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A posteriori error estimation with the finite element method of lines for a nonlinear parabolic equation in one space dimension*

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Summary. Convergence of a posteriori error estimates to the true error for the semidiscrete finite element method of lines is shown for a nonlinear parabolic initial-boundary value problem.

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1. Introduction

Adaptive methods for solving parabolic equations are mostly based on a posteriori error estimates [1, 2, 3, 5, 7, 8, 9, 10, 12]. One of the most common strategies for constructing such estimates is the finite element p-refinement, i.e. the computation of a second, higher order solution. The error estimate can be computed as a correction to the original solution on each element.

This approach is very suitable for solving parabolic partial differential equations by the method of lines. The analysis of the approximate solution at the actual time level based on the calculation of an a posteriori error estimate yields a new grid to be used for the time step leading to the next time level.

Experimental evidence indicates a high efficiency of this approach to linear as well as nonlinear problems. Convergence of the error estimates to the true error for a semidiscrete method has been shown for linear equations in [3, 8, 12]. The paper [9] is devoted to semidiscrete error estimation in the semilinear case and fully discrete error estimation (if SIRK or BDF methods are used) in the nonlinear case. In the present paper, we partially extend the semidiscrete error estimation to the nonlinear case. The results

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can be used as a basis for an adaptive numerical procedure that carries out the fully discrete computation with an arbitrary time discretization.

In general, we use notation of [9] and prove, for the same nonlinear problem, some stronger results than presented in [9]. In Sect. 2, the model problem and its weak formulation are stated. We prove some properties of the discrete solution in Sect. 3 and present results of the semidiscrete error estimation in Sect. 4. In Sect. 5, we briefly discuss applicability of the results to fully discrete adaptive numerical procedures.

2. Model problem

The principal ideas as well as algorithmic procedures connected with the use of an adaptive grid for solving nonlinear parabolic partial differential equations can be demonstrated with the help of a simple initial-boundary value model problem.

Consider the nonlinear equation

$$\partial_t u(x,t)+f(u) = \partial_x(a(u)\partial_x u(x,t)), \quad 0 < x < 1, \quad 0 < t \le T, \tag{2.1}$$

with a fixed $T > 0$ for an unknown function u where we use the notation $\partial_x = \partial/\partial x$ and $\partial_t = \partial/\partial t$. In (2.1), a and f are smooth functions with

$$0 < \mu \le a(s) \le M \text{ for all } s \in \mathbb{R} \tag{2.2}$$

that satisfy the global Lipschitz conditions

$$|a(r) - a(s)| \le L|r - s|, \tag{2.3}$$

$$|f(r) - f(s)| \le L|r - s| \text{ for all } r, s \in \mathbb{R} \tag{2.4}$$

with constants μ, M, and L. Possible further assumptions on the model problem will be formulated when necessary.

In addition, we impose the homogeneous Dirichlet boundary condition

$$u(0,t) = u(1,t) = 0,\ 0 \le t \le T, \tag{2.5}$$

and the initial condition

$$u(x,0) = u_0(x),\ 0 < x < 1, \tag{2.6}$$

where u_0 is a given smooth function. We assume that the boundary and initial conditions are consistent.

We present the weak formulation of the model problem which is the starting point for the finite element discretization.

Denote by

$$(v,w) = \int_0^1 v(x)w(x)\,dx$$

the L_2 inner product of two functions and $\|w\|_0$ the corresponding norm. Let k be a nonnegative integer. Then $H^k = H^k(0,1)$ is the Sobolev space of functions defined on the interval $(0,1)$ with the norm

$$\|w\|_k{}^2 = \sum_{i=0}^{k} \Big\| \frac{\partial^i w}{\partial x^i} \Big\|_0^2.$$

The case of $k = 1$ is important for the weak formulation. We introduce the usual subspace $H_0^1 = H_0^1(0,1)$ of functions $w \in H_0^1$ satisfying the homogeneous Dirichlet boundary conditions.

We will also use the space L_∞ with the corresponding norm $\|w\|_\infty =$ ess $\sup |v|$.

The constants C, C_1, C_2, etc. are generic in the paper, i.e., they may represent different constant quantities in different occurrences.

We say that a function $u(x,t)$ is the *weak solution* of the problem (2.1), (2.5), and (2.6) if $u \in H^1([0,T], H_0^1(0,1))$, if the identity

$$(\partial_t u, v) + (f(u), v) + (a(u)\partial_x u, \partial_x v) = 0 \tag{2.7}$$

holds for almost every $t \in (0,T]$ and all functions $v \in H_0^1$, and if the identity

$$(a(u_0)\partial_x u, \partial_x v) = (a(u_0)\partial_x u_0, \partial_x v), \tag{2.8}$$

where $u_0 \in H_0^1$, holds for $t = 0$ and all functions $v \in H_0^1$.

In this weak formulation as well as in the whole paper, the variable t appears as a parameter. Without explicitly stating, we assume that all the statements and, in particular, constants may depend on t.

Lemma 2.1 *Let* (2.2) *hold. Then*

$$\|v\|_1^2 \le C(a(v)\partial_x v, \partial_x v) \tag{2.9}$$

for all $v \in H_0^1$.

Proof. The inequality (2.9) is a simple consequence of the Friedrichs inequality and (2.2). □

3. Discretization

Let us choose a positive integer p. We solve the problem (2.1), (2.5), and (2.6) or, in the weak formulation, (2.7) and (2.8) by a finite element method with a piecewise polynomial hierarchical basis of degree $p \ge 1$. We introduce a partition

$$0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1$$

of the interval $[0,1]$ into N subintervals (x_{j-1},x_j), $j=1,\dots,N$. We further put $h_j=x_j-x_{j-1}$, $j=1,\dots,N$, and

$$h=\max_{j=1,\dots,N} h_j.$$

Remark 3.1 The following analysis can be modified to the case when the value of p is different for the individual subintervals (x_{j-1},x_j).

We suppose that there exist positive constants τ_1 and τ_2 such that

$$\tau_1 \le \frac{h_j}{h_{j+1}} \le \tau_2,\ j=1,\dots,N-1,$$

holds for all the partitions of the interval $[0,1]$.

Remark 3.2 The above grid regularity condition may be restrictive in the presence of adaptivity. However, there are adaptive procedures (cf. [9]) where refinement criteria satisfy this condition.

We construct a finite dimensional subspace $S_0^{N,p}\subset H_0^1$ in the following way. A function V belongs to $S_0^{N,p}$ if

$$V(x)=\sum_{j=1}^{N-1} V_{j1}\varphi_{j1}(x)+\sum_{j=1}^{N}\sum_{k=2}^{p} V_{jk}\varphi_{jk}(x),$$

where

$$\begin{aligned}\varphi_{j1}(x)&=(x-x_{j-1})/h_j,\ x_{j-1}\le x<x_j,\\ &=(x_{j+1}-x)/h_{j+1},\ x_j\le x\le x_{j+1},\\ &=0 \text{ otherwise}\end{aligned} \tag{3.1}$$

for $j=1,\dots,N-1$,

$$\begin{aligned}\varphi_{jk}(x)&=\frac{\sqrt{2(2k-1)}}{h_j}\int_{x_{j-1}}^{x} \mathrm{P}_{k-1}(y)\,dy,\ x_{j-1}\le x<x_j,\\ &=0 \text{ otherwise}\end{aligned} \tag{3.2}$$

for $j=1,\dots,N$ and $k=2,\dots,p$, and where V_{jk} are coefficients. The function $\mathrm{P}_k(x)$ is the kth degree Legendre polynomial scaled to subinterval $[x_{j-1},x_j]$. The functions (3.1) and (3.2) form a *hierarchical basis* of the subspace $S_0^{N,p}$ [15]. The piecewise linear polynomial portion (3.1) of the basis φ_{j1} is nodally based while the higher-degree portion (3.2) of the basis φ_{jk}, $k>1$, is elemental. To express a function $V(x,t)\in S_0^{N,p}$ in the basis (3.1) and (3.2) we put $V_{jk}=V_{jk}(t)$.

