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Convergence of nonstationary cascade algorithms

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Summary. A nonstationary multiresolution of $L^2(\mathbb{R}^s)$ is generated by a sequence of scaling functions $\phi_k \in L^2(\mathbb{R}^s),\ k \in \mathbb{Z}$. We consider (ϕ_k) that is the solution of the nonstationary refinement equations $\phi_k = |M| \sum_j h_{k+1}(j)\phi_{k+1}(M\,\cdot\ -j),\ k \in \mathbb{Z}$, where h_k is finitely supported for each k and M is a dilation matrix. We study various forms of convergence in $L^2(\mathbb{R}^s)$ of the corresponding nonstationary cascade algorithm $\phi_{k,n} = |M| \sum_j h_{k+1}(j)\phi_{k+1,n-1}(M\,\cdot\ -j)$, as k or n tends to ∞. It is assumed that there is a stationary refinement equation at ∞ with filter sequence h and that $\sum_k |h_k(j) - h(j)| < \infty$. The results show that the convergence of the nonstationary cascade algorithm is determined by the spectral properties of the transition operator associated with h.

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1. Introduction

In the applications of stationary wavelet decomposition and reconstruction, the filter sequences are independent of the resolution levels. In some cases, especially in multiwavelet decomposition and reconstruction, preprocessing is necessary. Preprocessing can be viewed as a step in the decomposition using a different filter sequence. Different filter sequences at different multiresolution levels give rise to different scaling functions and different wavelets at different multiresolution levels. This leads to nonstationary multiresolution. Stationary multiresolution does not exist in Hilbert spaces, such as

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Sobolev spaces and periodic L^2-spaces, where unitary dilation operators do not exist. Therefore, in the multiresolution decomposition of such a space, it is natural to consider nonstationary multiresolution.

In this paper we study the solution of nonstationary refinement equations associated with nonstationary multiresolution of $L^2(\mathbb{R}^s)$, where s is a positive integer. A characterization of $\phi_k \in L^2(\mathbb{R}^s)$, $k \in \mathbb{Z}$, that generate a nonstationary multiresolution has been established by deBoor, DeVore and Ron in [1]. We shall assume that $\phi_k = \phi_0$ for $k < 0$. Then their results can be succinctly stated as follows.

Theorem 1.1 (deBoor, DeVore and Ron) *Suppose that* $\{\phi_k(\,\cdot\, - 2^k j) : j \in \mathbb{Z}^s\}$ *is a Riesz basis of its closed linear span* V_k *and assume that* $V_k \subset V_{k+1},\ k \in \mathbb{Z}$. *Then* (V_k) *is a multiresolution of* $L^2(\mathbb{R}^s)$ *if and only if*

$$\bigcap_{k\in\mathbb{Z}} \left\{ u \in \mathbb{R}^s : \hat{\phi}_k(u) = 0 \right\} \quad \textit{is a null set.}$$

The above theorem assumes that $V_k \subset V_{k+1},\ k \in \mathbb{Z}$, which is equivalent to

$$\phi_k(x) = \sum_{j\in\mathbb{Z}^s} 2^s h_{k+1}(j)\phi_{k+1}(x - 2^{k+1}j), \quad x \in \mathbb{Z}^s,\ k = 0, 1, \ldots,$$

for some family of sequences $h_k \in \ell_2(\mathbb{Z}^s)$, $k = 1, 2, \ldots$.

We shall consider the following more general form of *nonstationary refinement equations*

$$\phi_k(x) = |M| \sum_{j\in\Omega} h_{k+1}(j)\phi_{k+1}(Mx - j),$$
$$x \in \mathbb{R}^s,\ k = 0, 1, 2, \ldots, \tag{1.1}$$

where for $k = 0, 1, 2, \ldots, \phi_k$ is a tempered distribution on $\mathbb{R}^s$, h_k is a finitely supported sequence with support in Ω, a bounded subset of $\mathbb{Z}^s$, and M is an $s \times s$ integer matrix with determinant $|M| \geq 2$ and $\lim_{n\to\infty} M^{-n} = 0$. We shall assume throughout that there is a sequence h supported on Ω such that for each $j \in \Omega$,

$$\sum_{k=1}^{\infty} |h_k(j) - h(j)| < \infty, \tag{1.2}$$

and

$$\sum_{j\in\Omega} h(j) = 1. \tag{1.3}$$

It follows that

$$\sum_{k=1}^{\infty} |\sum_{j} h_k(j) - 1| < \infty. \tag{1.4}$$

In the Fourier domain, (1.1) is equivalent to

$$\hat{\phi}_k(u) = H_{k+1}(Nu)\hat{\phi}_{k+1}(Nu), \quad u \in \mathbb{R}^s, \tag{1.5}$$

where $N = (M^{-1})^{\mathrm{T}}$, and

$$H_k(u) = \sum_{j\in\Omega} h_k(j)\mathrm{e}^{-\mathrm{i}ju}, \quad k = 0, 1, 2, \dots,\ u \in \mathbb{R}^s. \tag{1.6}$$

For convenience we shall call H_k the Fourier transform of h_k. A sequence (ϕ_k) is a *solution* of (1.1) if it satisfies (1.1) or (1.5) and

$$\hat{\phi}_{k+n}(N^n u) \to 1, \quad \text{as } n \to \infty, \tag{1.7}$$

uniformly in $k \geq 1$ and locally uniformly in $u \in \mathbb{R}^s$.

If $h_k = h$ is independent of k, then $\phi_k = \phi$ is also independent of k, and the equations in (1.1) reduce to the ordinary refinement equation

$$\phi(x) = |M| \sum_{j\in\Omega} h(j)\phi(Mx - j), \quad x \in \mathbb{R}^s. \tag{1.8}$$

In this case (1.7) is equivalent to $\hat{\phi}(0) = 1$. Equation (1.8) has been extensively studied recently in connection with wavelets analysis ([4, 6, 9–15, 19, 20, 22–30, 36–39]). Equation (1.8) can also be viewed as the limiting case of the nonstationary refinement equation (1.1), and we shall refer to it as the *ideal refinement equation* associated with (1.1).

Let Γ comprise coset representatives of $\mathbb{Z}^s/M\mathbb{Z}^s$. We say that h is *fundamental* if for any $\gamma \in \Gamma$,

$$\sum_j h(Mj + \gamma) = \frac{1}{|M|}. \tag{1.9}$$

Clearly h satisfies (1.3) if it is fundamental.

We choose a bounded set $K \subset \mathbb{R}^s$ satisfying

$$\bigcup_{j\in\Omega} M^{-1}(K + j) \subset K. \tag{1.10}$$

We now choose a sequence $\phi_{k,0}$, $k = 0, 1, 2, \dots$, in $L^2(\mathbb{R}^s)$ with support in K satisfying

$$\phi_{k,0} \to \tilde{\phi}_0 \text{ in } L^2(\mathbb{R}^s), \text{ as } k \to \infty, \tag{1.11}$$

and

$$\hat{\phi}_{k+n,0}(N^n u) \to 1, \text{ as } n \to \infty, \tag{1.12}$$

uniformly in k and locally uniformly in u. Trivially (1.11) is satisfied if $\phi_{k,0}$ is independent of k. We note that $\hat{\phi}_{k,0}$ is continuous and $\lim_{n\to\infty} N^n = 0$. Thus (1.12) is satisfied if $\phi_{k,0}$ is independent of k and $\hat{\phi}_{k,0}(0) = 1$.

Now for $n = 1, 2, \ldots,\ k = 0, 1, 2, \ldots$, we define $\phi_{k,n}$ in $L^2(\mathbb{R}^s)$ by

(1.13) $$\phi_{k,n}(x) = |M| \sum_{j\in\Omega} h_{k+1}(j)\phi_{k+1,n-1}(Mx - j), \quad x \in \mathbb{R}^s.$$

We call (1.13) a *nonstationary cascade algorithm.* By (1.10) we see by induction on n that $\phi_{k,n}$ has support in K for all k and n. Applying the relation (1.13) iteratively leads to

(1.14) $$\phi_{k,n}(x) = |M| \sum_{j\in\Omega} s_{k,n}(j)\phi_{k+n,0}(M^n x - j), \quad x \in \mathbb{R}^s,$$

where

$$s_{k,n} := S_{k+n} \cdots S_{k+1}\delta_0,$$

and $S_m : \ell^\infty(\mathbb{Z}^s) \to \ell^\infty(\mathbb{Z}^s)$ is the *subdivision operator* with mask h_m defined by

(1.15) $$(S_m b)(j) = |M| \sum_{\nu\in\mathbb{Z}^s} h_m(j - M\nu)b(\nu), \quad j \in \mathbb{Z}^s.$$

The sequence of subdivision operators $(S_m)_{m=1}^\infty$ is called a *nonstationary subdivision process.* The process is said to *converge* if for each $k = 0, 1, \ldots,$ there is a compactly supported function $\phi_k \in C(\mathbb{R}^s)$ such that

$$\lim_{n\to\infty} ||s_{k,n} - \phi_k(M^{-n}\,\cdot\,)||_\infty = 0.$$

If $\phi_{k,0}$ is stable, then the nonstationary subdivision process $(S_m)_{m=1}^\infty$ converges to (ϕ_k) if and only if

$$\lim_{n\to\infty} ||\phi_{k,n} - \phi_k||_\infty = 0.$$

Thus the limiting sequence of functions of a nonstationary subdivision process is the solution of the corresponding nonstationary refinement equations. Convergence of nonstationary subdivision processes with dilation $M = 2I$, and the smoothness of the limiting functions were recently studied in [13, 2, 14] and [11]. The stationary case in which $h_k = h$ for all k, has been studied earlier in [3], [17] and [22], and the corresponding stationary refinement equations are very much investigated in the theory of wavelets. In [14], Dyn and Levin showed that if (1.2) holds, the L^∞-convergence of a nonstationary subdivision process (S_k) with mask sequence (h_k) follows from that of the equivalent stationary subdivision process S with mask h. They also obtained sufficient conditions based on the convergence of (S_k) to S, in order that the solutions of the nonstationary refinement equations with mask sequence (h_k) would inherit the stability and smoothness from the solution of the ideal stationary refinement equation with mask h. In this paper we shall investigate the convergence in $L^2(\mathbb{R}^s)$ of the cascade sequences $(\phi_{k,n})$ to the solution (ϕ_k) of (1.1) as $n \to \infty$. We consider both weak and strong convergence of the cascade algorithms and also the weak convergence of the derivatives of the cascade sequences.

For f, g in $L^2(\mathbb{R}^s)$ we write $\langle f, g\rangle = \int_{\mathbb{R}^s} f\bar{g}$. For $k, n = 0, 1, \ldots, j \in \mathbb{Z}^s$, define a sequence $b_{k,n}(j)$ by

$$b_{k,n}(j) = \langle \phi_{k,n},\ \phi_{k,n}(\cdot - j)\rangle. \tag{1.16}$$

As $\phi_{k,n}$ has support in the bounded set K, we may choose a finite set $I \subset \mathbb{Z}^s$ so that $b_{k.n}(j) = 0$ for $j \notin I$ and for all $k, n = 0, 1, 2, \ldots$. In fact I can be chosen so that $K \cap (K + j) = \emptyset$ for $j \notin I$. Now for $n = 1, 2, \ldots,\ k = 0, 1, 2, \ldots$, (1.13) and (1.16) give

$$b_{k,n}(j) = |M| \sum_{\nu\in\mathbb{Z}^s} g_{k+1}(Mj - \nu) b_{k+1,n-1}(\nu), \quad j \in I, \tag{1.17}$$

where g_{k+1} is the autocorrelation of h_{k+1}, i.e.

$$g_{k+1}(j) := \sum_{\ell\in\Omega} h_{k+1}(\ell)\overline{h_{k+1}(\ell - j)}, \quad j \in \mathbb{Z}^s. \tag{1.18}$$

To simplify the relation (1.17) we introduce the *transition operators* $T_k : \mathbb{C}^I \to \mathbb{C}^I$, $k = 1, 2, \ldots$, by defining

$$T_k a(j) = |M| \sum_{\nu\in\mathbb{Z}^s} g_k(Mj - \nu)a(\nu), \quad j \in I,\ a \in \mathbb{C}^I. \tag{1.19}$$

Then (1.17) becomes

$$b_{k,n} = T_{k+1} b_{k+1,n-1}, \quad k = 0, 1, \ldots,\ n = 1, 2, \ldots. \tag{1.20}$$

Iterating (1.20) leads to

$$b_{k,n} = \left(\prod_{\ell=k+1}^{k+n} T_\ell\right) b_{k+n,0}, \quad k, n = 0, 1, 2, \ldots. \tag{1.21}$$

By (1.2), for $j \in \mathbb{Z}^s$,

$$\sum_{k=0}^{\infty} |g_k(j) - g(j)| < \infty, \tag{1.22}$$

where g is the autocorrelation of h. Let $T : \mathbb{C}^I \to \mathbb{C}^I$ be defined by g as T_k is defined by g_k in (1.19) so that

$$Ta(j) = |M| \sum_{\nu\in\mathbb{Z}^s} g(Mj - \nu)a(\nu), \quad j \in I,\ a \in \mathbb{C}^I. \tag{1.23}$$

Then (1.19), (1.22) and (1.23) lead to

$$\sum_{k=0}^{\infty} \|T_k - T\| < \infty. \tag{1.24}$$

We now introduce some terminology before we state our main results on the L^2-convergence of nonstationary cascade algorithms. A square matrix A is said to satisfy *Condition* E^{**} if it has unit spectral radius and all its eigenvalues on the unit circle are nondegenerate. If in addition, 1 is the only eigenvalue on the unit circle, then A is said to satisfy *Condition* E^*. We say that A satisfies *Condition E* if it satisfies Condition E^* and 1 is a simple eigenvalue.

