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Discrete Hodge operators*

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Summary. Many linear boundary value problems arising in computational physics can be formulated in the calculus of differential forms. Discrete differential forms provide a natural and canonical approach to their discretization. However, much freedom remains concerning the choice of discrete Hodge operators, that is, discrete analogues of constitutive laws. A generic discrete Hodge operator is introduced and it turns out that most finite element and finite volume schemes emerge as its specializations. We reap the possibility of a unified convergence analysis in the framework of discrete exterior calculus.

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1 Introduction

A huge body of literature in numerical analysis is devoted to devising and examining discretizations for the linear strongly elliptic reaction-diffusion equation

$$(1) \qquad -\mathrm{div}(a\,\mathbf{grad}\,u) + cu = f \quad \text{in } \Omega$$
$$u = 0 \text{ on } \Gamma_D \ , \ \langle a\,\mathbf{grad}\,u, \mathbf{n}\rangle = 0 \text{ on } \Gamma_N \ , \ \langle a\,\mathbf{grad}\,u, \mathbf{n}\rangle = \beta u \text{ on } \Gamma_M \ .$$

Usually, $\Omega \subset \mathbb{R}^n$ is some Lipschitz-domain with exterior unit normal vector field $\mathbf{n}$, and Γ_D, Γ_N, and Γ_M are to form a partition of the boundary $\partial\Omega$

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of Ω. On top of that, $a, c \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma_M)$, $\beta, c \geq 0$ a.e., and a is supposed to be uniformly positive. Then we know that there exists a unique solution $u \in H^1(\Omega)$, $u_{|\Gamma_D} = 0$. We also know very well, how to compute discrete approximations of u: Finite elements, finite volume methods, finite differences on all kinds of grids and meshes offer a vast array of viable schemes.

Almost the same holds true for the linear Maxwell's equations, here stated in the time domain with first order absorbing Silver-Müller boundary conditions

$$
\begin{aligned}
\mathbf{curl}\mathbf{E} &= -\tfrac{d}{dt}\mathbf{B} \quad , \mathbf{B} = \mu\mathbf{H} \\
\mathbf{curl}\mathbf{H} &= \tfrac{d}{dt}\mathbf{D} + \mathbf{J} \ , \mathbf{D} = \epsilon\mathbf{E}
\end{aligned}
\qquad \text{in } \Omega \subset \mathbb{R}^3 \ , \tag{2}
$$

$$\mathbf{E}\times\mathbf{n} = 0 \text{ on } \Gamma_D \ , \quad \mathbf{H}\times\mathbf{n} = 0 \text{ on } \Gamma_N \ , \quad \mathbf{E}\times\mathbf{n} = \sqrt{\frac{\mu}{\epsilon}}\,\mathbf{H_t} \text{ on } \Gamma_M \ .$$

Here, the material parameters $\mu, \epsilon \in L^\infty(\Omega)$ are uniformly positive and $\mathbf{J}$ is a solenoidal exciting current. Of course, some initial values at time $t = 0$ have to be provided to complete the statement of the problem. The existence of unique solutions for $\mathbf{E}$ and $\mathbf{H}$ in $C^1([0;T], \boldsymbol{H}(\mathbf{curl}; \Omega))$ can be established [56,43]. Scores of spatial and temporal discretization schemes for (2) and related models have been designed (see e.g. [62,41,68]).

At first glance the second order linear boundary value problems (1) and (2) have precious little in common. Yet, appearances are deceptive, because they both fit the framework provided by the calculus of differential forms. The tenet is that regarding the quantities occurring in (1) and (2) as plain vectorfields and functions is grossly misleading. Rather, physics tells us what their true nature is: Some, like $\mathbf{B}$, $\mathbf{D}$, and $a\mathbf{grad}u$ are measured as fluxes through surfaces. Others like $\mathbf{E}$ are expressed through integrals along paths, and, finally, for u point values make sense. In sum, all these quantities should be regarded as *differential forms*, mappings assigning values to oriented manifolds of different dimensions.

Among physicists this is widely appreciated, in particular in the case of Maxwell's equations. Among numerical analysts the statement might only prompt a shrug: What can we benefit from this theory that might supply us with a neat, trim, and unified way to state the problems, but offers little tangible hint on how to solve them numerically? The main message I want to send in this paper is that there is a substantial benefit both for the design, understanding, and error analysis of discretization procedures. I am by no means the first to make this point. I would like to mention Matiussi [45], Tonti [65], Dezin [28], Shashkov [38,39], Chew [21,64], and, most prominently, Bossavit [5–7,9–11,63]. Whereas the foundation is borrowed from these works, I am setting out to build upon it a *general unifying framework for the quantitative analysis of a wide class of finite element and finite* *volume schemes*. In a sense, the present paper supplements the previous conceptual works by what is cherished as "rigorous analysis". In addition, many relationships between different methods will be disclosed, widening the scope of techniques originally developed for only one type of method.

The abstract perspective can also help untangle different aspects of the problem of discretization that are often lumped together in the traditional analyses of special methods: First, topological and metric aspects are cleanly separated. Secondly, it is revealed that different sources of error, approximation errors, consistency errors,and lack of stability, lead to the total discretization error. Thirdly, different notions of discretization error are immediately suggested, nodal errors of degrees of freedom, energy norms of nodal errors, and errors in the total energies.

It should be mentioned that efforts are being made to exploit the approach of this paper for the sake of object oriented code development in computational electromagnetism [63]. This idea has motivated attempts to extend the abstract framework beyond exterior calculus toward a more general tensor calculus [34].

Admittedly, there is a price tag on generality. The results I am getting may be weaker than those obtained through more specialized techniques. In addition, quite a few cumbersome details may still be left to work out for specific schemes. However, the sheer scope of the method compensates for these drawbacks. It can also serve as a reliable guide to the construction of appropriate schemes.

The paper is organized as follows: In the next section I give a brief survey of exterior calculus. For details I refer the reader to the monographs [20] and [40] and to the lucid exposition in [11,12] and [19, Ch. IV]. We will learn how differential forms help separate topological and metric dependent aspects of the boundary value problems. The third section reviews discrete differential forms, i.e., finite elements for differential forms. The fourth section introduces the key concept of discrete Hodge operators and gives a purely algebraic characterization. Similar, though more restricted, approaches to the construction of discrete Hodge operators are pursued in [63] and [64, Sect. VII]. The fifth section studies examples of discrete Hodge operators. In particular, the focus is on finite volume methods. Using only a few basic algebraic properties of discrete Hodge operators, abstract bounds for the energies of the discretization errors are established in the sixth section. This is further pursued in the case of concrete schemes in the seventh section.

2 Exterior calculus

In the most general sense, Ω may be a (piecewise) smooth oriented and bounded n-dimensional Riemannian manifold, $n \in \mathbb{N}$, with a piecewise

Table 1. Relationship between differential forms (top line: 0-forms, bottom line: 3-forms) and vectorfields for $n = 3$ ($\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$) (cf. [12])

Differential form	Related function u/vectorfield $\mathbf{u}$
$\mathbf{x} \mapsto \{() \mapsto \omega(\mathbf{x})\}$	$\omega(\mathbf{x}) = u(\mathbf{x})$
$\mathbf{x} \mapsto \{\mathbf{v} \mapsto \omega(\mathbf{x})(\mathbf{v})\}$	$\omega(\mathbf{x})(\mathbf{v}) = \langle \mathbf{u}(\mathbf{x}), \mathbf{v} \rangle$
$\mathbf{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2) \mapsto \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2)\}$	$\omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2) = \langle \mathbf{u}(\mathbf{x}), \mathbf{v}_1 \times \mathbf{v}_2 \rangle$
$\mathbf{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \mapsto \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\}$	$\omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = u(\mathbf{x}) \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

smooth boundary. To stick to familiar ground I am considering only bounded domains in the oriented Euclidean affine space $A(\mathbb{R}^n)$. Everything can be easily adapted to manifolds. A differential form ω of order l, $0 \leq l \leq n$, is a mapping from Ω into the $\binom{n}{l}$-dimensional space of alternating l–multilinear forms on $\mathbb{R}^n$ [20, Sect. 2.1]. In the sequel, $\mathcal{D}^l_k(\Omega)$ stands for the space of l-forms on Ω of class C^k. After an orientation and inner product in $\mathbb{R}^n$ has been chosen, they can be used to define an isomorphism between l forms and vectorfields $\Omega \mapsto \mathbb{R}^N$, $N := \binom{n}{l}$. The usual identification in $\mathbb{R}^3$ is depicted in Table 1. Please note that forms of different order permit a description via vectorfields ("vector proxies") of the same type.

A fundamental concept in the theory of differential forms is the *integral* of a p-form over a piecewise smooth p-dimensional oriented manifold. Through integration a differential form assigns a value to each suitable manifold, modeling the physical procedure of measuring a field. We write $\mathcal{D}^l(\Omega)$ for the vector space of l-forms on Ω, whose integrals exist for any compact piecewise smooth oriented l-submanifold of Ω. Note that point values may not make sense for forms in $\mathcal{D}^l(\Omega)$.

From alternating l-multilinear forms differential l-forms inherit the exterior product $\wedge : \mathcal{D}^l_0(\Omega) \times \mathcal{D}^k_0(\Omega) \mapsto \mathcal{D}^{l+k}_0(\Omega)$, $0 \leq l, k, l+k \leq n$, defined in a pointwise sense. We note the property

(3) $$\omega \wedge \eta = (-1)^{lk} \eta \wedge \omega\,, \quad \omega \in \mathcal{D}^l_0(\Omega), \eta \in \mathcal{D}^k_0(\Omega)\;.$$

Another crucial device is the exterior derivative d, a linear mapping from differentiable l-forms into $l+1$-forms. In three dimensions, given the identification of forms and vectorfields from Table 1, the exterior derivative defines the conventional differential operators **grad**, div, and **curl** for 0-forms through 2-forms, respectively. A fundamental fact about exterior differentiation is that $d\,d\omega = 0$ for any sufficiently smooth differential form ω. In addition, Stokes' theorem makes it possible to define the exterior derivative $d\omega \in \mathcal{D}^{l+1}(\Omega)$ of $\omega \in \mathcal{D}^l(\Omega)$. A main result about the exterior derivative is the integration by parts formula [40, Sect. 3.2]

(4) $$\int_\Omega d\omega \wedge \eta + (-1)^l \int_\Omega \omega \wedge d\eta = \int_{\partial\Omega} \omega \wedge \eta$$

for $\omega \in \mathcal{D}^l_1(\Omega)$, $\eta \in \mathcal{D}^k_1(\Omega)$, $0 \le l, k < n-1$, $l+k = n-1$. Here, the boundary $\partial\Omega$ is endowed with the induced orientation.

