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# The approximation theory for the $p$-version finite element method and application to non-linear elliptic PDEs

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**Summary.** Approximation theoretic results are obtained for approximation using continuous piecewise polynomials of degree $p$ on meshes of triangular and quadrilateral elements. Estimates for the rate of convergence in Sobolev spaces $W^{m,q}(\Omega)$, $q \in [1, \infty]$ are given. The results are applied to estimate the rate of convergence when the $p$-version finite element method is used to approximate the $\alpha$-Laplacian.

It is shown that the rate of convergence of the $p$-version is always at least that of the $h$-version (measured in terms of number of degrees of freedom used). If the solution is very smooth then the $p$-version attains an exponential rate of convergence. If the solution has certain types of singularity, the rate of convergence of the $p$-version is twice that of the $h$-version.

The analysis generalises the work of Babuska and others to the case $q \neq 2$. In addition, the approximation theoretic results find immediate application for some types of spectral and spectral element methods.

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## 1. Introduction

The $h$-version of the finite element method is the standard version in which the degree of the elements is fixed and convergence is achieved by refining

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the mesh size $h$. The $p$-version fixes the *mesh* and achieves convergence by increasing the polynomial degree $p$ of the elements. The $p$-version retains the geometric flexibility of the finite element method while seeking the high rates of convergence of spectral methods.

Traditionally, it was thought that there is little point in using high order finite elements to approximate the solutions of partial differential equations since the rate of convergence of the $h$-version is limited by the smoothness of the solution. The classical error estimates for the $h$-version are of the form

$$\|e\|_{W^{1,2}(\Omega)} \leq C(p)h^{\mu} \|u\|_{W^{m,2}(\Omega)}$$

where

$$\mu = \min(m-1, p)$$

with $p$ being the polynomial degree of the elements and $m$ measuring the regularity of the solution of the partial differential equation. The estimate seems to indicate that there is no point in choosing $p$ larger than $m-1$. However, this argument ignores the dependence of the constant $C(p)$ on the polynomial degree.

The traditional viewpoint was challenged in the work of Babuska and others. The first analysis of the $p$-version was given by Babuska [4] and subsequently refined by Babuska and Suri [2]. It was shown that the corresponding estimate for the $p$-version is

$$\|e\|_{W^{1,2}(\Omega)} \leq C(h)p^{-(m-1)} \|u\|_{W^{m,2}(\Omega)} .$$

Consequently, when the true solution is smooth ($m$ large), the rate of convergence is similar to the rates for spectral methods. Of course, in practical problems the solution will generally have singularities that limit the regularity. However, the singular terms are known to have a very specific form and this fact was exploited by Babuska and others [2,4] who showed that even in the presence of singularities, the $p$-version will converge at twice the rate of the $h$-version. The chief purpose of the current work is to show that such conclusions are also valid more generally in the case of $L_q$-type norms with $q \neq 2$. Such results find immediate application to certain types of non-linear elliptic boundary value problems. In addition, the results are useful for the analysis of some types of spectral element method.

The major part of the analysis is devoted to obtaining approximation results for piecewise polynomial approximation in Sobolev spaces $W^{m,q}(\Omega)$ with $q \in [1, \infty]$. While the case $q = 2$ has received a great deal of attention, little is known for the general case. The reason for the lack of results in the general case seems to be largely due to the extensive use of orthogonal polynomials and their properties in the analysis. Preliminary results were obtained by Quarteroni [10] for polynomial approximation in $L_q$-type spaces on a single element in one dimension. The present work deals with approximation by *continuous piecewise polynomials in two dimensions*. The extra number of dimensions along with the continuity of the piecewise polynomials across the element boundaries requires special attention and poses a number of difficulties not present in one dimension or if there is only one element.

It is shown that the conclusions for the case $q = 2$ also hold in the general case. The results obtained in the present work are immediately applicable to spectral methods and to spectral element methods. One point of particular interest arises in our analysis of the approximation of singular functions. The original analysis in [4] resulted in an estimate of the form

$$\|e\|_{W^{1,2}(\Omega)} \leq C(\varepsilon)p^{-(m-1-\varepsilon)} \|u\|_{W^{m,2}(\Omega)}$$

where $\varepsilon > 0$ is arbitrary. The presence of the $\varepsilon$ in the exponent of $p$ is of little concern. However, the analysis suggested that the constant $C(\varepsilon)$ could blow up as $\varepsilon \to 0$. The need for the $\varepsilon$ was removed in the later analysis in [2] where a rather different method of proof was followed, involving the use of orthogonal families of polynomials. The current analysis follows the original method of proof in [4], and is applicable to the more general case of $q \neq 2$.

In conclusion, the analysis shows that the traditional view of avoiding the use of high order polynomial finite element methods is incorrect. The rate of convergence of the $p$-version is always at least that of the $h$-version (measured in terms of number of degrees of freedom used). If the solution is very smooth then the $p$-version attains an exponential rate of convergence. If the solution has certain types of singularity, the rate of convergence of the $p$-version is twice that of the $h$-version.

## 2. Preliminaries

Let $\mathbb{R}^2$ be the usual Euclidean space with $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$. Throughout, it is assumed that $\Omega$ is a bounded, polygonal domain in $\mathbb{R}^2$. For $q \in [1, \infty]$ the space $L^q(\Omega)$ is defined to be the usual space of classes of functions for which the norm

$$\|f\|_{L^q(\Omega)} = \begin{cases} \left(\int_\Omega |f|^q d\boldsymbol{x}\right)^{1/q}, & q < \infty \\ \operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega} |f(\boldsymbol{x})|, & q = \infty \end{cases}$$

is finite. For integer values of $s$, the Sobolev spaces $W^{s,q}(\Omega)$ are equipped with the norms

$$\|f\|_{W^{s,q}(\Omega)} = \begin{cases} \left\{\sum_{|\alpha|\leq s} \|D^\alpha f\|^q_{L^q(\Omega)}\right\}^{1/q}, & q < \infty \\ \max_{|\alpha|\leq s} \|D^\alpha f\|_{L^\infty(\Omega)}, & q = \infty \end{cases}.$$

For non-integer values of $s$, the Sobolev spaces $W^{s,q}(\Omega)$ are defined using the $K$-method of interpolation [5]. Thus, writing $s = m + \sigma$ where $m$ is an integer and $\sigma \in (0,1)$, the space $W^{s,q}(\Omega)$ is obtained by interpolating between the spaces $W^{m,q}(\Omega)$ and $W^{m+1,q}(\Omega)$. This process is indicated using the notation

$$W^{s,q}(\Omega) = [W^{m,q}(\Omega), W^{m+1,q}(\Omega)]_{\sigma,q}.$$

The subspaces $W_0^{s,q}(\Omega)$ are defined in the usual manner [1]. Equally well, Sobolev spaces may be defined on an interval $I = (a,b)$ and on curves $\gamma$. Let $S(\rho)$, $\rho > 0$ be the square

$$S(\rho) = \{(x_1, x_2) : |x_1| < \rho, |x_2| < \rho\}.$$

The space $W_{per}^{k,q}(S(\rho)) \subset W^{k,q}(S(\rho))$ consists of the periodic functions with period $2\rho$.

A partition $\mathcal{P}$ of the domain $\Omega$ consists of a finite number of open subdomains (or *elements*) $K \in \mathcal{P}$ such that:

- each element $K$ is either a triangle or a convex quadrilateral
- $\overline{\Omega} = \bigcup_{K \in \mathcal{P}} \overline{K}$
- for any distinct pair of elements $K$ and $J$, the intersection $\overline{K} \cap \overline{J}$ is either empty, a single common edge or a single common vertex.

Associated with each type of element is a reference domain given in the case of quadrilateral elements by

$$S(1) = \{(x,y) : -1 \leq x \leq 1; \quad -1 \leq y \leq 1\}$$

or, in the case of triangular elements

$$T(1) = \{(x,y) : -1 \leq x \leq 1; \quad -1 \leq y \leq x\}.$$

Polynomial spaces of degree $p \in \mathbb{N}$ are defined on the quadrilateral and triangular reference elements respectively by

$$\widehat{Q}(p) = \text{span}\left\{x^j\, y^k : 0 \leq j, k \leq p\right\}$$

and

$$\widehat{P}(p) = \text{span}\left\{x^j\, y^k : 0 \leq j + k \leq p\right\}.$$

For simplicity, it is assumed that there exists an invertible mapping $F_K : \widehat{K} \mapsto K$ that is affine for triangular elements and bilinear for quadrilateral elements. A polynomial space $P_K$ is taken to be either $\widehat{Q}(p)$ or $\widehat{P}(p)$ as appropriate for each type of element. The space $X_p$ is constructed using the partition $\mathcal{P}$

$$X_p = \left\{ v \in C(\Omega) : v|_K = \widehat{v} \circ F_K^{-1} \text{ for some } \widehat{v} \in P_K \text{ for all } K \in \mathcal{P} \right\}$$

and, with a slight abuse of the nomenclature, will be referred to as being a space of *piecewise polynomials*.

Suppose that the function $u$ belongs to the space $W^{m,q}(\Omega)$. One of the goals will be to obtain estimates for the rate of convergence that may be obtained using sequences $\{u_p\}$ of approximations $u_p \in X_p$ to $u$ in terms of the polynomial degree $p$.

Consider the $\alpha$-Laplacian

(1) $$-\nabla \cdot \left\{ |\nabla u|^{\alpha-2} \nabla u \right\} = f \text{ in } \Omega$$

where $\alpha \in (1, \infty)$ and $f$ is smooth given data. Even if the data $f$ is smooth the solution $u$ may be singular. For example, suppose the domain $\Omega$ has a single corner at the point $\boldsymbol{A}$ with internal angle $\omega \in (0, 2\pi]$. Letting $r$ and $\theta \in (0, \omega)$ be polar coordinates with origin at $\boldsymbol{A}$, it has been shown [7,13] that in the neighbourhood of the corner a positive solution $u$ of the $\alpha$-Laplacian has the structure

(2) $$u(\boldsymbol{x}) = c r^{\lambda} \Theta(\theta) + o(r^{\lambda})$$

where $c \in \mathbb{R}$, $\Theta$ is a smooth function with $\Theta(0) = \Theta(\omega) = 0$,

$$\lambda = \begin{cases} s + \sqrt{s^2 + 1/\beta}, & \text{if } 0 < \omega \leq \pi \\ s - \sqrt{s^2 + 1/\beta}, & \text{if } \pi \leq \omega < 2\pi \\ (\alpha - 1)/\alpha, & \text{if } \omega = 2\pi \end{cases}$$

with

$$\beta = (\omega/\pi - 1)^2 - 1$$

and

$$s = \frac{(\beta - 1)\alpha - 2\beta}{2\beta(\alpha - 1)}.$$

The lack of smoothness of the true solution may lead to a degradation in the rate of convergence of both $h$- and $p$-version finite element approximations of problem (1). Indeed, the degradation in the rate of convergence is often cited as a reason for avoiding the use of high order finite elements. One of the purposes of the current work is to show that such a conclusion is incorrect: a better rate of convergence is achieved by increasing the polynomial degree uniformly than is obtained by uniformly refining the partition. Before the claim can be proved, it will be necessary to study the approximation of singular functions of the form (2) by piecewise polynomials.

The Dirichlet boundary $\Gamma_{\mathrm{D}}$ is a closed subset of the boundary $\partial\Omega$. Frequently, one wishes to impose Dirichlet boundary conditions on the approximation. For instance, the trace $g$ of the function $u$ might be given on the Dirichlet boundary. Thus, one requires estimates for the rate of convergence of a sequence of approximations $u_p \in X_p$ satisfying $u_p = g_p$ on $\Gamma_{\mathrm{D}}$ where $g_p$ are appropriately chosen continuous piecewise polynomial approximations to $g$. To facilitate the construction of suitable approximations $g_p$, it will be assumed that the partition $\mathcal{P}$ is constructed so that element vertices are located at the endpoints of the Dirichlet boundary.

Throughout, $C$ will be used to denote positive constants that are independent of other quantities appearing in the same relation, and whose values need not be the same in any two places. The notation $a \approx b$ means that there exist positive constants $C_1$, $C_2$ such that $C_1 a \leq b \leq C_2 a$.

## 3. Piecewise polynomial approximation of smooth functions

This section deals with the approximation of smooth functions using the spaces $X_p$ of continuous piecewise polynomials on a fixed partition $\mathcal{P}$ of the domain. Suppose that a function $u$ belongs to the space $W^{m,q}(\Omega)$. The goal will be to obtain estimates for the rate of convergence that may be obtained using sequences $\{u_p\}$ of approximations $u_p \in X_p$ to $u$ in terms of the polynomial degree $p$.

The derivation consists of two main steps. To begin with, approximation by sequences of polynomials on a single reference element is considered. Estimates are then obtained for approximations from the spaces $X_p$ by piecing together functions from each element (obtained by mapping the approximations on the reference element) and making appropriate adjustments to satisfy continuity requirements.

### *3.1. Polynomial approximation on the reference element*

The Appendix consists of results concerning approximation by partial sums of Fourier series of functions $f$ belonging to the periodic Sobolev spaces $W^{l,q}_{\mathrm{per}}(S(\pi))$. These results will be used here to deduce approximation properties for sequences $\{\phi_p(u)\}$ of algebraic polynomial approximations to a function $u \in W^{l,q}(S(1))$ defined on the square $S(1)$. However, in general the approximation obtained by changing the variable in the partial Fourier series will generally fail to be an algebraic polynomial unless the function $f$ possesses certain symmetries. It is therefore necessary for the function $u$ to undergo some preliminary surgery [4].

