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A posteriori error estimators for convection-diffusion equations

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Summary. We derive a posteriori error estimators for convection-diffusion equations with dominant convection. The estimators yield global upper and local lower bounds on the error measured in the energy norm such that the ratio of the upper and lower bounds only depends on the local mesh-Peclet number. The estimators are either based on the evaluation of local residuals or on the solution of discrete local Dirichlet or Neumann problems.

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1. Introduction

We consider the convection-diffusion equation

$$
\begin{aligned}
-\varepsilon \Delta u + \underline{a} \cdot \nabla u + bu &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \Gamma_{\mathrm{D}} \\
\varepsilon \frac{\partial u}{\partial n} &= g \quad \text{on } \Gamma_{\mathrm{N}}
\end{aligned}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^n, n \geq 2$, is a bounded polygonal domain with Lipschitz boundary $\Gamma = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$ and $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$. We are interested in the convection dominated case and assume that:

(A1) $0 < \varepsilon << 1$,
(A2) $\underline{a} \in W^{1,\infty}(\Omega)^n, b \in L^\infty(\Omega)$, $\|\underline{a}\|_{L^\infty} + \|b\|_{L^\infty} = O(1)$,
(A3) $-\frac{1}{2}\nabla \cdot \underline{a} + b \geq 1$,
(A4) $\Gamma_- := \{x \in \Gamma : \underline{a}(x) \cdot \underline{n}(x) < 0\} \subset \Gamma_{\mathrm{D}}$.

Our aim is the construction of robust a posteriori error estimators for finite element discretizations (standard Galerkin or SUPG) of this problem. Here, robust means that the error estimators yield global upper and local lower bounds on the error measured in the energy norm

$$|||u||| := \left\{\varepsilon\|\nabla u\|_0^2 + \|u\|_0^2\right\}^{1/2} \tag{1.2}$$

which differ by multiplicative constants which depend at most on the local mesh-Peclet number. (As usual, $\|.\|_0$ refers to the norm of $L^2(\Omega)$.)

This problem is not as imple as it might seem at first sight. Standard approaches as presented in, e.g. [11, 12], for the case $\varepsilon = 1, \underline{a} = 0$, $b = 0$ yield upper and lower bounds which differ by a factor ε^{-1}. More careful estimates within the same approach reduce this gap to the factor $\varepsilon^{-1/2}$. In what follows we will present estimators such that the upper and lower bounds differ by a factor $c + \varepsilon^{-1/2}\|\underline{a}\|_{L^\infty} \min\{h\varepsilon^{-1/2}, 1\}$. Here, c is independent of ε and of any meshsize, and h is the local meshsize. Thus, the estimates are optimal if the local mesh-Peclet number is sufficiently small. This will in particular hold in the critical regions near an interior or a boundary layer.

The main tools in achieving the result are an appropriate trace theorem in deriving upper bounds and a judicious modification of the local cut-off functions of [11, 12] and a sharp estimate for the convection term in establishing lower bounds. The estimators are either based on residuals with weights depending on ε and the local meshsize or on the solution of local discrete Dirichlet or Neumann problems.

Our results should be compared with those of Angermann [2]. He constructs a posteriori error estimators for problem (1.1) which yield upper and lower bounds on the error such that their ratio is bounded independently of h and ε. The error, however, is measured in a norm which is only implicitely defined by an infinite dimensional variational problem. Hence, it can hardly be computed in practice. Moreover, the condition number of this norm with respect to the energy norm (1.2) or to the standard H^1-semi-norm behaves like $O(\varepsilon^{-1/2})$. Our results, on the other hand, hold for the energy norm (1.2) which is much more natural and easy to compute in practice.

The paper is organized as follows. In Sect. 2 we present the variational formulation of (1.1) and its finite element discretization. In Sect. 3 we collect some auxiliary results which are needed for deriving the upper and lower bounds. In Sects. 4 and 5 we present the error estimators and prove their robustness. In Sect. 6 we shortly show how our results extend to slightly nonlinear problems and thus complement the results of [9]. Finally, we present in Sect. 7 two sets of numerical examples which give an impression of the global and local effectivity indices which can be expected from the residual eror estimator of Sect. 4.

In what follows we always use the following *convention*:

$$a \preceq b \iff a \leq cb$$
$$a \simeq b \iff a \preceq b \quad \text{and} \quad b \preceq a.$$

Here, the constant c must be independent of any meshsize and of ε.

2. Finite element discretization

For any bounded open subset ω of Ω with polygonal boundary γ, we denote by $H^k(\omega), k \in \mathbb{N}, L^2(\omega) = H^0(\omega)$, and $L^2(\gamma)$ the usual Sobolev and Lebesgue spaces equipped with the standard norms $\|.\|_{k;\omega} := \|.\|_{H^k(\omega)}$ and $\|.\|_{0;\gamma} := \|.\|_{L^2(\gamma)}$ (cf. [1]). Similarly, $(.,.)_\omega$ and $(.,.)_\gamma$ denote the scalar products of $L^2(\omega)$ and $L^2(\gamma)$, respectively. If $\omega = \Omega$ we will omit the index Ω. On the other hand, $|||.|||_\omega$ denotes the canonical restriction of the energy norm (1.2) on $H^1(\omega)$.

Set

$$H^1_{\mathrm{D}}(\Omega) := \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_{\mathrm{D}}\}.$$

Then the standard variational formulation of problem (1.1) is to find $u \in H^1_{\mathrm{D}}(\Omega)$ such that

$$B(u,v) = (f,v) + (g,v)_{\Gamma_{\mathrm{N}}} \quad \forall v \in H^1_{\mathrm{D}}(\Omega) \tag{2.1}$$

where

$$B(u,v) := \varepsilon(\nabla u, \nabla v) + (\underline{a} \cdot \nabla u, v) + (bu, v). \tag{2.2}$$

Problem (2.1) admits a unique solution. Moreover, assumptions (A1) – (A4) and integration by parts imply that

$$B(v,v) \geq |||v|||^2 \qquad \forall v \in H^1_{\mathrm{D}}(\Omega) \tag{2.3}$$

and

$$\begin{aligned} B(v,w) \leq |||v|||\,|||w|||\{1 + \|b\|_{L^\infty}\} + |||v|||\,\|w\|_0 \varepsilon^{-1/2} \|\underline{a}\|_{L^\infty} \\ \forall v,w \in H^1_{\mathrm{D}}(\Omega). \end{aligned} \tag{2.4}$$

We denote by $\mathcal{T}_h, h > 0$, a family of partitions of Ω into n-simplices which satisfies the following two properties:

(1) *admissibility:* any two elements are either disjoint or share a complete k-face, $0 \leq k \leq n-1$,
(2) *shape regularity:* $\sup\limits_{h>0} \sup\limits_{T \in \mathcal{T}_h} h_T/\rho_T \preceq 1$.

Here, h_T and ρ_T denote the diameter of T and the diameter of the largest ball insribed into T. Note, that the shape regularity allows the use of locally refined meshes and that, in two dimensions, it is equivalent to a minimal angle condition.

For $k \in \mathbb{N}$ we denote by $\mathbb{P}_k$ the set of polynomials of degree at most k and set

$$\begin{aligned} S_h^{k,-1} &:= \{\varphi : \Omega \to \mathbb{R} : \varphi|_T \in \mathbb{P}_k \quad \forall T \in \mathcal{T}_h\}, \\ S_h^{k,0} &:= S_h^{k,-1} \cap C(\overline{\Omega}), \quad k \geq 1, \\ S_{h,D}^{k,0} &:= \{\varphi \in S_h^{k,0} : \varphi = 0 \text{ on } \Gamma_{\mathrm{D}}\}, \quad k \geq 1. \end{aligned} \tag{2.5}$$

We then consider the following discretization of problem (1.1): Find $u_h \in S_{h,D}^{k,0}$ such that

$$B_\delta(u_h, v_h) = l_\delta(u_h) \quad \forall v_h \in S_{h,D}^{k,0} \tag{2.6}$$

where

$$\begin{aligned} B_\delta(u_h, v_h) &:= B(u_h, v_h) \\ &\qquad + \sum_{T \in \mathcal{T}_h} \delta_T(-\varepsilon \Delta u_h + \underline{a} \cdot \nabla u_h + b u_h, \underline{a} \cdot \nabla v_h)_T, \\ l_\delta(v_h) &:= (f, v_h) + (g, v_h)_{\Gamma_{\mathrm{N}}} + \sum_{T \in \mathcal{T}_h} \delta_T(f, \underline{a} \cdot \nabla v_h)_T. \end{aligned} \tag{2.7}$$