The trace $\mathbf{t}_\Sigma\omega$ of an l-form $\omega \in \mathcal{D}^l(\Omega)$, $0 \le l < n$ with respect to some piecewise smooth $n-1$-dimensional submanifold $\Sigma \subset \bar{\Omega}$ yields an l-form on Σ [40, Sect. 1.10]. It commutes with the exterior product and exterior differentiation $d\mathbf{t}_\Sigma\omega = \mathbf{t}_\Sigma d\omega$ for $\omega \in \mathcal{D}^l_1(\Omega)$. Please note that for $n = 3$ the tangential traces $\mathbf{u} \times \mathbf{n}$ and normal traces $\langle \mathbf{u}, \mathbf{n} \rangle$ of a vectorfield $\mathbf{u}$ realize $\mathbf{t}_\Sigma$ for the vector representatives of 1-forms and 2-forms, respectively [13, Sect. 1.2].

It is important to be aware that all the concepts are genuinely affine and metric-invariant. This does not apply to the so-called *Hodge-operators* $\star_\alpha$, linear mappings of continuous l-forms into $(n-l)$-form. For their definition, which depends on a Riemannian metric α on Ω, I refer to [40, Sect. 1.4] or [9, Sect. 4.5]. As one of the many properties of Hodge operators let me mention that for continuous l-forms $\star_\alpha \circ \star_{1/\alpha} = (-1)^{ln-l}$ [40, Sect. 1.8], i.e. the Hodge operators are invertible.

Hodge operators define inner products on the continuous l-forms via $(\omega, \eta)_\alpha := \int_\Omega \omega \wedge \star_\alpha \eta$. Write $\|\cdot\|_0$ for the induced norm with respect to the Euclidean metric and define the Sobolev spaces $\mathcal{H}_l(d, \Omega)$ of l-forms on Ω as completion of the spaces of smooth l-forms with respect to the norm $(\|\cdot\|_0^2 + \|d\cdot\|_0^2)^{\frac{1}{2}}$ (see [40, Ch. 3]). From what has been explained above, it is immediate that these spaces are isometrically isomorphic to $H^1(\Omega)$, $\boldsymbol{H}(\mathbf{curl}; \Omega)$, and $\boldsymbol{H}(\mathrm{div}; \Omega)$, respectively, in the case $n = 3$. Moreover, Sobolev spaces of forms with vanishing traces on parts of $\partial\Omega$ can be defined as usual.

Eventually, we are in a position to cast the boundary value problems (1), (2) (and many more) into the calculus of differential forms: Various second order (initial)-boundary value problems turn out to be special cases of

$$
\begin{array}{ccc}
\text{I: Elliptic} & \text{II: Parabolic} & \text{III: Hyperbolic} \\
du = (-1)^l \boldsymbol{\sigma} & du = (-1)^l \boldsymbol{\sigma} & du = (-1)^l \frac{d}{dt}\boldsymbol{\sigma} \\
d\mathbf{j} = -\psi + \phi & d\mathbf{j} = -\frac{d}{dt}\psi + \phi & d\mathbf{j} = -\frac{d}{dt}\psi + \phi
\end{array}
$$

$$
\mathbf{t}_D u = 0 \quad \text{on } \Gamma_D \quad , \quad \mathbf{t}_N \mathbf{j} = 0 \quad \text{on } \Gamma_N
$$

$$
\mathbf{j} = \star_\alpha \boldsymbol{\sigma} \quad , \quad \psi = \star_\gamma u \quad \text{in } \Omega \tag{5}
$$

$$
\mathbf{t}_M \mathbf{j} = (-1)^{l-1} \star^\Gamma_\beta \mathbf{t}_M u \quad \text{on } \Gamma_M \ .
$$

Here α, γ are fixed Riemannian metrics on Ω and β is one on Γ_M. The operators $\mathbf{t}_D$, $\mathbf{t}_N$, and $\mathbf{t}_M$ designate the traces onto the respective parts of the boundary. The function u is to be a (time-dependent) $(l-1)$-form, $0 < l \le n$, and this fixes $\boldsymbol{\sigma}$ as an l-form, $\mathbf{j}$ as a m-form, $m := n - l$, and ϕ, ψ have to be $(m+1)$-forms. ϕ plays the role of a source term. Of course, a complete statement of the time-dependent problems in (5) entails specifying appropriate initial values. I am using different typefaces for forms of different order, yet no convention is introduced to designate, in particular, 0-forms, 1-forms, etc.

When I speak of solutions of the problems of (5), I have in mind weak solutions: For (5,I) we seek a unique $u \in \mathcal{H}_{l-1}(d,\Omega)$ such that for all $v \in \mathcal{H}_{l-1}(d,\Omega)$

$$(du,dv)_\alpha + (u,v)_\gamma + (\mathbf{t}_M u, \mathbf{t}_M v)_\beta = \int_\Omega \phi \wedge v\ , \tag{6}$$

where the boundary conditions for u have to be incorporated into $\mathcal{H}_{l-1}(d,\Omega)$. Having found u from (6), $\sigma \in \mathcal{H}_l(d,\Omega)$, $\mathbf{j} \in \mathcal{H}_m(d,\Omega)$, and $\psi \in \mathcal{H}_{m+1}(d,\Omega)$ are defined, too. For the time-dependent problems (5,II) and (5,III) weak solutions $u \in C^1([0;T],\mathcal{H}_{l-1}(d,\Omega))$ and $u \in C^2([0;T],\mathcal{H}_{l-1}(d,\Omega))$ can also be found, provided that the initial values are smooth enough.

The boundary value problem (1) emerges from (5,I) as the case $l = 1$, when we regard the function f as a representation of the 3-form ϕ and use the coefficient functions a and c to define the metrics α and γ, respectively. For $n = 3, l = 2$ (5,I) agrees with an elliptic problem in $\boldsymbol{H}(\mathbf{curl};\Omega)$, and for $n = 3, l = 3$ we recover $\boldsymbol{H}(\mathrm{div};\Omega)$-elliptic boundary value problems.

It is also straightforward how to express Maxwell's equations (2) in exterior calculus: The electric field $\mathbf{E}$ is a representative of the 1-form u, $\mathbf{B}$ is related to the 2-form $\boldsymbol{\sigma}$, $\mathbf{H}$ is a vector proxy for $\mathbf{j}$, and $\mathbf{D}$ corresponds to ψ in (5,III). The relationships of the metrics and the electrodynamic material parameters should be obvious. In the case $l = 0$ we get the standard wave equation from (5,III).

A key observation is that the top three lines in (5) do not involve any metric. They represent the topological relationships underlying the boundary value problems. I will refer to them as *equilibrium equations* (topological field equations in [64]). On top of that, the two equilibrium equations are not directly coupled. One may view them as equations for different kinds of differential forms: “ordinary” forms and twisted forms [19, 9, 64]. The link between the equilibrium equations is established by the *constitutive relations* (material laws, metric field equations [64]) that fundamentally depend on metrics.

Remark 1 Sometimes it makes sense to admit $\gamma = 0$, in particular, when $l = 1$, in order to cover pure diffusion problems. I am not going to dwell on this special case, but occasional remarks will hint how to adapt my considerations. It is important to note that $\gamma = 0$ might forfeit uniqueness of u in (6), but one still has a unique solution for σ and $\mathbf{j}$.

Remark 2 Maxwell's equations permit an even more concise formulation through an exterior calculus that treats space and time alike [27, 2], i.e.,

$n = 4$. Yet, then indefinite "metrics" pop up that are beyond the scope of the present paper.

3 Discrete differential forms

When asking for a discrete solution of (5) one should not settle for less than valid differential forms approximating some or all of the unknowns. Thus I opt for conforming approximation by discrete differential forms of appropriate order.

Definition 1 *A sequence of spaces $\mathcal{W}^l$, $0 \leq l \leq n$, provides* discrete differential forms *on Ω, if $\mathcal{W}^l \subset \mathcal{D}^l(\Omega)$ for all $0 \leq l \leq n$, $d\mathcal{W}^l \subset \mathcal{W}^{l+1}$ for $0 \leq l < n$, all the spaces $\mathcal{W}^l$ have finite dimension, and there is a linear mapping $I_h^l : \mathcal{D}^l(\Omega) \mapsto \mathcal{W}^l$ satisfying the* commuting diagram property *$dI_h^l \omega = I_h^{l+1}(d\omega)$ for all $\omega \in \mathcal{D}^l(\Omega)$, $0 \leq l < n$ (see e.g. [63, Sect. IV]).*

In the sequel I take for granted that the corresponding discrete differential forms comply with the boundary traces $\mathbf{t}_N \mathbf{j} = 0$ and $\mathbf{t}_D u = 0$ stated in (5).

In practice, the $\mathcal{W}^l$ are constructed as finite element spaces. In particular, the discrete differential forms are usually based upon some mesh (tessellation, cell complex) of Ω (cf. [64, Sect. IV])

Definition 2 *For any $0 \leq k \leq n$ denote by $\mathcal{F}_k$ a collection of piecewise smooth oriented contractible k-dimensional submanifolds (k-faces) contained in $\bar{\Omega}$ such that*

- *for distinct faces their interiors are disjoint regardless of dimension.*
- *the intersection of the closures of any two faces of any dimensions coincides with the closure of one and only one other face.*
- *the boundary of each k-face, $1 \leq k \leq n$, is the union of a finite number of $(k+1)$-faces.*
- *the union of the closures of all n-faces is equal to $\bar{\Omega}$.*

Then $\{\mathcal{F}_k\}_{k=0}^n$ forms a mesh *$\mathcal{T}_h$ of Ω, and $\mathcal{F}_n$ is the set of its cells.*

The boundary parts Γ_N, Γ_D, and Γ_M are to be composed of entire faces of elements. Thus, by plain restriction of a mesh of Ω we get meshes of Γ_N, Γ_D, and Γ_M. By dropping all faces contained in the closure of Γ_D (Γ_N) we end up with the so-called $\Gamma_D(\Gamma_N)$-active mesh. Examples of valid meshes are the customary finite element triangulations in the sense of [22].