We will also use the local inner product

$$(v,w)_j = \int_{x_{j-1}}^{x_j} v(x)w(x)\,dx$$

and the corresponding local norm $\|v\|_{s,j}$.

To start the analysis, we introduce an elliptic projection of the solution u. We say that a function $u^h(x,t)$ is the *elliptic projection* of the solution $u(x,t)$ of the problem (2.7) and (2.8) if $u^h \in H^1([0,T], S_0^{N,p})$, if the identity

$$(a(u)\partial_x u^h, \partial_x V) = (a(u)\partial_x u, \partial_x V) \tag{3.3}$$

holds for almost every $t \in (0,T]$ and all functions $V \in S_0^{N,p}$, and if the identity

$$(a(u_0)\partial_x u^h, \partial_x V) = (a(u_0)\partial_x u_0,\ \partial_x V)$$

holds for $t = 0$ and all functions $V \in S_0^{N,p}$.

We further denote by

$$\rho(x,t) = u(x,t) - u^h(x,t) \tag{3.4}$$

the error of the elliptic projection.

Our analysis relies on the properties of the elliptic projection u^h presented.

Lemma 3.1 *Let $u \in H^{p+1} \cap H_0^1$ and $u^h \in S_0^{N,p}$ be the elliptic projection. Then*

$$\|\rho\|_0 + h\|\partial_x \rho\|_0 \le C(u)h^{p+1}, \tag{3.5}$$

$$\|\partial_t \rho\|_0 \le C(u)h^{p+1}, \tag{3.6}$$

and if $\partial_{tt}\rho \in L_2$ then also

$$\|\partial_{tt}\rho\|_0 \le C(u)h^{p+1}. \tag{3.7}$$

Further

$$\|\partial_x u^h\|_\infty \le C(u) \tag{3.8}$$

independently of t and h.

Proof. See [16] Lemmas 13.2 and 13.3, and [9] Lemma 2.1. □

We now introduce the semidiscrete solution. We say that a function $\bar{U}(x,t)$ is the *semidiscrete approximate solution* of the problem (2.7) and (2.8) if $\bar{U} \in H^1([0,T], S_0^{N,p})$, if the identity

$$(\partial_t \bar{U}, V) + (f(\bar{U}), V) + (a(\bar{U})\partial_x \bar{U}, \partial_x V) = 0 \tag{3.9}$$

holds for almost every $t \in (0, T]$ and all functions $V \in S_0^{N,p}$, and if the identity

$$(a(u_0)\partial_x \bar{U}, \partial_x V) = (a(u_0)\partial_x u_0, \partial_x V) \tag{3.10}$$

holds for $t = 0$ and all functions $V \in S_0^{N,p}$.

Note that, in contrast to [9], we treat the problem (3.9) and (3.10) with a general coefficient $a(u)$.

We further denote by

$$e(x,t) = u(x,t) - \bar{U}(x,t) \tag{3.11}$$

the error of the semidiscrete solution.

Lemma 3.2 *Let $u \in H^{p+1} \cap H_0^1$ and $\bar{U} \in S_0^{N,p}$ be the semidiscrete solution. Then*

$$\|e\|_1 \leq C(u)h^p.$$

Proof. See [16] Theorem 13.1 and [9] formula (3.3). □

The phenomenon that $\bar{U}$ is a better approximation to u^h than to u is referred to as superconvergence in [16]. Putting

$$\theta(x,t) = u^h(x,t) - \bar{U}(x,t), \tag{3.12}$$

we formulate this statement as a lemma. Note that $e(x,t) = \rho(x,t) + \theta(x,t)$ according to (3.4), (3.11), and (3.12).

Lemma 3.3 *Let $u^h \in S_0^{N,p}$ and $\bar{U} \in S_0^{N,p}$ be the elliptic projection and semidiscrete solution. Let $\|\theta(.,t)\|_0 = \|\theta(t)\|_0$ be a nondecreasing function of the variable t. Then*

$$\|\theta\|_1 \leq C(u,L)h^{p+1}. \tag{3.13}$$

Proof. Recall that u, u^h, and $\bar{U}$ are given by (2.7), (3.3), and (3.9), respectively, for $t > 0$. Substituting from (3.3) into (2.7), then subtracting (2.7) from (3.9), and recalling further (3.4), (3.11), and (3.12), we find

$$\begin{aligned}(\partial_t\theta, V) + (a(\bar{U})\partial_x\theta, \partial_x V) = &-(\partial_t\rho, V) + (f(\bar{U}) - f(u), V)\\ &+ ([a(\bar{U}) - a(u)]\partial_x u^h, \partial_x V)\end{aligned} \tag{3.14}$$

for all functions $V \in S_0^{N,p}$. Putting $V = \theta$ in (3.14) and using the obvious formula

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_0^2 = (\partial_t\psi, \psi) \tag{3.15}$$

that holds for any sufficiently smooth function ψ, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta\|_0^2 + (a(\bar U)\partial_x\theta, \partial_x\theta) \\
&\qquad = -(\partial_t\rho, \theta) + (f(\bar U) - f(u^h), \theta) + (f(u^h) - f(u), \theta) \\
&\qquad\quad + ([a(\bar U) - a(u^h)]\partial_x u^h, \partial_x\theta) + ([a(u^h) - a(u)]\partial_x u^h, \partial_x\theta).
\end{aligned} \tag{3.16}$$

We now bound the individual terms in (3.16). We have

$$0 < \mu\|\partial_x\theta\|_0^2 \le (a(\bar U)\partial_x\theta, \partial_x\theta) \tag{3.17}$$

according to (2.2). On the right-hand part, we employ a positive constant K to be fixed later. From the Schwarz inequality, (2.3), (2.4), and (3.8) we then have

$$|(\partial_t\rho, \theta)| \le \tfrac{1}{2}\|\partial_t\rho\|_0^2 + \tfrac{1}{2}\|\theta\|_0^2, \tag{3.18}$$

$$\begin{aligned}
|(f(\bar U) - f(u^h), \theta)| &\le \tfrac{1}{2}\|f(\bar U) - f(u^h)\|_0^2 + \tfrac{1}{2}\|\theta\|_0^2 \\
&\le \tfrac{1}{2}(L^2 + 1)\|\theta\|_0^2,
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
|(f(u^h) - f(u), \theta)| &\le \tfrac{1}{2}\|f(u^h) - f(u)\|_0^2 + \tfrac{1}{2}\|\theta\|_0^2 \\
&\le \tfrac{1}{2}L^2\|\rho\|_0^2 + \tfrac{1}{2}\|\theta\|_0^2,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
|([a(\bar U) - a(u^h)]\partial_x u^h, \partial_x\theta)| &\le (C\sqrt{K^{-1}}|a(\bar U) - a(u^h)|,\ \sqrt{K}|\partial_x\theta|) \\
&\le \tfrac{1}{2}C^2L^2K^{-1}\|\theta\|_0^2 + \tfrac{1}{2}K\|\partial_x\theta\|_0^2,
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
|([a(u^h) - a(u)]\partial_x u^h, \partial_x\theta)| &\le (C\sqrt{K^{-1}}|a(u^h) - a(u)|,\ \sqrt{K}|\partial_x\theta|) \\
&\le \tfrac{1}{2}C^2L^2K^{-1}\|\rho\|_0^2 + \tfrac{1}{2}K\|\partial_x\theta\|_0^2.
\end{aligned} \tag{3.22}$$

Substituting now the bounds (3.17) to (3.22) into (3.16), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\theta\|_0^2 + (\mu - K)\|\partial_x\theta\|_0^2 &\le (C_1 + C_2K^{-1})\|\theta\|_0^2 \\
&\quad + (C_3 + C_4K^{-1})\|\rho\|_0^2 + \tfrac{1}{2}\|\partial_t\rho\|_0^2.
\end{aligned}$$

Note that

$$\|\theta\|_0 \le C(u)h^{p+1} \tag{3.23}$$

according to [16], proof of Theorem 13.1 and [9], proof of Lemma 3.1.

