Numer. Math. (1999) 84: 327–350
Digital Object Identifier (DOI) 10.1007/s002119900119

Numerische
Mathematik

Error analysis of the combination technique

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Received September 25, 1997 / Revised version received October 22, 1998 /
Published online September 7, 1999

Summary. The combination technique is a method to reduce the computational time in the numerical approximation of partial differential equations. In this paper, we present a new technique to analyze the convergence rate of the combination technique. This technique is applied to general second order elliptic differential equations in two dimensions. Furthermore, it is proved that the combination technique for Poisson's equation convergences in arbitrary dimensions.

Mathematics Subject Classification (1991): 65N15; 65N30

1. Introduction

The combination technique is a method for the discretization of partial differential equations. The important property of this technique is the high accuracy, while the computational time and the storage requirement are low.

Let us explain this for the 2-dimensional combination technique. For reasons of simplicity, let us restrict to the domain $\Omega =]0,1[^2$. Let $P_{h_x,h_y}(u)$ be the Ritz-Galerkin approximation of the solution u of a partial differential equation on a uniform grid of mesh size h_x in x-direction and h_y in y-direction. Then, the combination solution of depth n is defined by

$$u_h^{\mathrm{c}} = \sum_{i=1}^{n} P_{2^{-\mathrm{i}},2^{\mathrm{i}-n-1}}(u) - \sum_{i=1}^{n-1} P_{2^{-\mathrm{i}},2^{\mathrm{i}-n}}(u)$$

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where $h = 2^{-n}$.

The important observation is now that the approximation u_h^c of u is nearly as accurate as the approximation $P_{h,h}(u)$ (see [4]). But the calculation of u_h^c costs much less computational time. Let us precise this in case of a symmetric elliptic differential equation of second order and piecewise bilinear finite elements. Now, the computational time for the calculation of $P_{h,h}(u)$ is $O(h^{-2})$, if one uses a multigrid algorithm. But the computational time for the calculation of $u_h^c(u)$ is only $O(h^{-1}\log h^{-1})$. In three dimensions this difference of the computational times is even larger. Then, the computational time for the calculation of $P_{h,h,h}(u)$ is $O(h^{-3})$ while the computational time for the calculation of the 3D combination solution $u_h^c(u)$ is $O(h^{-1}(\log h^{-1})^2)$. A similar effect can be observed for the storage requirement.

Another important property of the combination technique is that it can be parallized in a very efficient way (see [3]). Here, we will analyze the combination technique on the unit cube for second order elliptic differential equations. But the combination technique can also be applied to nonlinear PDE on curvilinear bounded domains (see [5]).

Although, the implementation of the combination technique is not very difficult, the convergence of this method cannot be proved with standard arguments from finite element theory. In [1] it is proved that the combination technique converges pointwise of order $O(h^2 \log h^{-1})$ for Laplace's equation on the unit square. This proof uses a suitable asymptotic error expansion for the solutions $P_{h_x,h_y}(u)$ and Fourier analysis. Another proof of convergence in two dimensions is presented in [11] or [13] and uses Sobolev space techniques. The proof in [11] is restricted to second order elliptic differential equations in 2D, where the coefficients have to satisfy certain assumptions.

The aim of this paper is to present a new technique to prove the convergence of the combination technique. It can be applied to a much larger class of equations than the proofs in [1] or [11]. The advantages of this new technique are:

- In Sect. 3, we present a short proof for the convergence of the combination solution in the H^1-norm for Poisson's equation. This proof shows the main ideas how to prove the convergence of the combination solution in general. The new convergence proofs are shorter than the previous proofs.
- It is possible to prove the convergence of the combination solution for elliptic differential equations of second order in 2D under weaker assumptions to the coefficients than in [11]. Especially, we do not have to require that the normal derivative of some coefficients is zero at the boundary (see Theorem 5). This is significant, if one likes to prove the convergence of the combination solution on a curvilinear bounded do-

main. Then, it is necessary to divide the curvilinear bounded domain in several blocks and to transform each block onto the unit square. In case of one block and Poisson's equation the transformed equation on the unit square has variable coefficients. The normal derivatives of these coefficients are not zero at the boundary, in general. By the new technique presented in this paper, we can show for this case a convergence of order $O(h^2 \log h^{-1})$ in the L^2-norm. So, it is proved that the combination solution converges of order $O(h^2 \log h^{-1})$ in the L^2-norm on every curvilinear bounded domain which is a smooth transformation of the domain $]0,1[^2$. This is not possible by the results in [11].
Nevertheless, we have to assume that the variable coefficients are smooth. At least, the coefficients have to be in the space $W^1_\infty(\Omega)$ (see assumptions **A** to **C** in Sect. 4).

– By the new technique, we can prove the convergence of the combination technique in three or more dimensions (see Theorem 8). This is important, since the reduction of the computational time by the combination technique increases with the dimension of the problem. For simplicity, we restrict to Poisson's equation and the convergence in the H^1-norm in case of more than two dimensions. The problems of a generalization for variable coefficients are explained at the end of Sect. 3.

In this paper, we apply a superconvergence technique which is a modification of technique presented in [9,16] and [17]. More details about the superconvergence analysis for Ritz-Galerkin approximations can be found in [7,8,10,14] and references cited therein.

At the end of this introduction let us introduce the following notation:

Let d be the dimension of the space and $\Omega =]0,1[^d$. Let the indices α, β, γ be of the form

$$\alpha, \beta, \gamma \in \{0,1\}^d.$$

Furthermore denote $\mathbf{0} = (0,\ldots,0)$, $\mathbf{e} = (1,\ldots,1)$ and $e_i = (0,\ldots,0,1,0,\ldots,0)$. Let $\mathbf{H} = (h_1,\cdots,h_d)$ be mesh sizes $h_i = 2^{-k}$ with $k \in \mathbb{N}$. Let us write

$$\mathbf{H}^\alpha := h_1^{\alpha} \cdot \ldots \cdot h_d^{\alpha_d} \quad \text{and} \quad |\alpha| = \alpha_1 + \ldots + \alpha_d$$

where $\alpha = (\alpha_1,\ldots,\alpha_d)$. For $\alpha = (\alpha_1,\ldots,\alpha_d)$ and $\beta = (\beta_1,\ldots,\beta_d)$ we introduce the following ordering

$$\alpha \leq \beta \quad :\Longleftrightarrow \quad \alpha_i \leq \beta_i \quad \forall i = 1,\ldots,d.$$

In the 2-dimensional case we prefer to write h_x and h_y instead of h_1 and h_2.

2. Results from the basic theory of the combination formula

In this section, we describe the two and more dimensional combination formula and give some results from the general theory of this formula.

Let be $\mathbf{H} = (h_1, \ldots, h_d) := (2^{-\mathrm{i}_1}, \ldots, 2^{-\mathrm{i}_d})$, where $I = (i_1, \ldots, i_d) \in \mathbb{N}^d$ is a multiindex and let $h = 2^{-n}$. Furthermore, let $(u_{\mathbf{H}})_{I \in \mathbb{N}^d}$ be a sequence of approximations to $u \in H_0^1(\Omega)$. Such approximations can be constructed by an interpolation of u or by a projection on suitable spaces. The combined solution u_h^{c} of this sequence is defined by the following combination formula:

$$u_h^{\mathrm{c}} := \sum_{s=1}^{d} (-1)^{s+1} \binom{d-1}{s-1} \sum_{\substack{|I|=n+d-s \\ I \in \{1,\ldots,n\}^d}} u_{\mathbf{H}}. \tag{1}$$

This formula was already used in [2] for the construction of a d-variate Boolean interpolation.

For example, in two dimensions the combination formula is:

$$u_h^{\mathrm{c}} := \sum_{k=1}^{n} u_{(2^{-k}, 2^{-(n+1-k)})} - \sum_{k=1}^{n-1} u_{(2^{-k}, 2^{-(n-k)})}. \tag{2}$$

The convergence of the combined solution depends on the properties of the hierarchical surplus. The hierarchical surplus operator δ^α is defined by

$$\delta^\alpha(w_{\mathbf{H}}) := \sum_{\mathbf{0} \le \beta \le \alpha} (-1)^{|\beta|} w_{\mathbf{H}(\mathbf{e}+\beta)},$$

where $\mathbf{H}\gamma := (h_1\gamma_1, \ldots, h_d\gamma_d)$ is the product in each component. $\delta^\alpha(w_{\mathbf{H}})$ is called an $|\alpha|$-dimensional hierarchical surplus of $w_{\mathbf{H}}$. Let us make two simple examples of the operator δ^α. For $d = 2$ and $\alpha = (1, 0)$ it is

$$\delta^{(1,0)}(w_{(h_1,h_2)}) = w_{(h_1,h_2)} - w_{(2h_1,h_2)}.$$

For $d = 2$ and $\alpha = (1, 1)$ we get the 2-dimensional hierarchical surplus

$$\delta^{(1,1)}(w_{(h_1,h_2)}) = w_{(h_1,h_2)} - w_{(h_1,2h_2)} - w_{(2h_1,h_2)} + w_{(2h_1,2h_2)}.$$

The basic convergence theorem of the combined solution is the following theorem.

Theorem 1. *Assume that*

$$\|\delta^\alpha u_{\mathbf{H}}\|_{H^1} \le K(u)\mathbf{H}^\alpha \tag{3}$$

for every $\mathbf{0} \le \alpha \le \mathbf{e}$ *and*

$$\|u - u_{\mathbf{H}}\|_{H^1} \le K(u) \max\{h_1, \ldots, h_d\},$$

where $K(u)$ is a constant, which depends only on u. Then, it follows

$$\|u - u_h^c\|_{H^1} \quad \stackrel{\sim}{<} \quad K(u)h(\log h^{-1})^{d-1}.$$

At the end of this section, we will prove this theorem for the 2-dimensional case. The proof for more than 2 dimensions can be found in [13].

If $u_{\mathbf{H}}$ is the interpolant of u, then it is not very difficult to show the assumption (3).

