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Simplicial grid refinement: on Freudenthal's algorithm and the optimal number of congruence classes

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Summary. In the present paper we investigate Freudenthal's simplex refinement algorithm which can be considered to be the canonical generalization of Bank's well known *red* refinement strategy for triangles. Freudenthal's algorithm subdivides any given (n)-simplex into 2^n subsimplices, in such a way that recursive application results in a stable hierarchy of consistent triangulations. Our investigations concentrate in particular on the number of congruence classes generated by recursive refinements. After presentation of the method and the basic ideas behind it, we will show that Freudenthal's algorithm produces at most $n!/2$ congruence classes for any initial (n)-simplex, no matter how many subsequent refinements are performed. Moreover, we will show that this number is optimal in the sense that recursive application of any affine invariant refinement strategy with 2^n sons per element results in *at least* $n!/2$ congruence classes for *almost all* (n)-simplices.

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1. Introduction

Along with the growing acceptance of adaptive discretization methods for partial differential equations, a number of adaptive mesh refinement algorithms have been developed during the last years. Much attention has been paid in particular to the refinement of simplicial grids (*triangulations*), which have some advantages, compared to cubic-type meshes, concerning the approximation of curved domains and the preservation of consistency after adaptive refinements. Usually, simplicial meshes obtained by subsequent refinements are required to fulfill at least two conditions: *stability* and *consistency*.

The stability condition means that the simplices (*elements*) generated during the refinement process must not degenerate, i.e., the interior angles of all elements have to be bounded uniformly away from zero. Consistent triangulations are characterized by the fact that the intersection of each pair of adjacent elements is either a common vertex, a common edge, or, in general, a common lower-dimensional subsimplex. Note that in most cases consistency is required rather for the sake of convenience than on account of mathematical necessity. In contrast, stability is essential for example for the approximation properties of finite element spaces and the convergence behaviour of multigrid and multilevel algorithms.

Besides the numerical solution of partial differential equations there are many other fields where mesh refinement techniques are of particular interest (see [26] for an overview). In two and three space dimensions, for example, adaptively refined meshes are used for realistic surface and volume rendering in computer graphics or for the resolution of details in geometric design, cf. [15]. Higher-dimensional meshes are applied in combinatorical algorithms for the computation of fixed points or in financial mathematics to determine fair prices of options. In most of these cases stability and consistency are important properties.

For triangular and tetrahedral grids there exist several refinement algorithms satisfying both conditions for arbitrary consistent input triangulations. These algorithms can be divided into two major classes depending on the basic subdivision scheme and on the way how stability and consistency are preserved: *red/green refinement algorithms* [2,6,21,33] and *bisection methods* [1,3,20,22,25,27,28].

There also exist more general refinement algorithms for simplicial grids in n space dimensions. Most of these methods, however, are not fully satisfactory. The bisection methods of Maubach [23,24] and Traxler [31], for example, are based on certain rather restrictive assumptions concerning the initial triangulation that are often hard to fulfill in practice. For Rivara's longest edge bisection [28], on the other hand, which indeed applies to any consistent triangulation, it has not been proved yet that it is stable for $n > 2$.

Subject of the present paper is a refinement strategy which has been published already in 1942 by H. Freudenthal [13]. Freudenthal's algorithm subdivides any given (n)-simplex T into 2^n subsimplices of equal volume in such a way that recursive refinement of T yields stable and consistent triangulations of T. Moreover, the method can be applied to arbitrary consistent triangulations. Inspired by [2], we refer to such subdivisions with 2^n sons per element as *regular* (or *red*) refinements. In fact, the famous red refinement strategy for triangles, proposed by Bank [2], and also the corresponding 3D refinement strategy proposed by the author in [6], are special cases of Freudenthal's method.

Freudenthal's algorithm is frequently used in, for example, fixed point computations, cf. [30]. The method, however, is not well-known in the field of numerical solution of partial differential equations. The original paper of Freudenthal [13], which is indeed very interesting for historical reasons, is not very suitable as an introduction for engineers and application programmers. Therefore we will present a detailed description of the method and the basic idea behind it. The algorithm itself will be presented in a somewhat more modern form which comes closer to what is usually called a "ready-to-implement" formulation.

After presentation of the method we concentrate on the number of congruence classes generated in subsequent refinement steps. As a first result we will show that Freudenthal's algorithm produces at most $n!/2$ congruence classes for any initial (n)-simplex T, no matter how many subsequent refinement steps are performed. Of course, this result implies stability. We will also show that a finite number of congruence classes is not only sufficient but also necessary for stability, provided we restrict ourselves to regular refinement schemes. For practical reasons, however, it is often desirable that the number of congruence classes is not only finite but even as small as possible. In many applications such as finite element computations, for example, there are data depending on the element's congruence class and refinement level only. Such applications can often be significantly accelerated if data of this type are calculated and stored only once.

In this respect, the following result – which can be considered to be the main result of this paper – seems to be interesting: We will prove that recursive application of *any* regular and affine invariant refinement strategy produces at least $n!/2$ congruence classes for almost all (n)-simplices. Hence, Freudenthal's algorithm is optimal in that sense. In fact one can show that the bisection methods mentioned above generate up to $n! \cdot 2^{n-2}$ congruence classes if n subsequent bisection steps are considered to be one single regular refinement, cf. [1]. Consequently, Freudenthal's algorithm has at least two advantages compared to bisection: It applies to any given consistent triangulation and it produces significantly fewer congruence classes.

It should be mentioned at this point that Freudenthal's algorithm without modification can be used for *uniform* refinements only. In contrast, the bisection methods mentioned above yield consistent triangulations also in the case of *adaptive* refinements. In order to extend Freudenthal's method to a fully adaptive algorithm, it has to be combined with some suitable set of additional *irregular* refinement rules for the so called *green closure*. This topic, however, will be addressed in a forthcoming paper, cf. [7]. Here we are mainly concerned with recursive, uniform refinements.

The remainder of this paper is organized as follows: Sect. 2 contains basic definitions and some elementary geometric results. In Sect. 3 we give a short survey of existing grid refinement algorithms in two, three, and more space dimensions. In Sect. 4 we derive Freudenthal's algorithm and show that it generates at most $n!/2$ congruence classes for any initial element T. Finally, in Sect. 5, we prove that this number is optimal for any regular and affine invariant refinement strategy.

2. Basic definitions

We start with the definition of basic notions and recall some elementary results from geometry. Most of the material presented here can be found in [10]. One result (Theorem 2.1), however, appears to be new and will therefore be proved at the end of this section.

2.1. Simplices

A closed subset $T \subset \mathbb{R}^n$ is called a *(k)-simplex*, $0 \le k \le n$, if T is the convex linear hull of $k+1$ vertices $x^{(0)}, \ldots, x^{(k)} \in \mathbb{R}^n$:

$$
\begin{aligned}
(1) \quad T &= [\, x^{(0)}, \ldots, x^{(k)} \,] \\
&:= \Big\{ \, x = \sum_{j=0}^{k} \lambda_j x^{(j)} \;\Big|\; \sum_{j=0}^{k} \lambda_j = 1;\; \lambda_j \in [0,1],\, 0 \le j \le k \, \Big\}.
\end{aligned}
$$

The vertex ordering of each such (k)-simplex T is assumed to be fixed. Hence, two simplices $T = [\, x^{(0)}, \ldots, x^{(k)} \,]$, $T' = [\, y^{(0)}, \ldots, y^{(k)} \,]$ are defined to be equal, i.e. $T = T'$, if $x^{(j)} = y^{(j)}$ for $0 \le j \le k$. If the vertex numbering is different but T and T' still denote the same subset of $\mathbb{R}^n$, we say that T coincides with T' *in the sense of sets* and write $T \cong T'$.

If $k = n$ then T is simply called *simplex* or *element* of $\mathbb{R}^n$. (2)- and (3)-simplices are called *triangles* and *tetrahedra* as usual. Note that the vertex ordering plays an important role in many grid refinement algorithms and in particular in the algorithm considered in this paper. The boundary of a (k)-simplex consists of lower-dimensional subsimplices: An (ℓ)-simplex $S = [\, y^{(0)}, \ldots, y^{(\ell)} \,]$ is called an *(ℓ)-subsimplex* of $T = [\, x^{(0)}, \ldots, x^{(k)} \,]$, $0 \le \ell < k \le n$, if the vertices of S are vertices of T and if their ordering coincides with the ordering induced by the vertex numbering of T, i.e., if there are indices $0 \le i_0 < i_1 < \cdots < i_\ell \le k$ such that $y^{(j)} = x^{(i_j)}$ for $0 \le j \le \ell$. Obviously, the (0)- and (1)-subsimplices of T are just its vertices and edges, respectively. The number of (ℓ)-subsimplices of a (k)-simplex T is $\binom{k+1}{\ell+1}$.

The k-dimensional volume of a (k)-simplex T is denoted by $\mathrm{vol}(T)$ if $k = n$ and by $\mathrm{vol}_k(T)$ otherwise. If $\mathrm{vol}_k(T) = 0$ or, equivalently, if the vertices of T belong to a $(k-1)$-dimensional hyperplane, T is called *degenerate*. In many applications, for example in the discretization of partial differential equations, such degenerate simplices should be avoided. A more refined quality measure for the shape of (k)-simplices is the *measure of degeneracy*

$$\delta(T) \;:=\; h(T)\,/\varrho(T)\,, \tag{2}$$

where $h(T)$ denotes the length of the longest edge of T, and $\varrho(T)$ is the diameter of the biggest k-dimensional ball contained in T. It is easily seen that T is degenerate if and only if $\delta(T) = \infty$. Other shape measures and their relations are considered in [19], for example. Our choice is motivated by the finite element convergence analysis in [10]. To calculate $\delta(T)$ for a (k)-simplex T, in practice one may use the formula

$$\delta(T) \;:=\; \frac{h(T)\,\mathrm{vol}_{k-1}(\partial T)}{2\,n\,\mathrm{vol}_k(T)}\,,$$

which can be shown to hold using similar arguments as in the proof of the corresponding three-dimensional result in [33].