I should stress that it is by no means trivial to conceive finite element spaces satisfying the above requirements on discrete differential forms. Nevertheless, such spaces are known for a variety of meshes. Most prominently, I would like to cite the *Whitney forms* first introduced as a theoretical tool in differential geometry [69,50,29]. Whitney forms provide discrete differential forms of all orders on simplicial meshes in any dimension. Integrals over

l-faces of the mesh are used as degrees of freedom for $\mathcal{W}^l$, $0 \leq l \leq n$. In three dimensions they agree with the $\boldsymbol{H}(\text{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$-conforming finite elements introduced by Nédélec in [51], in 2D we get the so-called Raviart-Thomas elements [55,18]. Extensions of Whitney forms to Cartesian meshes (cf. [51]), and hybrid meshes [32,24,30] in 3D are known. The vector proxies of simplicial Whitney forms are at most linear in each component, but spaces with the same properties (but more complicated degrees of freedom) can be constructed that accommodate higher polynomial degrees [35,51,48,17,26]. Transformations permit us to extend the schemes to triangulations with curved faces. With all these constructions we may use nodal projections relying on the finite element degrees of freedom to defined the mappings I_h^l. By construction they possess the commuting diagram property.

In the sequel we only consider (generalized) Whitney-forms, i.e. discrete differential l-forms, whose degrees of freedom are supplied by integrals over the l-faces of the mesh. Mainly for the sake of simplicity, because the basic considerations carry over to higher order elements, as well. However, we admit rather general meshes according to definition 2. The only restriction should be that any cell can be split into a few (curved) simplices. Hence, for complicated shapes of elements we can come up with composite Whitney elements.

For each space $\mathcal{W}^l$ we pick the basis dual to the set of degrees of freedom and assume a numbering of the basis functions. Thus, a differential form $u_h \in \mathcal{W}^l$ is uniquely characterized by its coefficient vector $\mathbf{u} \in \mathcal{C}^l$, where we have abbreviated $\mathcal{C}^l := \mathbb{R}^{N_l}$, $N_l := \dim \mathcal{W}^l$ (number of active l-faces). Coefficient vectors will be written in bold print and discrete differential forms are tagged by a subscript h. The latter can also be seen as a mapping assigning a differential form in $\mathcal{W}^l$ to some coefficient vector $\in \mathcal{C}^l$. This induces a matrix representation for all linear operators on the spaces $\mathcal{W}^l$, $l = 0, \dots, n$. For instance, we write $\mathsf{D}^l \in \mathbb{R}^{N_{l+1} \times N_l}$ for the matrix representing the exterior derivative $d : \mathcal{W}^l \mapsto \mathcal{W}^{l+1}$. These matrices turn out to be the incidence matrices of the active mesh [7,52,54,13], e.g., D^l is the incidence matrix of oriented active l-faces and active $(l+1)$-faces. Both from this background and $d \circ d = 0$ we can conclude $\mathsf{D}^l \mathsf{D}^{l-1} = 0$, $l = 0, \dots, n-2$ [7, Ch. 5].

Whitney forms on the boundary meshes naturally emerge as traces of forms in $\mathcal{W}^l$. They inherit those degrees of freedom of $\mathcal{W}^l$ that are associated with l-faces of the respective part of the boundary [35]. It goes without saying that the (linear) trace operators $\mathbf{t}_D$, $\mathbf{t}_N$, and $\mathbf{t}_M$ possess matrix representations, too. Thanks to the construction, the values of d.o.f. for the trace $\mathbf{t}u_h$, $u_h \in \mathcal{W}^l$ agree with certain components of $\mathbf{u}$. Thus, the matrices related to the discrete traces have 1s only on particular locations on the main diagonal. I choose the symbol T for these matrices.

Now, we are already able to come up with the discrete equilibrium equations

$$(7)\qquad \begin{array}{lll} \text{(I)} : & \mathsf{D}^{l-1}\mathbf{u} = (-1)^l \boldsymbol{\sigma} & , \quad \mathsf{D}^m \mathbf{j} = -\boldsymbol{\psi} + \boldsymbol{\phi}\,, \\ \text{(II)} : & \mathsf{D}^{l-1}\mathbf{u} = (-1)^l \boldsymbol{\sigma} & , \quad \mathsf{D}^m \mathbf{j} = -\frac{d}{dt}\boldsymbol{\psi} + \boldsymbol{\phi}\,, \\ \text{(III)} : & \mathsf{D}^{l-1}\mathbf{u} = (-1)^l \frac{d}{dt}\boldsymbol{\sigma} & , \quad \mathsf{D}^m \mathbf{j} = -\frac{d}{dt}\boldsymbol{\psi} + \boldsymbol{\phi}\,. \end{array}$$

Here, $\boldsymbol{\phi}$ is the coefficient vector of some suitable interpolant of the source term $\phi \in \mathcal{D}^n(\Omega)$. The reader should be aware that whatever features of the equations arise from the equilibrium equations alone are preserved in the discrete setting. For instance, we get for the solutions of the discrete wave equation (7,III) $d\boldsymbol{\sigma}_h = 0$ for all times if $d\boldsymbol{\sigma}_{h|t=0} = 0$, and “charge conservation” $-\frac{d}{dt}d\psi_h = -d\phi$. By (4), “energy conservation” follows for the discrete quantities in the stationary case: For any oriented control volume $\Omega' \subset \Omega$

$$\int\limits_{\Omega'} (\boldsymbol{\sigma}_h \wedge \mathbf{j}_h + u_h \wedge \psi_h) = \int\limits_{\Omega'} u_h \wedge \phi_h + (-1)^l \int\limits_{\partial\Omega'} \boldsymbol{\sigma}_h \wedge \mathbf{j}_h\,.$$

In short, thanks to discrete differential forms we automatically achieve discrete models that inherit most of the global features of the original problem (cf. [42,21] for a discussion of Maxwell’s equations and [39] for discrete decomposition theorems).

Remark 3 The discrete equilibrium equations (7) could also have been derived as network or lattice equations [13,21,68] by applying Stokes’ theorem directly to faces of the mesh in the spirit of a finite volume approach. Yet, my point is that discrete differential forms are indispensable when trying to assess the approximation properties of discretization schemes. This will be elucidated in the remaining sections.

4 Discrete Hodge operators

The Hodge operators defy a straightforward discretization in the spaces of discrete differential forms: Consider the example of a discrete 1-form ω in three dimensions (“edge elements”): Its vector proxy $\mathbf{u}$ needs only sport only tangential continuity [51]. Its normal components on interelement faces are may jump. Given the relationship expressed in Table 1, the Hodge operator belonging to the Euclidean metric leaves vector proxies invariant, i.e. the (twisted) 2-form $\star\omega$ is also described by $\mathbf{u}$. If F is a face, at which the normal component of $\mathbf{u}$ has a jump, it is not possible to calculate the degree of freedom $\int_F \star\omega = \int_F \langle \mathbf{u}, \mathbf{n}_F \rangle \, dS$: This reveals that $\star\omega$ fails to supply a proper discrete 2-form on the same mesh.

Thus, when embracing discrete differential forms, one inevitably stumbles across the issue of a discrete Hodge operator [63, 42, 25, 14]. Its construction is outside the scope of the canonical discrete exterior calculus. This is not a nuisance, but leaves us with ample choices (cf. the introduction of [13]).

First, we recall that the two distinct equilibrium equations are not linked at all. So there is no reason, why both equilibrium equations should be discretized via the same family of discrete differential forms:

> In general, the two equilibrium equations can be discretized on different unrelated meshes of Ω, called *primary mesh* $\mathcal{T}_h$ and *secondary mesh* $\tilde{\mathcal{T}}_h$.

I stress that absolutely no relationship between these two meshes is stipulated. The terms "primary" and "secondary" must not even insinuate some precedence. I adopt the convention that all symbols related to the secondary mesh will be labeled by a tilde. This applies to matrices acting on coefficient vectors of discrete differential forms on the secondary mesh, too.

Basically, discrete versions of the Hodge operators occurring in (5) have to establish linear mappings between spaces of discrete differential forms based on possibly different meshes. Thus, we get the following generic discrete form for a material law linking an ordinary l-form w and a twisted m-form ω:

$$\left.\begin{array}{r} \star_\mu w = \omega \\ \Updownarrow \\ (-1)^{ln-l} \star_{1/\mu} \omega = w \end{array}\right\} \xrightarrow{\text{discretize}} \left\{\begin{array}{r} \mathsf{M}^l_\mu \mathbf{w} \;=\; \widetilde{\mathsf{K}}^l_m \boldsymbol{\omega} \\ \textbf{or} \\ (-1)^{ln-l}\, \widetilde{\mathsf{M}}^m_{1/\mu} \boldsymbol{\omega} \;=\; \mathsf{K}^m_l \mathbf{w} \; , \end{array}\right. \tag{8}$$

with yet obscure matrices $\mathsf{M}^l_\mu, \widetilde{\mathsf{M}}^m_{1/\mu}$, and $\mathsf{K}^m_l, \widetilde{\mathsf{K}}^l_m$. In an obvious fashion, the various indices are related to the order of differential forms on whose coefficient vectors the matrices act. Note that the discrete versions of the equivalent continuous material laws need not remain equivalent as explained in [64, Sect. VII]. I dub the upper discrete material law "primary", the lower "secondary".

These matrices have to satisfy no more than a few simple *algebraic requirements*:

A: Both $\mathsf{M}^l_\mu \in \mathbb{R}^{N_l,N_l}$ and $\widetilde{\mathsf{M}}^m_{1/\mu} \in \mathbb{R}^{\tilde{N}_m,\tilde{N}_m}$ are to be square, symmetric, and positive definite matrices, where $N_l := \dim \mathcal{W}^l$, $\tilde{N}_m := \dim \tilde{\mathcal{W}}^m$.