Problem (2.6) is the standard Galerkin approximation, if $\delta_T = 0 \quad \forall T \in \mathcal{T}_h$, and the SUPG discretization of [8], if $\delta_T > 0 \quad \forall T \in \mathcal{T}_h$. From assumptions (A1) – (A4) and a local inverse estimate it follows that problem (2.6) admits a unique solution if $\delta_T \preceq h_T^2 \varepsilon^{-1}$ (cf. [8]). In what follows we will always assume that

$$\delta_T \leq h_T \qquad \forall T \in \mathcal{T}_h. \tag{2.8}$$

Finally, we introduce some useful notations. $\mathcal{E}_h$ denotes the set of all $(n-1)$-faces in $\mathcal{T}_h$. It can be split in the form $\mathcal{E}_h = \mathcal{E}_{h,\Omega} \cup \mathcal{E}_{h,\mathrm{N}} \cup \mathcal{E}_{h,\mathrm{D}}$ where $\mathcal{E}_{h,\Omega}, \mathcal{E}_{h,\mathrm{N}}$, and $\mathcal{E}_{h,\mathrm{D}}$ refer to interior faces, faces on the Neumann boundary Γ_{N}, and faces on the Dirichlet boundary Γ_{D}, respectively. For $E \in \mathcal{E}_h$, h_E is the diameter of E. The shape regularity implies that $h_T \simeq h_E$ and $h_T \simeq h_{T'}$, whenever $E \subset \partial T$ and $T \cap T' \neq \emptyset$. For any piecewise continuous function φ and any $E \in \mathcal{E}_{h,\Omega}$, we denote by $[\varphi]_E$ the jump of φ across E in an arbitrary but fixed direction n_E orthogonal to E. The jump $[\varphi]_E$ of course depends on the orientation of n_E, but expressions like $[n_E \cdot \nabla \varphi]_E$ are independent of the orientation of n_E. For any $T \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$ we finally set

$$\omega_T := \bigcup_{\emptyset \neq T \cap T' \in \mathcal{E}_h} T', \quad \widetilde{\omega}_T := \bigcup_{T \cap T' \neq \emptyset} T', \quad \omega_E := \bigcup_{E \subset \partial T'} T'.$$

3. Auxiliary results

Denote by $e_i \in \mathbb{R}^n, 1 \le i \le n$, the i-th unit vector and set $e_{n+1} := 0 \in \mathbb{R}^n$. Let $\hat{T}$ be the standard n-simplex with vertices $e_1, ..., e_{n+1}$ and faces

$$\hat{E}_i := \hat{T} \cap \{x_i = 0\} \quad , 1 \le i \le n,$$
$$\hat{E}_{n+1} := \hat{T} \cap \{|x|_1 = 1\}.$$

Here, $|.|_1$ denotes the standard l_1-norm in $\mathbb{R}^n$.

3.1 Lemma *The following trace inequality holds for all $T \in \mathcal{T}_h, E \subset \partial T$, and $v \in H^1(T)$*

$$\|v\|_{0;E} \preceq h_T^{-1/2}\|v\|_{0;T} + \|v\|_{0;T}^{1/2}\|\nabla v\|_{0;T}^{1/2}.$$

Proof. Enumerate the vertices of T such that the vertices of E numbered first and denote by $\lambda_1, ..., \lambda_{n+1}$ the barycentric co-ordinates of T. Since $\lambda_1 + ... + \lambda_n = 1$ on E we have

$$\|v\|_{0;E} \le \sum_{i=1}^{n} \|\lambda_i v\|_{0;E}.$$

For $1 \le i \le n$ denote by $F_i : \hat{T} \to T$ the affine mapping which maps $\hat{E}_i$ onto E and set

$$\hat{v} := (\lambda_i v) \circ F_i.$$

From Lemma 3.2 in [13] we then know that

$$\|\hat{v}\|_{0;\hat{E}_i} \le \sqrt{2}\|\hat{v}\|_{0;\hat{T}}^{1/2}\|\nabla \hat{v}\|_{0;\hat{T}}^{1/2}.$$

This estimate and standard scaling arguments yield

$$\begin{aligned}
\|\lambda_i v\|_{0;E} &\preceq h_E^{(n-1)/2}\|\hat{v}\|_{0;\hat{E}_i} \\
&\le h_E^{(n-1)/2}\sqrt{2}\|\hat{v}\|_{0;\hat{T}}^{1/2}\|\nabla \hat{v}\|_{0;\hat{T}}^{1/2} \\
&\preceq \|\lambda_i v\|_{0;T}^{1/2}\|\nabla(\lambda_i v)\|_{0;T}^{1/2} \\
&\preceq \|v\|_{0;T}^{1/2}\|\nabla v\|_{0;T}^{1/2} + h_T^{-1/2}\|v\|_{0;T}.
\end{aligned}$$

Since $1 \le i \le n$ was arbitrary, this proves the desired result. □

Denote by $I_h : L^2(\Omega) \to S_{h,\mathrm{D}}^{1,0}$ the quasi-interpolation operator of Clément (cf. [7] and Exercise 3.2.3 in [6]).

3.2 Lemma *The following error estimates hold for all* $T \in \mathcal{T}_h$, $E \subset \partial T$, *and* $v \in H^1(\widetilde{\omega}_T)$

$$\begin{aligned}\|v - I_h v\|_{0;T} &\preceq \min\{h_T \varepsilon^{-1/2}, 1\} |||v|||_{\widetilde{\omega}_T},\\ \|v - I_h v\|_{0;E} &\preceq \varepsilon^{-1/4} \min\{h_T \varepsilon^{-1/2}, 1\}^{1/2} |||v|||_{\widetilde{\omega}_T},\\ |||I_h v|||_T &\preceq |||v|||_{\widetilde{\omega}_T}.\end{aligned}$$

Proof. The Lemma follows from the error estimate

$$\|\nabla^l (v - I_h v)\|_{0;T} \preceq h_T^{k-l} \|\nabla^k v\|_{0;\widetilde{\omega}_T} \qquad \forall 0 \le l \le k \le 1, T \in \mathcal{T}_h, v \in H^k(\widetilde{\omega}_T)$$

(cf. [7] and Exercise 3.2.3 in [6]), definition (1.2) of $|||.|||$, Lemma 3.1, and the obvious estimate

$$\begin{aligned}&h_T^{-1/2} \min\{h_T \varepsilon^{-1/2}, 1\} + \varepsilon^{-1/4} \min\{h_T \varepsilon^{-1/2}, 1\}^{1/2}\\ &\le 2\varepsilon^{-1/4} \min\{h_T \varepsilon^{-1/2}, 1\}^{1/2}.\end{aligned} \tag{3.1}$$

□

Denote by $\hat{\lambda}_i, 1 \le i \le n+1$, the i-th barycentric co-ordinate of $\hat{T}$, i.e., the linear function that takes the value 1 at the vertex e_i and that vanishes identically on the face $\hat{E}_i$. Set

$$\hat{\psi} := (n+1)^{n+1} \prod_{i=1}^{n+1} \hat{\lambda}_i.$$

Given any number $\theta \in (0, 1]$ denote by $\Phi_\theta : \mathbb{R}^n \to \mathbb{R}^n$ the transformation which maps $(x_1, ..., x_n)$ onto $(x_1, ..., x_{n-1}, \theta x_n)$. Let

$$\hat{T}_\theta := \Phi_\theta(\hat{T})$$

and denote by $\hat{\lambda}_{1,\theta}, ..., \hat{\lambda}_{n+1,\theta}$ its barycentric co-ordinates. Set

$$\hat{\psi}_\theta := \begin{cases} n^n \hat{\lambda}_{n+1,\theta} \prod_{i=1}^{n-1} \hat{\lambda}_{i,\theta} & \text{on } \hat{T}_\theta,\\ 0 & \text{on } \hat{T} \backslash \hat{T}_\theta.\end{cases}$$

For an arbitrary simplex $T \in \mathcal{T}_h$ denote by F_T an affine transformation which maps $\hat{T}$ onto T and set

$$\psi_T := \begin{cases} \widehat{\psi} \circ F_T^{-1} & \text{on } T,\\ 0 & \text{on } \Omega \backslash T.\end{cases} \tag{3.2}$$

Let $E \in \mathcal{E}_{h,\Omega}$ and denote by T_1, T_2 the two simplices which have E in common. Denote by $F_{E,i}, i = 1, 2$, the orientation preserving affine transformation which maps $\hat{T}$ onto T_i and $\hat{E}_n$ onto E. Set

$$\psi_{E,\theta} := \begin{cases} \hat{\psi}_\theta \circ F_{E;i}^{-1} & \text{on } T_i, i = 1, 2, \\ 0 & \text{on } \Omega \backslash \omega_E. \end{cases} \tag{3.3}$$

If $E \in \mathcal{E}_{h,\mathrm{D}} \cup \mathcal{E}_{h,\mathrm{N}}$ the function $\psi_{E,\theta}$ is defined in the same way with the obvious modifications.

Finally, we define a continuation operator $P_E : L^\infty(E) \to L^\infty(\omega_E), E \in \mathcal{E}_h$, which maps polynomials onto piecewise polynomials of the same degree. To this end, define $\hat{P}_E : L^\infty(\hat{E}_n) \to L^\infty(\hat{T})$ by

$$\hat{P}_E \hat{\sigma}(x_1, ..., x_{n-1}, x_n) := \hat{\sigma}(x_1, ..., x_{n-1}, 0)$$

and set

$$P_E \sigma|_{T_i} := [\hat{P}_E(\sigma \circ F_{E,i})] \circ F_{E,i}^{-1} \quad , i = 1, 2$$

with the obvious modifications for faces on the boundary Γ.