Theorem 1.2 *Suppose that* (1.2) *and* (1.3) *hold. The following are equivalent.*

(a) *T satisfies Condition E^{**}.*
(b) *For any $(\phi_{k,0})$ satisfying* (1.11) *and* (1.12)*, $\phi_{k,n}$ converges weakly to ϕ_k in $L^2(\mathbb{R}^s)$ as $n \to \infty$, uniformly in k, where (ϕ_k) is the L^2-solution of* (1.1)*, and ϕ_k converges weakly in $L^2(\mathbb{R}^s)$ to ϕ which is the solution of the ideal refinement equation* (1.8).
(c) *For any $(\phi_{k,0})$ satisfying* (1.11) *and* (1.12)*, $\phi_{k,n}$ converges weakly to $\tilde{\phi}_n$ in $L^2(\mathbb{R}^s)$ as $k \to \infty$, uniformly in n, where $(\tilde{\phi}_n)$ is the cascade sequence for h with starting function $\tilde{\phi}_0$, and $\tilde{\phi}_n$ converges weakly in $L^2(\mathbb{R}^s)$ to ϕ which is the solution of* (1.8).

Theorem 1.3 *Suppose that* (1.2) *and* (1.3) *hold. The following are equivalent.*

(a) *T satisfies condition E and h is fundamental.*
(b) *For any $(\phi_{k,0})$ satisfying* (1.11), (1.12)*, and*

$$\sum_{j\in\mathbb{Z}^s} \tilde{\phi}_0(\cdot - j) = 1, \quad \text{almost everywhere,} \tag{1.25}$$

the cascade sequence $\phi_{k,n}$ converges strongly to ϕ_k in $L^2(\mathbb{R}^s)$ as $n \to \infty$, uniformly in k, where (ϕ_k) is the solution of (1.1)*, and ϕ_k converges strongly in $L^2(\mathbb{R}^s)$ to ϕ which is the solution of* (1.8).
(c) *For any $(\phi_{k,0})$ satisfying* (1.11), (1.12) *and* (1.25) *the cascade sequence $\phi_{k,n}$ converges strongly to $\tilde{\phi}_n$ in $L^2(\mathbb{R}^s)$ as $k \to \infty$, uniformly in n, where $(\tilde{\phi}_n)$ is the cascade sequence for h with starting function $\tilde{\phi}_0$, and $\tilde{\phi}_n$ converges strongly in $L^2(\mathbb{R}^s)$ to ϕ which is the solution of* (1.8).

Remark 1. The conditions on $(\phi_{k,0})$ in Theorems 1.2 and 1.3 are easily satisfied if $(\phi_{k,0})$ is independent of k.

Corollary 1.1 *Suppose $h_k = h$ for all k. Then the following are equivalent.*

(a) *T satisfies Condition E^{**}.*

(b) *For any initial function $\tilde{\phi}_0$, with $\widehat{\tilde{\phi}}_0(0) = 1$, the cascade sequence $\tilde{\phi}_n$ converges weakly to ϕ in $L^2(\mathbb{R}^s)$ as $n \to \infty$, where ϕ is the solution of the ideal refinement equation* (1.8).

Corollary 1.2 *Suppose $h_k = h$ for all k. Then the following are equivalent.*

(a) *T satisfies Condition E and h is fundamental.*

(b) *For any initial function $\tilde{\phi}_0$, satisfying $\widehat{\tilde{\phi}}_0(0) = 1$ and* (1.25)*, the cascade sequence $\tilde{\phi}_n$ converges strongly to ϕ in $L^2(\mathbb{R}^s)$ as $n \to \infty$, where ϕ is the solution of the ideal refinement equation* (1.8).

The result (a) implies (b) in Corollary 1.1 was proved by Lawton, Lee and Shen [33]. In this case, the equivalence of (a) and (b) was subsequently established by Cohen, Gröchenig and Villemoes [7]. Corollary 1.2 was the main result in [33]. It was independently established by Han Bin and Jia [18] in a more general setting of L^p-convergence. In one dimension it was first considered by Strang [40]. The same result in a different form embedded in the theory of L^p-convergence of subdivision processes was obtained slightly earlier by Goodman, Micchelli and Ward [17] and also independently by Jia [22]. Recently, Shen [39] has extended Corollary 1.2 to the vector case.

In Sect. 2 we shall prove Theorem 1.2 and 1.3, while in Sect. 3 we consider the weak convergence of the derivatives of the nonstationary cascade algorithm.

2. Convergence of nonstationary cascade algorithms in $L^2(\mathbb{R}^s)$

Let B be any compact set in $\mathbb{R}^s$ containing the origin. Since (h_k) is uniformly bounded and $\lim_{n\to\infty} N^n = 0$, for $k = 0, 1, \ldots$, and $u \in B$,

$$|H_k(N^\ell u) - H_k(0)| \le C\eta^\ell,$$

for some $0 < \eta < 1$ and C independent of k and u. Since $|H_k(N^\ell u) - 1| \le C\eta^\ell + |H_k(0) - 1|$, we see from (1.4) that

$$\sum_{\ell=1}^{\infty} |H_{k+\ell}(N^\ell u) - 1| < \infty \tag{2.1}$$

uniformly for u in B and $k \ge 0$ and the sum is bounded in k and $u \in B$. It follows that the infinite product

$$P_k(u) := \prod_{\ell=1}^{\infty} H_{k+\ell}(N^\ell u), \quad u \in \mathbb{R}^s, \tag{2.2}$$

converges uniformly in $k \geq 0$ and u in B. A standard argument shows that for $k = 0, 1, \ldots,$ $P_k(u)$ is of polynomial growth as $|u| \to \infty$. Thus, it is the Fourier transform of a tempered distribution. By (2.1),

$$\left| \prod_{\ell=1}^{n} H_{k+\ell}(N^\ell u) \right| \leq C,$$

for all k and n and for all $u \in B$, where C is an absolute constant. Thus

$$\left| \prod_{\ell=1}^{\infty} H_{k+n+\ell}(N^{n+\ell} u) - 1 \right| \leq C \sum_{\ell=n+1}^{\infty} |H_{k+\ell}(N^\ell u) - 1| \to 0, \text{ as } n \to \infty, \tag{2.3}$$

uniformly in $k \geq 0$ and locally uniformly in $u \in \mathbb{R}^s$. It follows that $P_{k+n}(N^n u) \to 1$, as $n \to \infty$. Further,

$$P_k(u) = \prod_{\ell=1}^{\infty} H_{k+\ell}(N^\ell u) = H_{k+1}(Nu) P_{k+1}(Nu). \tag{2.4}$$

Thus P_k is the Fourier transform of a distributional solution of (1.1). The solution is unique. For, if (ϕ_k) is a solution of (1.1), iterating (1.5) leads to

$$\hat{\phi}_k(u) = \prod_{\ell=1}^{n} H_{k+\ell}(N^\ell u) \hat{\phi}_{k+n}(N^n u), \quad u \in \mathbb{R}^s, \tag{2.5}$$

for all $n \geq 1$. Taking the limit as $n \to \infty$, and using (1.7) we have

$$\hat{\phi}_k(u) = \prod_{\ell=1}^{\infty} H_{k+\ell}(N^\ell u) = P_k(u), \quad u \in \mathbb{R}^s. \tag{2.6}$$

Now, taking the Fourier transforms of the functions in (1.13) gives

$$\hat{\phi}_{k,n}(u) = H_{k+1}(Nu) \hat{\phi}_{k+1,n-1}(Nu), \tag{2.7}$$

where H_k is given in (1.6). Thus we have for $k, n = 0, 1, 2, \ldots,$

$$\hat{\phi}_{k,n}(u) = \prod_{\ell=1}^{n} H_{k+\ell}(N^\ell u) \hat{\phi}_{k+n,0}(N^n u), \quad u \in \mathbb{R}^s. \tag{2.8}$$

Then by (1.12) and (2.8),

$$\lim_{n \to \infty} \hat{\phi}_{k,n}(u) = P_k(u) \tag{2.9}$$

where the convergence is uniform over $k \geq 1$ and locally uniform in u.

The analysis on the convergence of the nonstationary cascade algorithm requires results on products of matrices. The space of all complex-valued $r \times r$ matrices will be denoted by $\mathbb{C}^{r \times r}$.

Proposition 2.1 *Let $\|\cdot\|$ be an operator norm on $\mathbb{C}^{r\times r}$, $A_\ell \in \mathbb{C}^{r\times r}$, $\ell = 1, 2, \ldots,$ and suppose there is an $A \in \mathbb{C}^{r\times r}$ of spectral radius $\rho(A) \geq 1$ satisfying*

$$\sum_{\ell=1}^{\infty} \|A_\ell - A\| < \infty. \tag{2.10}$$

Then $\left\|\prod_{\ell=k+1}^{k+n} A_\ell\right\|$ is bounded for all n and k if and only if A satisfies Condition E^{**}.

Proof. Suppose that A satisfies Condition E^{**}. Then we may choose an operator norm on $\mathbb{C}^{r\times r}$ with $\|A\| = 1$, and (2.10) still holds. Then

$$\begin{aligned}\|\prod_{\ell=k+1}^{k+n} A_\ell\| &\leq \prod_{\ell=k+1}^{k+n} (\|A_\ell - A\| + 1) \\ &\leq \prod_{\ell=k+1}^{k+n} \exp(\|A_\ell - A\|) \\ &\leq \exp(\sum_{\ell=1}^{\infty} \|A_\ell - A\|)\end{aligned}$$

for all k and n. Conversely, suppose that $\|\prod_{\ell=k+1}^{k+n} A_\ell\|$ is uniformly bounded in k and n. For each n,

$$\|A^n - \prod_{\ell=k+1}^{k+n} A_\ell\| \leq C_n \sum_{\ell=k+1}^{k+n} \|A_\ell - A\| \to 0, \text{ as } k \to \infty.$$

For all n and k,

$$\begin{aligned}\|A^n\| &\leq \|A^n - \prod_{\ell=k+1}^{k+n} A_\ell\| + \|\prod_{\ell=k+1}^{k+n} A_\ell\| \\ &\leq \|A^n - \prod_{\ell=k+1}^{k+n} A_\ell\| + C.\end{aligned}$$

Taking the limit as $k \to \infty$, gives $\|A^n\| \leq C$, for all n. Since the spectral radius of $\rho(A) \geq 1$, it follows that A satisfies Condition E^{**}. □

We also need the following theorem in [16].

