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The NGP-stability of Runge-Kutta methods for systems of neutral delay differential equations

Lin Qiu[1], **Biao Yang**[2], **Jiaoxun Kuang**[2]

[1] Graduate School of Human Informatics, Nagoya University, Nagoya 464-8601, Japan; e-mail: qiulin@torii.nuie.nagoya-u.ac.jp
[2] Department of Mathematics, Shanghai Normal University, Shanghai 200234, P.R.China

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Summary. This paper deals with the stability analysis of implicit Runge-Kutta methods for the numerical solutions of the systems of neutral delay differential equations. We focus on the behavior of such methods with respect to the linear test equations

$$y'(t) = Ly(t) + My(t-\tau) + Ny'(t-\tau), \quad t \geq 0,$$
$$y(t) = g(t), \quad -\tau \leq t \leq 0,$$

where $\tau > 0$, L, M and N are $d \times d$ complex matrices. We show that an implicit Runge-Kutta method is NGP-stable if and only if it is A-stable.

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1. Introduction

Consider the stability behavior in the numerical solution of neutral delay-differential equations (NDDEs)

$$y'(t) = f(t, y(t), y(t-\tau), y'(t-\tau)), t \geq 0, \tag{1.1}$$
$$y(t) = g(t), \quad -\tau \leq t \leq 0, \tag{1.2}$$

where τ is a given positive constant, f and g denote given vector-valued functions, and $y(t)$ is the vector-valued unknown function to be solved for $t \geq 0$.

Correspondence to: L. Qiu

We always assume that the unique solution $y(t)$ of (1.1)-(1.2) exists. (Compare Kamont and Kwapisz [11] or Jackiewicz [10]). The purpose of the present paper is to investigate the stability properties of implicit Runge-Kutta methods for NDDEs (1.1)-(1.2). We shall assess the stability properties of implicit Runge-Kutta methods (IRK) by analyzing the stability behavior in the numerical solution of the following system of the form

$$y'(t) = Ly(t) + My(t-\tau) + Ny'(t-\tau), \quad t \geq 0, \tag{1.3}$$

$$y(t) = g(t), \quad -\tau \leq t \leq 0, \tag{1.4}$$

where $\tau > 0$, L, M and N are constant matrices in $C^{d\times d}$. An application of numerical method for NDDEs to the test equations (1.3)-(1.4) usually leads to a difference equation with the matrix coefficients, but arbitrarily high order, since the order of the difference equation depends on the stepsize and the delay term τ. The difficulty leads to very few results about the linear stability of numerical methods for NDDEs (1.3)-(1.4) in the literature.

In 1967, Brayton and Willoughby [3] analyzed the stability properties of θ-methods for (1.1)-(1.2) in the case of the test equations (1.3)-(1.4) with symmetric real L, M and N, and positive definite $I \pm N$ and $-L \pm M$. In 1984, Jackiewicz [9] considered the numerical stability of one-step methods in the case when L, M and N reduce to scalar complex numbers. In 1988, Bellen, Jackiewicz and Zennaro [2] investigated the numerical stability of IRK for (1.3)-(1.4) with respect to scalar complex parameters L, M and N. But recently, in 1994, Kuang, Xiang and Tian [14] considered the numerical stability of θ-methods for (1.3)-(1.4). In 1995, Hu and Mitsui [7] also considered the numerical stability of Runge-Kutta methods for (1.3)-(1.4), and obtained an absolutely stable area of explicit Runge-Kutta methods.

For $N = 0$, in 1994, Koto [12] showed that an A-stable Runge-Kutta method $\{A, b, c\}$ with $\mathrm{Re}(\lambda_A) \geq 0$ preserves the asymptotic stability property of the analytical solutions of the system (1.3) - (1.4). In 1997, in 't Hout [5] also investigated the numerical stability of Runge-Kutta methods for the system (1.3)-(1.4) with $N = 0$, and obtained a better conclusion that an A-stable Runge-Kutta method preserves the asymptotic stability property of the analytical solutions of the system (1.3)-(1.4).