Let $\rho > 1$ be fixed. According to [12, Theorem 5] there exists an extension $U$ of the function $u$ onto the square $S(2\rho)$ such that $\mathrm{supp}(U) \subset S(3\rho/2)$

and $U \in W^{m,q}(S(2\rho))$ with

$$\|U\|_{W^{m,q}(S(2\rho))} \leq C \|u\|_{W^{m,q}(S(1))} .$$

Let $\Phi : S(\pi/2) \mapsto S(2\rho)$ be the bijective mapping

$$\widehat{\boldsymbol{x}} \mapsto \boldsymbol{x} = \Phi(\widehat{\boldsymbol{x}}) = 2\rho(\sin \widehat{x}_1, \sin \widehat{x}_2). \tag{3}$$

Furthermore, define a function $f \in W^{m,q}(S(\frac{\pi}{2}))$ by

$$f(\widehat{\boldsymbol{x}}) = (U \circ \Phi)(\widehat{\boldsymbol{x}})$$

and observe that the support of $f$ is contained in the square $S(\arcsin 3/4)$. Hence, $f$ may be extended to $S(\pi)$ as a smooth function such that it is symmetric across the lines $\widehat{x}_i = \pm\frac{\pi}{2}$. The estimate (3) shows $f \in W^{m,q}_{\mathrm{per}}(S(\pi))$ and

$$\|f\|_{W^{m,q}(S(\pi))} \leq C \|u\|_{W^{m,q}(S(1))} . \tag{4}$$

Let $s_p(f)$ denote the $p$-th partial sum of the Fourier series expansion for the function $f$ on $S(\pi)$. Each partial sum $s_p(f)$ inherits the symmetries of the function $f$. Therefore

$$s_p(f) = \phi_p(u) \circ \Phi \tag{5}$$

where $\phi_p(u)$ is an *algebraic* polynomial on the square $S(1)$ of degree at most $p$ in each variable.

**Lemma 1.** *Let $u \in W^{l,q}(S(1))$ where $q \in [1, \infty]$. Then there exists a sequence of algebraic polynomials $\phi_p(u) \in \widehat{Q}(p)$, $p \in \mathbb{N}$, which are independent of $q$, such that*
*1. for any $0 \leq k \leq l$*

$$\|u - \phi_p(u)\|_{W^{k,q}(S(1))} \leq Cp^{-(l-k)}(1 + \ln p)^{2|1-2/q|} \|u\|_{W^{l,q}(S(1))}$$

*2. for $l > k + 1/q$*

$$\|u - \phi_p(u)\|_{W^{k,q}(\gamma)} \leq Cp^{-(l-k-1/q)} \|u\|_{W^{l,q}(S(1))} \begin{cases} (1 + \ln p)^{(2/q-1)}, & q \in [1, 2] \\ (1 + \ln p)^{2(1-2/q)}, & q \in [2, \infty] \end{cases}$$

*where $\gamma$ is any edge or either principal diagonal of $S(1)$*
*3. for $l > k + 2/q$*

$$\|u - \phi_p(u)\|_{W^{k,\infty}(S(1))} \leq Cp^{-(l-k-2/q)} \|u\|_{W^{l,q}(S(1))} \begin{cases} 1, & q \in [1, 2] \\ (1 + \ln p)^{2(1-2/q)}, & q \in [2, \infty] \end{cases} .$$

*Proof.* Suppose first that $q \in [1,2]$.
1. By Lemma 15(1) for any $0 \le k \le l$

$$\begin{aligned}\|u-\phi_p(u)\|_{W^{k,q}(S(1))} &\le C\,\|f-s_p\|_{W^{k,q}(S(\pi))}\\ &\le Cp^{-(l-k)}(1+\ln p)^{2(2/q-1)}\,\|f\|_{W^{l,q}(S(\pi))}\\ &\le Cp^{-(l-k)}(1+\ln p)^{2(2/q-1)}\,\|U\|_{W^{l,q}(S(1))}\\ &\le Cp^{-(l-k)}(1+\ln p)^{2(2/q-1)}\,\|u\|_{W^{l,q}(S(1))}\end{aligned}$$

where (4) has been used.
2. Let $\widehat{\gamma} = \Phi^{-1}(\gamma)$. By Lemma 15(2) for any $0 \le k+1/q < l$

$$\begin{aligned}\|u-\phi_p(u)\|_{W^{k,q}(\gamma)} &\le C\,\|f-s_p\|_{W^{k,q}(\widehat{\gamma})}\\ &\le Cp^{-(l-k-1/q)}(1+\ln p)^{(2/q-1)}\,\|f\|_{W^{l,q}(S(\pi))}\\ &\le Cp^{-(l-k-1/q)}(1+\ln p)^{2}\,\|U\|_{W^{l,q}(S(1))}\\ &\le Cp^{-(l-k-1/q)}(1+\ln p)^{2}\,\|u\|_{W^{l,q}(S(1))}\,. \end{aligned} \tag{6}$$

3. By Lemma 15(3) for $0 \le k+2/q < l$

$$\begin{aligned}\|u-\phi_p(u)\|_{W^{k,\infty}(S(1))} &\le C\,\|f-s_p\|_{W^{k,\infty}(S(\pi))}\\ &\le Cp^{-(l-k-2/q)}\,\|f\|_{W^{l,q}(S(\pi))}\\ &\le Cp^{-(l-k-2/q)}\,\|u\|_{W^{l,q}(S(1))}\,. \end{aligned} \tag{7}$$

The proofs when $q \in (2,\infty]$ are essentially identical. □

Corresponding results hold for approximation on the triangular reference element:

**Lemma 2.** *Let $u \in W^{l,q}(T(1))$ where $q \in [1,\infty]$. Then there exists a sequence of algebraic polynomials $\phi_p(u) \in \widehat{P}(p)$, $p \in \mathbb{N}$, which are independent of $q$, such that*
*1. for any $0 \le k \le l$*

$$\|u-\phi_p(u)\|_{W^{k,q}(T(1))} \le Cp^{-(l-k)}(1+\ln p)^{2|1-2/q|}\,\|u\|_{W^{l,q}(T(1))}$$

*2. for $l > k+1/q$*

$$\|u-\phi_p(u)\|_{W^{k,q}(\gamma)} \le Cp^{-(l-k-1/q)}\,\|u\|_{W^{l,q}(T(1))}\begin{cases}(1+\ln p)^{(2/q-1)}, & q\in[1,2]\\ (1+\ln p)^{2(1-2/q)}, & q\in[2,\infty]\end{cases}$$

*where $\gamma$ is any edge of $T(1)$*

*3. for $l > k + 2/q$*

$$\|u - \phi_p(u)\|_{W^{k,\infty}(T(1))} \le Cp^{-(l-k-2/q)} \|u\|_{W^{l,q}(T(1))} \begin{cases} 1, & q \in [1,2] \\ (1+\ln p)^{2(1-2/q)}, & q \in [2,\infty] \end{cases} .$$

*Proof.* 1. Let $u \in W^{l,q}(T(1))$ be given. By [12, Theorem 5] there exists an extension $U$ of the function $u$ to the square $S(1)$ satisfying

$$\|U\|_{W^{l,q}(S(1))} \le C \|u\|_{W^{l,q}(T(1))} . \tag{8}$$

By Lemma 1 there exists a sequence $U_p \in \widehat{Q}(p)$ such that for any $0 \le k \le l$

$$\|U - U_p\|_{W^{k,q}(S(1))} \le Cp^{-(l-k)}(1+\ln p)^{2|1-2/q|} \|U\|_{W^{l,q}(S(1))} . \tag{9}$$

Now $\widehat{Q}(p) \subset \widehat{P}(2p)$ and therefore we may define the sequence by $\phi_{2p}(u) = U_p$ and $\phi_{2p+1}(u) = \phi_{2p}(u)$. Observing

$$\begin{aligned} \|u - \phi_{2p+1}(u)\|_{W^{k,q}(T(1))} &= \|u - \phi_{2p}(u)\|_{W^{k,q}(T(1))} \\ &= \|U - U_p\|_{W^{k,q}(T(1))} \le \|U - U_p\|_{W^{k,q}(S(1))} , \end{aligned} \tag{10}$$

the result then follows from (9) and (8). The remaining cases are similar. □

It is possible to generalize Lemmas 1 and 2 to cases when the norms on each side are based on different $L^q$ type spaces:

**Theorem 3.** *Let $u \in W^{m,r}(S(1))$ where $r \in [1,\infty]$. Then there exists a sequence of algebraic polynomials $\phi_p(u) \in \widehat{Q}(p)$, $p \in \mathbb{N}$ such that for $1 \le q \le r$ and $0 \le l \le m + 2/r - 2/q$*

$$\begin{aligned} &\|u - \phi_p(u)\|_{W^{l,r}(S(1))} \\ &\le Cp^{-(m-l+2/r-2/q)}(1+\ln p)^{2|1-1/r-1/q|} \|u\|_{W^{m,q}(S(1))} . \end{aligned} \tag{11}$$

*Moreover, analogous results hold for approximation on the triangle.*

*Proof.* By Lemma 1(1), for $0 \le l \le m$

$$\|u - \phi_p(u)\|_{W^{l,1}(S(1))} \le Cp^{-(m-l)}(1+\ln p)^2 \|u\|_{W^{m,1}(S(1))}$$

and by Lemma 1(3), for $0 \le l \le m-2$

$$\|u - \phi_p(u)\|_{W^{l,\infty}(S(1))} \le Cp^{-(m-l-2)} \|u\|_{W^{m,1}(S(1))} .$$

Applying interpolation gives for any $r \in [1,\infty]$ and $0 \le l \le m - 2 + 2/r$

$$\|u - \phi_p(u)\|_{W^{l,r}(S(1))} \le Cp^{-(m-l-2+2/r)}(1+\ln p)^{2/r} \|u\|_{W^{m,1}(S(1))} .$$

Moreover, by Lemma 1(1) if $r \in [1,2]$ and $0 \le l \le m$

$$\|u - \phi_p(u)\|_{W^{l,r}(S(1))} \le C p^{-(m-l)} (1 + \ln p)^{2(2/r-1)} \|u\|_{W^{m,r}(S(1))}$$

or if $r \in [2,\infty]$ and $0 \le l \le m$

$$\|u - \phi_p(u)\|_{W^{l,r}(S(1))} \le C p^{-(m-l)} (1 + \ln p)^{2(1-2/r)} \|u\|_{W^{m,r}(S(1))} .$$

Applying interpolation gives (11). □

If $1 \le r \le q$ then the following estimate is trivially obtained from Lemma 1:

$$\begin{aligned} &\|u - \phi_p(u)\|_{W^{l,r}(S(1))} \\ &\quad \le C p^{-(m-l)} (1 + \ln p)^{2|1-1/r-1/q|} \|u\|_{W^{m,q}(S(1))} . \end{aligned} \tag{12}$$

*3.2. Approximation using continuous piecewise polynomials*

The previous section dealt with approximation by algebraic polynomials on the reference element. These results will now be used to obtain approximation properties for the spaces $X_p$. The requirement that functions in the space $X_p$ be continuous means that one cannot trivially deduce such results directly from those on the reference element. The following deals with the basic process of constructing the continuous approximation when the function to be approximated belongs to the space $W^{m,q}(\Omega)$ with $m > 1 + 1/q$. Later, the result will be strengthened to cases when $m > 1$.

**Theorem 4.** *Let $u \in W^{m,q}(\Omega)$, $q \in [1,\infty]$, $m > 1 + 1/q$. Then there exists a sequence $u_p \in X_p$ of continuous piecewise polynomials, which are independent of $q$, such that on any element $J$ in the partition $\mathcal{P}$*

$$\|u - u_p\|_{W^{1,q}(J)} \le C p^{-(m-1)} (1 + \ln p)^{2|1-2/q|} \sum_{K \in \mathcal{P}: \overline{K} \cap \overline{J} \neq \emptyset} \|u\|_{W^{m,q}(K)} .$$

*Consequently the following global estimate holds*

$$\|u - u_p\|_{W^{1,q}(\Omega)} \le C p^{-(m-1)} (1 + \ln p)^{2|1-2/q|} \|u\|_{W^{m,q}(\Omega)} .$$

*Proof.* To begin with let $K$ be any quadrilateral element in the partition $\mathcal{P}$. The element $K$ is the image of the square reference element $S(1)$ under a bijective, bilinear mapping $F_K$. Define $\widehat{u}_K = u|_K \circ F_K$ and let $\widehat{w}_{K,p}$ be a sequence of approximations to $\widehat{u}_K$ as in Lemma 1. Let $w_{K,p} = \widehat{w}_K \circ F_K^{-1}$.

Transforming the estimates of Lemma 1 to the element $K$ leads to analogous estimates for the difference $e_{K,p} = u - w_{K,p}$ on $K$. In general, if elments $K$ and $J$ share a common edge $\gamma$ then the values of the approximations $w_{K,p}$ and $w_{J,p}$ will differ on the interface. Therefore, we shall adjust

$w_{K,p}$ and $w_{J,p}$ so that continuity is obtained whilst preserving the accuracy of the approximation. Consider the polynomial $\psi_p : [-1,1] \mapsto \mathbb{R}$ given by

$$\psi_p(s) = \left(\frac{1-s}{2}\right)^p$$

and note that for any $q \in [1,\infty]$

1. $\|\psi_p\|_{L^q(-1,1)} \le Cp^{-1/q}$
2. $|\psi_p|_{W^{1,q}(-1,1)} \le Cp^{1-1/q}$
3. $\psi_p(-1) = 1$ ; $\psi_p(1) = 0$.

