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Galerkin proper orthogonal decomposition methods for parabolic problems

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Summary. In this work error estimates for Galerkin proper orthogonal decomposition (POD) methods for linear and certain non-linear parabolic systems are proved. The resulting error bounds depend on the number of POD basis functions and on the time discretization. Numerical examples are included.

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1. Introduction

Proper orthogonal decomposition (POD) is a method for deriving low order models of dynamical systems. It was successfully used in different fields including signal analysis and pattern recognition (see e.g. [9]), fluid dynamics and coherent structures (see e.g. [6,16]) and more recently in control theory (see e.g. [4,5,11,14]). Surprisingly good approximation properties are reported for POD based schemes in several articles, see [8,19] for example. However, to the authors' knowledge convergence results have not yet been established. In this work error estimates for Galerkin POD based methods for parabolic systems are proved. The resulting error bounds depend on the number of POD basis functions and on the time discretization. First, linear evolution problems are studied. For the time integration the backward Euler, Crank -Nicolson as well as the forward Euler methods are analyzed. Secondly, the analysis is extended to certain non-linear problems: to semi-linear problems with Lipschitz non-linearity and to the Burgers equation.

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The paper is organized as follows. Sect. 2 is devoted to reviewing the POD method. Error estimates for linear problems are proved in Sect. 3. Non-linear problems are studied in Sect. 4. In Sect. 5 numerical examples are presented.

2. The proper orthogonal decomposition

Let X be a real Hilbert space endowed with inner product $(\cdot\,,\cdot)_X$ and norm $\|\cdot\|_X$. For $y_1,\dots,y_n \in X$ we set

$$\mathcal{V} = \operatorname{span}\,\{y_1,\dots,y_n\},$$

and refer to $\mathcal{V}$ as ensemble consisting of the snapshots $\{y_j\}_{j=1}^n$, at least one of which is assumed to be non-zero. Let $\{\psi_k\}_{k=1}^d$ denote an orthonormal basis of $\mathcal{V}$ with $d = \dim \mathcal{V}$. Then each member of the ensemble can be expressed as

$$y_j = \sum_{k=1}^{d} (y_j, \psi_k)_X \psi_k \text{ for } j = 1,\dots,n. \tag{1}$$

The method of proper orthogonal decomposition consists in choosing the orthonormal basis such that for every $\ell \in \{1,\dots,d\}$ the mean square error between the elements y_j, $1 \le j \le n$, and the corresponding ℓ-th partial sum of (1) is minimized on average:

$$\begin{aligned} &\min_{\{\psi_k\}_{k=1}^{\ell}} \frac{1}{n} \sum_{j=1}^{n} \Big\| y_j - \sum_{k=1}^{\ell} (y_j, \psi_k)_X \psi_k \Big\|_X^2 \\ &\text{subject to } (\psi_i, \psi_j)_X = \delta_{ij} \text{ for } 1 \le i \le \ell, 1 \le j \le i. \end{aligned} \tag{2}$$

A solution $\{\psi_k\}_{k=1}^{\ell}$ to (2) is called a POD-basis of rank ℓ. We introduce the correlation matrix $K = ((K_{ij})) \in \mathbb{R}^{n\times n}$ corresponding to the snapshots $\{y_j\}_{j=1}^n$ by

$$K_{ij} = \frac{1}{n}\,(y_j, y_i)_X.$$

The matrix K is positive semi-definite and has rank d. The solution of (2) can be found in [6, 16], for instance.

Proposition 1. *Let $\lambda_1 \ge \ldots \ge \lambda_d > 0$ denote the positive eigenvalues of K and $v_1,\dots,v_d \in \mathbb{R}^n$ the associated eigenvectors. Then a POD basis of rank $\ell \le d$ is given by*

$$\psi_k = \frac{1}{\sqrt{\lambda_k}} \sum_{j=1}^{n} (v_k)_j y_j,$$

where $(v_k)_j$ is the j-th component of the eigenvector v_k. Moreover, we have the error formula

$$\frac{1}{n}\sum_{j=1}^{n}\left\|y_j-\sum_{k=1}^{\ell}(y_j,\psi_k)_X\psi_k\right\|_X^2=\sum_{k=\ell+1}^{d}\lambda_k.$$

3. Pod approximation of evolution problems of first order in *t*

This section is devoted to error estimates for Galerkin-POD methods for linear parabolic problems. For the time integration we study the backward Euler, Crank-Nicolson as well as the forward Euler method.

3.1. Problem formulation

Let V and H be real, separable Hilbert spaces and suppose that V is dense in H with continuous injection so that, by identifying H and its dual H^*, we have

$$V\hookrightarrow H=H^*\hookrightarrow V^*,$$

each embedding being dense. In particular, there exists a constant $\alpha>0$ such that

(3) $$\|\varphi\|_H^2\le\alpha\,\|\varphi\|_V^2 \text{ for all } \varphi\in V.$$

For $T>0$ we denote the space of measurable functions which are square integrable in the sense of Bochner by $L^2(0,T;V)$. The space $W(0,T;V)$ is defined by

$$W(0,T;V)=\left\{\varphi\in L^2(0,T;V):\varphi_t\in L^2(0,T;V^*)\right\}.$$

It is a Hilbert space endowed with the common inner product, see for instance in [7]. It is well-known that every $\varphi\in W(0,T;V)$ is almost everywhere equal to an element of $C([0,T];H)$, the space of continuous functions from $[0,T]$ to H.

Let $a:V\times V\to\mathbb{R}$ be a continuous and V-elliptic bilinear form, i.e. there exist constants $\beta>0$ and $\kappa>0$ such that

(4) $$|a(\varphi,\psi)|\le\beta\,\|\varphi\|_V\,\|\psi\|_V \text{ for all } \varphi,\psi\in V$$

and

(5) $$a(\varphi,\varphi)\ge\kappa\,\|\varphi\|_V^2 \text{ for all } \varphi\in V.$$

Suppose that $\phi \in H$ and $f \in C([0,T];H)$. Then the problem

(6a)
$$\frac{d}{dt}\,(u(t),\varphi)_H + a(u(t),\varphi) = (f(t),\varphi)_H \text{ for all } \varphi \in V \text{ and } t \in (0,T)$$

and

(6b)
$$(u(0),\chi)_H = (\phi,\chi)_H \text{ for all } \chi \in H$$

admits a unique solution $u \in W(0,T;V)$. If moreover, $\phi \in V$, then $u \in C([0,T];V)$ and $u_t \in C([0,T];H)$. For the proofs of these results we refer to [15], for example.

3.2. Computation of the POD basis

Throughout this section we denote by $u \in C([0,T];V)$ a solution to (6) with $\phi \in V$. For $m \in \mathbb{N}$ we introduce the time step $\Delta t = \frac{T}{m}$ and the time instances $t_k = k\,\Delta t$, $k = 0,\dots,m$. In the context of Sect. 2 we set $n = 2m+1$ and choose

$$y_j = u(t_{j-1}),\ j = 1,\dots,m+1$$

and

$$y_j = \bar\partial u(t_{j-m-1}),\ j = m+2,\dots,2m+1,$$

where

$$\bar\partial u(t_k) = \frac{u(t_k) - u(t_{k-1})}{\Delta t}.$$

By construction all snapshots belong to the space V. We shall consider two different POD bases built up from the above snapshots. For the first one we choose $X = V$ and denote the corresponding POD basis by $\{\tilde\psi_k\}_{k=1}^d$. Due to Proposition 1 we have for any $\ell \le d$ the error formula

(7)
$$\begin{aligned}&\frac{1}{2m+1}\sum_{j=0}^{m}\Big\|u(t_j) - \sum_{k=1}^{\ell}(u(t_j),\tilde\psi_k)_V\tilde\psi_k\Big\|_V^2\\ &\quad+\frac{1}{2m+1}\sum_{j=1}^{m}\Big\|\bar\partial u(t_j) - \sum_{k=1}^{\ell}(\bar\partial u(t_j),\tilde\psi_k)_V\tilde\psi_k\Big\|_V^2 = \sum_{k=\ell+1}^{d}\tilde\lambda_k,\end{aligned}$$

where $\tilde\lambda_k$, $k = 1,\dots,n$, are the eigenvalues of the correlation matrix K with elements $K_{ij} = \frac{1}{2m+1}\,(y_j,y_i)_V$. The subspace spanned by the first ℓ POD basis functions is denoted by $\tilde V^\ell$.

Remark 1. It may come as a surprise at first that the finite difference quotients $\overline{\partial}u(t_k)$ are included into the set of snapshots. To motivate this choice let us point out that while the finite difference quotients are contained in the span of $\{u(t_{j-1})\}_{j=1}^{m+1}$, the POD bases differ depending on whether $\{\overline{\partial}u(t_k)\}_{j=1}^{m}$ are included or not. The linear dependence does not constitute a difficulty for the singular value decomposition which is required to compute the P OD basis. In fact, the snapshots themselves can be linearly dependent. The resulting POD basis is, in any case, maximally linearly independent in the sense expressed in (2) and Proposition 1. Secondly, in anticipation of the rate of convergence results that will be obtained further below, we note that the time derivative of u in (6a) must be approximated by the Galerkin-POD based scheme. In case the terms $\{\overline{\partial}u(t_k)\}_{j=1}^{m}$ are included in the snapshot ensemb le, we will be able to utilize the estimate

$$\text{(8)} \qquad \frac{1}{m}\sum_{j=1}^{m}\left\|\overline{\partial}u(t_j)-\sum_{k=1}^{\ell}(\overline{\partial}u(t_j),\tilde{\psi}_k)_V\tilde{\psi}_k\right\|_V^2 \le 3\sum_{k=\ell+1}^{d}\tilde{\lambda}_k.$$

Otherwise, if only the snapshots $y_j = u(t_{j-1})$ for $j = 1, \ldots, m+1$, are used, we obtain instead of (7) the error formula

$$\frac{1}{m+1}\sum_{j=0}^{m}\left\|u(t_j)-\sum_{k=1}^{\ell}(u(t_j),\tilde{\psi}_k)_V\tilde{\psi}_k\right\|_V^2 = \sum_{k=\ell+1}^{d}\tilde{\lambda}_k,$$

and (8) must be replaced by

$$\text{(9)} \qquad \frac{1}{m}\sum_{j=1}^{m}\left\|\overline{\partial}u(t_j)-\sum_{k=1}^{\ell}(\overline{\partial}u(t_j),\tilde{\psi}_k)_V\tilde{\psi}_k\right\|_V^2 \le \frac{8}{(\Delta t)^2}\sum_{k=\ell+1}^{d}\tilde{\lambda}_k,$$

which in contrast to (8) contains the factor $(\Delta t)^{-2}$ on the right-hand side.