3.3 Lemma *The following estimates hold for all $v \in \mathbb{P}_k$ and $T \in \mathcal{T}_h$*

$$\begin{aligned} \|v\|_{0;T}^2 &\preceq (v, \psi_T v)_T, \\ \|v\psi_T\|_{0;T} &\leq \|v\|_{0;T}, \\ |||v\psi_T|||_T &\preceq \min\{h_T \varepsilon^{-1/2}, 1\}^{-1} \|v\|_{0;T}. \end{aligned}$$

For $E \in \mathcal{E}_h$, set

$$\theta_E := \min\{\varepsilon^{1/2} h_E^{-1}, 1\}.$$

Then the following estimates hold for all $E \in \mathcal{E}_h$ and $\sigma \in \mathbb{P}_{k|E}$.

$$\begin{aligned} \|\sigma\|_{0;E}^2 &\preceq (\sigma, \psi_{E,\theta_E} P_E \sigma)_E, \\ \|\psi_{E,\theta_E} P_E \sigma\|_{0;\omega_E} &\preceq \varepsilon^{1/4} \min\{1, h_E \varepsilon^{-1/2}\}^{1/2} \|\sigma\|_{0;E}, \\ |||\psi_{E,\theta_E} P_E \sigma|||_{\omega_E} &\preceq \varepsilon^{1/4} \min\{1, h_E \varepsilon^{-1/2}\}^{-1/2} \|\sigma\|_{0;E}. \end{aligned}$$

Proof. The estimates concerning v follow from standard scaling arguments, the equivalence of norms on finite dimensional spaces (cf. Lemma 4.1 in [11]), and the obvious estimate

$$1 + \varepsilon^{1/2} h_T^{-1} \leq 2 \max\{1, \varepsilon^{1/2} h_T^{-1}\} = 2 \min\{1, h_T \varepsilon^{-1/2}\}^{-1}.$$

From Lemma 3.4 in [13] we know that the estimates

$$\begin{aligned} \|\hat\sigma\|^2_{0;\hat E_n} &\preceq (\hat\sigma, \hat\psi_\theta \hat P_E \hat\sigma)_{\hat E_n}, \\ \|\hat\psi_\theta \hat P_E \hat\sigma\|_{0;\hat T} &\preceq \theta^{1/2} \|\hat\sigma\|_{0;\hat E_n}, \\ \|\frac{\partial}{\partial x_i}(\hat\psi_\theta \hat P_E \hat\sigma)\|_{0;\hat T} &\preceq \theta^{1/2} \|\hat\sigma\|_{0;\hat E_n} \quad, 1 \le i \le n-1, \\ \|\frac{\partial}{\partial x_n}(\hat\psi_\theta \hat P_E \hat\sigma)\|_{0;\hat T} &\preceq \theta^{-1/2} \|\hat\sigma\|_{0;E_n} \end{aligned}$$

hold for all $\theta \in (0,1]$ and all $\hat\sigma \in \mathbb{P}_{k|\hat E_n}$. Now, the results concerning σ follow from these estimates, standard scaling arguments, and the obvious inequalities

$$\begin{aligned} h_E^{1/2}\theta_E^{1/2} &= \varepsilon^{1/4} \min\{1, h_E\varepsilon^{-1/2}\}^{1/2}, \\ \varepsilon^{1/2} h_E^{-1/2}\theta_E^{-1/2} &= \varepsilon^{1/4} \min\{1, h_E\varepsilon^{-1/2}\}^{-1/2}. \end{aligned}$$

□

4. A residual error estimator

Recall that u and u_h denote the unique solutions of problems (2.1) and (2.6), respectively. From (2.3) we conclude that

$$|||u - u_h||| \le \sup_{v \in H^1_{\mathrm{D}}(\Omega)\setminus\{0\}} \frac{B(u-u_h, v)}{|||v|||}. \tag{4.1}$$

Consider an arbitrary $v \in H^1_{\mathrm{D}}(\Omega)$ with $|||v||| = 1$. Obviously, we have

$$B(u-u_h, v) = B(u-u_h, v - I_h v) + B(u - u_h, I_h v). \tag{4.2}$$

Integration by parts elementwise yields for all $w \in H^1_{\mathrm{D}}(\Omega)$

$$\begin{aligned} B(u-u_h, w) &= \sum_{T\in\mathcal{T}_h} (f + \varepsilon\Delta u_h - \underline{a}\cdot\nabla u_h - bu_h, w)_T \\ &\quad + \sum_{E\in\mathcal{E}_{h,\Omega}} (-[\varepsilon\partial_{n_E} u_h]_E, w)_E \\ &\quad + \sum_{E\in\mathcal{E}_{h,\mathrm{N}}} (g - \varepsilon\partial_n u_h, w)_E \\ &= \sum_{T\in\mathcal{T}_h} (R_T(u_h), w)_T + \sum_{E\in\mathcal{E}_h} (R_E(u_h), w)_E \end{aligned} \tag{4.3}$$

where

$$R_T(u_h) := f + \varepsilon \Delta u_h - \underline{a} \cdot \nabla u_h - b u_h,$$
$$R_E(u_h) := \begin{cases} -[\varepsilon \partial_{n_E} u_h]_E & \text{if } E \in \mathcal{E}_{h,\Omega}, \\ g - \varepsilon \partial_n u_h & \text{if } E \in \mathcal{E}_{h,\mathrm{N}}, \\ 0 & \text{if } E \in \mathcal{E}_{h,\mathrm{D}}. \end{cases} \tag{4.4}$$

Inserting $w = v - I_h v$ im (4.3), invoking Lemma 3.2, and using Cauchy-Schwarz's inequality we obtain

$$\begin{aligned} B(u-u_h, v - I_h v) \preceq \Big\{ & \sum_{T \in \mathcal{T}_h} \min\{h_T \varepsilon^{-1/2}, 1\}^2 \|R_T(u_h)\|_{0;T}^2 \\ & + \sum_{E \in \mathcal{E}_h} \varepsilon^{-1/2} \min\{h_T \varepsilon^{-1/2}, 1\} \|R_E(u_h)\|_{0;E} \Big\}^{1/2}. \end{aligned} \tag{4.5}$$

From (2.1), (2.2), (2.6), and (2.7) we conclude that

$$B(u-u_h, w_h) = -\sum_{T \in \mathcal{T}_h} \delta_T (R_T(u_h), \underline{a} \cdot \nabla w_h)_T \tag{4.6}$$

holds for all $w_h \in S_{h,\mathrm{D}}^{k,0}$. A simple scaling argument shows for all $w_h \in S_{h,\mathrm{D}}^{k,0}$ that

$$\|\underline{a} \cdot \nabla w_h\|_{0;T} \preceq \|\underline{a}\|_{L^\infty(T)} h_T^{-1} \min\{h_T \varepsilon^{-1/2}, 1\} |||w_h|||_T. \tag{4.7}$$

Equation (4.6), estimate (4.7), Lemma 3.2, assumption (2.8), and Cauchy-Schwarz's inequality yield

$$B(u-u_h, I_h v) \preceq \Big\{ \sum_{T \in \mathcal{T}_h} \min\{h_T \varepsilon^{-1/2}, 1\}^2 \|R_T(u_h)\|_{0;T}^2 \Big\}^{1/2}. \tag{4.8}$$

From (4.1), (4.2), (4.5), and (4.8) we obtain an upper bound for the energy norm of the error:

$$\begin{aligned} |||u-u_h||| \preceq \Big\{ & \sum_{T \in \mathcal{T}_h} \min\{h_T \varepsilon^{-1/2}, 1\}^2 \|R_T(u_h)\|_{0;T}^2 \\ & + \sum_{E \in \mathcal{E}_h} \varepsilon^{-1/2} \min\{h_E \varepsilon^{-1/2}, 1\} \|R_E(u_h)\|_{0;E}^2 \Big\}^{1/2}. \end{aligned} \tag{4.9}$$

In order to derive lower bounds for the error, denote by f_h and g_h arbitrary approximations of f and g by piecewise polynomials of degree at most k with respect to $\mathcal{T}_h$ and with respect to the partition of Γ induced by $\mathcal{T}_h$, respectively. First, consider an arbitrary $T \in \mathcal{T}_h$ and set

$$w_T := \psi_T [f_h + \varepsilon \Delta u_h - \underline{a} \cdot \nabla u_h - b u_h]. \tag{4.10}$$

Inserting w_T in (4.3), we obtain

$$(4.11)\qquad \begin{aligned}&(f_h + \varepsilon\Delta u_h - \underline{a}\cdot\nabla u_h - bu_h, w_T)_T\\ &= B(u-u_h, w_T) + (f_h - f, w_T)_T.\end{aligned}$$

Estimate (2.4) and Lemma 3.3 imply that

$$\begin{aligned}&B(u-u_h, w_T)\\ &\le |||u-u_h|||_T\Big\{(1+\|b\|_{L^\infty(T)})|||w_T|||_T + \varepsilon^{-1/2}\|\underline{a}\|_{L^\infty(T)}\|w_T\|_{0;T}\Big\}\\ &\preceq |||u-u_h|||_T\Big\{(1+\|b\|_{L^\infty(T)})\min\{h_T\varepsilon^{-1/2},1\}^{-1}\\ (4.12)\qquad &\qquad + \varepsilon^{-1/2}\|\underline{a}\|_{L^\infty(T)}\Big\}\|f_h + \varepsilon\Delta u_h - \underline{a}\cdot\nabla u_h - bu_h\|_{0;T}.\end{aligned}$$