B: For all $0 \leq l, m \leq n$ such that $l + m = n$, the "pairing matrices" $\mathsf{K}^m_l \in \mathbb{R}^{\tilde{N}_m,N_l}$ and $\widetilde{\mathsf{K}}^l_m \in \mathbb{R}^{N_l,\tilde{N}_m}$ fulfill

$$\mathsf{K}^m_l = (-1)^{lm}(\widetilde{\mathsf{K}}^l_m)^{\mathrm{T}} \quad \Longleftrightarrow \quad \widetilde{\mathsf{K}}^l_m = (-1)^{lm}(\mathsf{K}^m_l)^{\mathrm{T}} \; . \tag{9}$$

C: The exterior derivatives and pairing matrices are connected by

(10) $$(\mathsf{D}^{l-1})^{\mathrm{T}}\widetilde{\mathsf{K}}^{l}_{m} = (-1)^{l}\,\widetilde{\mathsf{K}}^{l-1}_{m+1}\widetilde{\mathsf{D}}^{m} + (\mathsf{T}^{l-1}_{\Gamma})^{\mathrm{T}}\widetilde{\mathsf{K}}^{l-1}_{m,\Gamma}\widetilde{\mathsf{T}}^{m}_{\Gamma}$$

for all $0 < m, l < n$ with $l + m = n$. Here, we denote by $\widetilde{\mathsf{K}}^{l}_{m,\Gamma}$ a pairing matrix acting on degrees of freedom on the boundary part Γ_M. Using (9), (10) translates into

(11) $$(\widetilde{\mathsf{D}}^{m})^{\mathrm{T}}\mathsf{K}^{m+1}_{l-1} = (-1)^{m+1}\mathsf{K}^{m}_{l}\mathsf{D}^{l-1} + (\widetilde{\mathsf{T}}^{m}_{\Gamma})^{\mathrm{T}}\mathsf{K}^{m}_{l-1,\Gamma}\mathsf{T}^{l-1}_{\Gamma}\ .$$

A motivation for these requirements is based on the concept of weak solutions of (5), because this involves a "weak interpretation" of the Hodge operator, given by

(12)
$$\left.\begin{array}{c}\star_{\mu} w = \omega \\ \Updownarrow \\ (-1)^{ln-l}\star_{1/\mu}\,\omega = w\end{array}\right\} :\Longleftrightarrow \left\{\begin{array}{r}(w,\eta)_{\mu} = \int_{\Omega}\omega\wedge\eta \quad \forall\eta\in\mathcal{H}_l(d,\Omega) \\ \Updownarrow \\ (-1)^{ln-l}\,(\omega,v)_{1/\mu} = \int_{\Omega} w\wedge v \quad \forall v\in\mathcal{H}_m(d,\Omega)\end{array}\right. .$$

Obviously, the matrices M^{l}_{μ} and $\widetilde{\mathsf{M}}^{m}_{1/\mu}$ from (8) should give rise to discrete analogues of the inner products $(\cdot,\cdot)_{\mu}$ and $(\cdot,\cdot)_{1/\mu}$. In other words, they should resemble "mass matrices", in finite element parlance, or ensure reciprocity of the discrete formulation, in physical terminology, respectively. This justifies requirement **A**. The pairing matrices K^{m}_{l} and $\widetilde{\mathsf{K}}^{l}_{m}$ somehow approximate $\int_{\Omega}\omega\wedge\eta$ and $\int_{\Omega} w\wedge v$. Then (3) inspires **B**. Finally, accepting the preceding interpretations of the matrices, **C** is a discrete version of the integration by parts formula (4), since it means

(13) $$(\mathsf{D}^{l-1}\mathbf{u})^{\mathrm{T}}\widetilde{\mathsf{K}}^{l}_{m}\mathbf{v} = (-1)^{l}\,\mathbf{u}^{\mathrm{T}}\widetilde{\mathsf{K}}^{l-1}_{m+1}\widetilde{\mathsf{D}}^{m}\mathbf{v} + (\mathsf{T}^{l-1}_{\Gamma}\mathbf{u})^{\mathrm{T}}\widetilde{\mathsf{K}}^{l-1}_{m,\Gamma}\widetilde{\mathsf{T}}^{m}_{\Gamma}\mathbf{v}\ .$$

for all $\mathbf{u}\in\mathcal{C}^{l-1}$, $\mathbf{v}\in\tilde{\mathcal{C}}^{m}$.

Let us assume that we have found a discrete Hodge operator according to the above specifications. Nevertheless, we cannot be sure that the resulting linear system of equation has a solution at all. Most strikingly, the number of unknowns and equations need not agree. To end up with a square linear system of equations we have to get rid of some of the unknowns by means of (10), (11) and the material laws. I first consider (7) and discuss some variants of choosing the discrete Hodge operators. I start with listing the formal discrete constitutive laws that might be used in the discretization of (5):

(14) Primary:
$$\left\{\begin{array}{rcll}\mathsf{M}^{l}_{\alpha}\boldsymbol{\sigma} &=& \widetilde{\mathsf{K}}^{l}_{m}\mathbf{j} & \text{(a)} \\ \mathsf{M}^{l-1}_{\gamma}\mathbf{u} &=& \widetilde{\mathsf{K}}^{l-1}_{m+1}\boldsymbol{\psi} & \text{(b)} \\ \mathsf{M}^{l-1}_{\beta,\Gamma}\mathsf{T}^{l-1}_{\Gamma}\mathbf{u} &=& (-1)^{l-1}\,\widetilde{\mathsf{K}}^{l-1}_{m,\Gamma}\widetilde{\mathsf{T}}^{m}_{\Gamma}\mathbf{j} & \text{(c)}\end{array}\right.$$

$$\text{(15) Secondary:} \quad \begin{cases} \widetilde{\mathsf{M}}^m_{1/\alpha}\mathbf{j} \;=\; (-1)^{mn-m}\,\mathsf{K}^m_l\boldsymbol{\sigma} & \text{(a)} \\ \widetilde{\mathsf{M}}^{m+1}_{1/\gamma}\boldsymbol{\psi} \;=\; (-1)^{(l-1)(n-1)}\,\mathsf{K}^{m+1}_{l-1}\mathbf{u} & \text{(b)} \\ \widetilde{\mathsf{M}}^m_{1/\beta,\Gamma}\widetilde{\mathsf{T}}^m_\Gamma\mathbf{j} = (-1)^{(l-1)(n-1)}\,\mathsf{K}^m_{l-1,\Gamma}\mathsf{T}^{l-1}_\Gamma\mathbf{u} & \text{(c)} \end{cases}$$

I only discuss the stationary case. It goes without saying that exactly the same eliminations can be performed in the case of time-dependent problems.

1. *Primary elimination:* Using only primary discrete Hodge operators (14a), (14c) and (7), (10) we get

$$\begin{aligned}(\mathsf{D}^{l-1})^{\mathrm{T}}\mathsf{M}^l_\alpha\mathsf{D}^{l-1}\mathbf{u} &= (\mathsf{D}^{l-1})^{\mathrm{T}}(-1)^l\mathsf{M}^l_\alpha\boldsymbol{\sigma} = (-1)^l(\mathsf{D}^{l-1})^{\mathrm{T}}\widetilde{\mathsf{K}}^l_m\mathbf{j} = \\ &= \widetilde{\mathsf{K}}^{l-1}_{m+1}\widetilde{\mathsf{D}}^m\mathbf{j} + (-1)^l(\mathsf{T}^{l-1}_\Gamma)^{\mathrm{T}}\widetilde{\mathsf{K}}^{l-1}_{m,\Gamma}\widetilde{\mathsf{T}}^m_\Gamma\mathbf{j} = \\ &= \widetilde{\mathsf{K}}^{l-1}_{m+1}(-\boldsymbol{\psi}+\boldsymbol{\phi}) - (\mathsf{T}^{l-1}_\Gamma)^{\mathrm{T}}\mathsf{M}^{l-1}_{\beta,\Gamma}\mathsf{T}^{l-1}_\Gamma\mathbf{u} = \\ &= -\mathsf{M}^{l-1}_\gamma\mathbf{u} - (\mathsf{T}^{l-1}_\Gamma)^{\mathrm{T}}\mathsf{M}^{l-1}_{\beta,\Gamma}\mathsf{T}^{l-1}_\Gamma\mathbf{u} + \widetilde{\mathsf{K}}^{l-1}_{m+1}\boldsymbol{\phi}\,.\end{aligned}$$

We arrive at a linear system of equations

$$\text{(16)} \qquad (\mathsf{D}^{l-1})^{\mathrm{T}}\mathsf{M}^l_\alpha\mathsf{D}^{l-1}\mathbf{u} + \mathsf{M}^{l-1}_\gamma\mathbf{u} + (\mathsf{T}^{l-1}_\Gamma)^{\mathrm{T}}\mathsf{M}^{l-1}_{\beta,\Gamma}\mathsf{T}^{l-1}_\Gamma\mathbf{u} = \widetilde{\mathsf{K}}^{l-1}_{m+1}\boldsymbol{\phi}$$

for the unknown coefficients $\mathbf{u}$ with a symmetric positive definite coefficient matrix. If $\gamma \neq 0$, it has a unique solution. Even if $\gamma = 0$, at least $\mathsf{D}^{l-1}\mathbf{u}$ can be uniquely determined. Two more important facts have to mentioned: To begin with, the secondary spaces do *only* affect the right hand side $\widetilde{\mathsf{K}}^{l-1}_{m+1}\boldsymbol{\phi}$ of the final system. In other words, the choice of the secondary mesh is totally irrelevant as regards the ultimate system matrix. Secondly, in the process of elimination we irretrievable lost information about the secondary unknowns $\boldsymbol{\sigma}_h$ and ψ_h, unless the pairing matrices are invertible.