Fixing now $K < \mu$, and using (3.5), (3.6), and (3.23), we arrive at

$$\frac{d}{dt}\|\theta\|_0^2 + C_1\|\partial_x\theta\|_0^2 \le C_2h^{2p+2}. \tag{3.24}$$

Since we assumed $\|\theta(t)\|_0$ nondecreasing, $d\|\theta\|_0^2/dt \ge 0$ and we finally obtain (3.13) from (3.23) and (3.24). □

Remark 3.3 The assumption that $\|\theta\|_0$ is a nondecreasing function of t holds, e.g., for linear problems with the term f increasing in time. Consider, for example, the problem

$$\partial_t u - 2t(t + x(1-x)) = \partial_{xx} u$$

with $u(x,0) = 0$ where $u(x,t) = t^2 x(1-x)$ and the quantity θ can be easily calculated explicitly for $p = 1$.

Before we state further properties of the function θ we prove a lemma we are going to use. A weaker statement is presented in [6] Chap. 1.

Lemma 3.4 *Let* φ, ψ, *and* χ *be Lebesgue integrable functions on the interval* $[0,T]$, $\chi(t) \geq 0$ *almost everywhere in* $[0,T]$. *Let*

$$\varphi(t) \leq \psi(t) + \int_0^t \chi(\tau)\varphi(\tau)\, d\tau \tag{3.25}$$

hold for almost every $t \in [0,T]$.

Then

$$\varphi(t) \leq \psi(t) + \int_0^t \chi(\tau)\psi(\tau) \exp\left(\int_\tau^t \psi(\sigma)\, d\sigma \right) d\tau \tag{3.26}$$

for almost every $t \in [0,T]$.

Proof. We first show that if a function $R(t)$ satisfies the inequality

$$R'(t) - \chi(t)R(t) \leq \chi(t)\psi(t) \tag{3.27}$$

with the initial condition $R(0) = 0$ and a function $Z(t)$ is the solution of the equation

$$Z'(t) - \chi(t)Z(t) = \chi(t)\psi(t) \tag{3.28}$$

with the initial condition $Z(0) = 0$, then

$$R(t) \leq Z(t). \tag{3.29}$$

Putting $W(t) = Z(t) - R(t)$ and subtracting (3.27) from (3.28), we find out that $W(t)$ is the solution of the equation

$$W'(t) - \chi(t)W(t) = \alpha(t) \geq 0$$

with the initial condition $W(0) = 0$. The well-known formula then immediately gives

$$W(t) = \int_0^t \alpha(\tau) \exp\left(\int_\tau^t \chi(\sigma)\, d\sigma \right) d\tau \geq 0, \tag{3.30}$$

i.e., (3.29) holds.

Multiplying (3.25) by $\chi(t)$ and substituting then

$$R(t) = \int_0^t \chi(\tau)\varphi(\tau)\,d\tau,$$

we obtain just the inequality (3.27). Using now (3.25) and (3.29), and writing an analog of formula (3.30) for the solution $Z(t)$ of the equation (3.28), we arrive at (3.26). □

We will denote by $a'(r)$ and $f'(r)$ the derivatives of the functions $a(r)$ and $f(r)$ with respect to the only variable r.

Lemma 3.5 *Let $u^h \in S_0^{N,p}$ and $\bar{U} \in S_0^{N,p}$ be the elliptic projection and semidiscrete solution. Further assume the global Lipschitz conditions*

$$|a'(r) - a'(s)| \le L'|r - s|, \tag{3.31}$$

$$|f'(r) - f'(s)| \le L'|r - s| \text{ for all } r, s \in \mathbb{R} \tag{3.32}$$

and

$$|a'(s)| \le M', \tag{3.33}$$

$$|f'(s)| \le M' \text{ for all } s \in \mathbb{R} \tag{3.34}$$

with some positive constants L' and M'.

Moreover, let there be a constant $Q > 0$ such that

$$\|\partial_t u\|_\infty \le Q, \tag{3.35}$$

$$\|\partial_t \bar{U}\|_\infty \le Q, \tag{3.36}$$

$$\|\partial_{tx} u^h\|_\infty \le Q, \tag{3.37}$$

Let u^h and $\bar{U}$ depend on t in a sufficiently smooth way, let $\|\theta(.,t)\|_0$ be a nondecreasing function of the variable t, and let

$$\|\partial_t \theta(0)\|_0 \le Ch^{p+1}. \tag{3.38}$$

Then

$$\|\partial_t \theta\|_0 \le C(u)h^{p+1}. \tag{3.39}$$

Proof. Start again from the identity (3.14) that holds for all functions $V \in S_0^{N,p}$ and differentiate with respect to t. We have

$$\begin{aligned}(\partial_{tt}\theta, V) + (a'(\bar{U})\partial_t\bar{U}\partial_x\theta, \partial_x V) + (a(\bar{U})\partial_{tx}\theta, \partial_x V)\\ = -(\partial_{tt}\rho, V) + (f'(\bar{U})\partial_t\bar{U} - f'(u)\partial_t u, V)\\ + ([a'(\bar{U})\partial_t\bar{U} - a'(u)\partial_t u]\partial_x u^h, \partial_x V)\\ + ([a(\bar{U}) - a(u)]\partial_{tx}u^h, \partial_x V)\end{aligned}$$

and put $V = \partial_t\theta$. Using an analog of (3.15), we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_t\theta\|_0^2 + (a(\bar{U})\partial_{tx}\theta, \partial_{tx}\theta)\\
&\quad = -(a'(\bar{U})\partial_t\bar{U}\partial_x\theta, \partial_{tx}\theta) - (\partial_{tt}\rho, \partial_t\theta)\\
&\qquad - (f'(\bar{U})\partial_t\rho, \partial_t\theta) - (f'(\bar{U})\partial_t\theta, \partial_t\theta)\\
&\qquad - ([f'(u) - f'(\bar{U})]\partial_t u, \partial_t\theta)\\
&\qquad - (a'(\bar{U})\partial_x u^h\partial_t\rho, \partial_{tx}\theta) - (a'(\bar{U})\partial_x u^h\partial_t\theta, \partial_{tx}\theta)\\
&\qquad - ([a'(u) - a'(\bar{U})]\partial_x u^h\partial_t u, \partial_{tx}\theta)\\
&\qquad + ([a(\bar{U}) - a(u)]\partial_{tx}u^h, \partial_{tx}\theta).
\end{aligned}
\tag{3.40}
$$