Theorem 2.1 *Suppose that $A_\ell,\ \ell = 1, 2, \ldots,$ and A are matrices in $\mathbb{C}^{r\times r}$ satisfying* (2.10). *Then the following are equivalent.*

(a) *A satisfies Condition E^*.*

(b) *The matrix product* $\prod_{\ell=k+1}^{k+n} A_\ell$ *converges as* $n \to \infty$, *uniformly in* k.
Moreover, if (a) *or* (b) *holds, then*

$$\lim_{k\to\infty}\lim_{n\to\infty}\prod_{\ell=k+1}^{k+n} A_\ell = \lim_{n\to\infty}\lim_{k\to\infty}\prod_{\ell=k+1}^{k+n} A_\ell = \lim_{n\to\infty} A^n.$$

Proof of Theorem 1.2. Suppose T satisfies condition E^{**}. By Proposition 2.1, $\prod_{\ell=k+1}^{k+n} T_\ell$ is uniformly bounded in k and n. By (1.21) and (1.11), $\|b_{k,n}\|$ is bounded in k and n. In particular, $b_{k,n}(0) = \|\phi_{k,n}\|_2^2$ is bounded in k and n. So for each $k \geq 0$, there is a subsequence (ϕ_{k,n_j}) which converges weakly to some function ϕ_k in $L^2(\mathbb{R}^s)$ as $j \to \infty$. Thus $\hat{\phi}_{k,n_j} \to \hat{\phi}_k$ weakly as $j \to \infty$. On the other hand, by (2.9), $\hat{\phi}_{k,n_j}(u)$ converges to

$$P_k(u) := \prod_{\ell=1}^{\infty} H_{k+\ell}(N^\ell u),$$

as $j \to \infty$ uniformly in k and locally uniformly in u. Thus

$$\hat{\phi}_k(u) = P_k(u), \quad u \in \mathbb{R}^s, \tag{2.11}$$

and $\hat{\phi}_{k,n_j}(u)$ converges to $\hat{\phi}_k(u)$ as $j \to \infty$ uniformly in k and locally uniformly in u.

Note that for each k, $\langle \hat{\phi}_{k,n_j}, \hat{\phi}_k\rangle$ converges to $\langle \hat{\phi}_k, \hat{\phi}_k\rangle = \|\hat{\phi}_k\|_2^2$ as $j \to \infty$, and since $|\langle \hat{\phi}_{k,n_j}, \hat{\phi}_k\rangle| \leq \|\hat{\phi}_{k,n_j}\|_2 \|\hat{\phi}_k\|_2$ and $\|\hat{\phi}_{k,n_j}\|_2$ is bounded in k and j, it follows that $\|\hat{\phi}_k\|_2$ is bounded in k. Thus $\|\phi_k\|_2$ is bounded in k.

Now take $k \geq 0$ and f in $L^2(\mathbb{R}^s)$. Since $\|\hat{\phi}_{k,n}\|_2$ is bounded in k and n, and $\|\hat{\phi}_k\|_2$ is bounded in k, we can choose K with $\|\hat{\phi}_{k,n} - \hat{\phi}_k\|_2 \leq K$ for all k and n. Take $\epsilon > 0$ and choose N with $\int_{|u|\geq N} |f(u)|^2 du < \epsilon^2$. Then

$$\left|\int_{|u|\geq N} f(u)(\hat{\phi}_{k,n}(u) - \hat{\phi}_k(u))du\right| \leq \epsilon K \text{ for all } k, n.$$

Also

$$\left|\int_{|u|\leq N} f(u)(\hat{\phi}_{k,n}(u) - \hat{\phi}_k(u))du\right|$$
$$\leq \|f\|_2 \left(\int_{|u|\leq N} |\hat{\phi}_{k,n}(u) - \hat{\phi}_k(u))|^2 du\right)^{1/2},$$

which tends to zero uniformly in k as $n \to \infty$. Thus $\int_{\mathbb{R}^s} f(\hat{\phi}_{k,n} - \hat{\phi}_k)$ tends to zero uniformly in k as $n \to \infty$. So $\hat{\phi}_{k,n} \to \hat{\phi}_k$ weakly as $n \to \infty$ uniformly in k. Thus $\phi_{k,n} \to \phi_k$ weakly as $n \to \infty$ uniformly in k.

We note that $\|\phi_k\|_2$ is uniformly bounded and

$$\hat{\phi}_k(u) = \prod_{\ell=1}^{\infty} H_{k+\ell}(N^\ell u), \quad u \in \mathbb{R}^s.$$

By (2.4), (ϕ_k) satisfies (1.1). By (2.3), $\hat{\phi}_k$ satisfies (1.7). Hence, (ϕ_k) is the solution of (1.1).

Since $\|\phi_k\|_2$ is bounded there is a subsequence (ϕ_{n_j}) which converges weakly to φ in $L^2(\mathbb{R}^s)$. Now by (1.2),

$$\sum_{k=0}^{\infty} |H_k(u) - H(u)| < \infty, \qquad \text{for all } u \in \mathbb{R}^s, \tag{2.12}$$

where

$$H(u) := \sum_{j \in \Omega} h(j) \mathrm{e}^{-\mathrm{i}ju}. \tag{2.13}$$

The infinite product $\prod_{\ell=1}^{\infty} H(N^\ell u)$ is of power growth as $|u| \to \infty$. Thus it is the Fourier transform of a distribution ϕ which is the solution of the refinement equation (1.8). We now show that if T satisfies Condition E^{**} then $\hat{\phi}_k(u)$ converges locally uniformly in u to $\hat{\phi}(u)$, as $k \to \infty$. As in the derivation of (2.3), for u in any compact set B,

$$\begin{aligned} |\hat{\phi}_k(u) - \hat{\phi}(u)| &= |\prod_{\ell=1}^{\infty} H_{k+\ell}(N^\ell u) - \prod_{\ell=1}^{\infty} H(N^\ell u)| \\ &\le C \sum_{\ell=k+1}^{\infty} |H_\ell(N^{\ell-k} u) - H(N^{\ell-k} u)|. \end{aligned}$$

This leads to

$$|\hat{\phi}_k(u) - \hat{\phi}(u)| \le C \sum_{j \in \Omega} \sum_{\ell=k+1}^{\infty} |h_\ell(j) - h(j)|,$$

which tends to 0 uniformly in $u \in B$, as $k \to \infty$, by (1.2). By the same argument as above we conclude that $\phi = \varphi$ and that ϕ_k converges weakly to ϕ in $L^2(\mathbb{R}^s)$.

Now we assume that (b) is satisfied. Let $b(j) = \langle \phi, \phi(\cdot - j) \rangle,\ j \in I$. By (1.23) and (1.8), $Tb = b$. Thus b is an eigenvector of T with eigenvalue 1. In particular $\rho(T) \ge 1$. Without loss of generality we may assume that

$[0,1]^s \subset K$, and we choose any function ψ in $L^2(\mathbb{R}^s)$ with support in $[0,1]^s$ and $\hat{\psi}(0)=1$. Take an arbitrarily fixed $\ell \in I$ and for $k=0,1,2,\ldots$, define

$$\psi_{k,0}=\psi, \quad \phi_{k,0}=\psi(\cdot+\ell).$$

Note that $(\phi_{k,0})$ and $(\psi_{k,0})$ satisfy (1.11) and (1.12). For $k,n=0,1,2,\ldots$, we define $\phi_{k,n}$ by (1.13) and similarly we define $\psi_{k,n}$ by (1.13) with $\phi_{k,0}$ replaced by $\psi_{k,0}$. For $k,n=0,1,2,\ldots$, $j \in \mathbb{Z}^s$, we define $c_{k,n}(j)$ by

$$c_{k,n}(j)=\langle \psi_{k,n}, \phi_{k,n}(\cdot - j)\rangle.$$

Note that $c_{k,n}(j)=0$ for $j \notin I$ and

$$c_{k,0}(j)=\delta_{j,\ell}, \quad j \in I. \tag{2.14}$$

It follows from (1.13), as in the derivation of (1.21), that

$$c_{k,n}=\left(\prod_{\nu=k+1}^{k+n} T_\nu\right) c_{k+n,0}, \quad n,k=0,1,2,\ldots. \tag{2.15}$$

Now by (b), $\phi_{k,n} \to \phi_k$ and $\psi_{k,n} \to \phi_k$ weakly in $L^2(\mathbb{R}^s)$ as $n \to \infty$ uniformly in k. By the Uniform Boundedness Theorem, $\|\phi_{k,n}\|_2$ and $\|\psi_{k,n}\|_2$ are uniformly bounded in k and n. Since

$$|c_{k,n}(j)| \le \|\phi_{k,n}\|_2 \, \|\psi_{k,n}\|_2,$$

$c_{n,k}(j)$ is uniformly bounded in k and n. Recalling (2.14) and (2.15), it follows that $\|\prod_{\nu=k+1}^{k+n} T_\nu\|$ is bounded in k and n. Since $\rho(T) \ge 1$, it follows that T satisfies Condition E^{**}.

The equivalence of (a) and (c) is proved in the same way. $\square$

The proof of Theorem 1.3 requires the following lemma.

Lemma 2.1 *Suppose that A_ℓ, $\ell=1,2,\ldots$, and A are matrices in $\mathbb{C}^{r\times r}$ satisfying* (2.10)*, and A satisfies Condition E^*. Let E_1 be the eigenspace of A corresponding to the eigenvalue 1, and $\mathbb{C}^r = E_1 \oplus Q$, where Q is invariant under A. Then for any sequence (q_n) in $\mathbb{C}^r$ converging to $q \in Q$,*

$$\lim_{n\to\infty}\left(\prod_{\ell=k+1}^{k+n} A_\ell\right) q_n = 0, \quad \textit{uniformly in } k.$$

Proof. Choose an operator norm on $\mathbb{C}^{r\times r}$ with $\|A\|=1$. Then for $k,n \ge 0$, Proposition 2.1 gives

$$\left\|\prod_{\ell=k+1}^{k+n} A_\ell\right\| \le C,$$

where C is a constant independent of k and n. Then for any $k \geq 0$, $1 \leq m \leq n$,

$$\left\| \left(\prod_{\ell=k+1}^{k+n} A_\ell \right) - \left(\prod_{\ell=k+1}^{k+m} A_\ell \right) A^{n-m} \right\| \leq C^2 \sum_{\ell=k+m+1}^{k+n} \|A_\ell - A\|. \tag{2.16}$$

For any $q \in Q$,

$$\left\| \left(\prod_{\ell=k+1}^{k+n} A_\ell \right) q - \left(\prod_{\ell=k+1}^{k+m} A_\ell \right) A^{n-m} q \right\| \leq C^2 \|q\| \sum_{\ell=k+m+1}^{k+n} \|A_\ell - A\|.$$

Since the spectral radius $\rho(A|_Q) < 1$, $A^{n-m} q \to 0$ as $n - m \to \infty$. Letting $m \to \infty$ and $n - m \to \infty$, this implies that $\left(\prod_{\ell=k+1}^{k+n} A_\ell \right) q \to 0$, and hence $\left(\prod_{\ell=k+1}^{k+n} A_\ell \right) q_n \to 0$ as $n \to \infty$ uniformly in k. $\square$

Proof of Theorem 1.3. Suppose that (a) holds. Since T satisfies Condition E, Theorem 1.2 implies that the cascade sequence $\phi_{k,n}$ converges weakly to ϕ_k in $L^2(\mathbb{R})$ as $n \to \infty$ uniformly in k, for any starting sequence $(\phi_{k,0})$ satisfying (1.11) and (1.12). Further, (ϕ_k) is the solution of the nonstationary refinement equation (1.1), and (ϕ_k) converges weakly to the solution ϕ of (1.8) in $L^2(\mathbb{R}^s)$. We shall establish (b) by showing that for $k = 0, 1, \ldots,$ $\|\phi_{k,n}\| \to \|\phi_k\|$ as $n \to \infty$, and $\|\phi_k\| \to \|\phi\|$ as $k \to \infty$.

Recall that for $k, n = 0, 1, \ldots,$

$$b_{k,n}(j) = \langle \phi_{k,n}, \phi_{k,n}(\cdot - j) \rangle, \quad j \in \mathbb{Z}^s,$$

and $(b_{k,n})$ satisfies (1.20) and (1.21) and are finitely supported with common support $I \subset \mathbb{R}^s$. For $k = 0, 1, \ldots,$ define

$$b_k(j) := \langle \phi_k, \phi_k(\cdot - j) \rangle, \quad j \in \mathbb{Z}^s, \tag{2.17}$$

and

$$b(j) := \langle \phi, \phi(\cdot - j) \rangle, \quad j \in \mathbb{Z}^s, \tag{2.18}$$

which are all supported on I.