For this let us make a few definitions. Assume $\mathbf{0} \leq \alpha \leq \mathbf{e}$. Now, let $W^{G,\alpha}$ be the mixed Sobolev-space (similar spaces can be found in [6])

$$W^{G,\alpha} := \{w \in H^1(\Omega) \mid D^\beta w \in H^1(\Omega) \quad \text{for every} \quad \mathbf{0} \leq \beta \leq \alpha\}.$$

The norm on this space is

$$\|w\|_{W^{G,\alpha}} := \sqrt{\sum_{\mathbf{0} \leq \beta \leq \alpha} \|D^\beta w\|_{L^2}^2}.$$

Let $S_{\mathbf{H}}^\alpha$ be the space of functions $w \in H_0^1(\Omega)$ which are continuous and piecewise linear of mesh size h_i in the i-direction, if $\alpha_i = 1$ and arbitrary, if $\alpha_i = 0$. For example $S_{h_1}^1 \subset H_0^1(]0,1[)$ is the space of piecewise linear functions of mesh size h_1 in one dimension and $S_{(h_1,h_2)}^{(1,0)}$ is the space $S_{h_1}^1 \otimes H_0^1(]0,1[)$. In two dimensions we denote the spaces $S_{\mathbf{H}}^\alpha$ by

$$S_{h_x,h_y} := S_{\mathbf{H}}^{(1,1)}, \quad S_{h_x,0} := S_{\mathbf{H}}^{(1,0)}, \quad \text{and} \quad S_{0,h_y} := S_{\mathbf{H}}^{(0,1)}.$$

Now, let $I_{\mathbf{H}}^\alpha$ be the natural interpolation operator in the space $S_{\mathbf{H}}^\alpha$. Obviously

$$I_{\mathbf{H}}^\alpha = \prod_{\substack{\mathbf{0} \leq \beta \leq \alpha \\ |\beta|=1}} I_{\mathbf{H}}^\beta.$$

Let us make a few examples for the interpolation operator $I_{\mathbf{H}}^\alpha$. In the 2-dimensional case we prefer to write

$$I_{h_x,h_y} := I_{\mathbf{H}}^{(1,1)},$$
$$I_{h_x,0} := I_{\mathbf{H}}^{(1,0)}, \quad \text{and} \quad I_{0,h_y} := I_{\mathbf{H}}^{(0,1)}.$$

Then, I_{h_x,h_y} is the usual bilinear interpolation operator on a grid of mesh size h_x and h_y. $I_{h_x,0}$ is the interpolation operator which interpolates only in x-direction on lines of mesh size h_x etc. In the 1-dimensional case we simply write

$$I_h := I_{\mathbf{H}}^{(1)}.$$

Now, for $u_{\mathbf{H}} := I_{\mathbf{H}}^{\mathbf{e}}(u)$, the assumption (3) is contained in the following lemma.

Lemma 1. *If* $\mathbf{0} \leq \alpha \leq \mathbf{e}$ *and* $u \in W^{G,\alpha}$*, then it follows*

$$\|\delta^{\alpha} I^{\mathbf{e}}_{\mathbf{H}} u\|_{H^1} \ \overset{\sim}{<}\ \mathbf{H}^{\alpha} \|u\|_{W^{G,\mathbf{e}}} \quad \textit{and}$$
$$\|\delta^{\alpha} I^{\alpha}_{\mathbf{H}} u\|_{H^1} \ \overset{\sim}{<}\ \mathbf{H}^{\alpha} \|u\|_{W^{G,\alpha}}.$$

We want to give only a short hint how to prove this lemma. The complete proof can be found in [13]. For proving Lemma 1, first observe that the operators $\frac{\partial}{\partial x_i}$ and $I^{\alpha}_{\mathbf{H}}$ commutate if $\alpha_i = 0$. Then, apply the following basic facts of the 1-dimensional interpolation theory:

$$\|u - I^{e_i}_{\mathbf{H}}\|_{L^2} \ \overset{\sim}{<}\ h_i \|D^{e_i} u\|_{L^2} \quad \text{and} \tag{4}$$
$$\|D^{e_i}(u - I^{e_i}_{\mathbf{H}})\|_{L^2} \ \overset{\sim}{<}\ h_i \|D^{2e_i} u\|_{L^2}. \tag{5}$$

By Lemma 1 and Theorem 1, we obtain:

Corollary 1. *Assume* $u \in W^{G,\mathbf{e}}$ *and put* $u_{\mathbf{H}} := I^{\mathbf{e}}_{\mathbf{H}}(u)$*. Now, the function* u^{c}_h *defined by (1) is called the sparse grid interpolant of* u*. The error of this interpolation is*

$$\|u - u^{\mathrm{c}}_h\|_{H^1} \ \overset{\sim}{<}\ h(\log h^{-1})^{d-1} \|u\|_{W^{G,\mathbf{e}}}.$$

Theorem 1 can be generalized for other assumption to the hierarchical surplus (see [13]). Here, we apply such an generalization only to the 2-dimensional case. Therefore, we formulate it only for this case.

Theorem 2. *Assume* $d = 2$ *and let* $K(u)$ *be a constant, which depends only on* u*. Then, we get:*

- *If* $\|\delta^{\mathbf{e}} u_{h_x,h_y}\|_{H^1} \leq K(u) h_x h_y$ *and* $\|u - u_{h,h}\|_{H^1} \leq K(u) h$*, then it follows*

$$\|u - u^{\mathrm{c}}_h\|_{H^1} \ \overset{\sim}{<}\ K(u) h \log h^{-1}.$$

- *If* $\|\delta^{\mathbf{e}} u_{h_x,h_y}\|_{L^2} \leq K(u) h_x h_y \min(h_x, h_y)$ *and* $\|u - u_{h,h}\|_{L^2} \leq K(u) h^{\frac{3}{2}}$*, then it follows*

$$\|u - u^{\mathrm{c}}_h\|_{L^2} \ \overset{\sim}{<}\ K(u) h^{\frac{3}{2}}.$$

- *If* $\|\delta^{\mathbf{e}} u_{h_x,h_y}\|_{H^1} \leq K(u) h_x h_y \max(h_x, h_y)$ *and* $\|u - u_{h,h}\|_{H^1} \leq K(u) h$*, then it follows*

$$\|u - u^{\mathrm{c}}_h\|_{H^1} \ \overset{\sim}{<}\ K(u) h.$$

- *If* $\|\delta^{\mathbf{e}} u_{h_x,h_y}\|_{L^2} \leq K(u) h_x^2 h_y^2$ *and* $\|u - u_{h,h}\|_{L^2} \leq K(u) h^2$*, then it follows*

$$\|u - u^{\mathrm{c}}_h\|_{L^2} \ \overset{\sim}{<}\ K(u) h^2 \log h^{-1}.$$

By the first part of this theorem, we get Theorem 1 for the 2-dimensional case $d = 2$. Let us prove this part of the theorem. The rest can be proved in a similar way.

A simple calculation shows (see [15])

$$\sum_{i=2}^{n} \sum_{j=n-i+2}^{n} \delta^{\mathrm{e}} u_{2^{-\mathrm{i}},2^{-j}} = u_{h,h} - u_h^{\mathrm{c}}.$$

By the assumption $\|\delta^{\mathrm{e}} u_{h_x,h_y}\|_{H^1} \le K(u) h_x h_y$, we get

$$\left\| \sum_{i=2}^{n} \sum_{j=n-i+2}^{n} \delta^{\mathrm{e}} u_{2^{-\mathrm{i}},2^{-j}} \right\|_{H^1} \stackrel{\sim}{<} K(u) \sum_{i=2}^{n} \sum_{j=n-i+2}^{n} 2^{-\mathrm{i}} 2^{-j}$$
$$\stackrel{\sim}{<} K(u) \sum_{i=2}^{n} 2^{-\mathrm{i}} 2^{-n+i-1} \stackrel{\sim}{<} K(u) n h.$$

Hence $\left\|u_{h,h} - u_h^{\mathrm{c}}\right\|_{H^1} \stackrel{\sim}{<} K(u) h \log h^{-1}$. By the assumption $\|u - u_{h,h}\|_{H^1} \stackrel{\sim}{<} K(u) h$ and by the triangle inequality, we conclude

$$\|u - u_h^{\mathrm{c}}\|_{H^1} \stackrel{\sim}{<} K(u) h \log h^{-1}.$$

This completes the proof of the first part of Theorem 2 and the proof of Theorem 1 for the 2-dimensional case.

3. The idea of the convergence proofs

In this section, we explain the main idea of the new technique to prove the convergence of the combination solution. For reasons of simplicity, let us restrict to the 2-dimensional case.