From (4.10) – (4.12) and Lemma 3.3 we immediately get the lower bound

$$\begin{aligned}&\min\{h_T\varepsilon^{-1/2},1\}\|f_h + \varepsilon\Delta u_h - \underline{a}\cdot\nabla u_h - bu_h\|_{0;T}\\ &\quad\preceq |||u-u_h|||_T\Big\{1+\|b\|_{L^\infty(T)} + \varepsilon^{-1/2}\|\underline{a}\|_{L^\infty(T)}\min\{h_T\varepsilon^{-1/2},1\}\Big\}\\ (4.13)\qquad &\quad + \min\{h_T\varepsilon^{-1/2},1\}\|f-f_h\|_{0;T}.\end{aligned}$$

Next, consider an arbitrary $E\in\mathcal{E}_{h,\Omega}$ and set

$$(4.14)\qquad w_E := \psi_{E,\theta_E}P_E(-[\varepsilon\partial_{n_E}u_h]_E)$$

where

$$\theta_E = \min\{\varepsilon^{1/2}h_E^{-1},1\}.$$

Inserting w_E in (4.3), we obtain

$$\begin{aligned}&(-[\varepsilon\partial_{n_E}u_h]_E, w_E)_E\\ &\quad= B(u-u_h,w_E) - \sum_{T\subset\omega_E}(f_h + \varepsilon\Delta u_h - \underline{a}\cdot\nabla u_h - bu_h, w_E)_T\\ (4.15)\qquad &\qquad - \sum_{T\subset\omega_E}(f-f_h,w_E)_T.\end{aligned}$$

Estimate (2.4) and Lemma 3.3 now imply that

$$\begin{aligned}&B(u-u_h,w_E)\\ &\quad\le |||u-u_h|||_{\omega_E}\{(1+\|b\|_{L^\infty(\omega_E)})|||w_E|||_{\omega_E}\\ &\qquad\qquad + \varepsilon^{-1/2}\|\underline{a}\|_{L^\infty(\omega_E)}\|w_E\|_{0;\omega_E}\Big\}\\ &\quad\preceq |||u-u_h|||_{\omega_E}\Big\{(1+\|b\|_{L^\infty(\omega_E)})\varepsilon^{1/4}\min\{h_E\varepsilon^{-1/2},1\}^{-1/2}\\ &\qquad\qquad + \varepsilon^{-1/2}\|\underline{a}\|_{L^\infty(\omega_E)}\varepsilon^{1/4}\min\{h_E\varepsilon^{-1/2},1\}^{1/2}\Big\}\\ (4.16)\qquad &\quad\cdot\|-[\varepsilon\partial_{n_E}u_h]_E\|_{0;E}.\end{aligned}$$

Moreover, Lemma 3.3 and estimate (4.13) yield

$$\begin{aligned}&\sum_{T\subset\omega_E}(f_h+\varepsilon\Delta u_h-\underline{a}\cdot\nabla u_h-bu_h,w_E)_T\\ &\preceq\Big\{|||u-u_h|||_{\omega_E}\{1+\|b\|_{L^\infty(\omega_E)}+\varepsilon^{-1/2}\|\underline{a}\|_{L^\infty(\omega_E)}\min\{h_E\varepsilon^{-1/2},1\}\}\\ &\qquad\cdot\varepsilon^{1/4}\min\{h_E\varepsilon^{-1/2},1\}^{-1/2}\\ &\quad+\|f-f_h\|_{0;\omega_E}\varepsilon^{1/4}\min\{h_E\varepsilon^{-1/2},1\}^{1/2}\Big\}\|-[\varepsilon\partial_{n_E}u_h]_E\|_{0;E}.\end{aligned} \tag{4.17}$$

From (4.15) – (4.17) and Lemma 3.3 we immediately get the lower bound

$$\begin{aligned}&\varepsilon^{-1/4}\min\{h_E\varepsilon^{-1/2},1\}^{1/2}\|-[\varepsilon\partial_{n_E}u_h]_E\|_{0;E}\\ &\preceq|||u-u_h|||_{\omega_E}\Big\{1+\|b\|_{L^\infty(\omega_E)}+\varepsilon^{-1/2}\|\underline{a}\|_{L^\infty(\omega_E)}\min\{h_E\varepsilon^{-1/2},1\}\Big\}\\ &\quad+\min\{h_E\varepsilon^{-1/2},1\}\|f-f_h\|_{0;\omega_E}.\end{aligned} \tag{4.18}$$

With the same arguments we obtain the lower bound

$$\begin{aligned}&\varepsilon^{-1/4}\min\{h_E\varepsilon^{-1/2},1\}^{1/2}\|g_h-\varepsilon\partial_n u_h\|_{0;E}\\ &\quad\preceq|||u-u_h|||_T\Big\{1+\|b\|_{L^\infty(T)}+\varepsilon^{-1/2}\|\underline{a}\|_{L^\infty(T)}\min\{h_E\varepsilon^{-1/2},1\}\Big\}\\ &\qquad+\min\{h_E\varepsilon^{-1/2},1\}\|f-f_h\|_{0;T}\\ &\qquad+\varepsilon^{-1/4}\min\{h_E\varepsilon^{-1/2},1\}^{1/2}\|g-g_h\|_{0;E}.\end{aligned} \tag{4.19}$$

for all $E\in\mathcal{E}_{h,\mathrm{N}}$, where $T=\omega_E$. Thus, we have established the following a posteriori error estimate.

4.1 Proposition *Denote by u and u_h the unique solutions of problems (2.1) and (2.6), respectively. Let f_h and g_h be arbitrary approximations of f and g by piecewise polynomials of degree atmost k with respect to $\mathcal{T}_h$ and with respect to the partition of Γ induced by $\mathcal{T}_h$, respectively. Set*

$$\alpha_S:=\min\{h_S\varepsilon^{-1/2},1\},\quad S\in\mathcal{T}_h\cup\mathcal{E}_h$$

and

$$\begin{aligned}\eta_{\mathrm{R},T}^2:=&\ \alpha_T^2\|f_h+\varepsilon\Delta u_h-\underline{a}\cdot\nabla u_h-bu_h\|_{0;T}^2\\ &+\frac{1}{2}\sum_{E\subset\partial T\cap\Omega}\varepsilon^{-1/2}\alpha_E\|[\varepsilon\partial_{n_E}u_h]_E\|_{0;E}^2\\ &+\sum_{E\subset\partial T\cap\Gamma_{\mathrm{N}}}\varepsilon^{-1/2}\alpha_E\|g_h-\varepsilon\partial_{n_E}u_h\|_{0;E}^2.\end{aligned}$$

Then the following a posteriori error estimates are valid

$$
\begin{aligned}
&|||u-u_h||| \\
&\quad\preceq \Big\{\sum_{T\in\mathcal{T}_h}\eta_{\mathrm{R},T}^2\Big\}^{1/2} \\
&\qquad+\Big\{\sum_{T\in\mathcal{T}_h}\alpha_T^2\|f-f_h\|_{0;T}^2+\sum_{E\in\mathcal{E}_{h,\mathrm{N}}}\varepsilon^{-1/2}\alpha_E\|g-g_h\|_{0;E}^2\Big\}^{1/2}, \\
&\eta_{\mathrm{R},T} \\
&\quad\preceq \Big\{1+\|b\|_{L^\infty(\omega_T)}+\varepsilon^{-1/2}\|\underline{a}\|_{L^\infty(\omega_T)}\alpha_T\Big\}|||u-u_h|||_{\omega_T} \\
&\qquad+\alpha_T\|f-f_h\|_{0;\omega_T}+\Big\{\sum_{E\subset\partial T\cap\Gamma_\mathrm{N}}\varepsilon^{-1/2}\alpha_E\|g-g_h\|_{0;E}^2\Big\}^{1/2}, \\
&\Big\{\sum_{T\in T}\eta_{\mathrm{R},T}^2\Big\}^{1/2} \\
&\quad\preceq \Big\{1+\|b\|_{L^\infty(\Omega)}+\max_{T\in\mathcal{T}_h}\{\varepsilon^{-1/2}\|\underline{a}\|_{L^\infty(\omega_T)}\alpha_T\}\Big\}|||u-u_h||| \\
&\qquad+\Big\{\sum_{T\in\mathcal{T}_h}\alpha_T^2\|f-f_h\|_{0;T}^2+\sum_{E\subset\mathcal{E}_{h,\mathrm{N}}}\varepsilon^{-1/2}\alpha_E\|g-g_h\|_{0;E}^2\Big\}^{1/2}.
\end{aligned}
$$

4.2 Remark In the upper bounds for $|||u-u_h|||$ one may replace f_h and g_h by f and g respectively. The second term on the right-hand side of the first estimate of Proposition 4.1 then of course vanishes.