For the second choice we take $X = H$ and denote the corresponding POD basis by $\{\hat{\psi}_k\}_{k=1}^{d}$. Now (7) is replaced by

$$\text{(10)} \qquad \begin{aligned}&\frac{1}{2m+1}\sum_{j=0}^{m}\left\|u(t_j)-\sum_{k=1}^{\ell}(u(t_j),\hat{\psi}_k)_H\hat{\psi}_k\right\|_H^2\\ &+\frac{1}{2m+1}\sum_{j=1}^{m}\left\|\overline{\partial}u(t_j)-\sum_{k=1}^{\ell}(\overline{\partial}u(t_j),\hat{\psi}_k)_H\hat{\psi}_k\right\|_H^2 = \sum_{k=\ell+1}^{d}\hat{\lambda}_k\end{aligned}$$

for every $\ell \le d$. Here, $\hat{\lambda}_k$, $k = 1, \ldots, d$, are the eigenvalues of the correlation matrix K with the elements $K_{ij} = \frac{1}{2m+1}(y_j, y_i)_H$. Let $\hat{V}^\ell$ denote the linear subspace spanned by $\hat{\psi}_1, \ldots, \hat{\psi}_\ell$.

In the following we shall write $\{\psi_k\}_{k=1}^{\ell}$ and $V^\ell = \text{span}\,\{\psi_1, \dots, \psi_\ell\}$ if we do not distinguish between the two POD bases. Note that $V^d = \mathcal{V}$ holds.

It will be convenient to introduce the mass matrix

$$M = ((M_{ij})) \in \mathbb{R}^{d\times d} \text{ with } M_{ij} = (\psi_j, \psi_i)_H$$

and the stiffness matrix

$$S = ((S_{ij})) \in \mathbb{R}^{d\times d} \text{ with } S_{ij} = (\psi_j, \psi_i)_V. \tag{11}$$

The mass matrix for the POD basis in H as well as the stiffness matrix for the POD basis in V turn out to be the identity matrices.

On $\mathcal{V}$ we have the following estimates.

Lemma 2. *For all $u \in \mathcal{V}$ we have*

(12)

$$\|u\|_H \le \sqrt{\|M\|_2 \|S^{-1}\|_2}\, \|u\|_V \textit{ and } \|u\|_V \le \sqrt{\|S\|_2 \|M^{-1}\|_2}\, \|u\|_H,$$

where $\|\cdot\|_2$ denotes the spectral norm for symmetric matrices.

Proof. Let $u \in \mathcal{V}$ be an arbitrary element. Then

$$u = \sum_{k=1}^{d} (u, \psi_k)_X \psi_k.$$

Setting $x = ((u,\psi_1)_X, \dots, (u, \psi_d)_X)^T \in \mathbb{R}^d$ we obtain that

$$\begin{aligned}\|u\|_H^2 = x^T M x &\le \|M\|_2 x^T x \\ &\le \|S^{-1}\|_2 \|M\|_2\, x^T S x = \|S^{-1}\|_2 \|M\|_2\, \|u\|_V^2,\end{aligned}$$

which gives the first estimate. The second one follows analogously. □

Remark 2. a) In analogy to finite element approximation theory we refer to the second inequality in (12) as inverse estimate.

b) In case of the POD basis in V the inequalities in (12) lead to the estimates

$$\|u\|_H \le \sqrt{\|M\|_2}\, \|u\|_V \text{ and } \|u\|_V \le \sqrt{\|M^{-1}\|_2}\, \|u\|_H.$$

On the other hand for the POD basis in H we find for every $u \in \mathcal{V}$ that

$$\|u\|_H \le \sqrt{\|S^{-1}\|_2}\, \|u\|_V \text{ and } \|u\|_V \le \sqrt{\|S\|_2}\, \|u\|_H.$$

3.3. Backward Euler-Galerkin method

To study the backward Euler-POD-Galerkin method for (6), we introduce the Ritz-projection $P^\ell : V \to V^\ell$ by

$$a(P^\ell u, \psi) = a(u, \psi) \text{ for all } \psi \in V^\ell, \tag{13}$$

where $u \in V$. Due to (4) and (5) the linear operator P^ℓ is well-defined and bounded:

$$\|P^\ell u\|_V \le \frac{\beta}{\kappa} \|u\|_V \text{ for all } u \in V.$$

Lemma 3. *For every $\ell \in \{1, \dots, d\}$ the projection operators P^ℓ satisfy*

$$\frac{1}{m} \sum_{k=1}^{m} \|u(t_k) - P^\ell u(t_k)\|_V^2 \le \frac{3\beta}{\kappa} \sum_{k=\ell+1}^{d} \tilde{\lambda}_k \tag{14}$$

and

$$\frac{1}{m} \sum_{k=1}^{m} \|u(t_k) - P^\ell u(t_k)\|_V^2 \le \frac{3\beta \|S\|_2}{\kappa} \sum_{k=\ell+1}^{d} \hat{\lambda}_k, \tag{15}$$

where $\tilde{\lambda}_k$ and $\hat{\lambda}_k$ denote the eigenvalues of the correlation matrix K with the elements $\frac{1}{2m+1}\,(y_j, y_i)_V$ and $\frac{1}{2m+1}\,(y_j, y_i)_H$, respectively.

Proof. For arbitrary $u \in \mathcal{V}$ we deduce from (13) that

$$\begin{aligned} \kappa \|u - P^\ell u\|_V^2 &\le a(u - P^\ell u, u - P^\ell u) \\ &= a(u - P^\ell u, u - \psi) \quad \text{for all } \psi \in V^\ell \end{aligned}$$

so that

$$\|u - P^\ell u\|_V \le \frac{\beta}{\kappa} \|u - \psi\|_V \text{ for all } \psi \in V^\ell. \tag{16}$$

Using (16), (12) and (10) we obtain

$$\begin{aligned} &\frac{1}{m} \sum_{k=1}^{m} \|u(t_k) - P^\ell u(t_k)\|_V^2 \\ &\le \frac{\beta}{\kappa m} \sum_{k=1}^{m} \Big\| u(t_k) - \sum_{i=1}^{\ell} (u(t_k), \hat{\psi}_i)_H \hat{\psi}_i \Big\|_V^2 \\ &\le \frac{3\beta \|S\|_2}{\kappa(2m+1)} \sum_{k=1}^{m} \Big\| u(t_k) - \sum_{i=1}^{\ell} (u(t_k), \hat{\psi}_i)_H \hat{\psi}_i \Big\|_H^2 \\ &\le \frac{3\beta \|S\|_2}{\kappa} \sum_{k=\ell+1}^{d} \hat{\lambda}_k, \end{aligned}$$

which is estimate (15). The proof of (14) which does not rely on the inverse inequality is analogous. □

From the proof the following corollary immediately follows.

Corollary 4. *For the difference quotients we have the estimates*

$$\frac{1}{m}\sum_{k=1}^{m}\|\overline{\partial}u(t_k)-P^\ell\overline{\partial}u(t_k)\|_V^2 \le \frac{3\beta}{\kappa}\sum_{k=\ell+1}^{d}\tilde{\lambda}_k \tag{17}$$

and

$$\frac{1}{m}\sum_{k=1}^{m}\|\overline{\partial}u(t_k)-P^\ell\overline{\partial}u(t_k)\|_V^2 \le \frac{3\beta\|S\|_2}{\kappa}\sum_{k=\ell+1}^{d}\hat{\lambda}_k. \tag{18}$$

Now we describe the backward Euler-POD-Galerkin method for (6). It consists in finding a sequence $\{U_k\}_{k=0}^m$ in V^ℓ satisfying

$$(U_0,\psi)_H = (\phi,\psi)_H \text{ for all } \psi\in V^\ell \tag{19a}$$

and

$$(\overline{\partial}U_k,\psi)_H + a(U_k,\psi) = (f(t_k),\psi)_H \text{ for all } \psi\in V^\ell \tag{19b}$$

for $k=1,\dots,m$. Here, we have set

$$\overline{\partial}U_k = \frac{U_k-U_{k-1}}{\Delta t}.$$

Theorem 5. *There exist unique solution* $\{U_k\}_{k=0}^m$ *in* V^ℓ *to problem* (19). *Moreover, the estimate*

$$\|U_k\|_H \le \Big(1+\frac{\gamma T}{m}\Big)e^{-\frac{\gamma kT}{m}}\|\phi\|_H + \frac{1-e^{-\frac{\gamma kT}{m}}}{\gamma}\|f\|_{C([0,T];H)} \quad \textit{for } k=0,\dots,m, \tag{20}$$

with $\gamma=\kappa/\alpha$*, holds.*

Proof. We infer from (4) and (5) that there exists a unique solution $\{U_k\}_{k=0}^m$ of (19). Taking $\psi=U_k$ as the test function in (19b) we obtain from (3) and (5)

$$(1+\gamma\Delta t)\,\|U_k\|_H \le \|U_{k-1}\|_H + \Delta t\,\|f(t_k)\|_H,$$

which yields upon summation

(21)

$$\|U_k\|_H \le \Big(\frac{1}{1+\gamma\Delta t}\Big)^k\|U_0\|_H + \Delta t\,\|f\|_{C([0,T];H)}\sum_{j=1}^{k}\Big(\frac{1}{1+\gamma\Delta t}\Big)^j.$$

Note that $(1+\gamma\Delta t)^k \geq e^{\gamma k\Delta t}/(1+\gamma\Delta t)$. Moreover, setting $\zeta = 1/(1+\gamma\Delta t)$ we find

$$\Delta t \sum_{j=1}^{k} \Big(\frac{1}{1+\gamma\Delta t}\Big)^j = \Delta t \frac{1-\zeta^k}{\zeta^{-1}-1} = \frac{1-\zeta^k}{\gamma} \leq \frac{1-e^{-\gamma k\Delta t}}{\gamma}.$$

Inserting these two estimates and utilizing the fact that $\|U_0\|_H \leq \|\phi\|_H$ in (21) yield (20). □

It is simple to extend the previous theorem to a special class of non-linear problems.