2. *Secondary elimination:* We exclusively rely on secondary discrete Hodge operators (15b), (15c) along with (7), (11):

$$\begin{aligned}\widetilde{\mathsf{M}}^m_{1/\alpha}\mathbf{j} &= (-1)^{n(m+1)}\mathsf{K}^m_l\mathsf{D}^{l-1}\mathbf{u} \\ &= (-1)^{(n-1)(l-1)}((\widetilde{\mathsf{D}}^m)^{\mathrm{T}}\mathsf{K}^{m+1}_{l-1}\mathbf{u} - (\widetilde{\mathsf{T}}^m_\Gamma)^{\mathrm{T}}\mathsf{K}^m_{l-1,\Gamma}\mathsf{T}^{l-1}_\Gamma\mathbf{u}) \\ &= (\widetilde{\mathsf{D}}^m)^{\mathrm{T}}\widetilde{\mathsf{M}}^{m+1}_{1/\gamma}\boldsymbol{\psi} - (\widetilde{\mathsf{T}}^m_\Gamma)^{\mathrm{T}}\widetilde{\mathsf{M}}^m_{1/\beta,\Gamma}\widetilde{\mathsf{T}}^m_\Gamma\mathbf{j}\end{aligned}$$

Introducing the auxiliary unknown $\boldsymbol{\zeta} := \widetilde{\mathsf{M}}^{m+1}_{1/\gamma}\boldsymbol{\psi} = (-1)^{(l-1)(n-1)}\mathsf{K}^{m+1}_{l-1}\mathbf{u}$ we get the saddle point problem

$$\text{(17)} \qquad \begin{pmatrix} \widetilde{\mathsf{M}}^m_{1/\alpha} + (\widetilde{\mathsf{T}}^m_\Gamma)^{\mathrm{T}}\widetilde{\mathsf{M}}^m_{1/\beta,\Gamma}\widetilde{\mathsf{T}}^m_\Gamma & -(\widetilde{\mathsf{D}}^m)^{\mathrm{T}} \\ -\widetilde{\mathsf{D}}^m & -(\widetilde{\mathsf{M}}^{m+1}_{1/\gamma})^{-1} \end{pmatrix}\begin{pmatrix}\mathbf{j} \\ \boldsymbol{\zeta}\end{pmatrix} = \begin{pmatrix} 0 \\ -\boldsymbol{\phi}\end{pmatrix}\,.$$

As the diagonal blocks are positive and negative definite, respectively, this linear system has a unique solution (cf. [18]). In the case of vanishing γ we just define the auxiliary unknown as $\boldsymbol{\zeta} := \mathsf{K}_{l-1}^{m+1}\mathbf{u}$ and still get uniqueness for $\mathbf{j}$. Parallel to the purely primal case we observe that no trace of the primary discrete spaces is left. As above, in general we cannot solve for the primary unknowns $\mathbf{u}$ and $\boldsymbol{\sigma}$.

3. *Hybrid elimination:* Both primary and secondary discrete Hodge operators are used for the sake of eliminating unknowns. For example, if we use primary representations (14a), (14c) for the material laws $\mathbf{j} = \star_\alpha \boldsymbol{\sigma}$ and $\mathbf{t}_M \mathbf{j} = (-1)^{l-1} \star_\beta^\Gamma \mathbf{t}_M u$, but the secondary version (15b) for $\psi = \star_\gamma u$ we get

$$(\mathsf{D}^{l-1})^{\mathrm{T}} \mathsf{M}_\alpha^l \mathsf{D}^{l-1}\mathbf{u} + (\mathsf{T}_\Gamma^{l-1})^{\mathrm{T}} \mathsf{M}_{\beta,\Gamma}^{l-1} \mathsf{T}_\Gamma^{l-1}\mathbf{u} + \widetilde{\mathsf{K}}_{m+1}^{l-1} (\widetilde{\mathsf{M}}_{1/\gamma}^{m+1})^{-1} \mathsf{K}_{l-1}^{m+1}\mathbf{u} = \\ = \widetilde{\mathsf{K}}_{m+1}^{l-1}\boldsymbol{\phi} \; .$$

Again, we have obtained a positive semidefinite system of linear equations for the unknown vector $\mathbf{u}$, in which both meshes shine through. Yet, even if $\gamma \neq 0$ the system matrix need not be regular. At least a unique solution for $\boldsymbol{\sigma} := (-1)^l \mathsf{D}^{l-1}\mathbf{u}$ is guaranteed. On the other hand, the elimination might have squandered any information about $\mathbf{j}$.

5 Examples

The most natural way to define the matrices in (8) is to plug in the bases of the spaces of discrete differential forms in the variational definitions from (12). This may be dubbed the *finite element approach*: The matrices M_*^*, $\widetilde{\mathsf{M}}_*^*$ become exact mass matrices. I should point out that primary elimination yields (almost) the same system of linear equations (16) as the primal finite element Galerkin method. The only exception might be a modified right hand side. By secondary elimination we get the linear saddle point problem (17) of the dual mixed finite element Galerkin method (cf. [18]). Hitherto unknown problems arise from hybrid eliminations.

In the case of the finite element approach all the requirements stated for the matrices in the previous section are automatically satisfied. Since we are free to pick any primary and secondary mesh, the pairing matrices are not square in general. In particular, there is no reason, why they should be invertible and information about some eliminated unknowns cannot be recovered.

A bijective relationship between primary and secondary unknowns is the rationale behind the second class of methods. It can be achieved by using a *dual* secondary mesh [13].

Definition 3 *Two meshes $\widetilde{\mathcal{T}}_h$ and $\mathcal{T}_h$ covering an n-dimensional manifold are called (topologically) dual to each other if* $\mathsf{L}_l^{\mathrm{T}} = (-1)^l \mathsf{L}_{n-l+1}, 0 \leq l <$

n, where L_l and L_l are the incidence matrices of oriented l- and $(l+1)$-faces of $\mathcal{T}_h$ and $\tilde{\mathcal{T}}_h$, respectively.

In [64, Sect. V] the relationships between an externally oriented dual mesh and twisted differential forms is thoroughly discussed, but I will not dwell on this subject. Just note that for dual meshes the numbers of l-faces of one mesh and those of $(n-l)$-faces of the other mesh must be equal. More precisely, the secondary mesh $\tilde{\mathcal{T}}_h$ is chosen such that

1. the restriction of $\tilde{\mathcal{T}}_h$ to the interior of Ω is dual to the entire mesh $\mathcal{T}_h$.
2. the restriction of $\tilde{\mathcal{T}}_h$ and $\mathcal{T}_h$ to the boundary $\partial\Omega$ are dual to each other.

Thanks to duality, we can assume a one-to-one correspondence between l-faces of $\mathcal{T}_h$ and interior $n-l$-factes of $\tilde{\mathcal{T}}_h$. Similarly, we may associate boundary l-faces, $0 \leq l < n$, of $\mathcal{T}_h$ and $(n-1-l)$-faces of $\tilde{\mathcal{T}}_h$ on $\partial\Omega$. If $\mathcal{T}_h$ is Γ_D-active, an l-face of $\tilde{\mathcal{T}}_h$ is active, i.e., it bears a d.o.f., if one of the following alternatives applies:

1. Either it is contained in $\partial\Omega \setminus \bar{\Gamma}_N$ and associated with an active primary $(n-1-l)$-face of $\mathcal{T}_h$,
2. or it is located inside Ω and belongs to an active primary $(n-l)$-face.

Γ_D and Γ_N may switch roles depending on which unknowns are discretized on the primary mesh. Figure 1 sketches an example of two dual grids in two dimensions. It reveals that the dual mesh may fail to comply with the partitioning of the boundary. This can be cured by confining a degree of freedom to only a part of a boundary face. If $\Gamma_M = \emptyset$, there is no active $(n-1)$-face of $\tilde{\mathcal{T}}_h$ on the boundary. The numbering of interior l-faces of $\tilde{\mathcal{T}}_h$ is induced by the numbering of primary $(n-l)$-faces via duality. Boundary faces of $\tilde{\mathcal{T}}_h$ are numbered last. Then the intimate relationship between discrete exterior derivatives and incidence matrices shows that for $0 < m, l < n, l+m = m$

$$(18)\qquad (\mathsf{D}^{l-1})^{\mathrm{T}}\,(\mathsf{E}_{N_l}, \mathsf{O}) = (-1)^l \mathsf{E}_{N_{l-1}} \widetilde{\mathsf{D}}^m + (\mathsf{T}_\Gamma^{l-1})^{\mathrm{T}} \widetilde{\mathsf{T}}_\Gamma^m \,.$$

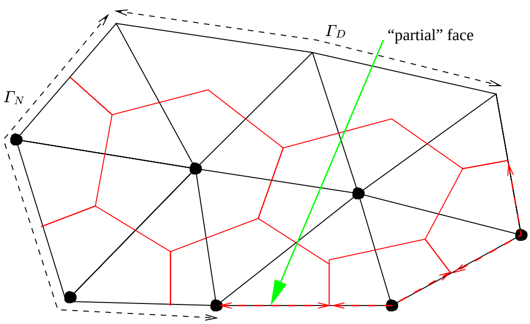

Fig. 1. Primal and dual grid in two dimensions. The bullets represent active vertices of the primary grid, the edges $\longleftrightarrow$ active boundary faces of the secondary grid

Here E_N stands for a $N \times N$ identity matrix, and O denotes a zero block of dimension $N_l \times \tilde{N}_m^{\Gamma}$, with $\tilde{N}_m^{\Gamma}$ the number of active boundary m-faces of $\tilde{\mathcal{T}}_h$. A glance at (13) and (10) shows that we can choose

$$\text{(19)} \qquad \widetilde{\mathsf{K}}_m^l := (\mathsf{E}_{N_l}, \mathsf{O}) \quad , \quad \widetilde{\mathsf{K}}_{m+1}^{l-1} := \mathsf{E}_{N_{l-1}} \quad , \quad \widetilde{\mathsf{K}}_{m,\Gamma}^{l-1} := \mathsf{E}_{\tilde{N}_m^{\Gamma}} \ ,$$

and abide by requirement **C** at the same time. If $\Gamma_M = \emptyset$, the zero block disappears and the pairing matrices reduce to (signed) identity matrices. In any case, the dual unknowns $\mathbf{j}$ and $\boldsymbol{\psi}$ can be calculated from $\boldsymbol{\sigma}$, $\mathbf{u}$, no matter which discrete Hodge oerator is used.

Examples for dual meshes are supplied by the usual covolumes (boxes) used in finite volume schemes [33]. Well known are the circumcentric dual Voronoi meshes of a Delauney tesselation and the barycentric dual meshes of the box method. This is why I chose to call the methods of this second class *generalized finite volume methods*. As a special subclass they include *covolume methods* distiguished by the use of diagonal approximate mass matrices and orthogonal dual meshes [53, 54].