5. Error estimators based on the solution of auxiliary local problems

We begin with an error estimator which is based on the solution of local Dirichlet problems and which is similar to the estimator of Babuška-Rheinboldt [3] for the Laplace equation. Given $T\in\mathcal{T}_h$, we set

$$
\begin{aligned}
V_T := \mathrm{span}\{&\psi_{T'}v,\psi_{E,\theta_E}P_E\sigma: \\
&T'\subset\omega_T, E\subset\partial T\backslash\Gamma_\mathrm{D}, v\in\mathbb{P}_k,\sigma\in\mathbb{P}_{k|E}\}
\end{aligned}
$$

where the functions $\psi_{T'}$ and ψ_{E,θ_E} are as in equations (3.2) and (3.3) and where $\theta_E:=\min\{\varepsilon^{1/2}h_E^{-1},1\}$. Denote by $v_T\in V_T$ the unique solution of

$$
\begin{aligned}
&\varepsilon(\nabla v_T,\nabla w)_{\omega_T}+(\underline{a}\cdot\nabla v_T,w)_{\omega_T}+(bv_T,w)_{\omega_T} \\
&\quad=(f,w)_{\omega_T}+(g,w)_{\partial T\cap\Gamma_\mathrm{N}}-\varepsilon(\nabla u_h,\nabla w)_{\omega_T} \\
&\qquad-(\underline{a}\cdot\nabla u_h,w)_{\omega_T}-(bu_h,w)_{\omega_T}\quad\forall w\in V_T
\end{aligned}
\tag{5.1}
$$

and set

$$\eta_{\mathrm{D},T} := |||v_T|||_{\omega_T}. \tag{5.2}$$

For practical computations, one will replace f and g by finite element approximations f_h and g_h as in the previous section. The function $u_h + v_T$ is a finite element approximation to the solution u_T of the local convection-diffusion problem

$$\begin{aligned} -\varepsilon \Delta u_T + \underline{a} \cdot \nabla u_T + b u_T = f &\quad \text{in } \omega_T \\ u_T = u_h &\quad \text{on } \partial\omega_T \setminus (\Gamma_{\mathrm{N}} \cap \partial T) \\ \varepsilon \partial_n u_T = g &\quad \text{on } \partial T \cap \Gamma_{\mathrm{N}}. \end{aligned}$$

Since the right-hand side of equation (5.1) equals

$$\varepsilon(\nabla(u-u_h), \nabla w)_{\omega_T} + (\underline{a}\cdot\nabla(u-u_h), w)_{\omega_T} + (b(u-u_h), w)_{\omega_T},$$

we obtain from inequality (2.4) the estimate

$$\begin{aligned} |||v_T|||^2_{\omega_T} &\le \{1 + \|b\|_{L^\infty(\omega_T)}\} |||u-u_h|||_{\omega_T} |||v_T|||_{\omega_T} \\ &\quad + \varepsilon^{-1/2} \|\underline{a}\|_{L^\infty(\omega_T)} |||u-u_h|||_{\omega_T} \|v_T\|_{0;\omega_T} \\ &\le \{1 + \|b\|_{L^\infty(\omega_T)} + \varepsilon^{-1/2}\|\underline{a}\|_{L^\infty(\omega_T)}\} |||u-u_h|||_{\omega_T} |||v_T|||_{\omega_T} \end{aligned}$$

and hence

$$\eta_{\mathrm{D},T} \le \{1 + \|b\|_{L^\infty(\omega_T)} + \varepsilon^{-1/2}\|\underline{a}\|_{L^\infty(\omega_T)}\} |||u-u_h|||_{\omega_T}.$$

This estimate is not satisfactory. In order to improve it, we observe that

$$\|w\|_{0;T'} \preceq h_{T'} \|\nabla w\|_{0;T'} \tag{5.3}$$

holds for all $T' \subset \omega_T$ and $w \in V_T$ since the functions in V_T vanish at the vertices of T. Hence, we have

$$\|v_T\|_{0;\omega_T} \preceq \min\{h_T \varepsilon^{-1/2}, 1\} |||v_T|||_{\omega_T}. \tag{5.4}$$

This yields the improved lower bound

$$\eta_{\mathrm{D},T} \preceq \{1 + \|b\|_{L^\infty(\omega_T)} + \varepsilon^{-1/2} \min\{h_T\varepsilon^{-1/2}, 1\} \|\underline{a}\|_{L^\infty(\omega_T)}\} |||u-u_h|||_{\omega_T}.$$

This estimate is of the same quality as the corresponding bound for $\eta_{\mathrm{R},T}$ of Proposition 4.1. In order to show that $\eta_{\mathrm{D},T}$ also yields upper bounds for the error, we compare it with the residual estimator $\eta_{\mathrm{R},T}$. Integration by parts elementwise of the right-hand side of equation (5.1) yields

$$\begin{aligned} &\varepsilon(\nabla v_T, \nabla w)_{\omega_T} + (\underline{a}\cdot\nabla v_T, w)_{\omega_T} + (b v_T, w)_{\omega_T} \\ &\quad = \sum_{T' \subset \omega_T} (f + \varepsilon \Delta u_h - \underline{a}\cdot\nabla u_h - b u_h, w)_{T'} \\ &\quad\quad + \sum_{E \subset \partial T \cap \Omega} (-\varepsilon [\partial_{n_E} u_h]_E, w)_E + \sum_{E \subset \partial T \cap \Gamma_{\mathrm{N}}} (g - \varepsilon \partial_n u_h, w)_E. \end{aligned} \tag{5.5}$$

Moreover, inequality (5.3), Lemma 3.1, and estimate (3.1) imply that

$$\|w\|_{0;E} \preceq \varepsilon^{-1/4} \min\{h_E \varepsilon^{-1/2}, 1\}^{1/2} |||w|||_{\omega_E} \quad \forall E \subset \partial T \backslash \Gamma_{\mathrm{D}}, w \in V_T. \tag{5.6}$$

Equation (5.5), estimates (5.4) and (5.6), and Cauchy-Schwarz's inequality immediately yield the estimate

$$\eta_{\mathrm{D},T} \preceq \Big\{ \sum_{T' \subset \omega_T} \eta_{\mathrm{R},T}^2 + \sum_{T' \subset \omega_T} \min\{h_{T'} \varepsilon^{-1/2}, 1\}^2 \|f - f_h\|_{0;T'}^2 + \sum_{E \subset \partial T \cap \Gamma_{\mathrm{N}}} \varepsilon^{-1/2} \min\{h_E \varepsilon^{-1/2}, 1\} \|g - g_h\|_{0;E}^2 \Big\}^{1/2},$$

where $\eta_{\mathrm{R},T}, f_h$, and g_h are as in Proposition 4.1. On the other hand, the functions w_T and w_E of equations (4.10) and (4.14) are contained in V_T. Hence, the proofs of estimates (4.13) and (4.18) yield the converse estimate

$$\eta_{\mathrm{R},T} \preceq \eta_{\mathrm{D},T} + \Big\{ \sum_{T' \subset \omega_T} \min\{h_{T'} \varepsilon^{-1/2}, 1\}^2 \|f - f_h\|_{0;T'}^2 + \sum_{E \subset \partial T \cap \Gamma_{\mathrm{N}}} \varepsilon^{-1/2} \min\{h_E \varepsilon^{-1/2}, 1\} \|g - g_h\|_{0;E}^2 \Big\}^{1/2}.$$

Thus, we have established the following result.

5.1 Proposition *Denote by u und u_h the unique solutions of problems (2.1) and (2.6), respectively. Let $\eta_{\mathrm{R},T}, f_h$, and g_h be as in Proposition 4.1 and $\eta_{\mathrm{D},T}$ be given by equation (5.2). Then the following estimates hold*

$$\eta_{\mathrm{R},T} \preceq \eta_{\mathrm{D},T} + \Big\{ \sum_{T' \subset \omega_T} \alpha_{T'}^2 \|f - f_h\|_{0;T'}^2 + \sum_{E \subset \partial T \cap \Gamma_{\mathrm{N}}} \varepsilon^{-1/2} \alpha_E \|g - g_h\|_{0;E}^2 \Big\}^{1/2},$$

$$\eta_{\mathrm{D},T} \preceq \Big\{ \sum_{T' \subset \omega_T} \eta_{R,T}^2 \Big\}^{1/2} + \Big\{ \sum_{T' \subset \omega_T} \alpha_{T'}^2 \|f - f_h\|_{0;T'}^2 + \sum_{E \subset \partial T \cap \Gamma_{\mathrm{N}}} \varepsilon^{-1/2} \alpha_E \|g - g_h\|_{0;E}^2 \Big\}^{1/2},$$

$$|||u - u_h||| \preceq \Big\{ \sum_{T \in \mathcal{T}_h} \eta_{\mathrm{D},T}^2 + \sum_{T \in \mathcal{T}_h} \alpha_T^2 \|f - f_h\|_{0;T}^2 + \sum_{E \in \mathcal{E}_{h,N}} \varepsilon^{-1/2} \alpha_E \|g - g_h\|_{0;E}^2 \Big\}^{1/2},$$

$$\eta_{\mathrm{D},T} \preceq \{1 + \|b\|_{L^\infty(\omega_T)} + \varepsilon^{-1/2}\alpha_T\|\underline{a}\|_{L^\infty(\omega_T)}\}|||u - u_h|||_{\omega_T},$$

$$\Big\{\sum_{T\in\mathcal{T}_h} \eta_{D,T}^2\Big\}^{1/2} \preceq \max_{T\in\mathcal{T}_h}\{1 + \|b\|_{L^\infty(\omega_T)} + \varepsilon^{-1/2}\alpha_T\|\underline{a}\|_{L^\infty(\omega_T)}\} \cdot|||u - u_h|||,$$

where

$$\alpha_S := \min\{h_S\varepsilon^{-1/2}, 1\}, \quad S \in \mathcal{T}_h \cup \mathcal{E}_h.$$