Let $\widehat{e}_{K,p} = e_{K,p} \circ F_K$. First, we shall adjust the function $\widehat{w}_{K,p}$ to produce a new polynomial $\widehat{v}_{K,p}$ interpolating $\widehat{u}|_K$ at the vertices on the reference element. The adjustment at the vertex $\widehat{A}_1 = (-1,-1)$ is given by

$$\widehat{\alpha}^{(1)}_{K,p}(x_1,x_2) = \widehat{e}_{K,p}(-1,-1)\psi_p(x_1)\psi_p(x_2).$$

and satisfies

$$\begin{aligned} \left\|\widehat{\alpha}^{(1)}_{K,p}\right\|_{W^{1,q}(S(1))} &\le C\left\|\widehat{e}_{K,p}\right\|_{L^\infty(S(1))}\left\|\psi_p\right\|_{L^q(-1,1)}\left|\psi_p\right|_{W^{1,q}(-1,1)} \\ &\le Cp^{1-2/q}\left\|\widehat{e}_{K,p}\right\|_{L^\infty(S(1))}. \end{aligned} \tag{13}$$

Similar functions are constructed for the remaining vertices. The polynomial

$$\widehat{v}_{K,p} = \widehat{w}_{K,p} + \sum_{j=1}^{4} \widehat{\alpha}^{(j)}_{K,p}$$

agrees with $\widehat{u}_K$ at the vertices and satisfies

$$\|\widehat{u} - \widehat{v}_{K,p}\|_{W^{1,q}(S(1))} \le |\widehat{e}_{K,p}|_{W^{1,q}(S(1))} + Cp^{1-2/q}\left\|\widehat{e}_{K,p}\right\|_{L^\infty(S(1))}.$$

Defining $v_{K,p} = \widehat{v}_{K,p} \circ F_K^{-1}$ and mapping back to the element $K$ gives

$$\|u - v_{K,p}\|_{W^{1,q}(K)} \le |e_{K,p}|_{W^{1,q}(K)} + Cp^{1-2/q}\left\|e_{K,p}\right\|_{L^\infty(K)}. \tag{14}$$

This process is repeated on each element.

The difference $v_{K,p} - v_{J,p}$ is still, in general, non-zero on the edge $\gamma$ but vanishes at the endpoints. Therefore, we use the difference to adjust the approximation on either one of the elements, say $K$, as follows. Suppose, without loss of generality, that $\gamma = F_K(\widehat{\gamma})$ where $\widehat{\gamma} = \{(x_1,-1) : -1 \le x_1 \le 1\}$. Let $\xi : \gamma \mapsto \mathbb{R}$ denote the restriction of $v_{K,p} - v_{J,p}$ to the edge $\gamma$. Then $\widehat{\xi} = \xi \circ F_K$ is a polynomial on the edge $\widehat{\gamma}$ vanishing at the endpoints.

The polynomial $\widehat{\beta} : S(1) \mapsto \mathbb{R}$ given by $\widehat{\beta} = \widehat{\xi}(x_1)\psi_p(x_2)$ is an extension of $\widehat{\xi}$ that vanishes on the remaining edges of $S(1)$. Transforming back to the element $K$ defines a function $\beta = \widehat{\beta} \circ F_K^{-1}$ satisfying

$$|\beta|_{W^{1,q}(S(1))} \leq C \left\{ p^{1-1/q} \|\xi\|_{L^q(\gamma)} + p^{-1/q} |\xi|_{W^{1,q}(\gamma)} \right\} \tag{15}$$

and for $j = 0, 1$

$$\|\xi\|_{W^{j,q}(\gamma)} \leq \|e_{K,p}\|_{W^{j,q}(\gamma)} + \|e_{J,p}\|_{W^{j,q}(\gamma)} . \tag{16}$$

The process is repeated for every interior edge in the partition.

The function $u_{K,p}$ is defined by subtracting the sum of the edge corrections $\beta$ applied to element $K$ from $v_{K,p}$. Thanks to the method of construction, $u_{K,p}$ agrees with $u_{J,p}$ on the edge $\gamma$. Consequently, we may define $u_p \in X_p$ to be the function whose restriction to any element $J \in \mathcal{P}$ is $u_{J,p}$ and satisfies

$$\|u - u_p\|_{W^{1,q}(J)} \leq \|u - v_{J,p}\|_{W^{1,q}(J)} + C \sum_{\gamma} \|\beta_\gamma\|_{W^{1,q}(J)} . \tag{17}$$

Hence, using (14), (15), (16), (17) and Lemma 1 completes the proof when the partition consists of quadrilateral elements.

The treatment of a triangular element $J$ is similar, except that the corrections at the vertices and edges are slightly different. The function $w_{J,p}$ is constructed as in the case of quadrilaterals using instead Lemma 2. The correction at the vertex $\widehat{A}_1 = (-1, -1)$ is given by

$$\widehat{\alpha}_{K,p}^{(1)}(x_1, x_2) = \frac{1}{2} \widehat{e}_{K,p}(-1, -1) \psi_s(x_1) \psi_s(x_2)(1 - x_1)$$

where $s = [(p-1)/2]$ and the extension $\widehat{\beta}$ associated with the edge $\widehat{\gamma} = \{(x_1, -1) : -1 \leq x_1 \leq 1\}$ is

$$\widehat{\beta}(x_1, x_2) = \frac{1}{2} \psi(x_2) \left\{ (x_1 - x_2) \widehat{\xi}(x_1) + (1 - x_1) \widehat{\xi}(x_1 - x_2 - 1) \right\} .$$

The remaining cases are similar. It is easily verified that the functions have the required properties. □

The restriction in Theorem 4 on the minimal smoothness of the function $u$ may be removed using the following standard argument:

**Theorem 5.** *Let $u \in W^{m,q}(\Omega)$, $q \in [1, \infty]$, $m > 1$. Then there exists a sequence $u_p \in X_p$ of continuous piecewise polynomials such that on any element $J$*

$$\|u - u_p\|_{W^{1,q}(J)} \leq C p^{-(m-1)} (1 + \ln p)^{2|1-2/q|} \sum_{K \in \mathcal{P} : \overline{K} \cap \overline{J} \neq \emptyset} \|u\|_{W^{m,q}(K)} .$$

*Hence, the following global estimates are valid*

$$\|u - u_p\|_{W^{1,q}(\Omega)} \leq Cp^{-(m-1)}(1 + \ln p)^{2|1-2/q|} \|u\|_{W^{m,q}(\Omega)} .$$

*Proof.* In view of Theorem 4 we need only consider the case $m \in (1, 1 + 1/q]$. From the characterisation

$$W^{m,q}(\Omega) = (W^{1,q}(\Omega), W^{2,q}(\Omega))_{\theta,q} \tag{18}$$

where $\theta = m - 1$, it follows from [5, Section 3.5] that for any $t > 0$, $u$ may be decomposed as $u = v_1(t) + v_2(t)$ with $v_1 \in W^{1,q}(\Omega)$ and $v_2 \in W^{2,q}(\Omega)$ satisfying

$$\|v_1\|_{W^{1,q}(\Omega)} \leq Ct^{m-1} \|u\|_{W^{m,q}(\Omega)} \tag{19}$$

$$\|v_2\|_{W^{2,q}(\Omega)} \leq Ct^{m-2} \|u\|_{W^{m,q}(\Omega)} \tag{20}$$

where $C$ is independent of $u$ and $t$. By Theorem 4 there exists a continuous piecewise polynomial $u_p$ such that

$$\|v_2 - u_p\|_{W^{1,q}(\Omega)} \leq Cp^{-1} \|v_2\|_{W^{2,q}(\Omega)} \leq Cp^{-1}t^{m-2} \|u\|_{W^{m,q}(\Omega)} .$$

Choosing $t = 1/p$ and applying the Triangle Inequality gives

$$\|u - u_p\|_{W^{1,q}(\Omega)} \leq Cp^{-(m-1)} \|u\|_{W^{m,q}(\Omega)}$$

as required. □

### *3.3. Non-homogeneous Dirichlet boundary data*

As mentioned earlier, it is often necessary to deal with approximations that must satisfy a supplementary condition on the Dirichlet boundary. In particular, the trace $g$ of the function $u \in W^{m,q}(\Omega)$ to be approximated may be specified as data on the Dirichlet boundary. Unless the trace itself happens to be a piecewise polynomial it becomes necessary to approximate the boundary data, thereby creating an additional source of error. The imposition of Dirichlet conditions consists of first *constructing* a sequence of polynomial approximations $g_p$ to the given Dirichlet data $g$. The problem is then to estimate the accuracy that may be obtained by approximating $u$ using sequences of piecewise polynomials $u_p \in X_p$ which, in addition, satisfy $u_p = g_p$ on the Dirichlet boundary.

Lemma 1 and Lemma 2 assert the *existence* of polynomial approximations $\phi_p(u)$ to the function $u$ that achieve certain rates of convergence. It appears that one can simply chose (subject to appropriate adjustments to obtain continuity between elements) the approximate Dirichlet data $g_p$ to be the values of the approximations $\phi_p(u)$ on the element boundaries. However, while this would ensure that the overall rate of convergence would not suffer any degradation, it is not a practical proposition since the polynomials $\phi_p(u)$ are not easily constructed on a machine. A full treatment of non-homogeneous Dirichlet conditions is a non-trivial matter even in the case $q = 2$, [3] (see also Maday [9]).

The approximate Dirichlet data for an element $K$ having an edge $\gamma = \Gamma_{\rm D} \cap \overline{K}$ on the Dirichlet boundary is constructed as follows. Without loss of generality, assume that $\gamma = (-1, 1)$ and denote the trace of the function $u$ on $\gamma$ by $g$. The $p$-th partial sum of the Chebyshev series expansion of $g$ is given by

$$\sigma_p(g; t) = \sum_{k=0}^{p} A_k T_k(t)$$

where $T_k$ is the $k$-th degree Chebyshev polynomial and the coefficients are given by

$$A_k = \frac{2}{\pi} \int_{-1}^{1} g(t) T_k(t) \frac{dt}{\sqrt{1-t^2}}.$$

Bounds on the rate of convergence of the partial sums are given in the following lemma.

**Lemma 6.** *Let $g \in W^{l,q}(-1,1)$ where $q \in [1,\infty]$. Then for $l > 2/q$*

$$\|g - \sigma_p(g)\|_{L^q(-1,1)} \le C(1+\ln p)p^{-l} \|g\|_{W^{l,q}(-1,1)}$$

*and for $l > 2 - 1/q$*

$$\|g - \sigma_p(g)\|_{W^{1,q}(-1,1)} \le C(1+\ln p)p^{-(l-2+1/q)} \|g\|_{W^{l,q}(-1,1)}.$$

*Proof.* According to [11, (3.29)]

$$\sigma_p(g; \cos\theta) = \frac{1}{2\pi} \int_0^{\pi} \{g[\cos(\alpha+\theta)] + g[\cos(\alpha-\theta)]\} D_p(\alpha) d\alpha \tag{21}$$

where

$$D_p(\alpha) = \frac{\sin(p+1/2)\alpha}{\sin(\alpha/2)}$$

and

$$\frac{1}{2\pi} \int_0^{\pi} |D_p(\alpha)| \, d\alpha \le C(1+\ln p). \tag{22}$$

1. Using (21) and (22)

$$\|\sigma_p(f)\|_{L^\infty(-1,1)} \le C(1+\ln p)\,\|f\|_{L^\infty(-1,1)}\,.$$

Let $g_p$ be any polynomial of degree $p$ and note that

$$\|g-\sigma_p(g)\|_{L^\infty(-1,1)} \le \|g-g_p\|_{L^\infty(-1,1)} + \|\sigma_p(g-g_p)\|_{L^\infty(-1,1)}\,.$$

Inserting $f = g - g_p$ into the above bound leads to

$$\|g-\sigma_p(g)\|_{L^\infty(-1,1)} \le C(1+\ln p)\,\|g-g_p\|_{W^{1,\infty}(-1,1)}$$

and then taking the infimum over $g_p$ gives

$$\|g-\sigma_p(g)\|_{L^\infty(-1,1)} \le C(1+\ln p)p^{-m}\,\|f\|_{W^{m,\infty}(-1,1)}\,.$$

2. Let $\theta = \arccos x$, $x \in [-1,1]$. Then

$$\frac{d}{dx}\sigma_p(g;x) = \frac{1}{\sin\theta}\frac{d}{d\theta}\sigma_p(g;\cos\theta).$$

Now

$$\begin{aligned}
&\left|\frac{d}{d\theta}\left\{g[\cos(\alpha+\theta)] + g[\cos(\alpha-\theta)]\right\}\right| \\
&\le \left|\cos\alpha\sin\theta\left\{g'[\cos(\alpha+\theta)] - g'[\cos(\alpha-\theta)]\right\}\right| \\
&+ \left|\sin\alpha\cos\theta\left\{g'[\cos(\alpha+\theta)] + g'[\cos(\alpha-\theta)]\right\}\right| \\
&\le 2|\sin\theta|\,\left\|g'\right\|_{L^\infty(-1,1)} + 2|\cos(\alpha+\theta)-\cos(\alpha-\theta)|\,\left\|g''\right\|_{L^\infty(-1,1)} \\
&\le 4|\sin\theta|\,\|g\|_{W^{2,\infty}(-1,1)}\,.
\end{aligned}$$

Using (21) and (22) and arguing as before gives

$$\|g-\sigma_p(g)\|_{W^{1,\infty}(-1,1)} \le Cp^{-(m-2)}(1+\ln p)\,\|g\|_{W^{m,\infty}(-1,1)}\,.$$

3. Observe that

$$\|\sigma_p(g)\|_{L^1(-1,1)} \le \|D_p\|_{L^1(-1,1)}\,\|g\|_{L^1(-1,1)}$$

and the result then follows as in the first case.
4. Applying the change of variable $x = \cos\theta$ gives

$$\left\|\sigma_p(g)'\right\|_{L^1(-1,1)} = \int_0^\pi \sin\theta\left|\frac{d}{dx}\sigma_p(g;x)\right|d\theta = \int_0^\pi\left|\frac{d}{d\theta}\sigma_p(g;\cos\theta)\right|d\theta$$

then using (21) and interchanging the order of integration leads to

$$\begin{aligned}
\left\|\sigma_p(g)'\right\|_{L^1(-1,1)} &\le \int_0^\pi d\alpha\left|\frac{\sin(p+1/2)\alpha}{\sin(\alpha/2)}\right| \\
&\int_0^\pi d\theta\left\{\left|\frac{d}{d\theta}g[\cos(\alpha+\theta)]\right| + \left|\frac{d}{d\theta}g[\cos(\alpha-\theta)]\right|\right\}.
\end{aligned} \tag{23}$$