We now bound the individual terms in (3.40). We have

$$0 < \mu\|\partial_{tx}\theta\|_0^2 \le (a(\bar{U})\partial_{tx}\theta, \partial_{tx}\theta) \tag{3.41}$$

according to (2.2). On the right-hand part, we will employ a constant K to be fixed later. From the Schwarz inequality, (2.3), (3.8), and (3.31) to (3.37), we then have

$$
\begin{aligned}
&|(a'(\bar{U})\partial_t\bar{U}\partial_x\theta, \partial_{tx}\theta)| \le (M'Q\sqrt{K^{-1}}|\partial_x\theta|, \sqrt{K}|\partial_{tx}\theta|)\\
&\quad \le \tfrac{1}{2}M'^2Q^2K^{-1}\|\partial_x\theta\|_0^2 + \tfrac{1}{2}K\|\partial_{tx}\theta\|_0^2,
\end{aligned}
\tag{3.42}
$$

$$|(\partial_{tt}\rho, \partial_t\theta)| \le \tfrac{1}{2}\|\partial_{tt}\rho\|_0^2 + \tfrac{1}{2}\|\partial_t\theta\|_0^2, \tag{3.43}$$

$$|(f'(\bar{U})\partial_t\rho, \partial_t\theta)| \le \tfrac{1}{2}M'^2\|\partial_t\rho\|_0^2 + \tfrac{1}{2}\|\partial_t\theta\|_0^2, \tag{3.44}$$

$$|(f'(\bar{U})\partial_t\theta, \partial_t\theta| \le M'^2\|\partial_t\theta\|_0^2, \tag{3.45}$$

$$
\begin{aligned}
&|([f'(u) - f'(\bar{U})]\partial_t u, \partial_t\theta)| \le (Q|f'(u) - f'(\bar{U})|,\ |\partial_t\theta|)\\
&\quad \le (QL'|\rho + \theta|, |\partial_t\theta|)\\
&\quad \le Q^2L'^2\|\rho\|_0^2 + Q^2L'^2\|\theta\|_0^2 + \tfrac{1}{2}\|\partial_t\theta\|_0^2,
\end{aligned}
\tag{3.46}
$$

$$
\begin{aligned}
&|(a'(\bar{U})\partial_x u^h\partial_t\rho, \partial_{tx}\theta)| \le (M'C\sqrt{K^{-1}}|\partial_t\rho|, \sqrt{K}|\partial_{tx}\theta|)\\
&\quad \le \tfrac{1}{2}M'^2C^2K^{-1}\|\partial_t\rho\|_0^2 + \tfrac{1}{2}K\|\partial_{tx}\theta\|_0^2,
\end{aligned}
\tag{3.47}
$$

$$
\begin{aligned}
&|(a'(\bar{U})\partial_x u^h\partial_t\theta, \partial_{tx}\theta)| \le (M'C\sqrt{K^{-1}}|\partial_t\theta|, \sqrt{K}|\partial_{tx}\theta|)\\
&\quad \le \tfrac{1}{2}M'^2C^2K^{-1}\|\partial_t\theta\|_0^2 + \tfrac{1}{2}K\|\partial_{tx}\theta\|_0^2,
\end{aligned}
\tag{3.48}
$$

$$
\begin{aligned}
&|([a'(u) - a'(\bar{U})]\partial_x u^h\partial_t u, \partial_{tx}\theta)|\\
&\quad \le (CQ\sqrt{K^{-1}}|a'(u) - a'(\bar{U})|, \sqrt{K}|\partial_{tx}\theta|)\\
&\quad \le C^2Q^2L'^2K^{-1}\|\rho\|_0^2 + C^2Q^2L'^2K^{-1}\|\theta\|_0^2 + \tfrac{1}{2}K\|\partial_{tx}\theta\|_0^2,
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
&|([a(\bar{U}) - a(u)]\partial_{tx}u^h, \partial_{tx}\theta)|\\
&\quad \le (Q\sqrt{K^{-1}}|a(\bar{U}) - a(u)|, \sqrt{K}|\partial_{tx}\theta|)\\
&\quad \le Q^2L^2K^{-1}\|\rho\|_0^2 + Q^2L^2K^{-1}\|\theta\|_0^2 + \tfrac{1}{2}K\|\partial_{tx}\theta\|^2{}_0.
\end{aligned}
\tag{3.50}
$$

Recalling now (3.5) to (3.7) and (3.13), and using the bounds (3.41) to (3.50), we rewrite (3.40) in the form

$$\frac{1}{2}\frac{d}{dt}\|\partial_t\theta\|_0^2 + (\mu - \tfrac{5}{2}K)\|\partial_{tx}\theta\|_0^2 \le C_1(t)\|\partial_t\theta\|_0^2 + C_2(t)h^{2p+2}. \tag{3.51}$$

Fixing now $K < \frac{2}{5}\mu$, we can omit the term $\|\partial_{tx}\theta\|_0^2$ and integrate on both parts of (3.51) to get

$$\|\partial_t\theta(t)\|_0^2 \le \|\partial_t\theta(0)\|_0^2 + C_3(t)h^{2p+2} + 2\int_0^t C_1(s)\|\partial_t\theta(s)\|_0^2\,ds$$

if the corresponding functions are Lebesgue integrable on the interval $[0,T]$.

Assuming again that the corresponding functions are Lebesgue integrable on the interval $[0,T]$, and employing (3.38) and Lemma 3.4, we finally arrive at the bound

$$\|\partial_t\theta(t)\|_0^2 \le C_5(t)h^{2p+2} + C_6(t)h^{2p+2} \le Ch^{2p+2}, \tag{3.52}$$

which is (3.39). □

Remark 3.4 As follows from the proof, such a smooth dependence of u^h and $\bar{U}$ on t is needed in Lemma 3.5 that guarantees the integration of (3.51) and the application of Lemma 3.4, including the upper bound for the expression in (3.52).

4. Semidiscrete error estimation

From the formula (3.11), we have

$$u(x,t) = \bar{U}(x,t) + e(x,t) \tag{4.1}$$

and replace u by (4.1) in (2.7) and (2.8). Then

$$\begin{aligned}(\partial_t e, v) &+ (a(\bar{U}+e)\partial_x e, \partial_x v)\\ &= -(f(\bar{U}+e), v) - (\partial_t\bar{U}, v) - (a(\bar{U}+e)\partial_x\bar{U}, \partial_x v)\end{aligned} \tag{4.2}$$

holds for almost every $t \in (0,T]$ and all functions $v \in H_0^1$, and

$$(a(u_0)\partial_x e, \partial_x v) = (a(u_0)\partial_x(u_0 - \bar{U}), \partial_x v) \tag{4.3}$$

holds for $t = 0$ and all functions $v \in H_0^1$.