Following Hu [7], we shall show that for a wider class of (1.3)-(1.4), an A-stable IRK method preserves the asymptotic stability property of the analytical solutions of the system (1.3)-(1.4).

2. The stability of the test equations

Define a function of two complex variables z, ω by

$$F(z, \omega) = \det[zI_d - (L + \omega^{-1}(zN + M))].$$

Then the characteristic equation of (1.3) is written as $F(z, \exp(\tau z)) = 0$. From the proof of Theorem 3.1 in [14], we can see that when $||N|| < 1$, $\pm i\infty$ are not accumulation points of the roots of the characteristic equation. Then the system (1.3)-(1.4) with $||N|| < 1$ is asymptotically stable, i. e., $\lim_{t\to\infty} y(t) = 0$ for any $\tau > 0$, if and only if the following condition (S) is satisfied

$(S) \qquad F(z, \exp(\tau z)) \neq 0$ for (τ, z) such as $\mathrm{Re}(z) \geq 0$ and $\tau > 0$.

Lemma 2.1. *(see [14]) Let $||N|| < 1$, the condition (S) is equivalent to the following three conditions*

(S_1) $\lambda \in \sigma[L] \Longrightarrow \mathrm{Re}(\lambda) < 0$,
(S_2) $\rho[(zI - L)^{-1}(M + zN)] < 1$ *(whenever* $\mathrm{Re}(z) = 0, z \neq 0)$,
(S_3) $-1 \notin \sigma[L^{-1}M]$,

where $||N|| = \sup_{||z||=1} ||Nz||$, $||z||^2 = \langle z, z\rangle$, $z \in \mathbb{C}^N$, and $\sigma[X]$ and $\rho[X]$ denote the spectrum and the spectral radius of the square matrix X, respectively.

Note that, under the condition (S_1), the identity

$$\omega^d F(z, \omega) = \det[(zI_d - L)] \det[\omega I_d - (zI_d - L)^{-1}(zN + M)]$$

holds for z such as $\mathrm{Re}(z) \geq 0$. The condition

$$(\tilde{S}_2) \quad F(z, \omega) \neq 0 \text{ for any } z (\neq 0)$$
$$\text{and } \omega \text{ which satisfy } \mathrm{Re}(z) \geq 0 \text{ and } |\omega| \geq 1$$

implies (S_2) if (S_1) is satisfied.

Lemma 2.2. *Let $||N|| < 1$. Then the condition (S) is satisfied if and only if (S_1), $(\tilde{S}_2)$ and (S_3) are satisfied.*

The proof is analogous to that of Theorem in [12].

Moreover, we have the following lemma.

Lemma 2.3.*(see [7])* *The system (1.3)-(1.4) is asymptotically stable if the conditions*

$$\text{Re}\{\lambda_i[(I-\xi N)^{-1}(L+\xi M)]\}<0 \quad \textit{for all } i \textit{ and } \xi\in\mathbb{C} \textit{ such as } |\xi|\le 1, \tag{2.1}$$

and

$$\rho(N)<1 \tag{2.2}$$

hold.

From the following Theorem 2.1, we can see that, (2.1)-(2.2) are much general conditions on L, M and N such that the system (1.3)-(1.4) with $||N||<1$ is asymptotically stable.

Define

$$Q(\xi)=(I-\xi N)^{-1}(L+\xi M).$$

Theorem 2.1. *Let $||N||<1$. Then the system (1.3)-(1.4) is asymptotically stable if and only if*

(S_1) $\lambda\in\sigma[L]\Longrightarrow\text{Re}(\lambda)<0,$

(S_2^*) $\lambda\in\sigma[Q(\xi)],\lambda\ne 0\Longrightarrow\text{Re}(\lambda)<0,$ (*whenever* $|\xi|\le 1$),

(S_3) $-1\notin\sigma[L^{-1}M].$

Proof. According to Lemma 2.2, we only need to prove that the condition (S_2^*) is equivalent to the condition $(\tilde S_2)$. Assume that $(\tilde S_2)$ is satisfied, but (S_2^*) dose not hold, i.e., there exist a certain $\xi_0\in\mathbb{C}$ with $|\xi_0|\le 1$ and $\lambda_0\in\sigma[Q(\xi_0)]$ with $\lambda_0\ne 0$ such that $\text{Re}(\lambda_0)\ge 0$.