2.2. Affine transformations

To carry over certain results from one simplex to another, affine transformations have proved to be a useful tool. An *affine transformation* in $\mathbb{R}^n$ is a mapping $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ of the form

$$F(x) \;=\; v + Bx\,, \qquad x \in \mathbb{R}^n\,,$$

where $v \in \mathbb{R}^n$ is an arbitrary translation vector and $B \in \mathbb{R}^{n\times n}$ is a non-singular transformation matrix. In most cases we write Fx instead of $F(x)$. For any subset $M \subset \mathbb{R}^n$, the transformed set $M' = F(M)$ is given by $F(M) := \{\, Fx \mid x \in M \,\}$.

Every affine transformation $F : x \mapsto v + Bx$ is one-to-one, and the inverse mapping F^{-1}, given by $F^{-1} : x \mapsto B^{-1}(x - v)$, $x \in \mathbb{R}^n$, is also an affine transformation. Furthermore, the image of any (k)-simplex $T = [\,x^{(0)}, \ldots, x^{(k)}\,] \subset \mathbb{R}^n$ under some affine transformation F is again a (k)-simplex. The vertex ordering of the transformed simplex $T' = F(T)$ is induced by the vertex ordering in T, i.e., $F(T)$ is defined by

$$F(T) \;:=\; [\,Fx^{(0)}, \ldots, Fx^{(k)}\,]\,.$$

For $F(T)$ we also use the notation $F(T) = v + B\,T$. If $B = cI$ for some $c \neq 0$, we write $F(T) = v + c\,T$.

Since both T and $F(T)$ have a given vertex ordering, it follows that for any two non-degenerate simplices $T, T' \subset \mathbb{R}^n$ there is a unique affine transformation F satisfying $T' = F(T)$. If T' equals T in the sense of sets, $T' \cong T$, then F is called *renumbering* of T. Renumberings can be used to replace relations of the form $T \cong T'$ by equations of the form $T' = F(T)$. Considering simplices as subsets of $\mathbb{R}^n$ makes sense if a certain simplex property is independent of the actual vertex ordering. In the present paper the most important property of this type is the congruence property: Two simplices $T, T' \subset \mathbb{R}^n$ are called *congruent* to each other if there exists a translation vector $v \in \mathbb{R}^n$, a scaling factor $c > 0$, and an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$T' \;\cong\; v + c\,Q\,T\,. \tag{3}$$

In this case T and T' are elements of the same *congruence class*. Since (3) reads $T' \cong v + c\,Q\,T$ but not $T' = v + c\,Q\,T$, the congruence class of a simplex is independent of its vertex ordering. Clearly, any two simplices of the same congruence class have the same measure of degeneracy.

2.3. Triangulations

A finite set $\mathcal{T}$ of non-degenerate simplices $T \subset \mathbb{R}^n$ with pairwise non-overlapping interior is called a *triangulation* (in $\mathbb{R}^n$). The vertices of $\mathcal{T}$ are just the vertices of its elements. A triangulation $\mathcal{T}$ is called *consistent*[1] if the intersection of any two distinct simplices $T, T' \in \mathcal{T}$ is either empty or – in the sense of sets – a common lower-dimensional subsimplex. If each such subsimplex is a common subsimplex not only in the sense of sets but even in the sense of equality of simplices (cf. Sect. 2.1), then we say that $\mathcal{T}$ is *consistently numbered*. More precisely: $\mathcal{T}$ is called consistently numbered if for any two simplices $T = [\,x^{(0)}, \ldots, x^{(n)}\,]$, $T' = [\,y^{(0)}, \ldots, y^{(n)}\,] \in \mathcal{T}$ with non-empty intersection $T \cap T' \neq \emptyset$ there are a number $0 \leq \ell \leq n$ and indices $i_0 < i_1 < \cdots < i_\ell,\; j_0 < j_1 < \cdots < j_\ell$ such that

$$T \cap T' \;\cong\; [\,x^{(i_0)}, \ldots, x^{(i_\ell)}\,] = [\,y^{(j_0)}, \ldots, y^{(j_\ell)}\,]\,.$$

Examples of consistently numbered triangulations in 2D and 3D are shown in Fig. 1. We note that the consistent-numbering-property is of great importance when applying Freudenthal's algorithm globally to all simplices of a given triangulation – at least if $n > 3$, cf. Sect. 4.4. In case of $n = 2$ or $n = 3$ it usually suffices to consider consistent triangulations.

[1] Some authors prefer the terms *regular*, *conforming*, or *compatible*

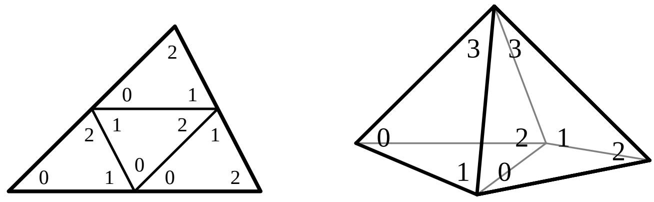

Fig. 1. Consistently numbered triangulations in 2D and 3D

The *measure of degeneracy* of a triangulation $\mathcal{T}$ is defined by

$$\delta(\mathcal{T}) \;:=\; \max_{T \in \mathcal{T}} \delta(T)\,.$$

For any triangulation $\mathcal{T}$ and any affine transformation F in $\mathbb{R}^n$, the transformed triangulation $F(\mathcal{T})$ is given by

$$F(\mathcal{T}) \;:=\; \{\, F(T) \mid T \in \mathcal{T} \,\}\,.$$

It is easily verified that $F(\mathcal{T})$ is consistent (consistently numbered) if and only if $\mathcal{T}$ has this property.

2.4. Refinements

Refinement is the key operation in various grid adaptation algorithms. Here we consider only such refinements where the new vertices coincide with edge midpoints of refined simplices. To be more precise, let $T \subset \mathbb{R}^n$ be a non-degenerate simplex. A *refinement* of T is a triangulation $\mathcal{R}(T)$ consisting of at least two elements such that each vertex of $\mathcal{R}(T)$ is either a vertex or an edge midpoint of T. The elements in $\mathcal{R}(T)$ are called *sons* of T while T is called *father* of the elements in $\mathcal{R}(T)$. It follows from the definition that each refined simplex can have at most 2^n sons. According to [2], we refer to refinements with 2^n sons as *regular refinements*. Note that the volume of any son T' of a regular refinement $\mathcal{R}(T)$ is given by $\mathrm{vol}(T') = 2^{-n}\,\mathrm{vol}(T)$.

A *(regular) refinement strategy* is a mapping $\mathcal{R}$ associating with each non-degenerate simplex $T \subset \mathbb{R}^n$ a (regular) refinement $\mathcal{R}(T)$. A refinement strategy $\mathcal{R}$ is called *affine invariant* if $F(\mathcal{R}(T)) = \mathcal{R}(F(T))$ for each non-degenerate simplex T and any affine transformation F. Affine invariant strategies are fully determined by the corresponding refinement of an arbitrary reference element $\hat{T}$. In contrast, non-affine invariant strategies often depend on some geometric property of the simplex under consideration, as for example the longest-edge-bisection method of Rivara [27,28] and the shortest-interior-edge strategy of Zhang [33]. For a description of these and other methods we refer to Sect. 3.

So far we have considered the refinement of single simplices only. Now we consider the refinement of a triangulation. Let $\mathcal{T} \neq \mathcal{T}'$ be two triangulations covering the same region $\overline{\Omega} := \bigcup \{ T \mid T \in \mathcal{T} \} = \bigcup \{ T' \mid T' \in \mathcal{T}' \}$. Then $\mathcal{T}'$ is called a *refinement* of $\mathcal{T}$ if for each simplex $T \in \mathcal{T}$ either $T \in \mathcal{T}'$ or there is a refinement $\mathcal{R}(T) \subset \mathcal{T}'$. Repeating the refinement process, we obtain a *(nested) hierarchy of triangulations*, i.e. a sequence $(\mathcal{T}_k)_{k\in\mathbb{N}_0}$ starting with some *initial triangulation* $\mathcal{T}_0$ such that $\mathcal{T}_{k+1}$ is a refinement of $\mathcal{T}_k$ for each $k \geq 0$. Such a hierarchy $(\mathcal{T}_k)_{k\in\mathbb{N}_0}$ is called *stable* if $\delta(\mathcal{T}_k)$ is bounded uniformly in k.

Hierarchies of triangulations are often produced by recursive application of some refinement strategy $\mathcal{R}$. Let $\mathcal{T}_0$ be an initial triangulation; then the hierarchy of triangulations $(\mathcal{T}_k)_{k\in\mathbb{N}_0}$, generated *by recursive application of* $\mathcal{R}$ *to* $\mathcal{T}_0$, is defined by

$$\mathcal{T}_{k+1} := \bigcup \{ \mathcal{R}(T) \mid T \in \mathcal{T}_k \} , \qquad k \geq 0 .$$

If $\mathcal{T}_0$ consists of a single simplex $\hat{T}$, we say that $(\mathcal{T}_k)_{k\in\mathbb{N}_0}$ is generated by recursive application of $\mathcal{R}$ to $\hat{T}$. Note that at this point we mainly consider uniform rather than adaptive refinements.