As in the derivation of (1.20), the equation (2.17) gives

$$b_k = T_{k+1} b_{k+1}, \quad k = 0, 1, \ldots. \tag{2.19}$$

Similarly, (2.18) gives $b = Tb$. Thus b is the eigenvector of T with eigenvalue 1. Since 1 is a simple eigenvalue of $T : \mathbb{C}^I \to \mathbb{C}^I$ and all its other eigenvalues lie inside the unit circle, we can decompose $\mathbb{C}^I = M \oplus Q$, where

$$M = \{\alpha b : \alpha \in \mathbb{C}\} \text{ and the spectral radius } \rho(T|_Q) < 1.$$

Then for $k = 0, 1, \dots,$

$$b_{k,0} = \alpha_k b + q_k, \quad \alpha_k \in \mathbb{C}, \text{ and } q_k \in Q. \tag{2.20}$$

By Theorem 1.2, $\hat{\phi}_k$ converges locally uniformly to $\hat{\phi}$ and by (1.7), $\int \phi(t)dt = 1$. Since h is fundamental, $\sum_j \phi(\cdot - j) = 1$ which implies that

$$\sum_j b(j) = 1. \tag{2.21}$$

The conditions (1.11) and (1.12) imply that $\widehat{\tilde{\phi}}_0(0) = 1$. This together with (1.25) gives

$$\lim_{k\to\infty} \sum_{j\in I} b_{k,0}(j) = \sum_{j\in I} \langle \tilde{\phi}_0, \tilde{\phi}_0(\,\cdot\, - j)\rangle = 1. \tag{2.22}$$

Since $q_k \in Q$, it follows (see for instance Lemma 3.4 of [33]) that

$$\sum_{j\in\mathbb{Z}^s} q_k(j) = 0, \quad k = 0, 1, \dots. \tag{2.23}$$

By (2.20), (2.21), (2.22) and (2.23), we deduce that $\alpha_k \to 1$ as $k \to \infty$. It follows from (1.11) and (2.20) that q_k converges, as $k \to \infty$, to some $q \in Q$. For $k = 0, 1, \dots,$

$$b_{k,n} = \left(\prod_{\ell=k+1}^{k+n} T_\ell\right) b_{k+n,0}. \tag{2.24}$$

By (2.20)

$$b_{k,n} = \alpha_{k+n}\left(\prod_{\ell=k+1}^{k+n} T_\ell\right) b + \left(\prod_{\ell=k+1}^{k+n} T_\ell\right) q_{k+n}. \tag{2.25}$$

Thus,

$$\lim_{n\to\infty} b_{k,n} = \left(\prod_{\ell=k+1}^{\infty} T_\ell\right) b, \text{ uniformly in } k, \tag{2.26}$$

by Lemma 2.1 and Theorem 2.1. Since T satisfies Condition E, (2.24), (1.11) and Theorem 2.1 give

$$\lim_{k\to\infty}\lim_{n\to\infty} b_{k,n} = \lim_{n\to\infty}\lim_{k\to\infty} b_{k,n}.$$

From (2.25) and Theorem 2.1, recalling $Tb = b$,

$$\lim_{n\to\infty}\lim_{k\to\infty} b_{k,n} = b + \lim_{n\to\infty} T^n q = b.$$

Therefore,

$$\lim_{k\to\infty}\lim_{n\to\infty} b_{k,n} = b. \tag{2.27}$$

In particular $\lim_{k\to\infty}\lim_{n\to\infty} b_{k,n}(0) = b(0)$, i.e.

$$\lim_{k\to\infty}\lim_{n\to\infty} \|\phi_{k,n}\|^2 = \|\phi\|^2.$$

But Fatou's Lemma gives

$$\lim_{k\to\infty}\lim_{n\to\infty} \|\phi_{k,n}\|^2 \geq \lim_{k\to\infty} \|\phi_k\|^2 \geq \|\phi\|^2.$$

Therefore,

$$\lim_{k\to\infty} \|\phi_k\|^2 = \|\phi\|^2,$$

from which we conclude that ϕ_k converges strongly to ϕ in $L^2(\mathbb{R}^s)$ as $k \to \infty$.

The strong convergence of ϕ_k to ϕ implies that $b_k \to b$. Now (2.19) gives

$$b_k = \left(\prod_{\ell=k+1}^{k+n} T_\ell\right) b_{k+n}, \quad k, n = 0, 1, \ldots. \tag{2.28}$$

Letting $n \to \infty$, we have

$$b_k = \left(\prod_{\ell=k+1}^{\infty} T_\ell\right) b, \quad k = 0, 1, \ldots. \tag{2.29}$$

Combining (2.26) and (2.29) gives $b_{k,n} \to b_k$ as $n \to \infty$, uniformly in k. In particular,

$$\|\phi_{k,n}\|_2^2 = b_{k,n}(0) \to b_k(0) = \|\phi_k\|_2^2.$$

Therefore, $(\phi_{k,n})$ converges strongly to ϕ_k as $n \to \infty$, uniformly in $k = 0, 1, \ldots$.

Conversely, suppose (b) holds. Let ψ be the characteristic function of $[0,1]^s \subset K$. For $\nu \in I$ and $k = 0, 1, \ldots,$ define $\phi_{k,0} := \psi(\cdot + \nu)$, as in the proof of Theorem 1.2. The sequence $\phi_{k,0}$ is independent of k and thus satisfies all the conditions in (b). With $(\phi_{k,0})$ as an initial sequence, define the cascade sequence $(\phi_{k,n})$ by (1.13). Then $(\phi_{k,n})$ converges in $L^2(\mathbb{R}^d)$ to ϕ_k as $n \to \infty$, uniformly in $k = 0, 1, \ldots$. The sequence $b_{k,n}$ defined in (1.16) by this cascade sequence satisfies

$$b_{k,n} = \left(\prod_{\ell=k+1}^{k+n} T_\ell\right) b_{k+n,0} \text{ and } b_{k,0}(j) = \delta_\nu(j)\ ,$$

for $k, n = 0, 1, \ldots$. Thus

$$b_{k,n} = \left(\prod_{\ell=k+1}^{k+n} T_\ell \right) \delta_\nu. \tag{2.30}$$

Since $b_{k,n}$ converges as $n \to \infty$ uniformly in k, it follows that $\left(\prod_{\ell=k+1}^{k+n} T_\ell \right)$ δ_ν, converges for all $\nu \in I$, as $n \to \infty$ uniformly in k. Thus $\prod_{\ell=k+1}^{k+n} T_\ell$ converges as $n \to \infty$ uniformly in k. By Theorem 2.1, T satisfies Condition E^*.

Since $\phi_{k,n}$ converges in norm to ϕ_k in $L^2(\mathbb{R}^s)$ as $n \to \infty$, uniformly in k, it follows that $b_{k,n} \to b_k$ as $n \to \infty$, uniformly in k, where b_k is defined in (2.17). Thus (2.30) gives

$$\left(\prod_{\ell=k+1}^{\infty} T_\ell \right) \delta_\nu = b_k, \tag{2.31}$$

for all $\nu \in I$. Since ϕ_k converges in norm to ϕ in $L^2(\mathbb{R}^s)$, b_k converges to b as $k \to \infty$, where $b(j) = \langle \phi, \phi(\cdot - j) \rangle$. We know that b is an eigenvector of T with eigenvalue 1. To show that T satisfies Condition E, it remains to show that b is unique up to a constant multiple.

Let $v \in \mathbb{C}^I$ be any eigenvector of T with eigenvalue 1. By (2.31)

$$\left(\prod_{\ell=k+1}^{\infty} T_\ell \right) v = \beta b_k, \tag{2.32}$$

where $\beta = \sum_{j \in I} v(j)$. We now show that

$$\lim_{k \to \infty} \left(\prod_{\ell=k+1}^{\infty} T_\ell \right) v = v. \tag{2.33}$$

To prove this, consider

$$\left\| \left(\prod_{\ell=k+1}^{k+n} T_\ell \right) v - v \right\| = \left\| \left(\prod_{\ell=k+1}^{k+n} T_\ell \right) v - T^n v \right\| \le C \|v\| \sum_{\ell=k+1}^{\infty} \|T_\ell - T\|.$$

It follows that

$$\left\| \left(\prod_{\ell=k+1}^{\infty} T_\ell \right) v - v \right\| \le C \|v\| \sum_{\ell=k+1}^{\infty} \|T_\ell - T\|,$$

which tends to 0 as $k \to \infty$. Thus (2.33) holds. Taking the limits as $k \to \infty$ of (2.32) gives $v = \beta b$. Thus T satisfies Condition E.

To show that h is fundamental, it suffices to show that its autocorrelation g defined in (1.18) is fundamental. For any $\nu \in I$, (2.31) implies

$$\lim_{k\to\infty}\left(\prod_{\ell=k+1}^{\infty} T_\ell\right)\delta_\nu = b. \tag{2.34}$$

By Theorem 2.1, the relation (2.34) implies that

$$\lim_{n\to\infty} T^n \delta_\nu = b. \tag{2.35}$$

Let $v \in \mathbb{C}^I$ be a left eigenvector of T with eigenvalue 1. Then for each $\nu \in I$,

$$v(\nu) = vT^n(\nu) = (vT^n)\delta_\nu = v(T^n\delta_\nu), \quad \text{for all } n = 0, 1, \dots .$$

It follows from (2.35) that

$$v(\nu) = vb = \sum_j v(j)b(j) =: \mu.$$

This means that $v = \mu e$ where $e = \sum_{\nu\in I}\delta_\nu$. Hence $eT = e$, which is equivalent to

$$|M|\sum_j g(Mj-\ell) = 1, \quad \text{for all } \ell \in \mathbb{Z}^s.$$

Therefore, g is fundamental. Hence h is fundamental.

The equivalence of (a) and (c) is established in the same way following the same argument as in [34]. □

Remark 2. Starting with a stationary mask or filter h that satisfies Condition E or E^{**}, one can construct many nonstationary masks h_k that satisfies *(1.2)*. By Theorem 1.2 (respectively Theorem 1.3) the corresponding nonstationary cascade algorithm converges weakly (respectively strongly) in $L^2(\mathbb{R}^s)$ to the solution (ϕ_k) of the nonstationary refinement equations with mask (h_k).

Example 1. Take $s = 1$ and $(h(j))_{j=0}^3 = \{1/8, 3/8, 3/8, 1/8\}$, the mask for the uniform quadratic B-spline. It is easy to see that the transition operator T for h satisfies Condition E. The nonstationary mask sequence h_k, $k = 2, 3, \dots$, with

$$h_k(0) = 1/8, \quad h_k(1) = 3/8 - 1/2^k, \quad h_k(2) = 3/8 - 1/2^k,$$
$$h_k(3) = 1/8, \quad h_k(j) = 0, \text{ otherwise},$$

clearly satisfies (1.2) and therefore defines a system of nonstationary refinement equations with solutions $\phi_k \in L^2(\mathbb{R}^s)$. The scaling sequence ϕ_k

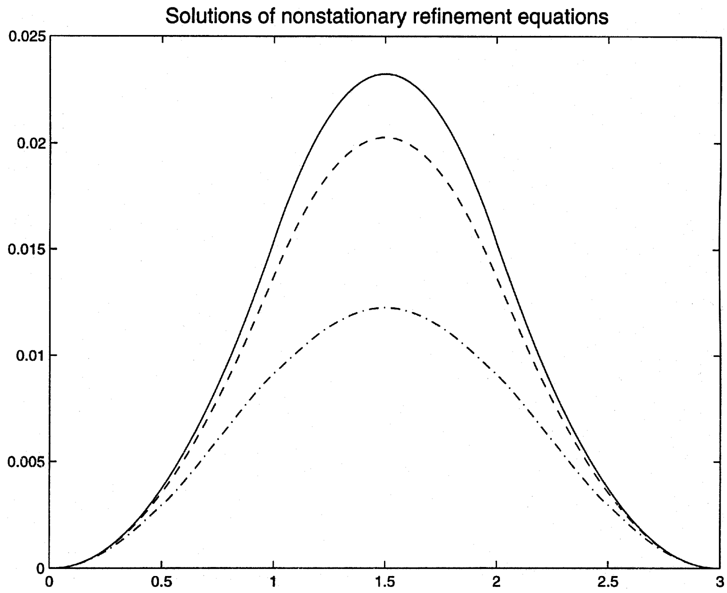

Fig. 1. Nonstationary scaling functions tending to the uniform quadratic B-spline- $\phi_2 - \cdot -$; $\phi_4 --$; ϕ_8 —

converges to the uniform quadratic B-spline. Figure 1 shows the graph of ϕ_2, ϕ_4 and ϕ_8.

Example 2. The most common examples of solutions of nonstationary refinement equations are the exponential B-splines and the up-function. These examples were discussed in detail in [14]. The up-function is not covered by our theory since the supports of its masks are not uniformly bounded.

The exponential B-splines ϕ_k, $k = 0, 1, \ldots$, whose Fourier transforms are

$$\widehat{\phi_k}(u) := \frac{\prod_{j=0}^{N}(\mathrm{e}^{-2^{-k}\lambda_j} - \mathrm{e}^{-\mathrm{i}u})}{\prod_{j=0}^{N}(\mathrm{i}u - 2^{-k}\lambda_j)},$$

where $\lambda_0, \ldots, \lambda_N \in \mathbb{R}$, are solutions of the nonstationary refinement equations with mask sequence (h_k) where

$$\sum_j h_k(j)z^j \equiv H_k(z) := \prod_{j=0}^{N} \left(\frac{\mathrm{e}^{-2^{-k}\lambda_j} + z}{2} \right).$$

Clearly,

$$\lim_{k\to\infty} H_k(z) = \left(\frac{1+z}{2} \right)^{N+1},$$

and it is easy to see that h_k and h satisfy (1.2).