The value of the inner integral is bounded by

$$\int_0^\pi d\theta \left\{ \left| \frac{d}{d\theta} g[\cos(\alpha+\theta)] \right| + \left| \frac{d}{d\theta} g[\cos(\alpha-\theta)] \right| \right\}$$
$$\leq 4 \int_0^\pi d\theta \left\{ \left| \frac{d}{d\theta} g[\cos(\theta)] \right| \right\} = 4 \left\| g' \right\|_{L^1(-1,1)}.$$

Inserting this into (23) and using (22) gives

$$\left\| \sigma_p(g)' \right\|_{L^1(-1,1)} \leq C(1+\ln p) \left\| g' \right\|_{L^1(-1,1)}$$

and arguing as before

$$\left\| \sigma_p(g)' \right\|_{L^1(-1,1)} \leq C(1+\ln p) p^{-(m-1)} \left\| g \right\|_{W^{m,1}(-1,1)}.$$

The claimed estimates are obtained by interpolating these results. □

The actual approximation $g_p \approx g$ is taken to be

$$g_p(t) = \{g(-1) - \sigma_p(g;-1)\}\psi_p(t)$$
$$+ \{g(1) - \sigma_p(g;1)\}\psi_p(-t) + \sigma_p(g;t).$$

The following result complements Theorem 4:

**Theorem 7.** *Let $u \in W^{m,q}(\Omega)$ and assume $g \in W^{m+1-2/q,q}(\Gamma_D)$ where $q \in [1,\infty]$, $m > 1$ and $g$ is the trace of $u$ on $\Gamma_D$. Then there exists a sequence $u_p \in X_p$ of continuous piecewise polynomials such that $u_p = g_p$ on the Dirichlet boundary $\Gamma_D$. Moreover, the following global estimate holds*

$$\|u - u_p\|_{W^{1,q}(\Omega)}$$
$$\leq Cp^{-(m-1)}(1+\ln p)^{2|1-2/q|}\{\|u\|_{W^{m,q}(\Omega)} + \|g\|_{W^{m+1-2/q,q}(\Gamma_D)}\}.$$

*Proof.* Let $\tilde{u}_p$ be a sequence of approximations to $u$ as in Theorem 4. Let $K$ be any element having an edge on the Dirichlet boundary. It suffices to consider the case when $K$ is the reference element with the edge $\gamma = \{(x_1,-1) : -1 \leq x_1 \leq 1\}$ on the boundary (the case for triangles is similar). Let $v_p$ be the polynomial

$$v_p(x_1,x_2) = \tilde{u}_p(x_1,x_2) + (\sigma_p(g;x_1) - \tilde{u}_p(x_1,-1))\psi_p(x_2).$$

Following steps similar to those in the proof of Theorem 4 and using Lemma 6 leads to the estimate

$$\|u - v_p\|_{W^{1,q}(K)}$$
$$\leq C\|u - \tilde{u}_p\|_{W^{1,q}(K)} + C(1+\ln p)p^{-(m-1)}\|g\|_{W^{m+1-2/q,q}(\gamma)}.$$

The proof is completed by applying exactly the same procedure used in the proof of Theorem 4 to adjust $v_p$ and obtain a continuous piecewise polynomial. □

Comparing the estimate with those obtained previously shows that the rates are optimal. However, the optimal rate is obtained at the expense of assuming slightly more regularity of the boundary data $g$ than one would hope for. Naturally, it is of some theoretical interest to consider the limiting case when the boundary data is of minimal regularity and ask what rate of convergence might be expected in such cases. In a practical situation, the boundary data will generally be quite smooth (even piecewise analytic). Therefore, assuming the data has more than minimal regularity is not a limitation in practice.

## 4. Piecewise polynomial approximation of singular functions

The analysis has hitherto presumed the function $u$ to be approximated is smooth. However, as remarked earlier, if the domain has a corner then the solution of the partial differential equation may contain singular components. The goal of this section is to obtain estimates for the rate of convergence of sequences of piecewise polynomials approximating functions of the form

$$u(\boldsymbol{x}) = c\,\chi(r)\,r^{\lambda}\,|\ln r|^{\gamma}\,\Theta(\theta) \tag{24}$$

where $(r, \theta)$ are polar coordinates with origin at the corner, with $\chi$ and $\Theta$ smooth ($C^{\infty}$) functions. The function $\Theta$ is assumed to vanish along the edges corresponding to the boundary of the domain while $\chi$ is a smooth cut-off function that vanishes when $r$ is large. In this way, the singular function is localized around the corner with which it is associated. Fortunately, since the singularities arise at corners, it is reasonable to assume that the partition $\mathcal{P}$ has been constructed so that the corner is located at the vertices (rather than, for instance, on the edges) of an element.

### *4.1. An approximation result*

Let $\widetilde{S}(t)$, $t > 0$, denote the square

$$\widetilde{S}(t) = \{(x_1, x_2) : 0 < x_1 < t; 0 < x_2 < t\}$$

and let $B(t)$ denote the ball of radius $t$ centred at the origin. For $\kappa > 1$, let $A(\kappa)$ denote the cone

$$A(\kappa) = \left\{(x_1, x_2) : 0 < \kappa^{-1} x_1 < x_2 < \kappa x_1\right\}$$

and, finally, for $\kappa > 1$ and $t > 0$, let $R(\kappa, t)$ denote the set

$$R(\kappa, t) = A(\kappa) \cap \widetilde{S}(t).$$

The purpose of this section is to obtain results on the attainable rate of convergence of polynomial approximations of a particular class of singular functions of the form

$$u(\boldsymbol{x}) = \chi(r)\, r^{\lambda}\, |\ln r|^{\gamma}\, \Theta(\theta)$$

where is assumed that:

(A1) $\chi$ is a smooth function satisfying

$$\chi(r) = 1, \quad r \leq \rho/3 \text{ and } \chi(r) = 0, \quad r \geq 2\rho/3$$

for some fixed $\rho \in (0, 1)$;

(A2) $\Theta$ is a smooth function such that for some fixed $\kappa > 1$, the function $u$ is assumed to vanish on the rays $x_1 = \kappa x_2$ and $x_2 = \kappa x_1$ emanating from the origin;

(A3) for some fixed $\kappa_0 > \kappa$, the function $u$ is supported in the set $A(\kappa_0)$.

An immediate consequence of these assumptions is that

$$\operatorname{supp}(u) \subset R(\kappa_0, 2/3).$$

The polynomial

$$\xi(\boldsymbol{x}) = (x_1 - \kappa x_2)(\kappa x_1 - x_2)$$

vanishes on the rays $x_1 = \kappa x_2$ and $x_2 = \kappa x_1$, and so

$$u_0(\boldsymbol{x}) = \frac{u(\boldsymbol{x})}{\xi(\boldsymbol{x})} = \chi(r)\, r^{\lambda-2}\, |\ln r|^{\gamma}\, \Theta_0(\theta)$$

shares properties (A1) and (A3) of the function $u$.

*4.1.1. Regularisation* Let $\zeta \in C^{\infty}[0, \infty)$ satisfy

$$\zeta(r) = \begin{cases} 0, \, r < 1 \\ 1, \, r > 2 \end{cases}$$

and for $\Delta \in (0, 1/2)$, define

$$\zeta^{\Delta}(r) = \zeta(r/\Delta).$$

Regularised approximations of the singular functions $u$ and $u_0$ are defined by

$$u^{\Delta} = \zeta^{\Delta} u \text{ and } u_0^{\Delta} = \zeta^{\Delta} u_0$$

and satisfy

– $u^{\Delta} = 0$ on the rays $x_1 = \kappa x_2$ and $x_2 = \kappa x_1$;

– $u^\Delta$ and $u_0^\Delta$ are supported on $R(\kappa_0, 2/3) - B(\Delta)$.

Elements of the family $\{u^\Delta\}$ approach the singular function $u$ in the following sense:

**Lemma 8.** *Let $q \in (0,\infty)$ and that $u$ is given by (24), where $\lambda > 1 - 2/q$. Then*

$$\left\| u - u^\Delta \right\|_{W^{1,q}(\widetilde{S}(1))} \le C |\ln \Delta|^\gamma \Delta^{\lambda-(1-2/q)} \tag{25}$$

*Proof.* By direct calculation using the above properties of $u^\Delta$

$$\begin{aligned}\left\| u - u^\Delta \right\|^q_{W^{1,q}(\widetilde{S}(1))} &\le C \int_0^\Delta \left\{ |u|^q (1 + C\Delta^{-q}) + |\nabla u|^q \right\} r \,\mathrm{dr} \\ &\le C \Delta^{\lambda q-(q-2)} |\ln \Delta|^\gamma\end{aligned}$$

and the result follows. □

*4.1.2. Trigonometric polynomial approximation* The algebraic polynomial approximations to the regularised singular functions are constructed by applying a trigonometric transformation and then developing trigonometric polynomial approximations. Therefore, let $\Phi : \widetilde{S}(\pi/2) \mapsto \widetilde{S}(1)$ be the bijective mapping

$$\Phi(\widehat{x}_1, \widehat{x}_2) = (\sin^2 \widehat{x}_1, \sin^2 \widehat{x}_2)$$

and set

$$\widehat{u}_0^\Delta(\widehat{\boldsymbol{x}}) = u_0^\Delta \circ \Phi(\widehat{\boldsymbol{x}}).$$

**Lemma 9.** *Let $q \in [2,\infty)$ and denote $T = R(\kappa_0, 2/3)$. Suppose $v \in W^{1,q}(T)$ and define $\widehat{v} = v \circ \Phi$ and $\widehat{T} = \Phi^{-1}(T)$. Then,*

$$\|v\|_{W^{2/q,q}(T)} \approx \|\widehat{v}\|_{W^{2/q,q}(\widehat{T})} \tag{26}$$

*and,*

$$\|v\|_{W^{1,q}(T-B(\Delta))} \le C\Delta^{-\frac{1}{2}(1-2/q)} \|\widehat{v}\|_{W^{1,q}(\widehat{T}-B(\widehat{\Delta}))} . \tag{27}$$

*where $\widehat{\Delta} = \arcsin \sqrt{\Delta}$.*

*Proof.* The norm on the space $W^{2/q,q}(T)$ satisfies ([1, Theorem 7.48])

$$\|v\|_{W^{2/q,q}(T)} \approx \left\{ \|v\|^q_{L^q(T)} + \int_T \int_T \frac{|v(\boldsymbol{x}) - v(\boldsymbol{y})|^q}{|\boldsymbol{x} - \boldsymbol{y}|^4} d\boldsymbol{x} d\boldsymbol{y} \right\}^{1/q} . \tag{28}$$

The identity

$$\int_T |v|^q d\boldsymbol{x} = \int_{\widehat{T}} |\widehat{v}|^q |\sin 2\widehat{x}_1 \sin 2\widehat{x}_2| d\widehat{\boldsymbol{x}}$$

immediately shows that $\|v\|_{L^q(T)} \leq \|\widehat{v}\|_{L^q(\widehat{T})}$. Moreover, by Hölder's Inequality, (with $t = q + 1$ and $1/t + 1/t' = 1$)

$$\int_{\widehat{T}} |\widehat{v}|^q d\widehat{\boldsymbol{x}} \leq \left( \int_T |v|^{qt} d\boldsymbol{x} \right)^{1/t} \left( \int_T |\sin 2\widehat{x}_1 \sin 2\widehat{x}_2|^{-t'} d\boldsymbol{x} \right)^{1/t'} .$$

In the neighbourhood of the origin $\sin 2\widehat{x}_1 \approx x_1^{1/2}$ and hence (since $t' = 1 + 1/q < 2$) the second term is bounded. Consequently,

$$\|\widehat{v}\|^q_{L^q(\widehat{T})} \leq C \left( \int_T |v|^{qt} d\boldsymbol{x} \right)^{1/t} \leq C \|v\|^q_{L^{q(q+1)}(T)}$$

and by the Sobolev Embedding Theorem [1, Theorem 7.57(b)]

$$\|v\|_{L^{q(q+1)}(T)} \leq C \|v\|_{W^{2/q,q}(T)} .$$

Hence

$$\|\widehat{v}\|_{L^q(\widehat{T})} \leq C \|v\|_{L^{q(q+1)}(T)} \leq C \|v\|_{W^{2/q,q}(T)} .$$

Consider the second term in (28). It suffices to show that for all $(\widehat{x}_1, \widehat{x}_2) \in \widehat{T}$

$$\frac{1}{|\widehat{\boldsymbol{x}} - \widehat{\boldsymbol{y}}|^4} \approx \frac{\sin 2\widehat{x}_1 \sin 2\widehat{x}_2 \sin 2\widehat{y}_1 \sin 2\widehat{y}_2}{((\sin^2 \widehat{x}_1 - \sin^2 \widehat{y}_1)^2 + (\sin^2 \widehat{x}_2 - \sin^2 \widehat{y}_2)^2)^2}.$$

which is easily verified using elementary arguments after observing that

$$\frac{\sin \widehat{x}_1}{\sin \widehat{x}_2} \approx 1 \tag{29}$$

for all $(\widehat{x}_1, \widehat{x}_2) \in \widehat{T}$. The first result then follows at once.

Observe that

$$\Phi^{-1}(T - B(\Delta)) \subset \widehat{T} - B(\widehat{\Delta}).$$

Hence, for any $v \in W^{1,q}(T)$, $q \in [2, \infty)$,

$$\int_{T-B(\Delta)} |v|^q d\boldsymbol{x} \leq \int_{\widehat{T}-B(\widehat{\Delta})} |\widehat{v}|^q |\sin 2\widehat{x}_1 \sin 2\widehat{x}_2| \, \mathrm{d}\widehat{\boldsymbol{x}}$$

and for $i = 1, 2$

$$\int_{T-B(\Delta)} \left| \frac{\partial v}{\partial x_i} \right|^q d\boldsymbol{x} \leq \int_{\widehat{T}-B(\widehat{\Delta})} \left| \frac{\partial \widehat{v}}{\partial \widehat{x}_i} \right|^q \frac{|\sin 2\widehat{x}_1 \sin 2\widehat{x}_2|}{|\sin 2\widehat{x}_i|^q} \, \mathrm{d}\widehat{\boldsymbol{x}}.$$

Combining these results with (29) and the fact that $\widehat{\Delta} \approx \Delta^{1/2}$ gives (27). □

A key result in the development is an estimate for the higher order Sobolev norms of the regularized singular functions in terms of the regularisation parameter $\Delta$.