Let us for simplicity assume $\Gamma_M = \emptyset$. Then the discrete material laws for generalized finite volume methods read

$$\text{(20)} \qquad \mathsf{M}_\alpha^l \boldsymbol{\sigma} = \mathbf{j} \quad \text{or} \quad (\widetilde{\mathsf{M}}_{1/\alpha}^m)^{-1} \boldsymbol{\sigma} = \mathbf{j} \ ,$$

$$\text{(21)} \qquad \mathsf{M}_\gamma^{l-1} \mathbf{u} = \boldsymbol{\psi} \quad \text{or} \quad (\widetilde{\mathsf{M}}_{1/\gamma}^{m+1})^{-1} \mathbf{u} = \boldsymbol{\psi} \ .$$

In short, all discrete material laws can be viewed as both a primary and secondary version. This makes it possible to proceed with both primary and secondary elimination. We end up with linear systems of equations for primary or secondary unknowns only and draw an important conclusion:

Corollary 1 *Generalized finite volume methods combined with either primary or secondary elimination lead to linear systems of equations that also arise from a primal or mixed-dual finite element discretization employing some approximation of the mass matrices and source term.*

Thus, the study of generalized finite volume method can exploit the powerful tools of finite element theory. This generalizes the results of [33, 3], where the case $n = 2$, $l = 1$ and its links with the primal Galerkin finite element method were thoroughly investigated. In [4, 57] the connection with a dual mixed Galerkin scheme with lumped mass matrix was explored. Covolume schemes for Maxwell's equations [67, 68, 70, 66] can also be analyzed from this perspective [15]. Eventually, knowledge about the underlying discrete differential forms offers a recipe for the natural reconstruction of fields from the degrees of freedom. Thanks to the canonical transformations of discrete differential forms [35] this is useful even for distorted elements as in [61]. Ultimately, awareness of basic requirements for discrete Hodge operators reveals causes for instability of finite volume schemes and leads to remedies [58].

Another conclusion is that a finite volume method can be completely specified by prescribing some procedure to compute the (approximate) mass matrices and the right hand side. Conversely, Galerkin finite element schemes can be viewed as finite volume methods (cf [63]). Thus we learn how to recover approximations to quantities that have been dumped in the process of primary or secondary elimination (cf. [8]).

Also the *method of support operators* (mimetic finite differences) [59, 60] can be seen as a special finite volume approach to the construction of discrete Hodge operators. Using only discrete differential forms on a primary grid [38], it focuses on Hodge codifferentials $d_* := (-1)^{nl-1} \star d \star : \mathcal{D}^l(\Omega) \mapsto \mathcal{D}^{l-1}(\Omega)$ [40, Sect. 3.2]. The construction of their discrete counterparts $\mathsf{D}^l_* : \mathcal{C}^l \mapsto \mathcal{C}^{l-1}$ is based on the variational characterization $\left(\mathbf{u}, \mathsf{D}^{l-1}\mathbf{v}\right)_0 = \left(\mathsf{D}^l_*\mathbf{u}, \mathbf{v}\right)_0$ for all $\mathbf{u} \in \mathcal{C}^l, \mathbf{v} \in \mathcal{C}^{l-1}$. Special approximations of the inner products are employed to this end [37].

Remark 4 The discussion illustrates the basic limitation of finite volume schemes to lowest order. If we had decided to use higher order discrete differential forms, the matrices of the exterior derivative could not have been identified as incidence matrices. Then it is very hard to come up with a suitable secondary mesh rendering the pairing matrices square and invertible.

6 Abstract error analysis

We have learned that often the error analysis can be carried out in a Galerkin setting involving variational crimes (cf. [16, Ch. 6]). Yet, this is not possible for all combinations of discrete material laws. Therefore, the error analysis presented in this section forgoes the Galerkin option.

First, the stationary problem is considered. I start from the premises that it is most natural to use the energy norm when gauging the discretization error. An ambiguity arises, because, given discrete solutions $u_h, \boldsymbol{\sigma}_h, \mathbf{j}_h, \psi_h$ we can either examine continuous energy norms of the error, e.g.

$$\|(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{\mathcal{E}} := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_\alpha + \|u - u_h\|^2_\gamma + \|u - u_h\|^2_\beta \ ,$$
$$\|(\mathbf{j} - \mathbf{j}_h, \psi - \psi_h)\|^2_{\tilde{\mathcal{E}}} := \|\mathbf{j} - \mathbf{j}_h\|^2_{1/\alpha} + \|\psi - \psi_h\|^2_{1/\gamma} + \|\mathbf{j} - \mathbf{j}_h\|^2_{1/\beta} \ ,$$

or discrete energy norms of the following *nodal errors*

$$\delta\mathbf{u} := \mathbf{u}^* - \mathbf{u}\,,\ \ \mathbf{u}^* := I_h^{l-1} u \quad , \quad \delta\boldsymbol{\sigma} := \boldsymbol{\sigma}^* - \boldsymbol{\sigma}\,,\ \ \boldsymbol{\sigma}^* := I_h^l \boldsymbol{\sigma}\ ,$$
$$\delta\mathbf{j} := \mathbf{j}^* - \mathbf{j}\,,\ \ \mathbf{j}^* := \tilde{I}_h^m \mathbf{j} \quad , \quad \delta\boldsymbol{\psi} := \boldsymbol{\psi}^* - \boldsymbol{\psi}\,,\ \ \boldsymbol{\psi}^* := \tilde{I}_h^{m+1} \psi\ .$$

What are meaningful discrete energy norms heavily hinges on the choice of the discrete material laws.

First, we tackle the case of the discrete stationary boundary value problem (5,I) when only primary discrete Hodge operators from (14) are used. This fixes the relevant discrete energy norm

$$|(\mathbf{u},\boldsymbol{\sigma})|^2_{\mathcal{E}} := |\boldsymbol{\sigma}|^2_\alpha + |\mathbf{u}|^2_\gamma + |\mathbf{u}|^2_\beta \ ,$$
$$|\cdot|^2_\alpha := \left\langle \mathsf{M}^l_\alpha \cdot, \cdot \right\rangle \ , \ |\cdot|^2_\gamma := \left\langle \mathsf{M}^{l-1}_\gamma \cdot, \cdot \right\rangle \ , \ |\cdot|^2_\beta := \left\langle \mathsf{M}^{l-1}_{\beta,\Gamma} \mathsf{T}^{l-1}_\Gamma \cdot, \mathsf{T}^{l-1}_\Gamma \cdot \right\rangle \ .$$

First, in the spirit of [54], we observe that thanks to the commuting diagram property of the nodal projection the discrete equilibrium laws are free of consistency errors:

$$\mathsf{D}^{l-1}\delta\mathbf{u} = (-1)^l\,\delta\boldsymbol{\sigma} \quad , \quad \widetilde{\mathsf{D}}^m\delta\mathbf{j} = -\delta\boldsymbol{\psi} + \delta\boldsymbol{\phi} \ . \tag{22}$$

In what follows I assume that $\delta\boldsymbol{\phi} = 0$, i.e. $\boldsymbol{\phi} = \tilde{I}^{m+1}_h\phi$. If this is not the case, one additional term enters the error bounds established below. In constrast to (22), consistency errors lurk in the discrete material laws (cf. [64, Sect. VII] and [67])

$$\mathsf{M}^l_\alpha\delta\boldsymbol{\sigma} = \widetilde{\mathsf{K}}^l_m\delta\mathbf{j} + \mathbf{R}_l \ , \tag{23}$$
$$\mathsf{M}^{l-1}_\gamma\delta\mathbf{u} = \widetilde{\mathsf{K}}^{l-1}_{m+1}\delta\boldsymbol{\psi} + \mathbf{R}_{l-1} \ , \tag{24}$$
$$\mathsf{M}^{l-1}_{\beta,\Gamma}\mathsf{T}^{l-1}_\Gamma\delta\mathbf{u} = (-1)^{l+1}\widetilde{\mathsf{K}}^{l-1}_{m,\Gamma}\widetilde{\mathsf{T}}^m_\Gamma\delta\mathbf{j} + \mathbf{R}_\Gamma \ , \tag{25}$$

with some residuals $\mathbf{R}_l \in \mathcal{C}^1$, $\mathbf{R}_{l-1} \in \mathcal{C}^0$, and $\mathbf{R}_\Gamma \in \mathcal{C}^0_\Gamma$. Based on (22) and (13), we can estimate the discrete energy of the nodal errors

$$\begin{aligned}
&\left\langle \mathsf{M}^l_\alpha\delta\boldsymbol{\sigma}, \delta\boldsymbol{\sigma}\right\rangle + \left\langle \mathsf{M}^{l-1}_\gamma\delta\mathbf{u}, \delta\mathbf{u}\right\rangle + \left\langle \mathsf{M}^{l-1}_{\beta,\Gamma}\mathsf{T}^{l-1}_\Gamma\delta\mathbf{u}, \mathsf{T}^{l-1}_\Gamma\delta\mathbf{u}\right\rangle = \\
&\quad = \left\langle \widetilde{\mathsf{K}}^l_m\delta\mathbf{j} + \mathbf{R}_l, (-1)^l\,\mathsf{D}^{l-1}\delta\mathbf{u}\right\rangle + \left\langle \widetilde{\mathsf{K}}^{l-1}_{m+1}\delta\boldsymbol{\psi} + \mathbf{R}_{l-1}, \delta\mathbf{u}\right\rangle + \\
&\qquad + \left\langle (-1)^{l-1}\widetilde{\mathsf{K}}^{l-1}_{m,\Gamma}\widetilde{\mathsf{T}}^m_\Gamma\delta\mathbf{j} + \mathbf{R}_\Gamma, \mathsf{T}^{l-1}_\Gamma\delta\mathbf{u}\right\rangle \\
&\quad = \langle \mathbf{R}_l, \delta\boldsymbol{\sigma}\rangle + \langle \mathbf{R}_{l-1}, \delta\mathbf{u}\rangle + \left\langle \mathbf{R}_\Gamma, \mathsf{T}^{l-1}_\Gamma\delta\mathbf{u}\right\rangle \ .
\end{aligned}$$

Please note that the second terms on both sides do not occur in the case $\gamma = 0$ and an error estimate for u remains elusive. Hardly surprising, because there might not be a unique solution for $\mathbf{u}$, unless special properties of D^{l-1} (injectivity) are known. By the Cauchy-Schwarz inequality

$$|(\delta\mathbf{u},\delta\boldsymbol{\sigma})|_{\mathcal{E}} \leq \left|(\mathsf{M}^l_\alpha)^{-1}\mathbf{R}_l\right|_\alpha + \left|(\mathsf{M}^{l-1}_\gamma)^{-1}\mathbf{R}_{l-1}\right|_\gamma + \left|(\mathsf{M}^{l-1}_{\beta,\Gamma})^{-1}\mathbf{R}_\Gamma\right|_\beta \ .$$