Let

$$z_0=\lambda_0 \text{ and } \omega_0=\xi_0^{-1}(\xi_0\ne 0), \tag{2.3}$$

then we get $|\omega_0|\ge 1$ and $||\omega_0^{-1}N||<1$, which implies $I_d-\omega_0^{-1}N$ is nonsinglar. Thus

$$\begin{aligned}F(z_0,\omega_0)&=\det[z_0I_d-(L+\omega_0^{-1}(z_0N+M))]\\&=\det[I_d-\omega_0^{-1}N]\det[z_0I_d-(I_d-\omega_0^{-1}N)^{-1}(L+\omega_0^{-1}M)]\\&=0,\end{aligned}$$

but this contradicts $(\tilde S_2)$, since $|\omega_0|\ge 1$ and $\text{Re}(z_0)\ge 0, z_0\ne 0$.

Conversely, assume (S_2^*) holds, but $(\tilde S_2)$ does not hold. Then for some $z^*(\ne 0)$ with $\text{Re}(z^*)\ge 0$ and some ω^* with $|\omega^*|\ge 1$,

$$\begin{aligned}F(z^*,\omega^*)&=\det[I_d-\omega^{*-1}N]\det[z^*I_d-(I_d-\omega^{*-1}N)^{-1}(L+\omega^{*-1}M)]\\&=0.\end{aligned}$$

This means that there exists $z^* \in \sigma[Q(\xi^*)]$ with $z^* \neq 0$ such that $\mathrm{Re}(z^*) \geq 0$ (where $\xi^* = \omega^{*-1}$), but it is impossible by the assumption (S_2^*) and this completes the proof of Theorem 2.1.

According to Theorem 2.1, we can see the condition (2.1) is sufficient, but it is not necessary for the asymptotic stability of the system (1.3)-(1.4) with $||N|| < 1$.

3. The NGP-stability of IRK

For the initial-value problem (1.1)-(1.2), consider the following implicit Runge-Kutta method

(3.1)
$$K_{n,i} = f(t_n + c_i h, y_n + h\sum_{j=1}^{\upsilon} a_{ij}K_{n,j}, y_{n-m+\delta} + h\sum_{j=1}^{\upsilon} a_{ij}K_{n-m+\delta,j}, K_{n-m+\delta,i}),$$

$$(i = 1, 2, \ldots, \upsilon),$$

(3.2)
$$y_{n+1} = y_n + h\sum_{i=1}^{\upsilon} b_i K_{n,i}, \quad n \geq 0,$$

where $\sum_{i=1}^{\upsilon} b_i = 1, c_i = \sum_{j=1}^{\upsilon} a_{ij}, 1 \leq i \leq \upsilon, y_n \approx y(t_n)$, $y_n = g(t_n)$ for $-\tau \leq t_n \leq 0, t_n = nh, (m-\delta)h = \tau, \delta \in [0,1), h > 0$ is a stepsize, $y_{n-m+\delta}$ and $K_{n-m+\delta,i}$ $(1 \leq i \leq \upsilon)$ are defined by some interpolations.

Define $r(\bar{h})$ by

$$r(\bar{h}) = 1 + \bar{h}\boldsymbol{b}^{\mathrm{T}}(I_\upsilon - \bar{h}A)^{-1}\boldsymbol{e},$$

where $A = (a_{ij})_{\upsilon\times\upsilon}, \boldsymbol{e} = (1, 1, \ldots, 1)^{\mathrm{T}}, \boldsymbol{b} = (b_1, b_2, \ldots, b_\upsilon)^{\mathrm{T}}$. It is well known that $r(\bar{h})$ can be also written

(3.3)
$$r(\bar{h}) = \frac{\det[I_\upsilon - \bar{h}A + \bar{h}\boldsymbol{e}\boldsymbol{b}^{\mathrm{T}}]}{\det[I_\upsilon - \bar{h}A]}.$$

We recall that a Runge-Kutta method is said to be A-stable if

(3.4) $(I - \bar{h}A)$ is regular and $|r(\bar{h})| < 1$ for any $\mathrm{Re}(\bar{h}) < 0$.