Let $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ a H_0^1-elliptic bilinear form and $f \in L^2(\Omega)$. Let $u \in H_0^1(\Omega)$ be the solution of

$$a(u, v) = \int_\Omega f v \, dz \quad \text{for every} \quad v \in H_0^1(\Omega).$$

The Ritz-Galerkin projection $P_{h_x,h_y} : H_0^1(\Omega) \longrightarrow S_{h_x,h_y}$ is defined by:

$$a(P_{h_x,h_y} u - u, v) = 0, \quad \forall v \in S_{h_x,h_y}.$$

Now, let us set

$$u_{h_x,h_y} := P_{h_x,h_y}(u)$$

and define u_h^{c} by the combination formula (1). u_h^{c} is called the combination solution. The aim of this paper is to analyze the error $\|u - u_h^{\mathrm{c}}\|$ in different norms. For this we want to apply Theorem 1 and Theorem 2, respectively. Therefore, we have to prove for example the assumption

$$\|\delta^{\mathbf{e}} u_{h_x,h_y}\|_{H^1} \leq K(u) h_x h_y.$$

The main idea for proving this inequality is to divide this inequality in four inequalities, which can be proved more easily. For this observe

$$I = I_{h_x,h_y} + (I - I_{h_x,0}) + (I - I_{0,h_y}) - (I - I_{h_x,0})(I - I_{0,h_y}). \tag{6}$$

Then, we get

$$\begin{aligned}
\|\delta^{\mathbf{e}} u_{h_x,h_y}\|_{H^1} &= \|\delta^{\mathbf{e}} P_{h_x,h_y}(u)\|_{H^1} \\
&\leq \|\delta^{\mathbf{e}} P_{h_x,h_y}(I_{h_x,h_y}(u))\|_{H^1} \\
&\quad + \|\delta^{\mathbf{e}} P_{h_x,h_y}(I - I_{h_x,0}(u))\|_{H^1} \\
&\quad + \|\delta^{\mathbf{e}} P_{h_x,h_y}(I - I_{0,h_y}(u))\|_{H^1} \\
&\quad + \|\delta^{\mathbf{e}} P_{h_x,h_y}(I - I_{h_x,0})(I - I_{0,h_y})(u)\|_{H^1}.
\end{aligned}$$

Therefore, we only have to prove

$$\|\delta^{\mathbf{e}} P_{h_x,h_y}(I_{h_x,h_y}(u))\|_{H^1} \leq K(u) h_x h_y, \tag{7}$$

$$\|\delta^{\mathbf{e}} P_{h_x,h_y}(I - I_{h_x,0}(u))\|_{H^1} \leq K(u) h_x h_y, \tag{8}$$

$$\|\delta^{\mathbf{e}} P_{h_x,h_y}(I - I_{0,h_y}(u))\|_{H^1} \leq K(u) h_x h_y, \quad \text{and} \tag{9}$$

$$\|\delta^{\mathbf{e}} P_{h_x,h_y}(I - I_{h_x,0})(I - I_{0,h_y})(u)\|_{H^1} \leq K(u) h_x h_y. \tag{10}$$

The first inequality (7) follows by Lemma 1 and

$$\|\delta^{\mathbf{e}} P_{h_x,h_y}(I_{h_x,h_y}(u))\|_{H^1} = \|\delta^{\mathbf{e}} I_{h_x,h_y}(u)\|_{H^1}.$$

The last inequality (10) can be obtained easily, too. By inequality (5), we get

$$\begin{aligned}
\|P_{h_x,h_y}(I - I_{h_x,0})(I - I_{0,h_y})(u)\|_{H^1} &\lesssim \|(I - I_{h_x,0})(I - I_{0,h_y})(u)\|_{H^1} \\
&\lesssim h_x \|\partial_x (I - I_{0,h_y})(u)\|_{H^1} \\
&\lesssim h_x \|(I - I_{0,h_y})(\partial_x u)\|_{H^1} \\
&\lesssim h_x h_y \|u\|_{H^{G,\mathbf{e}}}.
\end{aligned}$$

By the triangle inequality, we obtain (10).

The inequalities (8) and (9) are very similar. So let us prove only (8). For this we define the projection operator on the semi-discrete space $S_{h_x,0}$. Let $P_{h_x,0}(w) \in S_{h_x,0}$ be the solution of

$$a(P_{h_x,0}(w) - w, v) = 0 \quad \forall v \in S_{h_x,0}.$$

By Cea's Lemma, we get

$$\|P_{h_x,0}(w) - P_{h_x,h_y}(P_{h_x,0}(w))\|_{H^1} \ \stackrel{\sim}{<}\ h_y \|\partial_y P_{h_x,0}(w)\|_{H^1}.$$

By the triangle inequality, we get for $w = (I - I_{h_x,0})(u)$

$$\begin{aligned} \|\delta^{(0,1)} P_{h_x,h_y}(I - I_{h_x,0})(u)\|_{H^1} &= \|\delta^{(0,1)} P_{h_x,h_y} P_{h_x,0}(I - I_{h_x,0})(u)\|_{H^1} \\ &\stackrel{\sim}{<} h_y \|\partial_y P_{h_x,0}(I - I_{h_x,0})(u)\|_{H^1}. \end{aligned}$$

Now, let us assume that we can show

$$\|\partial_y P_{h_x,0}(w)\|_{H^1} \ \stackrel{\sim}{<}\ \|\partial_y w\|_{H^1}. \tag{11}$$

Then, by inequality (4), we obtain

$$\begin{aligned} \|\delta^{(0,1)} P_{h_x,h_y}(I - I_{h_x,0})(u)\|_{H^1} &\stackrel{\sim}{<} h_y \|\partial_y (I - I_{h_x,0})(u)\|_{H^1} \\ &\stackrel{\sim}{<} h_y \|(I - I_{h_x,0})(\partial_y u)\|_{H^1} \\ &\stackrel{\sim}{<} h_x h_y \|u\|_{W^{G,\mathbf{e}}}. \end{aligned}$$

By the triangle inequality, we get (8). By Theorem 1, we conclude:

Theorem 3. *Assume that $u \in W^{G,\mathbf{e}}$. Put $u_{h_x,h_y} = P_{h_x,h_y}(u)$. Then, the combination solution u_h^c converges in the H^1-norm with order*

$$\|u - u_h^c\| \ \stackrel{\sim}{<}\ \|u\|_{W^{G,\mathbf{e}}} h \log h^{-1}.$$

So, we only have to prove the inequality (11). For Poisson's equation inequality (11) is contained in Theorem 6. The proof of Theorem 3 can be generalized to arbitrary dimensions. This generalization is explained in the Sects. 8 to 10.

In case of general elliptic differential equations, it is difficult to prove inequality (11). Therefore, we have to modify the proof of Theorem 3 in case of variable coefficients. This can be done by a superconvergence technique, which is presented in [9] and [16]. This technique uses the special form of the function $w = I - I_{h_x,0}(u)$. Then, we get (see Proposition 2):

There is a $\alpha_{h_x} \in H_0^1(\Omega) \cap W_2^2(\Omega)$ such that

$$\begin{aligned} P_{h_x,h_y}((I - I_{h_x,0})(u)) &= P_{h_x,h_y}(\alpha_{h_x}), \\ \|\alpha_{h_x}\|_{W_2^2} &\stackrel{\sim}{<} h_x \|u\|_{W_2^{G,3}}. \end{aligned}$$

In this paper, we apply the superconvergence technique only for the 2D case. In our opinion, it is possible to use this technique also for the 3D case. Nevertheless, this would lead to a very long and very technical proof, since we have to avoid inequality (11), which we can prove only in case of Poisson's equation.

4. Special notations for the 2-dimensional case

Let us assume that $a(\cdot,\cdot)$ is the following bounded, H_0^1-elliptic bilinear form on $H_0^1(\Omega)\times H_0^1(\Omega)$

$$a(u,v)=\int_\Omega a_{11}\partial_x u\partial_x v+a_{12}\partial_x u\partial_y v+a_{21}\partial_y u\partial_x v+a_{22}\partial_y u\partial_y v \\ +b_1\partial_x uv+b_2\partial_y uv+cuv,$$

where $a_{ij}\in W^1_\infty(\Omega), b_i, c\in L^\infty(\Omega)$ and $f\in L^2(\Omega)$. Assume that T^{h_x} is a uniform mesh with mesh size h_x on $[0,1]$. This means that

$$T^{h_x}=\left\{[ih_x,(i+1)h_x]\;\middle|\; i=0,\cdots,\frac{1}{h_x}-1\right\}.$$

Now, denote by $T^{h_x,h_y}=T^{h_x}\times T^{h_y}$ the product mesh. Obviously, it is

$$S_{h_x}=\{v\in H_0^1(]0,1[):v\,|_e \text{ is linear}, \forall e\in T^{h_x}\}.$$

The operator $I_h: C_0[0,1]\longrightarrow S_h$ is the Lagrangian interpolation operator on the grid T^h. If we use this interpolation operator only in x- or y-direction, then we obtain the interpolation operators $I_{h_x,0}$ and I_{0,h_y}, respectively.

Let us abbreviate

$$\delta^x:=\delta^{(0,1)}\quad\text{and}\quad\delta^y:=\delta^{(1,0)}.$$

For proving the convergence of the combination technique, suitable spaces are spaces with bounded mixed derivatives. For our purpose the following spaces are useful

$$W_\infty^{G,2}:=\left\{w\in W^1_\infty(\Omega)\;\middle|\;\partial_{xy}(w)\in L^\infty(\Omega)\right\},$$
$$W_2^{G,3}:=\left\{w\in W^2_2(\Omega)\;\middle|\;\partial_{xxy}(w),\partial_{xyy}(w)\in L^2(\Omega)\right\}\quad\text{and}$$
$$W_2^{K,4}:=\left\{w\in W^3_2(\Omega)\;\middle|\;\partial_{xxyy}(w),\partial_{xyyy}(w),\partial_{xxxy}(w)\in L^2(\Omega)\right\}$$

with their natural norms $\|\cdot\|_{W_\infty^{G,2}}$, $\|\cdot\|_{W_2^{G,3}}$ and $\|\cdot\|_{W_2^{K,4}}$, respectively. Observe that $W_2^{G,3}=W^{G,\mathbf{e}}$.

We define different assumptions to u and the coefficients a_{ij}, b_i, and c, which lead to different convergence results (see Theorem 5):

Assumption A: $u\in W_2^{G,3}$, $a_{11},a_{12},a_{21},a_{22}\in W^1_\infty(\Omega)$, $b_1,b_2\in L^\infty(\Omega)$, and $c\in L^\infty(\Omega)$.

Assumption B: $u\in W_2^{G,3}$, $a_{11},a_{22}\in W^1_\infty(\Omega)$, $a_{12},a_{21}\in W_\infty^{G,2}$, $b_1,b_2\in L^\infty(\Omega)$, $\partial_x b_1,\partial_y b_2\in L^\infty(\Omega)$, and $c\in L^\infty(\Omega)$.

Assumption C: $u\in W_2^{K,4}$, $a_{11},a_{22}\in W^2_\infty(\Omega)$, $a_{12},a_{21}\in W_\infty^{G,2}$, $b_1,b_2\in L^\infty(\Omega)$, $\partial_x b_1,\partial_y b_2\in L^\infty(\Omega)$, and $c\in L^\infty(\Omega)$.