**Lemma 10.** *Let* $q \in (1, \infty)$ *and suppose* $k \geq 2(\lambda + 1/q)$. *Then,* $\widehat{u}^{\Delta} \in W^{k,q}(\widetilde{S}(\pi/2))$ *and there exists a constant* $C(k)$ *depending only on* $k$ *such that,*

(30) $$\left\|\widehat{u}^{\Delta}\right\|_{W^{k,q}(\widetilde{S}(\pi/2))} \leq C(k)|\ln \Delta|^{\gamma} \Delta^{-(k/2-\lambda-1/q)}.$$

*Proof.* Let $\widehat{\Delta} = \arcsin \sqrt{\Delta}$ and $\widehat{\rho} = \arcsin \sqrt{\Delta/\kappa}$. Then, for any multi-index $\alpha$,

$$\left|D^{\alpha}\widehat{u}^{\Delta}(\widehat{\boldsymbol{x}})\right| \leq \begin{cases} C\,|\ln \Delta|^{\gamma} \min(\widehat{x}_1, \widehat{x}_2)^{-\langle|\alpha|-2\lambda\rangle}, \\ \quad \text{for } \min(\widehat{x}_1, \widehat{x}_2) \geq \widehat{\rho} \\ C|\ln \Delta|^{\gamma} \sum_{j=1}^{\alpha_1} \sum_{l=1}^{\alpha_2} \widehat{x}_1^{\langle 2j-\alpha_1\rangle}\, \widehat{x}_2^{\langle 2l-\alpha_2\rangle}\, \Delta^{-\langle j+l-\lambda\rangle}, \\ \quad \text{for } \widehat{\boldsymbol{x}} \in \widetilde{S}(\widehat{\Delta}) \end{cases},$$

where $\langle t\rangle = \max(0, t)$. The first of these results is proved in the same way as Lemma 4.4 in [4] along with the observation that $\widehat{\rho} \approx \widehat{\Delta}$. The second result is obtained following arguments similar to those leading to the first equation on page 529 of [4].

The support of the function $\widehat{u}^{\Delta}$ satisfies $\operatorname{supp}(\widehat{u}^{\Delta}) \subset G_1 \cup G_2$, where

$$G_1 = \operatorname{supp}(\widehat{u}^{\Delta}) \cap \widetilde{S}(\widehat{\Delta})$$

and

$$G_2 = \operatorname{supp}(\widehat{u}^{\Delta}) \cap \left\{(\widehat{x}_1, \widehat{x}_2) : \min(\widehat{x}_1, \widehat{x}_2) \geq \widehat{\Delta}\right\}.$$

Applying the earlier estimates for the derivatives and applying the bound $\widehat{\Delta} \leq C\sqrt{\Delta}$ gives:

1. For any $|\alpha| \leq k$

$$\left\|D^{\alpha}\widehat{u}^{\Delta}\right\|_{L^q(G_1)}^q \leq C|\ln \Delta|^{q\gamma} \sum_{j=1}^{\alpha_1} \sum_{l=1}^{\alpha_2} \Delta^{-q\langle j+l-\lambda\rangle + q\langle j-\alpha_1/2\rangle + q\langle l-\alpha_2/2\rangle + 1}$$

and hence, since

$$-\langle j+l-\lambda\rangle + \langle j-\alpha_1/2\rangle + \langle l-\alpha_2/2\rangle \geq -\langle k/2-\lambda\rangle,$$

there follows

$$\left\|\widehat{u}^{\Delta}\right\|_{W^{k,q}(G_1)} \leq C(k)|\ln \Delta|^{\gamma} \Delta^{-(k/2-\lambda-1/q)}.$$

2. Let $G_2^+ = G_2 \cap \{(\widehat{x}_1, \widehat{x}_2) : \widehat{x}_2 < \widehat{x}_1\}$, then, for $\widehat{\boldsymbol{x}} \in G_2^+$,

$$\left|D^\alpha \widehat{u}^\Delta(\widehat{\boldsymbol{x}})\right| \leq C(\alpha)\,|\ln \Delta|^\gamma\, \widehat{x}_2^{-\langle|\alpha|-2\lambda\rangle}.$$

and so

$$\begin{aligned}\left\|D^\alpha \widehat{u}^\Delta\right\|^q_{L^q(G_2^+)} &\leq C(\alpha)\,|\ln \Delta|^{q\gamma} \int_{\widehat{\Delta}}^{\pi/6} \int_{\widehat{\rho}}^{\widehat{x}_1} \widehat{x}_2^{-q\langle|\alpha|-2\lambda\rangle} d\widehat{x}_1 d\widehat{x}_2 \\ &\leq C(\alpha)\,|\ln \Delta|^{q\gamma}\, \widehat{\Delta}^{-q\langle\langle|\alpha|-2\lambda\rangle-2/q\rangle}.\end{aligned}$$

The same estimate is obtained for the norm evaluated over the remaining part of the set $G_2$. Therefore, summing over multi-indices gives,

$$\left\|\widehat{u}^\Delta\right\|_{W^{k,q}(G_2)} \leq C(k)|\ln \Delta|^\gamma \Delta^{-(k/2-\lambda-1/q)}$$

and the claimed result follows. □

*4.1.3. Algebraic polynomial approximation* The next result is concerned with approximation by algebraic polynomials and generalises the corresponding result Theorem 5.1 from [2] to the case when $q \neq 2$.

**Lemma 11.** *Let $q \in [2, \infty)$ and suppose the function $u$ satisfies the conditions (A1)-(A3). Then, for $p > 1$, there exists $z_p \in Q(p)$ such that $z_p$ vanishes on the rays $x_2 = \kappa x_1$ and $x_1 = \kappa x_2$. Moreover, for any fixed $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that*

$$\begin{aligned}&\|u - z_p\|_{W^{1,q}(R(\kappa_0,2/3))} \\ &\quad \leq C(\varepsilon)(1+\ln p)^{2(1-2/q)}|\ln p|^\gamma p^{-2(\lambda-1+2/q)+\varepsilon}\end{aligned} \tag{31}$$

*provided that $\lambda > 1 - 2/q$.*

*Proof.* The notations described earlier are again adopted. First, extend the function $\widehat{u}_0^\Delta$ from $\widetilde{S}(\pi/2)$ to the square $S(\pi)$ as an even periodic function by reflecting in the lines $\widehat{x}_k = 0, \pm\pi/2$, $k = 1, 2$. Let $s_p(\widehat{u}_0^\Delta)$ be partial sums of the Fourier series expansion of $\widehat{u}_0^\Delta$. Then, by Lemma 15(1), for any $0 \leq m \leq k$

$$\begin{aligned}&\left\|\widehat{u}_0^\Delta - s_p(\widehat{u}_0^\Delta)\right\|_{W^{m,q}(\widetilde{S}(\pi/2))} \\ &\leq C(k)(1+\ln p)^{2(1-2/q)} p^{-(k-m)} \left\|\widehat{u}_0^\Delta\right\|_{W^{k,q}(\widetilde{S}(\pi/2))} \\ &\leq C(k)(1+\ln p)^{2(1-2/q)} p^{-(k-m)} |\ln \Delta|^\gamma \Delta^{-(k/2-\lambda+2-1/q)}\end{aligned} \tag{32}$$

since Lemma 10 applies equally well to the function $\widehat{u}_0^\Delta$.

Thanks to the symmetries of the the extended function, the inverse images of the partial Fourier sums, given by

$$u_{0,p}^\Delta = s_p(\widehat{u}_0^\Delta) \circ \Phi^{-1}$$

are, in fact, algebraic polynomials. We now develop estimates for their rate of convergence.
1. The estimate (32) in the case $m = 0$ is preserved under the transformation to the original domain,

$$\|u_0^\Delta - u_{0,p}^\Delta\|_{L^q(\widetilde{S}(1))} \le \|u_0^\Delta - u_{0,p}^\Delta\|_{L^q(\widetilde{S}(1))} \le C(k)(1+\ln p)^{2(1-2/q)} p^{-k} |\ln \Delta|^\gamma \Delta^{-(k/2-\lambda+2-1/q)}.$$

Moreover, since the polynomial $\xi$ and its first order derivatives are uniformly bounded on the domain $\widetilde{S}(1)$, the same estimate holds for both $\|\xi(u_0^\Delta - u_{0,p}^\Delta)\|_{L^q(\widetilde{S}(1))}$ and $\|\partial\xi/\partial x_1(u_0^\Delta - u_{0,p}^\Delta)\|_{L^q(\widetilde{S}(1))}$.
2. Consider now

$$\left\|\xi \frac{\partial}{\partial x_1}(u_0^\Delta - u_{0,p}^\Delta)\right\|^q_{L^q(\widetilde{S}(1))} \le \left\|\xi \frac{\partial}{\partial x_1}(u_0^\Delta - u_{0,p}^\Delta)\right\|^q_{L^q(\widetilde{S}(1)\cap B(\Delta))} + \left\|\xi \frac{\partial}{\partial x_1}(u_0^\Delta - u_{0,p}^\Delta)\right\|^q_{L^q(\widetilde{S}(1)-B(\Delta))}.$$

The first term is estimated using the bound $|\xi(\boldsymbol{x})| \le C\Delta^2$ on $B(\Delta)$ and the fact that $u_0^\Delta$ vanishes on $B(\Delta)$, as follows

$$\left\|\xi \frac{\partial}{\partial x_1}(u_0^\Delta - u_{0,p}^\Delta)\right\|_{L^q(\widetilde{S}(1)\cap B(\Delta))} \le C\Delta^2 \|u_{0,p}^\Delta\|_{W^{1,q}(\widetilde{S}(1)\cap B(\Delta))}.$$

Now, since $u_{0,p}^\Delta$ is an algebraic polynomial, an application of the inequalities of Markov and Schmidt, along with an interpolation argument leads to

$$\|u_{0,p}^\Delta\|_{W^{1,q}(\widetilde{S}(1)\cap B(\Delta))} \le C(p^2/\Delta)^{1-2/q} \|u_{0,p}^\Delta\|_{W^{2/q,q}(\widetilde{S}(1)\cap B(\Delta))},$$

and hence, again observing $u_0^\Delta$ vanishes on $B(\Delta)$, one arrives at the estimate

$$\left\|\xi \frac{\partial}{\partial x_1}(u_0^\Delta - u_{0,p}^\Delta)\right\|_{L^q(\widetilde{S}(1)\cap B(\Delta))} \le C\Delta^2(p^2/\Delta)^{1-2/q} \|u_0^\Delta - u_{0,p}^\Delta\|_{W^{2/q,q}(\widetilde{S}(1)\cap B(\Delta))}.$$

Applying Lemma 9 then yields

$$\|u_0^\Delta - u_{0,p}^\Delta\|_{W^{2/q,q}(\widetilde{S}(1)\cap B(\Delta))} \le C \|\widehat{u}_0^\Delta - s_p(\widehat{u}_0^\Delta)\|_{W^{2/q,q}(\widetilde{S}(\pi/2))}.$$

Applying the estimate (32) in the case $m = 2/q$ leads to the bound

$$\left\|\xi \frac{\partial}{\partial x_1}(u_0^\Delta - u_{0,p}^\Delta)\right\|_{L^q(\widetilde{S}(1)\cap B(\Delta))} \le C(k)(1+\ln p)^{2(1-2/q)}(p^2/\Delta)^{1-2/q} p^{-(k-2/q)} |\ln \Delta|^\gamma \Delta^{-(k/2-\lambda-1/q)}.$$

The second term is estimated using Lemma 8 as follows:

$$\begin{aligned}
&\left\|\xi \frac{\partial}{\partial x_1}(u_0^\Delta - u_{0,p}^\Delta)\right\|_{L^q(\widetilde{S}(1)-B(\Delta))} \\
&\le \|u_0^\Delta - u_{0,p}^\Delta\|_{W^{1,q}(\widetilde{S}(1)-B(\Delta))} \\
&\le C\Delta^{-\frac{1}{2}(1-2/q)} \|\widehat{u}_0^\Delta - s_p(\widehat{u}_0^\Delta)\|_{W^{1,q}(\widetilde{S}(\pi/2))}
\end{aligned}$$

and then, applying Lemma 10, gives

$$\begin{aligned}
&\left\|\xi \frac{\partial}{\partial x_1}(u_0^\Delta - u_{0,p}^\Delta)\right\|_{L^q(\widetilde{S}(1)-B(\Delta))} \\
&\le C(k)(1+\ln p)^{2(1-2/q)} \Delta^{-\frac{1}{2}(1-2/q)} p^{-(k-1)} \Delta^{-(k/2-\lambda+2-1/q)} |\ln \Delta|^\gamma.
\end{aligned}$$

Similar estimates hold for the $x_2$-derivatives, so that

$$\begin{aligned}
&\|\xi(u_0^\Delta - u_{0,p}^\Delta)\|_{W^{1,q}(\widehat{S}(1))} \\
&\le C(k)(1+\ln p)^{2(1-2/q)} |\ln \Delta|^\gamma \times \\
&\quad \Big\{ p^{-k} \Delta^{-(k/2-\lambda+2-1/q)} + p^{-(k-2+2/q)} \Delta^{-(k/2-\lambda+1-3/q)} \\
&\qquad + p^{-(k-1)} \Delta^{-(k/2-\lambda+5/2-2/q)} \Big\}.
\end{aligned}$$