5.2 Remark The second terms on the right-hand sides of the first and second estimate can be omitted if in equation (5.1) f and g are replaced by f_h and g_h, respectively. Proposition 5.1 also holds if the convection term $(\underline{a}\cdot\nabla v_T, w)_{\omega_T}$ is omitted on the left-hand side of equation (5.1). Similarly, a stabilization term

$$\sum_{T'\subset\omega_T} \delta_{T'}(f + \varepsilon\Delta v_T - \underline{a}\cdot\nabla v_T - bv_T, \underline{a}\cdot\nabla w)_T,$$

with $\delta_{T'} \le \min\{h_{T'}\varepsilon^{-1/2}, 1\}$ may be added on the left-hand side of equation (5.1) if

$$\Big\{\sum_{T'\subset\omega_T} \delta_{T'}^2\|\underline{a}\cdot\nabla v_T\|_{0;T'}^2\Big\}^{1/2}$$

is added on the right-hand side of equation (5.2). Note, that the functions ψ_{E,θ_E} introduce an extra scaling factor in Problem (5.1).

For completeness we also consider an error estimator which is based on the solution of local Neumann problems and which is similar to the estimator of Bank-Weiser [4] for the Laplace equation. For $T \in \mathcal{T}_h$, let

$$\widetilde{V}_T := \mathrm{span}\{\psi_T v, \psi_{E,\theta_E} P_E\sigma : E \subset \partial T\backslash\Gamma_{\mathrm{D}}, v \in \mathbb{P}_k, \sigma \in \mathbb{P}_{k|E}\},$$

denote by $\widetilde{v}_T \in \widetilde{V}_T$ the unique solution of

(5.7)
$$\begin{aligned}&\varepsilon(\nabla\widetilde{v}_T, \nabla w)_T + (\underline{a}\cdot\nabla\widetilde{v}_T, w)_T + (b\widetilde{v}_T, w)_T\\ &\quad = (R_T(u_h), w)_T + \sum_{E\subset\partial T}(R_E(u_h), w)_E \quad \forall w \in \widetilde{V}_T\end{aligned}$$

and set

(5.8)
$$\eta_{\mathrm{N},T} := |||\widetilde{v}_T|||_T.$$

Recall, that $R_T(u_h)$ and $R_E(u_h)$ are given by equation (4.4). The function $\tilde{v}_T$ is a finite element approximation to the solution $\tilde{u}_T$ of the local convection-diffusion equation

$$\begin{aligned}-\varepsilon\Delta\tilde{u}_T + \underline{a}\cdot\nabla\tilde{u}_T + b\tilde{u}_T &= R_T(u_h) && \text{in } T\\ \varepsilon\partial_{n_T}\tilde{u}_T &= R_E(u_h) && \text{on } \partial T\backslash\Gamma_{\mathrm{D}}\\ \tilde{u}_T &= 0 && \text{on } \partial T\cap\Gamma_{\mathrm{D}}.\end{aligned}$$

Here, n_T is the unit exterior normal of T. The analysis of $\eta_{D,T}$ immediately carries over to $\eta_{N,T}$ and yields:

5.3 Proposition *Denote by u und u_h the unique solutions of problems (2.1) and (2.6), respectively. Let $\eta_{R,T}$, f_h, and g_h be as in Proposition 4.1 and $\eta_{N,T}$ be given by equation (5.8). Then the following estimates hold*

$$\begin{aligned}
\eta_{\mathrm{R},T} \preceq \eta_{\mathrm{N},T} + \Big\{ \sum_{T' \subset \omega_T} \alpha_{T'}^2 \|f - f_h\|_{0;T'}^2 \\
+ \sum_{E \subset \partial T \cap \Gamma_{\mathrm{N}}} \varepsilon^{-1/2} \alpha_E \|g - g_h\|_{0;E}^2 \Big\}^{1/2}, \\
\eta_{\mathrm{N},T} \preceq \eta_{\mathrm{R},T} + \Big\{ \sum_{T' \subset \omega_T} \alpha_{T'}^2 \|f - f_h\|_{0;T'}^2 \\
+ \sum_{E \subset \partial T \cap \Gamma_{\mathrm{N}}} \varepsilon^{-1/2} \alpha_E \|g - g_h\|_{0;E}^2 \Big\}^{1/2}, \\
|||u - u_h||| \preceq \Big\{ \sum_{T \in \mathcal{T}_h} \eta_{\mathrm{N},T}^2 + \sum_{T \in \mathcal{T}_h} \alpha_T^2 \|f - f_h\|_{0;T}^2 \\
+ \sum_{E \in \mathcal{E}_{h,\mathrm{N}}} \varepsilon^{-1/2} \alpha_E \|g - g_h\|_{0;E}^2 \Big\}^{1/2}, \\
\eta_{\mathrm{N},T} \preceq \{1 + \|b\|_{L^\infty(\omega_T)} + \varepsilon^{-1/2} \alpha_T \|\underline{a}\|_{L^\infty(\omega_T)}\} |||u - u_h|||_T, \\
\Big\{ \sum_{T \in \mathcal{T}_h} \eta_{\mathrm{N},T}^2 \Big\}^{1/2} \preceq \max_{T \in \mathcal{T}_h} \{1 + \|b\|_{L^\infty(\omega_T)} + \varepsilon^{-1/2} \alpha_T \|\underline{a}\|_{L^\infty(\omega_T)}\} \\
\cdot |||u - u_h|||,
\end{aligned}$$

where

$$\alpha_S := \min\{h_S \varepsilon^{-1/2}, 1\}, \quad S \in \mathcal{T}_h \cup \mathcal{E}_h.$$

With the obvious modifications, Remark 5.2 applies to $\eta_{\mathrm{N},T}$, too.

6. Semilinear convection diffusion equations

In this section we consider the following semilinear analogue of equation (1.1)

$$\begin{aligned}
-\varepsilon \Delta u + \underline{a} \cdot \nabla u + bu = F(x, u) \quad & \text{in } \Omega, \\
u = 0 \quad & \text{on } \Gamma_{\mathrm{D}}, \\
\varepsilon \frac{\partial u}{\partial n} = g \quad & \text{on } \Gamma_{\mathrm{N}}.
\end{aligned} \tag{6.1}$$

We retain the assumptions of Sect. 1 concerning $\Omega, \Gamma_{\mathrm{D}}, \Gamma_{\mathrm{N}}, \varepsilon, \underline{a}$, and b. In addition, we assume that F is continuously differentiable with respect to u and satisfies the following assumptions:

(A5) There is a constant β with $0 \le \beta < 1$ and

$$\max\Big\{\frac{\partial}{\partial s}F(x,s),0\Big\} \le \beta \quad \forall x \in \Omega, s \in \mathbb{R}.$$

(A6) There is a constant $\gamma > 0$ with

$$\Big|\frac{\partial}{\partial s}F(x,s)\Big| \le \gamma(1+s^2) \quad \forall x \in \Omega, s \in \mathbb{R}.$$

These conditions are satisfied if, e.g., F is given by $F(x,s) = f(x) - s^3$ with $f \in L^2(\Omega)$. This is precisely the example which is considered in [9].

The weak formulation of problem (6.1) is obtained by replacing in equation (2.1) f by $F(.,u)$. Similarly, the finite element discretization of problem (6.1) is given by replacing in equation (2.7) f by $F(.,u_h)$.

Let u and u_h be solutions of the weak formulation of problem (6.1) and of its finite element discretization. We then have for all $v \in H^1_{\mathrm{D}}(\Omega)$

$$\begin{aligned} B(u-u_h,v) &= (F(.,u),v) - B(u_h,v) \\ &= (F(.,u) - F(.,u_h),v) - B(u_h,v) + (F(.,u_h),v) \\ &= \int_0^1 \Big(\frac{\partial F}{\partial s}(.,u_h + t(u-u_h))(u-u_h),v\Big)dt \\ &\quad -B(u_h,v) + (F(.,u_h),v). \end{aligned} \tag{6.2}$$

The results of Sect. 4 with f replaced by $F(.,u_h)$ provide us with robust and computable residual a posteriori error estimates for the residual $r(u_h)$ which is implicitely defined by

$$< r(u_h), v >:= -B(u_h,v) + (F(.,u_h),v) \quad \forall v \in H^1_{\mathrm{D}}(\Omega).$$

Here, $< .,. >$ denotes the duality pairing of $H^1_{\mathrm{D}}(\Omega)$ with its dual space. In order to obtain similar a posteriori error estimates for the non-linear problem (6.1) we must therefore balance the first term on the right-hand side of equation (6.2) which is a measure for the non-linearity.