5. Unidirectional difference for Galerkin approximations

Proposition 1. *Let* $w \in H_0^1(\Omega)$ *and* $h_y = 2^{-n_y}$, *where* $n_y \in \mathbb{N}$. *Then it follows*

$$\|\delta^y P_{h_x,h_y}(w)\|_{L^2} \ \tilde{<}\ h_y \|w\|_{H^1}.$$

If $w \in H_0^1(\Omega) \cap W_2^2(\Omega)$, *then*

$$\|\delta^y P_{h_x,h_y}(w)\|_{H^1} \ \tilde{<}\ h_y \|\partial_y w\|_{H^1} \quad \textit{and}$$
$$\|\delta^y P_{h_x,h_y}(w)\|_{L^2} \ \tilde{<}\ h_y^2 \|\partial_y w\|_{H^1}.$$

Proof. Let $g \in S_0^{h_x,0}(\Omega)$ be the solution of

$$a(v,g) = \int_\Omega v(P_{h_x,h_y}(w) - P_{h_x,0}(w)) \quad \forall v \in S_0^{h_x,0}(\Omega).$$

By the regularity of the semi-discrete solution (see Satz 4.3 in [11]), we obtain

$$\|\partial_y g\|_{H^1} \ \tilde{<}\ \|P_{h_x,h_y}(w) - P_{h_x,0}(w)\|_{L^2}.$$

Observe $I_{0,h_y}(g) \in S_{h_x,h_y}$. Thus, we get

$$\begin{aligned}
&\|P_{h_x,h_y}(w) - P_{h_x,0}(w)\|_{L^2}^2 = a(P_{h_x,h_y}(w) - P_{h_x,0}(w), g)\\
&= \ a(P_{h_x,h_y}(w) - P_{h_x,0}(w), g - I_{0,h_y}(g))\\
&\tilde{<}\ \|P_{h_x,h_y}(w) - P_{h_x,0}(w)\|_{H^1} \|g - I_{0,h_y}(g)\|_{H^1}\\
&\tilde{<}\ \|P_{h_x,h_y}(w) - P_{h_x,0}(w)\|_{H^1} h_y \|\partial_y g\|_{H^1}\\
&\tilde{<}\ h_y \|P_{h_x,h_y}(w) - P_{h_x,0}(w)\|_{H^1} \|P_{h_x,h_y}(w) - P_{h_x,0}(w)\|_{L^2}.
\end{aligned}$$

This implies

$$\text{(12)} \qquad \|P_{h_x,h_y}(w) - P_{h_x,0}(w)\|_{L^2} \ \tilde{<}\ h_y \|P_{h_x,h_y}(w) - P_{h_x,0}(w)\|_{H^1}$$
$$\text{and} \quad \|P_{h_x,h_y}(w) - P_{h_x,0}(w)\|_{L^2} \ \tilde{<}\ h_y \|w\|_{H^1}.$$

By the triangle inequality we get the first inequality. Now, let us assume $w \in W_2^2(\Omega)$. By the regularity of the semi-discrete solution (see Satz 4.3 in [11]) and a short calculation, we get

$$\|\partial_y P_{h_x,0}(w)\|_{H^1} \ \tilde{<}\ \|\partial_y w\|_{H^1}.$$

Now, Cea's Lemma gives

$$\begin{aligned}
\|P_{h_x,h_y}(w) - P_{h_x,0}(w)\|_{H^1} &= \|P_{h_x,h_y}(P_{h_x,0}(w)) - P_{h_x,0}(w)\|_{H^1}\\
&\tilde{<}\ h_y \|\partial_y P_{h_x,0}(w)\|_{H^1} \ \tilde{<}\ h_y \|\partial_y w\|_{H^1}.
\end{aligned}$$

By (12), we obtain

$$\|P_{h_x,h_y}(w) - P_{h_x,0}(w)\|_{L^2} \ \tilde{<}\ h_y^2 \|\partial_y w\|_{H^1}.$$

The triangle inequality completes the proof. □

6. Error resolution for integrals

Similar to [9, 16], and [17], in this section, we have to study error resolution for some integrals. Integration by parts leads to the following lemma.

Lemma 2. *For $e = [x_e - \frac{h_x}{2}, x_e + \frac{h_x}{2}] \in T^{h_x}$, we have*

$$\int_e w - I_{h_x} w = \int_e E_e w'',$$

where $E_e = \frac{1}{2}(x - x_e)^2 - \frac{1}{8}h_x^2$.

Lemma 3. *(i). If $\phi \in L^\infty(\Omega)$, then there exists $f_{h_x} \in L^2(\Omega)$ such that*

$$\int_\Omega \phi(w - I_{h_x,0}w)\partial_x v \;=\; (f_{h_x}, v), \;\; \forall v \in S_0^{h_x,0}(\Omega), \tag{13}$$

$$||f_{h_x}||_{L^2} \;\stackrel{\sim}{<}\; h_x \|\partial_x^2 w\|_{L^2}. \tag{14}$$

(ii). If $\phi, \partial_x\phi \in L^\infty(\Omega)$, then there exists $f_{h_x}^1, f_{h_x}^2 \in L^2(\Omega)$ such that

$$\int_\Omega \phi(w - I_{h_x,0}w)\partial_x v \;=\; (f_{h_x}^1, v), \;\; \forall v \in S_0^{h_x,0}(\Omega), \tag{15}$$

$$\int_\Omega \phi\partial_x(w - I_{h_x,0}w) v \;=\; (f_{h_x}^2, v), \;\; \forall v \in S_0^{h_x,0}(\Omega), \tag{16}$$

$$||f_{h_x}^i||_{L^2} \;\stackrel{\sim}{<}\; h_x^2 \|\partial_x^3 w\|_{L^2}, \;\; i = 1, 2. \tag{17}$$

Proof. It is simple to prove (13) with (14).

It is only necessary to prove (4), (5) with (6), or

$$\begin{aligned} &| \int_\Omega \phi(w - I_{h_x,0}w)\partial_x v \,| + |\int_\Omega \phi\partial_x(w - I_{h_x,0}w) v \,| \\ &\qquad \le C h_x^2 ||\partial_x^3 w||_{L^2} ||v||_{L^2} \quad \forall v \in S_0^{h_x,0}(\Omega). \end{aligned} \tag{18}$$

By Lemma 2, we have

$$\begin{aligned} &\int_\Omega I_{h_x,0}\phi(w - I_{h_x,0}w)\partial_x v \\ &= \int_\Omega (I_{h_x,0}\phi w - I_{h_x,0}(I_{h_x,0}\phi w) \\ &\qquad - (I_{h_x,0}\phi I_{h_x,0}w - I_{h_x,0}(I_{h_x,0}\phi I_{h_x,0}w)))\partial_x v \\ &= \sum_{e \in T^{h_x}} \int_{e\times[0,1]} E_e \partial_x^2((w - I_{h_x,0}w) I_{h_x,0}\phi)\partial_x v \end{aligned}$$

$$= \sum_{e\in T^{h_x}} \int_{e\times[0,1]} E_e \partial_x^2 w I_{h_x,0}\phi\partial_x v$$
$$+ 2 \sum_{e\in T^{h_x}} \int_{e\times[0,1]} E_e \partial_x (w - I_{h_x,0}w)\partial_x I_{h_x,0}\phi\partial_x v.$$

Note that

$$E_e = -\frac{h_x^2}{12} + \frac{1}{6}(E_e^2)''.$$

Then, we obtain

$$\int_\Omega I_{h_x,0}\phi(w - I_{h_x,0}w)\partial_x v$$
$$= \frac{h_x^2}{12}\int_\Omega \partial_x(\partial_x^2 w I_{h_x,0}\phi)v - \frac{1}{6}\sum_{e\in T^{h_x}}\int_{e\times[0,1]}(E_e^2)'\partial_x(\partial_x^2 w I_{h_x,0}\phi)\partial_x v$$
$$+2\sum_{e\in T^{h_x}}\int_{e\times[0,1]} E_e\partial_x(w - I_{h_x,0}w)\partial_x I_{h_x,0}\phi\partial_x v$$
$$= O(h_x^2)||\partial_x^3 w||_{L^2}||v||_{L^2},$$

which together with the following identities

$$\int_\Omega \phi(w - I_{h_x,0}w)\partial_x v$$
$$= -\int_\Omega \partial_x((\phi - I_{h_x,0}\phi)(w - I_{h_x,0}w))v + \int_\Omega I_{h_x,0}\phi(w - I_{h_x,0}w)\partial_x v,$$
$$\int_\Omega \phi\partial_x(w - I_{h_x,0}w)v$$
$$= \int_\Omega (\phi - I_{h_x,0}\phi)\partial_x(w - I_{h_x,0}w)v - \int_\Omega (w - I_{h_x,0}w)\partial_x(I_{h_x,0}\phi)v$$
$$- \int_\Omega (w - I_{h_x,0}w)I_{h_x,0}\phi\partial_x v$$

produce (7). This completes the proof. □

Proposition 2. *(i). If the assumption* **A** *holds, then there exists* $\alpha_{h_x} \in H_0^1(\Omega)\cap W_2^2(\Omega)$ *such that*

$$P_{h_x,h_y}((I - I_{h_x,0})(u)) = P_{h_x,h_y}(\alpha_{h_x}), \tag{19}$$

$$\|\alpha_{h_x}\|_{W_2^2} \stackrel{\sim}{<} h_x\|u\|_{W_2^{G,3}}. \tag{20}$$

(ii). If the assumption **B** *holds, then there exists* $\alpha_{h_x} \in H_0^1(\Omega)$ *such that*

$$P_{h_x,h_y}((I - I_{h_x,0})(u)) = P_{h_x,h_y}(\alpha_{h_x}), \tag{21}$$

$$\|\alpha_{h_x}\|_{H^1} \lesssim h_x^2 \|u\|_{W_2^{G,3}}. \tag{22}$$

(ii). If the assumption **C** *holds, then there exists* $\alpha_{h_x} \in H_0^1(\Omega) \cap W_2^2(\Omega)$ *such that*

$$P_{h_x,h_y}((I - I_{h_x,0})(u)) = P_{h_x,h_y}(\alpha_{h_x}), \tag{23}$$

$$\|\alpha_{h_x}\|_{W_2^2} \lesssim h_x^2 \|u\|_{W_2^{K,4}}. \tag{24}$$

Proof. Integration by parts yields for $v \in S_0^{h_x,0}(\Omega)$

$$\begin{aligned}
& a((I - I_{h_x,0})u, v) \\
& = - \int_\Omega \partial_x a_{11}(I - I_{h_x,0})u \partial_x v + \partial_y a_{12} \partial_x (I - I_{h_x,0})uv \\
& \quad + a_{12}\partial_x (I - I_{h_x,0})\partial_y uv \\
& \quad - a_{21}(I - I_{h_x,0})\partial_y u \partial_x v + \partial_y (a_{22}(I - I_{h_x,0})\partial_y u)v \\
& \quad - b_1 \partial_x (I - I_{h_x,0})uv - b_2 (I - I_{h_x,0})\partial_y uv - c(I - I_{h_x,0})uv.
\end{aligned}$$