Now, let $\lambda^* = \lambda - 1 + 2/q$, and choosing $\Delta = p^{-\mu}$ where $\mu > 0$ is determined below, allows the terms in parentheses to be rewritten as

$$\begin{aligned}
p^{-\mu\lambda^*} &\Big\{ p^{-k+\mu(k/2+1+1/q)} + p^{-(k-2+2/q)+\mu(k/2-1/q)} \\
&\quad + p^{-(k-1)+\mu(k/2+3/2)} \Big\}.
\end{aligned}$$

The value of $\mu$ is chosen so that each of the exponents of the terms inside the parentheses is non-positive, thus:

$$\mu = 2 \min \left\{ \frac{k}{k+2+2/q}, \frac{k-2+2/q}{k-2/q}, \frac{k-1}{k+3} \right\}$$

and then, for any given positive $\varepsilon$, $k$ may be chosen sufficiently large for the value of $\mu$ to satisfy

$$\mu \ge 2 - \varepsilon.$$

Hence, for any given $\varepsilon > 0$,

$$\begin{aligned}
&\|\xi(u_0^\Delta - u_{0,p}^\Delta)\|_{W^{1,q}(\widehat{S}(1))} \\
&\quad \le C(\varepsilon)(1+\ln p)^{2(1-2/q)} |\ln Cp|^\gamma p^{-2(\lambda-1+2/q)+\varepsilon}.
\end{aligned}$$

For $p > 1$, the polynomial $z_p$ is taken to be $\xi u^{\Delta}_{0,p-2}$. Obviously, $z_p$ vanishes on the lines $x_2 = \kappa x_1$ and $x_1 = \kappa x_2$ on which $\xi$ vanishes. Moreover, by the triangle inequality,

$$\|u - z_p\|_{W^{1,q}(\widetilde{S}(1))} \leq \left\|u - u^{\Delta}\right\|_{W^{1,q}(\widetilde{S}(1))} + \left\|\xi(u^{\Delta}_0 - u^{\Delta}_{0,p})\right\|_{W^{1,q}(\widetilde{S}(1))}$$

and the result follows from the previous estimates and Lemma 8. □

### *4.2. Piecewise polynomial approximation of singular functions*

Suppose that the domain has a re-entrant corner located at a vertex $\boldsymbol{A}$ of the partition, and that, relative to polar coordinates based at $\boldsymbol{A}$, the solution has a singularity of the form (24). Let $\Omega_0$ denote the domain consisting of the elements that have a vertex located at the corner $\boldsymbol{A}$. The main result of this section generalises Theorem 5.2 in [2] to the case when $q \neq 2$:

**Theorem 12.** *Let $u$ be the singular function given by (24) with the cut-off function $\chi$ supported on a sufficiently small ball. Then, for any $\varepsilon > 0$, there exists a sequence of piecewise polynomials $z_p \in X_p$ that vanish on $\partial\Omega_0$ and satisfy*

$$\|u - z_p\|_{W^{1,q}(\Omega)} \leq C(\varepsilon)|\ln p|^{\gamma} p^{-2(\lambda-1+2/q)+\varepsilon}$$

*provided that $\lambda > 1 - 2/q$, where $C$ is independent of $p$.*

*Proof.* The proof uses Lemma 11 in exactly the same way as the result was obtained in the case $q = 2$ in [2]. □

## 5. Application to finite element approximation of non-linear elliptic partial differential equations

Consider the $\alpha$-Laplacian

$$-\nabla \cdot \left\{|\nabla u|^{\alpha-2}\nabla u\right\} = f \text{ in } \Omega$$

where $\alpha \in (1, \infty)$ and $f$ is smooth given data. The boundary conditions are that $u = 0$ on the Dirichlet boundary $\Gamma_{\mathrm{D}}$ and prescribed normal flux $g$ on the Neumann boundary $\Gamma_{\mathrm{N}}$. The weak form of this problem is to find $u \in V$ such that

$$\int_{\Omega} |\nabla u|^{\alpha-2}\nabla u \cdot \nabla v d\boldsymbol{x} = \int_{\Omega} f v d\boldsymbol{x} + \int_{\Gamma_N} g v ds$$

for all $v \in V$, where $V$ is the space

$$V = \left\{v \in W^{1,\alpha}(\Omega) : v = 0 \text{ on } \Gamma_{\mathrm{D}}\right\}.$$

To approximate the problem using finite elements consists of constructing a finite dimensional subspace $X \subset V$ composed of piecewise polynomials on a partition $\mathcal{P}$. The finite element approximation $u_X \in X$ satisfies:

$$\int_\Omega |\nabla u_X|^{\alpha-2} \nabla u_X \cdot \nabla v_X d\boldsymbol{x} = \int_\Omega f v_X d\boldsymbol{x} + \int_{\Gamma_N} g v_X ds$$

for all $v_X \in X$. The accuracy of the approximation is controlled either by refining the size $h$ of the elements in the partition (the $h$-version) or increasing the polynomial degree $p$ on the elements (the $p$-version). The subspace for the $h$-version will be denoted by $X_h$.

We shall apply the approximation results to compare the rate of convergence of the $p$-version finite element method with the rate of the $h$-version finite element approximation. The basic tool we shall use to obtain the estimates is found in Chow [6]:

$$\|u - u_p\|_{W^{1,\alpha}(\Omega)} \leq \begin{cases} C \inf \|u - v_X\|^{\alpha/2}_{W^{1,\alpha}(\Omega)}, & \alpha \in (1, 2] \\ \inf C(\|v_X\|_{W^{1,\alpha}(\Omega)}) \|u - v_X\|^{2/\alpha}_{W^{1,\alpha}(\Omega)}, & \alpha \in (2, \infty) \end{cases} \tag{33}$$

where the infimum is taken over functions $v_X$ from the finite element subspace $X$.

### 5.1. Rate of convergence for smooth solutions

Suppose that the true solution $u$ of the model problem belongs to the space $W^{m,\alpha}(\Omega)$. The standard approximation results for the $h$-version imply that

$$\inf_{v \in X_h} \|u - v_h\|_{W^{1,\alpha}(\Omega)} \leq C h^\mu \|u\|_{W^{m,\alpha}(\Omega)}$$

where

$$\mu = \min(m - 1, p)$$

and $p$ is the (fixed) polynomial degree of the elements used to construct the $h$-version subspace. Theorem 5 shows that the corresponding result for the $p$-version is

$$\inf_{v \in X_p} \|u - v_p\|_{W^{1,\alpha}(\Omega)} \leq C p^{-(m-1)} (1 + \ln p)^{2|1-2/\alpha|} \|u\|_{W^{m,\alpha}(\Omega)} .$$

The basic difference between these estimates is that the rate for the $h$-version is limited by the polynomial degree of the elements used while for the $p$-version the rate is limited only by the regularity of the solution.

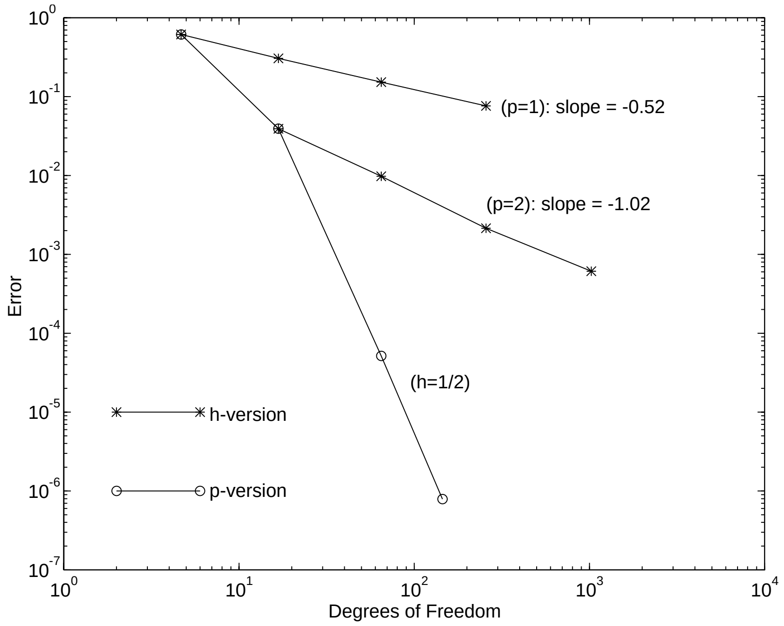


**Fig. 1.** Rate of convergence for smooth solution

As an example, consider the problem with true solution $u = \mathrm{e}^{(x+y)}$, $\alpha = 3/2$ and $\Omega = (0, 1) \times (0, 1)$. The problem is solved numerically using uniform $h$-refinement for fixed polynomial degree $p = 1$ and $p = 2$; and using the $p$-version on a partition consisting of four square elements. To compare the rate of convergence of the methods we consider the dimension $N$ of the subspaces $X_h$ and $X_p$. The approximation results for the $h$-version suggest that the rate will be $O(N^{-p/2})$ while for the $p$-version the rate will be greater than $O(N^{-r})$ for all values of $r$. That is, the $p$-version should exhibit an *exponential rate of convergence*. The results shown in Fig. 1 confirm these estimates.

### *5.2. Rate of convergence for singular solutions*

Consider the $\alpha$-Laplacian with $\alpha \in [2, \infty)$ and true solution of the form

$$u(\boldsymbol{x}) = c\,\chi(r)\,r^{\lambda}\,g(|\ln r|)\,\Theta(\theta) + w(\boldsymbol{x})$$

where $w$ is a smooth function and $\chi$, $\Theta$ and $g$ as in equation (24). One easily verifies that the solution belongs to the space $W^{\lambda+2/\alpha-\varepsilon,\alpha}(\Omega)$, where $\varepsilon > 0$ is arbitrarily small. The approximation results for the $h$-version indicate that

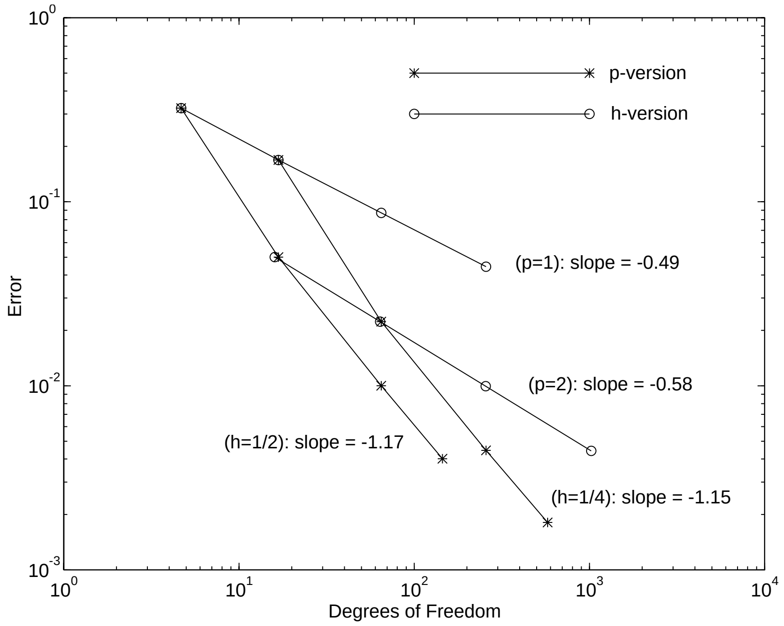


**Fig. 2.** Rate of convergence for singular solution

for elements of fixed degree $p$, the rate of convergence will be $O(N^{-\mu})$ where $\mu = \min(\lambda + 2/\alpha - 1 - \varepsilon, p)/2$.

For the $p$-version, one could apply Theorem 5 and obtain an estimate for the rate to be $O(N^{-(\lambda+2/\alpha-1)/2})$ as in the $h$-version. However, the more refined estimates obtained in Theorem 12 show that, in fact, the rate should be $O(N^{-(\lambda+2/\alpha-1)+\varepsilon)})$. Therefore, in such cases, *the rate of convergence of the $p$-version is twice that of the $h$-version*.

As a simple example, consider the $\alpha$-Laplacian with $\Omega = (0,1) \times (0,1)$, $\alpha = 3$ and true solution $u = r^{3/4}$. The solution belongs to the space $W^{13/6-\varepsilon,3}(\Omega)$. The approximation results for the $h$-version indicate that for elements of fixed degree $p$, the rate of convergence will be $O(N^{-\mu})$ where $\mu = \min(7/6 - \varepsilon, p)/2$. Consequently, in the case $p = 1$ the rate will be unaffected by the smoothness of the solution and should be $O(N^{-1/2})$. However, for elements of fixed degree $p = 2$ the rate should be degraded from the full order $O(N^{-1})$ to $O(N^{-7/12})$. Meanwhile, for the $p$-version one should observe a rate of $O(N^{-7/6})$. The results shown in Fig. 2 confirm these estimates.