An error estimate can be obtained by calculating another semidiscrete approximate solution of (2.7) and (2.8), $\bar{\bar{U}} \in S_0^{N,p+1}$, using (3.9) and (3.10). Since $\bar{\bar{U}}$ is higher-order than $\bar{U}$, $\|\bar{\bar{U}} - \bar{U}\|_1$ can provide an estimate for the error e. The computational efficiency of this approach is improved when superconvergence can be employed. Superconvergence implies that $\bar{\bar{U}} \approx \bar{U} + \bar{E}$ where

$$\bar{E}(x,t) = \sum_{j=1}^{N} \bar{E}_j(t)\varphi_{j,p+1}(x) \tag{4.4}$$

is the *error estimate*. Let us introduce the space $\widehat{S}_0^{N,p+1}$ of functions $\widehat{V}(x)$ such that

$$\widehat{V}(x) = \sum_{j=1}^{N} \widehat{V}_j \varphi_{j,p+1}(x).$$

There are several possibilities to define $\bar{E}$ with the help of (4.2) and (4.3). If no ambiguity can occur we denote all the error estimates introduced in what follows by the symbol $\bar{E}$ and omit the indices. Recall that $\varphi_{j,p+1}(x) = 0$ for $x \le x_{j-1}$ and $x_j \le x$ according to (3.2) and notice that the identity

$$(\psi, \widehat{V}) = 0 \text{ for all } \widehat{V} \in \widehat{S}_0^{N,p+1}$$

is equivalent to the system of identities

$$(\psi, \widehat{V})_j = 0 \text{ for all } \widehat{V} \in \widehat{S}_0^{N,p+1},\ j = 1, \dots, N.$$

We then say that a function $\bar{E} = \bar{E}_{\mathrm{PN}}$ of the form (4.4) is the *nonlinear parabolic error estimate* if the identity

$$\begin{aligned} &(\partial_t \bar{E}, \widehat{V})_j + (a(\bar{U}+\bar{E})\partial_x \bar{E}, \partial_x \widehat{V})_j \\ &\quad = -(f(\bar{U}+\bar{E}), \widehat{V})_j - (\partial_t \bar{U}, \widehat{V})_j - (a(\bar{U}+\bar{E})\partial_x \bar{U}, \partial_x \widehat{V})_j \end{aligned} \tag{4.5}$$

holds for almost every $t \in (0,T]$, $j = 1, \dots, N$, and all functions $\widehat{V} \in \widehat{S}_0^{N,p+1}$, and if the identity

$$(a(u_0)\partial_x \bar{E}, \partial_x \widehat{V})_j = (a(u_0)\partial_x(u_0 - \bar{U}), \partial_x \widehat{V})_j \tag{4.6}$$

holds for $t = 0$, $j = 1, \dots, N$, and all functions $\widehat{V} \in \widehat{S}_0^{N,p+1}$. Note that (4.5) and (4.6) are a series of N uncoupled local parabolic problems and, therefore, the solution costs are rather low.

To save some further computation we can neglect the time change of the error estimate. We say that a function $\bar{E} = \bar{E}_{\mathrm{EN}}$ of the form (4.4) is the *nonlinear elliptic error estimate* if the identity

$$\begin{aligned} &(a(\bar{U}+\bar{E})\partial_x \bar{E}, \partial_x \widehat{V})_j \\ &\quad = -(f(\bar{U}+\bar{E}), \widehat{V})_j - (\partial_t \bar{U}, \widehat{V})_j - (a(\bar{U}+\bar{E})\partial_x \bar{U}, \partial_x \widehat{V})_j \end{aligned} \tag{4.7}$$

holds for almost every $t \in (0,T]$, $j = 1, \dots, N$, and all functions $\widehat{V} \in \widehat{S}_0^{N,p+1}$, and if the identity (4.6) holds for $t = 0$ and the same functions $\widehat{V}$.

The problem (4.7) and (4.6) represents a series of N uncoupled local elliptic problems. The computational advantage of the elliptic estimate consists in the fact that, for practical reasons, it need not be computed for each t but only when needed.

Additional savings can be realized by neglecting the $\bar{E}$ term in $a(\bar{U}+\bar{E})$ and $f(\bar{U}+\bar{E})$, thereby reducing (4.5) to a linear parabolic problem

$$
\begin{aligned}
&(\partial_t \bar{E}, \widehat{V})_j + (a(\bar{U})\partial_x \bar{E}, \partial_x \widehat{V})_j \\
&\qquad = -(f(\bar{U}), \widehat{V})_j - (\partial_t \bar{U}, \widehat{V})_j - (a(\bar{U})\partial_x \bar{U}, \partial_x \widehat{V})_j
\end{aligned} \tag{4.8}
$$

and (4.7) to a linear elliptic problem

$$
\begin{aligned}
(a(\bar{U})\partial_x \bar{E}, \partial_x \widehat{V})_j = &-(f(\bar{U}), \widehat{V})_j - (\partial_t \bar{U}, \widehat{V})_j \\
&- (a(\bar{U})\partial_x \bar{U}, \partial_x \widehat{V})_j .
\end{aligned} \tag{4.9}
$$

The corresponding error estimates are called *linear parabolic* $(\bar{E}_{\mathrm{PL}})$ and *linear elliptic* $(\bar{E}_{\mathrm{EL}})$.

As an analog of the elliptic projection u^h of the solution u into $S_0^{N,p}$ (cf. (3.3)), let us introduce a function $e^h \in \widehat{S}_0^{N,p+1}$,

$$
e^h(x,t) = \sum_{j=1}^{N} e_j^h(t)\varphi_{j,p+1}(x),
$$

such that $u^h + e^h$ is an elliptic projection of u into $S_0^{N,p+1}$, i.e.,

$$
(a(u)\partial_x(u^h + e^h), \partial_x \widehat{V})_j = (a(u)\partial_x u, \partial_x \widehat{V})_j \tag{4.10}
$$

holds for almost every $t \in [0,T]$, $j = 1, \dots, N$, and all functions $\widehat{V} \in \widehat{S}_0^{N,p+1}$.

We further put

$$
\eta_{\mathrm{PN}}(x,t) = e^h(x,t) - \bar{E}_{\mathrm{PN}}(x,t), \tag{4.11}
$$

$$
\eta_{\mathrm{PL}}(x,t) = e^h(x,t) - \bar{E}_{\mathrm{PL}}(x,t), \tag{4.12}
$$

$$
\eta_{\mathrm{EN}}(x,t) = e^h(x,t) - \bar{E}_{\mathrm{EN}}(x,t),
$$

$$
\eta_{\mathrm{EL}}(x,t) = e^h(x,t) - \bar{E}_{\mathrm{EL}}(x,t), \tag{4.13}
$$

where, apparently, $\eta \in \widehat{S}_0^{N,p+1}$. We omit the indices of η if no ambiguity can occur.

Finally, we set

$$
\widehat{\rho}(x,t) = u(x,t) - u^h(x,t) - e^h(x,t) \tag{4.14}
$$

analogically to (3.4).

A bound for the quantity η_{PN} is provided by the following lemma.