Corollary 6. *Let us consider the non-linear evolution problem*

(22a) $$\frac{d}{dt}\,(u(t),\varphi)_H + a(u(t),\varphi) + (F(u(t)),\varphi)_H = (f(t),\varphi)_H$$

for all $\varphi \in V$ *and a.e.* $t \in (0,T)$, *with*

(22b) $$(u(0),\chi)_H = (\phi,\chi)_H \textit{ for all } \chi \in H,$$

where the non-linear term satisfies

(23) $$(F(\varphi),\varphi)_H \geq 0 \textit{ for all } \varphi \in V.$$

Then the backward Euler-Galerkin scheme for (22)

(24a) $$(U_0,\psi)_H = (\phi,\psi)_H \textit{ for all } \psi \in V^\ell$$

and

(24b)
$$(\overline{\partial} U_k,\psi)_H + a(U_k,\psi) + (F(U_k),\psi)_H = (f(t_k),\psi)_H \textit{ for all } \psi \in V^\ell$$

for $k = 1,\dots,m$ *has a unique solution* $\{U_k\}_{k=0}^m$ *satisfying the estimate* (20).

Let us present an example for non-linear problems, which satisfy assumption (23).

Example 1. In case of the Burgers equation the bilinear form and the non-linearity are given by

$$a(\varphi,\psi) = \nu \int_\Omega \varphi'\psi'\,dx \text{ for } \varphi,\psi \in V,$$
$$F(\varphi) = \varphi\varphi' \text{ for } \varphi \in V,$$

where $\nu > 0$, $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ and $\Omega = (0,1)$. In this case

$$\int_\Omega F(\varphi)\varphi\,dx = 0 \text{ for all } \varphi \in V.$$

Moreover, for $f \in C([0,T];H)$ and $\phi \in V$ there exists a unique solution $u \in W(0,T;V) \cap C([0,T];V)$, see [12], for instance.

Our next goal is to derive an error estimate for the term

$$\frac{1}{m}\sum_{k=1}^{m}\|U_k - u(t_k)\|_H^2,$$

where $u(t_k)$ is the solution of (6) at the time instances $t = t_k, k = 1, \ldots, m$. Throughout we shall use the decomposition

$$U_k - u(t_k) = U_k - P^\ell u(t_k) + P^\ell u(t_k) - u(t_k) = \vartheta_k + \varrho_k, \tag{25}$$

where $\vartheta_k = U_k - P^\ell u(t_k)$ and $\varrho_k = P^\ell u(t_k) - u(t_k)$. It follows that

$$\frac{1}{m}\sum_{k=1}^{m}\|U_k - u(t_k)\|_H^2 \le \frac{2}{m}\sum_{k=1}^{m}\|\vartheta_k\|_H^2 + \frac{2}{m}\sum_{k=1}^{m}\|\varrho_k\|_H^2. \tag{26}$$

Due to (3) and Lemma 3 we have

$$\frac{1}{m}\sum_{k=1}^{m}\|\varrho_k\|_H^2 \le \frac{3\alpha\beta}{\kappa}\sum_{k=\ell+1}^{d}\tilde{\lambda}_k \tag{27a}$$

and

$$\frac{1}{m}\sum_{k=1}^{m}\|\varrho_k\|_H^2 \le \frac{3\alpha\beta\|S\|_2}{\kappa}\sum_{k=\ell+1}^{d}\hat{\lambda}_k, \tag{27b}$$

where $\tilde{\lambda}_k$ and $\hat{\lambda}_k$ denote the eigenvalues of the correlation matrix with the elements $K_{ij} = \frac{1}{2m+1}\,(y_j, y_i)_V$ and $K_{ij} = \frac{1}{2m+1}\,(y_j, y_i)_H$, respectively. Using the notation $\overline{\partial}\vartheta_k = \frac{\vartheta_k - \vartheta_{k-1}}{\Delta t}$, $k = 1, \ldots, m$, we obtain

$$\begin{aligned}
(\overline{\partial}\vartheta_k, \psi)_H + a(\vartheta_k, \psi) &= (\overline{\partial}U_k, \psi)_H + a(U_k, \psi) \\
&\quad -(\overline{\partial}P^\ell u(t_k), \psi)_H - a(P^\ell u(t_k), \psi) \\
&= (f(t_k), \psi)_H - (\overline{\partial}P^\ell u(t_k), \psi)_H - a(u(t_k), \psi) \\
&= (v_k, \psi)_H,
\end{aligned} \tag{28}$$

where

$$v_k = u_t(t_k) - \overline{\partial}P^\ell u(t_k) = u_t(t_k) - \overline{\partial}u(t_k) + \overline{\partial}u(t_k) - \overline{\partial}P^\ell u(t_k).$$

We put $w_k = u_t(t_k) - \overline{\partial}u(t_k)$ and $z_k = \overline{\partial}u(t_k) - P^\ell\overline{\partial}u(t_k)$. Choosing $\psi = \vartheta_k \in V^\ell$ in (28) we infer that

$$\|\vartheta_k\|_H^2 - (\vartheta_k, \vartheta_{k-1}) + \Delta t\, a(\vartheta_k, \vartheta_k) \le \Delta t\,\|v_k\|_H\|\vartheta_k\|_H,$$

from which we derive the estimate

$$\|\vartheta_k\|_H \le \frac{1}{1+\frac{\kappa}{\alpha}\Delta t}\Big(\|\vartheta_{k-1}\|_H + \Delta t\,\|v_k\|_H\Big)$$

and upon summation

$$\|\vartheta_k\|_H \le \Big(\frac{1}{1+\frac{\kappa}{\alpha}\Delta t}\Big)^k \|\vartheta_0\|_H + \Delta t\sum_{j=1}^{k}\Big(\frac{1}{1+\frac{\kappa}{\alpha}\Delta t}\Big)^{k-j+1}\|v_j\|_H.$$

Hence, with $\gamma = \frac{\kappa}{\alpha} > 0$

$$\|\vartheta_k\|_H^2 \le 2\Big(\frac{1}{1+\gamma\Delta t}\Big)^{2k}\|\vartheta_0\|_H^2 + \frac{2T^2}{m^2}\Big(\sum_{j=1}^{k}\Big(\frac{1}{1+\gamma\Delta t}\Big)^{k-j+1}\|v_j\|_H\Big)^2.$$

To shorten notation we put $\zeta = \frac{1}{1+\gamma\Delta t}$. From

$$\Big(\frac{1}{\zeta}\Big)^{2m} = \big(1+\gamma\Delta t\big)^{2m} = \Big(1+\frac{2\gamma T}{2m}\Big)^{2m} \le e^{2\gamma T}$$

we infer that $1-\zeta^{2m} \le 1-e^{-2\gamma T}$ and

$$\frac{2}{m}\,\frac{\zeta^2-\zeta^{2m+2}}{1-\zeta^2} = \frac{2}{m}\,\frac{1-\zeta^{2m}}{\zeta^{-2}-1} \le \frac{2}{m}\,\frac{1-\zeta^{2m}}{2\gamma\Delta t} \le \frac{1-e^{-2\gamma T}}{\gamma T}.$$

Hence, we obtain that

$$\begin{aligned}
\frac{1}{m}\sum_{k=1}^{m}\|\vartheta_k\|_H^2 &\le \frac{2}{m}\sum_{k=1}^{m}\zeta^{2k}\|\vartheta_0\|_H^2 + \frac{2T^2}{m^3}\sum_{k=1}^{m}\Big(\sum_{j=1}^{k}\zeta^{k-j+1}\|v_j\|_H\Big)^2 \\
&\le \frac{2}{m}\,\frac{\zeta^2-\zeta^{2m+2}}{1-\zeta^2}\|\vartheta_0\|_H^2 \\
&\qquad + \frac{2T^2}{m^3}\sum_{k=1}^{m}\Big(\sum_{j=1}^{k}\zeta^{2(k-j+1)}\sum_{j=1}^{k}\|v_j\|_H^2\Big) \\
&= \frac{1-e^{-2\gamma T}}{\gamma T}\|\vartheta_0\|_H^2 + \frac{2T^2}{m^3}\sum_{k=1}^{m}\Big(\sum_{j=1}^{k}\zeta^{2j}\sum_{j=1}^{k}\|v_j\|_H^2\Big) \\
&\le \frac{1-e^{-2\gamma T}}{\gamma T}\|\vartheta_0\|_H^2 + \frac{T^2}{m}\sum_{j=1}^{m}\|v_j\|_H^2\sum_{k=1}^{m}\frac{2}{m^2}\,\frac{\zeta^2-\zeta^{2k+2}}{1-\zeta^2} \\
&\le \frac{1-e^{-2\gamma T}}{\gamma T}\Big(\|\vartheta_0\|_H^2 + \frac{T^2}{m}\sum_{k=1}^{m}\|v_k\|_H^2\Big).
\end{aligned}$$

Now we estimate the terms $\|v_k\|_H^2 = \|w_k + z_k\|_H^2$. From

$$\begin{aligned} w_k &= u_t(t_k) - \frac{u(t_k) - u(t_{k-1})}{\Delta t} \\ &= \frac{1}{\Delta t} \Big(\Delta t \, u_t(t_k) - \big(u(t_k) - u(t_{k-1}) \big) \Big) \\ &= \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} (s - t_{k-1}) u_{tt}(s) \, ds \end{aligned}$$

we infer that

$$\begin{aligned} \sum_{k=1}^{m} \|w_k\|_H^2 &\le \sum_{k=1}^{m} \frac{1}{\Delta t^2} \int_{t_{k-1}}^{t_k} (t - t_{k-1})^2 \, dt \int_{t_{k-1}}^{t_k} u_{tt}(t)^2 \, dt \\ &\le \frac{\Delta t}{3} \int_0^T \|u_{tt}(t)\|_H^2 \, dt \end{aligned}$$

holds. Thus $\|v_k\|_H^2 \le 2\,\|w_k\|_H^2 + 2\,\|z_k\|_H^2$ leads to

$$\begin{aligned} \frac{1}{m} \sum_{k=1}^{m} \|\vartheta_k\|_H^2 \le \frac{1 - e^{-2\gamma T}}{\gamma T} \Bigg(&\|\vartheta_0\|_H^2 + \frac{2T^3}{3m^2} \int_0^T \|u_{tt}(t)\|_H^2 \, dt \\ &+ \frac{2T^2}{m} \sum_{k=1}^{m} \|z_k\|_H^2 \Bigg). \end{aligned}$$