A refinement strategy $\mathcal{R}$ is called *stable* if for any non-degenerate simplex $\hat{T}$ the corresponding hierarchy of triangulations, generated by recursive application of $\mathcal{R}$ to $\hat{T}$, is stable. Clearly, if $\mathcal{R}$ generates only a finite number $N = N(\hat{T})$ of congruence classes for any initial element $\hat{T}$, then R is stable. Surprisingly, the reverse implication also holds, at least in the case of regular refinements:

Theorem 2.1 *A regular refinement strategy* $\mathcal{R}$ *is stable if and only if for each non-degenerate initial simplex* $\hat{T}$ *the number of congruence classes generated by recursive application of* $\mathcal{R}$ *to* $\hat{T}$ *is finite.*

Proof. We have to show that stability implies a finite number of congruence classes. We therefore consider an arbitrary regular and stable refinement strategy $\mathcal{R}$. Let $\hat{T} = [\, x^{(0)}, \ldots, x^{(k)} \,] \subset \mathbb{R}^n$ be a non-degenerate simplex and let $(\mathcal{T}_k)_{k\in\mathbb{N}_0}$ be the hierarchy of triangulations generated by recursive application of $\mathcal{R}$ to $\hat{T}$. Without loss of generality we can assume $x^{(0)} = 0$. Due to the regularity of $\mathcal{R}$, the volume of each element $T \in \mathcal{T}_k$ is given by

$$\mathrm{vol}(T) \;=\; 2^{-kn}\mathrm{vol}(\hat{T}) \,. \tag{4}$$

For the diameter of the biggest n-dimensional ball contained in T, denoted again by $\varrho(T)$, we obtain the estimate

$$\varrho(T) \;\leq\; C\,\mathrm{vol}(T)^{1/n} \;=\; C\,2^{-k}\,\mathrm{vol}(\hat{T})^{1/n} \;\leq\; C\,2^{-k}\,h(\hat{T}) \,, \tag{5}$$

where $C = C(n)$ is a positive constant depending on n only. The stability condition in combination with (2) implies that there exists (another) constant C independent of k such that for all $k \geq 0$

$$h(T) \leq C\, 2^{-k}\, h(\hat{T}), \qquad T \in \mathcal{T}_k . \tag{6}$$

Now we make use of the fact that all vertices of $\mathcal{T}_k$ are either vertices or edge midpoints of $\mathcal{T}_{k-1}$. Using $x^{(0)} = 0$, it follows by induction that the vertices of $\mathcal{T}_k$ are contained in the set

$$\mathcal{Z}_{n,k} := \left\{ x = \sum_{j=1}^{n} 2^{-k} \lambda_j x^{(j)} \;\middle|\; \lambda_j \in \mathbb{Z},\ 1 \leq j \leq n \right\}.$$

For each element $T = [\, x_T^{(0)}, \ldots, x_T^{(n)}\,] \in \mathcal{T}_k$ let T^* be the simplex obtained from T by translating its first vertex to the origin and scaling the resulting simplex by the factor 2^k :

$$T^* := 2^k (T - x_T^{(0)}), \qquad T \in \mathcal{T}_k,\ k \geq 0 . \tag{7}$$

Clearly, T^* is congruent to T. Moreover, $h(T^*) = 2^k\, h(T)$ holds for each element $T \in \mathcal{T}_k$. From (6) we conclude that

$$h(T^*) \leq C\, h(\hat{T}), \qquad T \in \mathcal{T}_k,\ k \geq 0 .$$

Since the first vertex of every simplex T^* is the origin, it follows that all simplices T^* are contained in a closed ball B of radius $C\, h(\hat{T})$ around the origin. Moreover, from the fact that the vertices of each element $T \in \mathcal{T}_k$ belong $\mathcal{Z}_{n,k}$ it follows that the vertices of the corresponding element T^* belong $\mathcal{Z}_{n,0}$. The number of points in the set $\mathcal{Z}_{n,0} \cap B$ depends on $\delta(\hat{T})$ but is finite. Hence the number of possible elements T^* with vertices in this set is also finite. Since for any simplex $T \in \mathcal{T}_k$, $k \in \mathbb{N}_0$, the corresponding element T^*, defined by (7), is congruent to T, we conclude that the number of congruence classes in the hierarchy $(\mathcal{T}_k)_{k \in \mathbb{N}_0}$ must be finite. □

3. Survey of grid refinement algorithms

Before we discuss and analyze Freudenthal's algorithm, we first give a short overview of existing refinement methods in two, three, and more space dimensions. According to the topic treated in this paper, the emphasis here is on recursive refinement and stability. One main goal in this section is to make clear which of these methods can be considered to be regular and affine invariant because these are the only assumptions for the main theorem in Sect. 5.2. As mentioned in the introduction, (adaptive) refinement algorithms can in general be divided into two classes: *red/green refinement algorithms*

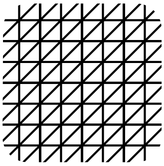

Fig. 2. Red/green refinement

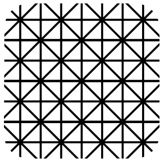

Fig. 3. Bisection

and *bisection methods*. In order to get a first impression of this classification, consider the triangulation snapshots in Fig. 2 and Fig. 3.

As a rule of thumb one can say that red/green refinement in 2D produces triangulations having a local structure as in Fig. 2 (at least in the uniformly refined regions and after some suitable affine transformation), while bisection methods usually lead to triangulations as in Fig. 3. In the higher-dimensional case a similar statement holds for the 2D-faces of refined simplices.

3.1. Red/green refinement algorithms

The most characteristic feature of red/green refinement is the strict distinction between refinement of marked simplices, selected for example by a suitable error estimator, and refinements that are applied for the sake of consistency preservation only. The latter procedure is usually called the *green closure*. In general, a red/green refinement algorithm consists of the following three components:

- A stable *regular* refinement strategy for the marked simplices,
- a set of additional *irregular* refinement rules for the green closure,
- a *global algorithm* combining regular and irregular refinements in such a way that the resulting triangulations are consistent and stable.

Prototype for this kind of method is Bank's well-known 2D refinement algorithm [2]. His method can be summarized as follows: Marked triangles are subdivided into four subtriangles by connecting the three edge midpoints, cf. Fig. 4. The four subtriangles have the same volume and are congruent to the original one. Hence, this red refinement strategy is stable. In order to preserve consistency, bisection is applied to triangles with one refined edge. Triangles with two or three refined edges are refined regularly. Elements resulting from bisection are not refined further but may be replaced by a regular refinement. Hence, stability is guaranteed even in the case of adaptive refinements.

The stable refinement of tetrahedral grids is more complex. In contrast to the 2D case, a tetrahedron can in general not be subdivided into eight (or 2^n) subtetrahedra of the same congruence class. Nevertheless, Bank's

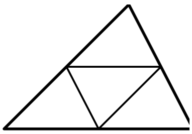

Fig. 4. Red refinement in 2D

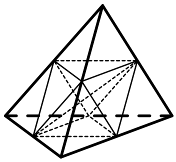

Fig. 5. Red refinement in 3D

red refinements have a canonical generalization in three dimensions which was found independently by Zhang [33] and the author [6]. The resulting 3D refinement strategy is illustrated in Fig. 5. It subdivides any given tetrahedron into eight subtetrahedra of equal volume. In general, however, only four of these subtetrahedra are congruent to their father.

After cutting off four subtetrahedra at the corners, the remaining octahedron can be subdivided further in three different ways each corresponding to one of three possible interior diagonals. In subsequent refinement steps, this interior diagonal has to be chosen carefully in order to satisfy the stability condition. Indeed it was shown by Zhang that always selecting the longest diagonal will in general lead to non-stable triangulations [33].

In [6] we proposed a simple algorithm which arranges the 3D red refinement in such a way that recursive application to any initial tetrahedron yields elements of at most three congruence classes, no matter how many subsequent refinements are performed. In the meantime this algorithm forms the backbone of various adaptive refinement algorithms for tetrahedral grids [4–6]. Let $T = [\, x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)} \,]$ be the tetrahedron to be refined and denote by $x^{(ij)}$ the edge midnode between $x^{(i)}$ and $x^{(j)}$. Then the algorithm reads as follows:

Algorithm *RedRefinement3D(* $T = [\, x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)} \,]$ *)*

{

$$
\begin{aligned}
T_1 &:= [x^{(0)}, x^{(01)}, x^{(02)}, x^{(03)}]; & T_5 &:= [x^{(01)}, x^{(02)}, x^{(03)}, x^{(13)}];\\
T_2 &:= [x^{(01)}, x^{(1)}, x^{(12)}, x^{(13)}]; & T_6 &:= [x^{(01)}, x^{(02)}, x^{(12)}, x^{(13)}];\\
T_3 &:= [x^{(02)}, x^{(12)}, x^{(2)}, x^{(23)}]; & T_7 &:= [x^{(02)}, x^{(03)}, x^{(13)}, x^{(23)}];\\
T_4 &:= [x^{(03)}, x^{(13)}, x^{(23)}, x^{(3)}]; & T_8 &:= [x^{(02)}, x^{(12)}, x^{(13)}, x^{(23)}];
\end{aligned}
$$

}

The diagonal chosen runs from $x^{(02)}$ to $x^{(13)}$ and hence is given implicitly by the vertex ordering of T. The vertex ordering of the sons T_i, $1 \le i \le 8$, is crucial for the stability of the algorithm. If we exchange the order of the vertices $x^{(01)}, x^{(02)}$ in T_1, for example, the resulting algorithm will in general not be stable anymore. By construction, algorithm RedRefinement3D is affine invariant.

Zhang also investigated a second strategy selecting always the shortest diagonal for refinement [33]. This so called *shortest-interior-edge* strategy turned out to be equivalent to Algorithm RedRefinement3D as long as it is applied to initial elements with non-obtuse faces and a suitable vertex ordering. For elements with at least one obtuse triangle, however, the stability of his method has not been proved yet. Note that shortest-interior-edge refinement is not affine invariant.