Lemma 4.1 *Let $\bar{E}_{\mathrm{PN}} \in \widehat{S}_0^{N,p+1}$ be the error estimate given by* (4.5) *and* (4.6)*, and $e^h \in \widehat{S}_0^{N,p+1}$ a function such that $u^h + e^h$ is an elliptic projection of u into $S_0^{N,p+1}$ according to* (4.10)*. Let $\|\theta(.,t)\|_0$ and $\|\eta_{\mathrm{PN}}(.,t)\|_0$ be nondecreasing functions of the variable t and let*

$$\|\eta(0)\|_0 \le Ch^{p+1} \tag{4.15}$$

hold for the function $\eta_{\mathrm{PN}}(x,0)$ given by (4.11)*. Let the assumptions of Lemma* 3.5 *be fulfilled, and let e^h and $\bar{E}_{\mathrm{PN}}$ depend on t in a sufficiently smooth way. Then there exists a constant $C > 0$ such that*

$$\|\eta_{\mathrm{PN}}\|_1 \le Ch^{p+1}.$$

Proof. Substituting from (4.10) into (2.7) and choosing test functions $\widehat{V} \in \widehat{S}_0^{N,p+1} \subset H_0^1$, we arrive at

$$(\partial_t u, \widehat{V})_j + (a(u)\partial_x(u^h + e^h), \partial_x \widehat{V})_j = -(f(u), \widehat{V})_j \tag{4.16}$$

for $j = 1, \dots, N$. Subtracting now (4.16) from (4.5) and substituting for u from (4.14), we have

$$\begin{aligned}
&(\partial_t \bar{E}, \widehat{V})_j + (a(\bar{U} + \bar{E})\partial_x \bar{E}, \partial_x \widehat{V})_j + (\partial_t \bar{U}, \widehat{V})_j \\
&+(a(\bar{U} + \bar{E})\partial_x \bar{U}, \partial_x \widehat{V})_j - (\partial_t \widehat{\rho}, \widehat{V})_j - (\partial_t u^h, \widehat{V})_j \\
&-(\partial_t e^h, \widehat{V})_j - (a(u)\partial_x(u^h + e^h), \partial_x \widehat{V})_j \\
&= -(f(\bar{U} + \bar{E}), \widehat{V})_j + (f(u), \widehat{V})_j.
\end{aligned}$$

We now rearrange the terms, and add and subtract terms with $a(\bar{U} + \bar{E})\partial_x e^h$ and $a(\bar{U} + \bar{E})\partial_x u^h$ to arrive at

$$\begin{aligned}
&(\partial_t \eta, \widehat{V})_j + (a(\bar{U} + \bar{E})\partial_x \eta, \partial_x \widehat{V})_j \\
&= ([a(\bar{U} + \bar{E}) - a(u)]\partial_x(u^h + e^h), \partial_x \widehat{V})_j \\
&\quad - (a(\bar{U} + \bar{E})\partial_x \theta, \partial_x \widehat{V})_j - (\partial_t \theta, \widehat{V})_j - (\partial_t \widehat{\rho}, \widehat{V})_j \\
&\quad + (f(\bar{U} + \bar{E}) - f(u), \widehat{V})_j.
\end{aligned} \tag{4.17}$$

A further course of the proof is analogous to that of Lemma 3.5. We substitute $\eta \in \widehat{S}_0^{N,p+1}$ for $\widehat{V}$ into (4.17) and use (3.15) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\eta\|_{0,j}^2 + (a(\bar{U} + \bar{E})\partial_x \eta, \partial_x \eta)_j \\
&= ([a(\bar{U} + \bar{E}) - a(u)]\partial_x(u^h + e^h), \partial_x \eta)_j \\
&\quad - (a(\bar{U} + \bar{E})\partial_x \theta, \partial_x \eta)_j - (\partial_t \theta, \eta)_j - (\partial_t \widehat{\rho}, \eta)_j \\
&\quad + (f(\bar{U} + \bar{E}) - f(u), \eta)_j
\end{aligned} \tag{4.18}$$

for $j = 1, \dots, N$.

We now bound the individual terms in (4.18). We have

$$0 < \mu \|\partial_x \eta\|_{0,j}^2 \le (a(\bar{U} + \bar{E})\partial_x \eta, \partial_x \eta)_j \tag{4.19}$$

according to (2.2). Note that

$$\|\partial_x (u^h + e^h)\|_\infty \le C(u) \tag{4.20}$$

according to (3.8) and (4.10). We employ a positive constant K independent of j to be fixed later on the right-hand part of (4.18). From the Schwarz inequality, (2.2) to (2.4), and (4.20), we then have

$$\begin{aligned}
&|(-[a(\bar{U} + \bar{E}) - a(u)]\partial_x (u^h + e^h) + a(\bar{U} + \bar{E})\partial_x \theta, \partial_x \eta)_j| \\
&\quad \le (C\sqrt{K^{-1}}|a(\bar{U} + \bar{E}) - a(u)| + \sqrt{K^{-1}} M |\partial_x \theta|, \sqrt{K}|\partial_x \eta|)_j \\
&\quad \le 4C^2 K^{-1} L^2 \|\eta\|_{0,j}^2 + 4C^2 K^{-1} L^2 \|\theta\|_{0,j}^2 + 2C^2 K^{-1} L^2 \|\widehat{\rho}\|_{0,j}^2 \\
&\quad + K^{-1} M^2 \|\partial_x \theta\|_{0,j}^2 + \tfrac{1}{2} K \|\partial_x \eta\|_{0,j}^2,
\end{aligned} \tag{4.21}$$

$$|(\partial_t \theta, \eta)_j| \le \tfrac{1}{2}\|\partial_t \theta\|_{0,j}^2 + \tfrac{1}{2}\|\eta\|_{0,j}^2, \tag{4.22}$$

$$|(\partial_t \widehat{\rho}, \eta)_j| \le \tfrac{1}{2}\|\partial_t \widehat{\rho}\|_{0,j}^2 + \tfrac{1}{2}\|\eta\|_{0,j}^2, \tag{4.23}$$

$$\begin{aligned}
|(f(\bar{U} + \bar{E}) - f(u), \eta)_j| &\le \tfrac{1}{2}\|f(\bar{U} + \bar{E}) - f(u)\|_{0,j}^2 + \tfrac{1}{2}\|\eta\|_{0,j}^2 \\
&\le C_1 \|\eta\|_{0,j}^2 + 2L^2 \|\theta\|_{0,j}^2 + L^2 \|\widehat{\rho}\|_{0,j}^2.
\end{aligned} \tag{4.24}$$

Substituting now the bounds (4.19) and (4.21) to (4.24) into (4.18), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\eta\|_{0,j}^2 + (\mu - \tfrac{1}{2}K)\|\partial_x \eta\|_{0,j}^2 \\
&\qquad \le (C_2 + C_3 K^{-1})\|\eta\|_{0,j}^2 + (C_4 + C_5 K^{-1})\|\theta\|_{1,j}^2 \\
&\qquad\quad + (C_6 + C_7 K^{-1})\|\widehat{\rho}\|_{0,j}^2 + \tfrac{1}{2}\|\partial_t \theta\|_{0,j}^2 + \tfrac{1}{2}\|\partial_t \widehat{\rho}\|_{0,j}^2
\end{aligned} \tag{4.25}$$

for $j = 1, \dots, N$. Summing up the inequalities (4.25) for $j = 1, \dots, N$, fixing K so that

$$K < 2\mu, \tag{4.26}$$

using (3.13) and (3.39), and applying Lemma 3.1 with $p+1$ instead of p to the function $\widehat{\rho}$ given by (4.14), we finally arrive at

$$\frac{1}{2}\frac{d}{dt}\|\eta\|_0^2 + S\|\partial_x \eta\|_0^2 \le C_8(t)\|\eta\|_0^2 + C_9(t)h^{2p+2} \tag{4.27}$$

with some $S > 0$. Omitting the term with $\|\partial_x \eta\|_0^2$ on the left-hand part and integrating on both parts of (4.27), we find

$$\|\eta(t)\|_0^2 \le \|\eta(0)\|_0^2 + C_{10}(t)h^{2p+2} + 2\int_0^t C_8(s)\|\eta(s)\|_0^2\, ds$$

if the corresponding functions are Lebesgue integrable on the interval $[0, T]$. Employing now (4.15), assuming that the corresponding functions are again Lebesgue integrable on $[0, T]$, and applying Lemma 3.4, we have the bound

$$\|\eta(t)\|_0^2 \le C_{11}(t)h^{2p+2} + C_{12}(t)h^{2p+2} \le Ch^{2p+2}. \tag{4.28}$$

Let us turn back to the inequality (4.27). Assuming that $\|\eta(t)\|_0$ is nondecreasing, we can write

$$\frac{d}{dt}\|\eta\|_0^2 \ge 0. \tag{4.29}$$

Using (4.26), (4.28), and (4.29), we finally obtain from (4.27) that

$$\|\partial_x \eta\|_0^2 \le Ch^{2p+2}. \tag{4.30}$$

The statement of the lemma follows from (4.28) and (4.30). □

Remark 4.1 As follows from the proof, such a smooth dependence of e^h and $\bar{E}_{\mathrm{PN}}$ on t is needed in Lemma 4.1 that guarantees the integration of (4.27) and the application of Lemma 3.4, including the upper bound for the expression in (4.28).