## A. Appendix

The Fourier series expansion of a sufficiently smooth function $f$ on the square $S(\pi)$ is denoted by

$$f(x_1, x_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} A_{mn} \mathrm{e}^{i(mx_1+nx_2)}$$

where $A_{mn}$ are the Fourier coefficients and the partial sum is denoted by

$$s_N(f) = \sum_{|m|<N} \sum_{|n|<N} A_{mn} \mathrm{e}^{\mathrm{i}(mx_1+nx_2)}.$$

For numbers $N \in \mathbb{N}$ and $r \in \mathbb{Z}^+$ let

$$\mathcal{C}_{N,r} = \frac{4}{\pi^2} N^{-r} \ln N + O(N^{-r})$$

and

$$\mathcal{D}_{N,r}(t) = \sum_{|m|>N} \frac{1}{m^r} \cos\left(mt - \frac{\pi r}{2}\right). \tag{34}$$

It will be useful to recall some results from classical Fourier analysis:
– [8, Theorem 4.3.1]: for any $r \in \mathbb{Z}^+$

$$\begin{aligned} &\sum_{|m|>N} \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s,t) \mathrm{e}^{\mathrm{i}m(x_1-s)} ds\, dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \mathcal{D}_{N,r}(x_1 - s) f^{(r,0)}(s,t) ds\, dt \end{aligned} \tag{35}$$

and for any fixed $t$

$$\int_{-\pi}^{\pi} |\mathcal{D}_{N,r}(t-s)| ds = \mathcal{C}_{N,r} \tag{36}$$

and if $r > 1$

$$\|\mathcal{D}_{N,r}\|_{L_\infty(-\pi,\pi)} \le \sum_{|m|>N} \frac{1}{m^r} \le C N^{1-r}. \tag{37}$$

– [8, Theorem 2.2.1, p. 54]

$$\begin{aligned} &\frac{1}{2\pi} \sum_{|n|\le N} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n(x_2-t)} f(s,t) dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(s, x_2 + t) \frac{\sin(n+1/2)t}{2 \sin t/2} dt \end{aligned} \tag{38}$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(n+1/2)t}{2\sin t/2}\right| dt \le C\mathcal{C}_{N,0}$$

so that

–

$$\left\| \frac{1}{2\pi}\sum_{|n|\le N}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n(x_2-t)} f(s,t)dt \right\|_{L^\infty(S(\pi))} \le \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(n+1/2)t}{2\sin t/2}\right| dt\, \|f\|_{L^\infty(S(\pi))}$$

(39) $$\le C\mathcal{C}_{N,0}\,\|f\|_{L^\infty(S(\pi))}$$

The first result deals with the rate of convergence of the partial Fourier sums in $L_\infty$ type norms.

**Lemma 13.** *If* $f \in W^{l,\infty}_{\mathrm{per}}(S(\pi))$ *then for* $0 \le k \le l$

(40) $$\|f - s_N(f)\|_{W^{k,\infty}(S(\pi))} \le C(1+\ln N)^2 N^{-(l-k)}\, \|f\|_{W^{l,\infty}(S(\pi))}\,.$$

*Proof.* Choose $\beta_1$, $\beta_2 \in \mathbb{Z}^+$ such that $\beta_1+\beta_2 \le k$. Then

$$D^{(\beta_1,\beta_2)}(f - s_N(f)) = \left(\sum_{|m|>N}\sum_{|n|>N} + \sum_{|m|>N}\sum_{|n|\le N} + \sum_{|m|\le N}\sum_{|n|>N}\right) \times A_{mn}(\mathrm{i}m)^{\beta_1}(\mathrm{i}n)^{\beta_2}\mathrm{e}^{\mathrm{i}(mx_1+nx_2)} = I + II + III.$$

Now fix $n$ and consider the term

$$\sum_{|m|>N} A_{mn}(\mathrm{i}m)^{\beta_1}(\mathrm{i}n)^{\beta_2}\mathrm{e}^{\mathrm{i}(mx_1+nx_2)} = \sum_{|m|>N}\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s,t)(\mathrm{i}m)^{\beta_1}(\mathrm{i}n)^{\beta_2}\mathrm{e}^{-\mathrm{i}(ms+nt)}\mathrm{e}^{\mathrm{i}(mx_1+nx_2)}ds\,dt.$$

Since $f \in W^{l,\infty}_{\mathrm{per}}(S(\pi))$

$$\sum_{|m|>N}\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s,t)(\mathrm{i}m)^{\beta_1}(\mathrm{i}n)^{\beta_2}\mathrm{e}^{-\mathrm{i}(ms+nt)}\mathrm{e}^{\mathrm{i}(mx_1+nx_2)}ds\,dt$$
$$= \sum_{|m|>N}\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f^{(\beta_1,\beta_2)}(s,t)\mathrm{e}^{-\mathrm{i}(ms+nt)}\mathrm{e}^{\mathrm{i}(mx_1+nx_2)}ds\,dt$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}n(x_2-t)}dt\sum_{|m|>N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f^{(\beta_1,\beta_2)}(s,t)\mathrm{e}^{\mathrm{i}m(x_1-s)}ds.$$

Using (35) for $\alpha_1 \in \mathbb{Z}^+$:

$$\sum_{|m|>N} A_{mn}(\mathrm{i}m)^{\beta_1}(\mathrm{i}n)^{\beta_2}\mathrm{e}^{\mathrm{i}(mx_1+nx_2)}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n(x_2-t)}dt \frac{1}{\pi}\int_{-\pi}^{\pi} \mathcal{D}_{N,\alpha_1}(x_1-s) f^{(\alpha_1+\beta_1,\beta_2)}(s,t)ds \tag{41}$$

Summing (41) over $n : |n| > N$ gives, for $\alpha_2 \in \mathbb{Z}^+$

$$\sum_{|m|>N}\sum_{|n|>N} A_{mn}(\mathrm{i}m)^{\beta_1}(\mathrm{i}n)^{\beta_2}\mathrm{e}^{\mathrm{i}(mx_1+nx_2)}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \mathcal{D}_{N,\alpha_1}(x_1-s)ds \frac{1}{\pi}\int_{-\pi}^{\pi} \mathcal{D}_{N,\alpha_2}(x_2-t) f^{(\alpha_1+\beta_1,\alpha_2+\beta_2)}(s,t)dt.$$

Hence for any $\alpha_1, \alpha_2 \in \mathbb{Z}^+ : \alpha_1 + \alpha_2 + \beta_1 + \beta_2 = l$

$$|I| \le C\mathcal{C}_{N,\alpha_1}\mathcal{C}_{N,\alpha_2} \left\| f^{(\alpha_1+\beta_1,\alpha_2+\beta_2)} \right\|_{L^\infty(S(\pi))}$$

$$\le C(1+\ln N)^2 N^{-(l-\beta_1-\beta_2)} |f|_{W^{l,\infty}(S(\pi))} . \tag{42}$$

Consider the second term $II$. Summing (41) over $n : |n| \le N$ gives for $r \in \mathbb{Z}^+$

$$II = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathcal{D}_{N,r}(x_1-s)ds \frac{1}{2\pi}\sum_{|n|\le N}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n(x_2-t)} f^{(r+\beta_1,\beta_2)}(s,t)dt.$$

Hence using (36)

$$|II| \le \mathcal{C}_{N,r} \left\| \frac{1}{2\pi}\sum_{|n|\le N}\int_{0}^{2\pi} \mathrm{e}^{\mathrm{i}n(x_2-t)} f^{(r+\beta_1,\beta_2)}(s,t)dt \right\|_{L^\infty(S(\pi))} . \tag{43}$$

Using (38) and (39), we obtain for $r + \beta_1 + \beta_2 = l$

$$|II| \le C\mathcal{C}_{N,0}\mathcal{C}_{N,r} \left\| f^{(r+\beta_1,\beta_2)} \right\|_{L^\infty(S(\pi))}$$

$$\le C(1+\ln N)^2 N^{-(l-\beta_1-\beta_2)} |f|_{W^{l,\infty}(S(\pi))} . \tag{44}$$

The third term is dealt with similarly. Therefore, combining (42) and (44) gives for $s \in \mathbb{Z}^+ : s + \beta_1 + \beta_2 = l$

$$\left\| D^{(\beta_1,\beta_2)}(f - s_N(f)) \right\|_{L^\infty(S(\pi))}$$

$$\le C(1+\ln N)^2 N^{-(l-\beta_1-\beta_2)} |f|_{W^{l,\infty}(S(\pi))} .$$

Summing over $\beta_1, \beta_2 : \beta_1 + \beta_2 \le k$

$$\|f - s_N(f)\|_{W^{k,\infty}(S(\pi))} \le C(1+\ln N)^2 N^{-(l-k)} \|f\|_{W^{l,\infty}(S(\pi))}$$

and the result follows. □

Consider now the rate of convergence in $L_1$ type norms.

**Lemma 14.** *If* $f \in W^{l,1}_{\text{per}}(S(\pi))$ *then*

1. *for* $0 \le k \le l$

(45) $$\|f - s_N(f)\|_{W^{k,1}(S(\pi))} \le C(1 + \ln N)^2 N^{-(l-k)} \|f\|_{W^{l,1}(S(\pi))} .$$

2. *for* $0 \le k + 1 < l$

(46) $$\|f - s_N(f)\|_{W^{k,1}(\gamma)} \le C(1 + \ln N) N^{-(l-1-k)} \|f\|_{W^{l,1}(S(\pi))}$$

*where* $\gamma$ *may be any line segment contained in* $S(\pi)$ *on which* $x_2$ *is constant or on which* $x_1 = \pm x_2$*.*

3. *for* $0 \le k + 2 < l$

(47) $$\|f - s_N(f)\|_{W^{k,\infty}(S(\pi))} \le C N^{-(l-k-2)} \|f\|_{W^{l,1}(S(\pi))} .$$

*Proof.* 1. Let $\beta_1, \beta_2 \in \mathbb{Z}^+$ satisfy $\beta_1 + \beta_2 \le k$. Then

$$\begin{aligned}
&\left\| D^{(\beta_1,\beta_2)}(f - s_N(f)) \right\|_{L^{l,1}(S(\pi))} \\
&\le \left\| \sum_{|m|>N} \sum_{|n|>N} A_{mn} (\mathrm{i}m)^{\beta_1} (\mathrm{i}n)^{\beta_2} \mathrm{e}^{\mathrm{i}(mx_1+nx_2)} \right\|_{L^{l,1}(S(\pi))} \\
&+ \left\| \sum_{|m|>N} \sum_{|n|\le N} A_{mn} (\mathrm{i}m)^{\beta_1} (\mathrm{i}n)^{\beta_2} \mathrm{e}^{\mathrm{i}(mx_1+nx_2)} \right\|_{L^{l,1}(S(\pi))} \\
&+ \left\| \sum_{|m|\le N} \sum_{|n|>N} A_{mn} (\mathrm{i}m)^{\beta_1} (\mathrm{i}n)^{\beta_2} \mathrm{e}^{\mathrm{i}(mx_1+nx_2)} \right\|_{L^{l,1}(S(\pi))} \\
&= I + II + III.
\end{aligned}$$

Using (35) gives for $\alpha_1, \alpha_2 \in \mathbb{Z}^+ : \alpha_1 + \alpha_2 + \beta_1 + \beta_2 = l$

$$\begin{aligned}
I = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Bigg| &\frac{1}{\pi} \int_{-\pi}^{\pi} \mathcal{D}_{N,\alpha_1}(x_1 - s) ds \\
&\frac{1}{\pi} \int_{-\pi}^{\pi} \mathcal{D}_{N,\alpha_2}(x_2 - t) f^{(\alpha_1+\beta_1,\alpha_2+\beta_2)}(s,t) dt \Bigg| d\boldsymbol{x}_1 d\boldsymbol{x}_2
\end{aligned}$$

and then, since $\mathcal{D}_{N,r}$ and $f$ are both periodic and continuous with period $2\pi$, recalling $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = l$ we obtain from (36)

$$\begin{aligned}
I &\le C \mathcal{C}_{N,\alpha_1} \mathcal{C}_{N,\alpha_2} \left\| f^{(\alpha_1+\beta_1,\alpha_2+\beta_2)} \right\|_{L^1(S(\pi))} \\
&\le C(1 + \ln N)^2 N^{-(l-\beta_1-\beta_2)} |f|_{W^{l,1}(S(\pi))} . \qquad (48)
\end{aligned}$$

Similarly, using (35) and (38) yields for any $r \in \mathbb{Z}^+$: $\beta_1 + \beta_2 + r = l$,

$$II = \Bigg\| \frac{1}{\pi} \int_{s=-\pi}^{\pi} \mathcal{D}_{N,r}(x_1 - s) ds \\ \frac{1}{\pi} \int_{t=-\pi}^{\pi} \frac{\sin(N+1/2)t}{2 \sin t/2} f^{(r+\beta_1,\beta_2)}(s, x_2 + t) dt \Bigg\|_{L^1(S(\pi))}$$

and then, since $f$ is a periodic function, from (36) and (39) we have

$$\begin{aligned} II &\leq C \mathcal{C}_{N,r} \int_{t=-\pi}^{\pi} \left| \frac{\sin(N+1/2)t}{2 \sin t/2} \right| dt \left\| f^{(\beta_1+r,\beta_2)} \right\|_{L^1(S(\pi))} \\ &\leq C \mathcal{C}_{N,r} \mathcal{C}_{N,0} \left\| f^{(r+\beta_1,\beta_2)} \right\|_{L^1(S(\pi))} \\ &\leq C(1 + \ln N)^2 N^{-(l-\beta_1-\beta_2)} |f|_{W^{l,1}(S(\pi))} . \end{aligned}$$

The third term, III, is treated similarly. Consequently, for $s \in \mathbb{Z}^+$: $\beta_1 + \beta_2 + s = l$

$$\begin{aligned} &\left\| D^{(\beta_1,\beta_2)}(f - s_N(f)) \right\|_{L^1(S(\pi))} \\ &\quad \leq C(1 + \ln N)^2 N^{-(l-\beta_1-\beta_2)} |f|_{W^{l,1}(S(\pi))} \end{aligned}$$

and summing over $\beta_1$, $\beta_2$: $\beta_1 + \beta_2 \leq k$ gives (45).
2. Let $\gamma$ be a line contained in $S(\pi)$ on which $x_2$ is constant and let $\beta \in \mathbb{Z}^+$: $\beta \leq k$. Then

$$\begin{aligned} \left\| D^{(\beta,0)}(f - s_N(f)) \right\|_{L^1(\gamma)} \leq & \left\| \sum_{|m|>N} \sum_{|n|>N} A_{mn} (\mathrm{i}m)^\beta \mathrm{e}^{\mathrm{i}(mx_1+nx_2)} \right\|_{L^1(\gamma)} \\ & + \left\| \sum_{|m|>N} \sum_{|n|\leq N} A_{mn} (\mathrm{i}m)^\beta \mathrm{e}^{\mathrm{i}(mx_1+nx_2)} \right\|_{L^1(\gamma)} \\ & + \left\| \sum_{|m|\leq N} \sum_{|n|>N} A_{mn} (\mathrm{i}m)^\beta \mathrm{e}^{\mathrm{i}(mx_1+nx_2)} \right\|_{L^1(\gamma)} \\ = & \; I + II + III. \end{aligned}$$