Bank's red refinements in 2D and Algorithm RedRefinement3D turned out to be special cases of a refinement algorithm which has been published already in 1942 by H. Freudenthal [13]. This method applies to simplices of arbitrary dimension. The method is regular and affine invariant. Moreover, it can be extended to a fully adaptive red/green refinement algorithm, cf. [7]. Since Freudenthal's algorithm will be discussed in detail in Sect. 4, we proceed at this point and turn to the second class of refinement algorithms.

3.2. Bisection methods

This second class contains those algorithms using only simplex bisection for subdivision. The main advantage of these methods is that bisection may result in more local refinements compared to regular subdivision because elements are divided into only two instead of 2^n sons. Therefore these methods may be preferable if n is large and a subdivision into 2^n sons yields a too strong refinement. On the other hand, as we shall see later, bisection methods also have some clear disadvantages.

Bisection methods differ by the way the bisection edge is selected and consistency is preserved. In 2D there are two basic techniques. The first one goes back to Mitchell and is referred to as *newest-vertex-bisection* [25]. In this method always the triangle edge opposite to the vertex created last (*newest vertex*) is used for refinement. As can be seen from Fig. 6, recursive application of this method generates at most four congruence classes for any initial triangle. Hence, the method is stable. Moreover, it is affine invariant. Consistency is obtained by a recursive process refining simultaneously pairs of triangles with a common bisection edge. If the bisection edges of the initial triangulation are chosen properly, this recursive procedure terminates after a finite number of steps. As Mitchell was able to show this can be done without further restrictions on $\mathcal{T}_0$. Consequently, newest-vertex-bisection in 2D can be applied to any consistent initial triangulation.

The second approach, due to Rivara, is based on using the triangle's longest edge for refinement [27]. The stability of this method follows from the fact that *longest-edge-bisection* changes into newest-vertex-bisection after a finite number of refinement steps [29]. Hence, the number of congruence classes is also finite but in general it depends on $\delta(\mathcal{T}_0)$. Consistency

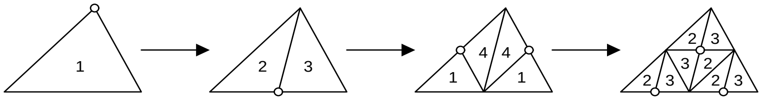

Fig. 6. Newest-Vertex-Bisection

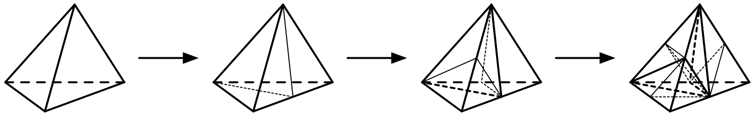

Fig. 7. Subsequent bisections in 3D

is achieved by a recursive process similar to the one of Mitchell. Rivara's method applies to any consistent triangulation. In contrast to newest-vertex-bisection it is not affine invariant.

Both the method of Rivara and the one of Mitchell have been generalized to three and even higher dimensions. For the longest-edge-bisection method this was done by Rivara [28]. Indeed, n-dimensional longest-edge-bisection can be applied to any consistent triangulation. The stability of this method, however, has not been proved yet for $n > 2$.

3D versions of Mitchell's algorithm have been developed by Bänsch [3], Maubach [22], Liu Joe [20], and Arnold et al. [1]. These methods are stable and apply to any consistent triangulation. In fact, they are all equivalent and affine invariant when applied recursively to a single element. In this case at most 36 congruence classes can appear, as was shown in [1,24]. From these papers we also know that this bound is sharp. Fig. 7 illustrates the result of three subsequent bisection steps. Here "subsequent" means that each but the first bisection step is applied to all elements generated in the previous step.

Generalizations of Mitchell's method to n dimensions have been developed by Maubach [23,24] and Traxler [31]. Both methods are equivalent and affine invariant when applied recursively to a single element. In this case at most $n \cdot n! \cdot 2^{n-2}$ congruence classes are generated, and this bound is sharp, cf. [1,24]. In order to satisfy the consistency condition, however, both algorithms make some restrictive assumptions on the initial triangulation $\mathcal{T}_0$. The method of Traxler, for example, requires the number of elements sharing any interior $(n–2)$-subsimplex of $\mathcal{T}_0$ to be even. Although this condition is easily checked in practice, it is not clear how to modify $\mathcal{T}_0$ if it is violated.

Remark 3.1 As illustrated in Fig. 6 and Fig. 7, n subsequent bisection steps (newest-vertex-bisection or one of its generalizations) can be regarded as one single regular refinement step. Hence, it is possible to compare bisection methods with regular refinement schemes, for example with respect to the number of congruence classes, cf. Theorem 5.1. It follows from the results in [1,24] that recursive application of such a regular bisection strategy produces at most $n! \cdot 2^{n-2}$ congruence classes for any initial element. Moreover, this bound can be shown to be sharp.

We briefly summarize the main points of this section. Bank's red refinement strategy in 2D, Algorithm RedRefinement3D, and, more general, Freudenthal's algorithm in n dimensions, are regular, affine invariant, and stable. If n subsequent bisection steps are considered as one refinement, then Mitchell's newest-vertex-bisection and its higher-dimensional variants are also regular, affine invariant, and stable. The higher-dimensional variants of Mitchell's method, however, can not be applied to arbitrary consistent triangulations if $n > 3$. Zhang's shortest-interior-edge strategy in 3D is regular but not affine invariant. Rivara's longest-edge-bisection in two and higher dimensions is neither regular nor affine invariant. Both methods apply to any consistent triangulation. Stability, however, has been proved for 2D longest-edge-bisection only.

4. Freudenthal's algorithm

Freudenthal's algorithm can be considered to be the canonical generalization of Bank's red refinement strategy and of Algorithm RedRefinement3D to n space dimensions. Though published already in 1942 [13], it seems that Freudenthal's algorithm is hardly known in the field of numerical methods for partial differential equations. Maybe this is explained by the fact that Freudenthal's paper was not motivated by finite element or multigrid methods. In effect, Freudenthal mentioned in [13] that his main motivation was a question of Brouwer concerning the construction of nested and stable hierarchies of triangulations. The intention of Brouwer, however, is not clear from [13]. Freudenthal only mentioned that his method may be of use in analysis and in the limit field between analysis and topology. Todd assumed in [30] that Brouwer intended to apply the method for the computation of fixed points. In fact, this seems to be the field of research where Freudenthal's algorithm is best known today.

4.1. The basic idea: Kuhn's triangulation

The basic idea of Freudenthal's algorithm can be explained best by considering the 2D case. Fig. 8 shows different triangulations of the unit square. The triangulation in the middle of the lower row can be obtained in two ways: by subdividing the square into two triangles each of which is then refined red or by subdividing the square into four equally sized subsquares each of which is then divided into two triangles. Here it is important that the diagonal subdividing each square has the same orientation as the diagonal

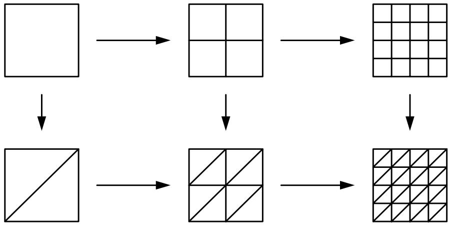

Fig. 8. Unit square triangulations

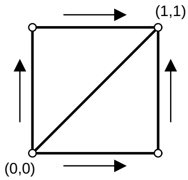

Fig. 9. How to reach vertex (1,1) ?

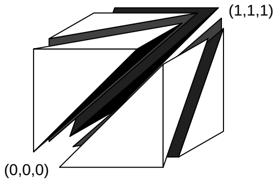

Fig. 10. Kuhn's triangulation in 3D

of the square at the lower left corner. Both procedures can be repeated to obtain equivalent triangulations of arbitrary refinement depth.

In order to derive from this observation a refinement strategy for (n)-simplices, we first have to construct a suitable triangulation of the n-dimensional unit cube $C := [\,0,1\,]^n$. Therefore, in Fig. 9, we consider the unit square triangulation once again. Starting from the origin, the vertices of the lower triangle are obtained by following first the edge in x- and then the edge in y-direction. In the same way the vertices of the upper triangle are obtained by going first in y- and then in x-direction. In both cases we end at the vertex $(1,1)^{\mathrm{T}}$.

In the n-dimensional case there are exactly $n!$ of such paths from the origin over the unit cube's edges to the vertex $(1,1,\ldots,1)^{\mathrm{T}}$, each one passing exactly $n+1$ vertices if start and endpoint are included. These vertices define an (n)-simplex and the set of all $n!$ simplices is in fact a triangulation of C. This follows from Lemma 4.1b) below. Due to [17], this triangulation is usually called *Kuhn's triangulation* of C. We note, however, that the same subdivision already appears in the paper of Freudenthal [13].

To be more precise, let $e^{(1)}, \ldots, e^{(n)}$ be the standard unit vectors of $\mathbb{R}^n$ and denote by S_n the group of permutations of $\{\,1,\ldots,n\,\}$. For $\pi \in S_n$, the simplex $T_\pi \;=\; [\,x_\pi^{(0)},\ldots,x_\pi^{(n)}\,]$ is defined by

$$(8)\quad x_\pi^{(0)} = (0,0,\ldots,0)^{\mathrm{T}}, \qquad x_\pi^{(j)} = x_\pi^{(j-1)} + e^{(\pi(j))}, \quad 1 \le j \le n.$$

The set $\mathcal{K}(C) = \{\,T_\pi \,|\, \pi \in S_n\,\}$ is called *Kuhn's triangulation* of C. For any affine transformation F, Kuhn's triangulation of $F(C)$ is defined by $\mathcal{K}(F(C)) := F(\mathcal{K}(C))$.