Remark 4.2 The assumption of Lemma 4.1 that $\|\eta_{\mathrm{PN}}\|_0$ is a nondecreasing function of t holds, e.g., for linear problems with the term f increasing in time. Considering the same problem as in Remark 3.3, we can easily calculate the quantity η_{PN} (equal to η_{PL} for linear problems) for $p = 1$ explicitly.

The same bound as for η_{PN} can be proved for η_{PL}, too.

Lemma 4.2 *Let $\bar{E}_{\mathrm{PL}} \in \widehat{S}_0^{N,p+1}$ be the error estimate given by (4.6) and (4.8), and $e^h \in \widehat{S}_0^{N,p+1}$ a function such that $u^h + e^h$ is an elliptic projection of u into $S_0^{N,p+1}$ according to (4.10). Let $\|\theta(.,t)\|_0$ and $\|\eta_{\mathrm{PL}}(.,t)\|_0$ be nondecreasing functions of the variable t and let (4.15) hold for the function $\eta_{\mathrm{PL}}(x,0)$ given by (4.12). Let the assumptions of Lemma 3.5 be fulfilled and let e^h and $\bar{E}_{\mathrm{PL}}$ depend on t in a sufficiently smooth way. Then there exists a constant $C > 0$ such that*

$$\|\eta_{\mathrm{PL}}\|_1 \le Ch^{p+1}.$$

Proof. Subtracting (4.16) from (4.8), we have

$$\begin{aligned}(\partial_t \bar{E}, \widehat{V})_j + (a(\bar{U})\partial_x \bar{E}, \partial_x \widehat{V})_j + (\partial_t \bar{U}, \widehat{V})_j + (a(\bar{U})\partial_x \bar{U}, \partial_x \widehat{V})_j \\ - (\partial_t u, \widehat{V})_j - (a(u)\partial_x(u^h + e^h), \partial_x \widehat{V})_j \\ = -(f(\bar{U}), \widehat{V})_j + (f(u), \widehat{V})_j.\end{aligned}$$

Proceeding in the same way as in the proof of Lemma 4.1, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\eta\|_{0,j}^2 + (a(\bar U)\partial_x\eta, \partial_x\eta)_j\\ &\quad = ([a(\bar U) - a(u)]\partial_x(u^h + e^h), \partial_x\eta)_j - (a(\bar U)\partial_x\theta, \partial_x\eta)_j\\ &\qquad -(\partial_t\theta, \eta)_j - (\partial_t\widehat{\rho}, \eta)_j + (f(\bar U) - f(u), \eta)_j.\end{aligned} \tag{4.31}$$

We now bound the individual terms in (4.31). We have

$$0 < \mu\|\partial_x\eta\|_{0,j}^2 \le (a(\bar U)\partial_x\eta, \partial_x\eta)_j \tag{4.32}$$

according to (2.2). A positive constant K will be fixed later. From the Schwarz inequality, (2.2), (2.4), and (4.20), we obtain

$$\begin{aligned}&|(-[a(\bar U) - a(u)]\partial_x(u^h + e^h) + a(\bar U)\partial_x\theta, \partial_x\eta)_j|\\ &\quad \le 2C^2K^{-1}L^2\|\theta\|_{0,j}^2 + 2C^2K^{-1}L^2\|\rho\|_{0,j}^2\\ &\qquad + K^{-1}M^2\|\partial_x\theta\|_{0,j}^2 + \tfrac{1}{2}K\|\partial_x\eta\|_{0,j}^2,\end{aligned} \tag{4.33}$$

$$|(f(\bar U) - f(u), \eta)_j| \le L^2\|\theta\|_{0,j}^2 + L^2\|\rho\|_{0,j}^2 + \tfrac{1}{2}\|\eta\|_{0,j}^2. \tag{4.34}$$

Substituting now the bounds (4.22), (4.23), and (4.32) to (4.34) into (4.31), we obtain an analog of (4.25), where $\|\widehat{\rho}\|_{0,j}^2$ is replaced by $\|\rho\|_{0,j}^2$. The same procedure as in the proof of Lemma 4.1 and, moreover, the application of Lemma 3.1 lead to (4.27) with K fixed by (4.26). The rest of the proof is the same as in Lemma 4.1. □

Remark 4.3 Comments analogous to Remarks 4.1 and 4.2 are concerned with the assumptions of Lemma 4.2, too.

The last case we are going to analyze is η_{EL}.

Lemma 4.3 *Let $\bar E_{\mathrm{EL}} \in \widehat{S}_0^{N,p+1}$ be the error estimate given by* (4.6) *and* (4.9), *and $e^h \in \widehat{S}_0^{N,p+1}$ a function such that $u^h + e^h$ is an elliptic projection of u into $S_0^{N,p+1}$ according to* (4.10). *Let $\|\theta(.,t)\|_0$ and $\|\eta_{\mathrm{EL}}(.,t)\|_0$ be nondecreasing functions of the variable t and let* (4.15) *hold for the function $\eta_{\mathrm{EL}}(x,0)$ given by* (4.13). *Let the assumptions of Lemma* 3.5 *be fulfilled. Than there exists a constant $C > 0$ such that*

$$\|\eta_{\mathrm{EL}}\|_1 \le Ch^{p+1}. \tag{4.35}$$

Proof. Subtracting (4.16) from (4.9), we have

$$\begin{aligned}&(a(\bar U)\partial_x\bar E, \partial_x\widehat{V})_j + (\partial_t\bar U, \widehat{V})_j + (a(\bar U)\partial_x\bar U, \partial_x\widehat{V})_j\\ &\quad -(\partial_t u, \widehat{V})_j - (a(u)\partial_x(u^h + e^h), \partial_x\widehat{V})_j\\ &\quad = -(f(\bar U), \widehat{V})_j + (f(u), \widehat{V})_j.\end{aligned}$$

Proceeding in the same way is in the proof of Lemma 4.2, we arrive at

$$(4.36)\quad (a(\bar U)\partial_x\eta, \partial_x\eta)_j \\ = ([a(\bar U) - a(u)]\partial_x(u^h + e^h), \partial_x\eta)_j - (a(\bar U)\partial_x\theta, \partial_x\eta)_j \\ -(\partial_t\theta, \eta)_j - (\partial_t\rho, \eta)_j + (f(\bar U) - f(u), \eta)_j$$

for $j = 1, \ldots, N$. Summing up the inequalities (4.36) for $j = 1, \ldots, N$, we can bound the individual terms in the sum. Using analogs of (4.32) and (4.33), and the consequences of the Schwarz inequality and (2.4), we conclude that

$$\mu\|\partial_x\eta\|_0^2 \le 2C^2K^{-1}L^2\|\theta\|_0^2 + 2C^2K^{-1}L^2\|\rho\|_0^2 + K^{-1}M^2\|\partial_x\theta\|_0^2 \\ + \tfrac{1}{2}K\|\partial_x\eta\|_0^2 + (\|\partial_t\theta\|_0 + \|\partial_t\rho\|_0 + L\|\rho\|_0 + L\|\theta\|_0)\|\eta\|_0.$$