Thus, by Lemma 2, there exists $f_{h_x} \in L^2(\Omega)$ such that

$$\begin{aligned}
a((I - I_{h_x,0})u, v) &= (f_{h_x}, v), \ \ \forall v \in S_0^{h_x,0}(\Omega), \\
\|f_{h_x}\|_{L^2} &\lesssim h_x \|u\|_{W_2^{G,3}}.
\end{aligned}$$

On the other hand, there exists $\alpha_{h_x} \in H_0^1(\Omega) \cap W_2^2(\Omega)$ satisfying

$$\begin{aligned}
a(\alpha_{h_x}, v) &= (f_{h_x}, v), \ \ \forall v \in H_0^1(\Omega), \\
\|\alpha_{h_x}\|_{W_2^2} &\lesssim \|f_{h_x}\|_{L^2}.
\end{aligned}$$

Therefore, combining these equations, we obtain (19) and (20). Analogously, we obtain (23) and (24).

Note that

$$\begin{aligned}
& \int_\Omega \partial_y a_{12} \partial_x (I - I_{h_x,0})uv + a_{12}\partial_x (I - I_{h_x,0})\partial_y uv - b_1 \partial_x (I - I_{h_x,0})uv \\
& = - \int_\Omega (I - I_{h_x,0})u \partial_x (\partial_y a_{12} v) + (I - I_{h_x,0})\partial_y u \partial_x (a_{12} v) \\
& \quad - (I - I_{h_x,0})u \partial_x (b_1 v).
\end{aligned}$$

This implies

$$a((I - I_{h_x,0})u, v) \ \tilde{<}\ h_x^2 \|u\|_{W_2^{G,3}} \|v\|_{H^1}, \ \ \forall v \in S_0^{h_x,0}(\Omega).$$

Hence, there exists $\alpha_{h_x} \in H_0^1(\Omega)$ satisfying (21) and (22). This completes the proof. □

Proposition 3. *Let* $r_{h_x,h_y}(u, v) = a((I - I_{h_x,0})(I - I_{0,h_y})u, v)$.
(i) If the assumption **A** *holds, then*

$$|r_{h_x,h_y}(u, v)| \ \tilde{<}\ h_x h_y \|u\|_{W_2^{G,3}} \|v\|_{H^1}, \ \ \forall v \in S_{h_x,h_y}(\Omega).$$

(ii) If the assumption **B** *holds, then*

$$|r_{h_x,h_y}(u, v)| \ \tilde{<}\ h_x h_y \min(h_x, h_y) \|u\|_{W_2^{G,3}} \|\partial_x \partial_y v\|_{L^2}, \qquad \forall v \in S_{h_x,h_y}(\Omega).$$

(iii) If the assumption **C** *holds, then*

$$\begin{aligned} |r_{h_x,h_y}(u, v)| &\ \tilde{<}\ h_x h_y \min(h_x, h_y) \|u\|_{W_2^{K,4}} \|v\|_{H^1}, \ \ \forall v \in S_{h_x,h_y}(\Omega), \\ |r_{h_x,h_y}(u, v)| &\ \tilde{<}\ h_x^2 h_y^2 \|u\|_{W_2^{K,4}} \|\partial_x \partial_y v\|_{L^2}, \ \ \forall v \in S_{h_x,h_y}(\Omega). \end{aligned}$$

Proof. (i) is obvious.

Set $w = (I - I_{h_x,0})(I - I_{0,h_y})u$. Integration by parts leads to

$$\begin{aligned} r_{h_x,h_y}(u, v) = -\int_\Omega & \partial_x a_{11} w \partial_x v + w \partial_x (a_{12} \partial_y v) + w \partial_y (a_{21} \partial_x v) \\ & + \partial_y a_{22} w \partial_y v + w \partial_x (b_1 v) + w \partial_y (b_2 v) - c w v. \end{aligned}$$

Hence, we have

$$\begin{aligned} |r_{h_x,h_y}(u, v)| &\ \tilde{<}\ h_x h_y^2 \|u\|_{W_2^{G,3}} \|\partial_x \partial_y v\|_{L^2}, \ \ \forall v \in S_{h_x,h_y}(\Omega), \\ |r_{h_x,h_y}(u, v)| &\ \tilde{<}\ h_x^2 h_y \|u\|_{W_2^{G,3}} \|\partial_x \partial_y v\|_{L^2}, \ \ \forall v \in S_{h_x,h_y}(\Omega), \\ |r_{h_x,h_y}(u, v)| &\ \tilde{<}\ h_x^2 h_y^2 \|u\|_{W_2^{K,4}} \|\partial_x \partial_y v\|_{L^2}, \ \ \forall v \in S_{h_x,h_y}(\Omega). \end{aligned}$$

By the inverse inequality, we obtain

$$\begin{aligned} |r_{h_x,h_y}(u, v)| &\ \tilde{<}\ h_x h_y^2 \|u\|_{W_2^{K,4}} \|\partial_y v\|_{L^2}, \ \ \forall v \in S_{h_x,h_y}(\Omega), \\ |r_{h_x,h_y}(u, v)| &\ \tilde{<}\ h_x^2 h_y \|u\|_{W_2^{K,4}} \|\partial_x v\|_{L^2}, \ \ \forall v \in S_{h_x,h_y}(\Omega). \end{aligned}$$

This completes the proof. □

7. Convergence of the combination solution in 2D

Let us first estimate the 2-dimensional hierarchical surplus.

For this, we prove two consequences of the last two sections.

Proposition 4. *(i) If the assumption* **A** *holds, then we obtain*

$$\|\delta^y P_{h_x,h_y}((I - I_{h_x,0})(u))\|_{H^1} \ \tilde{<}\ h_x h_y \|u\|_{W_2^{G,3}}.$$

(ii) If the assumption **B** *holds, then we obtain*

$$\|\delta^y P_{h_x,h_y}((I - I_{h_x,0})(u))\|_{L^2} \ \tilde{<}\ h_x h_y \min(h_x, h_y)\|u\|_{W_2^{G,3}}.$$

(iii) If the assumption **C** *holds, then we obtain*

$$\begin{aligned}
\|\delta^y P_{h_x,h_y}((I - I_{h_x,0})(u))\|_{H^1} \ &\tilde{<}\ h_x h_y \max(h_x, h_y)\|u\|_{W_2^{K,4}},\\
\|\delta^y P_{h_x,h_y}((I - I_{h_x,0})(u))\|_{L^2} \ &\tilde{<}\ h_x^2 h_y^2 \|u\|_{W_2^{K,4}}.
\end{aligned}$$

Proof. By Proposition 1 and 2, we obtain

$$\begin{aligned}
\|\delta^y P_{h_x,h_y}((I - I_{h_x,0})(u))\|_{H^1} &= \|\delta^y P_{h_x,h_y}(\alpha_{h_x})\|_{H^1}\\
&\tilde{<}\ h_y\|\partial_y(\alpha_{h_x})\|_{H^1} \ \tilde{<}\ h_x h_y \|u\|_{W_2^{G,3}},\\
\|\delta^y P_{h_x,h_y}((I - I_{h_x,0})(u))\|_{L^2} &= \|\delta^y P_{h_x,h_y}(\alpha_{h_x})\|_{L^2}\\
&\tilde{<}\ h_y^2\|\partial_y(\alpha_{h_x})\|_{H^1} \ \tilde{<}\ h_x h_y^2 \|u\|_{W_2^{G,3}},\\
\|\delta^y P_{h_x,h_y}((I - I_{h_x,0})(u))\|_{L^2} &= \|\delta^y P_{h_x,h_y}(\alpha_{h_x})\|_{L^2}\\
&\tilde{<}\ h_y\|\alpha_{h_x}\|_{H^1} \ \tilde{<}\ h_x^2 h_y \|u\|_{W_2^{G,3}},\\
\|\delta^y P_{h_x,h_y}((I - I_{h_x,0})(u))\|_{H^1} &= \|\delta^y P_{h_x,h_y}(\alpha_{h_x})\|_{H^1}\\
&\tilde{<}\ h_y\|\partial_y(\alpha_{h_x})\|_{H^1} \ \tilde{<}\ h_x^2 h_y \|u\|_{W_2^{K,4}},\\
\|\delta^y P_{h_x,h_y}((I - I_{h_x,0})(u))\|_{L^2} &= \|\delta^y P_{h_x,h_y}(\alpha_{h_x})\|_{L^2}\\
&\tilde{<}\ h_y^2\|\partial_y(\alpha_{h_x})\|_{H^1} \ \tilde{<}\ h_x^2 h_y^2 \|u\|_{W_2^{K,4}}.
\end{aligned}$$

This completes the proof. □

Proposition 5. *(i) If the assumption* **A** *holds, then we obtain*

$$\|P_{h_x,h_y}((I - I_{h_x,0})(I - I_{0,h_y})(u))\|_{H^1} \ \tilde{<}\ h_x h_y \|u\|_{W_2^{G,3}}.$$

(ii) If the assumption **B** *holds, then we obtain*

$$\|P_{h_x,h_y}((I - I_{h_x,0})(I - I_{0,h_y})(u))\|_{L_2} \ \tilde{<}\ h_x h_y \min(h_x, h_y)\|u\|_{W_2^{G,3}}.$$

(iii) If the assumption **C** *holds, then we obtain*

$$\begin{aligned}\|P_{h_x,h_y}((I - I_{h_x,0})(I - I_{0,h_y})(u))\|_{H^1} &\overset{\sim}{<} h_x h_y \max(h_x, h_y)\|u\|_{W_2^{K,4}}, \\ \|P_{h_x,h_y}((I - I_{h_x,0})(I - I_{0,h_y})(u))\|_{L^2} &\overset{\sim}{<} h_x^2 h_y^2 \|u\|_{W_2^{K,4}}.\end{aligned}$$

Proof. The estimates in the H^1-norm follow directly by Proposition 3.

