to delay equations. We investigated a number of collocation variants studied earlier in the context of DDE initial value problems. For the standard variant we do not need mesh-ratio restrictions. We illustrated our findings with numerical results.

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