Applying the implicit Runge-Kutta method (3.1)-(3.2) to (1.3)-(1.4), we have

$$K_{n,i} = L(y_n + h\sum_{j=1}^{\upsilon} a_{ij}K_{n,j}) + M(y_{n-m+\delta} + h\sum_{j=1}^{\upsilon} a_{ij}K_{n-m+\delta,j}) + NK_{n-m+\delta,i}, \qquad (i = 1, 2, \ldots, \upsilon), \tag{3.5}$$

$$y_{n+1} = y_n + h\sum_{i=1}^{\upsilon} b_i K_{n,i}, \tag{3.6}$$

for $n = 1, 2, \ldots$, where $y_0 = g(0), y_n = g(t_n)$ for $-\tau \le t_n \le 0, t_n = nh, (m-\delta)h = \tau, \delta \in [0, 1), y_{n-m+\delta}$ and $K_{n-m+\delta,j}$ are defined by the interpolation which was first introduced by in 't Hout [4]. That is

$$y_{n-m+\delta} = \sum_{p=-r}^{s} L_p(\delta) y_{n-m+p}, \tag{3.7}$$

$$K_{n-m+\delta,j} = \sum_{p=-r}^{s} L_p(\delta) K_{n-m+p,j}, \quad (1 \le j \le \upsilon), \tag{3.8}$$

where

$$L_p(\delta) = \prod_{k=-r,k\neq p}^{s} [(\delta - k)/(p - k)],$$
$$m \ge s + 1.$$

Define

$$\mathbf{K}_n = (K_{n,1}, K_{n,2}, \ldots, K_{n,\upsilon})^{\mathrm{T}}.$$

Then (3.5) and (3.6) become

$$\mathbb{K}_n = \boldsymbol{e} \otimes Ly_n + hA \otimes L\mathbb{K}_n + \boldsymbol{e} \otimes M(\sum_{p=-r}^{s} L_p(\delta) y_{n-m+p}) + hA \otimes M \sum_{p=-r}^{s} L_p(\delta)\mathbb{K}_{n-m+p} + I_\upsilon \otimes N \sum_{p=-r}^{s} L_p(\delta)\mathbb{K}_{n-m+p},$$

$$y_{n+1} = y_n + h\boldsymbol{b}^{\mathrm{T}} \otimes I_d \mathbb{K}_n,$$

or

$$\begin{pmatrix} I_{\upsilon\times d} - h(A\otimes L) & 0 \\ -h\boldsymbol{b}^{\mathrm{T}} \otimes I_d & I_d \end{pmatrix} \begin{pmatrix} \mathbb{K}_n \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{e}\otimes L \\ 0 & I_d \end{pmatrix} \begin{pmatrix} \mathbb{K}_{n-1} \\ y_n \end{pmatrix} + \begin{pmatrix} hA\otimes M + I_\upsilon \otimes N & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sum_{p=-r}^{s} L_p(\delta)\mathbb{K}_{n-m+p} \\ \sum_{p=-r}^{s} L_p(\delta) y_{n-m+p+1} \end{pmatrix} + \begin{pmatrix} 0 & \boldsymbol{e}\otimes M \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sum_{p=-r}^{s} L_p(\delta)\mathbb{K}_{n-m+p-1} \\ \sum_{p=-r}^{s} L_p(\delta) y_{n-m+p} \end{pmatrix}, \tag{3.9}$$

where the symbol $\otimes$ denotes the Kronecker product.