Let us define $r_{h_x,h_y}(u, v)$ like in Proposition 3. For the estimates in the L^2-norm we use a duality argument. Let $q := P_{h_x,h_y}((I - I_{h_x,0})(I - I_{0,h_y})(u))$.

Let us assume **B**. Let $w \in S_0^{h_x,0}$ be the solution of

$$a(v, w) = \int_\Omega qv \quad \forall v \in S_0^{h_x,0}.$$

Then we obtain $\|\partial_y w\|_{H^1} \overset{\sim}{<} \|q\|_{L^2}$. $I_{h_x,h_y}(w) = I_{0,h_y}(w)$ is the projection of w in the space $S_0^{h_x,h_y}$ with respect to the bilinear form $(v_1, v_2) \mapsto \int_\Omega \partial_x\partial_y v_1 \partial_x\partial_y v_2$. Thus, we get

$$\|\partial_x\partial_y I_{0,h_y}(w)\|_{L^2} \leq \|\partial_x\partial_y w\|_{L^2} \leq \|\partial_y w\|_{H^1}.$$

This shows

$$\begin{aligned}\|q\|_{L^2}^2 &= a(q, w) = a(q, w - I_{0,h_y}(w)) + r_{h_x,h_y}(u, I_{0,h_y}(w)) \\ &\overset{\sim}{<} \|q\|_{H^1}\|w - I_{0,h_y}(w)\|_{H^1} + |r_{h_x,h_y}(u, I_{0,h_y}(w))| \\ &\overset{\sim}{<} \|q\|_{H^1} h_y \|\partial_y w\|_{H^1} + |r_{h_x,h_y}(u, I_{0,h_y}(w))| \\ &\overset{\sim}{<} h_x h_y^2 \|u\|_{W_2^{G,3}} \|\partial_y w\|_{H^1} + h_x h_y^2 \|u\|_{W_2^{G,3}} \|\partial_x\partial_y I_{0,h_y}(w)\|_{L^2} \\ &\overset{\sim}{<} h_x h_y^2 \|u\|_{W_2^{G,3}} \|q\|_{L^2}.\end{aligned}$$

Therefore, we get

$$\|P_{h_x,h_y}((I - I_{h_x,0})(I - I_{0,h_y})(u))\|_{L^2} \overset{\sim}{<} h_x h_y^2 \|u\|_{W_2^{G,3}}.$$

Analogously, we obtain

$$\|P_{h_x,h_y}((I - I_{h_x,0})(I - I_{0,h_y})(u))\|_{L^2} \overset{\sim}{<} h_x^2 h_y^2 \|u\|_{W_2^{K,4}}.$$

This completes the proof. □

Now, we can estimate the 2-dimensional hierarchical surplus.

Theorem 4. *(i) If the assumption* **A** *holds, then we obtain*

$$\|\delta^{\mathbf{e}}_{h_x,h_y}(u)\|_{H^1} \ \tilde{<}\ h_x h_y \|u\|_{W_2^{G,3}}.$$

(ii) If the assumption **B** *holds, then we obtain*

$$\|\delta^{\mathbf{e}}_{h_x,h_y}(u)\|_{L^2} \ \tilde{<}\ h_x h_y \min(h_x,h_y) \|u\|_{W_2^{G,3}}.$$

(iii) If the assumption **C** *holds, then we obtain*

$$\begin{aligned}\|\delta^{\mathbf{e}}_{h_x,h_y}(u)\|_{H^1} &\ \tilde{<}\ h_x h_y \max(h_x,h_y) \|u\|_{W_2^{K,4}},\\ \|\delta^{\mathbf{e}}_{h_x,h_y}(u)\|_{L^2} &\ \tilde{<}\ h_x^2 h_y^2 \|u\|_{W_2^{K,4}}.\end{aligned}$$

Proof. A short calculation shows (see [12]):

If the assumption **A** holds, then we obtain

$$\|\delta^x \circ \delta^y I_{h_x,h_y}(u)\|_{H^1} \ \tilde{<}\ h_x h_y \|u\|_{W_2^{G,3}}.$$

If the assumption **B** holds, then we obtain

$$\|\delta^x \circ \delta^y I_{h_x,h_y}(u)\|_{L_2} \ \tilde{<}\ h_x h_y \min(h_x,h_y) \|u\|_{W_2^{G,3}}.$$

If the assumption **C** holds, then we obtain

$$\begin{aligned}\|\delta^x \circ \delta^y I_{h_x,h_y}(u)\|_{H^1} &\ \tilde{<}\ h_x h_y \max(h_x,h_y) \|u\|_{W_2^{K,4}},\\ \|\delta^x \circ \delta^y I_{h_x,h_y}(u)\|_{L^2} &\ \tilde{<}\ h_x^2 h_y^2 \|u\|_{W_2^{K,4}}.\end{aligned}$$

Observe that

$$I - I_{h_x,h_y} = (I - I_{h_x,0}) + (I - I_{0,h_y}) - (I - I_{h_x,0})(I - I_{0,h_y}).$$

Let $\|\cdot\|$ be the norm $\|\cdot\|_{H^1}$ or $\|\cdot\|_{L^2}$. Then we obtain

$$\begin{aligned}&\|\delta^{\mathbf{e}}_{h_x,h_y}(u)(u)\| = \|\delta^x \circ \delta^y P_{h_x,h_y}(u)\|\\ &\leq \|\delta^x \circ \delta^y (P_{h_x,h_y} - I_{h_x,h_y})(u)\| + \|\delta^x \circ \delta^y I_{h_x,h_y}(u)\|\\ &\leq \|\delta^x \circ \delta^y P_{h_x,h_y}(I - I_{h_x,h_y})(u)\| + \|\delta^x \circ \delta^y I_{h_x,h_y}(u)\|\\ &\leq \|\delta^x \circ \delta^y P_{h_x,h_y}(I - I_{h_x,0})(u)\| + \|\delta^x \circ \delta^y P_{h_x,h_y}(I - I_{0,h_y})(u)\|\\ &\quad + \|\delta^x \circ \delta^y P_{h_x,h_y}(I - I_{h_x,0})(I - I_{0,h_y})(u)\| + \|\delta^x \circ \delta^y I_{h_x,h_y}(u)\|\\ &\ \tilde{<}\ \max_{\tilde{h}_x = h_x, \tilde{h}_x = 2h_x} \|\delta^y P_{\tilde{h}_x,h_y}(I - I_{\tilde{h}_x,0})(u)\|\\ &\quad + \max_{\tilde{h}_y = h_y, \tilde{h}_y = 2h_y} \|\delta^x P_{h_x,\tilde{h}_y}(I - I_{0,\tilde{h}_y})(u)\|\\ &\quad + \max_{\substack{\tilde{h}_x = h_x, \tilde{h}_x = 2h_x\\ \tilde{h}_y = h_y, \tilde{h}_y = 2h_y}} \|P_{\tilde{h}_x,\tilde{h}_y}(I - I_{\tilde{h}_x,0})(I - I_{0,\tilde{h}_y})(u)\|\\ &\quad + \|\delta^x \circ \delta^y I_{h_x,h_y}(u)\|.\end{aligned}$$

Applying Proposition 4 and 5 completes the proof. □

Now, we can complete our analysis of the combination technique in 2D. By Theorem 4, Theorem 2 and well-know convergence properties of the finite element solution $P_{h,h}(u)$, we obtain the following theorem.

Theorem 5. *Put $u_{h_x,h_y} = P_{h_x,h_y}(u)$. Then, the the combination solution u_h^c has the following convergence properties:*
(i) If the assumption **A** *holds, then we obtain*

$$\|u - u_h^c\|_{H^1} \lesssim h \log(h^{-1}) \|u\|_{W_2^{G,3}}.$$

(ii) If the assumption **B** *holds, then we obtain*

$$\|u - u_h^c\|_{L^2} \lesssim h^{1.5} \|u\|_{W_2^{G,3}}.$$

(iii) If the assumption **C** *holds, then we obtain*

$$\begin{aligned} \|u - u_h^c\|_{H^1} &\lesssim h \|u\|_{W_2^{K,4}}, \\ \|u - u_h^c\|_{L^2} &\lesssim h^2 \log(h^{-1}) \|u\|_{W_2^{K,4}}. \end{aligned}$$

8. Special notations for the case of arbitrary dimensions

In the case of arbitrary dimensions, we restrict to the bilinear form

$$a(u,v) = \int_\Omega \langle \nabla u, \nabla v \rangle \, dx.$$

We have to define a few operators:

- Let $P_{\mathbf{H}}^\alpha : H_0^1(\Omega) \to S_{\mathbf{H}}^\alpha \cap H_0^1(\Omega)$ be the orthogonal projection operator with respect to the bilinear form a.
- Define the operator

$$\mathcal{F}_{\mathbf{H}}^\alpha := \prod_{\substack{\mathbf{0} \le \beta \le \alpha \\ |\beta| = 1}} (I - I_{\mathbf{H}}^\beta).$$

- Let Max^α be the following simple maximum operator:

$$\mathrm{Max}^\alpha(w_{\mathbf{H}}) := \max_{\mathbf{0} \le \beta \le \alpha} \left| w_{\mathbf{H}(\mathbf{e}+\beta)} \right|,$$

  where $w_{\mathbf{H}}$ is a real number which depends on $\mathbf{H}$.