If the POD basis is chosen in V then by (17)

$$\frac{1}{m} \sum_{k=1}^{m} \|z_k\|_H^2 = \frac{1}{m} \sum_{k=1}^{m} \big\|\overline{\partial} u(t_k) - P^\ell \overline{\partial} u(t_k)\big\|_V^2 \le \frac{3\alpha\beta}{\kappa} \sum_{k=\ell+1}^{d} \tilde{\lambda}_k,$$

otherwise, if the POD basis is chosen in H we have

$$\frac{1}{m} \sum_{k=1}^{m} \|z_k\|_H^2 = \frac{1}{m} \sum_{k=1}^{m} \big\|\overline{\partial} u(t_k) - P^\ell \overline{\partial} u(t_k)\big\|_H^2 \le \frac{3\alpha\beta \|S\|_2}{\kappa} \sum_{k=\ell+1}^{d} \hat{\lambda}_k.$$

Summarizing, we obtain

$$\begin{aligned} \frac{1}{m} \sum_{k=1}^{m} \|\vartheta_k\|_H^2 &\le \frac{1 - e^{-2\gamma T}}{\gamma T} \, \|\vartheta_0\|_H^2 \\ &\quad + \frac{1 - e^{-2\gamma T}}{\gamma T} \Big(\frac{2(\Delta t)^2}{3} \, \|u_{tt}\|_{L^2(0,T;H)}^2 + \frac{6\alpha\beta T^2}{\kappa} \sum_{k=\ell+1}^{d} \tilde{\lambda}_k \Big) \end{aligned} \tag{29a}$$

and

$$\frac{1}{m}\sum_{k=1}^{m}\|\vartheta_k\|_H^2 \le \frac{1-e^{-2\gamma T}}{\gamma T}\,\|\vartheta_0\|_H^2$$

$$+\frac{1-e^{-2\gamma T}}{\gamma T}\Big(\frac{2(\Delta t)^2}{3}\,\|u_{tt}\|^2_{L^2(0,T;H)}+\frac{6\alpha\beta T^2\,\|S\|_2}{\kappa}\sum_{k=\ell+1}^{d}\hat{\lambda}_k\Big) \tag{29b}$$

for the POD bases $\{\tilde{\psi}_k\}_{k=1}^d$ and $\{\hat{\psi}_k\}_{k=1}^d$, respectively. From (27) and (29) we conclude the following result.

Theorem 7. *Let u and $\{U_k\}_{k=0}^m$ be the solutions to* (6) *and* (19)*, respectively and suppose that $u_{tt}\in L^2(0,T;H)$. Then there exists a constant $C>0$ depending on u, α, β, κ, T, but independent of ℓ and m, such that*

$$\frac{1}{m}\sum_{k=1}^{m}\|u(t_k)-U_k\|_H^2 \le C\Big(\|\phi-P^\ell\phi\|_H^2+\sum_{k=\ell+1}^{d}\tilde{\lambda}_k+(\Delta t)^2\Big)$$

and

$$\frac{1}{m}\sum_{k=1}^{m}\|u(t_k)-U_k\|_H^2 \le C\Big(\|\phi-P^\ell\phi\|_H^2+\|S\|_2\sum_{k=\ell+1}^{d}\hat{\lambda}_k+(\Delta t)^2\Big),$$

where S denotes the stiffness matrix introduced in (11).

Remark 3. From the derivation leading to Theorem 7 we also obtain the following estimate exhibiting the dependence of C on α,β,κ and T:

$$\begin{aligned}&\frac{1}{m}\sum_{k=1}^{m}\|u(t_k)-U_k\|_H^2\\ &\le c(T)\,\|\phi-P^\ell\phi\|_H^2+\frac{2}{3}c(T)(\Delta t)^2\,\|u_{tt}\|^2_{L^2(0,T;H)}\\ &\quad+\frac{6\alpha\beta c(T)}{\kappa}(1+T^2)\sum_{k=\ell+1}^{d}\tilde{\lambda}_k,\end{aligned} \tag{30}$$

where $c(T)=2(1-e^{-2\gamma T})/(\gamma T)$, and the POD-basis is taken in V. In case the POD basis is taken in H, then $\tilde{\lambda}_k$ has to be replaced by $\|S\|_2\hat{\lambda}_k$ in (30). From (30) it follows that the influence of ϑ_0 decays with $T\to\infty$. The factor in front of $\sum_{k=\ell+1}^{d}\tilde{\lambda}_k$ behaves like T for large T. We must not forget, however, that the singular values $\tilde{\lambda}_k$ and $\hat{\lambda}_k$ themselves depend on T as well.

Remark 4. Up to now we have not made use of $\phi \in \mathcal{V}$. Since the initial condition is included in the set of snapshots, we are able to estimate $\|\phi - P^\ell \phi\|_H$. From (10) and (16) we conclude

$$\|\phi - P^\ell \phi\|_H^2 \leq \frac{\alpha\beta}{\kappa} \left\| \phi - \sum_{k=1}^{\ell} (\phi, \psi_k)_X \psi_k \right\|_V^2 = (2m+1)\frac{\alpha\beta}{\kappa} \sum_{k=\ell+1}^{d} \tilde{\lambda}_k,$$

respectively,

$$\|\phi - P^\ell \phi\|_H^2 \leq \frac{\alpha\beta}{\kappa} \left\| \phi - \sum_{k=1}^{\ell} (\phi, \psi_k)_X \psi_k \right\|_V^2$$
$$= (2m+1)\|S\|_2 \frac{\alpha\beta}{\kappa} \sum_{k=\ell+1}^{d} \hat{\lambda}_k.$$

Thus, if we choose $\ell = d$, we obtain

$$\frac{1}{m} \sum_{k=1}^{m} \|u(t_k) - U_k\|_H^2 \leq \frac{2}{3} c(T) (\Delta t)^2 \, \|u_{tt}\|_{L^2(0,T;H)}^2,$$

with $c(T)$ defined in Remark 3..

3.4. Crank-Nicolson scheme

In this subsection we investigate Crank-Nicolson-POD approximation to (6). Thus we consider the family of discrete problems of finding a sequence $\{U_k\}_{k=0}^m$ in V^ℓ satisfying

(31a) $$(U_0, \psi)_H = (\phi, \psi)_H \text{ for all } \psi \in V^\ell$$

and

(31b)
$$(\bar{\partial} U_k, \psi)_H + \frac{1}{2}\, a(U_k + U_{k-1}, \psi) = (f(t_k - \tfrac{\Delta t}{2}), \psi)_H \text{ for all } \psi \in V^\ell$$

for $k = 1, \ldots, m$.

Theorem 8. *There exists a unique solution $\{U_k\}_{k=0}^m$ in V^ℓ to* (31). *Moreover, the estimate*

$$\|U_k\|_H \leq \|\phi\|_H + T\, \|f\|_{C([0,T];H)} \textit{ for } k = 0, \ldots, m$$

holds.

Proof. Equation (31b) implies that

$$(U_k, \psi)_H + \frac{\Delta t}{2}\, a(U_k, \psi) = (U_{k-1}, \psi)_H - \frac{\Delta t}{2}\, a(U_{k-1}, \psi) + \big(f(t_k - \frac{\Delta t}{2}), \psi\big)_H \tag{32}$$

for all $\psi \in V^\ell$. Since the bilinear form a satisfies (4) and (5), there exists a unique solution U_k to (31b) for every $k \in \{1, \dots, m\}$. Taking $\psi = U_k + U_{k-1}$ as a test function in (31b) we obtain

$$\begin{aligned}&\big(\|U_k\|_H - \|U_{k-1}\|_H\big)\big(\|U_k\|_H + \|U_{k-1}\|_H\big)\\ &\qquad\leq \Delta t\, \|f(t_k - \tfrac{\Delta t}{2})\|_H \|U_k + U_{k-1}\|_H\end{aligned}$$

and hence

$$\|U_k\|_H \leq \|U_{k-1}\|_H + \Delta t\, \|f(t_k - \tfrac{\Delta t}{2})\|_H.$$

Summation with respect to k yields

$$\|U_k\|_H \leq \|\phi\|_H + \Delta t \sum_{j=1}^{k} \|f(t_j - \frac{\Delta t}{2})\|_H \leq \|\phi\|_H + T\, \|f\|_{C([0,T];H)}.$$

□

Remark 5. An analogous result to Corollary 6 also holds for the Crank-Nicolson scheme.

We shall require the following condition concerning the solution u of (6) and the bilinear form a.

$$(\mathbf{H}) \quad \begin{cases} \text{There exists a subspace } W \text{ of } V \text{ with continuous injection and a} \\ \text{constant } \hat{C} > 0 \text{ such that } u \in W^{2,2}(0,T;W) \text{ and} \\ \qquad a(\varphi, \psi) \leq \hat{C}\, \|\varphi\|_W \|\psi\|_H \text{ for all } \varphi \in W, \psi \in V. \end{cases}$$

Example 2. For $W = H^2(\Omega) \cap H_0^1(\Omega)$, $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, with Ω a bounded domain in $\mathbb{R}^l$ and

$$a(\varphi, \psi) = \int_\Omega \nabla\varphi \cdot \nabla\psi \, dx \text{ for all } \varphi, \psi \in H_0^1(\Omega)$$

we have $a(\varphi, \psi) \leq \|\varphi\|_W \|\psi\|_H$ for all $\varphi \in W$, $\psi \in V$ and the inequality in **(H)** holds with $\hat{C} = 1$.

Theorem 9. *Let u and $\{U_k\}_{k=0}^m$ be the solutions to* (6) *and* (31)*, respectively, and suppose that* **(H)** *holds and that $u_{ttt} \in L^2(0,T;H)$. Then there exists a constant $C > 0$ depending on u, α, β, κ, T, but independent of ℓ and m, such that*

$$\frac{1}{m}\sum_{k=1}^{m} \|u(t_k) - U_k\|_H^2 \le C\Big(\|\phi - P^\ell \phi\|_H^2 + \sum_{k=\ell+1}^{d} \tilde{\lambda}_k + (\Delta t)^4\Big)$$

and

$$\frac{1}{m}\sum_{k=1}^{m} \|u(t_k) - U_k\|_H^2 \le C\Big(\|\phi - P^\ell \phi\|_H^2 + \|S\|_2 \sum_{k=\ell+1}^{d} \hat{\lambda}_k + (\Delta t)^4\Big),$$

where S denotes the stiffness matrix introduced in (11).