Kuhn's triangulation of the 3D cube is shown in Fig. 10. Some basic properties of $\mathcal{K}(C)$ are stated in the following Lemma.

Lemma 4.1 *Kuhn's triangulation $\mathcal{K}(C)$ has the following properties:*

a) $(0,0,\ldots,0)^{\mathrm{T}}$ *and* $(1,1,\ldots,1)^{\mathrm{T}}$ *are common vertices of all elements* T_π, $\pi \in S_n$.

b) *For each element* T_π, $\pi \in S_n$, *the following representation holds:*

(9) $$T_\pi \;\cong\; \{\, x \in C \mid 0 \le x_{\pi(n)} \le \;\cdots\; \le x_{\pi(1)} \le 1 \,\}.$$

c) $\mathcal{K}(C)$ *is a consistent triangulation of* C.

d) $\mathcal{K}(C)$ *is compatible with translation, i.e., for each vector* $v \in \{\, 0,1 \,\}^n$ *the union of* $\mathcal{K}(C)$ *and* $\mathcal{K}(v+C)$ *is a consistent triangulation of the set* $C \cup (v+C)$.

A rigorous proof of these results, in particular those in Lemma 4.1c) and d), is hard to find though these properties are already known for a long time and used by many authors, cf. [11,17,30]. A proof of Lemma 4.1, which is indeed rather technical, can be found in [8].

4.2. Refinement of Kuhn's triangulation

Figure 8 suggests the next step towards Freudenthal's algorithm: We subdivide C into 2^n subcubes

$$C_v \;:=\; v \;+\; \tfrac{1}{2}\, C\,, \qquad v \in \{\, 0, \tfrac{1}{2} \,\}^n\,.$$

Then $\mathcal{K}(C_v)$ consists of the tetrahedra $T_{v,\pi} := v + \frac{1}{2}\, T_\pi$, $\pi \in S_n$. The union of all triangulations $\mathcal{K}(C_v)$, $v \in \{\, 0, \frac{1}{2} \,\}^n$, yields

(10) $$\mathcal{T}_1 \;:=\; \Big\{\, T_{v,\pi} \;\Big|\; v \in \{\, 0, \tfrac{1}{2} \,\}^n,\; \pi \in S_n \,\Big\}\,.$$

Clearly, $\mathcal{T}_1$ is a triangulation of C. The consistency of $\mathcal{T}_1$ follows from Lemma 4.1c), d). We shall now show that $\mathcal{T}_1$ is in fact a refinement of $\mathcal{T}_0 := \mathcal{K}(C)$. The proof of the following Lemma represents the main part of Freudenthal's paper.

Lemma 4.2 *The triangulation $\mathcal{T}_1$ in (10) is a refinement of $\mathcal{K}(C)$.*

Proof. Given any $v \in \{\, 0, \frac{1}{2} \,\}^n$ and $\pi \in S_n$, we will show that there exists a unique permutation $\hat\pi = \hat\pi(v,\pi)$ such that $T_{v,\pi} \subset T_{\hat\pi}$. To this end, let $0 \le k \le n$ be the number of entries v_i of v such that $v_i = 1/2$. It follows that there are k unique indices $i_1,\ldots,i_k \in \{\, 1,\ldots,n \,\}$ satisfying

(11) $$1 \le i_1 < \cdots < i_k \le n, \qquad v_{\pi(i_1)} = \cdots = v_{\pi(i_k)} = \tfrac{1}{2},$$

and the remaining $n-k$ indices $i_{k+1},\dots,i_n \in \{1,\dots,n\} \setminus \{i_1,\dots,i_k\}$ can be ordered such that

$$1 \le i_{k+1} < \dots < i_n \le n, \qquad v_{\pi(i_{k+1})} = \dots = v_{\pi(i_n)} = 0. \tag{12}$$

Here and in the sequel, for the cases $k = 0$ and $k = n$, we skip those parts of the corresponding (in)equalities that make no sense. We now define a permutation $\hat{\pi}$ by $\hat{\pi}(j) = \pi(i_j)$, $1 \le j \le n$. From the equality results in (11), (12) we conclude

$$v_{\hat{\pi}(1)} = \dots = v_{\hat{\pi}(k)} = \tfrac{1}{2}, \qquad v_{\hat{\pi}(k+1)} = \dots = v_{\hat{\pi}(n)} = 0. \tag{13}$$

Now assume $x \in \frac{1}{2}\,T_\pi$ or, equivalently, $0 \le x_{\pi(n)} \le \dots \le x_{\pi(1)} \le \frac{1}{2}$, cf. Lemma 4.1b). Using the inequality results in (11), (12), we obtain

$$0 \le x_{\hat{\pi}(k)} \le \dots \le x_{\hat{\pi}(1)} \le \tfrac{1}{2}, \quad 0 \le x_{\hat{\pi}(n)} \le \dots \le x_{\hat{\pi}(k+1)} \le \tfrac{1}{2}. \tag{14}$$

Combination of (13) and (14) yields

$$\begin{array}{cccccccc} 0 & \le & v_{\hat{\pi}(n)} + x_{\hat{\pi}(n)} & \le & \cdots & \le & v_{\hat{\pi}(k+1)} + x_{\hat{\pi}(k+1)} & \le \tfrac{1}{2} \\ & \le & v_{\hat{\pi}(k)} + x_{\hat{\pi}(k)} & \le & \cdots & \le & v_{\hat{\pi}(1)} + x_{\hat{\pi}(1)} & \le 1\,, \end{array}$$

or, equivalently, $v + x \in T_{\hat{\pi}}$. Hence we have proved $T_{v,\pi} = v + \frac{1}{2}\,T_\pi \subset T_{\hat{\pi}}$. By construction, the vertices of $T_{v,\pi}$ are either vertices or edge midpoints of $T_{\hat{\pi}}$. Since $T_{v,\pi} \in \mathcal{T}_1$ was chosen arbitrarily, we conclude that $\mathcal{T}_1$ is a refinement of $\mathcal{K}(C)$. $\square$

4.3. Freudenthal's algorithm

The proof of Lemma 4.2 already contains the main idea underlying Freudenthal's algorithm. In this section we want to present this algorithm in a somewhat more modern form compared to [13]. To this end, for any simplex $T_{\hat{\pi}} \in \mathcal{K}(C)$, let $\mathcal{R}(T_{\hat{\pi}}) = \bigcup \{ T_{v,\pi} \in \mathcal{T}_1 \,|\, T_{v,\pi} \subset T_{\hat{\pi}} \}$ be the refinement induced by $\mathcal{T}_1$. It then follows from the proof of Lemma 4.2 that $\mathcal{R}(T_{\hat{\pi}})$ contains precisely those simplices $T_{v,\pi} \in \mathcal{T}_1$ for which there is a number $0 \le k \le n$ such that the pair v, π satisfies (13) and

$$(\pi^{-1} \circ \hat{\pi})(1) \quad < \dots < \; (\pi^{-1} \circ \hat{\pi})(k)\,, \tag{15a}$$

$$(\pi^{-1} \circ \hat{\pi})(k+1) < \dots < \; (\pi^{-1} \circ \hat{\pi})(n)\,. \tag{15b}$$

In particular $\mathcal{R}(T_{\pi_{\mathrm{id}}})$, $\pi_{\mathrm{id}} = (1, 2, \dots, n)$, consists of all simplices $T_{v,\pi}$ such that v takes the form

$$v = (\underbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}_{k}, \underbrace{0, \dots, 0}_{n-k})^{\mathrm{T}} \tag{16}$$

for some $0 \le k \le n$, and π satisfies

$$\pi^{-1}(1) < \cdots < \pi^{-1}(k)\,, \qquad \pi^{-1}(k+1) < \cdots < \pi^{-1}(n)\,. \tag{17}$$

We will now show that $\mathcal{T}_1$ can be generated by an affine invariant refinement strategy. To this end, let $F_{\hat\pi} : x \mapsto P_{\hat\pi} x$ be the affine transformation defined by the *permutation matrix* $P_{\hat\pi} = (\,\delta_{i,\hat\pi(j)}\,)_{i,j=1}^{n}$, and this for each $\hat\pi \in S_n$. A simple calculation shows $T_{\hat\pi} = F_{\hat\pi}(T_{\pi_{\rm id}})$, $\hat\pi \in S_n$. Now fix any $\hat\pi \in S_n$ and consider an arbitrary element $T_{v,\pi} \in \mathcal{R}(T_{\pi_{\rm id}})$. Using $\hat v = P_{\hat\pi} v$ and $\pi^{-1} = (\hat\pi \circ \pi)^{-1} \circ \hat\pi$, (16), (17) can be written as

$$\hat v_{\hat\pi(1)} = \cdots = \hat v_{\hat\pi(k)} = \tfrac{1}{2}\,, \qquad \hat v_{\hat\pi(k+1)} = \cdots = \hat v_{\hat\pi(n)} = 0\,,$$

and

$$\begin{array}{lcccl} ((\hat\pi \circ \pi)^{-1} \circ \hat\pi)(1) & < & \cdots & < & ((\hat\pi \circ \pi)^{-1} \circ \hat\pi)(k)\,, \\ ((\hat\pi \circ \pi)^{-1} \circ \hat\pi)(k+1) & < & \cdots & < & ((\hat\pi \circ \pi)^{-1} \circ \hat\pi)(n)\,. \end{array}$$

From (13), (15) we conclude that $T_{v,\pi} \in \mathcal{R}(T_{\pi_{\rm id}})$ holds if and only if $T_{\hat v,\hat\pi\circ\pi} \in \mathcal{R}(T_{\hat\pi})$. Using the relation $P_{\pi'\circ\pi} = P_{\pi'}\,P_{\pi}$, it is not difficult to show that $T_{\hat v,\hat\pi\circ\pi} = F_{\hat\pi}(T_{v,\pi})$. It follows that $\mathcal{R}(T_{\hat\pi}) = F_{\hat\pi}\,\mathcal{R}(T_{\pi_{\rm id}})$ holds for $\hat\pi \in S_n$. Hence, $\mathcal{T}_1$ can be generated from $\mathcal{K}(C)$ by an affine invariant refinement strategy.