  These operators have the following properties.

Lemma 4.

$$\|\delta^\alpha(q_{\mathbf{H}})\| \le 2^{|\alpha|} \mathrm{Max}^\alpha(\|q_{\mathbf{H}}\|).$$

Lemma 5.

$$I - I_{\mathbf{H}}^{\beta} = - \sum_{\substack{\mathbf{0} \le \alpha \le \beta \\ |\alpha| \ge 1}} (-1)^{|\alpha|} \mathcal{F}_{\mathbf{H}}^{\alpha}$$

Proof. Obviously, it is

$$\mathcal{F}_{\mathbf{H}}^{\alpha} = \prod_{\substack{\mathbf{0} \le \beta \le \alpha \\ |\beta| = 1}} (I - I_{\mathbf{H}}^{\beta}) = \sum_{\gamma \le \alpha} (-1)^{|\gamma|} I_{\mathbf{H}}^{\gamma}.$$

Therefore, we get

$$\begin{aligned} - \sum_{\substack{\mathbf{0} \le \alpha \le \beta \\ |\alpha| \ge 1}} (-1)^{|\alpha|} \mathcal{F}_{\mathbf{H}}^{\alpha} &= - \sum_{\substack{\mathbf{0} \le \alpha \le \beta \\ |\alpha| \ge 1}} (-1)^{|\alpha|} \sum_{\gamma \le \alpha} (-1)^{|\gamma|} I_{\mathbf{H}}^{\gamma} \\ &= - \sum_{\gamma \le \beta} (-1)^{|\gamma|} I_{\mathbf{H}}^{\gamma} \sum_{\substack{\gamma \le \alpha \le \beta \\ |\alpha| \ge 1}} (-1)^{|\alpha|} \\ &= I - I_{\mathbf{H}}^{\beta}. \end{aligned}$$

□

Lemma 6. *Assume $\alpha, \beta, \alpha + \beta \in \{0,1\}^d$ and $w \in W^{G,\alpha+\beta}$. Then, the following inequality holds:*

$$\|\mathcal{F}_{\mathbf{H}}^{\alpha}(w)\|_{W^{G,\beta}} \ \tilde{<}\ \mathbf{H}^{\alpha} \|w\|_{W^{G,\alpha+\beta}}.$$

For the proof of this lemma apply the inequalities (4) and (5).

9. Regularity results

$H_0^1(\Omega)$ is a Hilbert space with scalar product a. The orthogonal projection $P_{\mathbf{H}}^{\alpha}(w)$ onto the space $S_{\mathbf{H}}^{\alpha}$ has the following regularity property:

Theorem 6. *If $\gamma \in \{0,1\}^d$, then it follows*

$$\|P_{\mathbf{H}}^{\mathbf{e}-\gamma}(w)\|_{W^{G,\gamma}} \ \tilde{<}\ \|w\|_{W^{G,\gamma}}$$

for every $w \in W^{G,\gamma} \cap H_0^1(\Omega)$.

Proof. Let us first introduce the spaces

$$\begin{aligned} \Gamma_i &:= \left\{ (x_1, \ldots, x_d) \in \bar{\Omega} \;\middle|\; x_i = 0 \vee x_i = 1 \right\}, \\ \mathcal{C}_{\alpha}^{\infty} &:= \left\{ \varphi \in \mathcal{C}^{\infty}(\bar{\Omega}) \;\middle|\; \varphi|_{\Gamma_i} = 0 \quad \text{if} \quad \alpha_i = 1 \right\} \quad \text{and} \\ H_{\alpha}^1 &:= \overline{\mathcal{C}_{\alpha}^{\infty}}^{H^1}. \end{aligned}$$

Now, we prove the following general statement by induction to $k = 0, \ldots, |\gamma|$:

Assume $|\alpha| = k$ *and* $\alpha \leq \gamma$*. Then, it follows*

$$\|P_{\mathbf{H}}^{\mathbf{e}-\gamma}(w)\|_{W^{G,\alpha}} \ \stackrel{\sim}{<} \ \|w\|_{W^{G,\alpha}} \tag{25}$$

Beginning of the induction $k = 0$*:* Trivial.
Induction $0 \leq k-1 \mapsto k \leq |\gamma|$*:*
Assume $|\alpha| = k$ and $\alpha \leq \gamma$. The projection $P_{\mathbf{H}}^{\mathbf{e}-\gamma}(w)$ is the unique function $P_{\mathbf{H}}^{\mathbf{e}-\gamma}(w) \in S_{\mathbf{H}}^{\mathbf{e}-\gamma} \cap H_0^1(\Omega)$ such that

$$a(P_{\mathbf{H}}^{\mathbf{e}-\gamma}(w), v) = a(w, v) \quad \text{for every } v \in S_{\mathbf{H}}^{\mathbf{e}-\gamma} \cap H_0^1(\Omega).$$

Let $\beta \leq \alpha$ and $|\beta| = |\alpha| - 1$. By (25), we obtain $P_{\mathbf{H}}^{\mathbf{e}-\gamma}(w) \in W^{G,\beta}$. Thus, by partial integration, we get for $D^\beta P_{\mathbf{H}}^{\mathbf{e}-\gamma}(w) \in S_{\mathbf{H}}^{\mathbf{e}-\gamma} \cap H_{\mathbf{e}-\beta}^1(\Omega)$

$$a(D^\beta P_{\mathbf{H}}^{\mathbf{e}-\gamma}(w), v) = a(D^\beta w, v) \quad \text{for every } v \in S_{\mathbf{H}}^{\mathbf{e}-\gamma} \cap H_{\mathbf{e}-\beta}^1(\Omega).$$

Without loss of generality, we assume $\alpha - \beta = (1, 0, 0, \ldots, 0)$. Now we use a finite difference operator to prove regularity. Let δ_τ^1 be the symmetric difference operator in the x_1-direction. To apply this operator to functions on the domain Ω, we extend each function q on Ω in a point-symmetric way to the function $\tilde{q}$ on a band. For more details about the properties of the extension operator $\tilde{\ }$ and the finite difference operator δ_τ^1 see [11]. Then, we get by discrete partial integration

$$\begin{aligned}
|\delta_\tau^1 D^\beta P_{\mathbf{H}}^{\mathbf{e}-\gamma}(\tilde{w})|_{H^1}^2 &= a(\delta_\tau^1 D^\beta P_{\mathbf{H}}^{\mathbf{e}-\gamma}(\tilde{w}), \delta_\tau^1 D^\beta P_{\mathbf{H}}^{\mathbf{e}-\gamma}(\tilde{w})) \\
&= -a(D^\beta P_{\mathbf{H}}^{\mathbf{e}-\gamma}(w), \delta_\tau^1 \delta_\tau^1 D^\beta P_{\mathbf{H}}^{\mathbf{e}-\gamma}(\tilde{w})) \\
&= -a(D^\beta w, \delta_\tau^1 \delta_\tau^1 D^\beta P_{\mathbf{H}}^{\mathbf{e}-\gamma}(\tilde{w})) \\
&= a(\delta_\tau^1 D^\beta \tilde{w}, \delta_\tau^1 D^\beta P_{\mathbf{H}}^{\mathbf{e}-\gamma}(\tilde{w})) \\
&\leq \|\delta_\tau^1 D^\beta \tilde{w}\|_{H^1} |\delta_\tau^1 D^\beta P_{\mathbf{H}}^{\mathbf{e}-\gamma}(\tilde{w})|_{H^1} \\
&\leq \|w\|_{H^{G,\alpha}} |\delta_\tau^1 D^\beta P_{\mathbf{H}}^{\mathbf{e}-\gamma}(\tilde{w})|_{H^1}.
\end{aligned}$$

This implies

$$|\delta_\tau^1 D^\beta P_{\mathbf{H}}^{\mathbf{e}-\gamma}(\tilde{w})|_{H^1} \leq \|w\|_{H^{G,\alpha}}.$$

Thus, we get

$$|D^\alpha P_{\mathbf{H}}^{\mathbf{e}-\gamma}(w)|_{H^1} \leq \|w\|_{H^{G,\alpha}}.$$

Therefore, we get (25). □

10. Convergence of the combination solution in arbitrary dimensions

Let us first estimate the hierarchical surplus.

Theorem 7. *Assume that* $\mathbf{0} \leq \alpha \leq \mathbf{e}$ *and* $u \in W^{G,\mathbf{e}} \cap H_0^1(\Omega)$*. Then, it follows*

$$\|\delta^\alpha P_{\mathbf{H}}^{\mathbf{e}}(u)\|_{H^1} \stackrel{\sim}{<} \mathbf{H}^\alpha \|u\|_{W^{G,\alpha}}. \tag{26}$$

Proof. Let us first prove the following statement by induction to $|\alpha| = k = 1, \ldots, d$.

If $w \in S_{\mathbf{H}}^{\mathbf{e}-\alpha} \cap W^{G,\alpha} \cap H_0^1(\Omega)$, then it follows

$$\|\delta^\alpha P_{\mathbf{H}}^{\mathbf{e}}(w)\|_{H^1} \stackrel{\sim}{<} \mathbf{H}^\alpha \|w\|_{W^{G,\alpha}}. \tag{27}$$

Beginning of the induction $k = 1$*:* Let $|\alpha| = k = 1$. By the interpolation theory, we obtain

$$\|w - I_{\mathbf{H}}^\alpha(w)\|_{H^1} \stackrel{\sim}{<} \mathbf{H}^\alpha \|w\|_{W^{G,\alpha}}.$$

By $w \in S_{\mathbf{H}}^{\mathbf{e}-\alpha} \cap H_0^1(\Omega)$, we get $I_{\mathbf{H}}^\alpha(w) \in S_{\mathbf{H}}^{\mathbf{e}} \cap H_0^1(\Omega)$. Thus, by Cea's Lemma, we obtain

$$\|w - P_{\mathbf{H}}^{\mathbf{e}}(w)\|_{H^1} \stackrel{\sim}{<} \mathbf{H}^\alpha \|w\|_{W^{G,\alpha}}.$$

By the triangle inequality, we get (27).