Proof. We proceed as in Sect. 3.3. The term ϱ_k can be estimated as before. For ϑ_k we obtain

$$\begin{aligned}(\overline{\partial}\vartheta_k, \psi)_H &+ \frac{1}{2}\, a(\vartheta_k + \vartheta_{k-1}, \psi)\\ &= (w_k + z_k, \psi)_H + \frac{1}{2}\, a\big(2u(t_k - \tfrac{\Delta t}{2}) - u(t_k) - u(t_{k-1}), \psi\big),\end{aligned}$$

where w_k and z_k are given by

$$w_k = u_t(t_k - \tfrac{\Delta t}{2}) - \overline{\partial} u(t_k) \text{ and } z_k = \overline{\partial} u(t_k) - P^\ell \overline{\partial} u(t_k).$$

Let us choose $\psi = \vartheta_k + \vartheta_{k-1}$. Due to **(H)** we arrive at

$$\begin{aligned}(\overline{\partial}\vartheta, \vartheta_k + \vartheta_{k-1})_H &\le (\|\vartheta_k\|_H + \|\vartheta_{k-1}\|_H)(\|w_k\|_H + \|z_k\|_H)\\ &+ (\|\vartheta_k\|_H + \|\vartheta_{k-1}\|_H)\Big(\frac{\hat{C}\Delta t}{2}\int_{t_{k-1}-\frac{\Delta t}{2}}^{t_k}\int_{t-\frac{\Delta t}{2}}^{t} \|u_{tt}(s)\|_W \, ds dt\Big).\end{aligned} \tag{33}$$

From (33) we conclude that

$$\|\vartheta_k\|_H \le \|\vartheta_{k-1}\|_H + \Delta t\Big(\|w_k\|_H + \|z_k\|_H + \frac{\hat{C}\Delta t}{2}\int_{t_{k-1}}^{t_k} \|u_{tt}(t)\|_W \, dt\Big).$$

Summation implies that

$$\begin{aligned}\|\vartheta_k\|_H \le \|\vartheta_0\|_H + \Delta t \sum_{j=1}^{k} \Big(\|w_j\|_H + \|z_j\|_H\Big)\\ + \frac{\hat{C}(\Delta t)^2}{2}\int_0^{t_k} \|u_{tt}(t)\|_W \, dt.\end{aligned}$$

Hence,

$$\frac{1}{m}\sum_{k=1}^{m}\|\vartheta_k\|_H^2 \le 3\,\|\vartheta_0\|_H^2 + \frac{3(\Delta t)^2}{m}\sum_{k=1}^{m}\Big(\sum_{j=1}^{k}\|w_j\|_H + \|z_j\|_H\Big)^2$$
$$+\frac{3\hat{C}^2(\Delta t)^4}{4m}\sum_{k=1}^{m}\Big(\int_0^T \|u_{tt}(t)\|_W\,dt\Big)^2$$
$$\le 3\,\|\vartheta_0\|_H^2 + 6(\Delta t)^2\sum_{k=1}^{m}\sum_{j=1}^{m}\Big(\|w_j\|_H^2 + \|z_j\|_H^2\Big)$$
$$+\frac{3\hat{C}^2T(\Delta t)^4}{4}\int_0^T \|u_{tt}(t)\|_W^2\,dt.$$

From Sect. 3.3 we know that

$$\frac{1}{m}\sum_{k=1}^{m}\|z_k\|_H^2 \le \frac{3\alpha\beta}{\kappa}\sum_{k=\ell+1}^{d}\tilde{\lambda}_k$$

and

$$\frac{1}{m}\sum_{k=1}^{m}\|z_k\|_H^2 \le \frac{3\alpha\beta\|S\|_2}{\kappa}\sum_{k=\ell+1}^{d}\hat{\lambda}_k$$

depending on the choice of the basis in V or H. As in the proof of Theorem 1.6 in [18] we estimate

$$(\Delta t)^2\sum_{k=1}^{m}\|w_k\|_H^2 = \sum_{k=1}^{m}\|u(t_k)-u(t_{k-1})-\Delta t\,u_t(t_{k-\frac{1}{2}})\|_H^2$$
$$\le \sum_{k=1}^{m}\Big(\tilde{C}(\Delta t)^2\int_{t_{k-1}}^{t_k}\|u_{ttt}(t)\|_H\,dt\Big)^2$$
$$\le \tilde{C}^2(\Delta t)^4\sum_{k=1}^{m}\Big(\int_{t_{k-1}}^{t_k}\|u_{ttt}(t)\|_H\,dt\Big)^2$$
$$\le \tilde{C}^2(\Delta t)^5\int_0^T\|u_{ttt}(t)\|_H^2\,dt$$

for a constant $\tilde{C} > 0$ independent of m. Thus, for $C^* = \max(6\tilde{C}^2T, \frac{3}{4}\hat{C}^2T^2)$ we have

$$\frac{1}{m}\sum_{k=1}^{m}\|\vartheta_k\|_H^2 \le 3\,\|\vartheta_0\|_H^2 + \frac{18\alpha\beta T^2}{\kappa}\sum_{k=\ell+1}^{d}\tilde{\lambda}_k$$
$$+C^*(\Delta t)^4\Big(\|u_{ttt}\|_{L^2(0,T;H)}^2 + \|u_{tt}\|_{L^2(0,T;W)}^2\Big)$$

or

$$\frac{1}{m}\sum_{k=1}^{m}\|\vartheta_k\|_H^2 \le 3\,\|\vartheta_0\|_H^2 + \frac{18\alpha\beta T^2\|S\|_2}{\kappa}\sum_{k=\ell+1}^{d}\hat\lambda_k$$
$$+C^*(\Delta t)^4\Big(\|u_{ttt}\|^2_{L^2(0,T;H)} + \|u_{tt}\|^2_{L^2(0,T;W)}\Big).$$

Combining the estimates for ϑ_k with those for ϱ_k in (27) we obtain

$$\frac{1}{m}\sum_{k=1}^{m}\|U_k - u(t_k)\|_H^2$$
$$\le 6\,\|\phi - P^\ell\phi\|_H^2 + \frac{6\alpha\beta}{\kappa}(1+6T\Delta t)\sum_{k=\ell+1}^{d}\tilde\lambda_k$$
$$+2C^*(\Delta t)^4\Big(\Delta t\,\|u_{ttt}\|^2_{L^2(0,T;H)} + \|u_{tt}\|^2_{L^2(0,T;W)}\Big) \tag{34}$$

and similarly for the basis taken in H. The assertion of the theorem now follows. □

3.5. Forward Euler-Galerkin scheme

The forward Euler-Galerkin-POD method for problem (6) leads to the problem

$$(U_0,\psi)_H = (\phi,\psi)_H \text{ for all } \psi\in V^\ell \tag{35a}$$

and

$$(\bar\partial U_k,\psi)_H + a(U_{k-1},\psi) = (f(t_{k-1}),\psi)_H \text{ for all } \psi\in V^\ell \tag{35b}$$

for $k = 1,\dots,m$. Taking $\psi = U_k$ as a test function in (35b) and using (3) and (4) we obtain

$$(1-\eta\Delta t)\,\|U_k\|_H \le \|U_{k-1}\|_H + \Delta t\,\|f(t_{k-1}\|_H \tag{36}$$

with $\eta = \beta/\alpha$. To guarantee stability of the scheme we have to assume that the step size Δt is sufficiently small, i.e.,

$$\Delta t < \frac{1}{\eta}. \tag{37}$$

Then, (36) yields upon summation

$$\|U_k\|_H \le \Big(\frac{1}{1-\eta\Delta t}\Big)^k\,\|U_0\|_H$$
$$+\Delta t\,\|f\|_{C([0,T];H)}\sum_{j=1}^{k}\Big(\frac{1}{1-\eta\Delta t}\Big)^j. \tag{38}$$

Note that $(1-\eta\Delta t)^k \geq e^{-\eta k \Delta t}$. Moreover, setting $\zeta = 1/(1-\eta\Delta t)$ we find

$$\Delta t \sum_{j=1}^{k} \Big(\frac{1}{1-\eta\Delta t}\Big)^j = \Delta t \frac{1-\zeta^k}{\zeta^{-1}-1} = \frac{\zeta^k - 1}{\eta} \leq \frac{e^{\eta k \Delta t}-1}{\eta}.$$

Inserting these two estimates and utilizing the fact that $\|U_0\|_H \leq \|\phi\|_H$ in (38) we obtain the estimate

$$\|U_k\|_H \leq e^{\frac{\eta k T}{m}} \, \|\phi\|_H^2 + \frac{e^{\frac{\eta k T}{m}} - 1}{\eta} \, \|f\|_{C([0,T];H)} \text{ for } k = 0, \ldots, m.$$

Moreover, with (37) holding an error estimate of the form (30) follows, where now $c(T) = 2(e^{2\eta T} - 1)/\eta T)$.

3.6. Remarks

From (30), (34) and the discussion and Sect. 3.5 it follows that the influence of the error in the initial condition either decays exponentially, is bounded by a constant, or is bounded by an exponential expression in T depending on whether the implicit Euler, the Crank–Nicolson, or the explicit Euler methods are used. This is analogous to estimates for fully discrete schemes based on the finite element methods, see for example [18].

Let us comment on the scope of the approximation results of this section. The POD technique for discretization of (6) requires snapshots, which can be obtained by an independent numerical method or by the appropriate technological means related to a specific applications. One of the strengths of POD-based methods are their good approximation properties with a relatively small number of basis elements. They are therefore frequently used as a method for system reduction. On the basis of the POD-reduced system further issues can be addressed, as for instance control or optimal control related problems. Due to model reduction, system theoretic problems become accessible that could otherwise be beyond the scope of computing power or alternatively the computing time can be significantly reduced when compared to finite element or finite difference based approximations. Our results specify the quality of the approximation of the POD-reduced system to the solutions of the continuous system. — In these results we estimate the difference of the snapshots to the solution of the dynamical system in the case where the snapshots are taken from a system whose parameters and inhomogeneities coincide with that of the system itself. In the alternative situation where the snapshots are taken from a system with one set of parameters and inhomogeneities (controls) and utilized as basis elements in a system with a different set of parameters the problem of unmodeled dynamics arises.

On a computational level progress was made on this topic in the context of optimal control of the Navier–Stokes equation where a dynamic update of the basis elements was performed, see [1,2,10].

4. Non-linear problems

In this section we extend our analysis to certain non-linear evolution problems. First we investigate a semi-linear problem with local Lipschitz non-linearity. Secondly, we study the Burgers equation, where the non-linear term does not satisfy a Lipschitz-condition on H.