Fixing K so that $K < 2\mu$ and using (3.5), (3.6), (3.13), and (3.39), we finally obtain

$$(4.37)\qquad C\|\partial_x\eta\|_0^2 \le C_2h^{p+1}\|\eta\|_0 + C_3h^{2p+2}.$$

According to Lemma 2.1, we have from (4.37)

$$(4.38)\qquad C_1\|\eta\|_0^2 \le C_2h^{p+1}\|\eta\|_0 + C_3h^{2p+2}$$

with some positive constants C_1, C_2, and C_3, i.e.,

$$(4.39)\qquad C_1\|\eta\|_0^2 - C_2h^{p+1}\|\eta\|_0 - C_3h^{2p+2} \le 0.$$

The left-hand part can be expressed as a product

$$(4.40)\qquad (\|\eta\|_0 - \xi_1)(\|\eta\|_0 - \xi_2) \le 0,$$

where

$$\xi_{1,2} = h^{p+1}\frac{C_2 \pm \sqrt{C_2^2 + 4C_1C_3}}{2C_1} \gtrless 0.$$

Since $\|\eta\|_0 \ge 0$ cannot be less than $\xi_2 < 0$ the only way to satisfy (4.40) (and (4.38) and (4.39) as well) is

$$(4.41)\qquad \|\eta\|_0 \le \xi_1 = C_4h^{p+1}.$$

Substituting (4.41) into (4.37), we finally obtain

$$\|\partial_x\eta\|_0 \le C_5h^{p+1},$$

which together with (4.41) implies (4.35). The lemma has been proved. □

We introduce the four quantities, $\Theta_{\mathrm{PN}}, \Theta_{\mathrm{EN}}, \Theta_{\mathrm{PL}}$, and Θ_{EL}, called the *effectivity index* of the respective error estimate and given by the formula

$$\Theta = \frac{\|\bar{E}\|_1}{\|e\|_1}.$$

The principal result that generalizes the statements of [9] Theorem 3.1, Corollary 3.1, and Theorem 3.2 is the following theorem.

Theorem 4.1 *Let* $u \in H^{p+1} \cap H_0^1$ *and* $\bar{U} \in S_0^{N,p}$ *be solutions of* (2.7), (2.8) *and* (3.9), (3.10). *Let* $\bar{E} \in \widehat{S}_0^{N,p+1}$ *be the solution of* (4.5), (4.6) (*for* $\bar{E}_{\mathrm{PN}}$), (4.6), (4.8) (*for* $\bar{E}_{\mathrm{PL}}$), *or* (4.6), (4.9) (*for* $\bar{E}_{\mathrm{EL}}$). *Let* $\|\theta(.,t)\|_0$ *and* $\|\eta(.,t)\|_0$ *be nondecreasing functions of the variable* t *and let* (4.15) *hold. Let the assumptions of Lemma* 3.5 *be fulfilled, and let* e^h, $\bar{E}_{\mathrm{PN}}$, *and* $\bar{E}_{\mathrm{PL}}$ *depend on* t *in a sufficiently smooth way* (*cf. Remarks* 4.1 *and* 4.3).

Further let

$$\|e\|_1 \geq Ch^p. \tag{4.42}$$

Then

$$\lim_{h\to 0} \Theta = 1 \tag{4.43}$$

for almost every $t \in [0,T]$, *where* Θ *is* $\Theta_{\mathrm{PN}}, \Theta_{\mathrm{PL}}$, *or* Θ_{EL}.

Proof. Rewrite e given by (4.1) as

$$e = u - (u^h + e^h) + (u^h - \bar{U}) + (e^h - \bar{E}) + \bar{E}.$$

Then

$$\bar{E} = e - \widehat{\rho} - \theta - \eta \tag{4.44}$$

and

$$\|\bar{E}\|_1 \leq \|e\|_1 + \|\widehat{\rho}\|_1 + \|\theta\|_1 + \|\eta\|_1 \leq \|e\|_1 + C_2 h^{p+1} \tag{4.45}$$

with respect to (4.14) and Lemma 3.1 with $p+1$ instead of p, Lemma 3.3, and Lemmas 4.1, 4.2, and 4.3.

Further, the equation (4.44) implies that $e = \bar{E} + \widehat{\rho} + \theta + \eta$, i.e.,

$$\|e\|_1 \leq \|\bar{E}\|_1 + \|\widehat{\rho}\|_1 + \|\theta\|_1 + \|\eta\|_1 \leq \|\bar{E}\|_1 + C_1 h^{p+1}$$

according to the same statements as above and

$$\|e\|_1 - C_1 h^{p+1} \leq \|\bar{E}\|_1. \tag{4.46}$$

Finally, according to (4.45) and (4.46), we arrive at

$$\|e\|_1 - C_1\, h^{p+1} \leq \|\bar{E}\|_1 \leq \|e\|_1 + C_2 h^{p+1} \tag{4.47}$$

and, dividing (4.47) by $\|e\|_1$ and taking (4.42) into account, at

$$1 - C_1 h \leq \Theta \leq 1 + C_2 h.$$

Then (4.43) holds for $h \to 0$. $\square$

5. Conclusions

Linear and nonlinear parabolic and linear elliptic a posteriori error estimates for nonlinear parabolic equations are shown to converge to the true error for semidiscrete methods. This fact has been indicated by results of many numerical experiments carried out in [1, 3, 9, 11, 13, 14, 15], and many other papers.

Every adaptive numerical procedure for solving parabolic equations is based on some fully discrete scheme. If the method of lines is chosen for the space discretization of the equation (2.1) a posteriori error estimates for the fully discrete case are covered by [9] only when the singly implicit Runge-Kutta (SIRK) method or the backward difference formula are used for the time discretization.

Theorem 4.1 is concerned with the semidiscrete case and can be used as a basis for an adaptive numerical procedure that uses any other time discretization, too, on the assumption that the error resulting from the numerical solution in time is, as usually required, sufficiently small as compared with the total error tolerance set. The relation between the error tolerances for the time and space has been intensively studied recently.

An example of numerical values of the effectivity index Θ for the nonlinear equation (2.1) can be found, e.g., in [9] Example 5.3. The equation

$$\partial_t u = \partial_x(3u^2\partial_x u)$$

is solved there for $0 < x < 1$ and $1 < t$ with boundary and initial conditions chosen so that the exact solution is

$$u(x,t) = t^{-1/4}\sqrt{1 - \tfrac{1}{12}x^2 t^{-1/2}}.$$

The method of lines with time discretization by the SIRK method is used. The results obtained for several values of p, and different space and time steps are given in [9] Table 5.3. The effectivity indices depend on the relation between p and the time step and range from 0.94 to 1.00.

If the diffusion term in the equation (2.1) is dominated by a reaction term the quality of the effectivity index may be expected to deteriorate.

Babuška et al. [4] showed that the error of the finite element solution of an elliptic partial differential equation has two parts, the local error and the pollution error. The investigation of the effect of the pollution error on the quality of local error indicators in the case of parabolic equations is a subject of further research.

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