To find such a strategy, we reformulate (16), (17) in an affine invariant way. Let v, π be such that (16), (17) hold. It follows from (8) that the first vertex of $T_{v,\pi} = v + \frac{1}{2}\,T_\pi$ is given by $v^{(0)} = v = \frac{1}{2}\,(x_{\pi_{\rm id}}^{(0)} + x_{\pi_{\rm id}}^{(k)})$. The remaining vertices $v^{(1)}, \ldots, v^{(n)}$ can be expressed in terms of the vertices $x_{\pi_{\rm id}}^{(j)}$ as follows:

$$\begin{aligned} v^{(\ell)} &= v + \tfrac{1}{2}\,x_{\pi}^{(\ell)} \;=\; v + \tfrac{1}{2}\sum_{j=1}^{\ell} e^{(\pi(j))} \\ &= v^{(\ell-1)} + \tfrac{1}{2}\,e^{(\pi(\ell))} \;=\; v^{(\ell-1)} + \tfrac{1}{2}\,(x_{\pi_{\rm id}}^{(\pi(\ell))} - x_{\pi_{\rm id}}^{(\pi(\ell)-1)})\,. \end{aligned}$$

Replacing $x_{\pi_{\rm id}}^{(0)}, \ldots, x_{\pi_{\rm id}}^{(n)}$ by the vertices of an arbitrary simplex T we end up with Freudenthal's algorithm:

Algorithm *RedRefinementND(* $T = [\, x^{(0)}, \ldots, x^{(n)} \,]$ *)*
{
for ($0 \le k \le n$) do *(1)*
{
$v^{(0)} := \frac{1}{2}\,(x^{(0)} + x^{(k)})$; *(2)*
for ($\pi \in S_n$ **) do** *(3)*
if ($\pi^{-1}(1) < \cdots < \pi^{-1}(k)$ **)** *(4)*
and ($\pi^{-1}(k+1) < \cdots < \pi^{-1}(n)$ **) then** *(5)*
{
for ($1 \le \ell \le n$ **) do** *(6)*
$v^{(\ell)} := v^{(\ell-1)} + \frac{1}{2}(x^{(\pi(\ell))} - x^{((\pi(\ell)-1)})$; *(7)*
$T_{v^{(0)},\pi} := [\, v^{(0)}, \ldots, v^{(n)} \,]$; *(8)*
}
}
}

By construction, Algorithm RedRefinementND is affine invariant. Note that for each k there are precisely $\binom{n}{k}$ permutations π satisfying the if-condition in line (4), (5). It follows that 2^n sons are generated in line (8) and hence, Algorithm RedRefinementND is a regular refinement strategy. The stability condition is proved in the following Theorem which also provides an upper bound for the number of congruence classes. This upper bound is in fact the main new result of this section.

Theorem 4.1 *For any non-degenerate simplex $T \subset \mathbb{R}^n$, recursive application of Algorithm RedRefinementND results in a stable hierarchy of consistent triangulations of T. Moreover, the number of generated congruence classes is at most $n!/2$.*

Proof. Let $(\mathcal{T}_k)_{k\in\mathbb{N}_0}$ be the hierarchy of triangulations generated by recursive application of Algorithm RedRefinementND to $\mathcal{T}_0 := \mathcal{K}(C)$. Clearly, $\mathcal{T}_1$ has the representation (10). By induction we obtain

$$\mathcal{T}_k \;=\; \{\; T_{v,\pi} := v + 2^{-k}\, T_\pi \mid v \in \mathcal{Z}_{n,k};\; \pi \in S_n \;\}\,, \qquad k \ge 0\,,$$

where $\mathcal{Z}_{n,k} := \{\, v \in [0,1)^n \mid 2^k\, v_i \in \mathbb{Z},\; 1 \le i \le n \,\}$ is a subset of the vertices of $\mathcal{T}_k$. From Lemma 4.1c), d) we conclude that $\mathcal{T}_k$ is consistent for all $k \ge 0$.

Now let $T = [\, x^{(0)}, \ldots, x^{(n)} \,]$ be any non-degenerate simplex and let $(\mathcal{T}_k(T))_{k\in\mathbb{N}_0}$ be the hierarchy of triangulations generated by recursive application of Algorithm RedRefinementND to T. The affine invariance of Algorithm RedRefinementND leads us to the representation

$$\mathcal{T}_k(T) \;=\; \{\; F(T') \mid T' \in \mathcal{T}_k,\; T' \subset T_{\pi_{\mathrm{id}}} \;\}\,, \qquad k \ge 0\,,$$

where $F : T_{\pi_{\rm id}} \longrightarrow T$ is the unique affine transformation between $T_{\pi_{\rm id}}$ and T. The consistency of $\mathcal{T}_k$ implies the consistency of $\mathcal{T}_k(T)$ for $k \geq 0$. Moreover, since any element $T' \in \mathcal{T}_k$ is of the form $T' = v + 2^{-k}T_\pi$ for some $v \in \mathcal{Z}_{n,k}$, $\pi \in S_n$, it follows that $F(T')$ is congruent with $F(T_\pi)$ for some $\pi \in S_n$. Hence, the number of congruence classes in the hierarchy $(\mathcal{T}_k(T))_{k\in\mathbb{N}_0}$ is at most $n!$. This bound can be improved: Let $\pi, \pi' \in S_n$ be permutations such that $\pi'(j) = \pi(n+1-j)$, $j = 1, \ldots, n$. Using the vector $\mathbb{1} := (\,1, 1, \ldots, 1\,)^{\rm T}$, the vertices of T_π, $T_{\pi'}$ are related by

$$x_{\pi'}^{(i)} = \sum_{j=1}^{i} e^{(\pi'(j))} = \sum_{j=1}^{i} e^{(\pi(n+1-j))} = \mathbb{1} - \sum_{j=1}^{n-i} e^{(\pi(j))} = \mathbb{1} - x_{\pi}^{(n-i)}$$

for $0 \leq i \leq n$. This implies $T_{\pi'} \cong \mathbb{1} - T_\pi$. It follows that $F(T_{\pi'})$ is congruent to $F(T_\pi)$. Hence, the number of congruence classes in $(\mathcal{T}_k(T))_{k\in\mathbb{N}_0}$ is bounded by $n!/2$. This implies stability. $\square$

Remark 4.1 Condition (17) means that $\pi = (\,\pi(1), \ldots, \pi(n)\,)$ contains the numbers $1, 2, \ldots, k$ and the numbers $k+1, \ldots, n$ in their natural order, that is, 1 appears somewhere on the left of 2, 2 appears somewhere on the left of 3, and so on, up to $k-1$ which appears somewhere on the left of k, and likewise for $k+1, \ldots, n$. If, for example, $n=3$ and $k=1$, then condition (17) is satisfied for all permutations $\pi \in \{\,(1,2,3),\ (2,1,3),\ (2,3,1)\,\}$.

Remark 4.2 One may easily verify that Algorithm RedRefinementND for $n=3$ is equivalent to Algorithm RedRefinement3D.

4.4. Global refinements

In the previous subsection we have seen that recursive application of Freudenthal's algorithm to a single simplex T produces consistent and stable triangulations of T. In the 3D case Freudenthal's algorithm can in fact be applied to any consistent triangulation, due to the symmetric subdivision of triangular faces, cf. Fig. 5. In higher dimensions, however, the refinement of adjacent simplices may give rise to consistency problems at their common lower-dimensional subsimplex. Nevertheless, as the following theorem shows, Freudenthal's algorithm can be successfully applied to any *consistently numbered* initial triangulation, cf. Sect. 2.3. The result appears already in Freudenthal's paper [13] but he did not prove it. For a proof of Theorem 4.2 we refer to [8].

Theorem 4.2 *Let $\mathcal{T}_0$ be a consistently numbered triangulation in $\mathbb{R}^n$. Then the triangulations $\mathcal{T}_k$, $k > 0$, obtained by recursive application of Algorithm RedRefinementND to $\mathcal{T}_0$, are consistently numbered as well.*

Note that, by definition, “consistently numbered” implies consistency. Also note that any consistent triangulation can be made consistently numbered by fitting the local vertex ordering of each element to an arbitrary global numbering of the vertices of the triangulation. Hence, in contrast to the bisection methods mentioned in Sect. 3.2 Freudenthal’s algorithm applies to *any* consistent initial triangulation. Note, however, that the basic form of Freudenthal’s algorithm can be used to produce *uniform* refinements only. In order to obtain a fully *adaptive* refinement algorithm it has to be combined with additional irregular refinement rules for the green closure. This will be the subject of a forthcoming paper, cf. [7].

5. The optimal number of congruence classes

Theorem 4.1 shows that recursive application of Freudenthal’s algorithm to any initial element T produces simplices of at most $n!/2$ congruence classes. In special cases, of course, less than $n!/2$ congruence classes may appear. All elements obtained by subsequent refinement of Kuhn’s triangulation $\mathcal{K}(C)$, for example, belong to the same congruence class. Sufficient conditions for tetrahedra having this property can be found in [14]. On the other hand, there are several refinement schemes producing significantly more than $n!/2$ congruence classes. For the bisection methods of Maubach and Traxler, for example, it is known that these may generate up to $n! \cdot 2^{n-2}$ congruence classes for general initial elements if n subsequent bisection steps are regarded as a single regular refinement step, cf. Remark 3.1.