Inserting $v := u - u_h$ in equation (6.2), recalling inequality (2.3), and invoking assumption (A5), we obtain

$$\begin{aligned} &|||u-u_h|||^2 \\ &\quad \le B(u-u_h,u-u_h) \\ &\quad \le \sup_{x\in\Omega, s\in\mathbb{R}} \max\Big\{\frac{\partial F}{\partial s}(x,s),0\Big\}\|u-u_h\|_0^2 \\ &\qquad + |||u-u_h||| \sup_{v\in H^1_{\mathrm{D}}(\Omega)\setminus\{0\}} \frac{-B(u_h,v)+(F(.,u_h),v)}{|||v|||} \\ &\quad \le \beta|||u-u_h|||^2 + |||u-u_h||| \sup_{v\in H^1_{\mathrm{D}}(\Omega)\setminus\{0\}} \frac{-B(u_h,v)+(F(.,u_h),v)}{|||v|||}. \end{aligned}$$

Hence, Proposition 4.1 yields the upper bound

$$|||u-u_h||| \preceq \Big\{\sum_{T\in\mathcal{T}_h}\eta_{\mathrm{R},T}^2\Big\}^{1/2}$$

where $\eta_{\mathrm{R},T}$ is defined as in Proposition 4.1 with f replaced by $F(.,u_h)$.

In order to obtain lower bounds on the error we observe that assumption (A6) and Hölder's inequality imply for all $v\in H^1_{\mathrm{D}}(\Omega)$ the estimate

$$\begin{aligned}
&\Big|\int_0^1\Big(\frac{\partial F}{\partial s}(.,u_h+t(u-u_h))(u-u_h),v\Big)dt\Big| \\
&\quad\le \int_\Omega\int_0^1\gamma\Big\{1+[(1-t)u_h+tu]^2\Big\}|u-u_h||v|dtdx \\
&\quad\le \frac{1}{2}\int_\Omega\gamma\{2+|u_h|^2+|u|^2\}|u-u_h||v|dx \\
&\quad\preceq \|u-u_h\|_0\|v\|_0 \\
&\qquad+\Big\{\|u_h\|^2_{L^4(\Omega)}+\|u\|^2_{L^4(\Omega)}\Big\}\|u-u_h\|_{L^4(\Omega)}\|v\|_{L^4(\Omega)}.
\end{aligned}\tag{6.3}$$

On the other hand we know from Chapter III Lemma 3.3 and Lemma 3.5 in [10] that

$$\|v\|_{L^4(\Omega)}\le 2^{\frac{d-1}{4}}\|v\|_0^{\frac{1}{2d-2}}\|\nabla v\|_0^{\frac{2d-3}{2d-2}}\le 2^{\frac{d-1}{4}}\varepsilon^{-\frac{2d-3}{2(2d-2)}}|||v|||\tag{6.4}$$

holds for $d\in\{2,3\}$ and $v\in H^1_{\mathrm{D}}(\Omega)$. Let $\tilde v$ be any of the functions w_T and w_E of equations (4.10), (4.14) with f_h replaced by some approximation $F_h(.,u_h)$ of $F(.,u_h)$. Inequalities (6.3), (6.4) and a standard inverse estimate for $\|\nabla\tilde v\|_0$ then imply that

$$\begin{aligned}
&\Big|\int_0^1\Big(\frac{\partial F}{\partial s}(.,u_h+t(u-u_h))(u-u_h),\tilde v\Big)dt\Big| \\
&\quad\preceq |||u-u_h|||_{\mathrm{supp}(\tilde v)}\|\tilde v\|_{0;\mathrm{supp}(\tilde v)} \\
&\qquad+\{|||u|||^2+|||u_h|||^2\}|||u-u_h|||_{\mathrm{supp}(\tilde v)}\|\tilde v\|_{0;\mathrm{supp}(\tilde v)}(\varepsilon^{3/2}h_T)^{-\frac{2d-3}{2d-2}}.
\end{aligned}$$

This estimate, Lemma 3.3 and Proposition 4.1 yield the lower bound

$$\begin{aligned}
&\eta_{\mathrm{R},T} \\
&\quad\preceq\Big\{1+\|b\|_{L^\infty(\omega_T)}+\varepsilon^{-1/2}\|\underline{a}\|_{L^\infty(\omega_T)}\min\{h_T\varepsilon^{-1/2},1\} \\
&\qquad+(\varepsilon^{3/2}h_T)^{-\frac{2d-3}{2d-2}}[|||u|||^2+|||u_h|||^2]\min\{h_T\varepsilon^{-1/2},1\}\Big\}|||u-u_h|||_{\omega_T} \\
&\qquad+\Big\{\sum_{T'\subset\omega_T}\min\{h_{T'}\varepsilon^{-1/2},1\}^2\|F(.,u_h)-F_h(u_h)\|^2_{0;T'} \\
&\qquad+\sum_{E\subset\partial T\cap\Gamma_{\mathrm{N}}}\varepsilon^{-1/2}\min\{h_E\varepsilon^{-1/2},1\}\|g-g_h\|^2_{0;E}\Big\}^{1/2}.
\end{aligned}$$

Here, $\eta_{\mathrm{R},T}$ is as in Proposition 4.1 with f_h replaced by some finite element approximation $F_h(.,u_h)$ of $F(.,u_h)$; g_h is as in Proposition 4.1. If in particular $F(x,s) = f(x) - s^3$ as in [9], we may choose $F_h(x,u_h) = f_h(x) - u_h^3$ with f_h as before. Thus our results complement those of [9], where the dependence of the constants on ε is not made explicit and where error estimates are only given for "sufficiently small h" without quantifying this notion.

7. Numerical examples

In order to get an impression of the local and global effectivity indices which can be expected from the residual error estimator $\eta_{\mathrm{R},T}$ we consider two sets of examples. Both have the following data in common

$$\Omega = (0,1)^2, \underline{a} = (2,1), b = 1, f = 0, \varepsilon \in \{1, 10^{-2}, 10^{-4}, 10^{-10}\}.$$

The first set of examples, to which we will refer as *Problem* N, has homogeneous Dirichlet boundary conditions on the left vertical edge of Ω, constant Dirichlet boundary conditions 1 on the lower horizontal edge of Ω, and homogeneous Neumann boundary conditions on the remaining edges of Ω. The solutions exhibit an interior layer along the line $x = 2y$ and a corner singularity at the origin due to incompatible boundary conditions. The influence of this singularity will diminish with decreasing ε.
The second set of examples, to which we will refer as *Problem* D, has homogeneous Dirichlet boundary conditions on the left vertical and upper horizontal edges of Ω and constant Dirichlet boundary conditions 1 on the lower horizontal and right vertical edges of Ω. The solutions exhibit an interior layer along the line $x = 2y$, a boundary layer at the right vertical edge of Ω, and two corner singularities at the origin and the top right corner of Ω due to incompatible bondary conditions. The influence of these singularities will again diminish with decreasing ε. The boundary layer will be stronger than the interior layer.

The discrete problem is given by equations (2.6) and (2.7) with

$$\delta_T = \frac{h_T}{2|\underline{a}|}\Big\{\coth(\frac{|\underline{a}|h_T}{2\varepsilon}) - \frac{2\varepsilon}{|\underline{a}|h_T}\Big\}$$

(cf. [5]; $|.|$ denotes the Euclidean norm in $\mathbb{R}^2$). All triangulations are created from the same initial triangulation $\mathcal{T}_0$ which consists of eight right-angled, isosceles triangles with short sides of length $\frac{1}{2}$ and long sides parallel to the line $x = y$. For the computation of errors and effectivity indices the exact solution u of problem (1.1) is always approximated by the solution of problem (2.6) corresponding to the triangulation $\mathcal{T}_9^{(u)}$ which is obtained by 9 steps of a uniform refinement of $\mathcal{T}_0$. This triangulation consists of

Table 1. Global and local effectivity indices for Problem N

ϵ	NT	E	$E_{\min}$	$E_{\max}$
1	4132	16.4	0.6	8.8
10^{-2}	1620	6.5	1.8	26.7
10^{-4}	1608	24.8	0.9	7.1
10^{-10}	1615	27.8	0.6	7.7

right-angled, isosceles triangles with short sides of length 2^{-10}. The discrete solution u_h is always computed by solving problem (2.6) on a locally refined triangulation $\mathcal{T}_5^{(l)}$ which is obtained as follows:

(1) Set $\mathcal{T}_0^{(l)} := \mathcal{T}_0$.
(2) Given $\mathcal{T}_k^{(l)}$ solve problem (2.6) on $\mathcal{T}_k^{(l)}$ and compute $\eta_{\mathrm{R},T}, T \in \mathcal{T}_k^{(l)}$, and $\eta_k := \max\limits_{T\in\mathcal{T}_k^{(l)}} \eta_{\mathrm{R},T}$.
(3) If $T \in \mathcal{T}_k^{(l)}$ satisfies $\eta_{T,R} \geq 0.5\eta_k$ it is cut into four new triangles by joining the midpoints of its edges (red refinement). Hanging nodes are eliminated by adding additional green and blue refinements as described in Sect. 4.1 of [12]. This gives $\mathcal{T}_{k+1}^{(l)}$.