Remark 3. More examples, including a nonstationary analogue of Daubechies orthonormal scaling functions, can be found in [35].

3. Convergence of derivatives

For any function f on $\mathbb{R}^s$ and $u \in \mathbb{Z}^s$, we define the difference operator

$$\Delta_u f(x) := f(x) - f(x-u), \quad x \in \mathbb{R}^s.$$

For $u_1, \ldots, u_m \in \mathbb{Z}^s$, let $\pi(u_1, \ldots, u_m)$ denote all polynomials in the null space of $\Delta_{u_1} \cdots \Delta_{u_m}$. We note that Δ_{u_m} maps $\pi(u_1, \ldots, u_m)$ onto $\pi(u_1, \ldots, u_{m-1})$.

Let V_0 be the space of all sequences on $\mathbb{Z}^s$ with finite support, and for $u_1, \ldots, u_m \in \mathbb{Z}^s$, we define

$$V(u_1, \ldots, u_m) := \left\{ v \in V_0 : \sum_{j \in \mathbb{Z}^s} p(j) v(j) = 0, \ \forall p \in \pi(u_1, \ldots, u_m) \right\}. \tag{3.1}$$

Similarly we define F_0 to be all functions f in $L^2(\mathbb{R}^s)$ with compact support and $F(u_1, \cdots, u_m)$ to be all functions in F_0 satisfying

$$\sum_{j \in \mathbb{Z}^s} p(j) f(\cdot - j) = 0, \quad \forall p \in \pi(u_1, \ldots, u_m). \tag{3.2}$$

If $m = 0$, we define $F(u_1, \ldots, u_m) = F_0$.

Lemma 3.1 *For $u_1, \ldots, u_m \in \mathbb{Z}^s$, $\Delta_{u_m} : F(u_1, \ldots, u_{m-1}) \to F(u_1, \cdots, u_m)$ is a bijection. In particular $F(u_1, \ldots, u_m) = \Delta_{u_1} \cdots \Delta_{u_m} F_0$.*

Proof. Take $f \in F(u_1, \ldots, u_{m-1})$, $p \in \pi(u_1, \ldots, u_m)$. Then

$$\begin{aligned} \sum_{i \in \mathbb{Z}^s} p(i) \Delta_{u_m} f(x-i) &= \sum_{i \in \mathbb{Z}^s} p(i)(f(x-i) - f(x-i-u_m)) \\ &= \sum_{i \in \mathbb{Z}^s} \Delta_{u_m} p(i) f(x-i) = 0, \end{aligned}$$

since $\Delta_{u_m} p \in \pi(u_1, \ldots, u_{m-1})$. So Δ_{u_m} maps $F(u_1, \ldots, u_{m-1})$ into $F(u_1, \ldots, u_m)$. If $\Delta_{u_m} f = 0$ and f has compact support, then $f = 0$. It remains to show that Δ_{u_m} maps $F(u_1, \ldots, u_{m-1})$ onto $F(u_1, \ldots, u_m)$.

Take $f \in F(u_1, \ldots, u_m)$. Let $g(x) = \sum_{\ell=0}^{\infty} f(x - \ell u_m)$. Since $f \in F(u_m)$, $\sum_{\ell=-\infty}^{\infty} f(x - \ell u_m) = 0$ and since f has compact support, g also has compact support and is in $L^2(\mathbb{R}^s)$. Also $\Delta_{u_m} g = f$. It remains to show that $g \in F(u_1, \ldots, u_{m-1})$. Take $p \in \pi(u_1, \ldots, u_{m-1})$. Then $p = \Delta_{u_m} q$ for some q in $\pi(u_1, \ldots, u_m)$ and

$$\begin{aligned}\sum_{i\in\mathbb{Z}^s} p(i)g(x-i) &= \sum_{i\in\mathbb{Z}^s} \Delta_{u_m} q(i)g(x-i)\\ &= \sum_{i\in\mathbb{Z}^s} q(i)\Delta_{u_m} g(x-i)\\ &= \sum_{i\in\mathbb{Z}^s} q(i)f(x-i) = 0. \qquad \square\end{aligned}$$

We now construct functions f with $D_{u_1} \cdots D_{u_m} f \in F(u_1, \ldots, u_m)$. Take any function f_0 in F_0 and set $f(\cdot|\emptyset) = f_0$. We may recursively define $f(\cdot|u_1, \ldots, u_m)$ for any $u_1, \ldots, u_m \in \mathbb{Z}^s$ by

$$f(x|u_1, \ldots, u_m) = \int_0^1 f(x - tu_m|u_1, \ldots, u_{m-1})dt.$$

By induction we see that $f(\cdot|u_1, \ldots, u_m) \in F_0$. Clearly

$$D_{u_m} f(\cdot|u_1, \ldots, u_m) = \Delta_{u_m} f(\cdot|u_1, \ldots, u_{m-1}).$$

Hence $D_{u_1} \cdots D_{u_m} f = \Delta_{u_1} \cdots \Delta_{u_m} f_0$ and by Lemma 3.1, $D_{u_1} \cdots D_{u_m} f \in F(u_1, \ldots, u_m)$. As a particular example we can take f_0 to be the characteristic function of $[0,1]^s$. Then $f(\cdot|u_1, \ldots, u_m)$ is a box spline of degree m.

Now for $v \in V_0$ and $u \in \mathbb{Z}^s$, we define the difference operator

$$\nabla_u v(i) = v(i) - v(i-u), \quad i \in \mathbb{Z}^s.$$

Let δ denote the sequence in V_0 given by $\delta(j) = \delta_{j,0}$.

Lemma 3.2 *If f_0 has support in $[0,1]^s$ with $\int |f_0|^2 = \sigma$, $f(\cdot|u_1, \ldots, u_m)$ is defined as above, and $f(\cdot|v_1, \ldots, v_m)$ is similarly defined with respect to vectors $v_1, \ldots, v_m \in \mathbb{Z}^s$, then for $\ell \in \mathbb{Z}^s$,*

$$\begin{aligned}&\langle D_{u_1} \cdots D_{u_m} f(\cdot|u_1, \ldots, u_m), D_{v_1} \cdots D_{v_m} f(\cdot - \ell|v_1, \ldots, v_m)\rangle\\ &= (-1)^m \sigma \nabla_{v_1} \cdots \nabla_{v_m} \nabla_{u_1} \cdots \nabla_{u_m} \delta(l + v_1 + \cdots + v_m).\end{aligned}$$

Proof.

$$\begin{aligned}&\langle D_{u_1} \cdots D_{u_m} f(\cdot|u_1, \ldots, u_m), D_{v_1} \cdots D_{v_m} f(\cdot - \ell|v_1, \cdots, v_m)\rangle\\ &= \langle \Delta_{u_1} \cdots \Delta_{u_m} f_0, \Delta_{v_1} \cdots \Delta_{v_m} f_0(\cdot - \ell)\rangle\\ &= (-1)^m \langle \Delta_{v_1} \cdots \Delta_{v_m} \Delta_{u_1} \cdots \Delta_{u_m} f_0, f_0(\cdot - \ell - v_1 - \cdots - v_m)\rangle\\ &= (-1)^m \nabla_{v_1} \cdots \nabla_{v_m} \nabla_{u_1} \cdots \nabla_{u_m} \langle f_0, f_0(\cdot - \ell - v_1 - \cdots - v_m)\rangle\\ &= (-1)^m \sigma \nabla_{v_1} \cdots \nabla_{v_m} \nabla_{u_1} \cdots \nabla_{u_m} \delta(\ell + v_1 + \cdots + v_m). \qquad \square\end{aligned}$$

In analogy with Lemma 3.1 we have the following

Lemma 3.3 *For* $u_1, \ldots, u_m \in \mathbb{Z}^s$, $\nabla_{u_m} : V(u_1, \ldots, u_{m-1}) \to V(u_1, \cdots, u_m)$ *is a bijection. In particular* $V(u_1, \ldots, u_m) = \nabla_{u_1} \cdots \nabla_{u_m} V_0$.

Proof. The proof is very similar to that of Lemma 3.1. Take $v \in V(u_1, \ldots, u_{m-1})$ and $p \in \pi(u_1, \ldots, u_m)$. Then

$$\sum_{i \in \mathbb{Z}^s} p(i) \nabla_{u_m} v(i) = -\sum_{i \in \mathbb{Z}^s} \Delta_{u_m} p(i + u_m) v(i) = 0.$$

So ∇_{u_m} maps $V(u_1, \ldots, u_{m-1})$ into $V(u_1, \ldots, u_m)$. Since ∇_{u_m} is injective on the space $V(u_1, \ldots, u_{m-1})$, we need only show that it maps onto $V(u_1, \ldots, u_m)$. Take $v \in V(u_1, \ldots, u_m)$ and let

$$w(i) = \sum_{\ell=0}^{\infty} v(i - \ell u_m).$$

Then w has finite support and $\nabla_{u_m} w = v$. For $p \in \pi(u_1, \ldots, u_{m-1})$, $p = \Delta_{u_m} q$ for some $q \in \pi(u_1, \ldots, u_m)$, and

$$\sum_{i \in \mathbb{Z}^s} p(i{+}u_m) w(i) = \sum_{i \in \mathbb{Z}^s} \Delta_{u_m} q(i{+}u_m) w(i) = -\sum_{i \in \mathbb{Z}^s} q(i) \nabla_{u_m} w(i) = 0.$$

So $w \in V(u_1, \ldots, u_{m-1})$. $\square$

Lemma 3.4 *Suppose that* $f \in F(u_1, \ldots, u_m)$, $g \in F(v_1, \ldots, v_m)$, *and for* $\ell \in \mathbb{Z}^s$, $b(\ell) := \langle f, g(\cdot - \ell) \rangle$. *Then* $b \in V(u_1, \ldots, u_m, v_1, \ldots, v_m)$.

Proof. Take $f \in F(u_1, \ldots, u_m)$ and $g \in F(v_1, \ldots, v_m)$. By Lemma 3.1, there are f_0, g_0 in F_0 with

$$f = \Delta_{u_1} \cdots \Delta_{u_m} f_0$$

and

$$g = \Delta_{v_1} \cdots \Delta_{v_m} g_0.$$

Then for $p \in \pi(u_1, \ldots, u_m, v_1, \ldots, v_m)$,

$$\begin{aligned}
\sum_{\ell \in \mathbb{Z}^s} p(\ell) b(\ell) &= \sum_{\ell \in \mathbb{Z}^s} p(\ell) \langle \Delta_{u_1} \cdots \Delta_{u_m} f_0, \Delta_{v_1} \cdots \Delta_{v_m} g_0(\cdot - \ell) \rangle \\
&= \sum_{\ell \in \mathbb{Z}^s} p(\ell) (-1)^m \langle f_0(\cdot - u_1 - \cdots - u_m), \\
&\qquad \Delta_{u_1} \cdots \Delta_{u_m} \Delta_{v_1} \cdots \Delta_{v_m} g_0(\cdot - \ell) \rangle \\
&= (-1)^m \langle f_0(\cdot - u_1 - \cdots - u_m),
\end{aligned}$$

$$\sum_{\ell\in\mathbb{Z}^s} p(\ell)\Delta_{u_1}\cdots\Delta_{u_m}\Delta_{v_1}\cdots\Delta_{v_m}g_0(\cdot-\ell)\rangle$$
$$= (-1)^m\langle f_0(\cdot-u_1-\cdots-u_m),$$
$$\sum_{\ell\in\mathbb{Z}^s}\Delta_{v_1}\cdots\Delta_{v_m}\Delta_{u_1}\cdots\Delta_{u_m}p(\ell)g_0(\cdot-\ell)\rangle = 0,$$

since $\Delta_{v_1}\cdots\Delta_{v_m}\Delta_{u_1}\cdots\Delta_{u_m}p = 0$. □

Recall that M is an integer matrix with $\det(M)\geq 2$, and Γ comprises a coset representatives of $\mathbb{Z}^s/M\mathbb{Z}^s$. For any vector space π of polynomials on $\mathbb{R}^s$ we say that h in V_0 satisfies the *sum rules for* π if for all γ in Γ and p in π,

$$\sum_{\beta\in\mathbb{Z}^s} h(M\beta)p(M\beta) = \sum_{\beta\in\mathbb{Z}^s} h(M\beta+\gamma)p(M\beta+\gamma). \tag{3.3}$$

When π comprises all polynomials of degree $\leq k-1$, the above definition becomes the sum rules of order k as defined in [24]. Recall that if h satisfies (1.3), then h is fundamental if and only if it satisfies the sum rules of order 1.