4.1. A semi-linear equation

Consider the non-linear evolution problem

$$\frac{d}{dt}\,(u(t),\varphi)_H + a(u(t),\varphi) + (F(u(t)),\varphi)_H = (f(t),\varphi)_H \tag{39a}$$

for all $\varphi \in V$ and a.e. $t \in (0,T)$, with

$$(u(0),\chi)_H = (\phi,\chi)_H \text{ for all } \chi \in H, \tag{39b}$$

where $\phi \in V$ and $f \in L^2(0,T;H)$. The backward Euler-Galerkin-POD scheme for (39) is given by

$$(U_0,\psi)_H = (\phi,\psi)_H \text{ for all } \psi \in V^\ell \tag{40a}$$

and

$$\begin{aligned}(\bar\partial U_k,\psi)_H &+ a(U_k,\psi) + (F(U_k),\psi)_H\\ &= (f(t_k),\psi)_H \text{ for all } \psi \in V^\ell.\end{aligned} \tag{40b}$$

Theorem 10. *Assume that* (39) *has a unique solution* $u \in C([0,T];V)$ *with* $u \in W^{2,2}(0,T;H)$ *and that* $\{U_k\}_{k=0}^m$ *is the unique solution to* (40) *satisfying*

$$\max_{0\le k\le m} \|U_k\|_H \le \tilde C$$

for a constant $\tilde C > 0$ *independent of* m*. If* F *is locally Lipschitz-continuous on* H *and* Δt *is sufficiently small, then there exists a constant* $C > 0$ *depending on* u, α, β, κ, T*, but independent of* ℓ *and* m*, such that*

$$\frac{1}{m}\sum_{k=1}^{m} \|u(t_k) - U_k\|_H^2 \le C\Big(\|\phi - P^\ell\phi\|_H^2 + \sum_{k=\ell+1}^{d} \tilde\lambda_k + (\Delta t)^2\Big)$$

and

$$\frac{1}{m}\sum_{k=1}^{m}\|u(t_k)-U_k\|_H^2 \le C\Big(\|\phi - P^\ell\phi\|_H^2 + \|S\|_2 \sum_{k=\ell+1}^{d}\hat\lambda_k + (\Delta t)^2\Big),$$

where S denotes the stiffness matrix introduced in (11).

Proof. Since the term ϱ_k can be estimated as before, we only have to consider ϑ_k. Using the notation introduced in Sect. 3.3 we obtain

$$(41)\qquad (\overline{\partial}\vartheta_k,\psi)_H + a(\vartheta_k,\psi) = (v_k,\psi)_H + (F(u(t_k)) - F(U_k),\psi)_H.$$

Setting $\psi=\vartheta_k$ and using (5) we infer that

$$\begin{aligned}\|\vartheta_k\|_H^2 - (\vartheta_k,\vartheta_{k-1})_H + \kappa\Delta t\,\|\vartheta_k\|_V^2&\\ \le \Delta t\|\vartheta_k\|_H\left(\|v_k\|_H + \|F(U_k)-F(u(t_k))\|_H\right).&\end{aligned}$$

As F is locally Lipschitz-continuous and $u(t_k)$ and U_k are bounded independently of k, there exists a constant $C^\star>0$ such that

$$\|F(U_k)-F(u(t_k))\|_H \le C^\star\,\|U_k - u(t_k)\|_H.$$

Inserting into (41) we find

$$\begin{aligned}\Big(1+\frac{2\kappa\Delta t}{\alpha}\Big)\,\|\vartheta_k\|_H^2 \le \|\vartheta_{k-1}\|_H^2 + \Delta t\,\Big(\|v_k\|_H^2 + (1+3C^\star)\,\|\vartheta_k\|_H^2&\\ +C^\star\,\|\varrho_k\|_H^2\Big).&\end{aligned}$$

Defining $\bar C = 1+3C^\star - \frac{2\kappa}{\alpha}$ this implies that

$$(1-\bar C\Delta t)\,\|\vartheta_k\|_H^2 \le \|\vartheta_{k-1}\|_H^2 + \Delta t\,\Big(\|v_k\|_H^2 + C^\star\,\|\varrho_k\|_H^2\Big).$$

Let $\Delta t>0$ be sufficiently small so that $\bar C\Delta t<1$ is satisfied. Then there exists a constant $\widehat C>0$ such that

$$(1-\bar C\Delta t)^{-1} \le 1+\widehat C\Delta t.$$

Thus, for small Δt

$$\|\vartheta_k\|_H^2 \le (1+\widehat C\Delta t)\,\Big(\|\vartheta_{k-1}\|_H^2 + \Delta t\,\Big(\|v_k\|_H^2 + C^\star\,\|\varrho_k\|_H^2\Big)\Big)$$

holds. Summation on k implies

$$\|\vartheta_k\|_H^2 \le e^{\widehat C T}\Big(\|\vartheta_0\|_H^2 + \Delta t\sum_{j=1}^{k}\left(\|v_j\|_H^2 + C^\star\,\|\varrho_j\|_H^2\right)\Big).$$

Using the estimates for v_k and ϱ_k derived in Sect. 3.3 we arrive at

$$\frac{1}{m}\sum_{k=1}^{m}\|\vartheta_k\|_H^2 \le e^{\widehat{C}T}\,\|\vartheta_0\|_H^2$$

$$+e^{\widehat{C}T}\left(\frac{2(\Delta t)^2}{3}\,\|u_{tt}\|^2_{L^2(0,T;H)} + \frac{3(2+C^\star)\alpha\beta T}{\kappa}\sum_{k=\ell+1}^{d}\tilde{\lambda}_k\right) \tag{42a}$$

and

$$\frac{1}{m}\sum_{k=1}^{m}\|\vartheta_k\|_H^2 \le e^{\widehat{C}T}\,\|\vartheta_0\|_H^2$$

$$+e^{\widehat{C}T}\left(\frac{2(\Delta t)^2}{3}\,\|u_{tt}\|^2_{L^2(0,T;H)} + \frac{3(2+C^\star)\alpha\beta T\|S\|_2}{\kappa}\sum_{k=\ell+1}^{d}\hat{\lambda}_k\right) \tag{42b}$$

for the POD bases $\{\tilde{\psi}_k\}_{k=1}^d$ and $\{\hat{\psi}_k\}_{k=1}^d$, respectively. From (42a) the theorem follows. □

Remark 6. We refer the reader to [15], for example, for sufficient conditions such that (39) admits a unique solution. Due to the fact that F is locally Lipschitz-continuous on H, problem (40) is uniquely solvable, provided that a solution exists.

4.2. Burgers' equation

We consider Burgers equation that was introduced in Example 1.. It is assumed that $\phi \in V = H_0^1(\Omega)$ and that $f \in L^2(0,T;H)$ with $H = L^2(\Omega)$ so that a solution $u \in W(0,T;V) \cap C([0,T];V)$ exists.

Theorem 11. *Let u and $\{U_k\}_{k=0}^m$ be the solutions to the Burgers equation and its backward Galerkin-POD scheme, respectively. Suppose that $u_{tt} \in L^2(0,T;H)$. If Δt is sufficiently small there exists a constant $C > 0$ depending on u and T but independent of ℓ and m such that*

$$\frac{1}{m}\sum_{k=1}^{m}\|u(t_k)-U_k\|_H^2 \le C\Big(\|\phi - P^\ell\phi\|_H^2 + \sum_{k=\ell+1}^{d}\tilde{\lambda}_k + (\Delta t)^2\Big)$$

and

$$\frac{1}{m}\sum_{k=1}^{m}\|u(t_k)-U_k\|_H^2 \le C\Big(\|\phi - P^\ell\phi\|_H^2 + \|S\|_2\sum_{k=\ell+1}^{d}\hat{\lambda}_k + (\Delta t)^2\Big),$$

where S denotes the stiffness matrix introduced in (11).

Proof. To obtain the desired estimate we utilize the error equation (41), from the previous proof. The non-linearity must be estimated in a different manner, however. Due to Corollary 6 and since $u \in C([0,T];V)$ there exists a constant $C_1 > 0$ such that

$$\|u(t_k)\|_V \le C_1 \text{ and } \|U_k\|_H \le C_1 \text{ for } 1 \le k \le m.$$

Using Hölder's inequality we have

$$\begin{aligned} \big|(F(u(t_k)) - F(U_k), \vartheta_k)_H\big| &\le \|u(t_k)\|_V \|u(t_k) \\ &- U_k\|_{L^4} \|\vartheta_k\|_{L^4} + \|U_k\|_H \|u(t_k) - U_k\|_V \|\vartheta_k\|_{L^\infty}, \end{aligned} \tag{43}$$

and hence, since $U_k - u(t_k) = \vartheta_k + \varrho_k$,

$$\begin{aligned} &\big|(F(u(t_k)) - F(U_k), \vartheta_k)_H\big| \\ &\le C_1 \Big(\|\vartheta_k\|_{L^4}^2 + \|\varrho_k\|_{L^4} \|\vartheta_k\|_{L^4} + \|\vartheta_k\|_V \|\vartheta_k\|_{L^\infty} + \|\varrho_k\|_V \|\vartheta_k\|_{L^\infty} \Big). \end{aligned} \tag{44}$$

Let $C_2 > 0$ denote the common embedding constant of V in $L^4(\Omega)$ and $L^\infty(\Omega)$ in $L^4(\Omega)$. Then

$$\begin{aligned} \big|(F(u(t_k)) - F(U_k), \vartheta_k)_H\big| \le C_3 \Big(&\|\vartheta_k\|_V \|\vartheta_k\|_{L^\infty} \\ &+ \|\varrho_k\|_V \|\vartheta_k\|_{L^\infty} \Big), \end{aligned} \tag{45}$$

where $C_3 = C_1(1 + C_2^2)$. By Agmon's inequality (see [17]) there exists a constant $C_4 > 0$ such that

$$\|\varphi\|_{L^\infty} \le C_4 \, \|\varphi\|_V^{1/2} \|\varphi\|_H^{1/2} \text{ for all } \varphi \in V,$$

and consequently Young's inequality implies the existence of a constant $C_5 > 0$ such that

$$\begin{aligned} C_3 \, \|\vartheta_k\|_V \|\vartheta_k\|_{L^\infty} &\le C_3 C_4 \, \|\vartheta_k\|_V^{3/2} \|\vartheta_k\|_H^{1/2} \\ &\le \frac{\nu}{2} \, \|\vartheta_k\|_V^2 + C_5 \, \|\vartheta_k\|_H^2 \end{aligned} \tag{46}$$

for $1 \le k \le m$. Moreover, there exists a constant $C_6 > 0$ such that

$$C_3 \, \|\varrho_k\|_{L^4} \|\vartheta_k\|_{L^4} \le \frac{\nu}{2} \, \|\vartheta_k\|_V^2 + C_6 \, \|\varrho_k\|_V^2 \text{ for } 1 \le k \le m. \tag{47}$$