Given u and u_h we can define global and local effectivity indices E and E_T, $T \in \mathcal{T}_5^{(l)}$, by

$$E := \left\{ \sum_{T\in\mathcal{T}_5^{(l)}} \eta_{\mathrm{R},T}^2 \right\}^{1/2} / |||u-u_h||| \quad , \quad E_T := \eta_{\mathrm{R},T} / |||u-u_h|||_T .$$

Set

$$E_{\min} := \min_{T\in\mathcal{T}_5^{(l)}} E_T \quad , \quad E_{\max} := \max_{T\in\mathcal{T}_5^{(l)}} E_T .$$

Figure 1 shows the triangulations $\mathcal{T}_5^{(l)}$ for Problem N. Table 1 gives the quantities $NT, E, E_{\min}$, and $E_{\max}$ for this set of examples. Here, NT is the number of triangles in $\mathcal{T}_5^{(l)}$. For all parameters ε we observed that

(1) E_T is maximal close to the origin,
(2) $E_T \sim 0.5E_{\max}$ in the vicinity of the interior layer,
(3) $E_T \sim 1$ away from the origin and the interior layer.

Figure 2 and Table 2 show the corresponding results for Problem D. For all parameters ε we observed that

(1) E_T is maximal close to the origin and the top right corner of Ω,
(2) $E_T \sim 0.9E_{\max}$ close to the boundary layer,

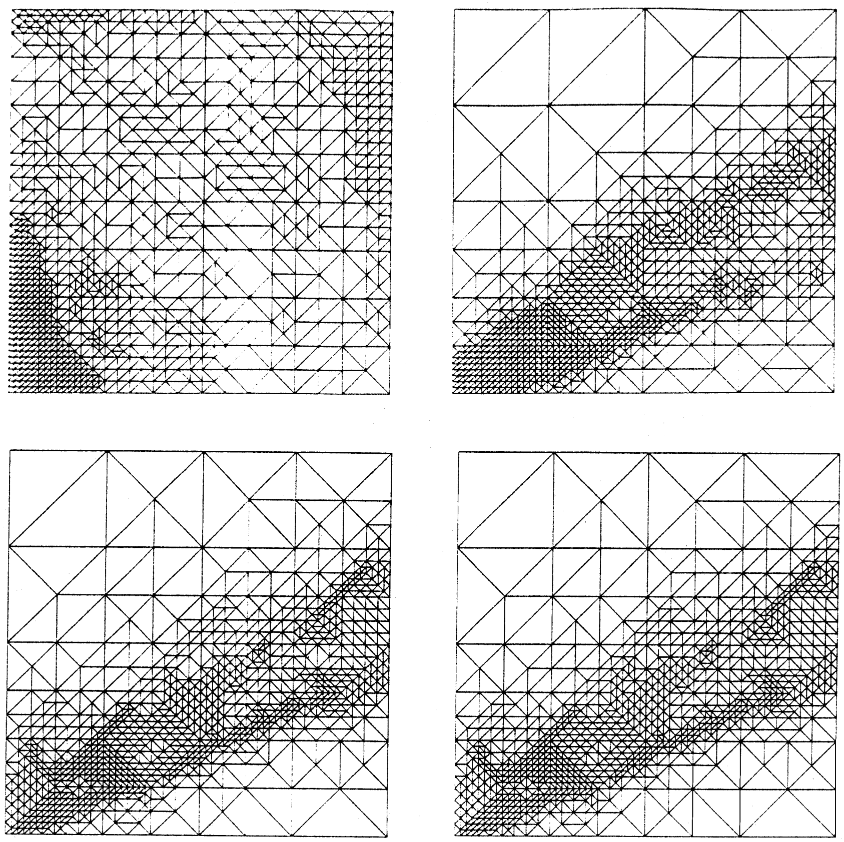

Fig. 1. Triangulations $\mathcal{T}_5^{(l)}$ for Problem N (top left: $\varepsilon = 1$; top right: $\varepsilon = 10^{-2}$; bottom left: $\varepsilon = 10^{-4}$; bottom right: $\varepsilon = 10^{-10}$)

Table 2. Global and local effectivity indices for Problem D

ϵ	NT	E	$E_{\min}$	$E_{\max}$
1	1464	42.8	1.1	16.9
10^{-2}	901	12.9	1.1	16.6
10^{-4}	956	128.6	1.0	15.8
10^{-10}	956	134.0	1.0	16.4

(3) $E_T \sim 0.3 E_{\max}$ close to the interior layer,
(4) $E_T \sim 1$ away from the singularities.

Consequently the triangulations are mostly refined at the boundary layer and at the two critical corners. The influence of the interior layer is not yet strong enough to enforce a considerable refinement there.

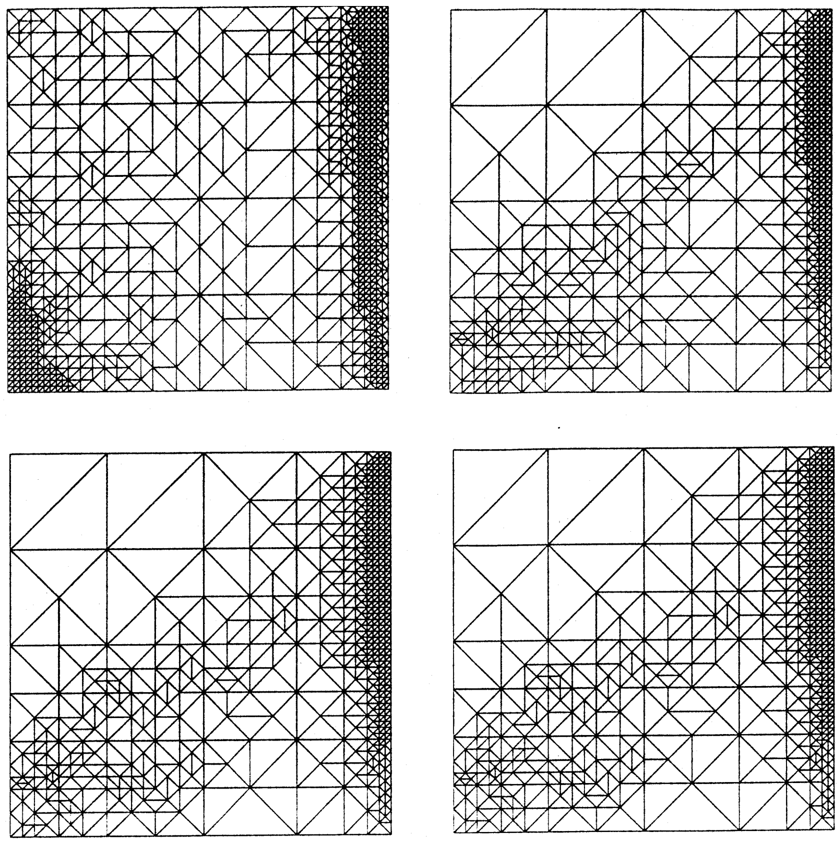

Fig. 2. Triangulations $\mathcal{T}_5^{(l)}$ for Problem D (top left: $\varepsilon = 1$; top right: $\varepsilon = 10^{-2}$; bottom left: $\varepsilon = 10^{-4}$; bottom right: $\varepsilon = 10^{-10}$)

Both sets of examples show that the estimator tends to over-estimate the error and that it is rather insensitive to variations of ε.

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References

1. Adams, R.A. (1975): Sobolev spaces. Academic Press, New York
2. Angermann, L. (1994): A posteriori Fehlerabschätzungen für Lösungen gestörter Operatorengleichungen. Habilitationsschrift, Universität Erlangen-Nürnberg
3. Babuška, I., Rheinboldt, W.C. (1978): Error estimates for adaptive finite element computations. SIAM J. Numer. Anal. 15, 736–754
4. Bank, R.E., Weiser, A. (1985): Some a posteriori error estimators for elliptic partial differential equations. Math. Comput. 44, 283–301

5. Christie, J., Griffiths, D.F., Mitchell, A.R., Zienkiewicz, O.C. (1976): Finite element methods for second order differential equations with significant first order derivatives. Int. J. Numer. Meth. Engrg 10, 1389–1396
6. Ciarlet, P.G. (1978): The finite element method for elliptic problems. North Holland, Amsterdam
7. Clément, P. (1975): Approximation by finite element functions using local regularization. RAIRO Anal. Numér. 9, 77–84
8. Franca, L.P., Frey, S.L., Hughes, T.J.R. (1992): Stabilized finite element methods I: Application to the advective-diffusive model. Comput. Math. Appl. Mech. Engrg. 95, 253–271
9. Medina, J., Picasso, M., Rappaz, J. (1996): Error estimates and adaptive finite elements for nonlinear diffusion-convection problems. Math. Models and Math. in Appl. Sci. 6, 689–712
10. Temam, R. (1984): Navier-Stokes equations. North Holland, Amsterdam
11. Verfürth, R. (1994): A posteriori error estimation and adaptive mesh-refinement techniques. J. Comput. Appl. Math. 50, 67–83
12. Verfürth, R. (1996): A review of posteriori error estimation and adaptive mesh-refinement techniques. Teubner-Wiley, Stuttgart
13. Verfürth, R.: Robust a posteriori error estimators for a singularly perturbed reaction-diffusion equation. Numer. Math. (to appear)