Lemma 3.5 *If h satisfies the sum rules for π and*

$$a(\ell) = \sum_{j\in\mathbb{Z}^s} h(j)h(j-\ell), \quad \ell\in\mathbb{Z}^s,$$

then a satisfies the sum rules for $\pi' := \{p : p(x+y) = \sum_{i\in K} q_i(x)r_i(y)$, where K is finite and for each $i\in K$, $q_i\in\pi$ or $r_i\in\pi\}$.

Proof. Take any γ in Γ and p in π'. Then

$$\begin{aligned}
&\sum_{\beta\in\mathbb{Z}^s} p(M\beta+\gamma)a(M\beta+\gamma) \\
&= \sum_{j,\beta\in\mathbb{Z}^s} p(M\beta+\gamma)h(j)h(j-M\beta-\gamma) \\
&= \sum_{\beta,\ell\in\mathbb{Z}^s}\sum_{\eta\in\Gamma} p(M\beta+\gamma)h(M\ell+\eta)h(M\ell+\eta-M\beta-\gamma) \\
&= \sum_{\ell,m\in\mathbb{Z}^s}\sum_{\eta\in\Gamma} p(M\ell-Mm+\gamma)h(M\ell+\eta)h(Mm+\eta-\gamma) \\
&= \sum_{i\in K}\sum_{\eta\in\Gamma}\sum_{\ell\in\mathbb{Z}^s} q_i(M\ell+\eta)h(M\ell+\eta) \\
&\quad\times\sum_{m\in\mathbb{Z}^s} r_i(Mm+\eta-\gamma)h(Mm+\eta-\gamma),
\end{aligned}\tag{3.4}$$

where K is a finite set, and for each $i \in K$, either $q_i \in \pi$ or $r_i \in \pi$. If $q_i \in \pi$, then by the sum rules

$$\sum_{\eta\in\Gamma}\sum_{\ell\in\mathbb{Z}^s} q_i(M\ell+\eta)h(M\ell+\eta)\sum_{m\in\mathbb{Z}^s} r_i(Mm+\eta-\gamma)h(Mm+\eta-\gamma)$$
$$=\sum_{\ell\in\mathbb{Z}^s} q_i(M\ell)h(M\ell)\sum_{\eta\in\Gamma}\sum_{m\in\mathbb{Z}^s} r_i(Mm+\eta-\gamma)h(Mm+\eta-\gamma)$$
$$=\sum_{\ell\in\mathbb{Z}^s} q_i(M\ell)h(M\ell)\sum_{j\in\mathbb{Z}^s} r_i(j)h(j).$$

Similarly if $r_i \in \pi$, then

$$\sum_{\eta\in\Gamma}\sum_{\ell\in\mathbb{Z}^s} q_i(M\ell+\eta)h(M\ell+\eta)\sum_{m\in\mathbb{Z}^s} r_i(Mm+\eta-\gamma)h(Mm+\eta-\gamma)$$
$$=\sum_{m\in\mathbb{Z}^s} r_i(Mm)h(Mm)\sum_{j\in\mathbb{Z}^s} q_i(j)h(j).$$

So for each i in K, the terms in the sum in (3.4) are independent of γ. Thus a satisfies the sum rules for π'. $\square$

For a space π of polynomials on $\mathbb{R}^s$, let

$$V_\pi := \{v \in V_0 : \sum_j p(j)v(j) = 0, \text{ for all } p \in \pi\}.$$

According to our previous notation, for $u_1, \ldots, u_m \in \mathbb{Z}^2$, $V_{\pi(u_1,\ldots,u_m)} = V(u_1, \ldots, u_m)$.

Lemma 3.6 *Suppose that π is a space of polynomials such that for any p in π, $p(x) = q(Mx)$, where q satisfies $q(x+y) = \sum_{i\in K} q_i(x)r_i(y)$, where K is finite and for each $i \in K$, $q_i, r_i \in \pi$. Suppose that a in V_0 satisfies the sum rules for π and define the matrix $A = (A_{ij})_{i,j\in\mathbb{Z}^s}$ by $A_{ij} = a(Mi-j)$. Then $AV_\pi \subset V_\pi$.*

Proof. For $p \in \pi$, $v \in V_\pi$,

$$\sum_{i\in\mathbb{Z}^s} p(i)Av(i)$$
$$=\sum_{i\in\mathbb{Z}^s} p(i)\sum_{j\in\mathbb{Z}^s} a(Mi-j)v(j)$$
$$=\sum_{i,\ell\in\mathbb{Z}^s}\sum_{\alpha\in\Gamma} q(Mi)a(Mi-M\ell-\alpha)v(M\ell+\alpha)$$
$$=\sum_{j,\ell\in\mathbb{Z}^s}\sum_{\alpha\in\Gamma} q(Mj+M\ell)a(Mj-\alpha)v(M\ell+\alpha)$$

$$= \sum_{i\in K}\sum_{\alpha\in\Gamma}\sum_{j\in\mathbb{Z}^s} q_i(Mj-\alpha)a(Mj-\alpha)\sum_{\ell\in\mathbb{Z}^s} r_i(M\ell+\alpha)v(M\ell+\alpha)$$
$$= \sum_{i\in K}\sum_{j\in\mathbb{Z}^s} q_i(Mj)a(Mj)\sum_{\ell\in\mathbb{Z}^s} r_i(\ell)v(\ell) = 0,$$

since $r_i \in \pi$ for $i \in K$ and $v \in V_\pi$. So $Av \in V_\pi$ and so $AV_\pi \subset V_\pi$. □

We now consider some examples of π to illustrate Lemmas 3.5 and 3.6. First let $\pi = \pi_{m-1}$, the space of all polynomials of total degree $\leq m-1$. Then clearly the conditions of Lemma 3.6 are satisfied. In this case the result was proved in Jia [24]. If $\pi = \pi_{m-1}$, then π' (as in Lemma 3.5) equals π_{2m-1}.

Next suppose that $\pi = \pi(u_1,\dots,u_m)$, where $u_1,\dots,u_m$ are eigenvectors of M. We may suppose $(u_1,\dots,u_m) = (v_1,\dots,v_1,v_2,\dots,v_2,\dots,v_r,\dots,v_r)$, where $v_1,\dots,v_r$ are linearly independent and for $i = 1,\dots,r$, v_i occurs with multiplicity $\alpha_i \geq 1$. Choose $v_{r+1},\dots,v_s$ so that $v_1,\dots,v_s$ are linearly independent and take a basis $w_1,\dots,w_s$ so that for $j = 1,\dots,s$, w_j is orthogonal to v_i, $1 \leq i \leq s$, $i \neq j$. We claim that $\pi(u_1,\dots,u_m)$ is the linear span of all polynomials of form $p(x) = (w_1x)^{j_1}\cdots(w_sx)^{j_s}$ where for some $\ell = 1,\dots,r$, $j_\ell \leq \alpha_\ell - 1$, a space we denote temporarily by $\tilde{\pi}(u_1,\dots,u_m)$. We prove by induction on m that $\pi(u_1,\dots,u_m) = \tilde{\pi}(u_1,\dots,u_m)$. Clearly it is true for $m = 1$. Suppose that for some $m \geq 2$, $\pi(u_1,\dots,u_{m-1}) = \tilde{\pi}(u_1,\dots,u_{m-1})$. Clearly $\tilde{\pi}(u_1,\dots,u_m) \subset \pi(u_1,\dots,u_m)$. Take $p \in \pi(u_1,\dots,u_m)$ and, without loss of generality, suppose $u_m = v_1$. Then $\Delta_{v_1}p \in \pi(u_1,\dots,u_{m-1})$. Since $\Delta_{v_1}p \in \tilde{\pi}(u_1,\dots,u_{m-1})$, there is a polynomial q in $\tilde{\pi}(u_1,\dots,u_m)$ with $\Delta_{v_1}q = \Delta_{v_1}p$. Then $p-q$ is a polynomial in $w_2x,\dots,w_sx$ and so p is in $\tilde{\pi}(u_1,\dots,u_m)$. Thus our claim is proved. We can now see that $\pi(u_1,\dots,u_m)$ satisfies the conditions of Lemma 3.6. Note that in this case $V_\pi = V(u_1,\dots,u_m)$. Also $\pi' = \pi(u_1,\dots,u_m,u_1,\dots,u_m)$.

For our final example take $u_1,\dots,u_m$ as above with $r = s$ and let π be $\pi(v_1,\dots,v_1)\cap\cdots\cap\pi(v_s,\dots,v_s)$, where for $i = 1,\dots,s$, v_i occurs with multiplicity $\alpha_i \geq 1$. From our above characterisation of $\pi(u_1,\dots,u_m)$ we see that π is the linear span of all polynomials of the form $p(x) = (w_1x)^{j_1}\cdots(w_sx)^{j_s}$, where for $\ell = 1,\dots,s$, $j_\ell \leq \alpha_\ell - 1$. It follows that π satisfies the conditions of Lemma 3.6. In this case π' is the linear span of all polynomials of the form $p(x) = (w_1x)^{j_1}\cdots(w_sx)^{j_s}$, where for some $n = 1,\dots,s$, $j_n \leq 2\alpha_n - 1$, and $j_\ell \leq \alpha_\ell - 1$ for all $\ell = 1,\dots,s$, $\ell \neq n$. We see that π' also satisfies the conditions of Lemma 3.6.

We choose K as before and choose a bounded set $I \subset \mathbb{Z}^s$ so that $K \cap (K+j) = \emptyset$ for $j \notin I$. For $\phi_{k,0} \in L^2(\mathbb{R}^s)$ with support in K satisfying (1.11) and (1.12), and for $n = 0,1,\dots$, define $\phi_{k,n}$ as in (1.13).

Theorem 3.1 *Suppose that* (1.2) *and* (1.3) *hold. Let* $u_1, \ldots, u_m$ *be eigenvectors of* M *in* $\mathbb{Z}^s$ *with corresponding eigenvalues* $\lambda_1, \ldots, \lambda_m$. *Let* $W \subset \mathbb{C}^I$ *be the least subspace containing* $\mathbb{C}^I \cap V(u_1, \ldots, u_m, u_1, \ldots, u_m)$ *that is invariant under* T_k *for all* $k \geq 0$, *and suppose that* T *satisfies Condition* E^{**}. *If*
(a) $(\lambda_1 \cdots \lambda_m)^2 T$ *restricted to* W *satisfies Condition* E^{**}, *then*
(b) *for any* $\phi_{k,0}$ *satisfying* (1.12) *with* $D_{u_1} \cdots D_{u_m} \phi_{k,0} \in F(u_1, \ldots, u_m)$, *the sequence* $(D_{u_1} \cdots D_{u_m} \phi_{k,n})$ *converges weakly in* $L^2(\mathbb{R}^s)$ *to* $D_{u_1} \cdots D_{u_m} \phi_k$ *as* $n \to \infty$, *uniformly in* k, *where* ϕ_k *is the solution of* (1.1), *and* $D_{u_1} \cdots D_{u_m} \phi_k$ *converges weakly to* $D_{u_1} \cdots D_{u_m} \phi$ *as* $k \to \infty$, *where* ϕ *is the solution of* (1.8).

Conversely if (b) *holds, then* $(\lambda_1 \cdots \lambda_m)^2 T$ *restricted to* $\widetilde{W}$ *satisfies Condition* E^{**}, *where* $\widetilde{W} \subset \mathbb{C}^I$ *is the least invariant subspace of* T *containing* $\mathbb{C}^I \cap V(u_1, \ldots, u_m, u_1, \ldots, u_m)$.

Remark 4. By our remarks after Lemma 1 we may choose K large enough so that there are functions $\phi_{k,0}$ satisfying the conditions in *(b)*, for instance, $\phi_{k,0} = \phi_0$ for all k and ϕ_0 is the box-spline of degree m with directions $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1), u_1, \ldots, u_m$.