The characteristic equation of the above difference equation turns out to

$$P_m(z, \overline{L}, \overline{M}, N, \delta) = \det \begin{bmatrix} T_1(z) & T_2(z) \\ T_3(z) & T_4(z) \end{bmatrix} = 0, \tag{3.10}$$

where

$$\begin{aligned}
T_1(z) &= (I_{v\times d} - A \otimes \overline{L})z^{m+1} - (A \otimes \overline{M} + I_v \otimes N) \sum_{p=-r}^{s} L_p(\delta) z^{p+1} \\
&= z^{m+1}[I_v \otimes (I_d - N \sum_{p=-r}^{s} L_p(\delta) \cdot z^{p-m}) \\
&\quad - A \otimes (\overline{L} + \overline{M} \sum_{p=-r}^{s} L_p(\delta) z^{p-m})], \\
T_2(z) &= -z^m[\boldsymbol{e} \otimes L + \boldsymbol{e} \otimes M \sum_{p=-r}^{s} L_p(\delta) z^{p-m}], \\
T_3(z) &= -h\, \boldsymbol{b}^{\mathrm{T}} \otimes I_d \cdot z^{m+1}, \quad T_4(z) = I_d z^{m+1} - I_d z^m, \\
\overline{L} &= hL, \quad \overline{M} = hM.
\end{aligned}$$

Definition 3.1. A numerical method for NDDEs is called NP-stable if and only if for all coefficients L, M and N satisfying (2.1)-(2.2), the numerical solution y_n of (1.3)-(1.4) at the mesh point $t_n = nh$ satisfies

$$\lim_{n\to\infty} y_n = 0,$$

for every stepsize h such that $mh = \tau$, where $m \geq 1$ is a positive integer.

Definition 3.2. A numerical method for NDDEs is called NGP-stable if and only if the numerical solution y_n of (1.3)-(1.4) tends to zero as $n \to \infty$ for every stepsize $h > 0$.

Now we focus on the following polynomial

$$\gamma(z, \delta) = \sum_{p=-r}^{s} L_p(\delta) z^{p+r}.$$

Consider the condition

(3.11) $$|\gamma(z,\delta)| \le 1 \text{ whenever } |z| = 1, 0 \le \delta < 1.$$

From Strang [17] and Iserles and Strang [8], it follows that the condition (3.11) can be characterized in terms of the integer r, s.

Lemma 3.1. *The condition (3.11) is equivalent to the condition* $r \le s \le r+2$*. Moreover, when* $r+s > 0$, $r \le s \le r+2$, $|z| = 1$, $0 \le \delta < 1$*, then* $|\gamma(z,\delta)| = 1$ *if and only if* $z = 1$.

Theorem 3.1. *Assume that the interpolation procedure (3.7)-(3.8) satisfies* $r \le s \le r+2$*. Then the implicit Runge-Kutta method (3.1)-(3.2) for NDDEs is NGP-stable if and only if it is A-stable for ODEs.*

Proof. Assume that an implicit Runge-Kutta method is A-stable. In order to show (3.1)-(3.2) is NGP-stable, we must show that every root z of the characteristic equation (3.10) satisfies $|z| < 1$ for any $\delta \in [0,1)$.

Let
$$R(z,\delta) = \sum_{p=-r}^{s} L_p(\delta) \cdot z^{p-m}.$$

When $|z| = 1$, we get $|R(z,\delta)| \le 1$ for $\delta \in [0,1)$ by Lemma 3.1; when $z = \infty$, we have $|R(\infty,\delta)| = 0$, since $m \ge s+1$. Thus we employ the maximum modulus principle for analytic functions to obtain

(3.12) $$|R(z,\delta)| \le 1 \text{ for } |z| \ge 1, \delta \in [0,1).$$

Noticing the conditions (2.1) and (3.12), we have

(3.13) $$\mathrm{Re}(\lambda_i(Q(z,\delta))) < 0 \ \text{ for all } i, 0 \le \delta < 1,$$

where $Q(z,\delta) = (I_d - NR(z,\delta))^{-1} \cdot (\overline{L} + \overline{M}R(z,\delta))$.

The remaining part of this proof is analogous to that of Theorem in [7] and we omit it here.

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