Induction $1 \leq k-1 \mapsto k \leq d$*:* Let $|\alpha| = k$. By $w \in S_{\mathbf{H}}^{\mathbf{e}-\alpha} \cap H_0^1(\Omega)$, we get $I_{\mathbf{H}}^\alpha(w) \in S_{\mathbf{H}}^{\mathbf{e}} \cap H_0^1(\Omega)$. Thus, by Lemma 5, we obtain

$$\begin{aligned}
\delta^\alpha P_{\mathbf{H}}^{\mathbf{e}}(w) &= \delta^\alpha P_{\mathbf{H}}^{\mathbf{e}}(w - I_{\mathbf{H}}^\alpha(w)) + \delta^\alpha P_{\mathbf{H}}^{\mathbf{e}} I_{\mathbf{H}}^\alpha(w) \\
&= -\delta^\alpha P_{\mathbf{H}}^{\mathbf{e}} \left(\sum_{\substack{\beta \leq \alpha \\ |\beta| \geq 1}} (-1)^{|\beta|} \mathcal{F}_{\mathbf{H}}^\beta(w) \right) + \delta^\alpha I_{\mathbf{H}}^\alpha(w).
\end{aligned}$$

By Lemma 1, we obtain

$$\|\delta^\alpha I_{\mathbf{H}}^\alpha(w)\|_{H^1} \stackrel{\sim}{<} \mathbf{H}^\alpha \|w\|_{W^{G,\alpha}}.$$

Therefore it is enough to show

$$\|\delta^\alpha P_{\mathbf{H}}^{\mathbf{e}}(\mathcal{F}_{\mathbf{H}}^\beta(w))\|_{H^1} \stackrel{\sim}{<} \mathbf{H}^\alpha \|w\|_{W^{G,\alpha}}$$

for $|\beta| \geq 1$ and $\beta \leq \alpha$. Assume $|\beta| \geq 1$ and $\beta \leq \alpha$. This implies $|\alpha - \beta| < |\alpha| \leq k$ and we can apply inequality (27) for the index $\alpha - \beta$. Thus, by Lemma 4, it follows

$$\begin{aligned} \|\delta^\alpha P_{\mathbf{H}}^{\mathbf{e}}(\mathcal{F}_{\mathbf{H}}^\beta(w))\|_{H^1} &= \|\delta^\beta \delta^{\alpha-\beta} P_{\mathbf{H}}^{\mathbf{e}}(\mathcal{F}_{\mathbf{H}}^\beta(w))\|_{H^1} \\ &\stackrel{\sim}{<} \mathrm{Max}^\beta \left(\|\delta^{\alpha-\beta} P_{\mathbf{H}}^{\mathbf{e}}(\mathcal{F}_{\mathbf{H}}^\beta(w))\|_{H^1} \right) \\ &\stackrel{\sim}{<} \mathrm{Max}^\beta \left(\|\delta^{\alpha-\beta} P_{\mathbf{H}}^{\mathbf{e}} P_{\mathbf{H}}^{\mathbf{e}-(\alpha-\beta)}(\mathcal{F}_{\mathbf{H}}^\beta(w))\|_{H^1} \right) \\ &\stackrel{\sim}{<} \mathrm{Max}^\beta \left(\mathbf{H}^{\alpha-\beta} \|P_{\mathbf{H}}^{\mathbf{e}-(\alpha-\beta)} \mathcal{F}_{\mathbf{H}}^\beta(w)\|_{W^{G,\alpha-\beta}} \right) \end{aligned}$$

By Theorem 6 and Lemma 6, we conclude

$$\begin{aligned} \|\delta^\alpha P_{\mathbf{H}}^{\mathbf{e}}(\mathcal{F}_{\mathbf{H}}^\beta(w))\|_{H^1} &\stackrel{\sim}{<} \mathrm{Max}^\beta \left(\mathbf{H}^{\alpha-\beta} \|P_{\mathbf{H}}^{\mathbf{e}-(\alpha-\beta)} \mathcal{F}_{\mathbf{H}}^\beta(w)\|_{W^{G,\alpha-\beta}} \right) \\ &\stackrel{\sim}{<} \mathrm{Max}^\beta \left(\mathbf{H}^{\alpha-\beta} \|\mathcal{F}_{\mathbf{H}}^\beta(w)\|_{W^{G,\alpha-\beta}} \right) \\ &\stackrel{\sim}{<} \mathrm{Max}^\beta \left(\mathbf{H}^{\alpha-\beta} \mathbf{H}^\beta \|w\|_{W^{G,\alpha}} \right) \\ &\stackrel{\sim}{<} \mathbf{H}^\alpha \|w\|_{W^{G,\alpha}}. \end{aligned}$$

Now we have proved (27). By Theorem 6, we conclude

$$\begin{aligned} \|\delta^\alpha P_{\mathbf{H}}^{\mathbf{e}}(u)\|_{H^1} &= \|\delta^\alpha P_{\mathbf{H}}^{\mathbf{e}} P_{\mathbf{H}}^{\mathbf{e}-\alpha}(u)\|_{H^1} \\ &\stackrel{\sim}{<} H^\alpha \|P_{\mathbf{H}}^{\mathbf{e}-\alpha}(u)\|_{H^{G,\alpha}} \\ &\stackrel{\sim}{<} H^\alpha \|u\|_{H^{G,\alpha}}. \qquad \square \end{aligned}$$

Now, we can prove the convergence of the combination solution for Poisson's equation in arbitrary dimensions.

By Theorem 7 and Theorem 1 and by the convergence of the finite element solution $P_{\mathbf{H}}(u)$, we obtain the following theorem.

Theorem 8. *Put* $u_{\mathbf{H}} = P_{\mathbf{H}}^{\mathbf{e}}(u)$. *Assume that* $u \in W^{G,\mathbf{e}} \cap H_0^1(\Omega)$. *Then, the combination solution* u_h^c *converges in the* H^1*-norm with the following order:*

$$\|u - u_h^c\|_{H^1} \stackrel{\sim}{<} h \log(h^{-1}) \|u\|_{W^{G,\mathbf{e}}}.$$

Acknowledgements. The authors are grateful to Professor C. Zenger for fruitful discussions. The second author thanks Max-Planck Society for sponsoring his visit in Munich.

References

1. Bungartz, H., Griebel, M., Röschke, D., Zenger, C. (1993): A Proof of Convergence for the Combination Technique for the Laplace Equation Using Tools of Symbolic Computation. In: Jacob G., Oussous N.E., Steinberg S. (eds.), Proceedings of the international IMACS Symposium on Symbolic Computation, Lille, 14.-17.6.1993. IMACS/USTL
2. Delvos, F.-J. (1982): d-Variate Boolean Interpolation Journal of Approximation Theory, 99–114
3. Griebel, M., Huber, W., Rüde, U., Störtkuhl, T. (1992): The combination technique for parallel sparse-grid-preconditioning and -solution of PDEs on multiprocessor machines and workstation networks. In: Bougé L., Cosnard M., Robert Y., Trystram, D., (eds.), Proceedings of the Second Joint International Conference on Vector and Parallel Processing CONPAR/VAPP V 92, 634, pp. 217-228. Springer Verlag, Berlin Heidelberg New York
4. Griebel, M., Schneider, M., Zenger, C. (1992): A combination technique for the solution of sparse grid problems. In: P. de Groen, R. Beauwens (eds.), Proceedings of the IMACS International Symposium on Iterative Methods in Linear Algebra, Brüssel, April, 1991. Elsevier, Amsterdam
5. M. Griebel, V. Thurner (1995): The efficient solution of fluid dynamics problems by the combination technique. International Journal of Numerical Methods for Heat and Fluid Flow, **5**(3):251–269
6. Hemker, P., Pflaum, C. (1997): Approximation on partially ordered sets of regular grids. Applied Numerical Mathematics, volume 25, number 1, pages 41–54
7. Krizek, M. (1994): Superconvergence Phenomena in the Finite Element Method. Comput. Method Appl. Mech. Engrg., **116**, 157–163
8. Krizek, M., Neittaanmaki, P. (1987): On Superconvergence Techniques. Acta Appl. Math., **9**, 175–198
9. Lin, Q. (1990): An Integral Identity and Interpolated Postprocess in Superconvergence. Research Report 90-07, Inst. of Sys. Sci., Academia Sinica
10. Lin, Q., Zhu, Q. (1994): The Preprocessing and Postprocessing for the Finite Element Method. Shanghai Sci. and Tech. Publishers, in Chinese
11. Pflaum, C. (1997): Convergence of the combination technique for second-order elliptic differential equation. SIAM J. Numer. Anal. **34**, No. 6, 2431–2455
12. Pflaum, C. (1994): Discretization of Second Order Elliptic Differential Equations on Sparse Grids. In: Bandle C., Bemelmans J., Chipot M., Saint Jean Paulin J., Shafrir I. (eds.), Progress in partial differential equations, Pont-á-Mousson 1994, vol. 2, calculus of variations, applications and computations, Pitman Research Notes in Mathematics Series. Longman
13. Pflaum, C. (1995): Diskretisierung elliptischer Differentialgleichungen mit dünnen Gittern. Dissertation, Technische Universität München
14. Wahlbin, L. B. (1995): Superconvergence in Galerkin Finite Element Methods. Number 1605 in Lecture notes in Math. Springer
15. Zenger, C. (1991): Sparse grids. In: Hackbusch W. (ed.), Parallel Algorithms for Partial Differential Equations: Proceedings of the Sixth GAMM-Seminar, Kiel, January 1990, volume 31 of Notes on Numerical Fluid Mechanics. Vieweg, Braunschweig
16. Zhou, A. (1996): Global Superconvergence Approximations of the Mixed Finite Element Method for the Stokes Problem and the Linear Elasticity Equation. RAIRO Mathematical Modelling & Numerical Analysis, **30**, No. 4, 401–411
17. Zhou, A., Li, J. (1994) The Full Approximation Accuracy for the Stream Function-Vorticity-Pressure method. Numer. Math., **68**, 427–435