Proof. We first prove (a) implies (b). For $k, n = 0, 1, 2, \ldots$, define $\phi_{k,n}$ as in (1.13) and let

$$b_{k,n}(\ell) = \langle D_{u_1} \cdots D_{u_m} \phi_{k,n}, D_{u_1} \cdots D_{u_m} \phi_{k,n}(\cdot - \ell) \rangle, \quad \ell \in \mathbb{Z}^s.$$

Differentiating (1.13) gives

$$D_{u_1} \cdots D_{u_m} \phi_{k,n}(x) = |M| \lambda_1 \cdots \lambda_m \sum_{j \in \Omega} h_{k+1}(j) D_{u_1} \cdots D_{u_m} \times \phi_{k+1,n-1}(Mx - j), \quad x \in \mathbb{R}^s,$$

and hence

$$b_{k,n} = (\lambda_1 \cdots \lambda_m)^2 T_{k+1} b_{k+1,n-1}.$$

It follows that

$$b_{k,n} = (\lambda_1 \cdots \lambda_k)^{2n} \left(\prod_{\ell=k+1}^{k+n} T_\ell \right) b_{k+n,0}.$$

Since $D_{u_1} \cdots D_{u_m} \phi_{k,0} \in F(u_1, \ldots, u_m)$, by Lemma 3.4,

$$b_{k,0} \in V(u_1, \ldots, u_m, u_1, \ldots, u_m).$$

Since $D_{u_1} \cdots, D_{u_m} \phi_{k,0}$ has support in K, $D_{u_1} \cdots, D_{u_m} \phi_{k,n}$ also has support in K. Hence $b_{k,n}$ has support in I for all n. Thus

$$b_{k,0} \in \mathbb{C}^I \cap V(u_1, \ldots, u_m, u_1, \ldots, u_m).$$

It follows that for $k, n = 0, 1, \ldots$, $b_{k,n}$ lies in W.

By Proposition 2.1, the product $(\lambda_1 \cdots \lambda_m)^{2n} \prod_{\ell=k+1}^{k+n} T_\ell$ restricted to W is uniformly bounded in n and k, and so $b_{k,n}$ is uniformly bounded in n and k. In particular $b_{k,n}(0) = \|D_{u_1} \cdots D_{u_m} \phi_{k,n}\|_2^2$ is uniformly bounded in n and k. So for each k, there is a subsequence $(D_{u_1} \cdots D_{u_m} \phi_{k,n_j})$ which converges weakly in $L^2(\mathbb{R}^s)$ to some function ψ_k in $L^2(\mathbb{R}^s)$ as $j \to \infty$ uniformly in k. So $(D_{u_1} \cdots D_{u_m} \phi_{k,n_j})\widehat{}$ converges weakly to $\hat{\psi}_k$ as $j \to \infty$ uniformly in k.

Now

$$(D_{u_1} \cdots D_{u_m} \phi_{k,n})\widehat{}\,(u) = (-\mathrm{i}uu_1) \cdots (-\mathrm{i}uu_m)\hat{\phi}_{k,n}(u)$$

which, by (2.9), converges to $(-\mathrm{i}uu_1) \cdots (-\mathrm{i}uu_m)P_k(u)$ as $n \to \infty$ uniformly in k and locally uniformly in u. Thus

$$\hat{\psi}_k(u) = (-\mathrm{i}uu_1) \cdots (-\mathrm{i}uu_m)P_k(u), \quad u \in \mathbb{R}^s, \tag{3.5}$$

and $(D_{u_1} \cdots D_{u_m} \phi_{k,n})\widehat{}\,(u)$ converges to $\hat{\psi}_k(u)$ as $n \to \infty$ locally uniformly in u and uniformly in k. By the argument in the proof of Theorem 1.2, it follows that $(D_{u_1} \cdots D_{u_m} \phi_{k,n})\widehat{}$ converges weakly to $\hat{\psi}_k$ as $n \to \infty$, and so $D_{u_1} \cdots D_{u_m} \phi_{k,n}$ converges weakly to ψ_k as $n \to \infty$ uniformly in k.

Since T satisfies Condition E^{**}, $\phi_{k,n}$ converges weakly to ϕ_k as $n \to \infty$ uniformly in k, where ϕ_k is the solution of (1.1), and ϕ_k converges weakly to ϕ as $k \to \infty$, where ϕ is the solution of (1.8). Further, $\hat{\phi}_k(u) = P_k(u),\ u \in \mathbb{R}^s$. By (3.5) $\hat{\psi}_k(u) = (-\mathrm{i}uu_1) \cdots (-\mathrm{i}uu_k)\hat{\phi}_k(u)$, which gives $\psi_k = D_{u_1} \cdots D_{u_m} \phi_k$. As in the proof of Theorem 1.2, $\hat{\phi}_k$ converges locally uniformly to $\hat{\phi}$, as $k \to \infty$. Therefore, $(D_{u_1} \cdots D_{u_m} \phi_k)\widehat{}$ converges locally uniformly to $(D_{u_1} \cdots D_{u_m} \phi)\widehat{}$, as $k \to \infty$. As in the proof of Theorem 1.2, $\|D_{u_1} \ldots D_{u_m} \phi_k\|_2$ is uniformly bounded and it follows that $(D_{u_1} \cdots D_{u_m} \phi_k)\widehat{}$ converges weakly in $L^2(\mathbb{R}^s)$ to $(D_{u_1} \cdots D_{u_m} \phi)\widehat{}$, as $k \to \infty$. Thus $D_{u_1} \cdots D_{u_m} \phi_k$ converges weakly in $L^2(\mathbb{R}^s)$ to $D_{u_1} \cdots D_{u_m} \phi$, as $k \to \infty$.

Now we assume that (b) is satisfied. By Lemma 3.3, $V(u_1, \ldots, u_m, u_1, \ldots, u_m)$ is spanned by $\nabla_{u_1}^2 \cdots \nabla_{u_m}^2 \delta(\cdot + j),\ j \in \mathbb{Z}^s$. Thus $\mathbb{C}^I \cap V(u_1, \ldots, u_m, u_1, \ldots, u_m)$ is spanned by those $\nabla_{u_1}^2 \cdots \nabla_{u_m}^2 \delta(\cdot + j),\ j \in \mathbb{Z}^s$, whose supports lie in I. Choose $j \in \mathbb{Z}^s$ so that this holds. Take $f(\cdot|u_1, \ldots, u_m)$ as in Lemma 3.2. We note that by choosing f_0 smooth enough we can make $f(\cdot|u_1, \ldots, u_m)$ arbitrarily smooth. Moreover if $f_0 \geq 0$ then $f(\cdot|u_1, \ldots, u_m) \geq 0$ and so if f_0 does not vanish identically then $\int f(\cdot|u_1, \ldots, u_m) \neq 0$, and we normalize f_0 so that $\int f(\cdot|u_1, \ldots, u_m) = 1$.

Let $\phi_{k,0} = f(\cdot - \alpha|u_1, \ldots, u_m)$, $\psi_{k,0} = f(\cdot - \beta|u_1, \ldots, u_m)$ for some $\alpha, \beta \in \mathbb{Z}^s$, and for all k. By Lemma 3.2,

$$\begin{aligned}&\langle D_{u_1} \cdots D_{u_m} \phi_{k,0}, D_{u_1} \cdots D_{u_m} \psi_{k,0}(\cdot - \ell)\rangle \\ &\quad = (-1)^m \sigma \nabla_{u_1}^2 \cdots \nabla_{u_m}^2 \delta(\ell + u_1 + \cdots + u_m + \beta - \alpha).\end{aligned}$$

We can then choose α, β so that $u_1 + \cdots + u_m + \beta - \alpha = j$ and $\phi_{k,0}, \psi_{k,0}$ have support in K. Now we define $\phi_{k,n}, \psi_{k,n}, n = 1, 2, \ldots$, as in (1.13) and for $n = 0, 1, 2, \ldots$,

$$c_{k,n}(\ell) = \langle D_{u_1} \cdots D_{u_m} \phi_{k,n}, D_{u_1} \cdots D_{u_m} \psi_{k,n}(\cdot - \ell)\rangle, \quad \ell \in \mathbb{Z}^s.$$

Thus

$$c_{k,0} = (-1)^m \sigma \nabla_{u_1}^2 \cdots \nabla_{u_m}^2 \delta(\cdot + j) \in \mathbb{C}^I \cap V(u_1, \ldots, u_m, u_1, \ldots, u_m)$$

and

$$c_{k,n} = (\lambda_1 \cdots \lambda_m)^{2n} \left(\prod_{\ell=k+1}^{k+n} T_\ell \right) c_{k+n,0}, \quad n = 1, 2, \ldots.$$

By (b) we know $D_{u_1} \cdots D_{u_m} \phi_{k,n}$ and $D_{u_1} \cdots D_{u_m} \psi_{k,n}$ converge weakly in $L^2(\mathbb{R}^s)$ as $n \to \infty$, uniformly in k. Hence $b_{k,n}$ is uniformly bounded in n and k. Thus by Lemma 3.3, for each v in $\mathbb{C}^I \cap V(u_1, \ldots, u_m, u_1, \ldots, u_m)$, $(\lambda_1 \cdots \lambda_m)^{2n} \left(\prod_{\ell=k}^{k+n} T_\ell \right) v$ is bounded in n and k. It follows that $(\lambda_1 \cdots \lambda_m)^{2n} T^n v$ is bounded in n for each v in $\mathbb{C}^I \cap V(u_1, \ldots, u_m, u_1, \ldots, u_m)$. Hence $(\lambda_1 \cdots \lambda_m)^{2n} T^n$ restricted to $\widetilde{W}$ is bounded in n.

It remains to show that $(\lambda_1 \cdots \lambda_m)^{2n} T^n$ restricted to $\widetilde{W}$ has spectral radius 1. Take any starting sequence $\phi_0 = \phi_{k,0}$ for all k, satisfying the conditions of (b), and let

$$b_{k,n}(\ell) = \langle D_{u_1} \cdots D_{u_m} \phi_{k,n}, D_{u_1} \cdots D_{u_m} \phi_{k,n}(\cdot - \ell)\rangle, \quad \ell \in \mathbb{Z}^s.$$

Then

$$b_{k,n}(0) = \|D_{u_1} \cdots D_{u_m} \phi_{k,n}\|_2^2 = \|(D_{u_1} \cdots D_{u_m} \phi_{k,n})\widehat{}\,\|_2^2.$$

Since for each k,

$$\lim_{n \to \infty} (D_{u_1} \cdots D_{u_m} \phi_{k,n})\widehat{} = (D_{u_1} \cdots D_{u_m} \phi_k)\widehat{} \quad \text{locally uniformly },$$

by Fatou's Lemma

$$\underline{\lim}_{n \to \infty} b_{k,n}(0) \geq \|D_{u_1} \cdots D_{u_m} \phi_k\|_2^2 > 0.$$

We note that

$$b_{k,n} = (\lambda_1 \cdots \lambda_m)^{2n} \left(\prod_{\ell=k+1}^{k+n} T_\ell \right) b_0, \quad n = 1, 2, \ldots. \tag{3.6}$$

Further, since $D_{u_1} \cdots D_{u_m} \phi_0 \in F(u_1, \ldots, u_m)$, by Lemma 3.4

$$b_0 \in \mathbb{C}^I \cap V(u_1, \ldots, u_m, u_1, \ldots, u_m) \subset \widetilde{W}.$$

If the spectral radius of $(\lambda_1 \cdots \lambda_m)^{2n} T^n$ is strictly less than 1, it follows from (3.6) by taking the limits as $n \to \infty$, that $b_{k,n} \to 0$, which is a contradiction. $\square$

Imposing various conditions on h_k gives various invariant subspaces of T_k in $\mathbb{C}^I$ which contain W. We consider three examples. First suppose that h_k satisfies the sum rules for $\pi(u_1,\dots,u_m)$ for all $k \geq 0$. Then by Lemma 3.5 and the remarks after Lemma 3.6, the autocorrelation g_k of h_k satisfies the sum rules for $\pi(u_1,\dots,u_m,u_1,\dots,u_m)$. By Lemma 3.6, $V(u_1,\dots,u_m,u_1,\dots,u_m)$ is invariant under T_k and hence

$$W = \mathbb{C}^I \cap V(u_1,\dots,u_m,u_1,\dots,u_m).$$

Next suppose that h_k satisfies the sum rules for π_{m-1}. Then by Lemma 3.5, g_k satisfies the sum rules for π_{2m-1}. By Lemma 3.6, $V_{\pi_{2m-1}}$ is invariant under T_k. Since $\pi_{2m-1} \subset \pi(u_1,\dots,u_m,u_1,\dots,u_m)$, we see that $V(u_1,\dots,u_m,u_1,\dots,u_m) \subset V_{\pi_{2m-1}}$. Thus $W \subset \mathbb{C}^I \cap V_{\pi_{2m-1}}$.