We emphasize that the number of congruence classes is of great practical importance. In many applications there are data which depend on the element’s congruence class and refinement level only. These applications can often be considerably accelerated if such data are calculated and stored only once. The assembling of stiffness matrices in finite element computations is a typical example. Also note that, if the number of congruence classes is small, one may expect better shaped triangulations in terms of $\delta(\mathcal{T})$. Hence, it is beneficial to have the number of congruence classes as small as possible.

At this point the following question arises: Is it possible to construct a refinement algorithm which generates less than $n!/2$ congruence classes for any initial element T ? We will show in the remainder of this section that the answer is negative – at least if we restrict our considerations to refinement strategies which are regular and affine invariant. More precisely: we will prove that recursive application of *any* regular and affine invariant refinement strategy must produce *at least* $n!/2$ congruence classes *for almost all* (n)-simplices T.

The formulation “for almost all” allows exceptions as the application of Freudenthal’s algorithm to the elements of $\mathcal{K}(C)$. The precise meaning of this formulation is the following: Every (n)-simplex represents a point in the Euclidean space $\mathbb{R}^{n\,(n+1)}$. A certain assertion is said to hold *for almost all simplices* $T \subset \mathbb{R}^n$ if the set of simplices violating this assertion corresponds to a set of Lebesgue measure zero in $\mathbb{R}^{n\,(n+1)}$. Note that the set of degenerate simplices in $\mathbb{R}^n$ corresponds to a set of measure zero in $\mathbb{R}^{n\,(n+1)}$. Hence, the formulations “for almost all simplices” and “for almost all non-degenerate simplices” are equivalent.

5.1. A special property of the polar decomposition

The proof of our theorem is based on a certain regularity property of the polar decomposition. In this subsection we will formulate and prove this property. Here and in the sequel, for $n \in \mathbb{N}$ fixed, we use the following matrix set notation:

$$\begin{aligned} \mathcal{GL} &:= \{\, A \in \mathbb{R}^{n\times n} \mid \det A \neq 0 \,\}\,, \\ \mathcal{O} &:= \{\, Q \in \mathbb{R}^{n\times n} \mid Q^{\mathrm{T}} Q = I \,\}\,, \\ \mathcal{S} &:= \{\, A \in \mathbb{R}^{n\times n} \mid A = A^{\mathrm{T}} \,\}\,, \\ \mathcal{S}_+ &:= \{\, A \in \mathcal{S} \mid x^{\mathrm{T}} A x > 0,\ x \in \mathbb{R}^n \setminus \{0\} \,\}\,. \end{aligned}$$

The set $\mathcal{GL}$ is usually referred to as *general linear group*. It is known, cf. [16], that each element A of this group has a unique *polar decomposition* of the form

$$A \;=\; Q X^{1/2}\,, \qquad Q \in \mathcal{O},\ X \in \mathcal{S}_+. \tag{18}$$

Clearly, the matrix X in (18) satisfies $X = A^{\mathrm{T}} A$. As we shall see later, this polar decomposition plays an important role in the proof of our theorem. In particular the following regularity property will be used:

Lemma 5.1 *Let $\mathcal{M} \subset \mathcal{S}$ be a proper subspace of dimension $k < \frac{n}{2}(n+1)$. Then the set*

$$\mathcal{B} \;:=\; \{\; A = Q X^{1/2} \mid Q \in \mathcal{O};\ X \in \mathcal{M} \cap \mathcal{S}_+ \;\} \tag{19}$$

is a set of Lebesgue measure zero in $\mathbb{R}^{n\times n}$.

Proof. This Lemma can be proved elegantly by using the concept of real manifolds, cf. [9,18].

It is known that $\mathcal{O}$ is a real manifold of dimension $\frac{n}{2}(n-1)$, cf. [9]. The set $\mathcal{S}_+$, which is an open subset of $\mathcal{S}$, can be regarded as a real manifold of dimension $\frac{n}{2}(n+1)$. It follows that the *product manifold* $\mathcal{O} \times \mathcal{S}_+$ has dimension n^2. Clearly, the open subset $\mathcal{GL} \subset \mathbb{R}^{n\times n}$ is also a real manifold of dimension n^2.

The mapping $X \mapsto X^{1/2}$ is the inverse of the C^∞-bijection $X \mapsto X^2$ on $\mathcal{S}_+$. From the inverse function theorem (cf. [12]) it follows that the mapping $X \mapsto X^{1/2}$ is a C^∞-bijection on $\mathcal{S}_+$, see [16]. Hence, the mapping $F : \mathcal{O} \times \mathcal{S}_+ \longrightarrow \mathcal{GL}$, defined by $F(Q,X) = Q\,X^{1/2}$, is C^∞, too. Moreover, from the existence and uniqueness of the polar decomposition for all matrices $A \in \mathcal{GL}$ it follows that F is one-to-one. We conclude that F is a C^∞-bijection from the real manifold $\mathcal{O} \times \mathcal{S}_+$ onto the real manifold $\mathcal{GL}$.

Now suppose that $\mathcal{M}$ is a proper subspace of dimension $k < \frac{n}{2}(n+1)$ in $\mathcal{S}$. Then for the set $\mathcal{B}$ from (19) we have $\mathcal{B} = F(\mathcal{O} \times (\mathcal{M} \cap \mathcal{S}_+))$. Hence, $\mathcal{B}$ is a *submanifold* of dimension $< n^2$ in $\mathcal{GL}$, cf. [9,18]. This implies that $\mathcal{B}$ is a set of Lebesgue measure zero in $\mathbb{R}^{n \times n}$. $\square$

5.2. The optimality result

We are now in a position to prove the main theorem of this paper:

Theorem 5.1 *Let $\mathcal{R}$ be a regular and affine invariant refinement strategy for simplices in $\mathbb{R}^n$. Then for almost all simplices $T \subset \mathbb{R}^n$, recursive application of $\mathcal{R}$ to T generates at least $n!/2$ congruence classes.*

Proof. Let again $T_{\pi_{\mathrm{id}}}$ be the reference simplex from Kuhn's triangulation $\mathcal{K}([\,0,1\,]^n)$ and let $(\mathcal{T}_k)_{k \in \mathbb{N}_0}$ be the hierarchy of triangulations obtained by recursive application of $\mathcal{R}$ to $T_{\pi_{\mathrm{id}}}$. For $k \geq 0$ fixed we divide the simplices $T \in \mathcal{T}_k$ into equivalence classes

$$[T] := \{\, T' \in \mathcal{T}_k \mid T' \cong v + T \text{ for some } v \in \mathbb{R}^n \,\}, \quad T \in \mathcal{T}_k.$$

A second set of equivalence classes can be defined by

$$[T]^\pm := \{\, T' \in \mathcal{T}_k \mid T' \cong v + T \text{ or } T' \cong v - T \text{ for some } v \in \mathbb{R}^n \,\}$$

for all $T \in \mathcal{T}_k$. The number of different equivalence classes $[T]$ and $[T]^\pm$, $T \in \mathcal{T}_k$, is denoted by μ_k, $\mu_k^\pm$. Clearly, $\mu_k \leq 2\,\mu_k^\pm$. Note that an arbitrary but fixed affine transformation F maps all simplices of the same *equivalence class* $[T]$ (or $[T]^\pm$) into the same *congruence class*. Also note that affine transformations can be regarded as points in $\mathbb{R}^n \times \mathbb{R}^{n \times n}$. Hence, the Lebesgue measure of a set of affine transformations is well defined. We are now going to prove the following assertions:

Assertion 1. *For k sufficiently large: $\mu_k \geq n!$ and hence, $\mu_k^\pm \geq n!/2$.*
Assertion 2. *Assume $T, T' \in \mathcal{T}_k$ such that $[T]^\pm \neq [T']^\pm$. Then the set of affine transformations F mapping T and T' into the same congruence class is a set of Lebesgue measure zero in $\mathbb{R}^n \times \mathbb{R}^{n \times n}$.*

It follows from these two assertions and from the finiteness of $\mu_k^\pm$ for each k that for sufficiently large k the elements $T \in \mathcal{T}_k$ are mapped to at least $n!/2$

congruence classes by almost all affine transformations F. Since $\mathcal{R}$ is affine invariant, we conclude that for almost all affine transformations F recursive application of R to $F(T_{\pi_{\rm id}})$ produces at least $n!/2$ congruence classes. We note that there is a one-to-one correspondence between affine transformations and non-degenerate simplices, which is in addition easily shown to be Lipschitz-continuous. This implies that a zero measure set of affine transformations corresponds to a zero measure set of simplices, cf. [32]. Hence, if Assertion 1 and 2 hold true, it follows that recursive application of $\mathcal{R}$ produces at least $n!/2$ congruence classes for almost all initial simplices $T \subset \mathbb{R}^n$.