Using (46) and (47) in (45) we arrive at

$$\begin{aligned} \big|(F(u(t_k)) - F(U_k), \vartheta_k)_H\big| &\le \nu \, \|\vartheta_k\|_V^2 \\ &+ C_5 \, \|\vartheta_k\|_H^2 + C_6 \, \|\varrho_k\|_V^2. \end{aligned} \tag{48}$$

Setting $\psi = \vartheta_k$ in (41) we obtain from (48)

$$\|\vartheta_k\|_H^2 - (\vartheta_k, \vartheta_{k-1})_H \le \Delta t \left(\|v_k\|_H \, \|\vartheta_k\|_H + C_5 \, \|\vartheta_k\|_H^2 + C_6 \, \|\varrho_k\|_V^2 \right)$$

and further

$$\|\vartheta_k\|_H^2 \le \|\vartheta_{k-1}\|_H^2 + \Delta t \left(\|v_k\|_H^2 + (1 + 2C_5) \, \|\vartheta_k\|_H^2 + 2C_6 \, \|\varrho_k\|_V^2 \right).$$

It follows that

$$(1 - (1 + 2C_5)\Delta t) \, \|\vartheta_k\|_H^2 \le \|\vartheta_{k-1}\|_H^2 + \Delta t \left(\|v_k\|_H^2 + 2C_6 \, \|\varrho_k\|_V^2 \right).$$

Thus, for small Δt there exists a constant $C_7 > 0$ such that

$$\|\vartheta_k\|_H^2 \le (1 + C_7 \Delta t) \left(\|\vartheta_{k-1}\|_H^2 + \Delta t \left(\|v_k\|_H^2 + 2C_6 \, \|\varrho_k\|_V^2 \right) \right)$$

holds. By summation on k we have

$$\|\vartheta_k\|_H^2 \le e^{C_7 T} \Big(\|\vartheta_0\|_H^2 + \Delta t \sum_{j=1}^{k} \left(\|v_j\|_H^2 + 2C_6 \, \|\varrho_j\|_V^2 \right) \Big).$$

Let V be endowed with $\|\varphi'\|_{L^2}$ as norm. Then $\beta = \kappa = 1$, and together with Lemma 3

$$\frac{1}{m} \sum_{k=1}^{d} \|\varrho_k\|_V^2 \le 3 \sum_{k=\ell+1}^{m} \tilde{\lambda}_k \text{ and } \frac{1}{m} \sum_{k=1}^{d} \|\varrho_k\|_V^2 \le 3\|S\|_2 \sum_{k=\ell+1}^{m} \hat{\lambda}_k,$$

where $\tilde{\lambda}_k$ and $\hat{\lambda}_k$ denote the eigenvalues of the correlation matrix with the elements $K_{ij} = \frac{1}{2m+1} \, (y_j, y_i)_V$ and $K_{ij} = \frac{1}{2m+1} \, (y_j, y_i)_H$, respectively. Using the estimate for v_k deduced in Sect. 3.4 we arrive at

$$\begin{aligned} \frac{1}{m} \sum_{k=1}^{m} \|\vartheta_k\|_H^2 &\le e^{C_7 T} \, \|\vartheta_0\|_H^2 \\ &\quad + e^{\widehat{C}T} \Big(\frac{2(\Delta t)^2}{3} \, \|u_{tt}\|_{L^2(0,T;H)}^2 + 6C_6 T \sum_{k=\ell+1}^{d} \tilde{\lambda}_k \Big) \end{aligned} \tag{49a}$$

and

$$\begin{aligned} \frac{1}{m} \sum_{k=1}^{m} \|\vartheta_k\|_H^2 &\le e^{C_7 T} \, \|\vartheta_0\|_H^2 \\ &\quad + e^{\widehat{C}T} \Big(\frac{2(\Delta t)^2}{3} \, \|u_{tt}\|_{L^2(0,T;H)}^2 + 6C_6 T \, \|S\|_2 \sum_{k=\ell+1}^{d} \hat{\lambda}_k \Big) \end{aligned} \tag{49b}$$

for the POD bases $\{\tilde{\psi}_k\}_{k=1}^d$ and $\{\hat{\psi}_k\}_{k=1}^d$, respectively. From (49) the claim follows directly. □

5. Numerical experiments

We present three examples for the approximation of parabolic equations by Galerkin-POD based schemes. The results confirm the good approximation properties of such schemes which was already reported in [3–6, 8, 11, 13, 14, 19], for example. We also compare V- and H-norm based schemes and schemes based on POD-ensembles with and without difference quotients. For the examples that we tested we found no significant difference between the V- and the H-norm implementations. Inc luding the difference quotients can improve the numerical result in case the snapshots are taken only from a coarse grid. For the numerical realization we used MATLAB version 5.3 executed on a DIGITAL Alpha 21264 computer.

Run 1. In our first test example we consider heat flow in a rectangular metal block with a rectangular cavity. The block is heated on one side, heat is flowing from the block to the surrounding air at a constant rate on the opposite side and the block is isolated at the remaining walls. This leads the the initial boundary value problem

$$\begin{aligned} u_t - \Delta u &= 0 \ \text{ in } \ Q = (0,T) \times \Omega, \\ u &= 1 \ \text{ on } \ \Gamma_1 = \{(-0.5, y) : -0.8 \le y \le 0.8\}, \\ \frac{\partial u}{\partial n} &= -0.1 \ \text{ on } \ \Gamma_2 = \{(0.5, y) : -0.8 \le y \le 0.8\}, \\ \frac{\partial u}{\partial n} &= 0 \ \text{ on } \ \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2), \\ u(0,\cdot) &= \phi \ \text{ in } \ \Omega, \end{aligned}$$

where $T = 5$,

$$\Omega = ((-0.5, 0.5) \times (-0.8, 0.8)) \setminus ((-0.05, 0.05) \times (-0.4, 0.4))$$

and $\phi = 0$. We computed an approximate solution to the heat equation by using the backward Euler-Galerkin finite element scheme. The spatial discretization was carried out by linear finite elements with respect to a triangular grid and 1844 degrees of freedom. Further Δt was chosen to be 0.04. Let

$$V^h = \text{span}\,\{\varphi_1, \ldots, \varphi_{1844}\} \subset H^1(\Omega)$$

denote the finite element space. Then the reference numerical solution was characterized by a coefficient matrix $U \in \mathbb{R}^{1844 \times 126}$ in the following way

$$u_{FE}(t_{j-1}) = \sum_{i=1}^{1844} U_{ij} \varphi_i \text{ for } j = 1, \ldots, 126.$$

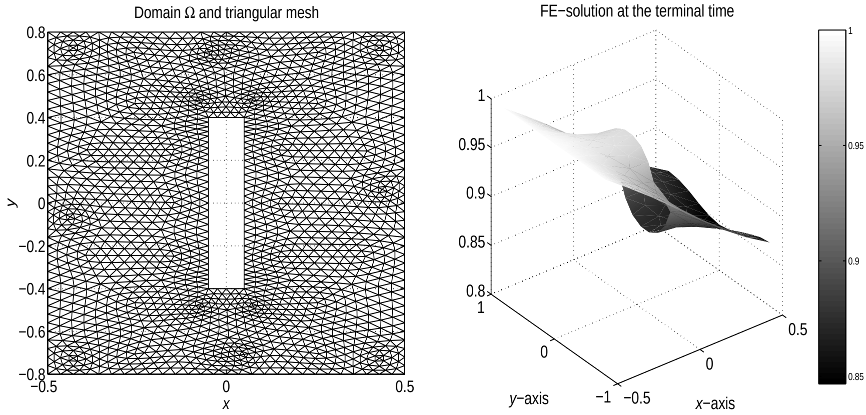

Fig. 1.

The triangular mesh as well as the finite element solution at time T are shown in Fig. 1. As snapshots we took the finite element solution at the discrete time instances. Thus, our snapshots were

$$y_j^h = \sum_{i=1}^{1844} Y_{ij}\varphi_i \text{ for } j = 1, \dots, 251,$$

where $Y \in \mathbb{R}^{1844\times 251}$ was given by

$$Y_{\cdot,j} = U_{\cdot,j} \text{ for } j = 1, \dots, 126$$

and

$$Y_{\cdot,j+126} = \frac{1}{\Delta t}\left(Y_{\cdot,j+1} - Y_{\cdot,j}\right) \text{ for } j = 1, \dots, 125.$$

Numerically, $\operatorname{rank} Y = 17$. Let $\Psi \in \mathbb{R}^{1844\times 17}$ denote the coefficient matrix of the POD basis functions $\{\psi_k\}_{k=1}^{17}$ satisfying

$$\psi_k = \sum_{i=1}^{1844} \Psi_{ik}\varphi_i \text{ for } k = 1, \dots, 17.$$

Due to Proposition 1 the matrix Ψ can be determined as follows. First compute the eigenvalues $\lambda_1 \geq \dots \geq \lambda_{17} > 0$ and the corresponding eigenvectors $v_1, \dots, v_d \in \mathbb{R}^{251}$ of the correlation matrix $K = \frac{1}{251}\, Y^T \Phi Y$, where $\Phi \in \mathbb{R}^{1844\times 1844}$ denotes the mass matrix with the elements $\Phi_{ij} = (\varphi_j, \varphi_i)_X$ for $1 \leq i, j \leq 1844$. Then the k-th column of Ψ is obtained by setting

$$\Psi_{\cdot,k} = \frac{1}{\sqrt{\lambda_k}}\, Y v_k.$$

Utilizing the MATLAB function `eig`, each solution of the eigenvalue problem for $X = V$ and for $X = H$ needed 3 seconds, whereas the CPU time was about 1 second if we computed only the first 5 eigenvalues and their corresponding eigenvectors by applying the iterative MATLAB eigenvalue solver `eigs`. The rapid decay of the eigenvalues is presented in Fig. 2. For instance, $\tilde{\lambda}_6 \approx 0.000002$ and $\hat{\lambda}_6 \approx 0.00000005$.