Similar considerations apply, when solely secondary discrete Hodge operators from (15) are employed. Then the suitable discrete energy norm is given by

$$|(\mathbf{j},\boldsymbol{\psi})|^2_{\tilde{\mathcal{E}}} := |\mathbf{j}|^2_{1/\alpha} + |\boldsymbol{\psi}|^2_{1/\gamma} + |\mathbf{j}|^2_{1/\beta} \ ,$$

$$|\cdot|^2_{1/\alpha} := \left\langle \widetilde{\mathsf{M}}^m_{1/\alpha}\cdot,\cdot\right\rangle \; , \; |\cdot|^2_{1/\gamma} := \left\langle \widetilde{\mathsf{M}}^{m+1}_{1/\gamma}\cdot,\cdot\right\rangle \; , \; |\cdot|^2_{1/\beta} := \left\langle \widetilde{\mathsf{M}}^m_{1/\beta,\Gamma}\widetilde{\mathsf{T}}^m_\Gamma\cdot, \widetilde{\mathsf{T}}^m_\Gamma\cdot\right\rangle .$$

Partly retaining the notations for the consistency errors, we can write

$$\widetilde{\mathsf{M}}^m_{1/\alpha}\delta\mathbf{j} = (-1)^{mn-m}\,\mathsf{K}^m_l\delta\boldsymbol{\sigma} + \mathbf{R}_m \tag{26}$$

$$\widetilde{\mathsf{M}}^{m+1}_{1/\gamma}\delta\boldsymbol{\psi} = (-1)^{(l-1)(n-1)}\,\mathsf{K}^{m+1}_{l-1}\delta\mathbf{u} + \mathbf{R}_{m+1} \tag{27}$$

$$\widetilde{\mathsf{M}}^m_{1/\beta,\Gamma}\widetilde{\mathsf{T}}^m_\Gamma\delta\mathbf{j} = (-1)^{(l-1)(n-1)}\,\mathsf{K}^m_{l-1,\Gamma}\mathsf{T}^{l-1}_\Gamma\delta\mathbf{u} + \mathbf{R}_\Gamma \; , \tag{28}$$

with $\mathbf{R}_m \in \tilde{\mathcal{C}}^m$, $\mathbf{R}_{m+1} \in \tilde{\mathcal{C}}^{m+1}$, and $\mathbf{R}_\Gamma \in \tilde{\mathcal{C}}^m_\Gamma$. Since (22) remains valid, the following identity is established as above:

$$\begin{aligned}\left\langle \widetilde{\mathsf{M}}^l_{1/\alpha}\delta\mathbf{j},\delta\mathbf{j}\right\rangle + \left\langle \widetilde{\mathsf{M}}^{l-1}_{1/\gamma}\delta\boldsymbol{\psi},\delta\boldsymbol{\psi}\right\rangle + \left\langle \widetilde{\mathsf{M}}^{l-1}_{1/\beta,\Gamma}\widetilde{\mathsf{T}}^m_\Gamma\delta\mathbf{j},\widetilde{\mathsf{T}}^m_\Gamma\delta\mathbf{j}\right\rangle =\\ = \langle\mathbf{R}_m,\delta\mathbf{j}\rangle + \langle\mathbf{R}_{m+1},\delta\boldsymbol{\psi}\rangle + \left\langle\mathbf{R}_\Gamma,\widetilde{\mathsf{T}}^m_\Gamma\delta\mathbf{j}\right\rangle \; .\end{aligned}$$

Other combinations of discrete material laws are treated alike. For the sake of brevity, I am not elaborating on this. In sum, estimating the consistency errors of the material laws is the key to controlling discrete energy norms of nodal errors. The analysis of finite volume methods [53, 54, 49] is often content with this goal, but I am not. Discrete energies might lack any physical meaning, so that the focus should be on the exact energy norm. In the case of primary discrete Hodge operators

$$\|(u-u_h,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{\mathcal{E}} \le \|(u-u_h^*,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^*)\|_{\mathcal{E}} + \|(\delta u_h,\delta\boldsymbol{\sigma}_h)\|_{\mathcal{E}}$$

tells us that it is essential to have *stability*

$$\|(u_h,\boldsymbol{\sigma}_h)\|_{\mathcal{E}} \le C\,|(\mathbf{u},\boldsymbol{\sigma})|_{\mathcal{E}} \quad \forall \mathbf{u}\in\mathcal{C}^{l-1},\; \boldsymbol{\sigma}\in\mathcal{C}^l \; , \tag{29}$$

in order to get information about the energy of the total discretization error. Here, the constant $C > 0$ should be independent of as much geometric parameters of the mesh as possible. In the case of secondary discrete constitutive laws, we proceed as above, replacing $\|\cdot\|_{\mathcal{E}}$ by $\|\cdot\|_{\tilde{\mathcal{E}}}$ and $|\cdot|_{\mathcal{E}}$ by $|\cdot|_{\tilde{\mathcal{E}}}$.

In addition, the energy of the *projection error* $\|(u-u_h^*,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^*)\|_{\mathcal{E}}$ has to be bounded. This is the objective of asymptotic finite element interpolation estimates. Those rely on a Sobolev space setting and are largely based on the Bramble-Hilbert lemma and affine equivalence techniques depending on families of quasiuniform and shape regular meshes [16, Ch. 4], [22, Ch. 3]. In particular, for results on discrete 2-forms in two and three dimensions the reader should consult [18]. Estimates for discrete 1-forms ($n = 3$) can be found in [51], [31], [23], and [47]. It is important to be aware that all estimates hinge on assumptions on the smoothness of the continuous solutions.

In the case of finite difference or finite volume methods, one might object that the spaces of discrete differential forms are "artificial" and so is the notion of a total discretization error. Yet, an approximation of the total energy must always be available and the error in the approximation of the total energy is well defined at any rate. For this error we get in the case of primary discrete Hodge operators

$$\begin{aligned}\|(u,\boldsymbol{\sigma})\|_{\mathcal{E}}^2 - |(\mathbf{u},\boldsymbol{\sigma})|_{\mathcal{E}}^2 &= \|(u,\boldsymbol{\sigma})\|_{\mathcal{E}}^2 - \|(u_h^*,\boldsymbol{\sigma}_h^*)\|_{\mathcal{E}}^2 + \|(u_h^*,\boldsymbol{\sigma}_h^*)\|_{\mathcal{E}}^2 - \\ &\quad - |(\mathbf{u}^*,\boldsymbol{\sigma}^*)|_{\mathcal{E}}^2 + |(\mathbf{u}^*,\boldsymbol{\sigma}^*)|_{\mathcal{E}}^2 - |(\mathbf{u},\boldsymbol{\sigma})|_{\mathcal{E}}^2 \\ &\leq \|(u-u_h^*,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^*)\|_{\mathcal{E}}\,\|(u+u_h^*,\boldsymbol{\sigma}+\boldsymbol{\sigma}_h^*)\|_{\mathcal{E}} + \\ &\quad + |(\mathbf{u}^*-\mathbf{u},\boldsymbol{\sigma}^*-\boldsymbol{\sigma})|_{\mathcal{E}}\,|(\mathbf{u}^*+\mathbf{u},\boldsymbol{\sigma}^*+\boldsymbol{\sigma})|_{\mathcal{E}} + \\ &\quad + \|(u_h^*,\boldsymbol{\sigma}_h^*)\|_{\mathcal{E}}^2 - |(\mathbf{u}^*,\boldsymbol{\sigma}^*)|_{\mathcal{E}}^2\ .\end{aligned}$$

Even if the discretization error, the nodal error, and the projection error tend to zero, the error in the energy need not, owing to the quantity $\|(u_h^*,\boldsymbol{\sigma}_h^*)\|_{\mathcal{E}}^2 - |(\mathbf{u}^*,\boldsymbol{\sigma}^*)|_{\mathcal{E}}^2$. It can be regarded as a consistency error in the approximation of the mass matrices. Hence, the quality of the approximation of the discrete energy can serve as an acid test for the efficacy of a discretization scheme.

Remark 5 Primary Hodge operators do not permit us to get error estimates for dual quantities. However, the generalized finite volume methods are an exception. For instance, in the case $\Gamma_M = \emptyset$ the bound for $|(\delta\mathbf{u},\delta\boldsymbol{\sigma})|_{\mathcal{E}}$ also applies to

$$\left\langle (\widetilde{\mathsf{M}}_\alpha^l)^{-1}\delta\mathbf{j},\delta\mathbf{j}\right\rangle + \left\langle (\widetilde{\mathsf{M}}_\gamma^{l-1})^{-1}\delta\boldsymbol{\psi},\delta\boldsymbol{\psi}\right\rangle + \left\langle (\widetilde{\mathsf{M}}_{\beta,\Gamma}^{l-1})^{-1}\widetilde{\mathsf{T}}_\Gamma^m\delta\mathbf{j},\widetilde{\mathsf{T}}_\Gamma^m\delta\mathbf{j}\right\rangle\ .$$

This paves the way for coming to terms with $\|(\mathbf{j}-\mathbf{j}_h,\psi-\psi_h)\|_{\tilde{\mathcal{E}}}^2$.