Finally suppose that $(u_1,\dots,u_m,u_1,\dots,u_m) = (v_1,\dots,v_1,\dots,v_s,\dots,v_s)$ where for $i = 1,2\dots,$ v_i occurs with multiplicity $\alpha_i \geq 1$ and $v_1,\dots,v_s$ are linearly independent. Suppose that h_k satisfies the sum rules for π, where $\pi = \pi(v_1,\dots,v_1)\cap\cdots\cap\pi(v_s,\dots,v_s)$, where for $i = 1,\dots,s$, v_i occurs with multiplicity $\alpha_i \geq 1$. By Lemma 3.5, g_k satisfies the sum rules for π' described in the final example after Lemma 3.6. By Lemma 3.6, $V_{\pi'}$ is invariant under T_k. For $s \geq 2$, $\pi' \subset \pi(u_1,\dots,u_m,u_1,\dots,u_m)$ and hence $V(u_1,\dots,u_m,u_1,\dots,u_m) \subset V_{\pi'}$. Thus $W \subset \mathbb{C}^I \cap V_{\pi'}$.

For $s = 1$, the nonstationary refinement equation can be written in the form

$$\phi_k(x) = M\sum_{j=0}^{N} h_{k+1}(j)\phi_{k+1}(Mx-j), \quad x \in \mathbb{R}, \tag{3.7}$$

where $M \geq 2$ is a integer. Here $\Omega = \{0,1,\dots,N\}$ and we can take $K = [0,N]$, $I = \{-N+1,\dots,N-1\}$. In this case Theorem 3.1 leads to the following:

Corollary 3.1 *Take $m \geq 0$. Let $W \subset \mathbb{C}^I$ be the least subspace containing $\mathbb{C}^I \cap V_{\pi_{2m-1}}$ that is invariant under T_k for all $k \geq 0$, and suppose that T satisfies Condition E^{**}. If*
(a) *$M^{2m}T$ restricted to W satisfies condition E^{**}, then*
(b) *for any sequence of functions $\phi_{k,0}$ with support in $[0,N]$ satisfying* (1.11) *with $\phi_{k,0}^{(m)} \in L^2(\mathbb{R})$, and*

$$\sum_{\ell=-\infty}^{\infty} p(\ell)\phi_{k,0}(\cdot-\ell) \in \pi_{m-1}, \quad \textit{for all } p \in \pi_{m-1}, \tag{3.8}$$

$\phi_{k,n}^{(j)}$ converges weakly in $L^2(\mathbb{R})$ to $\phi_k^{(j)}$ as $n \to \infty$, uniformly in k, for $j = 0,\dots,m$, where (ϕ_k) is the solution of (3.7)*, and $\phi_k^{(j)}$ converges weakly*

in $L^2(\mathbb{R})$ to $\phi^{(j)}$ as $k \to \infty$, for $j = 0, \dots, m$, where ϕ is the solution of

$$\phi(x) = M \sum_{j=0}^{N} h(j)\phi(Mx - j), \quad x \in \mathbb{R}. \tag{3.9}$$

Conversely if (b) *holds, then $M^{2m}T$ restricted to $\widetilde{W}$ satisfies Condition E^{**}, where $\widetilde{W} \subset \mathbb{C}^I$ is the least invariant subspace of T containing $\mathbb{C}^I \cap V_{\pi_{2m-1}}$.*

Remark 5. If h_k satisfies the sum rules for π_{m-1} for all $k \geq 0$, then $V_{\pi_{2m-1}}$ is invariant under T_k and $W = \mathbb{C}^I \cap V_{\pi_{2m-1}}$. If $N \geq m+1$ we can find a sequence of functions $\phi_{k,0}$ satisfying the conditions in *(b)*, e.g. the B-spline of degree m with knots $0, 1, \dots, m+1$.

Proof of Corollary 3.1. That (b) implies (a) follows immediately from Theorem 3.1. On the other hand, if (a) holds, it follows from Theorem 3.1 that $\phi_{k,n}^{(m)}$ converges weakly to $\phi_k^{(m)}$, as $n \to \infty$, uniformly in k, and $\phi_k^{(m)}$ converges weakly to $\phi^{(m)}$, as $k \to \infty$, where (ϕ_k) is the solution of (3.7) and ϕ is the solution of (3.9).

Now take $k, n \geq 0$ and $0 \leq j < m$. For $|u| \geq 1$, $|(\phi_{k,n}^{(m)})\widehat{\ }(u)| = |u|^{m-j}|(\phi_{k,n}^{(j)})\widehat{\ }(u)| \geq |(\phi_{k,n}^{(j)})\widehat{\ }(u)|$. For $|u| \leq 1$, $(\phi_{k,n}^{(j)})\widehat{\ }(u)$ converges to $(\phi_k^{(j)})\widehat{\ }(u)$ as $n \to \infty$, uniformly in k and u. Since $\|(\phi_{k,n}^{(m)})\widehat{\ }\|$ is uniformly bounded in k and n, it follows that $\|(\phi_{k,n}^{(j)})\widehat{\ }\|_2$ is bounded in k and n. So by our previous argument, $(\phi_{k,n}^{(j)})\widehat{\ }$ converges weakly to $(\phi_k^{(j)})\widehat{\ }$ as $n \to \infty$ uniformly in k, and so $\phi_{k,n}^{(j)}$ converges weakly to $\phi_k^{(j)}$ as $n \to \infty$, uniformly in k. Similarly for $j = 1, 2 \dots, m-1$, $\phi_k^{(j)} \to \phi^{(j)}$ weakly as $k \to \infty$. □

Theorem 3.2 *Suppose that* (1.2) *and* (1.3) *hold, and that M has s linearly independent eigenvectors in $\mathbb{Z}^s$ with eigenvalues $\lambda_1, \dots, \lambda_s$, h_k satisfies the sum rules for π_{m-1} for some $m \geq 1$, for all $k \geq 0$, and T satisfies Condition E^{**}. Then $V_{\pi_{2m-1}}$ is an invariant subspace of T_k, and* (a) *implies* (b)*, where* (a) *and* (b) *are as follows.*

(a) *$\rho(M)^{2m}T$ restricted to $\mathbb{C}^I \cap V_{\pi_{2m-1}}$ satisfies Condition E^{**}.*

(b) *For any eigenvectors $u_1, \dots, u_m$ of M and any sequence of functions $\phi_{k,0}$ with support in K satisfying* (1.11)*, $D_{u_1} \cdots D_{u_m}\phi_{k,0} \in L^2(\mathbb{R}^s)$, and*

$$\sum_{\ell \in \mathbb{Z}^s} p(\ell) D_{u_1} \cdots D_{u_m}\phi_{k,0}(\cdot - \ell) = 0, \quad \textit{for all } p \in \pi_{m-1}, \tag{3.10}$$

$D_{u_1} \cdots, D_{u_m}\phi_{k,n}$ converges weakly in $L^2(\mathbb{R}^s)$ to $D_{u_1} \cdots, D_{u_m}\phi_k$ as $n \to \infty$, uniformly in k, where ϕ_k is the solution of (1.1)*, and*

$D_{u_1} \cdots D_{u_m} \phi_k$ *converges weakly to* $D_{u_1} \cdots D_{u_m} \phi$ *as* $k \to \infty$, *where* ϕ *is the solution of* (1.8).

Moreover if $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_s|$, *then* (b) *implies* (a).

Proof. To prove that (a) implies (b) we follow the proof of (a) implies (b) in Theorem 3.1 except that we have $b_{k,0}$ lies in $V_{\pi_{2m-1}}$ rather than $V(u_1, \ldots, u_m, u_1, \ldots, u_m)$.

Now suppose that $|\lambda_1| = |\lambda_2| \cdots = |\lambda_s| = \rho(M)$ and (b) holds. We apply the method of the proof that (b) implies (a) in Theorem 3.1 except that now we take $\phi_{k,0} = f(\cdot - \alpha | u_1, \ldots, u_m)$, $\psi_{k,0} = f(\cdot - \beta | u_{m+1}, \ldots, u_{2m})$ for any choice of the eigenvectors $u_1, \ldots, u_{2m}$. We see that $(\rho(M)^{2m} T)^n$ restricted to $\mathbb{C}^I \cap V(u_1, \ldots, u_{2m})$ is bounded in n. Since the eigenvectors of M span $\mathbb{R}^s$, the intersection of $\pi(u_1, \ldots, u_{2m})$ over all choices of $u_1, \ldots, u_{2m}$ equals π_{2m-1}. Thus $V_{\pi_{2m-1}}$ is the linear span of the spaces $V(u_1, \ldots, u_{2m})$ over all choices of $u_1, \ldots, u_{2m}$. Thus $(\rho(M)^{2m} T)^n$ restricted to $\mathbb{C}^I \cap V_{\pi_{2m-1}}$ is bounded. A similar argument as in the proof of Theorem 3.1 shows that 1 is an eigenvalue of $\rho(M)^{2m} T$ restricted to $\mathbb{C}^I \cap V_{\pi_{2m-1}}$. Thus (a) holds. □

Corollary 3.2 *Assume the conditions of Theorem 3.2 and suppose that* (a) *of Theorem 3.2 holds. Take any function* $\phi_{k,0}$ *with support in* K *satisfying* (1.11) *and* (1.12) *with* $D^\alpha \phi_{k,0} \in L^2(\mathbb{R}^s)$ *for all* $\alpha \in \mathbb{Z}^s$ *with* $|\alpha| = m$, *and*

$$(3.11) \qquad \sum_{\ell \in \mathbb{Z}^s} p(\ell) \phi_{k,0}(\cdot - \ell) \in \pi_{m-1}, \quad \text{for all } p \in \pi_{m-1}.$$

Then for any $\beta \in \mathbb{Z}^s$ *with* $|\beta| \le m$, $D^\beta \phi_{k,n}$ *converges weakly in* $L^2(\mathbb{R}^s)$ *to* $D^\beta \phi_k$ *as* $n \to \infty$, *uniformly in* k, *where* (ϕ_k) *is the solution of* (1.1), *and* $D^\beta \phi_k$ *converges weakly to* $D^\beta \phi$ *as* $k \to \infty$, *where* ϕ *is the solution of* (1.8).

Proof. Suppose that $\phi_{k,0}$ satisfies the conditions of Corollary 3.2. Then for any eigenvectors $u_1, \ldots, u_m$ of M,

$$\sum_{\ell \in \mathbb{Z}^s} p(\ell) D_{u_1} \cdots D_{u_m} \phi_{k,0}(\cdot - \ell) = 0, \quad \text{for all } p \in \pi_{m-1}.$$

So by Theorem 3.2, the sequence $(D_{u_1} \cdots D_{u_m} \phi_{k,n})$ converges weakly in $L^2(\mathbb{R}^s)$ to $(D_{u_1} \cdots D_{u_m} \phi_k)$ as $n \to \infty$, uniformly in k. Since the eigenvectors of M span $\mathbb{R}^s$, it follows that for any α in $\mathbb{Z}^s$ with $|\alpha| = m$, $(D^\alpha \phi_{k,n})$ converges weakly in $L^2(\mathbb{R}^s)$ to $(-\mathrm{i}u)^\alpha \phi_k(u)$. It then follows, as in the proof of Corollary 3.1, that for all β with $|\beta| \le m$, $(D^\beta \phi_{k,n})$ converges weakly in $L^2(\mathbb{R}^s)$ to $D^\beta \phi_k$ as $n \to \infty$, uniformly in k. Similarly, $(D^\beta \phi_k)$ converges weakly in $L^2(\mathbb{R}^s)$ to $D^\beta \phi$ as $k \to \infty$. □

Remark 6. For a function $\phi_{k,0}$ that satisfies the conditions of Corollary 3.2, we can take a suitable translate of $N(x_1,\dots,x_s) := N_m(x_1)\cdots N_m(x_s)$, where N_m is the univariate B-spline of degree m with knots at $0,\dots,m+1$. If $p(x) = x_1^{\alpha_1}\cdots x_s^{\alpha_s}$, $0 \le \alpha_j \le m$, $j = 1,\dots,s$, then

$$\sum_{\ell\in\mathbb{Z}^s} p(\ell)N(x-\ell) = \sum_{\ell\in\mathbb{Z}} x_1^{\alpha_1} N_m(x_1-\ell)\cdots\sum_{\ell\in\mathbb{Z}} x_s^{\alpha_s} N_m(x_s-\ell)$$
$$= P_1(x_1)\cdots P_s(x_s),$$

where for $\ell = 1,\dots,s$, P_ℓ has degree $\le \alpha_\ell$. In particular, if $\sum_{j=1}^s \alpha_j \le m-1$, then $P_1(x_1)\cdots P_s(x_s)$ lies in π_{m-1}. Thus *(3.11)* is satisfied for $\phi_{k,0} = N$ for all k.

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