Using (35) gives for $\alpha_1$, $\alpha_2 \in \mathbb{Z}^+$: $\alpha_1 + \alpha_2 + \beta = l$ and $\alpha_2 > 1$,

$$I \leq C \mathcal{C}_{N,\alpha_1} \left\| \frac{1}{\pi} \mathcal{D}_{N,\alpha_2}(x_2 - \cdot) \right\|_{L^\infty(\gamma)} \left\| f^{(\alpha_1+\beta,\alpha_2)} \right\|_{L^1(S(\pi))}$$

where (36) has been used. Recalling (37) we have for all $\beta \le k$

$$I \le C\mathcal{C}_{N,\alpha_1} N^{1-\alpha_2} \left\| f^{(\alpha_1+\beta,\alpha_2)} \right\|_{L^1(S(\pi))}$$

$$\le C(1+\ln N)N^{-(l-1-\beta)} \left| f \right|_{W^{l,1}(S(\pi))} . \tag{49}$$

Equally well, (35) and (39) give for $\nu \in \mathbb{Z}^+$: $\nu = l - \beta$

$$II = \left\| \frac{1}{\pi}\int_{s=-\pi}^{\pi} \mathcal{D}_{N,\nu}(x_1-s) \sum_{|n|\le N} \frac{1}{2\pi}\int_{t=-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n(x_2-t)} f^{(l,0)}(s,t)\,ds\,dt \right\|_{L^1(\gamma)}$$

$$\le C\mathcal{C}_{N,\nu} \int_{t=-\pi}^{\pi}\int_{s=-\pi}^{\pi} \frac{1}{2\pi} \sum_{|n|\le N} \left| \mathrm{e}^{\mathrm{i}n(x_2-t)} \right| \left| f^{(l,0)}(s,t) \right| ds\,dt$$

$$\le CN\mathcal{C}_{N,\nu} \left\| f^{(l,0)} \right\|_{L^1(S(\pi))}$$

$$\le C(1+\ln N)N^{-(l-1-\beta)} \left| f \right|_{W^{l,1}(S(\pi))} . \tag{50}$$

Again, since $\mathcal{D}_{N,\sigma}$ and $f$ are both $2\pi$-periodic, (35) and (38) give for $\sigma \in \mathbb{Z}^+$: $\sigma = l - \beta$

$$III = \left\| \frac{1}{\pi}\int_{s=-\pi}^{\pi}\int_{t=-\pi}^{\pi} \frac{\sin(N+1/2)s}{2\sin s/2} \mathcal{D}_{N,\sigma}(x_2-t) f^{(\beta,\sigma)} (s+x_1,t)\,ds\,dt \right\|_{L^1(\gamma)}$$

$$\le \left\| \frac{1}{\pi}\mathcal{D}_{N,\alpha_2}(x_2-\cdot) \right\|_{L^\infty(-\pi,\pi)} \int_{s=-\pi}^{\pi} \frac{1}{\pi} \left| \frac{\sin(N+1/2)s}{2\sin s/2} \right| ds \times \left\| f^{(\beta,\sigma)} \right\|_{L^1(S(\pi))}$$

$$\le CN^{1+\beta-l}\mathcal{C}_{N,0} \left\| f^{(\beta,\sigma)} \right\|_{L^1(S(\pi))}$$

$$= C(1+\ln N)N^{-(l-1-\beta)} \left| f \right|_{W^{l,1}(S(\pi))} . \tag{51}$$

Combining (49), (50) and (51) and summing over $\beta \le k$ gives the result for the case when $\gamma$ is a line on which $x_2$ is constant. Now let $\gamma$ be the line contained in $S(\pi)$ given by $x_1 = x_2 = \tau$ and let $\beta \in \mathbb{Z}^+$: $\beta \le k$. Then

$$\left\| \left(\frac{\partial}{\partial\tau}\right)^{\beta} (f - s_N(f)) \right\|_{L^1(\gamma)}$$

$$\le \left\| \sum_{|m|>N}\sum_{|n|>N} A_{mn}[i(m+n)]^{\beta} \mathrm{e}^{\mathrm{i}(m+n)\tau} \right\|_{L^1(\gamma)}$$

$$
\begin{aligned}
&+ \left\| \sum_{|m|>N} \sum_{|n|\le N} A_{mn}[i(m+n)]^{\beta} \mathrm{e}^{\mathrm{i}(m+n)\tau} \right\|_{L^1(\gamma)} \\
&+ \left\| \sum_{|m|\le N} \sum_{|n|>N} A_{mn}[i(m+n)]^{\beta} \mathrm{e}^{\mathrm{i}(m+n)\tau} \right\|_{L^1(\gamma)} \\
&= I + II + III.
\end{aligned}
$$

Since $f$ is a periodic function, using (35), (36), (49) and the binomial expansion

$$
I = \int_{\tau=-\pi}^{\pi} \left| \frac{1}{4\pi^2} \sum_{j=0}^{\beta} \binom{\beta}{j} \sum_{|m|>N} \sum_{|n|>N} \int_{t=-\pi}^{\pi} \int_{s=-\pi}^{\pi} f^{(j,\beta-j)}(s,t) \mathrm{e}^{-\mathrm{i}(ms+nt)} \mathrm{e}^{\mathrm{i}(m+n)\tau} ds\, dt \right| d\tau
$$

$$
\le 2^{\beta} C(1+\ln N) N^{-(l-1-\beta)} \, |f|_{W^{l,1}(S(\pi))} . \tag{52}
$$

Similarly, using (35), (39) and (50)

$$
II \le 2^{\beta} C(1+\ln N) N^{-(l-1-\beta)} \, |f|_{W^{l,1}(S(\pi))} . \tag{53}
$$

Finally using (35), (37) and (51)

$$
III \le 2^{\beta} C(1+\ln N) N^{-(l-1-\beta)} \, |f|_{W^{l,1}(S(\pi))} . \tag{54}
$$

Combining (52), (53), (54) and summing over $\beta \le k$ gives the result when $\gamma$ is the line $x_1 = x_2$. The case when $x_1 = -x_2$ follows in a similar fashion.

3. Suppose $\beta_1$, $\beta_2 \in \mathbb{Z}^+$: $\beta_1 + \beta_2 \le k$. Then for any point $\boldsymbol{x} = (x_1, x_2) \in \overline{S(\pi)}$

$$
\begin{aligned}
&\left| D^{(\beta_1,\beta_2)}(f - s_N(f))(x_1,x_2) \right| \\
&\le \left| \sum_{|m|>N} \sum_{|n|>N} A_{mn} (\mathrm{i}m)^{\beta_1} (\mathrm{i}n)^{\beta_2} \mathrm{e}^{\mathrm{i}(mx_1+nx_2)} \right| \\
&+ \left| \sum_{|m|>N} \sum_{|n|\le N} A_{mn} (\mathrm{i}m)^{\beta_1} (\mathrm{i}n)^{\beta_2} \mathrm{e}^{\mathrm{i}(mx_1+nx_2)} \right| \\
&+ \left| \sum_{|m|\le N} \sum_{|n|>N} A_{mn} (\mathrm{i}m)^{\beta_1} (\mathrm{i}n)^{\beta_2} \mathrm{e}^{\mathrm{i}(mx_1+nx_2)} \right| \\
&= I + II + III.
\end{aligned}
$$

Let $\alpha_1, \alpha_2 \in \mathbb{Z}^+$: $\alpha_1 > 1$, $\alpha_2 > 1$ and $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = l$. Using (35) and (37) gives

$$I \le CN^{1-\alpha_1} N^{1-\alpha_2} \left\| f^{(\alpha_1+\beta_1,\alpha_2+\beta_2)} \right\|_{L^1(S(\pi))}$$

$$\le CN^{2+\beta_1+\beta_2-l} |f|_{W^{l,1}(S(\pi))} . \tag{55}$$

Let $\nu \in \mathbb{Z}^+$: $\nu + \beta_1 + \beta_2 = l$ and note that $\nu \ge l - k > 2$. Using (35) and (37)

$$II = \left| \frac{1}{\pi} \int_{s=-\pi}^{\pi} \mathcal{D}_{N,\nu}(x_1 - s) ds \frac{1}{2\pi} \sum_{|n|\le N} \int_{t=-\pi}^{\pi} \mathrm{e}^{in(x_2-t)} f^{(\nu,0)}(s,t) dt \right|$$

$$\le CNN^{1-\nu} \left\| f^{(\nu+\beta_1,\beta_2)} \right\|_{L^1(S(\pi))}$$

$$\le CN^{2+\beta_1+\beta_2-l} |f|_{W^{l,1}(S(\pi))} . \tag{56}$$

The third term III is dealt with similarly. Gathering these estimates gives

$$\left\| D^{(\beta_1,\beta_2)}(f - s_N(f)) \right\|_{L^\infty(S(\pi))} \le CN^{-(l-2-\beta_1-\beta_2)} |f|_{W^{l,1}(S(\pi))}$$

and taking the maximum over $\beta_1$, $\beta_2$: $\beta_1 + \beta_2 \le k$ gives the result claimed. □

Finally, the results for the $L_1$ and $L_\infty$ cases are combined to obtain estimates in the general norm $L_q$.

**Lemma 15.** *Let* $f \in W^{l,q}_{\mathrm{per}}(S(\pi))$ *then*
*1. for* $0 \le k \le l$

$$\|f - s_N(f)\|_{W^{k,q}(S(\pi))}$$

$$\le CN^{-(l-k)} (1 + \ln N)^{|2(1-2/q)|} \|f\|_{W^{l,q}(S(\pi))} . \tag{57}$$

*2. for* $0 \le k + 1/q < l$

$$\|f - s_N(f)\|_{W^{k,q}(\gamma)}$$

$$\le CN^{-(l-k-1/q)} \|f\|_{W^{l,q}(S(\pi))} \begin{cases} (1 + \ln N)^{(2/q-1)}, & q \in [1,2] \\ (1 + \ln N)^{2(1-2/q)}, & q \in [2,\infty] \end{cases} . \tag{58}$$

*where* $\gamma$ *is any line contained in* $S(\pi)$ *on which* $x_2$ *is constant or on which* $x_1 = \pm x_2$.
*3. for* $0 \le k + 2/q < l$

$$\|f - s_N(f)\|_{W^{k,\infty}(S(\pi))}$$

$$\le CN^{-(l-k-2/q)} \|f\|_{W^{l,q}(S(\pi))} \begin{cases} 1, & q \in [1,2] \\ (1 + \ln N)^{2(1-2/q)}, & q \in [2,\infty] \end{cases} . \tag{59}$$

*Proof.* 1. First, recall the following result proved in [4]

$$\|f - s_N(f)\|_{W^{k,2}(S(\pi))} \leq CN^{-(l-k)} \|f\|_{W^{l,2}(S(\pi))}$$

Combining this with the Lemma 14(1) and Lemma 13 and applying a standard interpolation argument [5] gives (57) for $q \in [1,2]$ and $q \in [2,\infty]$ respectively.
2. Using [4, equation 3.19] we have for $m > 1/2$

$$\|f - s_N(f)\|_{L^2(\gamma)} \leq CN^{-(m-\frac{1}{2})} \|f\|_{W^{m,2}(S(\pi))} . \tag{60}$$

Since $f$ is periodic, for any $\beta_1$, $\beta_2 \in \mathbb{Z}^+$

$$D^{(\beta_1,\beta_2)} s_N(f) = s_N\left(D^{(\beta_1,\beta_2)} f\right) . \tag{61}$$

Using (60) and (61) we have, for $m > 1/2$ and $\beta_1$, $\beta_2 \in \mathbb{Z}^+$: $\beta_1 + \beta_2 \leq k$

$$\begin{aligned} \left\|D^{(\beta_1,\beta_2)}(f - s_N f)\right\|_{L^2(\gamma)} &= \left\|D^{(\beta_1,\beta_2)} f - s_N D^{(\beta_1,\beta_2)} f\right\|_{L^2(\gamma)} \\ &\leq CN^{-(m-\frac{1}{2})} \|f\|_{W^{m+\beta_1+\beta_2,2}(S(\pi))} . \end{aligned} \tag{62}$$

Choosing $m + \beta_1 + \beta_2 = l$ and summing over all $\beta_1$, $\beta_2$: $\beta_1 + \beta_2 \leq k$

$$\|f - s_N(f)\|_{W^{k,2}(\gamma)} \leq CN^{-(l-k-\frac{1}{2})} \|f\|_{W^{l,2}(S(\pi))} . \tag{63}$$

Combining (63) with Lemma 14(2), Lemma 13 and applying an interpolation argument gives (58) for $q \in [1,2]$ and $q \in [2,\infty]$ respectively.
3. From [4, equation 3.29] we have for $m > 1$ and $(x_1, x_2) \in S(\pi)$

$$|(f - s_N(f))(x_1, x_2)| \leq CN^{-(m-1)} \|f\|_{W^{m,2}(S(\pi))} . \tag{64}$$

Using (61) and (64) we obtain for any $\beta_1$, $\beta_2 \in \mathbb{Z}^+$ and $m > 1$

$$|D^{(\beta_1,\beta_2)}(f - s_N(f))(x_1, x_2)| \leq CN^{-(m-1)} \|f\|_{W^{m+\beta_1+\beta_2,2}(S(\pi))} . \tag{65}$$

Choosing $m + \beta_1 + \beta_2 = l$ and summing over all $\beta_1$, $\beta_2$: $\beta_1 + \beta_2 \leq k$

$$|D^{(\beta_1,\beta_2)}(f - s_N(f))(x_1, x_2)| \leq CN^{-(l-\beta_1-\beta_2-1)} \|f\|_{W^{m+\beta_1+\beta_2,2}(S(\pi))} . \tag{66}$$

Combining (66) with the (47) and (40) and applying an interpolation argument gives (59) for $q \in [1,2]$ and $q \in [2,\infty]$ respectively. □

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