An abstract error analysis is also possible in the case of the wave equation following the strategy from [1]. (cf. [53], [49, Lemma 3.8]), and [46, Thm. 2.1] for the treatment of Maxwell's equations). Here, only the semidiscrete case without discretization in time is treated. Fully discrete schemes are examined, e.g., in [44], [23], and [1]. I assuming that there is no nodal error in the intial values, i.e. $\delta\mathbf{u}(0)=0$ and $\delta\mathbf{j}(0)=0$. In addition, a hybrid choice $\mathsf{M}_\gamma^{l-1}\mathbf{u} = \widetilde{\mathsf{K}}_{m+1}^{l-1}\boldsymbol{\psi}$, $\widetilde{\mathsf{M}}_{1/\alpha}^m\mathbf{j} = (-1)^{mn-m}\mathsf{K}_l^m\boldsymbol{\sigma}$, and $\mathsf{M}_{\beta,\Gamma}^{l-1}\mathsf{T}_\Gamma^{l-1}\mathbf{u} = (-1)^{l-1}\widetilde{\mathsf{K}}_{m,\Gamma}^{l-1}\widetilde{\mathsf{T}}_\Gamma^m\mathbf{j}$ of discrete material laws is employed. Then, using the discrete equilibrium laws, the definitions of the consistency errors, integration w.r.t. time and applying the Cauchy-Schwarz inequality several times, we end up with

$$\begin{aligned}\max_{0\leq t\leq T}\frac{1}{2}\left(|\delta\mathbf{j}(t)|_{1/\alpha}^2 + |\delta\mathbf{u}(t)|_\gamma^2\right) \leq\ 3\int_0^{\mathrm{T}}\left|(\widetilde{\mathsf{M}}_{1/\alpha}^m)^{-1}\dot{\mathbf{R}}_m\right|_{1/\alpha}^2 + \\ + \int_0^{\mathrm{T}}\left|(\mathsf{M}_\gamma^{l-1})^{-1}\dot{\mathbf{R}}_{l-1}\right|_\gamma^2 dt + \frac{3}{8}\int_0^{\mathrm{T}}\left|(\mathsf{M}_{\beta,\Gamma}^{l-1})^{-1}\mathbf{R}_\Gamma\right|_\beta^2 dt\ ,\end{aligned}$$

where $\dot{\mathbf{R}}_m := \frac{d}{dt}\mathbf{R}_m$, $\dot{\mathbf{R}}_{l-1} := \frac{d}{dt}\mathbf{R}_{l-1}$. For details the reader is referred to the papers cited above and [36]. For the parabolic case and purely primal discrete material laws we find

$$\max_{0<t<T} |\delta \mathbf{u}(t^*)|_\gamma^2 \leq 6 \int_0^T \left|(\mathsf{M}_\gamma^{l-1})^{-1}\dot{\mathbf{R}}_{l-1}\right|_\gamma^2 + \left|(\mathsf{M}_\alpha^{l})^{-1}\mathbf{R}_{l}\right|_\alpha^2 + + \left|(\mathsf{M}_{\beta,\Gamma}^{l-1})^{-1}\mathbf{R}_{\Gamma}\right|_\beta^2 dt \; .$$

The bottom line is that error analysis has to zero in on the consistency errors, projection errors, and stability issues concerning the approximate mass matrix in all cases.

7 Estimation of consistency errors

Let us study the consistency error term $\mathbf{R}_l$ from (23) and find bounds for the relevant primary norm $\left|(\mathsf{M}_\alpha^l)^{-1}\mathbf{R}_l\right|_\alpha$. As a consequence of (22), it is immediate that

$$\left|(\mathsf{M}_\alpha^l)^{-1}\mathbf{R}_l\right|_\alpha^2 = \left\langle \mathsf{M}_\alpha^l(\boldsymbol{\sigma}^* - \boldsymbol{\zeta}), \boldsymbol{\sigma}^* - \boldsymbol{\zeta} \right\rangle \; , \qquad (30)$$

where I set $\boldsymbol{\zeta} := (\mathsf{M}_\alpha^l)^{-1}\widetilde{\mathsf{K}}_m^l\mathbf{j}^*$. Note that by the definition of the weak solution (cf. (12)) $(\boldsymbol{\sigma}, \boldsymbol{\eta})_\alpha = \int_\Omega \mathbf{j} \wedge \boldsymbol{\eta}$ for all $\boldsymbol{\eta} \in \mathcal{H}_l(d, \Omega)$. Then

$$\begin{aligned}
\left|(\mathsf{M}_\alpha^l)^{-1}\mathbf{R}_l\right|_\alpha &= \sup_{\boldsymbol{\eta}\in\mathcal{C}^l} \frac{\left\langle \mathsf{M}_\alpha^l(\boldsymbol{\sigma}^* - \boldsymbol{\zeta}), \boldsymbol{\eta}\right\rangle}{|\boldsymbol{\eta}|_\alpha} = \\
&= \sup_{\boldsymbol{\eta}\in\mathcal{C}^l} \frac{1}{|\boldsymbol{\eta}|_\alpha} \left(\left\langle \mathsf{M}_\alpha^l \boldsymbol{\sigma}^*, \boldsymbol{\eta} \right\rangle - \left\langle \widetilde{\mathsf{K}}_m^l \mathbf{j}^*, \boldsymbol{\eta} \right\rangle \right) \\
&= \sup_{\boldsymbol{\eta}\in\mathcal{C}^l} \frac{1}{|\boldsymbol{\eta}|_\alpha} \Bigg(\left\langle \mathsf{M}_\alpha^l \boldsymbol{\sigma}^*, \boldsymbol{\eta} \right\rangle - (\boldsymbol{\sigma}^*, \boldsymbol{\eta}_h)_\alpha + (\boldsymbol{\sigma}^* - \boldsymbol{\sigma}, \boldsymbol{\eta}_h)_\alpha + \\
&\qquad + \int_\Omega \mathbf{j} \wedge \boldsymbol{\eta}_h - \int_\Omega \mathbf{j}_h^* \wedge \boldsymbol{\eta}_h + \int_\Omega \mathbf{j}_h^* \wedge \boldsymbol{\eta}_h - \left\langle \widetilde{\mathsf{K}}_m^l \mathbf{j}^*, \boldsymbol{\eta} \right\rangle \Bigg) \\
&\leq \sup_{\boldsymbol{\eta}\in\mathcal{C}^l} \frac{\left\langle \mathsf{M}_\alpha^l \boldsymbol{\sigma}^*, \boldsymbol{\eta} \right\rangle - (\boldsymbol{\sigma}_h^*, \boldsymbol{\eta}_h)_\alpha}{|\boldsymbol{\eta}|_\alpha} + \sup_{\boldsymbol{\eta}\in\mathcal{C}^l} \frac{\int_\Omega \mathbf{j}_h^* \wedge \boldsymbol{\eta}_h - \left\langle \widetilde{\mathsf{K}}_m^l \mathbf{j}^*, \boldsymbol{\eta} \right\rangle}{|\boldsymbol{\eta}|_\alpha} + \\
&\qquad + \sup_{\boldsymbol{\eta}\in\mathcal{C}^l} \frac{\|\boldsymbol{\eta}_h\|_\alpha}{|\boldsymbol{\eta}|_\alpha} \cdot \left(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*\|_\alpha + \|\mathbf{j} - \mathbf{j}_h^*\|_{1/\alpha} \right) \; .
\end{aligned}$$

The first two terms are typical consistency errors, as they occur in estimates for finite element schemes with numerical quadrature [16, Ch. 8]. The factor in front of the third term reflects the stability of the approximate mass matrix M_α^l, whereas the third term itself incorporates approximation errors of the nodal projection. In the case of an exact Galerkin approach, the consistency terms and the stability factor can be dropped. Then, in combination with the results of the previous section, a standard finite element error estimate pops up.

There is a more direct approach to bounding the norm from (30), which is particularly useful in the case of diagonal finite volume methods (cf. [53, 54]). So let us assume that $\widetilde{\mathsf{K}}_m^l$ is an identity matrix and M_α^l is diagonal. For an l-face F let $\tilde{F}$ stand for its associated dual $(n-l)$-face. Remember that the components of vectors in $\mathcal{C}^l$ can be indexed by the l-faces in $\mathcal{F}_l$. By definition of $\mathbf{R}_l$

$$(31)\qquad \left|(\mathsf{M}_\alpha^l)^{-1}\mathbf{R}_l\right|_\alpha^2 = \sum_{F\in\mathcal{F}_l} m_F \mathbf{R}_{l,F}^2\ , \quad \mathbf{R}_{l,F} := m_F^{-1}\boldsymbol{\sigma}_F^* - \mathbf{j}_{\tilde{F}}^*\ ,$$

with m_F the diagonal element of M_α^l belonging to F, and a subscript F acting as a selector for vector components. Plugging in the canonical degrees of freedom for Whitney forms, we arrive at

$$(32)\qquad \mathbf{R}_{l,F} = m_F^{-1}\int_F \boldsymbol{\sigma} - \int_{\tilde{F}} \mathbf{j} = m_F^{-1}\int_F \boldsymbol{\sigma} - \int_{\tilde{F}} \star_\alpha \boldsymbol{\sigma}\ .$$

In the spirit of finite difference methods, the final term may be tackled based on a Taylor's expansion of $\boldsymbol{\sigma}$. If possible, it should be around a suitable point in space, provided by the intersection of F and $\tilde{F}$. Sufficient smoothness of $\boldsymbol{\sigma}$ is tacitly assumed. An alternative to Taylor's expansion are Bramble-Hilbert techniques, which impose less stringent requirements on smoothness. However, shape-regularity of the meshes is indispensable then [53].

It is worth noting that (32) offers a prescription for a viable choice of M_α^l. For instance, one could try to fix all m_F such that $\mathbf{R}_{l,F}$, $F\in\mathcal{F}_l$, vanishes for all constant $\boldsymbol{\sigma}$. However, the space of constant l-forms has dimension $\binom{n}{l}$. As $0<l<n$, this objective cannot be achieved in general. Consider the case of a constant metric α and flat faces. If $\boldsymbol{\sigma}$ is constant, too, (32) means

$$(33)\qquad \mathbf{R}_{l,F} = m_F^{-1}\mathrm{vol}_\alpha(F)\boldsymbol{\sigma}(\mathbf{t}_1,\dots,\mathbf{t}_l) - \mathrm{vol}_\alpha(\tilde{F})\boldsymbol{\sigma}(\mathbf{n}_1,\dots,\mathbf{n}_l)\ ,$$

where $\{\mathbf{t}_1,\dots,\mathbf{t}_l\}$ and $\{\mathbf{n}_1,\dots,\mathbf{n}_l\}$ are α-orthonormal (oriented) bases of the tangent space of F and of the orthogonal complement of the tangent space of $\tilde{F}$. Only if $\mathrm{Span}\,\{\mathbf{t}_1,\dots,\mathbf{t}_l\} = \mathrm{Span}\,\{\mathbf{n}_1,\dots,\mathbf{n}_l\}$, i.e. if F and $\tilde{F}$ are α-orthogonal, we can make $\mathbf{R}_{l,F}$ vanish for all alternating l-linear forms $\boldsymbol{\sigma}$. This highlights the necessity of *orthogonal* dual meshes if diagonal approximate mass matrices are desired.

8 Conclusion

I have developed a basically algebraic approach to the discretization of linear initial boundary value problems that fit the exterior calculus of differential forms. Discrete analogues are introduced as linear equations in finite dimensional vector spaces of suitable discrete differential forms. This general description covers most practical finite element and finite volume schemes. Besides establishing a link between these methods, minimal algebraic requirements even permit an abstract analysis of the discretization error.

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