Proof of Assertion 1. We choose $k \geq 0$ large enough such that some (n)-cube C with axis-parallel edges of length 2^{-k} is contained in $T_{\pi_{\rm id}}$. Then we fix an arbitrary element $T \in \mathcal{T}_k$. Intersection of C with the union of all simplices $T' \in [T]$ yields

$$C \cap [T] \;:=\; \bigcup_{T' \in [T]} (C \cap T')\,, \qquad T \in \mathcal{T}_k\,.$$

We are going to estimate the volume of $C \cap [T]$. Since different elements in $\mathcal{T}_k$ overlap at most at their boundaries, we can write

$$\mathrm{vol}(C \cap [T]) \;=\; \sum_{T' \in [T]} \mathrm{vol}(C \cap T')\,.$$

The vertices of $\mathcal{T}_k$ belong to the set of grid points

$$\mathcal{Z}_{n,k} \;:=\; \{\; v \in \mathbb{R}^n \mid 2^k v_i \in \mathbb{Z},\; 1 \leq i \leq n \;\}\,.$$

It follows that for any simplex $T' \in [T]$ there is a vector $v \in \mathcal{Z}_{n,k}$ such that $T' \cong v + T$. Hence we have the estimate

$$\mathrm{vol}(C \cap [T]) \;\leq\; \sum_{v \in \mathcal{Z}_{n,k}} \mathrm{vol}\Big(\, C \cap (v+T) \Big)\,.$$

The intersection of C and the translates of T is illustrated in Fig. 11. The shaded regions are parts of translates of T which have been clipped at the larger square's boundary. Of course, instead of T also C might be moved. Using $C \cap (v+T) = (-v+C) \cap T$ for $v \in \mathbb{R}^n$ we obtain

$$\begin{aligned}\sum_{v \in \mathcal{Z}_{n,k}} \mathrm{vol}\Big(\, C \cap (v+T) \Big) &= \sum_{v \in \mathcal{Z}_{n,k}} \mathrm{vol}\Big(\, (-v+C) \cap T \,\Big) \\ &= \sum_{v \in \mathcal{Z}_{n,k}} \mathrm{vol}\Big(\, (v+C) \cap T \Big).\end{aligned}$$

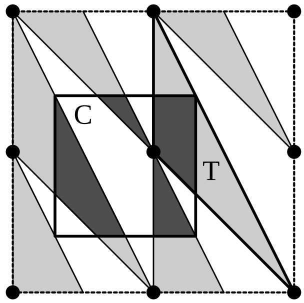

Fig. 11. Intersection of C and the translates of T

The translates $v+C$, $v \in \mathcal{Z}_{n,k}$, intersect pairwise at most at their boundaries. Hence we conclude

$$\sum_{v \in \mathcal{Z}_{n,k}} \mathrm{vol}\Big((v+C) \cap T \Big) = \mathrm{vol}\Big(T \cap \Big(\bigcup_{v \in \mathcal{Z}_{n,k}} (v+C) \Big) \Big)$$
$$= \mathrm{vol}(T \cap \mathrm{I\!R}^n) \; = \; \mathrm{vol}(T).$$

Since $T \in \mathcal{T}_k$ was chosen arbitrarily we end up with the estimate

$$\mathrm{vol}(C \cap [T]) \; \leq \; \mathrm{vol}(T)\,, \qquad T \in \mathcal{T}_k\,.$$

Now we use the assumption that $\mathcal{R}$ is a regular refinement strategy. It follows that all elements $T \in \mathcal{T}_k$ have the same volume

$$\mathrm{vol}(T) \; = \; 2^{-kn}\,\mathrm{vol}(\hat{T}) \; = \; \frac{2^{-kn}}{n!}\,.$$

By construction, $C \subset T_{\pi_{\mathrm{id}}}$ is completely covered by simplices $T \in \mathcal{T}_k$. We therefore obtain

$$2^{-kn} \; = \; \mathrm{vol}(C) \; = \; \sum_{[T]} \mathrm{vol}(C \cap [T]) \; \leq \; \sum_{[T]} \mathrm{vol}(T) \; = \; \mu_k\,\frac{2^{-kn}}{n!}\,,$$

that is $\mu_k \geq n!$ and hence, $\mu_k^{\pm} \geq n!/2$. This proves Assertion 1.

Proof of Assertion 2. We have to show that different equivalence classes $[T]^{\pm} \neq [T']^{\pm}$ are mapped into different congruence classes by almost all affine transformations F. To this end, consider two simplices $T, T' \in \mathcal{T}_k$ such that $[T]^{\pm} \neq [T']^{\pm}$. Let further $G : x \mapsto v + Ax$ be the unique affine transformation satisfying $T' = G(T)$. Then $\mathrm{vol}(T) = \mathrm{vol}(T')$ implies $|\det A| = 1$ and, due to $[T]^{\pm} \neq [T']^{\pm}$, neither A nor $-A$ can be the matrix of a renumbering of T, cf. Sect. 2.2. Now let $F : x \mapsto w + Bx$ be an arbitrary affine transformation. As the congruence class of $F(T)$ and of $F(T')$ is independent of w, we can assume $w = 0$. By definition, $F(T)$ and $F(T')$ are congruent to each other if and only if

$$BT' \;\cong\; z + cQBT \tag{20}$$

holds for some $c > 0$, $z \in \mathbb{R}^n$, and $Q \in \mathcal{O}$. From $\mathrm{vol}(BT) = \mathrm{vol}(BT')$ we conclude $c = 1$. Using a suitable renumbering $x \mapsto u + Ux$ of T', (20) can be rewritten as

$$B\,(u + UT') \;=\; z + QBT\,.$$

Inserting $T' = G(T)$ we obtain

$$B\,u + BUv \;+\; BUAT \;=\; z + QBT\,.$$

Now $\mathrm{vol}(T) > 0$ implies $z = B\,u + B\,Uv$ and hence, $B\,U\,A = Q\,B$. It follows that $F(T)$, $F(T')$ are congruent if and only if there exists some renumbering of T' with matrix U such that $B\,U\,A\,B^{-1}$ is orthogonal. Using the notation $\tilde{A} = UA$ it follows that

$$(B\,\tilde{A}\,B^{-1})^{\mathrm{T}} \;=\; (B\,\tilde{A}\,B^{-1})^{-1}$$

should hold or, equivalently

$$B^{\mathrm{T}}B \;=\; \tilde{A}^{\mathrm{T}}B^{\mathrm{T}}B\,\tilde{A}\,.$$

Thus, $F(T)$ and $F(T')$ are congruent to each other if and only if the polar decomposition of B, i.e. $B = VX^{1/2}$ with $V \in \mathcal{O}$, $X \in \mathcal{S}_+$, yields a symmetric positive definite matrix X satisfying

$$X \;=\; \tilde{A}^{\mathrm{T}}X\,\tilde{A} \tag{21}$$

with $\tilde{A} = UA$ and a suitable renumbering U. Let $\mathcal{M} \subset \mathcal{S}$ be the set of all matrices $X \in \mathcal{S}$ (not necessary $X \in \mathcal{S}_+$) satisfying (21). Clearly, $\mathcal{M}$ is a linear subspace of $\mathcal{S}$. A simple comparison of coefficients shows that $\mathcal{M} = \mathcal{S}$ implies $\tilde{A} = \pm I$ and hence $A = \pm U^{-1}$. In this case either A or $-A$ would be the matrix of a renumbering of T, in contradiction to $[T]^{\pm} \neq [T']^{\pm}$. Thus, $\mathcal{M}$ must be a proper subspace of $\mathcal{S}$ and we can apply Lemma 5.1. It follows that the set

$$\mathcal{B} \;:=\; \{\; B = VX^{1/2} \mid V \in \mathcal{O};\; X \in \mathcal{M} \cap \mathcal{S}_+ \;\}$$

is a set of Lebesgue measure zero in $\mathbb{R}^{n\times n}$. Hence, the set of all affine transformations F with transformation matrix $B \in \mathcal{B}$ is of Lebesgue measure zero in $\mathbb{R}^n \times \mathbb{R}^{n\times n}$. This, however, proves Assertion 2 because $B \in \mathcal{B}$ and the congruence of $F(T)$, $F(T')$ are equivalent. $\square$

Remark 5.1 It can be shown by induction that an (n)-cube C with axis-parallel edges of length 2^{-k} fits into $T_{\pi_{\text{id}}}$ if $k \geq n-1$. From this observation it follows that Freudenthal's algorithm stops generating new congruence classes after $n-1$ refinement steps.

Remark 5.2 Theorem 5.1 in combination with Theorem 4.1 shows that Freudenthal's algorithm is optimal with respect to the number of congruence classes. This, of course, does not imply that Freudenthal's algorithm is also optimal with respect to the measure of degeneracy of the resulting triangulations. Compared to the bisection algorithms of Maubach [23,24] and Traxler [31], however, we can make the following observation: The $n \cdot n! \cdot 2^{n-2}$ congruence classes (or $n! \cdot 2^{n-2}$ if n subsequent bisections are regarded as one regular refinement) generated by recursive bisection contain those $n!/2$ congruence classes generated by Freudenthal's algorithm. Hence, assuming identical initial triangulations, the measure of degeneracy of the triangulation hierarchy generated by Freudenthal's algorithm cannot be worse than the one obtained by recursive bisection.

Remark 5.3 Although Theorem 5.1 applies to affine invariant strategies only, it is in our opinion very unlikely that there exists a non-affine invariant recursive refinement strategy generating less than $n!/2$ congruence classes for any initial element. For Rivara's longest edge bisection [27,28] and for Zhang's shortest interior edge refinement [33] it is in fact known that in general the number of congruence classes depends on $\delta(T)$ and may be arbitrarily large. Hence, with respect to minimizing the number of congruence classes affine invariant methods are preferable.

The proof of Theorem 5.1 is based on the fact that the vertices of $\mathcal{T}_k$ belong to some rectangular grid $\mathcal{Z}_{n,k}$. The refinement assumption, however, is not used at all. In fact, we have proved the following more general result:

Corollary 5.1 *Let $\mathcal{T}$ be a triangulation of some region $\overline{\Omega} \subset \mathrm{IR}^n$. Assume there is some $h > 0$ such that (i) the vertices of $\mathcal{T}$ belong to a rectangular grid of meshwidth h, (ii) some (n)-cube with axis-parallel edges of length h fits into $\bar{\Omega}$, and (iii), all elements $T \in \mathcal{T}$ have the same volume vol$(T) = h^n/n!$. Then almost every affine transformation F results in a triangulation $F(\mathcal{T})$ with elements in at least $n!/2$ different congruence classes.*

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