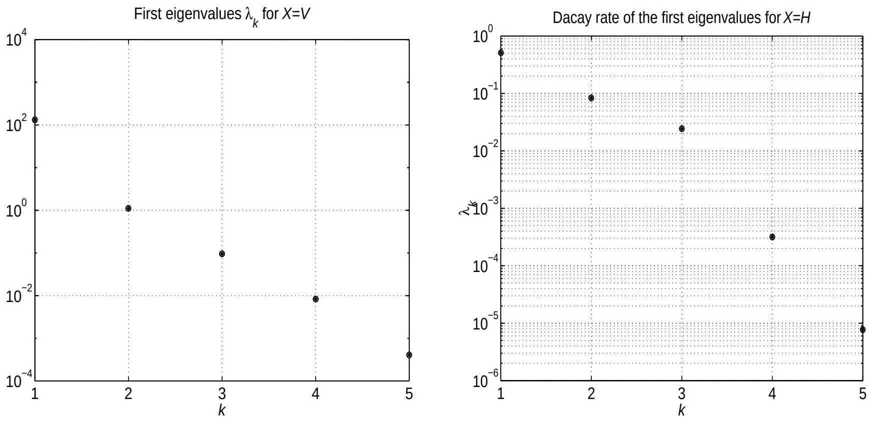

Fig. 2.

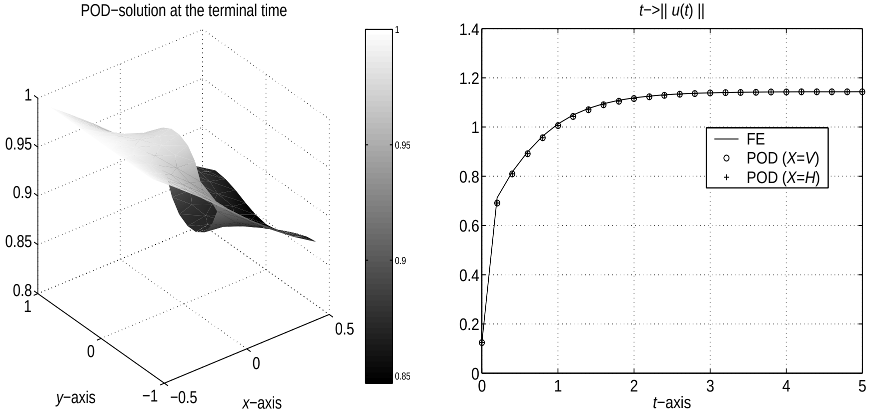

Fig. 3.

For the results presented below we took $\ell = 5$ POD basis functions. Then for the summation over the eigenvalues in the error formulas (7) and

(10) we found

$$\sum_{k=6}^{251} \tilde{\lambda}_k < 0.000002 \text{ and } \sum_{k=6}^{251} \hat{\lambda}_k < 0.00000006,$$

respectively. The solution of the heat equation with the backward Euler-Galerkin-POD method needed less than 0.1 seconds for each of the POD bases in V and H. The POD solution at $t = T$ is plotted at the left in Fig. 3, whereas on the right we find the L^2-norms of the finite element and the POD solution. We observe that the L^2-norms nearly coincide. In Table 1 we compare the error

$$e(m) = \frac{1}{m} \sum_{k=1}^{m} \|u_{FE}(t_k) - U_k\|_H^2$$

with $m = 125$ for different POD bases. For these simulations we took five POD basis functions. It turns out that the choice of the space V or H as well as the inclusion of the difference quotients made no difference for this example.

Table 1.

	$e(m)$
POD basis in V including $\overline{\partial} u(t_k)$	0.00011786
POD basis in H including $\overline{\partial} u(t_k)$	0.00011785
POD basis in V	0.00011755
POD basis in H	0.00011754

Run 2. Our second test is carried out for the Burgers equation. Here, we chose the snapshots on a relatively coarse grid and we shall see that inclusion of the difference equations leads to a smaller error. We took $T = 1$, $\nu = 0.5$,

$$\phi = \begin{cases} 1 \text{ in } (0, \frac{1}{2}], \\ 0 \text{ otherwise,} \end{cases}$$

and $f(t) = \phi$ in $(0, T)$. We computed an approximate solution to Burgers' equation by using the backward Euler-Galerkin finite element scheme. For the spatial discretization we used linear finite elements with 198 degrees of freedom, and as time grid we took $t_k = k\Delta t$ with $\Delta t = 5 \cdot 10^{-3}$. The finite element solution at time T are shown in Fig. 4. We chose snapshots on a

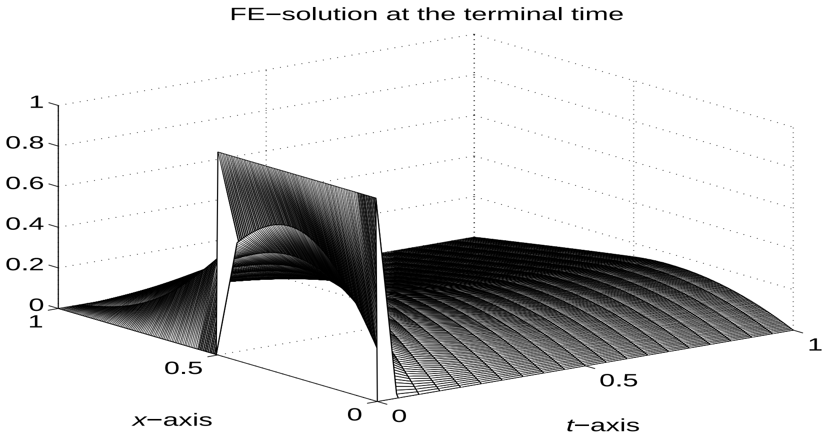

Fig. 4.

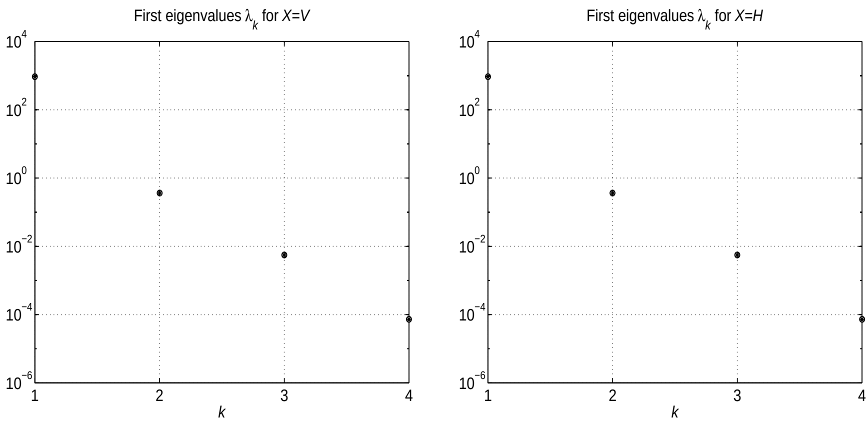

Fig. 5.

uniform time-grid with $\Delta t = \frac{1}{5}$. Thus, our snapshots were

$$y_j^h = \sum_{i=1}^{198} Y_{ij}\varphi_i \text{ for } j = 1, \dots, 11,$$

where $Y \in \mathbb{R}^{198\times 11}$ was given by

$$Y_{\cdot,j} = U_{\cdot,1000(j-1)+1} \text{ for } j = 1, \dots, 6$$

and

$$Y_{\cdot,j+6} = \frac{1}{\Delta t}\left(Y_{\cdot,j+1} - Y_{\cdot,j}\right) \text{ for } j = 1, \dots, 5.$$

Numerically, we had $\operatorname{rank} Y = 6$. Each solution of the eigenvalue problem for $X = V$ and for $X = H$ needed less than 0.01 seconds. The decay rate of the eigenvalues is presented in Fig. 5. In our computations we took $\ell = 2$ POD basis functions. Then we have in the error formulas (7) and (10)

$$\sum_{k=3}^{6} \tilde{\lambda}_k < 0.00564 \text{ and } \sum_{k=3}^{6} \hat{\lambda}_k < 0.00003,$$

respectively. Furthermore, the initial condition satisfies $\|\phi - P^\ell \phi\|_H \leq 0.45$. The solution of the Burgers equation with the backward Euler-Galerkin-POD method needed less than 0.02 seconds for each POD basis. The error $e(m)$ is shown in Table 2. Finally, computing the finite element solution $\tilde{u}_{FE}$

Table 2.

	$e(m)$
POD basis in V, including $\overline{\partial} u(t_k)$	0.0012796
POD basis in H, including $\overline{\partial} u(t_k)$	0.0012689
POD basis in V	0.0016033
POD basis in H	0.0015921

on the coarse time grid, we obtained $\frac{1}{5} \sum_{k=1}^{5} \|u_{FE}(t_k) - \tilde{u}_{FE}(t_k)\|_H^2 = 0.0205496$.

Run 3. In our third run we compare the numerical error to the estimate of Theorem 11. For that purpose let $y(t, x) = (x^2 - x)\sin(2\pi x t)$ be the exact solution of the Burgers equation and compute f accordingly. The spatial grid was chosen in such a way that the error between y and its approximation is about 10^{-10}. Then we took snapshots for different values of Δt and solved the Burgers equation with $\ell = 6$ POD basis functions. Since

$$\sum_{k=7}^{d} \tilde{\lambda}_k < 10^{-12}$$

and $y(0, \cdot) = 0$ hold, the error depends almost exclusively on terms of the form $C(\Delta t)^2$ where the constant $C > 0$ is independent of Δt and ℓ. We computed for $m_i = 2^i m$ with $i = 0, \ldots, 9$ and $m = 5$. Then, $m_i^2 = 4m_{i-1}^2$ and we infer that $\frac{e(m_{i-1})}{e(m_i)}$ ought to be close to 4 for $i = 1, \ldots, 9$. The numerical results are presented in Table 3.

6. Conclusion

Error estimates for Galerkin POD methods for parabolic problems were presented. The error depends on the number of POD basis functions and on the time discretization. To obtain the desired estimates the difference quotients were added to the snapshot ensemble. Without them the error bounds contain terms of the form $Cm \sum_{k=\ell+1}^{d} \lambda_k$, with C a positive constant independent of ℓ and m. Inclusion of the difference quotients showed little effect on the practical approximation properties of POD based discretization of the linear heat and the Burgers equation in numerical tests, unless the time discretization for the snapshots was very coarse. In this case the ensemble containing the difference quotients gave more favorable results.

Table 3.

i	m_i	$e(m_i)$	$\frac{e(m_{i-1})}{e(m_i)}$
0	5	0.00279337991	—
1	10	0.00090061392	3.11
2	20	0.00025387076	3.55
3	40	0.00006741209	3.77
4	80	0.00001737545	3.88
5	160	0.00000441201	3.94
5	320	0.00000111210	3.96
5	640	0.00000027939	3.97
6	1280	0.00000007013	3.98
7	2560	0.00000001763	3.98
8	5120	0.00000000445	3.96
9	10240	0.00000000113	3.92

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